Project #2: Orbital Dynamics

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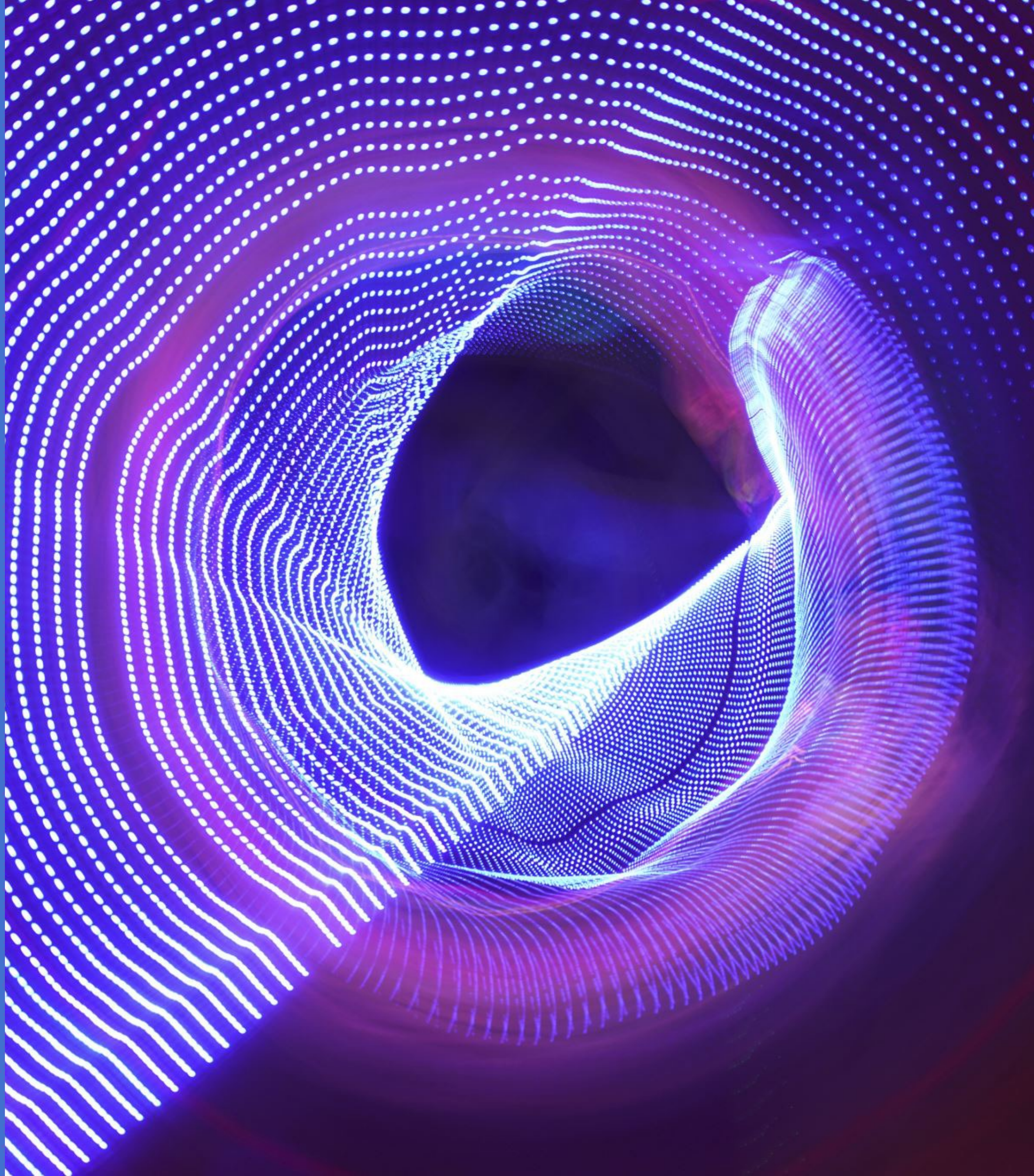


Code Architecture

```python
def analysis(e, n, h, t0, t1, r_min = 1, gif=True):
    """
    inputs:
    - e: is the eccentricity of the orbit
    - n: is the number of massless particles around a galaxy
    - h: is the stepsize used for the simulation
    - t0: is the starting time
    - t1: is the final time
    - r_min: the distance of closest aproach (default 1)
    - gif: If True records position of the system over time as png files and
           saves them in the 'test_snaps' directory to make a gif
        *** Note that to produce a gif 'make_gif.ipynb' notebook should be
           run after running this notebook

    outputs:
    - Plots the motion of two body system with massless particles
    - Saves png files for every time step to 'test_snaps' directory to make a gif
    """


    phi_0 = 10    #Initial condition for the orbit angle
    T = 30    #Orbital period

    a = (G*M*(T**2)/(4*(pi**2)))**(1/3)    #semi-major axis


    if 0<=e<1:    #Elliptical orbit
        p = a*(1-(e**2))    #semi-latus rectum
    elif e == 1:    #Parabolic orbit
        p = 2*r_min    #semi-latus rectum

    r_0 = p/(1 + e*cos(phi_0))    #Initial condition for r
    phi_dot_0 = np.sqrt(G*M*p) / np.power(r_0, 2)    #Initial condition for phi_dot_0
    r_dot_0 = (e * np.power(r_0, 2) * np.sin(phi_0) * phi_dot_0 )/ p    #Initial condition for r_dot_0


    #Finding the coordinates of the galaxies from the coordinates of the reduced mass
    #The mass/distance ratio is used since Galaxy1, Galaxy2, and proxy(reduced mass) are collinear
    r1_0 = r_0*m2/M
    r2_0 = -r_0*m1/M
    r1_dot_0 = r_dot_0*m2/M
    r2_dot_0 = -r_dot_0*m1/M

    #Converting polar coordinates to cartesian coordinates for Galaxy1
    x1_0 = r1_0*np.cos(phi_0)
    y1_0 = r1_0*np.sin(phi_0)
    x1_dot_0 = r1_dot_0 * np.cos(phi_0) - phi_dot_0 * r1_0*np.sin(phi_0)
    y1_dot_0 = r1_dot_0 * np.sin(phi_0) + phi_dot_0 * r1_0*np.cos(phi_0)

    #Converting polar coordinates to cartesian coordinates for Galaxy2
    x2_0 = r2_0*np.cos(phi_0)
    y2_0 = r2_0*np.sin(phi_0)
    x2_dot_0 = r2_dot_0 * np.cos(phi_0) - phi_dot_0 *r2_0*np.sin(phi_0)
    y2_dot_0 = r2_dot_0 * np.sin(phi_0) + phi_dot_0 *r2_0*np.cos(phi_0)

    d_0 = []    #Creating an empty state array (1D) for the inital states of both galaxies
    d1_0 = np.array([x1_0, y1_0, x1_dot_0, y1_dot_0], float)
    d_0 = np.append(d_0,d1_0)
    d2_0 = np.array([x2_0, y2_0, x2_dot_0, y2_dot_0], float)
    d_0 = np.append(d_0,d2_0)
```

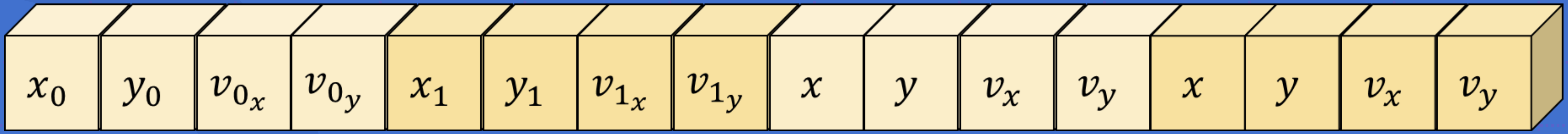
Analysis - function:

- **Define initial conditions here for:**
    - **Reduced mass (proxy)**
- **Convert initial conditions (IC) of the reduced mass to:**
    - **IC of Galaxy1**
    - **IC of Galaxy2**
- **Convert IC to cartesian coordinates**
- **Append all the IC to d_0 array of dimension 1**

```
#Finding the initial states of massless particle(s)
for i in range(n):
    phi_i0 = np.random.random()*2*pi    #randomly chosen direction
    e_i = 0    #all test particles have circular orbits
    T_i = np.random.random()*0.5*T + 1  #Orbital period is defined relative to orbital period of the Galaxy1
    a_i = (G*m1*(T_i**2)/(4*(pi**2)))**(1/3)    #semi-major axis
    p_i = a_i    #semi-latus rectum
    ri_0 = p_i    #since eccentricity is 0

    #Setting up the initial conditions
    phi_dot_0_i = np.sqrt(G*m1*p_i) / np.power(ri_0, 2)
    r_dot_0_i = e_i * np.power(ri_0, 2) * np.sin(phi_i0) * phi_dot_0_i / p_i

    #Placing the massless particles around the first galaxy
    xi_0 = ri_0*np.cos(phi_i0) + x1_0
    yi_0 = ri_0*np.sin(phi_i0) + y1_0
    xi_dot_0 = r_dot_0_i * np.cos(phi_i0) - phi_dot_0_i * ri_0*np.sin(phi_i0) + x1_dot_0
    yi_dot_0 = r_dot_0_i * np.sin(phi_i0) + phi_dot_0_i * ri_0*np.cos(phi_i0) + y1_dot_0
    di_0 = np.array([xi_0, yi_0, xi_dot_0, yi_dot_0], float)
    d_0 = np.append(d_0,di_0)

#Select the integration method to use
#t,rf = rk4(F,t0,t1,h,d_0,n)
t, rf = leap_frog(F,t0,t1,h,d_0,n)
```

Continuous of the Analysis - function:

- **Define initial conditions (IC) for the massless test particles by:**
  - **random phi_0**
  - **Eccentricity = 0**
  - **r_0 is defined by the orbital period which is randomly chosen by using orbital period of the Galaxy1**
- **Convert IC to cartesian coordinates**
- **Append all the IC to d_0 array of dimension 1**
- **Call rk4 (or any other ODE solved) with input chosen as the ICs predefined**
- **Unpack rk4 to get time and rf which is a 3D array of states over time**



```python
def rk4(f,t0,t1,h,d_0,n):
    """
    4th-order Runge_Kutta method
    inputs:
    - f: ODE function
    - t0: initial time
    - t1: final time
    - h = step size for Runge Kutta
    - d_0: 1D array with initial states of Galaxy1, Galaxy2, and massless particles(s)_i
    - n: number of massless particles around a galaxy

    outputs:
    - Time steps
    - 3D array of states over time
    """
    ri = d_0
    rf = np.array([ri])
    t = np.arange(t0,t1+h,h)

    for ti in t[:-1]:
        k1 = h*f(ri,n)
        k2 = h*f(ri+0.5*k1,n)
        k3 = h*f(ri+0.5*k2,n)
        k4 = h*f(ri+k3,n)
        ri += (k1 + 2*k2 + 2*k3 + k4)/6
        rf = np.append(rf,np.array([ri]),axis=0)

    return t,rf
```

rk4- function:

Intakes:

- **initial conditions of states = d_0**

Outputs:

- **Time steps**
- **3D array of states over time**

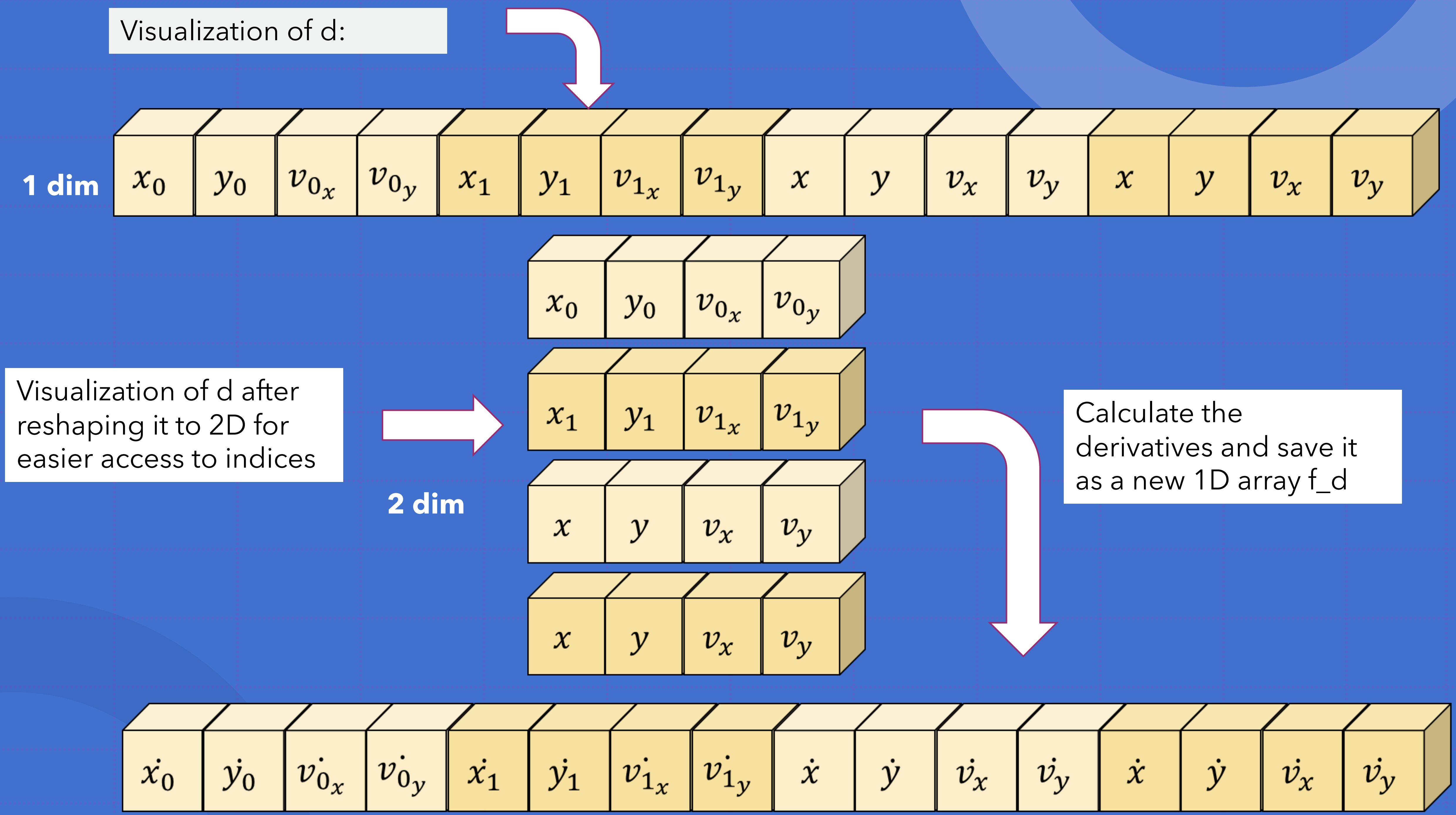
```python
def F(d,n):
    """

    inputs:
    - d: 1D array with elements x1, y1, x1_dot, y1_dot, x2, y2, x2_dot, y2_dot, xi, yi, xi_dot, yi_dot...
    - n: is the number of massless particles around a galaxy

    outputs:
    - 1D array output of the time derivative of the aforementioned elements
    """

    d = d.reshape(2+n,4)    #reshaping 1D state array to a 2D state array

    r = np.sqrt(np.power(d[1][0] - d[0][0], 2) + np.power(d[1][1] - d[0][1], 2))   #Distance between the galaxies

    # Derivatives for Galaxy1
    f_x1 = d[0][2]
    f_y1 = d[0][3]
    f_vx1 = -((G*m2)/np.power(r, 3)) * (d[0][0] - d[1][0])    #EoM for Galaxy1 - in x direction
    f_vy1 = -((G*m2)/np.power(r, 3)) * (d[0][1] - d[1][1])    #EoM for Galaxy1 - in y direction

    # Derivatives for Galaxy2
    f_x2 = d[1][2]
    f_y2 = d[1][3]
    f_vx2 = -((G*m1)/np.power(r, 3)) * (d[1][0] - d[0][0])    #EoM for Galaxy1 - in x direction
    f_vy2 = -((G*m1)/np.power(r, 3)) * (d[1][1] - d[0][1])    #EoM for Galaxy1 - in y direction

    f_d = []    # 1D array to store derivative states of both galaxies and massless particles

    f_d1 = np.array([f_x1, f_y1, f_vx1, f_vy1], float)   #Derivative states array for Galaxy1
    f_d = np.append(f_d,f_d1)
    f_d2 = np.array([f_x2, f_y2, f_vx2, f_vy2], float)   #Derivative states array for Galaxy1
    f_d = np.append(f_d,f_d2)

    # Derivatives for the massless particles
    for i in range(2, np.shape(d)[0]):

        #Distance between a massless particle and Galaxy1
        r1i = np.sqrt(np.power(d[i][0] - d[0][0], 2) + np.power(d[i][1] - d[0][1], 2))

        #Distance between a massless particle and Galaxy1
        r2i = np.sqrt(np.power(d[i][0] - d[1][0], 2) + np.power(d[i][1] - d[1][1], 2))
        f_xi = d[i][2]
        f_yi = d[i][3]

        #EoM for a massless particle - in x direction
        f_vxi = -((G*m1)/np.power(r1i, 3)) * (d[i][0] - d[0][0]) - ((G*m2)/np.power(r2i, 3)) * (d[i][0] - d[1][0])

        #EoM for a massless particle - in y direction
        f_vyi = -((G*m1)/np.power(r1i, 3)) * (d[i][1] - d[0][1]) - ((G*m2)/np.power(r2i, 3)) * (d[i][1] - d[1][1])

        f_di = np.array([f_xi, f_yi, f_vxi, f_vyi], float) #Derivative states array for massless particles
        f_d = np.append(f_d,f_di)

    return f_d    #1D array output of the derivative states of Galaxy1, Galaxy2, massless particle(s)_i...
```

