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Coplanarity of rooted spanning-tree vectors

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Abstract

Employing a recent technology of tree surgery, we prove a "deletion–constriction" formula for products of rooted spanning-trees on weighted directed graphs that generalizes deletion–contraction on undirected graphs. The formula implies that, letting $\tau_{\mathrm{x}}^{\varnothing}$, τ_{x}^{+}, and τ_{x}^{-} be the rooted spanning-tree polynomials obtained, respectively, by removing both directed edges between two vertices, or by forcing the tree to pass through either edge, the vectors $(\tau_{\mathrm{x}}^{\varnothing}, \tau_{\mathrm{x}}^{+}, \tau_{\mathrm{x}}^{-})$ are coplanar for all roots x. We deploy the result to give an alternative derivation of a recently found mutual linearity of stationary currents of Markov chains. We generalize deletion–constriction and current linearity for two pairs of edges and conjecture that similar results may hold for arbitrary subsets of edges.

Keywords Weighted directed graphs · Rooted spanning-trees · Deletion–contraction · Markov chains

Mathematics Subject Classification 05C05 · 05C20 · 05C31 · 60J27

1 Introduction

Spanning-trees and the properties of their associated polynomials are ubiquitous in physics, from electrical circuit theory [1, 2] to equilibrium statistical mechanics [3,

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4] and quantum field theory [5–7]. In the context of the statistical mechanics of irreversible systems based on Markov processes, undirected unweighted spanning-trees can be used to define basis of observables that characterize stationary and transient behaviour [8]; spanning-trees of (unweighted) digraphs allow one to derive the statistical large deviation rate function of the currents [9, 10]. More importantly for what follows, spanning-trees of weighted directed graphs provide the stationary probability distribution [11–14] (see also [15, 16] for recent reviews and [17–23] for recent applications).

In the present work, we will be concerned with this latter class of graphs, with the edges connecting vertices carrying different weights in either direction, which may represent transition rates of a Markov process among vertices.

For undirected graphs (same weight in both directions), the deletion–contraction formula [24, 25] allows one to express the spanning-tree polynomial of a graph in terms of those of smaller graphs obtained by removing edges or by shrinking them down to a vertex. An analogy can be made to electrical resistor networks where a resistance is either put to 0 or to $+\infty$ [26]. For directed graphs, spanning-tree polynomials need to be rooted to account for the directionality of the weights, and such "topological" contraction becomes contrived in that upon contraction of edges, we lose information about its direction. In the electrical network analogy, this case is similar to the inclusion of diodes, which conduct electricity preferentially in one direction [27], in contrast to resistors or coils. In this context, short-circuiting the extremal vertices of the diode would result in the loss of information about its directionality.

Our first, smaller contribution is to prove that properly constrained rooted spanning-tree polynomials for weighted directed graphs (as per Definition 1) satisfy a generalized "logical" acceptation of deletion contraction compatible with (but less powerful than) the above acceptation (Theorem 1). We call this formula deletion–constriction. While the deletion–contraction formula in undirected weighted graphs can be represented in terms of graphs where the edge is either removed or shrunk to a unique vertex, no such visual representation is possible for constriction.

Our second and most important contribution is a "second-order" deletion–constriction formula involving products of rooted spanning-tree polynomials (Theorem 2). The formula is graphically represented in Fig. 4 and explained in its caption. To prove it, we employ the technology of tree surgery introduced in [28, App. D] in an analysis of nonequilibrium response for Markov processes.

A suggestive corollary of the formula is that certain 3-vectors representing deletion–constriction through either directed edges between two vertices span a 2-dimensional space independently of the topology of the graph and the number of vertices (Theorem 3, see Fig. 1). Before proving the main results, this latter fact is shown in the next section by a simple example.

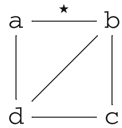
As a third contribution, we use this formulation to rederive a recent result by the authors [29] about mutual linearity of stationary mean currents of continuous-time Markov chains. We then expand on these ideas, showing that certain 9-vectors representing the deletion–constriction of directed edges between two pairs of vertices span a 3-dimensional space, and use this fact to suggest the validity of mutual linearity among three Markov currents, under technical assumptions whose necessity remains to be fully understood.

Finally, we conjecture that the 3^n-vectors representing the deletion–constriction through directed edges between n pairs of vertices span a $(n+1)$-dimensional space.

Below, the scalar product $\cdot$ is the standard Euclidean.

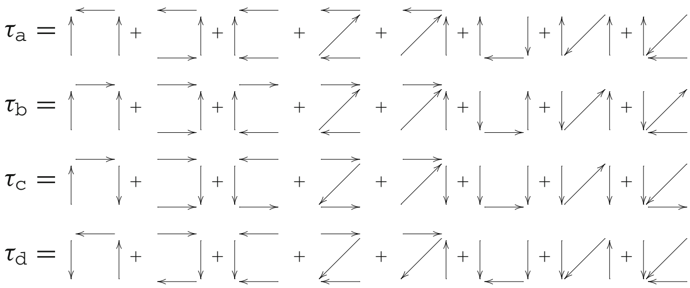
1.1 Example of coplanarity

Let us first give a very simple explicit example of coplanarity. Consider the graph



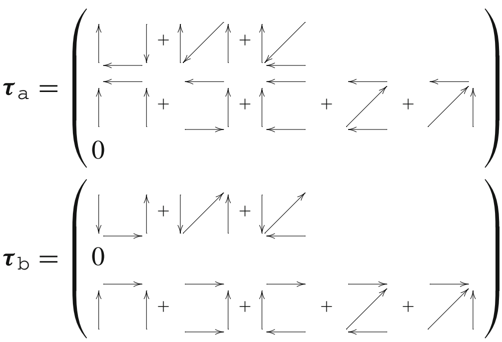
(1)

where $\star$ denotes the edges ($\mathtt{a} \to \mathtt{b}$ and $\mathtt{b} \to \mathtt{a}$) that will be pinned and subjected to deletion–constriction. The rooted spanning-tree polynomials are, in words (see below for mathematical definitions), the sum of the product of the weights along the directed edges of a sub-graph, the sum being over all sub-graphs where exactly one path connects any vertex to a fixed root vertex. In the present example, they are given by the diagrammatic representations:

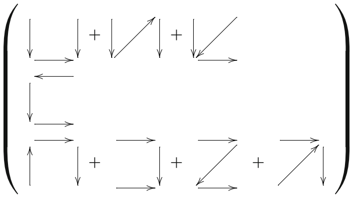


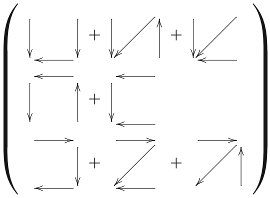
(2a) (2b) (2c) (2d)

For each rooted spanning-tree polynomial, we collect the terms according to the conditions: those that do not contain any pinned edges; those that contain $\mathtt{a} \leftarrow \mathtt{b}$; those that contain $\mathtt{a} \to \mathtt{b}$. We collect the sub-polynomials so obtained in 3-vectors. Using the diagrammatic representation, we obtain:

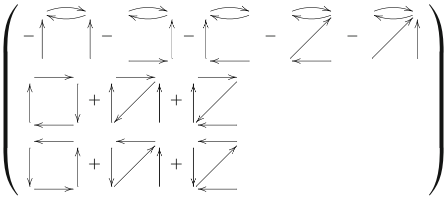


(3a) (3b)

$$\boldsymbol{\tau}_{\mathtt{c}} =$$  (3c)

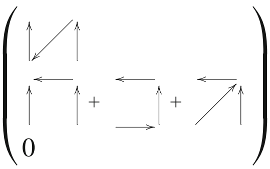
$$\boldsymbol{\tau}_{\mathtt{d}} =$$  . (3d)

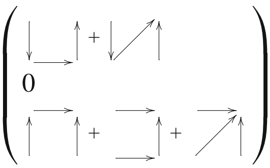
These vectors might appear to be quite arbitrary. Yet, given that there are four vectors spanning a three-dimensional space, at least one of them is not linearly independent of the others. In fact, two of them are not: the four vectors are coplanar. This can be shown by defining the vector

$$\boldsymbol{\sigma}_{\mathtt{a}\leftrightarrow\mathtt{b}} :=$$  (4)

and showing by a tedious hand calculation that $\boldsymbol{\sigma}_{\mathtt{a}\leftrightarrow\mathtt{b}} \cdot \boldsymbol{\tau}_{\mathtt{x}} = 0$, $\forall \mathtt{x} \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$.

