Empirical Asset Pricing
Assignment 3

Andi Boçari[1]

Selami Doğan Akansu[2]

email, student number, program

[1]andi.bocari@student.uni-tuebingen.de, 6303279, MSc Economics and Finance

[2]selami-dogan.akansu@student.uni-tuebingen.de, 6315870, MSc Economics and Finance

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MSc Economics and Finance

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

1 Introduction

The generalized method of moments (GMM) is an estimation technique in the field of asset pricing that evaluates how much deviation there is between sample averages of price and the discounted payoffs (also referred to as pricing errors). The GMM estimation method gives a statistical test that the underlying population means are zero (Cochrane, 2005b). The following paper presents three different approaches to applying the GMM in order to estimate the two parameters (β and γ) of the Consumption-Based basic asset pricing equation (BAPE) which employs a power utility function and gross returns. The derivation from the general BAPE is stated in Equation 4.5. A more detailed derivation is in the Appendix.

$$E\left[m_{t+1}X_{t+1}\right] = P_t \tag{1.1}$$

$$E\left[\beta\left[\frac{c_{t+1}}{c_t}\right]^{-\gamma}R_{t+1}-1\right] = 0 \tag{1.2}$$

The three approaches that aim to solve the minimization problem of the Objective-Based GMM are (1) performing on a grid search that relies on a fixed set of values for β and γ, (2) first-stage and (3) second-stage GMM that utilize the GMM and MINZ program libraries (Cliff, 2003). Following that is an assessment of how well each method fits the data, includes computing correlation between the Stochastic Discount Factor (SDF) of each method with the consumption growth. Different approaches include plotting model-implied mean returns against the market returns. Conducting hypothesis testing for the most convectional values for β =1 and γ =10 and commenting on the spread range of the confidence interval especially for γ. The conclusion will summarize all the points made and provide a critical evaluation that highlights the assumptions of the paper.

1.1 Methodology

1.1.1 Data Collection

The data used in this analysis consists of consumption growth from the 2^{nd} quarter 1959 to 4^{th} quarter 2022 and return data for ten portfolios (1^{st} size decile to 10^{th} size decile) of the same time period. The time-span of the data includes a range of economic events such as the oil crisis of 1973 and 1979, the Dot-Com bubble of 1999, the Financial Crisis of 2008, the COVID-19 pandemic of 2020/2021 and time-periods of economic upturn such as the 1960s to name a few. The impact of such a large data set is discussed in greater detail in the hypothesis testing section. These are the real world data required to solve the Objective-Based GMM optimisation problem, because the Consumption-Based BAPE only takes into consideration consumption growth and asset returns.

1.1.2 Objective-Based GMM Setup

The objective based GMM minimization problem is defined as the following

$$\hat{\beta} = \arg\min_{b} \; Q(\tilde{\beta}, \tilde{\gamma}) \tag{1.3}$$

$$Q(\tilde{\beta}, \tilde{\gamma}) \equiv \mathbf{g}_T(\tilde{\beta}, \tilde{\gamma})' \mathbf{W}_T \mathbf{g}_T(\tilde{\beta}, \tilde{\gamma}) \tag{1.4}$$

where $\mathbf{W}_T$ is the identity matrix and

$$\mathbf{g}_T(\tilde{\beta}, \tilde{\gamma}) = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\beta}(\frac{c_{t+1}}{c_t})^{-\tilde{\gamma}} R^1_{t+1} - 1\right] \\ \vdots \\ \frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\beta}(\frac{c_{t+1}}{c_t})^{-\tilde{\gamma}} R^{10}_{t+1} - 1\right] \end{bmatrix}_{(10\times 1)} \tag{1.5}$$

where, $g_T(\beta, \gamma)$ = Consumption Based basic asset pricing model with b_t = subjective discount factor (Cochrane, 2005a),

c_{t+1}/c_t = the consumption growth,

$b(c_{t+1}/c_t)$ = the stochastic discount factor,

R_t = Returns on investments.