system of ODE- function:

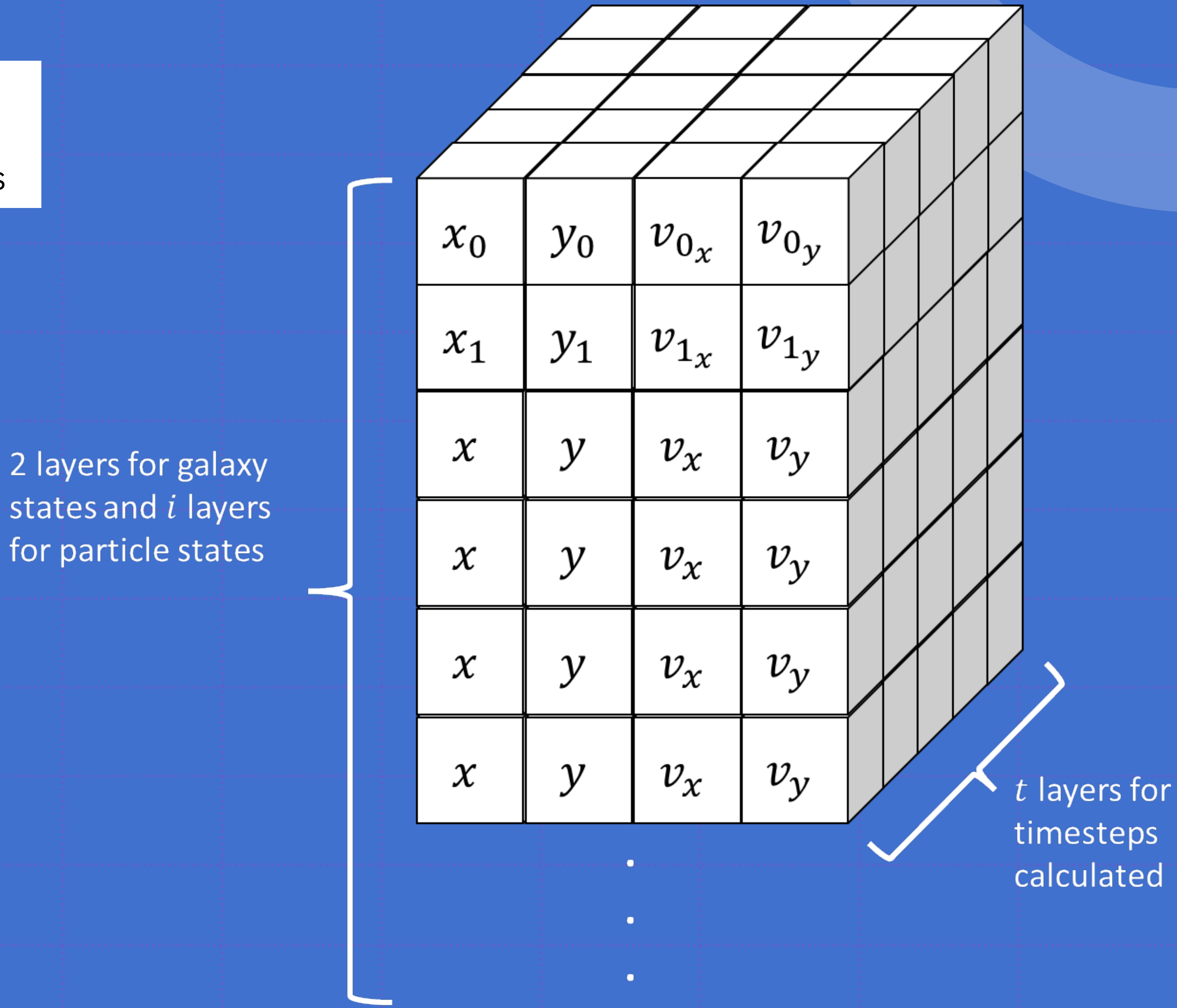
Intakes initial conditions = d
Outputs derivative states of d = f_d

This is done for:

- **Galaxy1**
  - **x1**
  - **y1**
  - **Vx1**
  - **Vy1**
- **Galaxy2**
  - **x2**
  - **y2**
  - **Vx2**
  - **Vy2**
- **For "n" number of test particles**
  - **xi**
  - **yi**
  - **Vxi**
  - **Vyi**


Visualization of d:
1 dim
x_0 y_0 v_{0_x} v_{0_y} x_1 y_1 v_{1_x} v_{1_y} x y v_x v_y x y v_x v_y
Visualization of d after reshaping it to 2D for easier access to indices
2 dim
x_0 y_0 v_{0_x} v_{0_y}
x_1 y_1 v_{1_x} v_{1_y}
x y v_x v_y
x y v_x v_y
Calculate the derivatives and save it as a new 1D array f_d
$\dot{x}_0$ $\dot{y}_0$ $\dot{v}_{0_x}$ $\dot{v}_{0_y}$ $\dot{x}_1$ $\dot{y}_1$ $\dot{v}_{1_x}$ $\dot{v}_{1_y}$ $\dot{x}$ $\dot{y}$ $\dot{v}_x$ $\dot{v}_y$ $\dot{x}$ $\dot{y}$ $\dot{v}_x$ $\dot{v}_y$

Visualization of rf after reshaping it to 3D for easier access to indices



```python
#Reshape rf into a 3D array of states across time
nt = t.size     #Number of timesteps(depth of time dimension)
rf = rf.reshape(nt,n+2,4)

#Splice rf into the states at each time for just the two galaxies
x1 = rf[:,0,0]
y1 = rf[:,0,1]
vx1 = rf[:,0,2]
vy1 = rf[:,0,3]
x2 = rf[:,1,0]
y2 = rf[:,1,1]
vx2 = rf[:,1,2]
vy2 = rf[:,1,3]

#Plotting the paths of all objects
for i in range(n):
        plt.scatter(rf[:,i+2,0],rf[:,i+2,1],s=0.5)    #Plots all particles
plt.plot(x1,y1,c='r')    #Plots first galaxy
plt.plot(x2,y2,c='g')    #Plots second galaxy
plt.xlabel('x')
plt.ylabel('y')
plt.title('Motion of the galaxies')
plt.xlim(-8,8)
plt.ylim(-8,8)
plt.show()

#Calculating the total energy at each timestep
r = np.sqrt(np.power(x1 - x2, 2) + np.power(y1 - y2, 2))    #distance
U = -G*m1*m2/r
T = 0.5*(m1*(np.square(vx1) + np.square(vy1)) + m2*(np.square(vx2) + np.square(vy2)))
E = T+U    #Array of energy values at each timestep

plt.figure(facecolor='#FFFFFF')
plt.plot(t,E)
plt.xlabel('t')
plt.ylabel('E')
plt.title('Total Energy')
plt.annotate('Timestep ' + str(h), xy=(0.77, 0.95),  xycoords='axes fraction')
plt.savefig('Energy')
print("The standard deviation of the total energy is " ,sqrt(np.var(E)))
```

Continuous of the Analysis - function:

- **From 'rf' slice the states of (in cartesian):**
  - **Galaxy1**
  - **Galaxy2**
  - **'n' number of massless test particles**
- **Plot the motion of the two-body system with massless particles**

```python
def saveSnapshot(ti, skip):
    """
    Saves snapshots of a given time ti to folder test_snaps
    inputs:
    - ti: index of the time we want to plot
    - skip: number of timesteps skipped (needed to format the file name)
    """
    #Plots two galaxies and their paths until ti
    plt.scatter(x1[ti],y1[ti],c='r')
    plt.plot(x1[:ti+1],y1[:ti+1],'--', c='r', alpha = 0.5)
    plt.scatter(x2[ti],y2[ti],c='g')
    plt.plot(x2[:ti+1],y2[:ti+1],'--', c='g', alpha=0.5)

    for i in range(n):
        #Plots all particles as a white dot
        plt.scatter(rf[ti,i+2,0],rf[ti,i+2,1],s=1.5, c = 'w')

    #Bounds for the image
    plt.xlim(-20,20)
    plt.ylim(-20,20)
    plt.axis('off')
    plt.axis('equal')

    numString = str(int(ti/skip))    #Name of the image

    #Saving the image with the proper name
    if len(numString) == 1:
        plt.savefig("test_snaps/snapshot_000" + numString + ".png", bbox_inches='tight',pad_inches = 0)
    elif len(numString) == 2:
        plt.savefig("test_snaps/snapshot_00" + numString + ".png", bbox_inches='tight',pad_inches = 0)
    elif len(numString) == 3:
        plt.savefig("test_snaps/snapshot_0" + numString + ".png", bbox_inches='tight',pad_inches = 0)
    elif len(numString) == 4:
        plt.savefig("test_snaps/snapshot_" + numString + ".png", bbox_inches='tight',pad_inches = 0)

    plt.clf()    #Clears the plot for the next image to be saved

skip = 50    #Skip this many timesteps at a time when making gif (1 if none)
if gif == True:
    plt.figure(figsize=(13.333,7.5))
    plt.style.use('dark_background')    #Makes the background black
    for ti in range(len(t)):
        if ti%skip == 0:
            saveSnapshot(ti, skip)

analysis(e = 1, n = 10, h = 0.001, t0 = 0, t1 = 20, r_min = 1, gif=False)
```

saveSnapshot- function:

- **Plotting and saving snapshots of the motion over time**

```python
launched = 0
maxDist = (G*m1*((T*0.5)**2)/(4*(pi**2)))**(1/3)
for i in range(n):
    dist = np.sqrt(np.square(rf[-1,i+2,0] - rf[-1, 0, 0]) + \
                   np.square(rf[-1,i+2,1] - rf[-1, 0, 1]))
    if dist > maxDist:
        launched += 1
print(launched/n)
```

Launched calculation:

- **Number of particles that are outside of the max initial radius of the first galaxy at t1**

```python
def leap_frog(f,t0,tf,h,d_0,n):
    """
    Leap frog method
    inputs:
    - f: ODE function
    - t0: initial time
    - t1: final time
    - h = step size for Runge Kutta
    - d_0: 1D array with initial states of Galaxy1, Galaxy2, and massless particles(s)_i
    - n: number of massless particles around a galaxy

    outputs:
    - Time steps
    - 3D array of states over time
    """


    t = np.arange(t0, tf+h, h)

    ri = d_0     #1 dimensional state array
    rf = np.array([ri], float)     #2 dimensional


    for j in range(n+2):
            f_r = f(ri, n)
            ri[j*4+2] += h/2*f_r[j*4+2]
            ri[j*4+3] += h/2*f_r[j*4+3]

    for i in t[1:]:
        rf = np.append(rf,np.array([ri]),axis=0)
        for j in range(n+2):
            ri[j*4] += h * ri[j*4+2]
            ri[j*4+1] += h* ri[j*4+3]
            f_r = f(ri, n)
            ri[j*4+2] += h*f_r[j*4+2]
            ri[j*4+3] += h*f_r[j*4+3]

    return t, rf
```

Leap_frog- function:

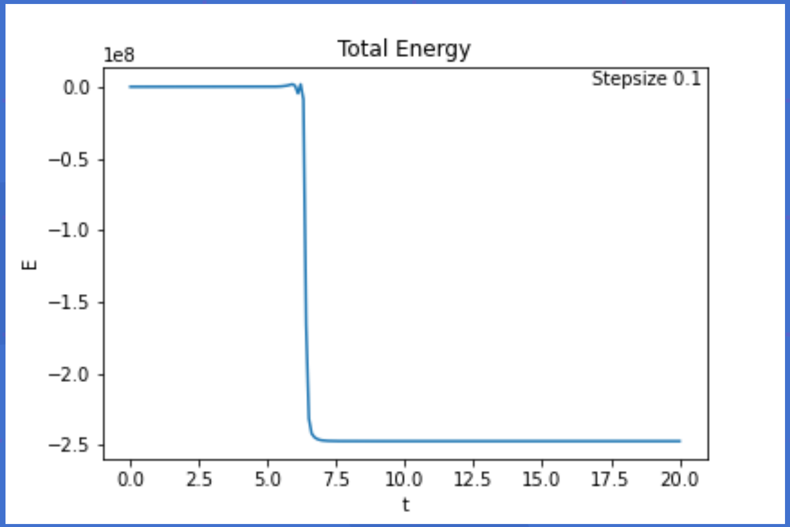
- **Alternative to rk4 as an ODE solver**

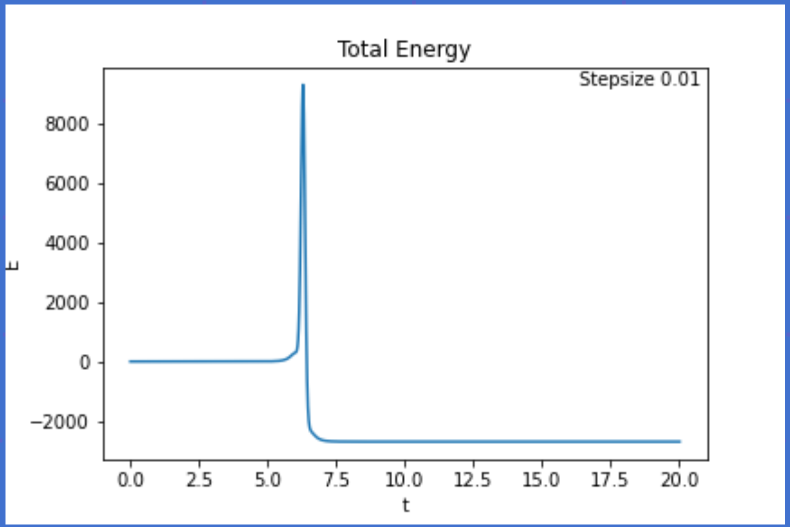
Parameters we have explored

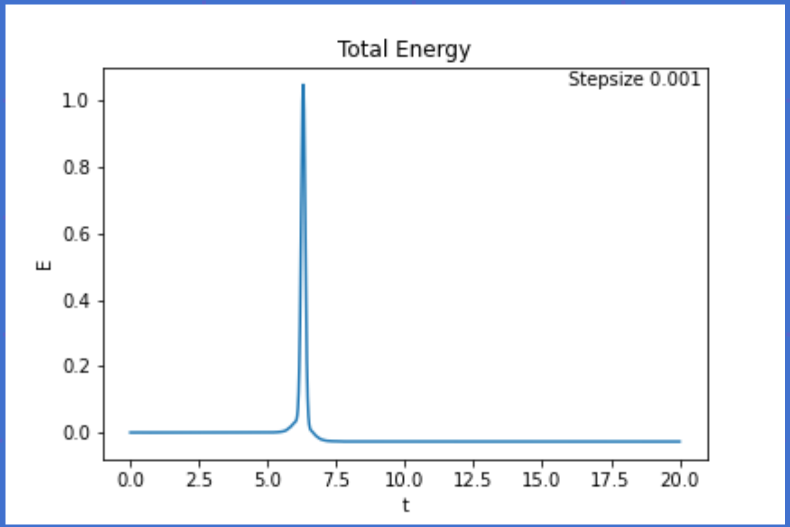
- Total energy
- Number of massless test particles(n)
- Step size(h)
- Integration method
- Eccentricity (parabolic vs eccentric)
- Minimum distance (r_min)
- Initial conditions
- Masses of the galaxies(consider unequal case)

Runge Katta Timestep (h)

The accuracy of the model is measured by the energy of the system.
Gravity is a conservative force, so energy should stay constant throughout the simulation.
With smaller timesteps, the energy spikes became smaller.

