

Formulae

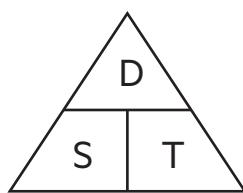
Compound measures

Distance, speed, time

$$\text{Distance} = \text{speed} \times \text{time}$$

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

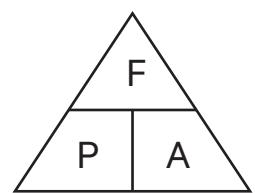


Force, pressure, area

$$\text{Force} = \text{pressure} \times \text{area}$$

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

$$\text{Area} = \frac{\text{Force}}{\text{Pressure}}$$

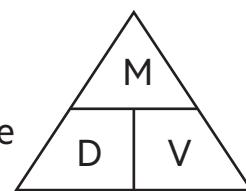


Mass, density, volume

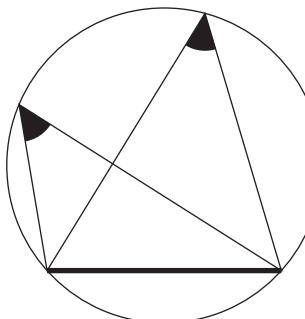
$$\text{Mass} = \text{density} \times \text{volume}$$

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

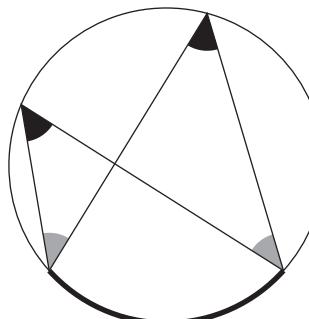
$$\text{Volume} = \frac{\text{Mass}}{\text{Density}}$$



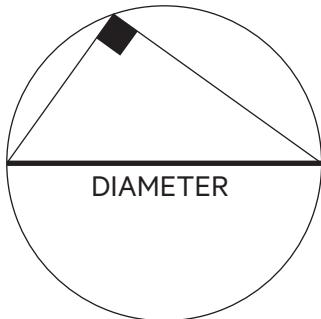
Circle theorem



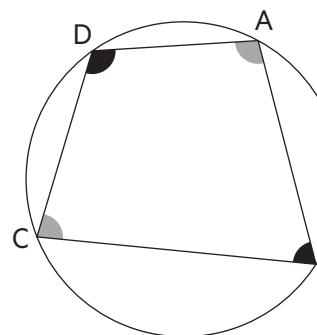
Angles in the same segment and standing on the same chord are **always equal**



Angles subtended from the same arc in the same segment are **always equal**

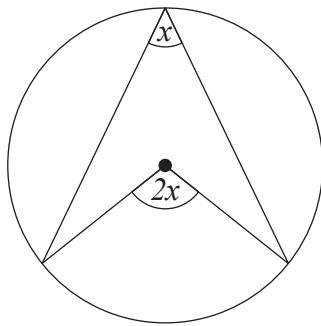


The angle in a semicircle is always **right angle**

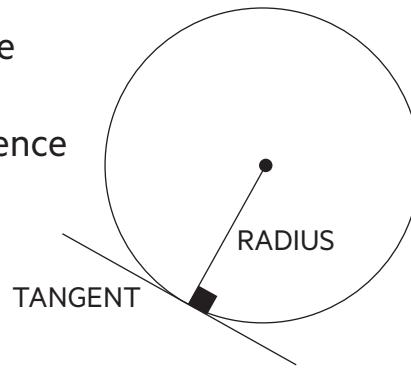


ABCD is a cyclical quadrilateral with all vertices on the circumference of the circle. **Diagonally opposite angles add up to 180°**

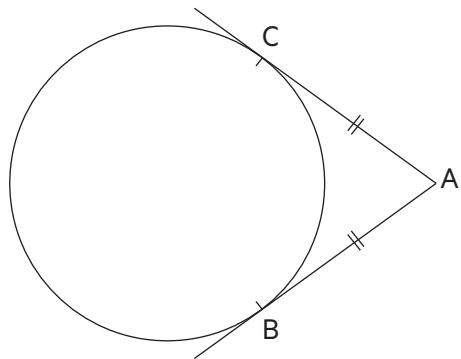
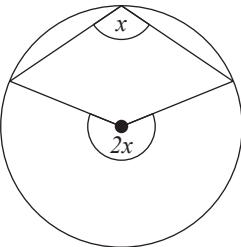
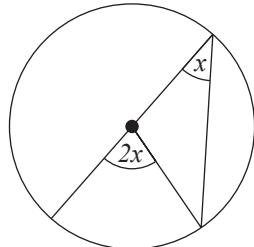
Circle theorem continued



