Dynamical properties of the J- and M- Sets?

* Julia Set?

A J-Set is a set that is associated with points $\boldsymbol{z = x + iy}$ of the complex plane for which the series $\boldsymbol{z(n+1) = z(n)^2 + c}$, where the constant complex c does not tend to infinity. In this, x and y are image pixel coordinates in a given range. In general, the accepted range is -2 and 2. The series expressed above is repeatedly updated with c another complex number that gives a specific J-Set. After a considerable number of iterations, if the magnitude of z is less than 2 one can state that pixel is in the J-Set and color accordingly.

* Mandelbrot Set?

An M-Set can be defined as the set of all the complex numbers z for which the series defined by the iteration $\boldsymbol{z(n+1) = z(n)^2 + z}$, for n = 0,1,2,3 ... where $z(0) = z$, remains bounded. By bounded we mean that there exists a number M such that for any n, absolute value of $z(n)$ is less than M. Plainly said, the M-Set can be looked at as the set of all the choices of c we can find (where z is initialized with 0) such that the iteration does not assume value bigger than 2.

The M-Set also well known as a map of all J-Sets uses a different c at each location as if transforming from one J-Set to another across space. For, c differs for each pixel and is

$x + iy$ where x and y are seen as image coordinates.

* What is the difference between the two sets?

Not only reside the differences between the two in the formula, but also in the type of iteration involved in their computation. In general, an M-set is plotted in the space centered at a given parameter c, which can be expressed in terms of $0, c, c^2, c^2 + c, (c^2 + c)^2 + c, \dots$. In contrast, a Julia-set is plotted in the orbit space z.

To be more precise, a plot of the M-set is obtained by iterating over many different values of c. In contrast, a plot of the J-set is obtained by iterating over many different values of z for a fixed c.

Computing J-sets and M-sets by computer is very easy by using the brute force methods. Despite having the same formula for iteration, two major changes can be noticed in the computation. With the J-set, we first start fixing a complex number c which never changes.

Second, we pick a complex point z in the plane. Third, we iterate and check if bounded. Fourth, count z as being in the J-set once founded bounded.

On the other hand, with the M-set we first start with an initial condition where z = 0. Second, we pick a point in the c plane. Third, we iterate and see if bounded, if so c should be counted as a point in the M-set.

With the M-set we are mainly plotting a picture of all the complex numbers c such that the mapping z to z^2+c is bounded with a certain initial condition. Opposed to that, with the Jset we are plotting a single value c while changing the initial condition

To resume, we can say that the M-Set :

a- gives a picture in parameter space
b- records the fate of the orbit 0
c- The Julia Set:
d- displays a picture in dynamical plane
e- records the fate of all orbits

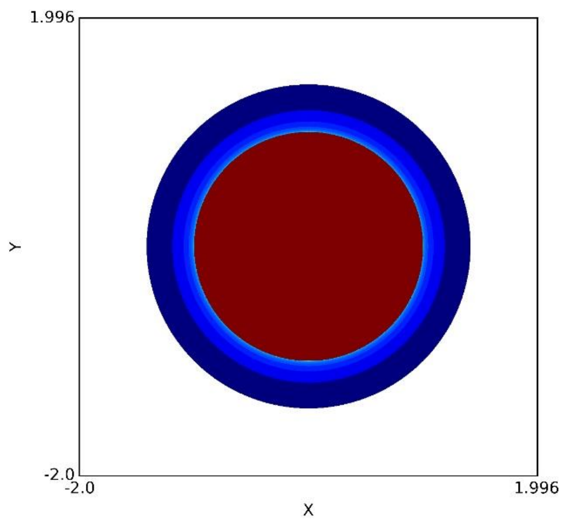
Trying to well understand the dynamic properties of the J- and the M- Sets begs the questions "What really happens to complex points as we iterate continuously on them via the function

For the investigations that will be conducted, let's consider the parametrized function

$$\boldsymbol{f(z) = z^2 + c}$$

Despite that a panoply of functions that can be produced by varying the complex c, let's mainly focus on the most well-known family of functions. In doing so, one should be aware that different features can be seen according to what the modulus of z can assume as value. For example, let's suppose c be a constant and If the modulus of z assumes larger values, then the modulus of z^2 will also be considerably larger. As result, the constant c would not have any major impact and the points always scape to infinity.