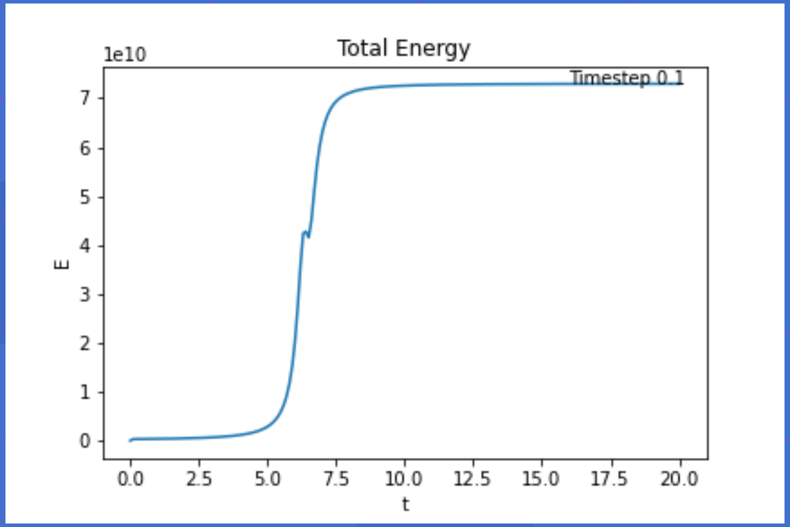


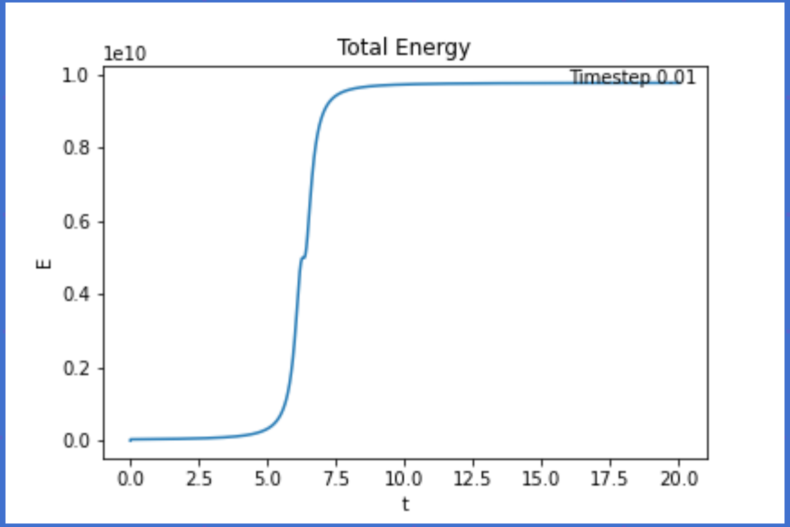


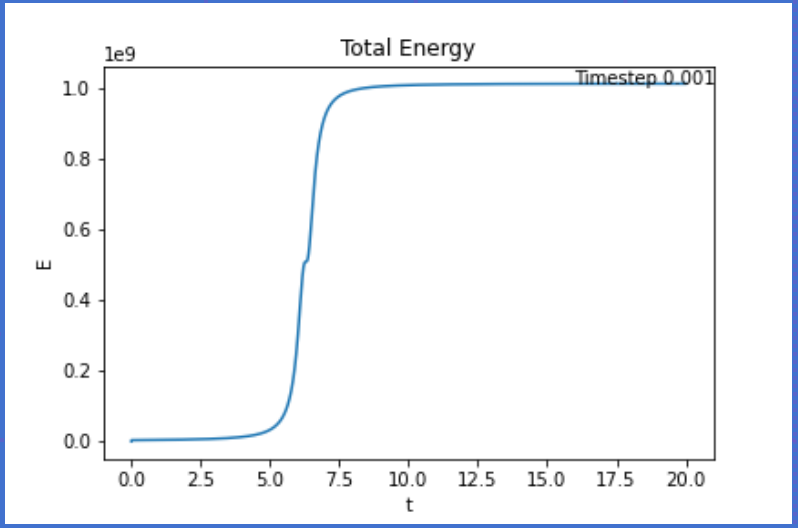


Leapfrog Timestep

Energy does not remain constant throughout the simulation
When energy spikes up, it does not come back down
Runge Katta is a better method for this simulation







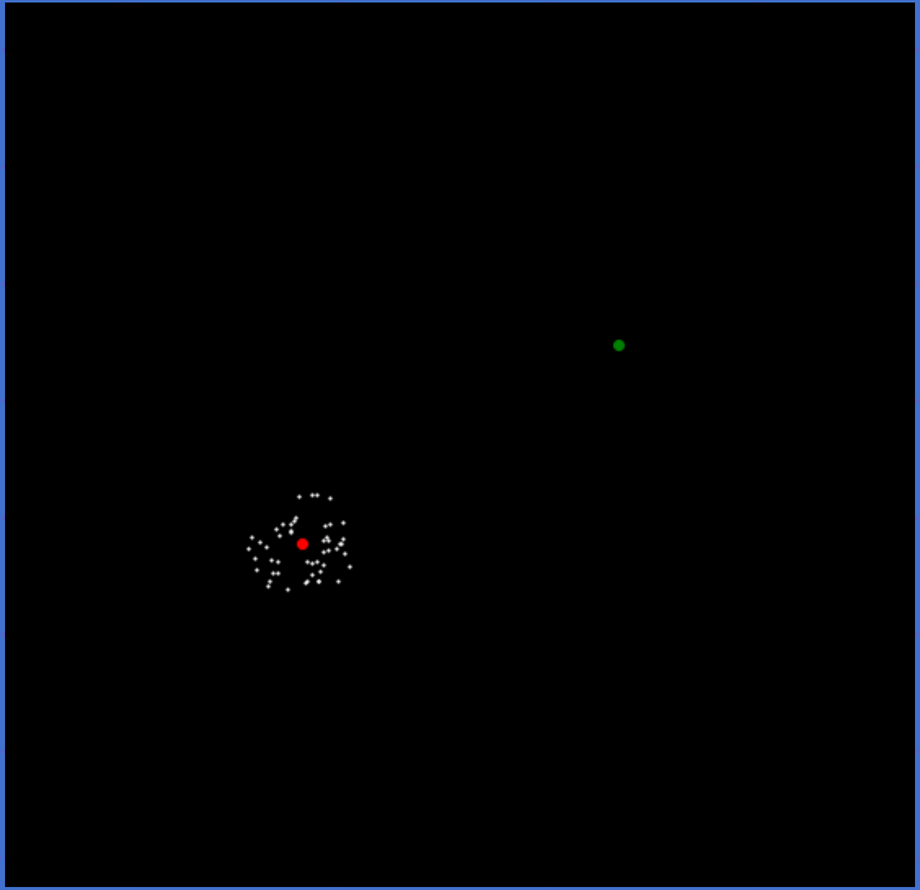
Number of massless test particles

- n = 20

vs

n =50

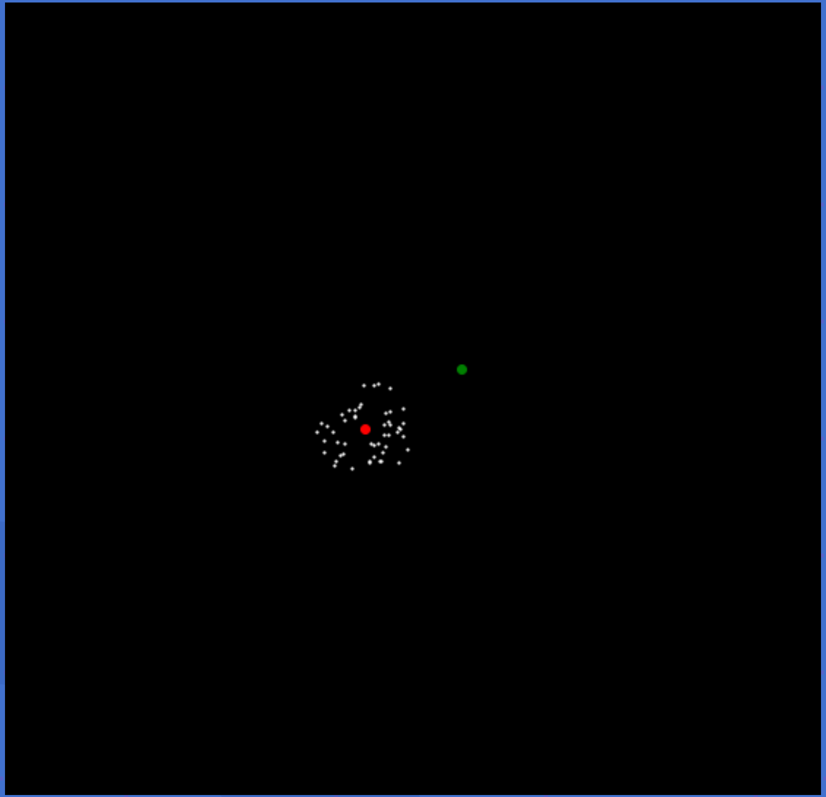
Fraction left the orbit 0.9 = 18 stars



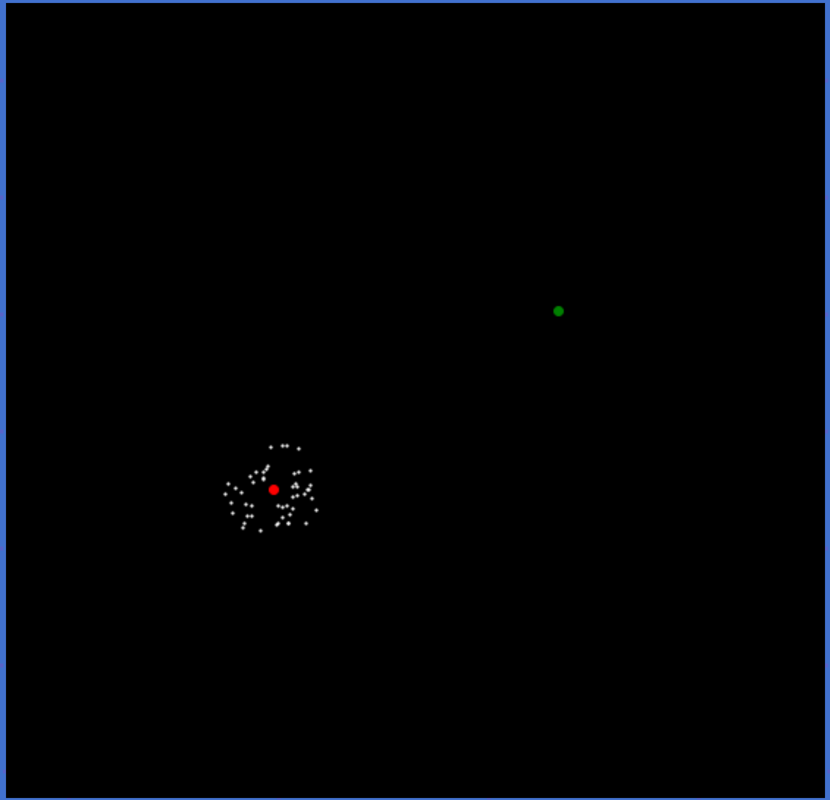
Fraction launched 0.88 = 44 stars

Eccentricity

- Eccentric orbit of the galaxies vs parabolic (0.5 vs 1)