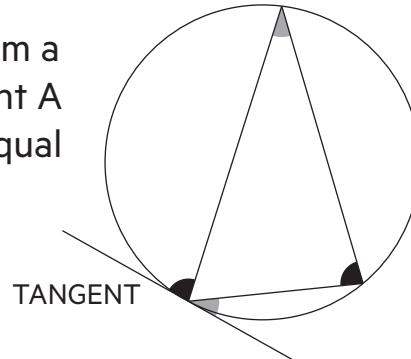
The angle at the centre
of a circle is **twice** the
angle at the circumference



The angle between
the tangent and the
radius is always **90°**



Tangents from a
common point A
are always equal
in length
AB = AC



The angle between a
tangent and a chord is
equal to the angle in
the alternate segment
(angle opposite the
chord)

Pythagoras theorem

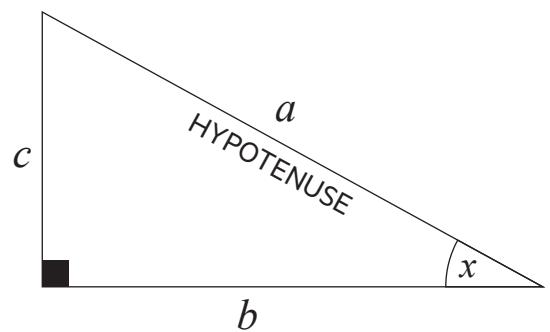
In a right-angled triangle the square of the hypotenuse
is equal to the sum of the squares of the other two sides

$$a^2 = b^2 + c^2$$

$$a = \sqrt{b^2 + c^2}$$

$$b = \sqrt{a^2 - c^2}$$

$$c = \sqrt{a^2 - b^2}$$



SOH CAH TOA

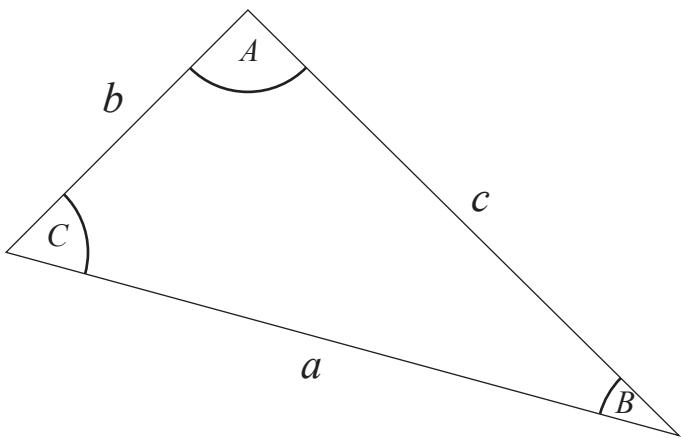
$$\sin(x) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos(x) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan(x) = \frac{\text{Opposite}}{\text{Adjacent}}$$

In a right-angled triangle the **hypotenuse is always opposite the right angle**

Trigonometry rules



Area of triangle

$$\frac{ab\sin(C)}{2}$$

Sine rule

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

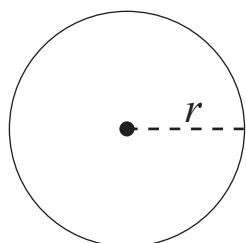
$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