- **When the complex number $c = 0$**



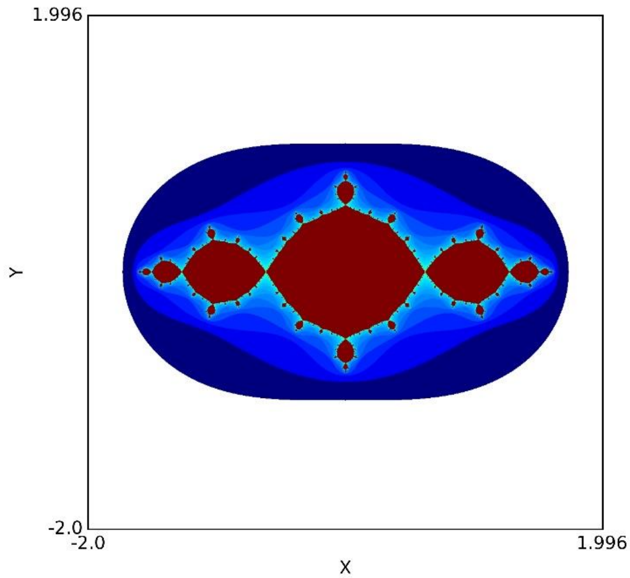
In c assumes the value of zero ($c = 0$), our famous equation will be reduced to$f(z) = z^2$. In this case, the iterations consists of squaring the radius of the complex point we are iterating. In the picture above, the dark red region contains the points that do not escape is called the Julia Set. Such a Julia Set is commonly named the 0-Julia Set. Despite being not very interesting, it constitutes a very good starting point for understanding Julia Sets.

Those points have radius that are less than 1. As learned from Number Theory, those points will keep being smaller through the entire iteration process till the exit condition is satisfied. The points of radius ie the ones on the unit circle will remain on the unit circle while moving around the origin; and the points with a radius bigger than 1 will always keep being larger in absolute value.

a- When C is a non-zero complex number

A small consideration of a non-zero complex number will produces a Julia set with amazing features.

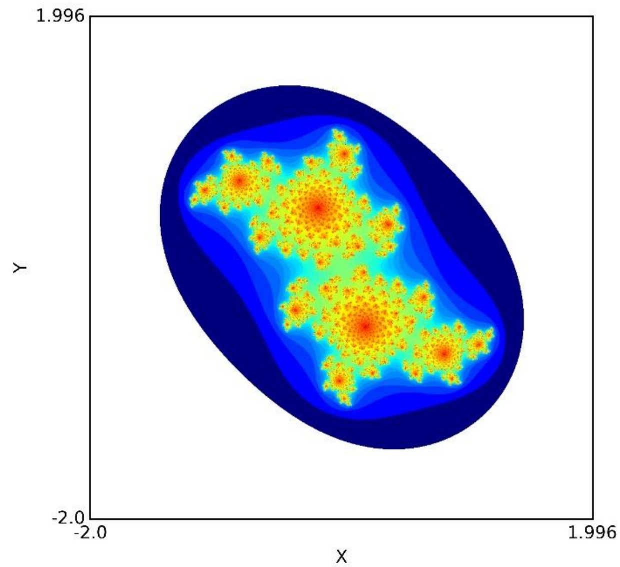
- $c = -1.0 + 0.0i$



In the above picture, the dark red region envelops complex numbers that give us iterates that do not escape to infinity. In the same picture, we can see some light blue and dark blue colored regions. The light blue colored regions informs us about how quickly the iterates escapes towards infinity. On the other hand, the darker blue colored region gives us an idea about how fast the sequences of numbers try to go to infinity.

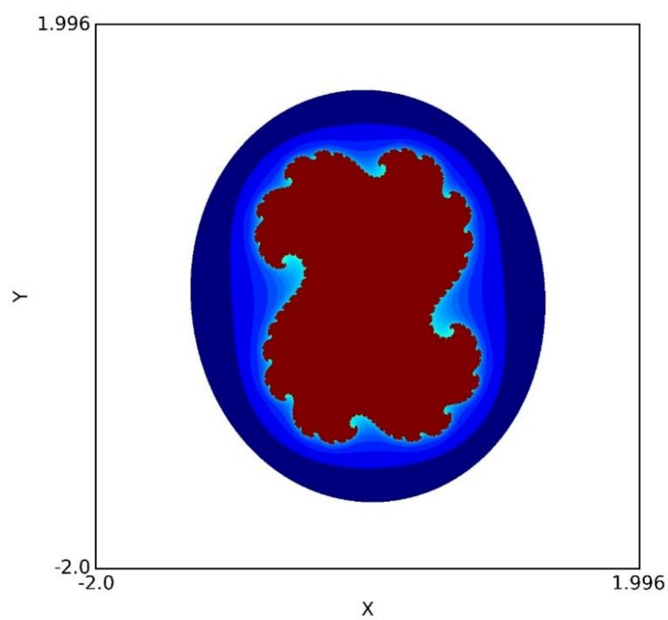
Empirically, we can see that the part of the set below the real axis can be found by reflecting the part above the real axis. Such a feature gives the Julia Set the reflective symmetry property.