A useful variant of this result is obtained when we further constrain spanning-trees to pass through or not pass through a given directed edge. Let us, for example, consider the subset of spanning-trees that pass through the directed edge $\mathtt{c} \to \mathtt{b}$. We now obtain the new vectors

$$\boldsymbol{\tau}_{\mathtt{a}}^{/\!/\mathtt{c}\to\mathtt{b}} =$$  (5a)

$$\boldsymbol{\tau}_{\mathtt{b}}^{/\!/\mathtt{c}\to\mathtt{b}} =$$  (5b)

$$\boldsymbol{\tau}_{\mathtt{c}}^{/\!/\mathtt{c}\to\mathtt{b}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{5c}$$

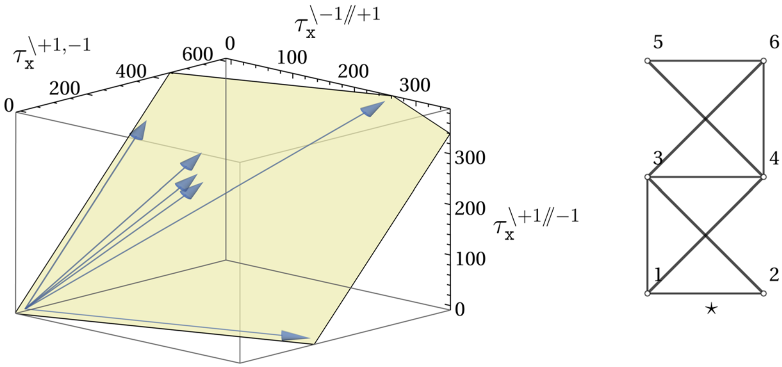
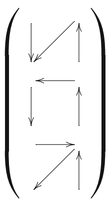


Fig. 1 Coplanarity of spanning-tree vectors defined in Eq. (17), for the graph depicted on the right. The pinned edges are $1 \leftarrow 2$ and $1 \rightarrow 2$ and the depicted vectors $\boldsymbol{\tau}_{\mathtt{x}}$ are rooted at each vertex $\mathtt{x}$ of the graph, all of them lying on the plane in yellow. Here the weights were drawn randomly between 0.5 and 3, so that the vectors $\boldsymbol{\tau}_{\mathtt{x}}$ can be visualized in a 3D plot

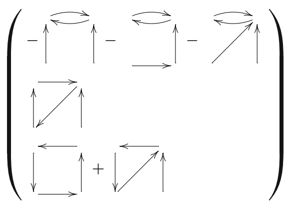
$$\boldsymbol{\tau}_{\mathtt{d}}^{/\!\!/\mathtt{c}\to\mathtt{b}} =$$



. (5d)

Similarly, we find a vector that is orthogonal to all three of them starting from $\boldsymbol{\sigma}_{\mathtt{a}\leftrightarrow\mathtt{b}}$ and removing all entries that do not pass through $\mathtt{c} \to \mathtt{b}$, yielding:

$$\boldsymbol{\sigma}_{\mathtt{a}\leftrightarrow\mathtt{b}}^{/\!\!/\mathtt{c}\to\mathtt{b}} =$$



. (6)

Once again it can be checked by hand calculation that this latter vector is orthogonal to the ones above.

Figure 1 suggests that coplanarity holds also for more complicated graphs, as we are going to prove.

2 Setting, methods and results

2.1 Rooted spanning-trees and deletion–constriction

We consider a weighted directed graph $G = (X, E, r)$, where $\mathtt{x} \in X$ are the vertices and each edge $\mathtt{x} \to \mathtt{y} \in E \subseteq X \times X$ is an ordered pair of vertices, and where

$r : E \to \mathbb{R}_{\geq 0}$ is a non-negative weight. For notational simplicity, we assume that if $\mathtt{x} \to \mathtt{y}$ belongs to E, $\mathtt{y} \to \mathtt{x}$ also belongs to E (if it were not present initially, one adds it to E and sets its weight to 0). For each pair of edges[1] $\mathtt{x} \to \mathtt{y}$ and $\mathtt{y} \to \mathtt{x}$, we choose a reference forward orientation and denote $\pm e$ the directed edges connecting a source vertex $\mathtt{s}(\pm e) \in X$ to a target vertex $\mathtt{s}(\mp e)$. In general, different weights are associated with the edges $+e$ and $-e$. By extension, r will also be the function that takes the product of weights in a subset of edges, on the assumption $r(\varnothing) = 1$.

We assume the graph to be irreducible, that is, there exists a path $\mathtt{x} \rightsquigarrow \mathtt{y}$ of non-zero weight between any two vertices. Notice that in general we do not need to assume reversibility, that is, that every edge $+e$ with a positive weight (positive transition rate) has a pair $-e$ with nonvanishing weight.

We use below a reference case: when $r(+e) = r(-e)$ for all edges in E, then the graph is equivalent to an undirected graph with same vertices and (twice less) edges $\bar{E} = \{x—y, x \to y \in E\}$. In this case, we denote by $\bar{e}$ the edges and by $r(\bar{e}) = r(+e) = r(-e)$ their weight.

In an undirected graph, a tree T is a subset of edges that contains no cycle; it is spanning if it connects all vertices. In a directed graph, a rooted spanning-tree $T_{\mathtt{x}}$ with root $\mathtt{x}$ is a spanning-tree such that each edge is directed along the unique path that leads to the root. For a given graph, we denote $\mathscr{T}$ the set of spanning-trees and $\mathscr{T}_{\mathtt{x}}$ the set of directed spanning-trees rooted at $\mathtt{x}$. Notice that the removal of one edge from a rooted spanning-tree disconnects it into two non-spanning-trees in a so-called forest of the graph; we call basins the two connected sets of vertices spanned by these trees.

We now define a generalization of the rooted spanning-tree polynomial.

Definition 1 Let A and B be two nonintersecting subsets of E and $\mathscr{T}_{\mathtt{x}}$ be the set of directed spanning-trees rooted at $\mathtt{x}$. *The conditioned rooted spanning-tree polynomial* is defined as:

$$\tau_{\mathtt{x}}^{\setminus A /\!/ B} = \sum_{\substack{T_{\mathtt{x}} \in \mathscr{T}_{\mathtt{x}} \\ A \cap T_{\mathtt{x}} = \varnothing \\ B \subseteq T_{\mathtt{x}}}} r(T_{\mathtt{x}}). \tag{7}$$

Clearly it is a function of some of the weights. In particular, the sum spans over trees that do not contain edges in A and that must contain edges in B. We refer to A as being deleted and B as being constricted. Notice that the above polynomial is non-negative. Furthermore, $\tau_{\mathtt{x}}^{\setminus A /\!/ B}$ does not depend on any of the weights in A and is multilinear in all the weights in B. Consequently, we can factorize the weights in B and rewrite

$$\tau_{\mathtt{x}}^{\setminus A /\!/ B} = \dot{\tau}_{\mathtt{x}}^{\setminus A /\!/ B} r(B) \tag{8}$$

[1] For simplicity, we assume that a directed edge $\mathtt{x} \to \mathtt{y}$ is only present once in E. Multiple directed edges e and e' between the same two vertices of the graph could be considered, but since our results refer to linear relations, it turns out that this is equivalent to considering single edges with weights $r_{\pm e} + r_{\pm e'}$.

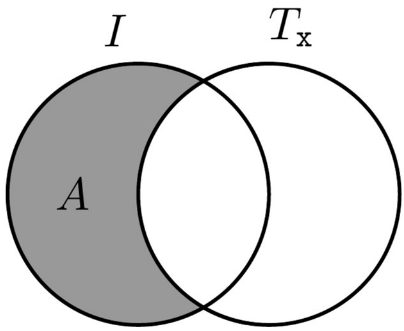


Fig. 2 The right circle is the set of edges of the rooted spanning-tree T_{x}, which by deletion does not contain the edges in A (grey area), which is a subset of I; due to constriction, $I\backslash A$ must belong to T_{x}. It follows that, given I and T_{x}, A is uniquely determined (possibly empty)

where now the rescaled spanning-tree polynomial $\dot{\tau}_{\mathrm{x}}^{\backslash A/\!\!/B}$, defined by this relation, does not depend on any of the weights in A and B, and we remind that $r(B)$ represents the product of the weights of the edges in subset B.