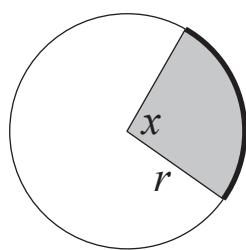
Area and volumes

Circle



$$\text{Area} = \pi r^2$$

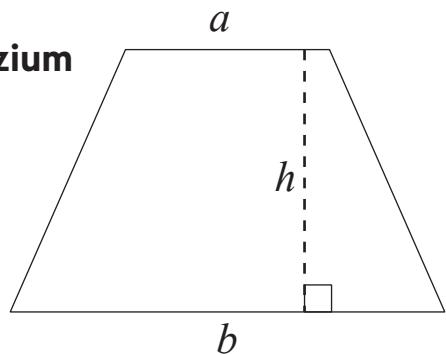
Sector



$$\text{Area} = \pi r^2 \times \frac{x^\circ}{360}$$

$$\text{Arc length} = 2\pi r \times \frac{x^\circ}{360}$$

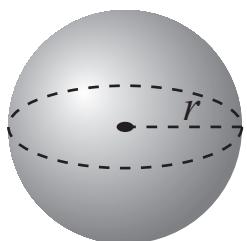
Trapezium



$$\text{Area} = \frac{(a + b)h}{2}$$

Area and volumes continued

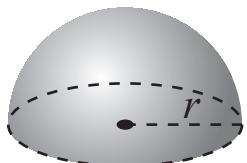
Sphere



$$\text{Surface area} = 4\pi r^2$$

$$\text{Volume} = \frac{4\pi r^3}{3}$$

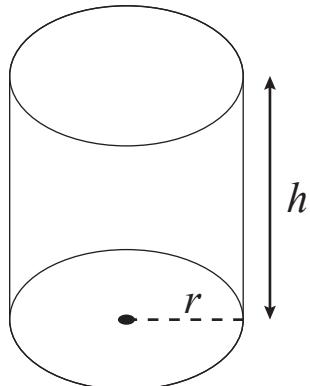
Hemisphere



$$\text{Surface area} = 2\pi r^2$$

$$\text{Volume} = \frac{2\pi r^3}{3}$$

Cylinder

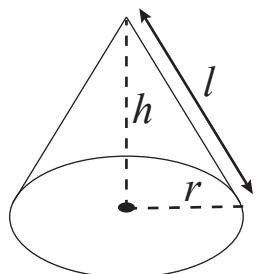


$$\text{Curved surface area} = 2\pi rh$$

$$\text{Total surface area} = 2\pi rh + 2\pi r^2$$

$$\text{Volume} = \pi r^2 h \text{ (base area x height)}$$

Cone

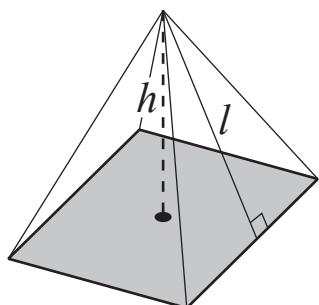


$$\text{Curved surface area} = \pi rl \text{ (where } l \text{ is the slant length)}$$

$$\text{Total surface area} = \pi r^2 + \pi rl$$

$$\text{Volume} = \frac{\pi r^2 h}{3}$$

Pyramid



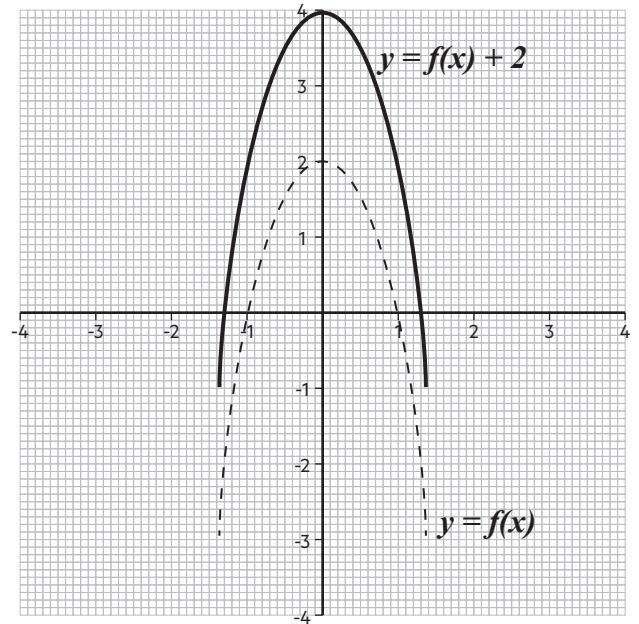
$$\text{Volume} = \frac{\text{base area} \times h}{3}$$

Transformations of functions

$f(x) \pm a \longrightarrow$ Translation by $\begin{pmatrix} 0 \\ \pm a \end{pmatrix}$

same sign up or down;
y changes

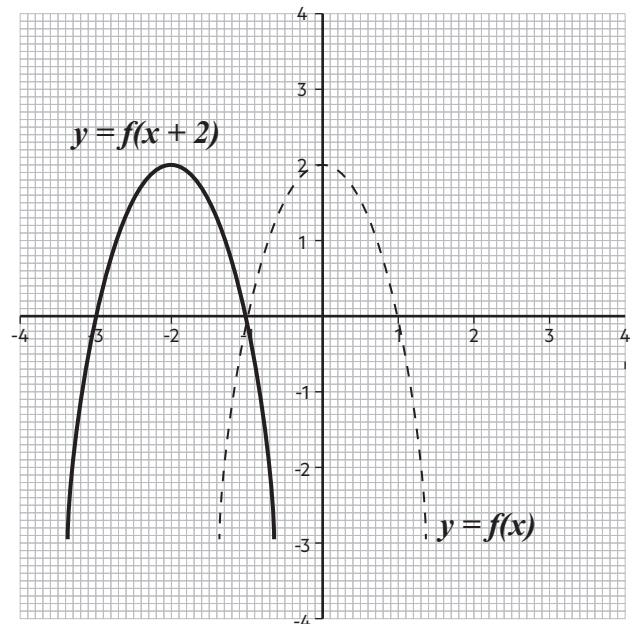
eg $y = f(x) + 2$



$f(x + a) \longrightarrow$ Translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$

left or right; **x changes**
 $+5 \longrightarrow$ 5 units **left**
 $-5 \longrightarrow$ 5 units **right**