- $c = 0.0 + 0.65i$



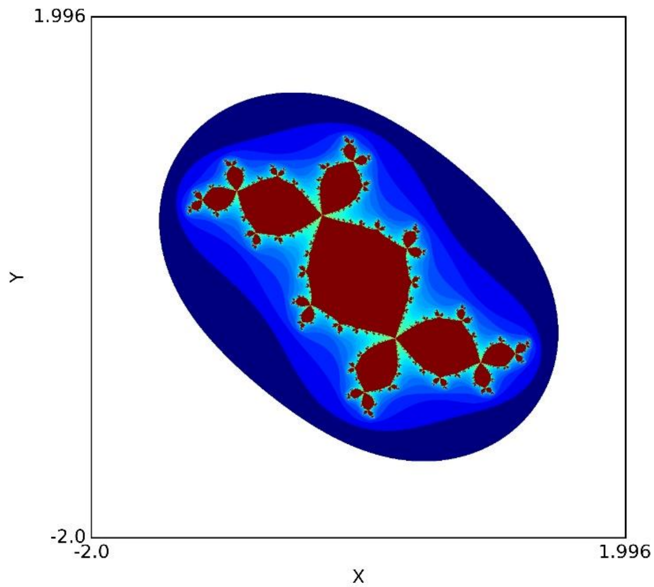
The above picture displays the output in the case of a pure imaginary complex number. We naked eyes, we can see a plot of two parts with an appearance of a rotational symmetry. Such feature gives the Julia Set the rotational symmetry property as well.

- $c = 0.295 + 0.055i$



The above picture shows the case where we a non-zero complex number. The shaded regions have the same interpretations as in the previous case. In addition to that, we can see an example of Julia set with a rotational symmetry.

- **Connectedness and disconnectedness of the Julia Set**

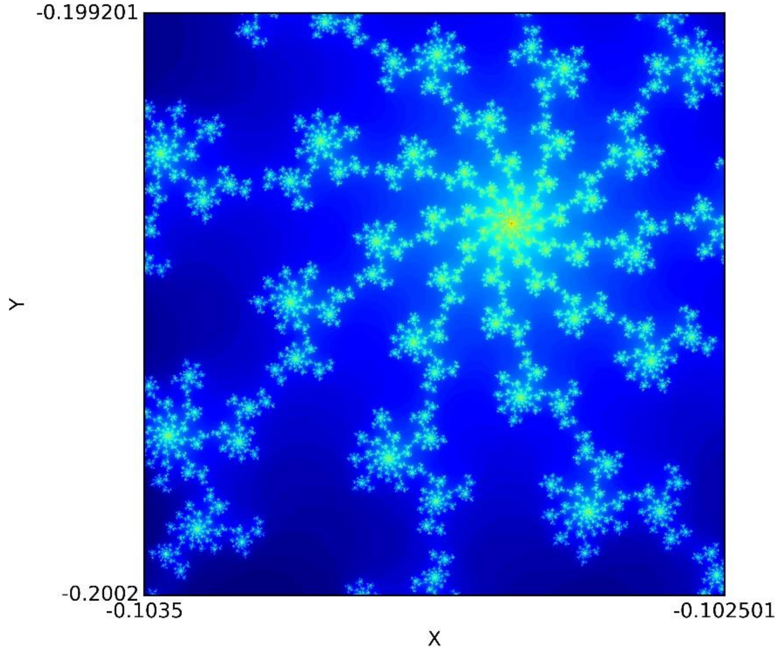


The above picture also falls in the case of a non-zero complex number $a + bi \neq 0$, where a and b are both non zero ($c = 0.125 + 0.731i$). Such a Julia set, well known as Douady's Rabbit, is a connected set. The points do not escape towards infinity are clearly noticeable.