1.1.3 Consumption-Based Asset Pricing Model

The consumption based asset pricing model revolves around the idea that an investor can either consume or invest into assets. The first-order condition of that decision-making process should result in the marginal utility loss of consuming less today but buying more of an asset, being equal to the marginal utility gain of consuming the assets' payoff in the future. If prices and payoffs don't conform to that rule then an investor should adjust their asset purchase accordingly. Therefore, an investor adjusts the consumption until the consumption lines up with the payoffs and the price. In this process, the marginal utility is used to discount the future payoff to equal the present price (Cochrane, 2005a).

The expression in each entry of the $g_t(\beta, \gamma)$ comes from its theoretical counterpart of gross returns (see Equation 4.5). The pricing error u_t is serially uncorrelated by assuming it is a martingale difference sequence. Additionally the assumption of stationary and ergodicity of the data generating process, allows replacing expectation with the sample means due to the law of large numbers property.

1.1.4 Moment Conditions and Identification Issues

The utility of the GMM estimation techniques is that it allows any expression for moment conditions as long as there is only one unique combination of parameters that fulfills those moment conditions. One could implement the same estimation approach in this paper using excess returns instead of gross returns, but as (Cochrane, 1996) realized there needs to be at least one more moment condition such as one additional risk-free-rate return in order to remedy the identification problem of over-fitting. The gross return approach in this paper, (*specifically the -1 in the expression for* $g_t(\beta, \gamma)$) prevents scaling up the expression. Hence there is no concern about not being able to uniquely identify the parameters in the given GMM estimation approach.

2 Evaluating the Objective Function

2.1 Grid Search Method

One method to find which values of β and γ minimize the objective function is to perform a grid search for a given range of plausible values for both parameters. The range of values for β is between 0.98 and 1.5 in steps of 0.004, and for values of γ it is from 5 to 30 in steps of 0.5.

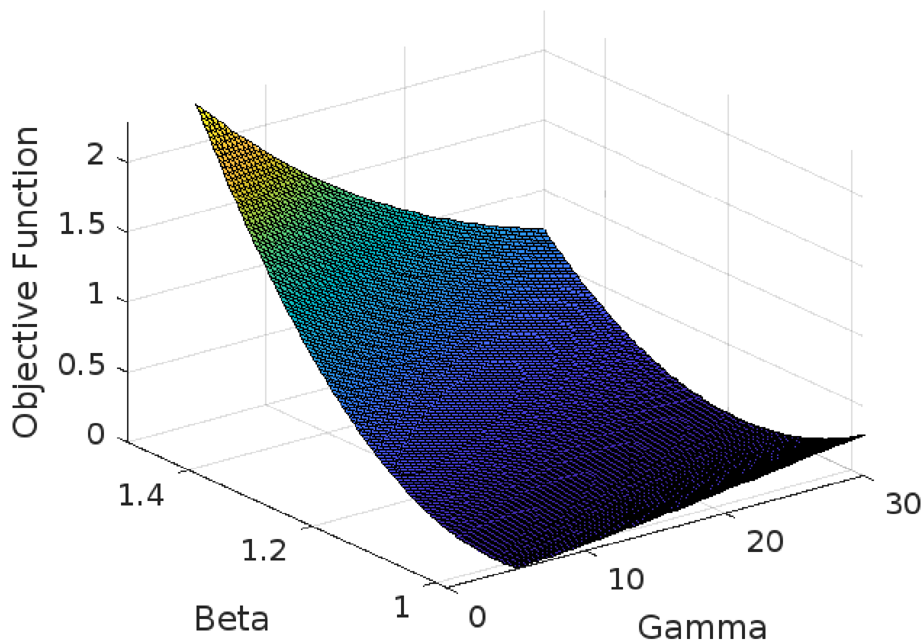


Figure 2.1: Objective Function 3D Span

The graphical representation is expressed in Figure 2.1 and its minimum value corresponds of 8.0674×10^{-6} to the following combination $\beta = 1.16$ and $\gamma = 26.5$ The value for b seems plausible, however the value for γ falls outside what is considered to be plausible which is the interval from 1 to 10, but still offers reasonable intuition for an investor behavior. The risk aversion formula in Equation 2.1 is one method to gauge the credibility a given value for γ.

$$\pi_z = 1/2 \times (\sigma_z)^2 \times \gamma \tag{2.1}$$

Even in the case of the most volatile return which had a standard deviation of 0.1325 the risk-aversion measures at most a value of 0.23. Meaning that the investor needs to remove 23 percent from their investment to protect themselves against this asset. This behaviour is certainly considered reasonable behaviour. A possible explanation for such a relatively high estimated value for γ could be the wide range of economic downturns and growth captured by our dataset. The numerous economic events are able to adequately capture the volatility of consumption growth in response to market changes.