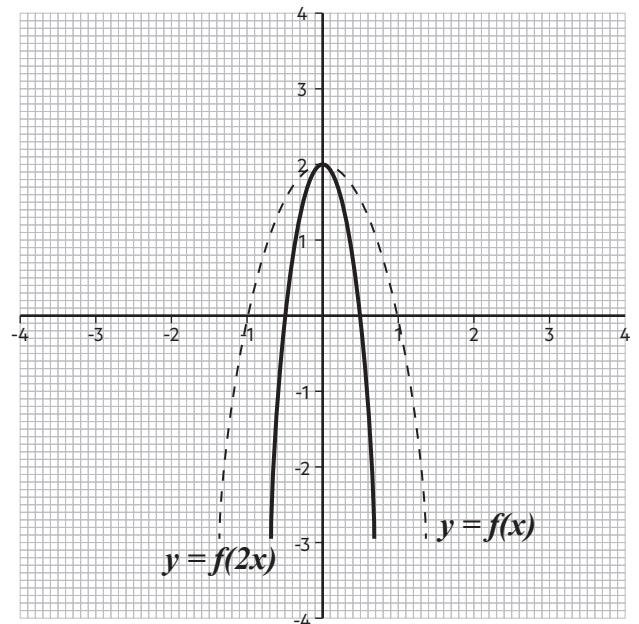
eg $y = f(x + 2)$



$f(ax) \longrightarrow$ Shrink by a scale factor of $\frac{1}{a}$; **x changes**

Period changes,
becomes smaller

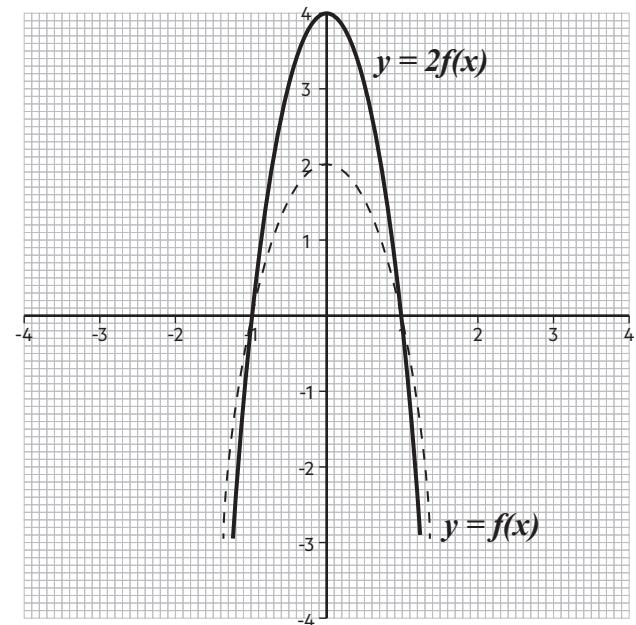
eg $y = f(2x)$



Transformations of functions continued

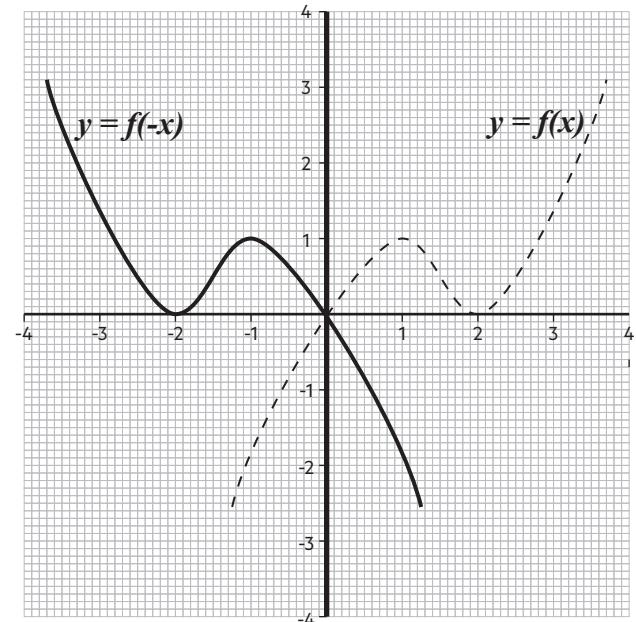
$af(x) \longrightarrow$ Enlargement by scale factor a ; y changes

eg $y = 2f(x)$



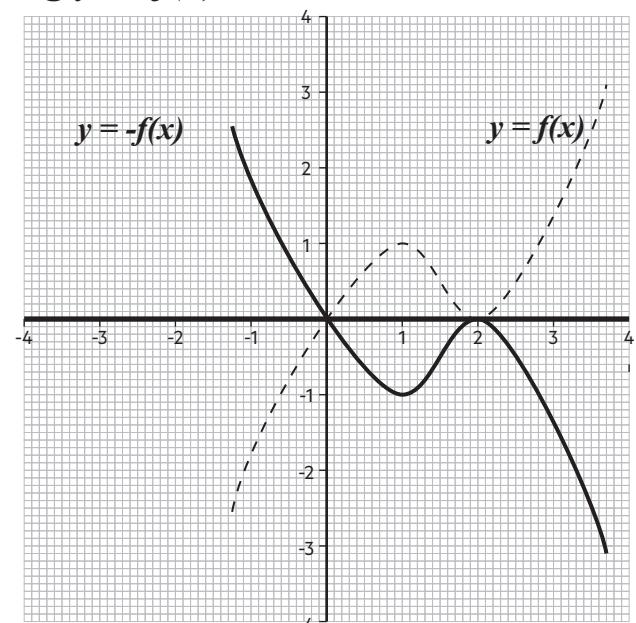
$f(-x) \longrightarrow$ Reflection in the y-axis; x changes