However, not only are connected Julia sets obtained with the variation of the complex number c. Indeed, there a considerable number of complex points that leads to disconnected Julia Sets. The earlier picture obtained in the case of $c = 0.0 + 0.65i$ where the graph composed of disconnected sets with each of the set being connected is a perfect illustration.

Without doubt, this double appearance of the Julia Sets that contradict each other raises questions about the Julia Sets. However, this contradiction can be easily explained by the fact that anytime a point escapes toward infinity, its future iterates and all of its inverse images also escape towards infinity.

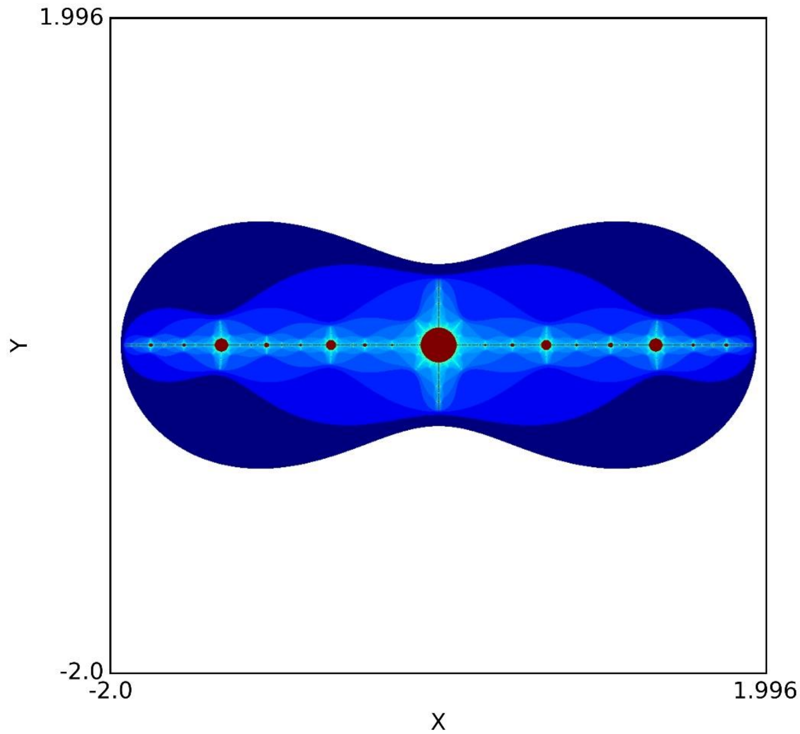
To better understand the structure of the Julia Set further steps were taken. In this, we fed the parameters that set the computation space that allowed us zoom a party of a given Julia Set. For, we set $\boldsymbol{xmin = -0.1035}$, $\boldsymbol{xmax = -0.1025}$, $\boldsymbol{ymin = -0.2002}$, $\boldsymbol{ymax = -0.1992}$, and $\boldsymbol{c = -0.70176 - 0.3842i}$



Zooming on different parts of a Julia Set gives us amazing pictures like the one displayed above. When we zoom again on a piece of the above plots, we can see that piece displays features that are quasi-similar to the first part that was zoomed before. Zoom it again. Same thing. So on and so forth, we can notice the presence of some features they share together. Thus, we can state that they all display self-similarity at different magnitudes. Such a feature is commonly called by Mathematicians as the Uniqueness of the Julia Set, which tells us that each Julia Set is unique.

By letting c assumes millions of values, we can millions of Julia sets. Among many others, the most well know Julia Sets are Douady's rabbit fractal ($\boldsymbol{c = -0.123 + 0.745i}$), the San Marco fractal ($\boldsymbol{c = -0.75}$), the Siegel disk fractal ($\boldsymbol{c = -0.391 - 0.587}$)… and the airplane that is displayed below :

- **C = -1.755 +0.0i**



Dynamic Properties of the Mandelbrot Set

For the M-Set, the mathematical formula is:

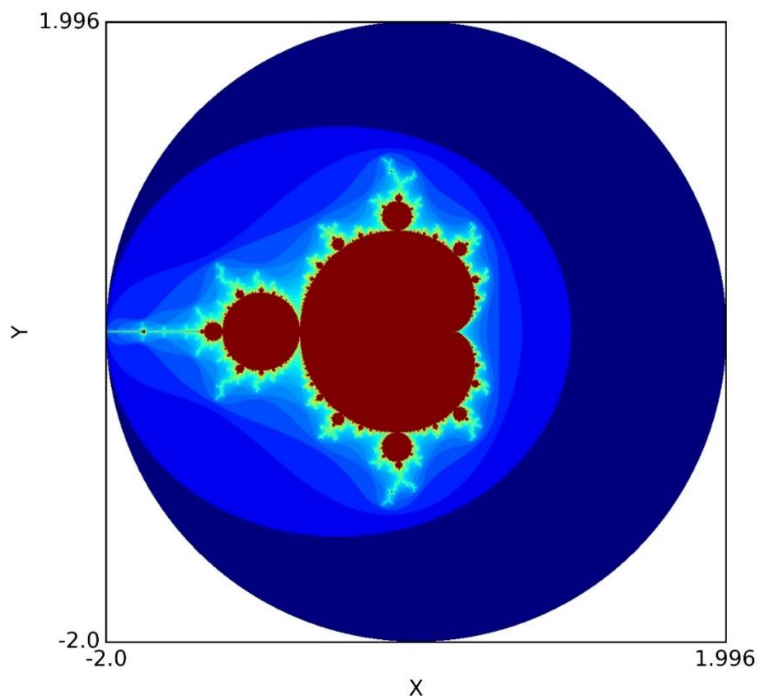
$$z = z^2 + c,$$

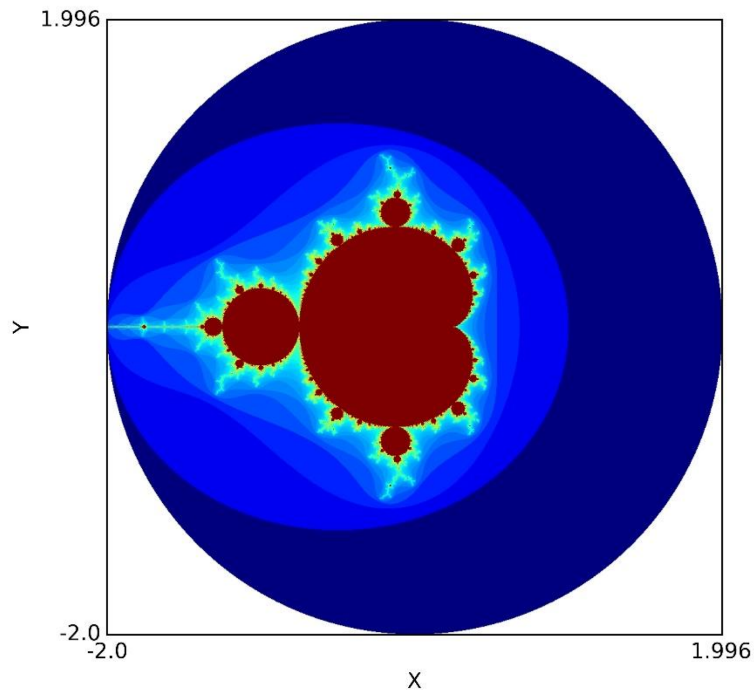
Where c is a fixed complex number?

In order to determine the M-Set, one should repeatedly iterate with the above equation. To do so, we start with given values for z and c. Once those values, one should plug them in the iteration equation to get a new z. Then, one will plug that new z to get a newer z.