Stochastic Discount Factor Correlation with Consumption Growth

The stochastic discount factor itself is a fundamental concept in the field of asset pricing. From the basic asset pricing formula (without considering a power utility function) $E[m_{t+1}X_{t+1}] = P_t$, the stochastic discount factor m_{t+1} is associated with incorporating all the risk associated with the certain payoff of the asset X_{t+1}. m_{t+1} is random variable because it remains unknown in time t. In the more specific context of the consumption based model with a power utility, the stochastic discount factor refers to the marginal rate of substitution. m_{t+1} reveals how much the investor is willing to substitute the consumption in $t+1$ for consumption in time t (Cochrane, 2005a).

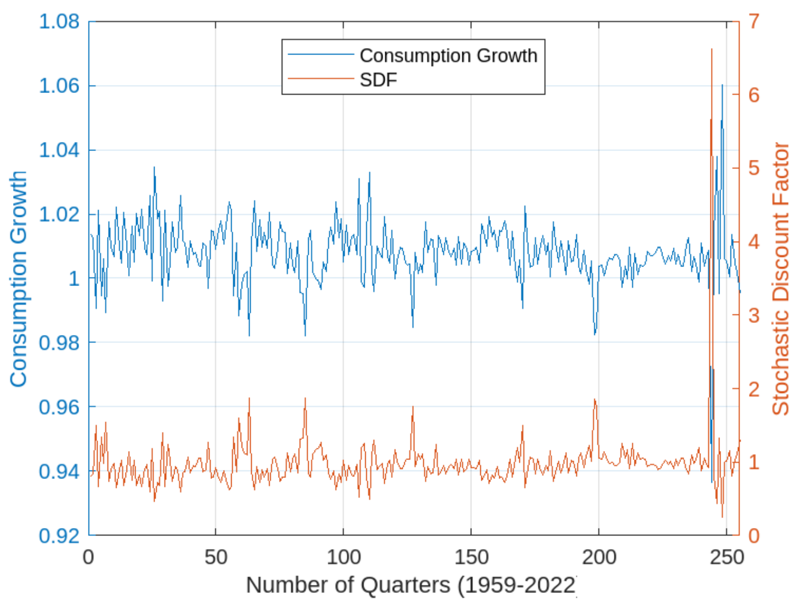


Figure 2.2: Objective Function 3D Span

As seen in Figure 2.2, the stochastic discount factor captures the consumption growth patter as measured by the high negative correlation value of -0.84. Since the SDF of the consumption-based BAPE with power utility function corresponds to the marginal rate of substitution, which in turn is inversely correlated with consumption growth.

Potential Drawbacks

One potential drawback of this methodology is how susceptible it can be to user's input. Had the range of values been more restrictive, there could be risk of failing to find any combination that provides a global minimum for the objective function. In addition to that, while reducing the amount of step to go run the objective function from one value to another would provide a more thorough result, it would come so at the cost at higher computing power.

2.1.1 One-Stage and Two-Stage GMM

Following the documentation by Cliff (2003), One-Stage and Two-Stage GMM estimations are performed. A noticeable difference when comparing the results with the grid search method is how more detailed parameter estimates can be. The grid search method is limited not only by the range but also by the size of the step from one value to another. Hence with the one and two-stage GMM approach the result extend up to five decimal points.

Hypothesis Testing

The GMM using the BAPE with utility function considers the value of $\beta = 1$ and $\gamma = 10$ to be optimal. A such a t-test statistic is performed to test the two different hypothesis $H_0 : \beta = 1$ and $H_0 : \gamma = 10$.