eg $y = f(-x)$



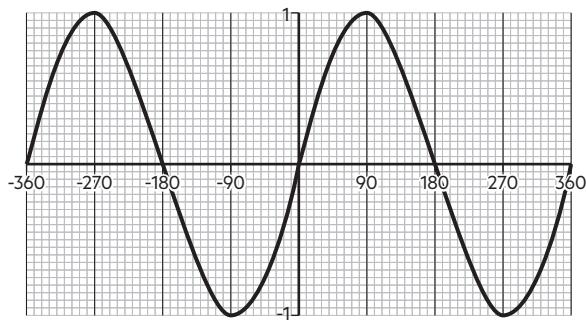
$-f(x) \longrightarrow$ Reflection in the x-axis; y changes

eg $y = -f(x)$



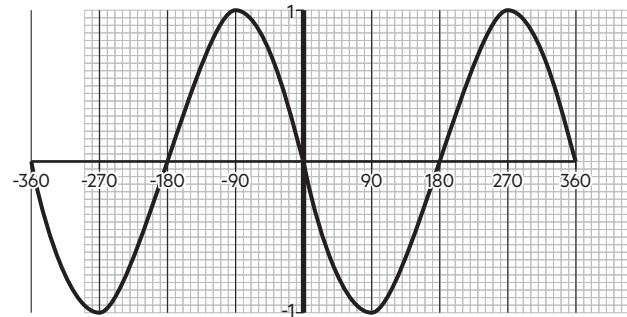
Sin, cos and tan graphs

Sin graph

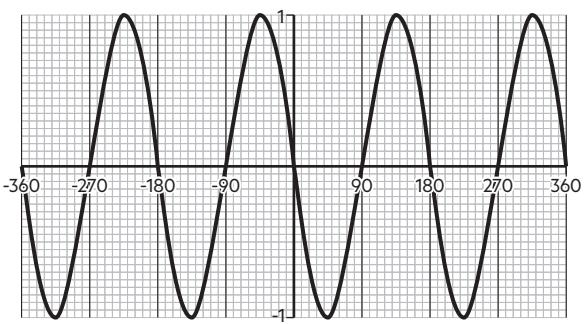


$y = \sin(-x)$ → Reflection in the **y-axis; x changes**

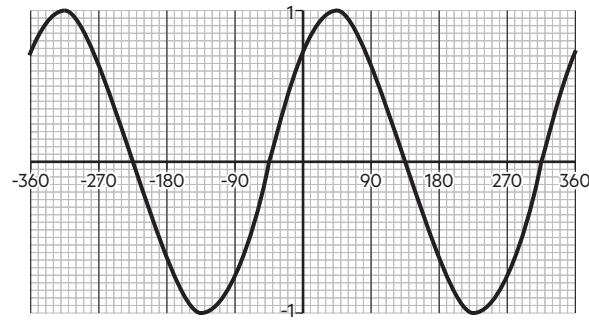
$y = \sin(-x)$ → Reflection in the **y-axis; x changes**



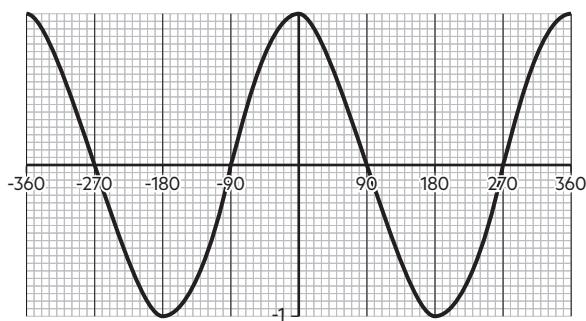
Period changes, becomes smaller



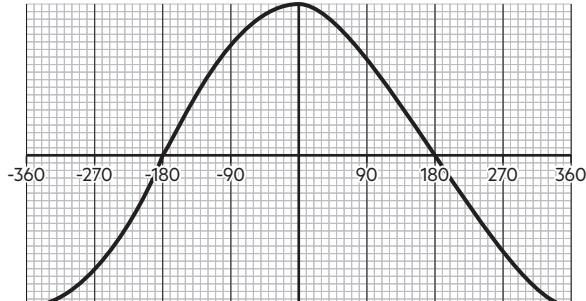
$y = \sin(x + 45^\circ)$ → Translate by -45° ; **x changes**