The (unconditioned) rooted spanning-tree polynomial $\tau_{\mathrm{x}} = \sum_{T_{\mathrm{x}}} r(T_{\mathrm{x}})$ is obtained for $A = B = \varnothing$. Notice that in the case the graph is undirected, $\tau_{\mathrm{x}}^{\backslash A/\!\!/B} = \tau^{\backslash A/\!\!/B}$ does not depend on the root. Furthermore, notice that one can obtain $\tau_{\mathrm{x}}^{\backslash A/\!\!/B}$ from τ_{x} by taking derivatives with respect to $\log r(e)$ for all $e \in B$, and by evaluating at $r(e') \equiv 0$ for all $e' \in A$.

The following result holds for such polynomials.

Theorem 1 *Let $I \subseteq E$ be any subset of the edge set. Then, the unconditioned rooted spanning-tree polynomial can be given in terms of conditioned rooted spanning-tree polynomials as*

$$\tau_{\mathrm{x}} = \sum_{A \subseteq I} \tau_{\mathrm{x}}^{\backslash A/\!\!/(I\backslash A)} \tag{9}$$

where A ranges among all subsets of I (including $\emptyset$ and I itself).

Proof Using Definition 1 with $B = I\backslash A$ and swapping sums, we obtain

$$\sum_{A \subseteq I} \tau_{\mathrm{x}}^{\backslash A/\!\!/(I\backslash A)} = \sum_{T_{\mathrm{x}}} r(T_{\mathrm{x}}) \sum_{\substack{A \subseteq I \\ A \cap T_{\mathrm{x}} = \varnothing \\ I\backslash A \subseteq T_{\mathrm{x}}}} 1\,. \tag{10}$$

The formula follows from the fact that, for given T_{x} and I, there is a unique subset A that meets the criteria of the right-hand sum $A = I\backslash(I \cap T_{\mathrm{x}})$ (see diagram in Fig. 2), in such way that $\sum_{\substack{A \subseteq I \\ A \cap T_{\mathrm{x}} = \varnothing \\ I\backslash A \subseteq T_{\mathrm{x}}}} 1 = 1$. □

The notations \ and // are borrowed from the deletion–contraction paradigm (\, /) for undirected graphs, by which new graphs are obtained either by deleting an edge (removing it entirely) or by contracting it (shrinking its extremal vertices as a single vertex and removing any resulting loop). The inverse operations involve adding an edge between existing vertices or splitting a vertex into two vertices connected by an edge. Denoting $\dot{\tau}$[2] the undirected spanning-tree polynomials, the deletion–contraction formula [25] states that

$$\dot{\tau} = \dot{\tau}^{\setminus \bar{e}} + r(\bar{e})\dot{\tau}^{/\bar{e}} \tag{11}$$

where the first term in the sum represents the spanning-tree polynomial of the graph obtained by deleting $\bar{e}$ [3] and the second term represents the spanning-tree polynomial of the graph obtained by contracting it. Provided that the spanning-tree polynomial of a graph consisting only of disconnected vertices is 1, this formula is constructive (it functions as recursive definition). In fact, it is a specialization to trees of a more general result applying to the Tutte polynomial, that also accounts for other combinatorial properties of graphs beyond trees (e.g. one of its evaluations yields the chromatic polynomial) [24, 30].

Taking $I = \{+e\}$ in Theorem 1, we find that (up to a factor $r(+e)$) the same formula applies to the rooted spanning-tree polynomial

$$\tau_{\mathrm{x}} = \tau_{\mathrm{x}}^{\setminus +e} + \tau_{\mathrm{x}}^{/\!\!/ +e}, \tag{12}$$

simply stating the obvious fact that a spanning-tree either contains or does not contain a given edge. We propose for this and similar formulas the term deletion–constriction. While the deletion operation is conceptually the same as in the standard deletion–contraction paradigm (i.e. it refers to the complete removal of an edge, whether directed or undirected), the constriction operation differs fundamentally from contraction. Constriction corresponds to forcing the spanning-tree to pass through a given directed edge, which ensures that its associated weight appears in the spanning-tree. In contrast, contraction refers to the topological merging of the two vertices connected by the edge; in this case, the resulting spanning-tree becomes independent of the edge weight. As such, we term “logical" the operation of deletion–constriction as opposed to the topological acceptation of deletion–contraction for undirected graphs, in that it does not entail the existence of a graph where some “shrinking” produces the desired polynomials.

There is one special case in which constriction becomes contraction: when the root is the source or target of the deleted-constricted edge itself. In this case, we have that

$$r(+e)\tau_{\mathrm{s}(+e)}^{/\!\!/ -e} = r(-e)\tau_{\mathrm{s}(-e)}^{/\!\!/ +e} = r(-e)r(+e)\dot{\tau}_{\mathrm{e}}, \tag{13}$$

where $\dot{\tau}_{\mathrm{e}}$ is the rooted spanning-tree polynomial of the graph obtained by shrinking edges $+e$ and $-e$ to a unique vertex e, rooted at that vertex (the above equality is intuitive, but see [31] for an explicit algebraic proof).

[2] We emphasize that the dot in $\dot{\tau}$ is not an operator, rather it is notation. This polynomial is not the same as $\dot{\tau}_{\mathrm{x}}^{\setminus A /\!\!/ B}$ in Eq. (8), nor the subsequent $\dot{\tau}_{\mathrm{e}}$.

[3] We remind that $\bar{e}$ is an undirected edge with equal weights in both directions.

A consequence of the fact that constriction does not have a graphical representation is that, differently from undirected graphs, to the best of our understanding, deletion–constriction is not recursive and constructive, i.e. one cannot build up the spanning-tree polynomial of a directed graph starting from those of smaller graphs, at least not in the usual intuitive way.

Finally, let us mention that a Tutte-type polynomial for directed graphs has been proposed in [32] satisfying a variant of the deletion–contraction formula. Deletion–contraction is established for signed graphs where each edge carries weight ± 1, and which is related to knots and their invariants [33].

2.2 Tree surgery and edge swaps

To obtain our main result, we will operate some tree surgery, as introduced in [28]. We will only be concerned with edge swaps between two rooted spanning-trees $T_{\mathtt{x}} \in \mathscr{T}_{\mathtt{x}}$ and $T_{\mathtt{y}} \in \mathscr{T}_{\mathtt{y}}$. The idea is that, by repeatedly swapping edges in a systematic manner, one obtains two new spanning-trees $T'_{\mathtt{x}}$ and $T'_{\mathtt{y}}$ with the roots interchanged, in such a way that the product of their weights is preserved

$$r(T'_{\mathtt{x}})r(T'_{\mathtt{y}}) = r(T_{\mathtt{y}})r(T_{\mathtt{x}}). \tag{14}$$

In the process, one of the two trees temporarily becomes a doubly rooted spanning-tree $S_{\mathtt{x},\mathtt{y};\mathtt{z}}$, defined as a spanning-tree where the orientation of the paths starting from a branching point $\mathtt{z}$ is towards root $\mathtt{x}$ on the one side and towards $\mathtt{y}$ on the other. Notice that a rooted spanning-tree can be seen as a degenerate doubly rooted spanning-tree with branching point one of the two roots, e.g. $T_{\mathtt{x}} = S_{\mathtt{x},\mathtt{y};\mathtt{y}}$. A useful characterization of a rooted spanning-tree is as a minimal set of edges connecting all vertices such that each vertex has exactly one outgoing edge (out-degree 1) apart from one vertex which has none (the root, with out-degree 0). A non-degenerate doubly rooted spanning-tree is such that one vertex has out-degree 2 (the branching vertex), two vertices have out-degree 0 (the roots), and all others have out-degree 1.

Let us now review the swapping map. See Fig. 3 for an illustration of the passages in a simple example. We view $T_{\mathtt{x}}$ as a degenerate doubly rooted spanning-tree $S_{\mathtt{x},\mathtt{y};\mathtt{y}}$. In $T_{\mathtt{x}}$ there is a unique path $\mathtt{y} \rightsquigarrow \mathtt{x}$. Take the first edge e along it, let $\mathtt{k}$ be its target. The removal of e from $T_{\mathtt{x}}$ induces a partition of X into what we call two basins of vertices, that is, connected sets of vertices, disconnected one to another. Each basin is spanned by a tree in the so-called forest of the original graph, one with root $\mathtt{y}$ and one with root $\mathtt{x}$. Now identify the same two basins in $T_{\mathtt{y}}$. Because $\mathtt{k}$ and $\mathtt{y}$ belong to different basins, along the unique path $\mathtt{k} \rightsquigarrow \mathtt{y}$ in $T_{\mathtt{y}}$ there is at least one edge that reconnects the two basins[4]. Take f as the last of these edges, on the temporary assumption that its source vertex $\mathtt{z}$ is different from $\mathtt{x}$. We now swap edges e and f among the two trees. First we map $T_{\mathtt{y}} \to \{T_{\mathtt{y}} \setminus f\} \cup e$. Notice that because of the removal of f, $\mathtt{z}$ has no outgoing edge. Furthermore, $\mathtt{y}$ now has out-degree 1 provided by the outgoing edge e. The degrees of all other vertices are untouched, and because

[4] One may be tempted to intuitively think that there is only one such edge: this is true for the path $\mathtt{y} \rightsquigarrow \mathtt{x}$ in $T_{\mathtt{x}}$, but not for the path $\mathtt{x} \rightsquigarrow \mathtt{y}$ in $T_{\mathtt{y}}$.