Table 2.1: One-Stage GMM Summary

	Coefficient	Standard Error	Null	t-stat	p-val	Confidence Interval
Beta	1.1707	0.0969	1.00	1.76	0.0782	[0.9807 , 1.3609]
Gamma	27.7188	14.8863	10.00	1.19	0.2339	[-1.4583 , 56.8960]

As shown in Table 2.1, the t-statistic for $H_0 : \beta = 1$ is 1.76 (and corresponding p-value of 0.0782). Since the t-statistic is smaller than the critical value of 1.96 (p-value is greater than 0.005), we fail to reject the null hypothesis, $H_0 : \beta = 1$ at a significance level of 0.05 using a two sided t-test. The t-statistic for $H_0 : \gamma = 10$ 1.19 (p-value=0.2339). Since $|t - statistic| < 1.96$ (p-value >0.005) we fail to reject the null hypothesis, $H_0 : \gamma = 10$ at a significance level of 0.05 using a two sided t-test.

Table 2.2: Two-Stage GMM Summary

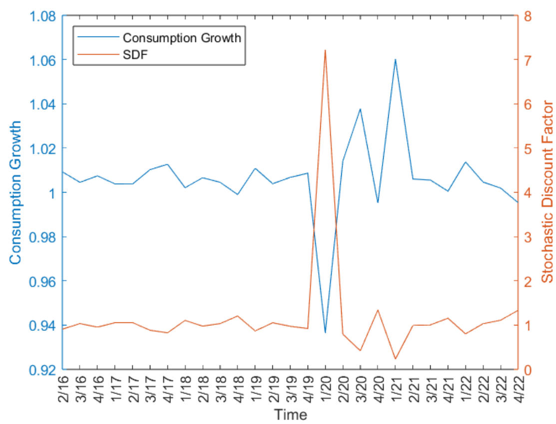
	Coefficient	Standard Error	Null	t-stat	p-val	Confidence Interval
Beta	1.1467	0.0982	1.00	1.49	0.1351	[0.9567 , 1.3369]
Gamma	23.1437	14.1520	10.00	0.93	0.3530	[-6.0334 , 52.3209]

As shown in Table 2.2 the t-statistic for $H_0 : \beta = 1$ is 1.49 (p-value= 0.1351), hence we fail to reject the null hypothesis, $H_0 : \beta = 1$ at a significance level of 0.05 using a two sided t-test. The t-statistic for

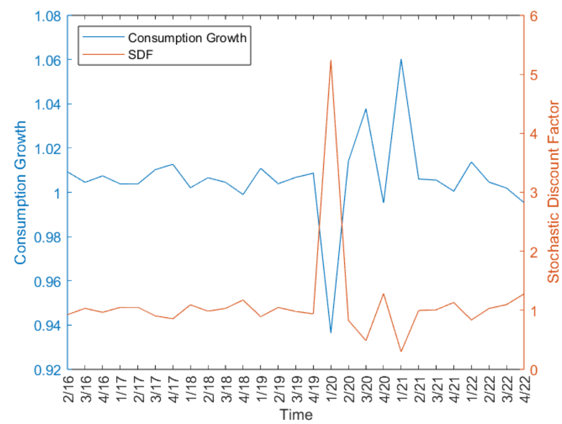
$H_0 : \gamma = 10$ is 0.93 (p-value=0.3530). Since $|t - statistic| < 1.96$ (p-value >0.005) we fail to reject the null hypothesis, $H_0 : \gamma = 10$ at a significance level of 0.05 using a two sided t-test.

Confidence Interval Interpretation

The values considered reasonable for β and γ are in the non-rejection region of the conducted which falls in line with prior research in the field. The confidence interval reveals the given values for $\bar{\beta}_k$ for which $H_0 : \beta = \bar{\beta}_k$ cannot be rejected at the significance level α =0.05 using a two-sided t-test. The bounds of the confidence interval for β for both one and two-stage GMM as seen in Table 2.1 and 2.2, can still be considered reasonable within the context of BAPE with a power utility function. The confidence interval for γ is extended beyond what can be considered reasonable, as a result the given model will not be able to reject a null of an unrealistically risk averse($\gamma > 30$) or an investor so risk-neutral that they are willing to afford to give up money without any compensation($0 < \gamma$). Having said that, the high confidence interval for γ shouldn't come as a surprise considering the wide time span of the data.