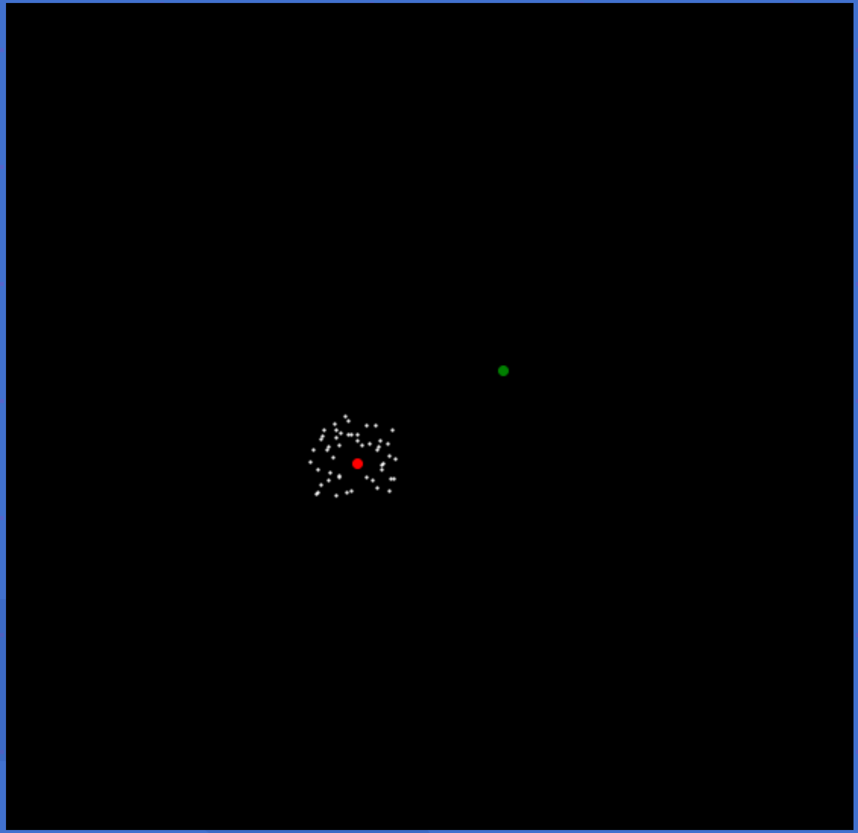
Fraction launched 0.78 = 39 stars



Fraction launched 0.88 = 44 stars

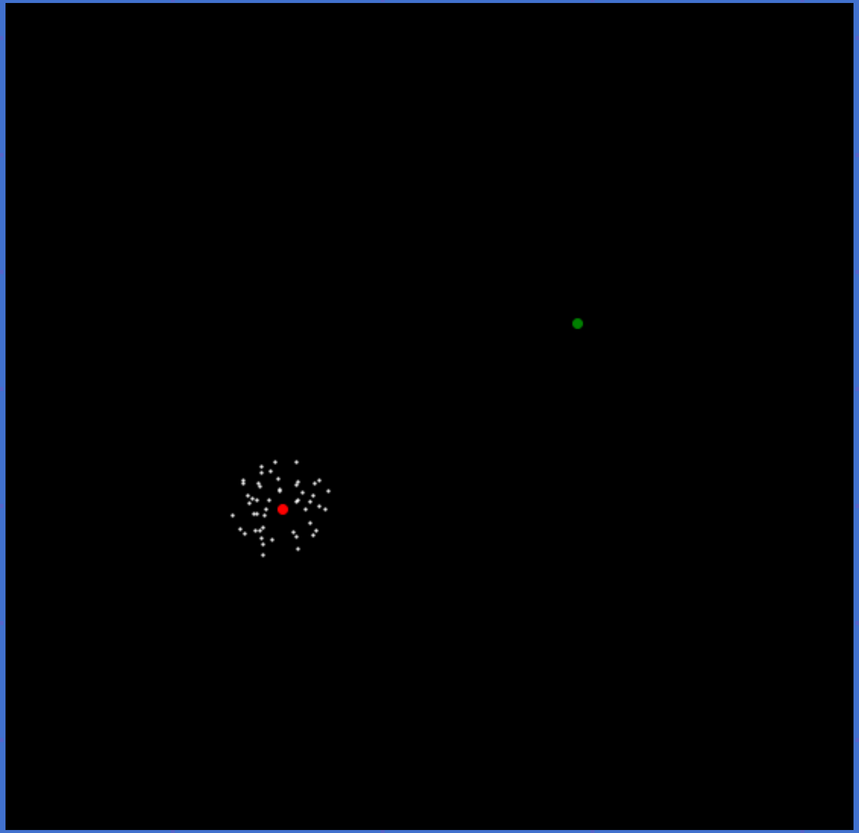
Minimum distance (r_min)

R_min = 0.5



Fraction launched 0.84 = 42 stars

r_min = 1



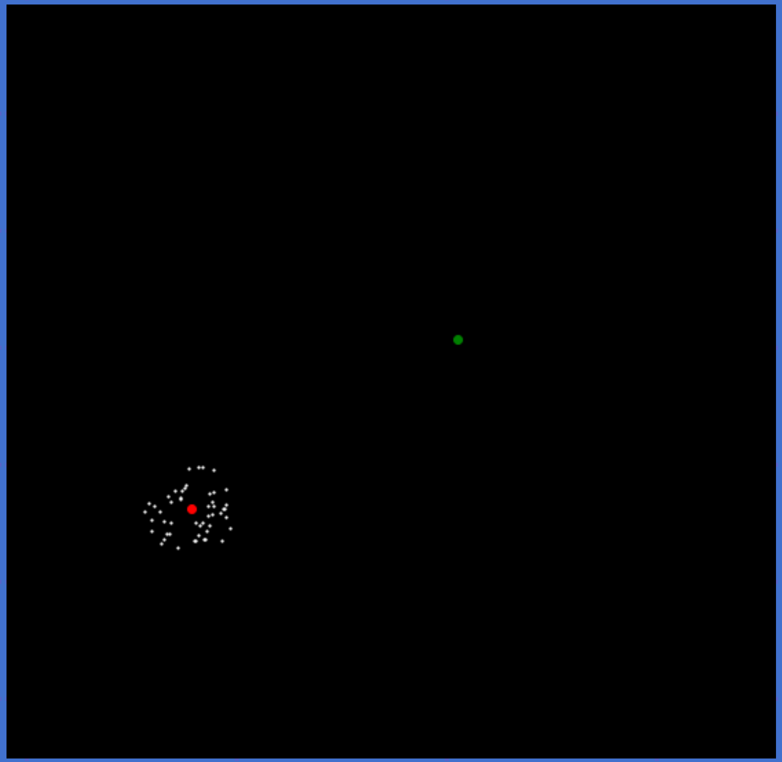
Fraction launched 0.86 = 43 stars

Mass Ratios

Galaxy 2 is a fourth the mass of Galaxy 1

Fraction launched 0.6= 30 stars

Galaxy 2 is three times the mass of Galaxy 1



Fraction launched 0.98 = 49 stars

Questions we have addressed:

- What ratio of the test particles are launched far away leaving either of the orbits around the galaxies?