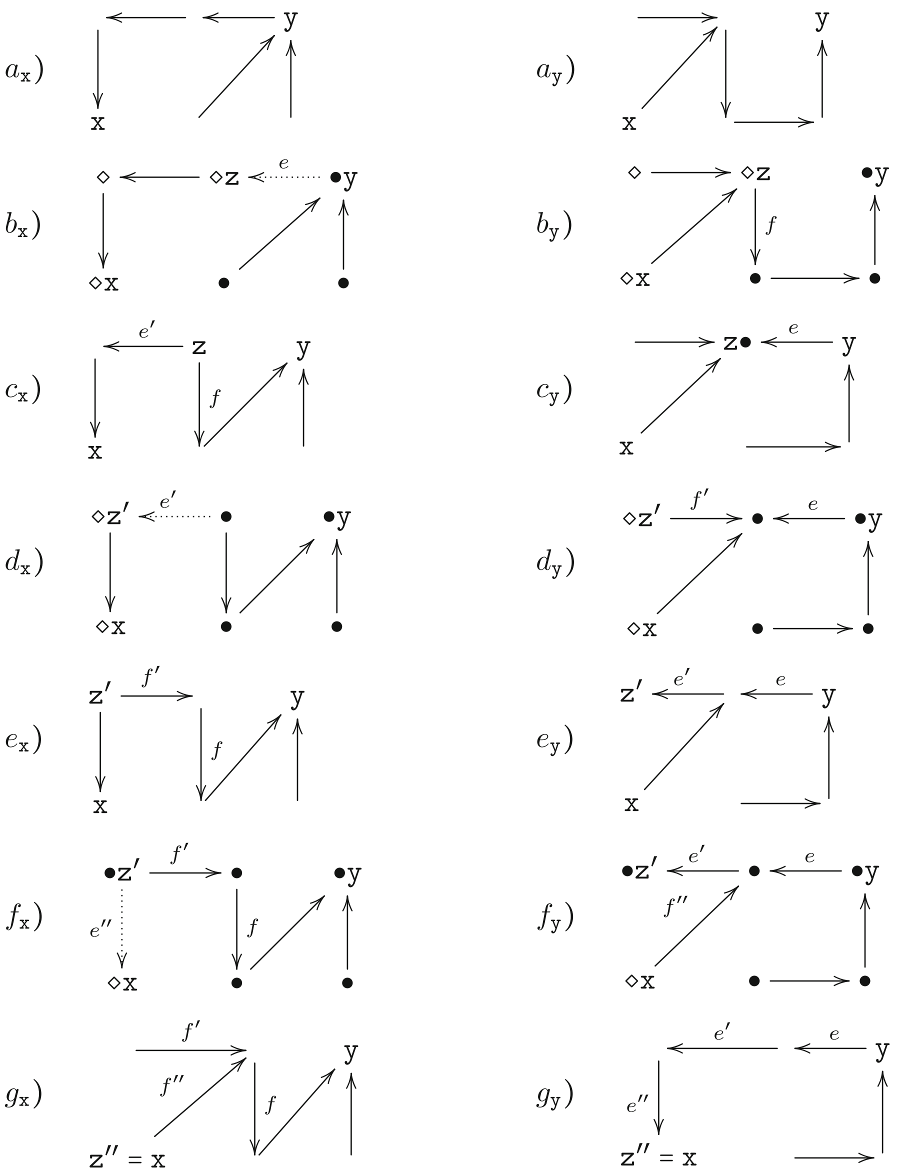


Fig. 3 $a_{\mathrm{x}}, a_{\mathrm{y}}$) Rooted spanning-trees $T_{\mathrm{x}}, T_{\mathrm{y}}$ with roots respectively x, y; in each there is a unique directed path from y to x (resp. x to y). b_{x}) In T_{x}, we label the first edge along such path as e with target node k (for these trees, $\mathrm{k} = \mathrm{z}$) and remove e; this disconnects the spanning-tree and the vertex set into two basins, one connected to x (diamonds) and one connected to y (bullets). b_{y}) We identify the same two basins in T_{y} and identify f as the last edge in the spanning-tree starting from z that reconnects the two basins. c_{x}), c_{y}): we swap e and f; notice that while T_{y} remains a spanning-tree with root in z, T_{x} has become a doubly rooted spanning-tree with branching point z. d_{x}) We repeat the procedure: in particular in this latter double-rooted spanning-tree we identify the first edge e' that goes from the branching point to x, move the z to its target, remove it, identify two disconnected basins; in the spanning-tree on the right we identify the first edge f' in the path from the new z that reconnects the two basins. e_{x}), e_{y}) We swap e' and f'. f_{x}), f_{y}), g_{x}, g_{y}) We repeat the procedure until the moving root / branching point coincides with x

it is connected, this object is a spanning-tree with "moving" root $\mathtt{z}$. Second we map $T_{\mathtt{x}} \to \{T_{\mathtt{x}} \setminus e\} \cup f$. The removal of e deprives $\mathtt{y}$ of its outgoing edge, so now both $\mathtt{x}$ and $\mathtt{y}$ are roots. The addition of f gives $\mathtt{z}$ a second outgoing edge. Thus this object is a doubly rooted spanning-tree $S_{\mathtt{x},\mathtt{y};\mathtt{z}}$ with "moving" branching vertex $\mathtt{z}$. Repeat the operation. Now let e' be the first edge along the unique path $\mathtt{z} \rightsquigarrow \mathtt{x}$ in $S_{\mathtt{x},\mathtt{y};\mathtt{z}}$. On the second iteration when removing e', identifying $\mathtt{k}'$ as its target vertex, finding new basins and identifying f' as the last edge that reconnects $\mathtt{k}'$ to $\mathtt{z}$, the out-degree of $\mathtt{z}$ in $S_{\mathtt{x},\mathtt{y};\mathtt{z}}$ goes back from 2 to 1 while that of the source $\mathtt{z}'$ of f' goes to 2, unless $\mathtt{z}'$ is $\mathtt{x}$, in which case its out-degree goes to 1 and it is no longer a root, and the doubly rooted spanning-tree degenerates into a rooted spanning-tree with root $\mathtt{y}$. But this is bound to happen, as the unique path $\mathtt{z}' \rightsquigarrow \mathtt{x}$ is strictly contained in the previous path $\mathtt{z} \rightsquigarrow \mathtt{x}$, and if we are not yet at $\mathtt{z}' = \mathtt{x}$ we can just repeat the procedure.

Importantly, edge swaps (1) are invertible (see [28] for a constructive proof); (2) do not swap an edge whose target is the root of the other tree (in fact, by construction, at any step e, e' cannot target $\mathtt{y}$ as they belong to the unique path $\mathtt{y} \rightsquigarrow \mathtt{x}$; f, f' cannot target $\mathtt{x}$ as $\mathtt{x}$ always belongs to the basin from which they lead out); (3) do not add an edge that is already present in a (doubly rooted) tree. An important consequence is that the above procedure extends to spanning-trees $T_{\mathtt{x}}$, $T_{\mathtt{y}}$ conditioned to pass through a subset of the edge space B and not to pass through a non-intersecting subset A (with $\pm 1 \notin A, B$). An example of the consequence of coplanarity in this case was given at the end of Sect. 1.1, with $B = \mathtt{c} \to \mathtt{b}$. This generalization would be trivial for non-directed graphs, where there always is a graphical representation of contraction. Here it is slightly more subtle.