(a) One-Stage GMM



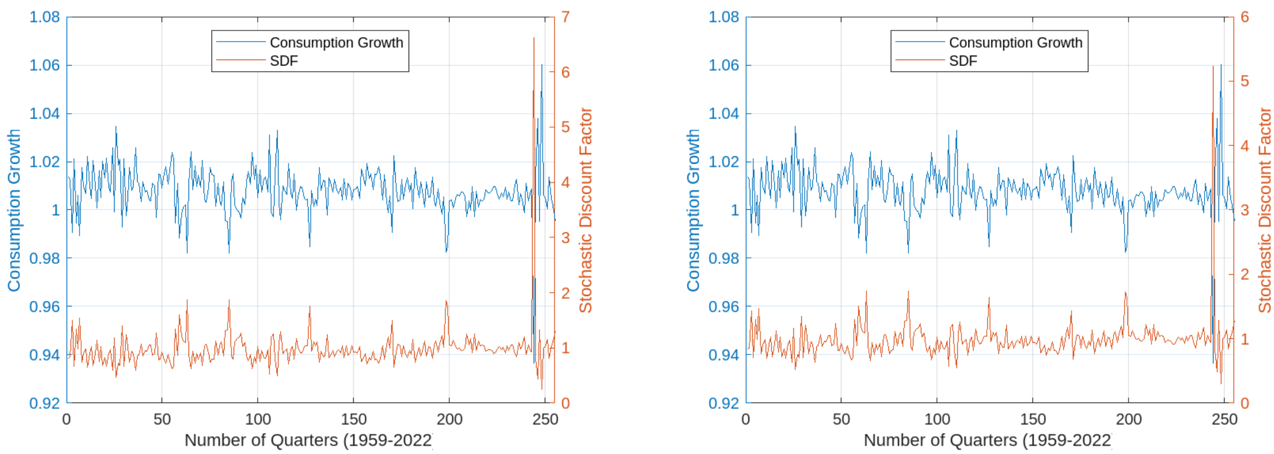
(b) Two-Stage GMM

Figure 2.3: Consumption Growth vs SDF During Covid-19 Pandemic

It can be argued consumption growth is not as volatile as market returns. A possible explanation could be that individual expenses cannot be as easily adjusted as market behaviour. An example could be rental, cellular, insurance or utility expenses that need a longer period to adjust due to contracts that fix future prices in the present. More recently it has been showed how that consumption is slow to recover after recessions such as the 2008 financial crisis (Verick and Islam, 2010). As a result while market responds quickly to macro-economic events, consumption level needs a longer time span to reflect the economics recessions or growth into the consumption path. The provided data encompasses numerous economic events as laid out in the introduction which is able to capture consumer growth response to said events. As a result the data is able to record how the risk-aversion is increased in economic downturns, and diminished in periods of economic growth, which is then reflected in the bounds of the confidence interval.

That is especially particular in the time of the Covid-19 pandemic as seen in Figure 2.3. Consumption growth severely diminishes in the first quarter of 2020 while the stochastic discount factor is greatly increased, as a result of a higher γ (*and additionally higher* β) as seen in graph.