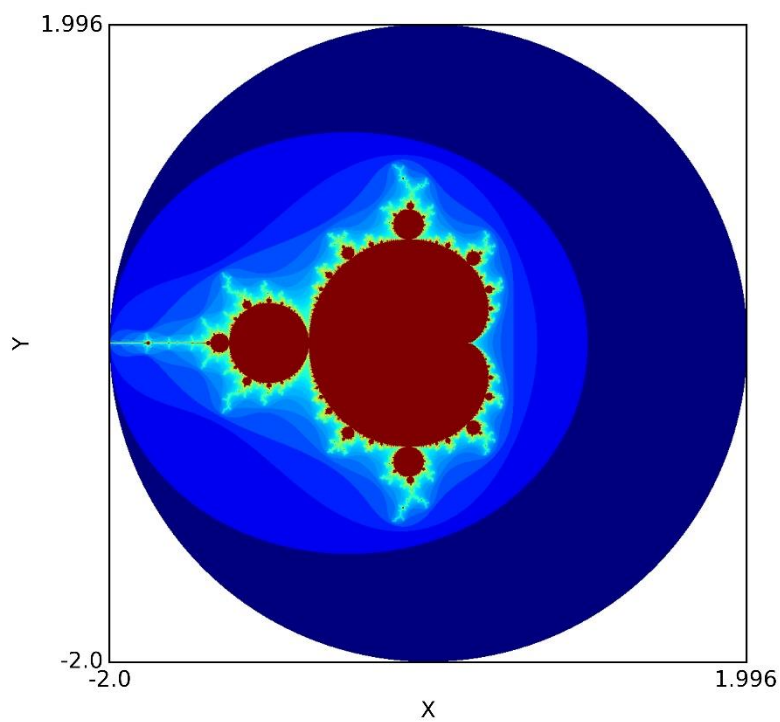
For a better understanding, let's take a look at a family of M-Sets while keeping in mind that the M- Set does not iterate over simple numbers but over complex numbers