2.3 Second-order deletion–constriction formula for a single edge

Without loss of generality, we now choose a pair of vertices and "pin" the two directed edges between them (which we refer to as $+1$ and -1), both with nonvanishing weight, assuming that their removal does not disconnect the graph into two subgraphs. We will be concerned with the following conditioned rooted spanning-tree polynomials: $\tau_{\mathtt{x}}^{\setminus +1,-1}$ rooted in $\mathtt{x}$ not containing either of the two pinned edges; $\tau_{\mathtt{x}}^{\setminus \mp 1 /\!\!/ \pm 1}$ rooted in $\mathtt{x}$ containing ± 1 but not ∓ 1; $\tau_{\mathtt{s}(\mp 1)}^{\setminus +1,-1}$ rooted at the source or target of the pinned edges but not containing them; $\dot{\tau}_1$ rooted in vertex $\mathtt{1}$, obtained by contracting the pinned edges to a unique vertex 1.

From Theorem 1, taking $I = \{+1, -1\}$, the following "first-order" deletion–constriction formula holds

$$\tau_{\mathtt{x}} = \tau_{\mathtt{x}}^{\setminus +1,-1} + \tau_{\mathtt{x}}^{\setminus -1 /\!\!/ +1} + \tau_{\mathtt{x}}^{\setminus +1 /\!\!/ -1}, \tag{15}$$

which intuitively states that a spanning-tree can contain only edge $+1$, only -1, or neither. In fact, a spanning-tree cannot pass through both $+1$ and -1.

We can now formulate the result illustrated in Fig. 4.

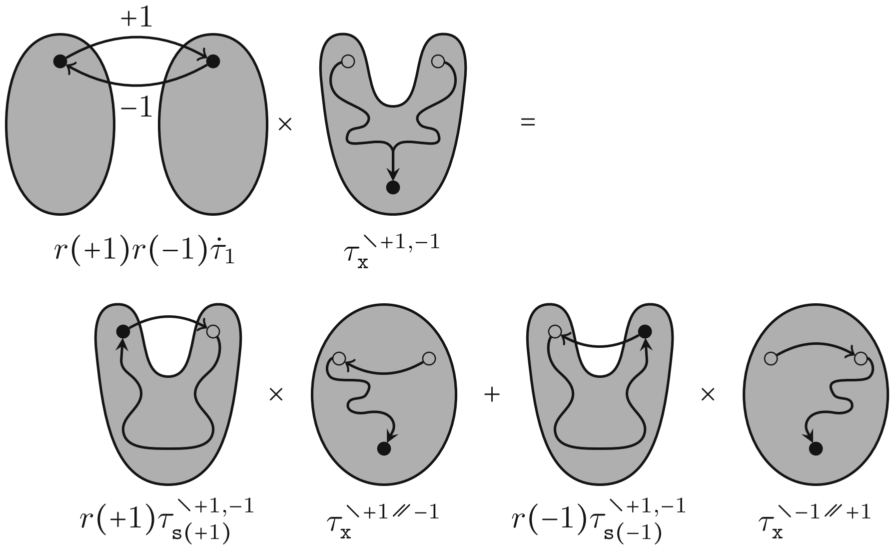


Fig. 4 Illustration of the main result, Theorem 2. Shaded areas represent spanning-trees with root in the fully bulleted vertex, while the circles are other vertices different from the root. Directed arrows are specific paths: if they exit the shaded area, their weights are multiplied to the overall tree polynomial; if they are within the shaded area, it means that the tree is forced to pass through them. Wiggled arrows denote specific paths within the spanning-tree. The reason the lower circled vertex is not represented in some of the terms is because it is undecided whether or not it falls on the wiggled paths (on the lower line), or to which grey area it belongs (on the upper line)

Theorem 2 *The conditioned spanning-tree polynomials satisfy the following "second-order" deletion–constriction relation:*

$$r(+1)r(-1)\dot{\tau}_1 \tau_{\mathtt{x}}^{\setminus +1,-1} = r(+1)\tau_{\mathtt{s}(+1)}^{\setminus +1,-1} \tau_{\mathtt{x}}^{\setminus +1 /\!\!/ -1} + r(-1)\tau_{\mathtt{s}(-1)}^{\setminus +1,-1} \tau_{\mathtt{x}}^{\setminus -1 /\!\!/ +1}. \quad (16)$$

Proof We proceed by swapping terms between the left-hand side and the right-hand side, but we need to be careful due to edges that we need to pass by, or to avoid.

In the first product on the right-hand side, consider the term $r(T_{\mathtt{s}(+1)})r(T_{\mathtt{x}})$, where $T_{\mathtt{s}(+1)} \not\ni (+1, -1)$ and $T_{\mathtt{x}} \ni -1$. Following the procedure outlined in Sect. 2.2, we want to turn T into T' by edge swaps. To do so, we treat $T_{\mathtt{x}}$ as a degenerate doubly rooted spanning-tree $S_{\mathtt{x},\mathtt{s}(+1);\mathtt{s}(+1)}$ and identify the first edge connecting $\mathtt{s}(+1) \rightsquigarrow \mathtt{x}$ as the edge to be swapped. We must make sure that edge -1 is not swapped. But this cannot happen, as the unique path $\mathtt{s}(+1) \rightsquigarrow \mathtt{x}$ cannot contain edge -1 (as edge -1 points towards $\mathtt{s}(+1)$), and thus, the branching vertex $\mathtt{z}$ is ensured to reach $\mathtt{x}$. Thus, we end up with $T'_{\mathtt{x}} \not\ni (+1, -1)$ and $T'_{\mathtt{s}(+1)} \ni -1$. Now notice that $r(T'_{\mathtt{x}})r(T'_{\mathtt{s}(+1)})$ is a term in the left-hand-side product according to the first representation of τ_1 in Eq. (13). The same treatment works specularly for the second term on the right-hand side by considering a term $r(T_{\mathtt{s}(-1)})r(T_{\mathtt{x}})$ with $T_{\mathtt{s}(-1)} \not\ni (+1, -1)$ and $T_{\mathtt{x}} \ni +1$.

The other way around, let us now consider a term of the left-hand-side product. Equation (13) provides two alternative representations of $\dot{\tau}_1$, respectively as a spanning-tree rooted in $\mathtt{s}(+1)$ with constricted edge -1 or rooted in $\mathtt{s}(-1)$ with constricted edge $+1$. Before performing the swapping, we choose one of these representations, specifically the one which grants that the unique path $\mathrm{x} \rightsquigarrow \mathrm{s}(\mp 1)$ does not contain edge ± 1. With this precaution in mind, we can safely apply the swapping map, and we end up with a term belonging to the right-hand-side product. But since the swapping map is invertible, we have established Eq. (16). $\square$

2.4 Coplanarity of spanning-tree vectors

We now define $\sigma_{\varnothing 1} := -r(+1)r(-1)\dot{\tau}_1$ and $\sigma_{\pm 1} = r(\pm 1)\tau_{\mathrm{s}(\pm 1)}^{\setminus +1,-1}$ and form the vectors

$$\boldsymbol{\tau}_{\mathrm{x}} = \begin{pmatrix} \tau_{\mathrm{x}}^{\setminus +1,-1} \\ \tau_{\mathrm{x}}^{\setminus -1 /\!\!/ +1} \\ \tau_{\mathrm{x}}^{\setminus +1 /\!\!/ -1} \end{pmatrix}, \qquad \boldsymbol{\sigma}_1 = \begin{pmatrix} \sigma_{\varnothing 1} \\ \sigma_{+1} \\ \sigma_{-1} \end{pmatrix}, \qquad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{17}$$

The deletion–constriction formula in Eq. (15) reads

$$\tau_{\mathrm{x}} = \mathbf{1} \cdot \boldsymbol{\tau}_{\mathrm{x}}. \tag{18}$$

Theorem 3 *Vectors $\boldsymbol{\tau}_{\mathrm{x}}$, for $\mathrm{x} \in X$, are coplanar.*

Proof Equation (16) can be rewritten as:

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\tau}_{\mathrm{x}} = 0 \tag{19}$$

and we conclude. $\square$

Remark Theorems 2 and 3 remain valid if one of the two weights $r(\pm 1)$ is sent to zero.

2.5 Mutual linearity among two Markov currents

As an application of coplanarity, we now reproduce a recent result obtained by the authors in [29] by means of a linear algebra approach developed in [34].