Stochastic Discount Factor Correlation with Consumption Growth



(a) One-Stage GMM (b) Two-Stage GMM

Figure 2.4: Consumption Growth vs SDF

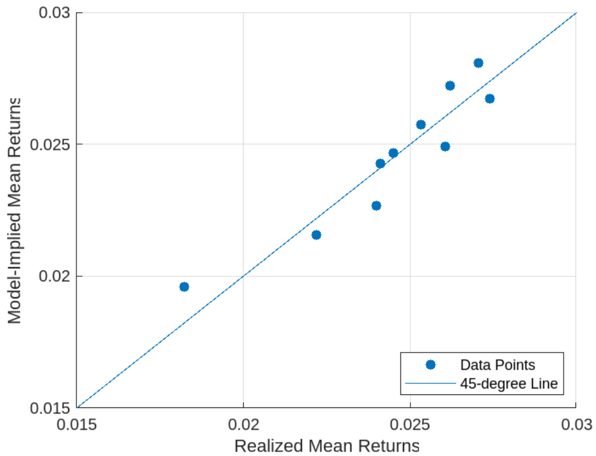
The correlation of the stochastic discount factor with the consumption growth in One-Stage and Two-Stage GMM follow in the same line as the grid search method, resulting in correlation values of -0.8268 and -0.87563 as seen in Figure 2.5. Again, the negative correlation come from the fact that the utility function of consumption is an upward-sloping with diminishing returns curve $u'(c_t) > 0$ *and* $u''(c_t) < 0$, while its marginal rate of substitution (or the SDF) with respect to the consumption growth will then be a convex curve. An increase in either direction will result in diminishing returns for the other. The empirical results for the three methods fall in line with that theory shows, suggesting that the SDF of the Consumption-Based BAPE is a reasonable method to gauge consumption growth.

Model Implied Returns

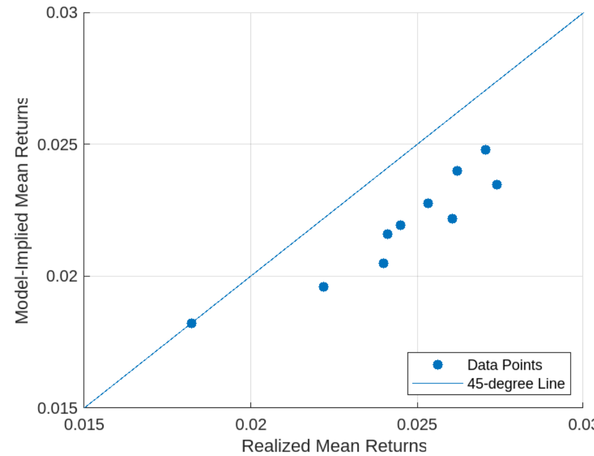
A robust way to see how the model fits the data is by comparing the mean returns of the sample with the model-implied returns, which can be computed as in Equation 2.2,

$$E(R^i) = \frac{1 - cov(m(\hat{\beta},\hat{\gamma}),R^i)}{E(m(\hat{\beta},\hat{\gamma}))} \tag{2.2}$$

where $m(\hat{\beta},\hat{\gamma})$is the stochastic discount factor.



(a) Result of one-stage GMM



(b) Result of two-Stage GMM

Figure 2.5: Realized vs Model-Implied Returns

Figure 2.5a and 2.5b display the realized mean returns against the model implied returns for both One-Stage and Two-Stage GMM. Each point corresponds to each individual return of the ten returns of the dataset. It is clearly seen that the One-Stage GMM model provides a better fit of the data to the BAPE with power utility function, as the points fall more closely on the 45-degree line than in the Two-Stage GMM. The Two-Stage GMM seems to underestimate the model-implied mean returns against the empirical returns, resulting in the points being shifted to the right of the 45-degree line.

3 Conclusion

The BAPE with power utility is an approach to price assets. By relying on the GMM which assumes that the pricing error is a martingale difference sequence, stationarity and ergodicity of the data generating process the law of large numbers can be invoked which turn theoretical expectations into sample means. As a result the moment conditions can be computed with readily available data on consumption growth and asset returns. The issue of uniquely identifying the parameters of the moment conditions is mitigated by taking gross returns.

The methods to run the GMM for BAPE with utility function that were discussed in this paper were the grid search, One-Stage and Two-Stage GMM. The grid search is a straightforward approach where different combinations of β and γ give the lowest value of the objective function but is unable to pinpoint the exact values that minimizes the objective, nor produce standard errors to be used for hypothesis testing as is the case for One-Stage and Two-Stage GMM.

The hypothesis testing showed that normally accepted values for the parameters lied in the confidence interval but so did values which are less intuitive. Having said that the wide spread of data was able to adequately capture the volatility of consumer in response to the market changes as a result of macroeconomic events taking place.

The One-Stage GMM in particular was able to provide implied mean returns that more or less fell in the line with the data shows.