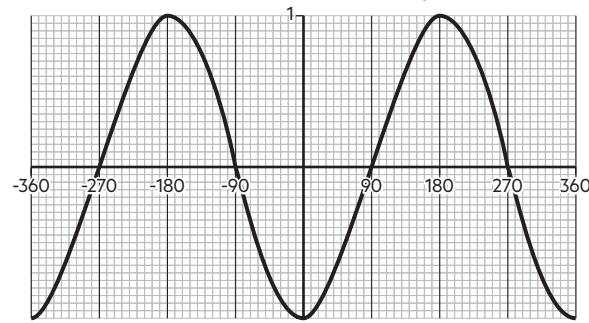
Cos graph



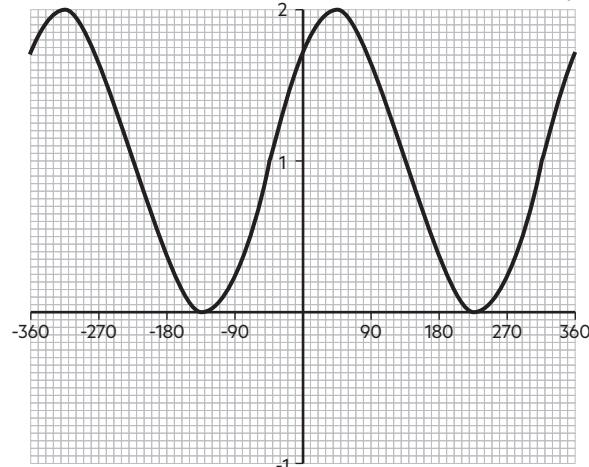
$y = \cos(\frac{1}{2}x)$ Scale by a scale factor of 2; **x changes**