Consider a continuous-time Markov chain on a graph G with time-independent rates (from now on: weights) $r(\pm e)$ of performing transitions $\pm e \in E$ between vertices $\mathrm{x} \in X$ in a short-time interval. We are interested in the mean stationary currents

$$\jmath_e = r(+e)\, p_{\mathrm{s}(+e)} - r(-e)\, p_{\mathrm{s}(-e)} \tag{20}$$

where $(p_{\mathrm{x}})_{\mathrm{x} \in X}$ is the unique stationary probability of being at a vertex, whose existence and uniqueness are granted by the assumption of irreducibility. We promote the arbitrarily chosen $\jmath_1$ as input current and study the dependence of all other currents $\jmath_e$ on the transition weights $\boldsymbol{r}_1 = (r(+1), r(-1))$ of the input current, while keeping all other weights fixed. We also assume that the removal of edges ± 1 does not disconnect the graph. In general, $\jmath_e(\boldsymbol{r}_1)$ is a nonlinear function of $\boldsymbol{r}_1$.

Lemma 1 *There exist parameters λ_e^0 and λ_e^1, not dependent on $\boldsymbol{r}_1$ (but possibly dependent on all other weights), such that all currents are linear-affine in the input current*

$$\jmath_e(\boldsymbol{r}_1) = \lambda_e^0 + \lambda_e^1\, \jmath_1(\boldsymbol{r}_1). \tag{21}$$

Proof We will apply Theorem 3 on the coplanarity of spanning-tree vectors, choosing as pinned edges those of the input current $\jmath_1$. Notice also, since any two currents $\jmath_e$ and $\jmath_{e'}$ are linear-affine in $\jmath_1$, Eq. (21) implies that they are linear-affine among themselves with respect to variation of the weights of an arbitrary pair ± 1.

By the Markov-chain matrix-tree theorem [11–14] (see also [15, 16] for recent approaches), the stationary probability is given by:

$$p_{\mathrm{x}} = \frac{\tau_{\mathrm{x}}}{\sum_{\mathrm{y}} \tau_{\mathrm{y}}}. \tag{22}$$

Plugging into the second expression of Eq. (20), one finds that the stationary currents are ratios of a homogeneous polynomial of degree $|X|$ in the weights over the normalization, which is a homogeneous polynomial of degree $|X| - 1$. Importantly, the polynomials in Eq. (22) contain $r(+1)$, $r(-1)$, or neither, and no quadratic terms are allowed. Plugging it into Eq. (20), all terms containing products $r(+1)r(-1)$ cancel out by application of Eqs. (12) and (13). We can then express the dependency of the currents on $\boldsymbol{r}_1$ as

$$\jmath_e(\boldsymbol{r}_1) = \frac{z_e^{\varnothing} + z_e^{+} r(+1) + z_e^{-} r(-1)}{z_0^{\varnothing} + z_0^{+} r(+1) + z_0^{-} r(-1)} \tag{23}$$

where, reminding the definition of the rescaled spanning-tree polynomial in Eq. (8), we defined

$$\begin{aligned}
z_e^{\varnothing} &= r(+e)\tau_{\mathrm{s}(+e)}^{\setminus +1,-1} - r(-e)\tau_{\mathrm{s}(-e)}^{\setminus +1,-1} && \\
z_e^{\pm} &= r(+e)\dot{\tau}_{\mathrm{s}(+e)}^{\setminus \mp 1 /\!\!/ \pm 1} - r(-e)\dot{\tau}_{\mathrm{s}(-e)}^{\setminus \mp 1 /\!\!/ \pm 1} && \\
z_1^{\varnothing} &= 0 && \\
z_1^{\pm} &= \pm\tau_{\mathrm{s}(\pm 1)}^{\setminus +1,-1} && \gtrless 0, \neq 0 \\
z_0^{\varnothing} &= \sum_{\mathrm{x}} \tau_{\mathrm{x}}^{\setminus +1,-1} && > 0 \\
z_0^{\pm} &= \sum_{\mathrm{x}} \dot{\tau}_{\mathrm{x}}^{\setminus \mp 1 /\!\!/ \pm 1} && > 0.
\end{aligned} \tag{24}$$

The strict inequalities on the right-hand side are due to the assumption we made that removal of edges ± 1 does not disconnect the graph, so that there always exists at least one rooted spanning-tree.

Multiplying Eq. (21) by the denominator in Eq. (23), we obtain

$$z_e^{\varnothing} + z_e^{+} r(+1) + z_e^{-} r(-1) = \lambda_e^0 [z_0^{\varnothing} + z_0^{+} r(+1) + z_0^{-} r(-1)] + \lambda_e^1 [z_1^{+} r(+1) + z_1^{-} r(-1)]. \tag{25}$$

Equation (25) holds true for any value of $\boldsymbol{r}_1$ if and only if there is a unique solution to the following overdetermined system of linear equations:

$$\overbrace{\begin{pmatrix} z_0^{\varnothing} & 0 \\ z_0^{+} & z_1^{+} \\ z_0^{-} & z_1^{-} \end{pmatrix}}^{(\boldsymbol{z}_0, \boldsymbol{z}_1)} \overbrace{\begin{pmatrix} \lambda_e^0 \\ \lambda_e^1 \end{pmatrix}}^{\boldsymbol{\lambda}_e} = \overbrace{\begin{pmatrix} z_e^{\varnothing} \\ z_e^{+} \\ z_e^{-} \end{pmatrix}}^{\boldsymbol{z}_e}. \tag{26}$$

By the Rouché–Capelli theorem, Eq. (26) admits a unique solution if and only if $(\boldsymbol{z}_0, \boldsymbol{z}_1)$ is full rank and the augmented matrix $(\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_e)$ has the same rank as $(\boldsymbol{z}_0, \boldsymbol{z}_1)$. In the present case, the rank of $(\boldsymbol{z}_0, \boldsymbol{z}_1)$ is 2, because the determinant of the upper 2×2 block is nonvanishing, $z_0^{\varnothing} z_1^{+} > 0$. The second condition means

$$\det(\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_e) = 0, \tag{27}$$

which is granted by Theorem 3. Indeed, vectors $\boldsymbol{z}_e$ and $\boldsymbol{z}_0$ are easily seen to be linear combinations of vectors $\boldsymbol{\tau}_{\mathrm{x}}$, and therefore, by coplanarity [Eq. (19)] $\boldsymbol{\sigma}_1$ is orthogonal to all of them. Furthermore, it can be checked directly that $\boldsymbol{\sigma}_1 \cdot \boldsymbol{z}_1 = 0$. Therefore, matrix $(\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_e)$ has $\boldsymbol{\sigma}_1$ as left-null vector and its determinant vanishes, granting Eq. (27). □

We can find $\boldsymbol{\lambda}_e$ satisfying Eq. (21) by inverting any 2×2 block of the above system. Choosing the upper 2×2 block, we obtain

$$\begin{pmatrix} \lambda_e^0 \\ \lambda_e^1 \end{pmatrix} = \begin{pmatrix} z_0^{\varnothing} & 0 \\ z_0^{+} & z_1^{+} \end{pmatrix}^{-1} \begin{pmatrix} z_e^{\varnothing} \\ z_e^{+} \end{pmatrix} = \frac{1}{z_0^{\varnothing} z_1^{+}} \begin{pmatrix} z_1^{+} z_e^{\varnothing} \\ z_0^{\varnothing} z_e^{+} - z_0^{+} z_e^{\varnothing} \end{pmatrix}. \tag{28}$$

This expression allows us a consistency check, as for vanishing $\boldsymbol{r}_1 = 0$ the input current stalls and we find for the output currents $\jmath_e(0) = \lambda_e^0 = z_e^{\varnothing}/z_0^{\varnothing}$, consistently with Eq. (23) and with what previously found e.g. in [31]. Furthermore, this also implies that all affine coefficients λ_e^0 vanish if the graph where $\boldsymbol{r}_1 = 0$ satisfies detailed balance.

3 Towards multiple pinned edges

3.1 Co-hyper-planarity for two pairs of pinned edges

An obvious question is whether the above results generalize to the deletion–constriction through multiple edges. Let us consider here the case of two pairs of pinned edges ± 1 and ± 2 with nonvanishing weights in both directions. Here we are not so much interested in the most general setup, and we introduce assumptions that allow us to generalize the approach above to the case of two pairs of pinned edges with no additional difficulties. Specifically, we assume that the graph (i) remains connected after the removal of pinned edges, and (ii) is large enough (at least $|X| = 9$ vertices—the reason being that, as we shall soon see, the spanning-tree vectors have dimension 9, and we want to have enough of them.