Answer: Around 80% or more are scattered for every case!

- Is this ratio affected by the previously tested parameter?

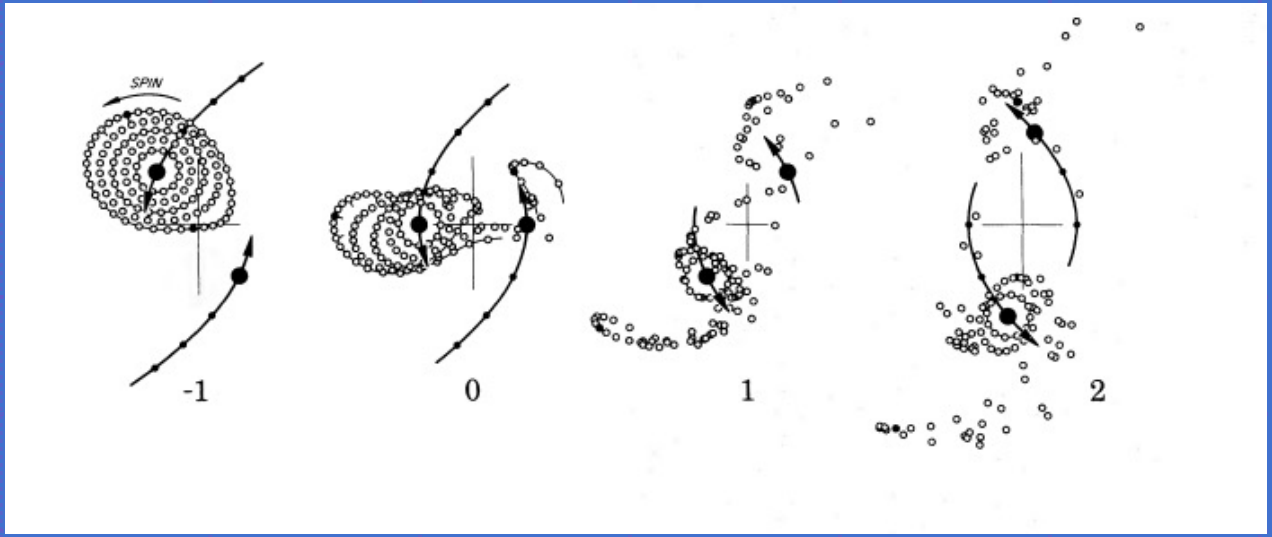
Answer: number of test particles, and minimum distance do not seem to affect the ratio remarkably. However, mass ratios of the galaxies and eccentricity do.

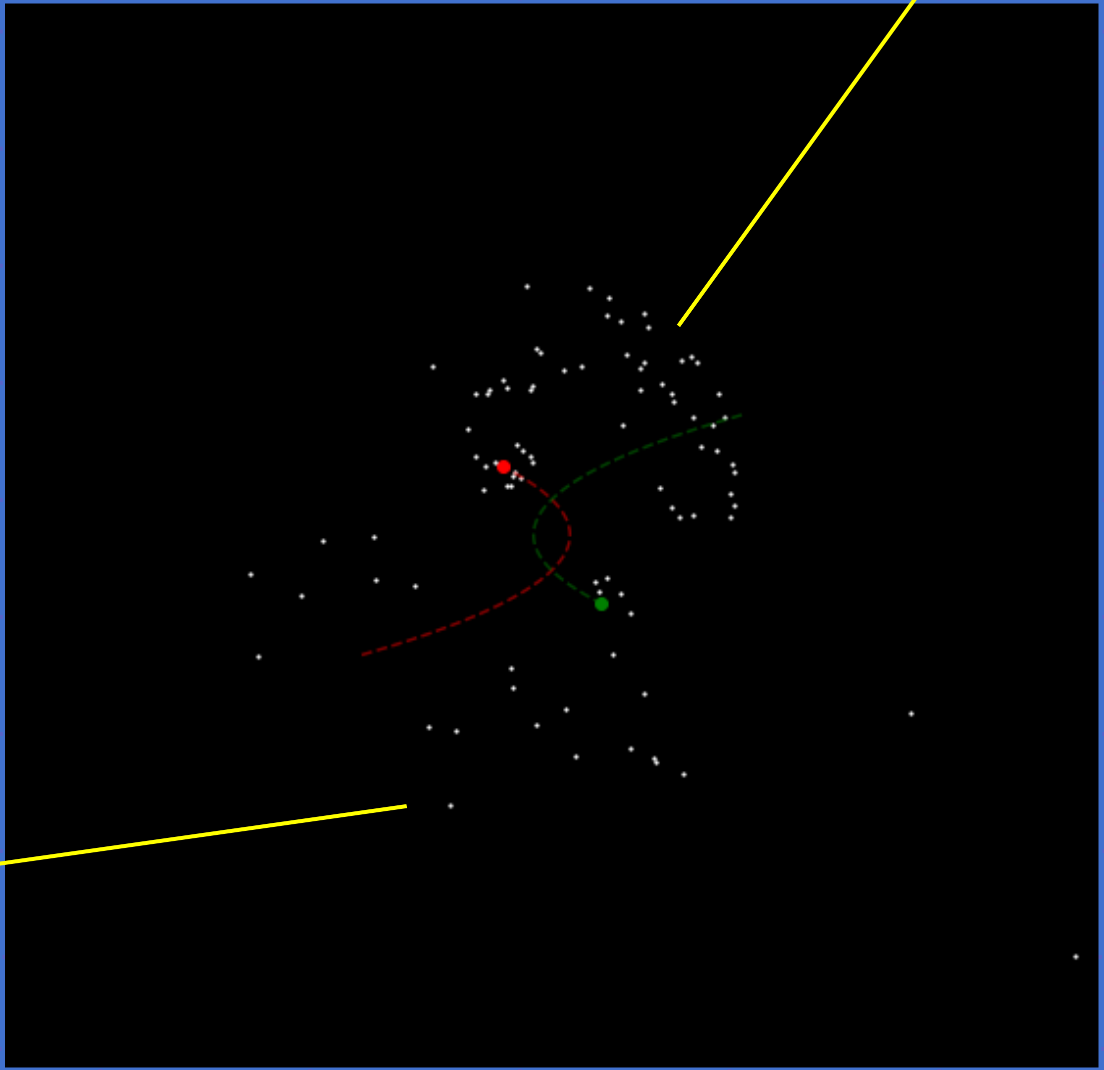
- Do our results visually agree with the Toomre paper – Galactic Bridges and Tails?

Answer: Look at the next slide :)

4. Do our results visually agree with Toomre paper – Galactic Bridges and Tails?

Galactic Tail





Galactic Tail

4. Do our results visually agree with actual collisions?