3.0.1 Critical evaluation

The GMM estimation of the Consumption-Based BAPE with a power utility is a practical approach when it comes to determining its parameters β and γ. Nonetheless the entire analysis is based on the following set of assumptions.

Data Availability

The model is able to fit the data properly as long as there is enough information to capture consumption's volatility in response to asset returns. Had the data included only certain years like 2005-2010, the parameters would then become too specified around the 2008 Financial Crisis. As a result even higher estimates for γ *and consequently even wider confidence interval* would be observed due to the omission of data in periods of economic growth which would counteract the risk-averse behaviour observed during economics crises.

Pricing Error and Data Generating Process

One set of important assumptions that underpin the entire analysis is the assumption that the pricing errors behave as a martingale difference sequence and the data generating process for the returns and consumption is stationary and ergodic. Should this assumptions be proven to be wrong then the entire analysis would be inconclusive as then the law of large numbers cannot be invoked to replace the theoretical moment expectation with sample means.

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Verick, S. and Islam, I. (June 2010). The great recession of 2008-2009: Causes, consequences and policy responses. *SSRN Electronic Journal* [online], p. 8. DOI: 10.2139/ssrn.1631069.

4 Appendix

4.1 Deriving The Consumption-Based BAPE Equation for Gross Returns

$$E\left[m_{t+1}X_{t+1}\right] = P_t \tag{4.1}$$

$$E\left[m_{t+1}\frac{X_{t+1}}{P_t}\right] = 1 \tag{4.2}$$

$$E\left[m_{t+1}R_{t+1}\right] = 1 \tag{4.3}$$

$$E\left[m_{t+1}R_{t+1} - 1\right] = 0 \tag{4.4}$$

$$E\left[\beta\left[\frac{c_{t+1}}{c_t}\right]^{-\gamma}R_{t+1} - 1\right] = 0 \tag{4.5}$$

4.2 One-Stage GMM Result

```
================================================================
                     GMM ESTIMATION PROGRAM
================================================================

2 Parameters, 10 Moment Conditions
10 Equation Model, 1 Instruments
255 Observations
1 Passes, Max., 100 Iterations/Pass
Search Direction:          Gauss-Newton
Derivatives:               Numerical
Initial Weighting Matrix: I
Weighting Matrix:          N/A
Spectral Density Matrix:  White
```

```
                    STARTING GMM ITERATION  1
  Weights Attached to Moments
        Moment  1   Moment  2   Moment  3   Moment  4   Moment  5   Moment  6
Var1       0.1002      0.1001      0.1003      0.1001      0.1001      0.1000
Var2       0.1018      0.1012      0.1009      0.1004      0.1002      0.0997

       Moment  7   Moment  8   Moment  9   Moment 10
Var1      0.1000      0.1000      0.0998      0.0994
Var2      0.1000      0.0993      0.0987      0.0979

Line Minimization Using STEP2.M
Ill-Conditioning Tolerance Set to 1000

 ---------------------------------------------------------------

        ITER      cond(H)  *    Step                 Obj Fcn
  1       1.46e+04 *  0.810000     0.0013884312
  2       5.99e+03 *  1.000000     0.0000077726
  3       6.12e+03 *  1.000000     0.0000077720
  4       6.12e+03 *  1.000000     0.0000077720
  CONVERGENCE CRITERIA MET: Change in Objective Function

            EVALUATING S at FINAL PARAMETER ESTIMATES

  ------------------  GMM PARAMETER ESTIMATES  -----------------
  Parameter          Coeff     Std Err   Null   t-stat    p-val
  parameter 1     1.170797    0.096981   1.00     1.76   0.0782
  parameter 2    27.718862   14.886304   5.00     1.53   0.1270

  -------------------  GMM MOMENT CONDITIONS  ------------------
                     Moment     Std Err   t-stat    p-val
     Moment  1    -0.000943    0.002372    -0.40   0.6909
     Moment  2    -0.000952    0.001175    -0.81   0.4175
     Moment  3     0.000711    0.000995     0.71   0.4748
     Moment  4    -0.000390    0.001096    -0.36   0.7219
     Moment  5     0.001171    0.000993     1.18   0.2380
```

```
    Moment  6    -0.000119   0.001201    -0.10   0.9212
    Moment  7    -0.000112   0.001379    -0.08   0.9351
    Moment  8     0.001319   0.000904     1.46   0.1442
    Moment  9     0.000638   0.000906     0.70   0.4812
    Moment 10    -0.001331   0.001252    -1.06   0.2879

     J-stat = 5.9569    Prob[Chi-sq.(8) > J] = 0.6521
===============================================================
```