$y = -\cos(x)$ → Reflection in the **x-axis; y changes**

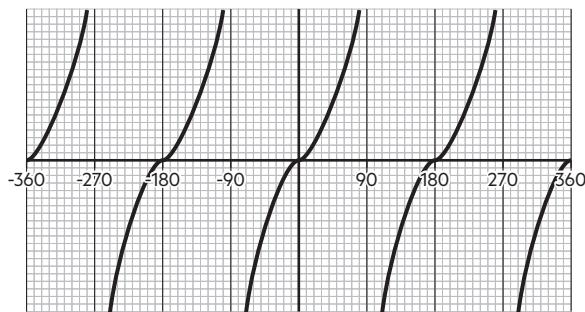


$y = \cos(x - 45^\circ) + 1$ → translate x by $+ 45^\circ$; translate y by $+ 1$



Sin, cos and tan graphs continued

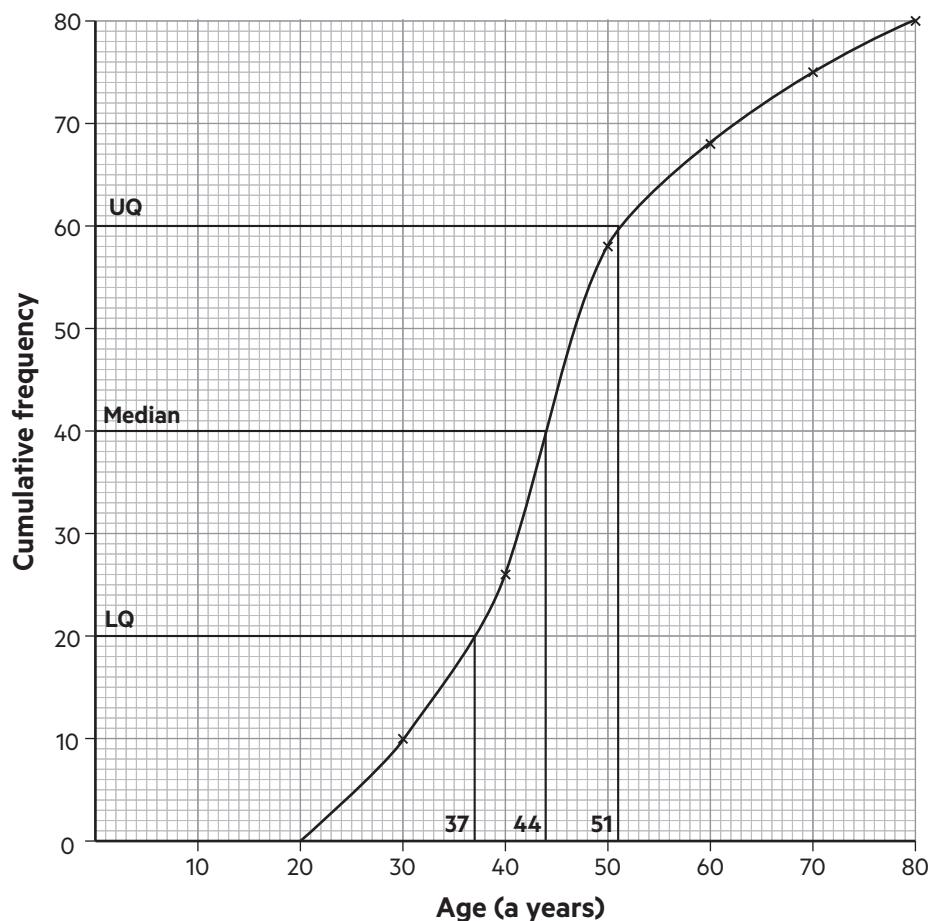
Tan graph



Cumulative frequency and box plots

Age (a years)	Frequency
$20 < a \leq 30$	10
$30 < a \leq 40$	16
$40 < a \leq 50$	32
$50 < a \leq 60$	10
$60 < a \leq 70$	7
$70 < a \leq 80$	5

When plotting a cumulative frequency graph, plot the marks at the **upper bound** of each range. Then join the points in a smooth curve



Cumulative frequency

Age (a years)	Cumulative frequency
$20 < a \leq 30$	10
$20 < a \leq 40$	26 ($10+16$)
$20 < a \leq 50$	58 ($26+32$)
$20 < a \leq 60$	68 ($58+10$)
$20 < a \leq 70$	75 ($68+7$)
$20 < a \leq 80$	80 ($75+5$)

To find the **median value** divide the total cumulative frequency (80) by 2, in this case 40. Draw a line across from the y-axis and down to the x-axis to get the median age → **44 years**

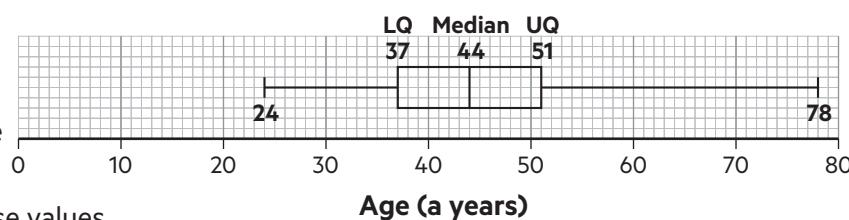
To draw a **box plot** of this graph we need to know the **Interquartile range (IQR)** – which is the range between the **lower quartile (LQ)** and the **upper quartile (UQ)** – the **maximum** and **minimum** values and the **median**

The minimum age was **24** and the maximum was **78**

To find the lower quartile divide the total cumulative frequency (80) by 4 → **20**

To find the upper quartile divide the total cumulative frequency (80) by 4, then multiply by 3 → **60**

Draw lines from y-axis and down to x-axis to get these values



Histograms

Bar charts and frequency diagrams show data grouped in **equal** class intervals.
For data grouped **unequal** class intervals, you need a **histogram**

In a histogram the area of the bar represents the frequency. The height of each bar is the frequency density

$$\text{Frequency density} = \frac{\text{Frequency}}{\text{Class width}}$$

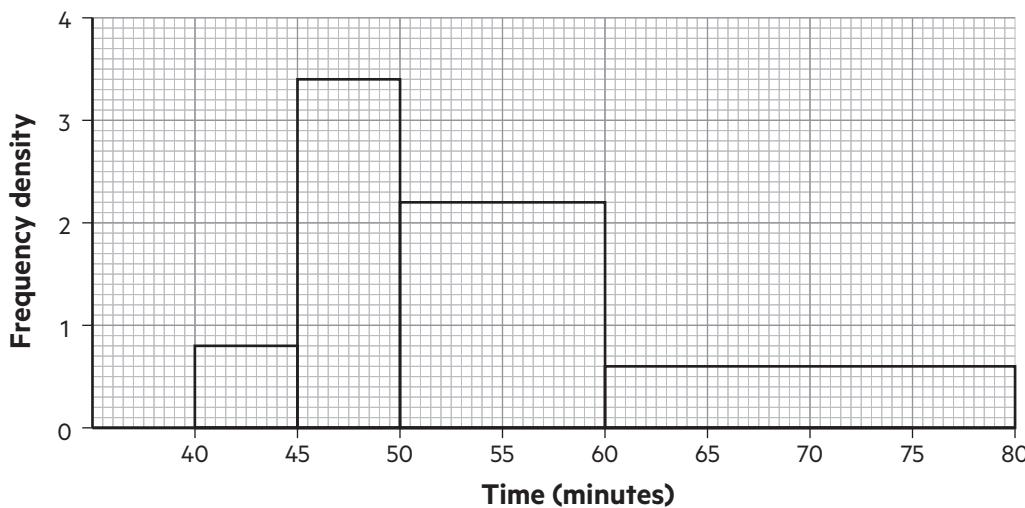
Example

This table shows the times taken for 55 runners to complete a fun run. Draw a histogram for this data