The deletion–constriction formula in Theorem 1 reads $\tau_{\mathrm{x}} = \mathbf{1} \cdot \boldsymbol{\tau}_{\mathrm{x}}$ with $\mathbf{1}$ a vector with all unit entries and

$$\boldsymbol{\tau}_{\mathrm{x}} = \begin{pmatrix} \tau_{\mathrm{x}}^{\setminus +1,-1,+2,-2} \\ \tau_{\mathrm{x}}^{\setminus -1,+2,-2 /\!/ +1} \\ \tau_{\mathrm{x}}^{\setminus +1,+2,-2 /\!/ -1} \\ \tau_{\mathrm{x}}^{\setminus +1,-1,-2 /\!/ +2} \\ \tau_{\mathrm{x}}^{\setminus +1,-1,+2 /\!/ -2} \\ \tau_{\mathrm{x}}^{\setminus -1,-2 /\!/ +1,+2} \\ \tau_{\mathrm{x}}^{\setminus +1,-2 /\!/ -1,+2} \\ \tau_{\mathrm{x}}^{\setminus -1,+2 /\!/ +1,-2} \\ \tau_{\mathrm{x}}^{\setminus +1,+2 /\!/ -1,-2} \end{pmatrix}. \tag{29}$$

Notice that there are no further nonvanishing vector entries in $\boldsymbol{\tau}_{\mathrm{x}}$ since constriction through both $+e$ and $-e$ yields a vanishing entry, as there do not exist spanning-trees containing both (otherwise they would contain a loop, against their definition).

The question we pose is the dimension of the span of $\{\boldsymbol{\tau}_{\mathrm{x}}\}_{\mathrm{x} \in X}$. Notice that for $\mathrm{x} = \mathrm{s}(+1)$, $\mathrm{x} = \mathrm{s}(-1)$ and $\mathrm{x} = \mathrm{s}(+2)$ we have

$$(\boldsymbol{\tau}_{\mathrm{s}(+1)}, \boldsymbol{\tau}_{\mathrm{s}(-1)}, \boldsymbol{\tau}_{\mathrm{s}(+2)}) = \begin{pmatrix} \tau_{\mathrm{s}(+1)}^{\setminus +1,-1,+2,-2} & \tau_{\mathrm{s}(-1)}^{\setminus +1,-1,+2,-2} & \tau_{\mathrm{s}(+2)}^{\setminus +1,-1,+2,-2} \\ 0 & \tau_{\mathrm{s}(-1)}^{\setminus -1,+2,-2 /\!/ +1} & \tau_{\mathrm{s}(+2)}^{\setminus -1,+2,-2 /\!/ +1} \\ \tau_{\mathrm{s}(+1)}^{\setminus +1,+2,-2 /\!/ -1} & 0 & \tau_{\mathrm{s}(+2)}^{\setminus +1,+2,-2 /\!/ -1} \\ \tau_{\mathrm{s}(+1)}^{\setminus +1,-1,-2 /\!/ +2} & \tau_{\mathrm{s}(-1)}^{\setminus +1,-1,-2 /\!/ +2} & 0 \\ \tau_{\mathrm{s}(+1)}^{\setminus +1,-1,+2 /\!/ -2} & \tau_{\mathrm{s}(-1)}^{\setminus +1,-1,+2 /\!/ -2} & \tau_{\mathrm{s}(+2)}^{\setminus +1,-1,+2 /\!/ -2} \\ 0 & \tau_{\mathrm{s}(-1)}^{\setminus -1,-2 /\!/ +1,+2} & 0 \\ \tau_{\mathrm{s}(+1)}^{\setminus +1,-2 /\!/ -1,+2} & 0 & 0 \\ 0 & \tau_{\mathrm{s}(-1)}^{\setminus -1,+2 /\!/ +1,-2} & \tau_{\mathrm{s}(+2)}^{\setminus -1,+2 /\!/ +1,-2} \\ \tau_{\mathrm{s}(+1)}^{\setminus +1,+2 /\!/ -1,-2} & 0 & \tau_{\mathrm{s}(+2)}^{\setminus +1,+2 /\!/ -1,-2} \end{pmatrix}. \tag{30}$$

The lower 3×3 block has determinant

$$\tau_{s(+1)}^{\setminus +1,-2/\!\!/-1,+2} \tau_{s(-1)}^{\setminus -1,+2/\!\!/+1,-2} \tau_{s(+2)}^{\setminus +1,+2/\!\!/-1,-2} > 0, \tag{31}$$

therefore at least 3 of the $\boldsymbol{\tau}_x$ are linearly independent. Notice that if the removal of the two pairs of pinned edges disconnected the graph, the above determinant would vanish.

We now argue that 3 is indeed the dimension of the span of $\{\boldsymbol{\tau}_x\}_{x \in X}$.

As briefly mentioned at the end of Sect. 2.2, coplanarity can be generalized to the case the spanning-tree 3-vectors having additional constrictions. First we claim that there exist vectors $\boldsymbol{\sigma}_1^\bullet$, $\boldsymbol{\sigma}_2^\bullet$ of the following form (collected in a matrix) that are orthogonal to all of the $\boldsymbol{\tau}_x$:

$$(\boldsymbol{\sigma}_1^{\setminus +2,-2}, \boldsymbol{\sigma}_2^{\setminus +1,-1}, \boldsymbol{\sigma}_1^{\setminus +2/\!\!/-2}, \boldsymbol{\sigma}_2^{\setminus +1/\!\!/-1}, \boldsymbol{\sigma}_1^{\setminus -2/\!\!/+2}, \boldsymbol{\sigma}_2^{\setminus -1/\!\!/+1}) \tag{32}$$

$$= \begin{pmatrix} \sigma_{\varnothing 1}^{\setminus +2,-2} & \sigma_{\varnothing 2}^{\setminus +1,-1} & 0 & 0 & 0 & 0 \\ \sigma_{+1}^{\setminus +2,-2} & 0 & 0 & 0 & 0 & \sigma_{\varnothing 2}^{\setminus -1/\!\!/+1} \\ \sigma_{-1}^{\setminus +2,-2} & 0 & 0 & \sigma_{\varnothing 2}^{\setminus +1/\!\!/-1} & 0 & 0 \\ 0 & \sigma_{+2}^{\setminus +1,-1} & 0 & 0 & \sigma_{\varnothing 1}^{\setminus -2/\!\!/+2} & 0 \\ 0 & \sigma_{-2}^{\setminus +1,-1} & \sigma_{\varnothing 1}^{\setminus +2/\!\!/-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{+1}^{\setminus -2/\!\!/+2} & \sigma_{+2}^{\setminus -1/\!\!/+1} \\ 0 & 0 & 0 & \sigma_{+2}^{\setminus +1/\!\!/-1} & \sigma_{-1}^{\setminus -2/\!\!/+2} & 0 \\ 0 & 0 & \sigma_{+1}^{\setminus +2/\!\!/-2} & 0 & 0 & \sigma_{-2}^{\setminus -1/\!\!/+1} \\ 0 & 0 & \sigma_{-1}^{\setminus +2/\!\!/-2} & \sigma_{-2}^{\setminus +1/\!\!/-1} & 0 & 0 \end{pmatrix}. \tag{33}$$

The vector entries can be found as per the definitions at the beginning of Sect. (2.4), but with the additional constraints specified as superscript. A way to impose such constraints starting from the definitions of Sect. (2.4) is by taking derivatives with respect to the (logarithm of the) weights of the constricted edges and evaluating at zero with respect to the weights of the deleted edges:

$$\sigma_e^{\setminus e_1, e_2, \ldots /\!\!/ e_1', e_2', \ldots} = \frac{\partial}{\partial \log r(e_1')} \frac{\partial}{\partial \log r(e_2')} \ldots \sigma_e \bigg|_{r(e_1)=r(e_2)=\ldots=0}. \tag{34}$$

Let us now choose (say) the last one, $\boldsymbol{\sigma}_2^{\setminus -1/\!\!/+1}$. When taking the scalar product with by $\boldsymbol{\tau}_x$ we obtain

$$\sigma_{\varnothing 2}^{\setminus -1,/\!\!/+1} \tau_x^{\setminus -1+2,-2/\!\!/+1} + \sigma_{+2}^{\setminus -1/\!\!/+1} \tau_x^{\setminus -1,-2/\!\!/+1,+2} + \sigma_{-2}^{\setminus -1/\!\!/+1} \tau_x^{\setminus -1,+2/\!\!/+1,-2}. \tag{35}$$

As commented in Sect. 2.2, the swapping procedure is successful also when dealing with additional constraints. Therefore, Theorem 2 applies, and the above expression vanishes.