- $c = 0$

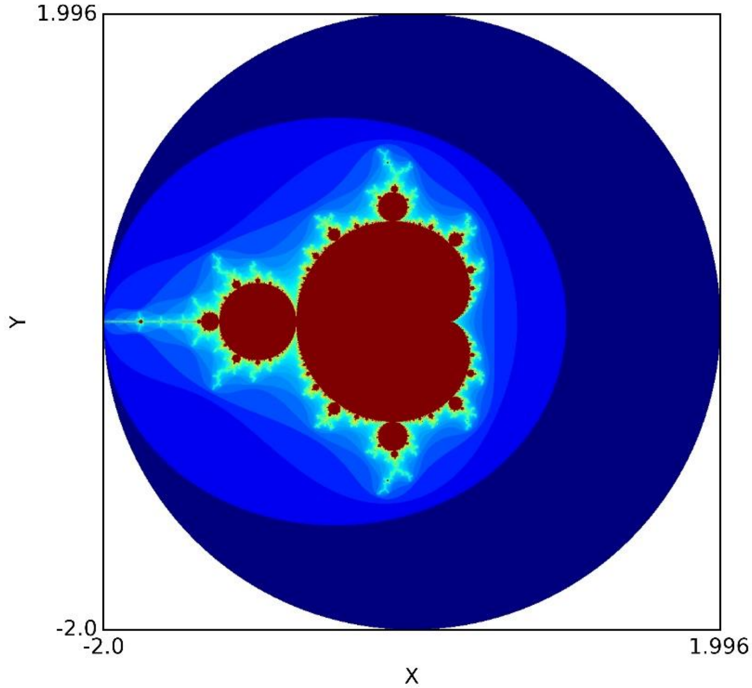




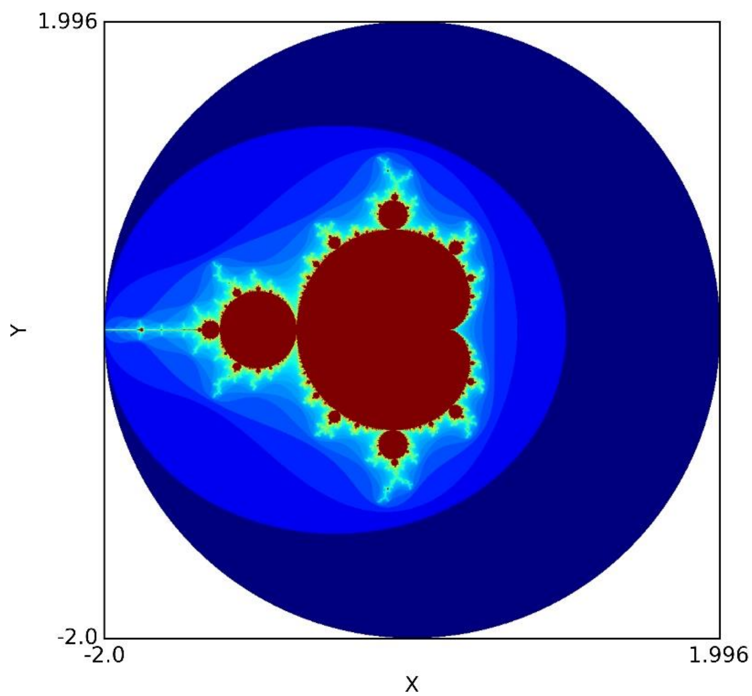
- $c = 0.0 - 0.65i$



- $c = 0.295 + 0.055i$



- $c = -1.755 + 0.00i$



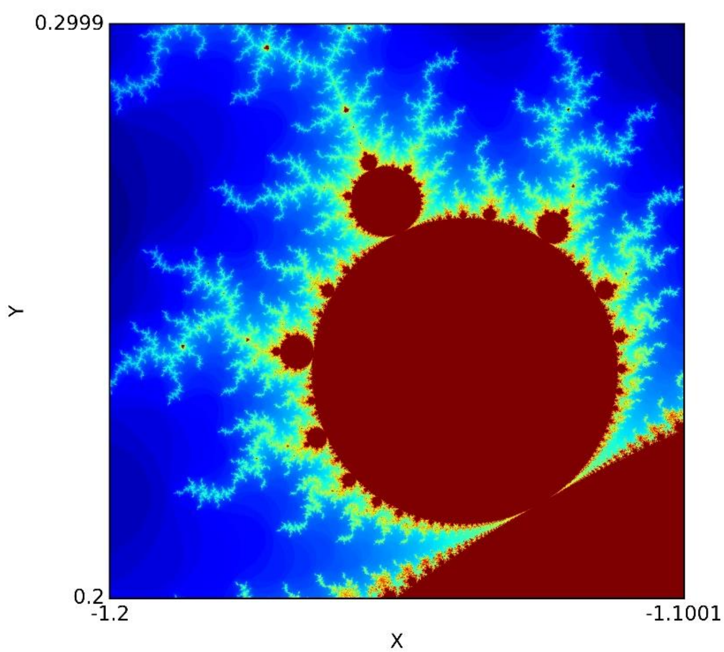
The above pictures show the Mandelbrot Sets achieved by considering the same complex numbers that allowed us earlier find the different Julia Sets. In each of plot, dark red colored glob located at the center of the computational space. On the periphery of the main glob can we also see a collection of small globs that look like the main one. From each of those small globes, mainly called pixels, emerge some yellow colored trajectories which indicate how many iterations it took for each of those pixels until the principal condition for not being in the Mandelbrot Set is satisfied. As mentioned while investigating the Julia Sets earlier, the light and dark blue regions tell us not only about the points that are not in the Mandelbrot Set, but also how quickly the point 0 escapes from a bounded region.

Also, in each of the pictures we can see that the Mandelbrot Sets are all symmetric with respect to the real axis. Such a feature informs us that if a complex number z is in the Mandelbrot Set then its complex conjugate is also in the same Mandelbrot Set. Furthermore, we can empirically see that in all the cases the Mandelbrot Sets lie in the interior of the circle radius 2. Thus, we can say that Mandelbrot Set is bounded.

To gain more information about the Mandelbrot Set, let's try to explore the complicated looking boundary of a given Mandelbrot Set. For, let's consider parameters as shown below:

$$xmin = -1.2, xmax = -1.1, ymin = 0.2, ymax = 0.3$$

Then, let's zoom a chosen portion of the boundary.



By naked eyes, one could be misled by the structure of the M-Sets allowing them say that M-Sets are exactly similar.

Above we can see a magnification of a given region of the dark red colored cardioid. On the top of this cardioid, there is another cardioid that is also studded with another small cardioid. When we zoom again on the last cardioid, we can realize that piece also contains, and consists of, another cardioid. A similar feature will appear again if we zoom in. By doing this, if possible, forever we will always notice the presence of another cardioid on the top of the other one.

With such an infinity cluster of cardioids connected by a filament, the question is to know whether or not those cardioids are similar?

In reality, by paying close attention and zooming through different angles they can notice that some of those cardioids, or those M-Sets, are kind of distorted. This underscores the **Quasi-self-similar feature** of the Mandelbrot Set.

Results

As results, we found that:

a- Julia Sets with purely real complex numbers are reflection symmetric
b- Julia Sets with complex c exhibit rotational symmetry
c- Depending on the value assumed by c, the resultant Julia Set may be connected or disconnected.
d- Each Julia Set is Unique
e- The Mandelbrot Set is symmetric with respect to the real axis
f- The Mandelbrot Set is bounded.
g- The Mandelbrot Set is Quasi- self-similar
h- The Mandelbrot Set is locally connected but not totally connected.