4.3 Two-Stage GMM Result

```
===============================================================
                     GMM ESTIMATION PROGRAM
===============================================================

2 Parameters, 10 Moment Conditions
10 Equation Model, 1 Instruments
255 Observations
2 Passes, Max., 100 Iterations/Pass
Search Direction:          Gauss-Newton
Derivatives:               Numerical
Initial Weighting Matrix: I
Weighting Matrix:          Optimal
Spectral Density Matrix:  White


                   STARTING GMM ITERATION  1
 Weights Attached to Moments
       Moment  1   Moment  2   Moment  3   Moment  4   Moment  5   Moment  6
Var1      0.1002      0.1002      0.1003      0.1001      0.1002      0.1000
Var2      0.1017      0.1011      0.1008      0.1003      0.1002      0.0997

      Moment  7   Moment  8   Moment  9   Moment 10
Var1     0.1000      0.0999      0.0998      0.0994
Var2     0.0999      0.0994      0.0988      0.0981
```

```
---------------------------------------------------------------

      ITER      cond(H)  *    Step                  Obj Fcn
  1       1.53e+04 *  0.729000     0.0051959501
  2       7.31e+03 *  1.000000     0.0000313995
  3       6.10e+03 *  1.000000     0.0000077720
  4       6.12e+03 *  1.000000     0.0000077720
  CONVERGENCE CRITERIA MET: Change in Parameter Vector
         Var1      Var2
 b1    1.1708   27.7189

                   STARTING GMM ITERATION  2
  Weights Attached to Moments
        Moment  1   Moment  2   Moment  3   Moment  4   Moment  5   Moment  6
Var1      -0.1213     -0.1905      2.7982     -0.7355     -2.6065      1.0156
Var2       0.0939     -0.1449      3.1035     -1.0922     -2.6155      0.4083

       Moment  7   Moment  8   Moment  9   Moment 10
Var1      5.4145     -4.2707      0.0834     -0.3872
Var2      7.3958     -4.9777     -0.5455     -0.6257


---------------------------------------------------------------

      ITER      cond(H)  *    Step                  Obj Fcn
  1       6.25e+02    1.000000     0.0233231022
  2       7.29e+02    1.000000     0.0232993290
  3       7.61e+02    1.000000     0.0232979538
  4       7.69e+02    1.000000     0.0232978597
  5       7.72e+02    1.000000     0.0232978532
  6       7.72e+02    1.000000     0.0232978528
  CONVERGENCE CRITERIA MET: Change in Objective Function

            EVALUATING S at FINAL PARAMETER ESTIMATES

  ------------------  GMM PARAMETER ESTIMATES  -----------------
```

Parameter	Coeff	Std Err	Null	t-stat	p-val
parameter 1	1.146786	0.098224	1.00	1.49	0.1351
parameter 2	23.143715	14.152023	10.00	0.93	0.3530

------------------- GMM MOMENT CONDITIONS -------------------

	Moment	Std Err	t-stat	p-val
Moment 1	0.002276	0.008395	0.27	0.7863
Moment 2	0.002209	0.007506	0.29	0.7686
Moment 3	0.003912	0.006413	0.61	0.5418
Moment 4	0.002559	0.006753	0.38	0.7047
Moment 5	0.003878	0.007315	0.53	0.5960
Moment 6	0.002511	0.006413	0.39	0.6954
Moment 7	0.002562	0.006715	0.38	0.7028
Moment 8	0.003486	0.007428	0.47	0.6388
Moment 9	0.002584	0.007133	0.36	0.7172
Moment 10	0.000037	0.007933	0.00	0.9963

J-stat = 5.9410   Prob[Chi-sq.(8) > J] = 0.6538

==