It remains to prove that the above σ-vectors are linearly independent. Rescaling each one of them in such a way that the first entry is 1 the above matrix now reads

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 0 & 1 \\ B & 0 & 0 & 1 & 0 & 0 \\ 0 & C & 0 & 0 & 1 & 0 \\ 0 & D & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E & F \\ 0 & 0 & 0 & G & H & 0 \\ 0 & 0 & I & 0 & 0 & J \\ 0 & 0 & K & L & 0 & 0 \end{pmatrix} \tag{36}$$

with properly defined $A, \dots, L$. If this matrix were not full rank, it would admit a right-null vector $\boldsymbol{\omega}$. Applying the upper 5-block to $\boldsymbol{\omega}$, we can constrain its form to

$$\boldsymbol{\omega} = \begin{pmatrix} 1 \\ -1 \\ D \\ -B \\ C \\ -A \end{pmatrix}. \tag{37}$$

Taking its scalar product the vector by the last four rows in Eq. (36), we obtain four consistency requirements for $\boldsymbol{\omega}$ to be a null vector. For example, multiplying by the sixth row, we find

$$0 = EC - FA = \frac{\sigma_{+1}^{\backslash -2 /\!/ +2}}{\sigma_{\varnothing 1}^{\backslash -2 /\!/ +2}} \frac{\sigma_{+2}^{\backslash +1,-1}}{\sigma_{\varnothing 2}^{\backslash +1,-1}} - \frac{\sigma_{+2}^{\backslash -1 /\!/ +1}}{\sigma_{\varnothing 2}^{\backslash -1 /\!/ +1}} \frac{\sigma_{+1}^{\backslash +2,-2}}{\sigma_{\varnothing 1}^{\backslash +2,-2}}. \tag{38}$$

At this stage, a general proof of the inexistence of such a null vector ω is still lacking. To proceed further, we make the technical (and, we believe, unnecessary) assumption that the graph is fully connected and that all weights are mutually incommensurable, so that there do not happen cancellations just due to a peculiar choice of the weights, unless there is a general law valid for all graphs. If Eq. (38) and similar ones were true, then under our technical assumptions each one of them would imply additional second-order deletion–constriction formulas among spanning-trees. A rapid randomized numerical check shows that these conditions are violated. Therefore, most often the σ-vectors are all independent, and the dimension of the span of the rooted tree vectors is 3. We do not pursue a complete proof of this in the present manuscript.

3.2 Linearity when controlling two Markov input currents

We will now show that if the co-hyper-planarity for two pairs of pinned edges discussed above holds, then it implies a linear-affine relation in similarity with Eq. (21) when fixing two input currents. Namely, choosing edges ± 1 and ± 2 as input currents, we will show that there exists unique parameters μ_e^0, μ_e^1, μ_e^2 not dependent on $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ (but possibly dependent on all other weights), such that all currents are linear-affine in the input currents:

$$\jmath_e(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mu_e^0 + \mu_e^1 \jmath_1(\boldsymbol{r}_1, \boldsymbol{r}_2) + \mu_e^2 \jmath_2(\boldsymbol{r}_1, \boldsymbol{r}_2). \tag{39}$$

The edges ± 1 and ± 2 of the input currents are chosen arbitrarily, provided that their removal does not disconnect the transition graph. The procedure is the same as in Sect. 2.5.

A generic current can be expressed as:

$$\jmath_e(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{\boldsymbol{w}_e \cdot \mathbf{1}}{\boldsymbol{w}_0 \cdot \mathbf{1}} \tag{40}$$

where now for the sake of elegance we incorporated the explicit dependency on $\boldsymbol{r}_1, \boldsymbol{r}_2$ in the $\boldsymbol{w}$'s defined by

$$\boldsymbol{w}_0 = \sum_{\mathrm{x}} \boldsymbol{\tau}_{\mathrm{x}} \tag{41}$$

$$\boldsymbol{w}_e = r(+e)\boldsymbol{\tau}_{\mathrm{s}(e)} - r(-e)\boldsymbol{\tau}_{\mathrm{s}(-e)}. \tag{42}$$

Equation (39) holds true if and only if

$$(\boldsymbol{w}_e - \mu_e^0 \boldsymbol{w}_0 - \mu_e^1 \boldsymbol{w}_1 - \mu_e^2 \boldsymbol{w}_2) \cdot \mathbf{1} = 0. \tag{43}$$

Under the same assumptions of Sect. 3.1, we now apply the co-hyper-planarity to the pinned edges ± 1 and ± 2. Given that its columns are linear combinations of the τ-vectors, matrix $(\boldsymbol{w}_e, \boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2)$ has rank 3; matrix $(\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2)$ is also easily shown to have maximal rank. Then, the Rouché–Capelli theorem grants that the system $\boldsymbol{w}_e = \mu_e^0 \boldsymbol{w}_0 + \mu_e^1 \boldsymbol{w}_1 + \mu_e^2 \boldsymbol{w}_2$ admits a unique solution, and therefore, linearity holds.

4 Conclusion and discussion

Before discussing possible generalizations and connections to other ideas, let us recap our results. We introduced the concept of deletion–constriction for directed graphs on n pairs of pinned edges in both directions, generalizing the well-known concept of deletion–contraction for undirected graphs. This leads to the construction of spanning-tree vectors $\boldsymbol{\tau}_{\mathrm{x}}$ for all roots $\mathrm{x} \in X$ satisfying the deletion–constriction formula $\tau_{\mathrm{x}} = \mathbf{1} \cdot \boldsymbol{\tau}_{\mathrm{x}}$. We suggested the idea that such vectors span a vector space of dimension $n + 1$, much smaller than their own dimension, and proved this fact for $n = 1$ in full generality and for $n = 2$, under certain conditions. We used these results to provide an alternative proof of a recently found mutual linearity of two currents for Markov chains on the graph [29], and we discussed its possible generalization to three currents.

Under the simplifying assumption that all edges have non-vanishing weights, for n pairs of pinned edges the spanning-tree vectors have dimension

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n, \tag{44}$$

where the summation takes into account the fact that no tree can contain two pinned edges within the same pair of vertices. Given the cases $n = 1$ and $n = 2$, it is tempting to speculate that the dimension of the linear space spanned by them is $n+1$. We collected numerical evidence supporting this conjecture in the case $n = 3$ (in which case the spanning-tree vectors have 27 entries), for graphs up to 13 vertices and randomized values of the weights of the unpinned edges (best would be to test it on graphs with $|X| > 27$ vertices, but this goes beyond our computational capabilities). In this case, a direct approach such as the one employed for the $n = 2$ case appears to be prohibitive. Using the product deletion–constriction formula, we can produce $n3^{n-1}$ vectors that are orthogonal to all of the $\boldsymbol{\tau}_{\mathrm{X}}$. The rationale behind this latter number is: there are n σ-vectors which are unconditioned; there are $2(n-1)$ σ-vectors conditioned on passing through a given edge, $2(n-2)$ through two edges, etc. Hence, overall we have

$$\sum_{k=0}^{n} 2^k \binom{n}{k}(n-k) = n \sum_{k=0}^{n} 2^k \binom{n-1}{k} = n3^{n-1} \tag{45}$$

possible σ-vectors (in fact for $n = 1$ we had 1 and for $n = 2$ we had 6). But for $n \geq 3$ this is way too many: for example for $n = 3$ the τ-vectors have dimension 27, but the above procedure would produce $3 \times 3^2 = 27$ σ-vectors, 4 of which should then be linearly dependent. We do not have an intuition on how to tame the unindependent ones.

Proving mutual linearity would have been straightforward if instead of varying both forward and backward weights $\boldsymbol{r}_1$ we only varied one of them, e.g. $r(+1)$. In that case in place of Eq. (25), we would have found

$$\frac{z'^{\varnothing}_{e} + z^{+}_{e} r(+1)}{z'^{\varnothing}_{0} + z^{+}_{0} r(+1)} = \lambda'^{0}_{e} + \lambda'^{1}_{e} \frac{z'^{\varnothing}_{1} + z^{+}_{1} r(+1)}{z'^{\varnothing}_{0} + z^{+}_{0} r(+1)}, \tag{46}$$

and the conclusion follows by noticing that the two ratios are Möbius transformations, and by exploiting properties of the Möbius group. It is particularly fitting that the Möbius group structure arises here as in other areas of physics [35] that deal with objects that are intrinsically projective (as are probabilities). One mathematical question, independent of the interpretation of the objects involved as “currents” of a Markov process, could then be whether our more general result could be the blueprint to frame some sort of multi-linear Möbius group structure.

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Data Availability The data supporting the findings of this study are available within the paper.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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