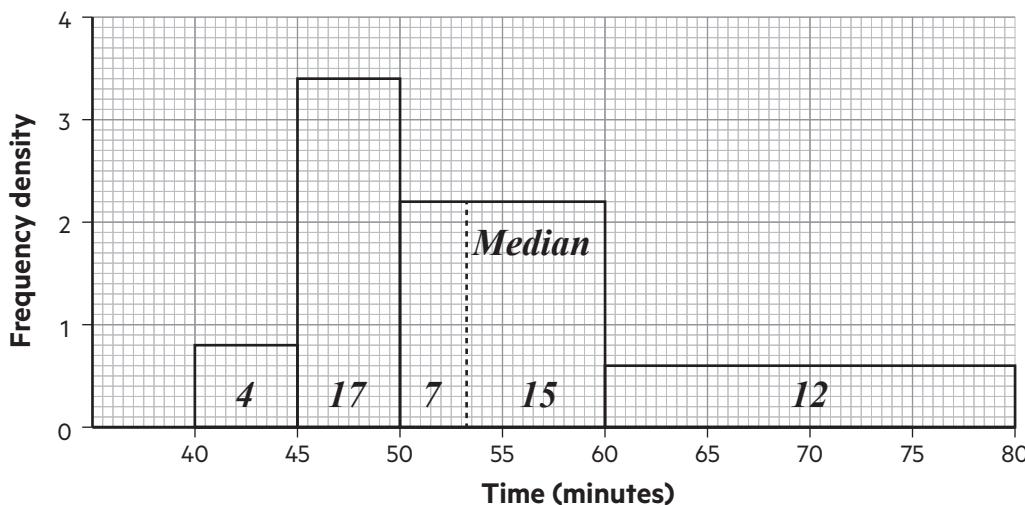
Time, t (minutes)	$40 < t \leq 45$	$45 < t \leq 50$	$50 < t \leq 60$	$60 < t \leq 80$
Frequency	4	17	22	12

Firstly you need to calculate the **frequency density**

Time, t (minutes)	$40 < t \leq 45$	$45 < t \leq 50$	$50 < t \leq 60$	$60 < t \leq 80$
Frequency	4	17	22	12
Frequency density	0.8	3.4	2.2	0.6



Estimating the median value of a histogram



Step 4

An estimate for the median is $50 + 3.18 = 53.2$ minutes (1dp)

Step 1

Find the median. The median is the 28th value...

$$\frac{55 + 1}{2} = 28$$

Step 2

You now know the median lies in the class $50 < t \leq 60$

Step 3

Use the frequency density formula to find the width of the class from 50 to the median.

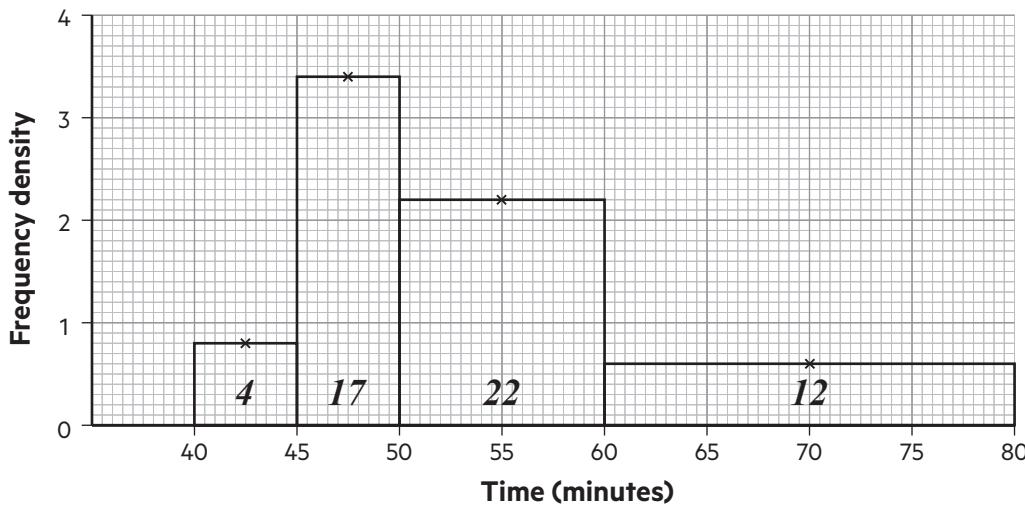
$$\text{Frequency} = 28 - (17 + 4) = 7$$

$$\text{Frequency density} = 2.2$$

$$\text{Class width} = 7 \div 2.2 = 3.18$$

Histograms continued

Estimating the mean value of a histogram



To find the mean of a histogram use the midpoint of each class interval and multiply by its frequency

$$4 \times 42.5 = 170$$

$$17 \times 47.5 = 807.5$$

$$22 \times 55 = 1,210$$

$$12 \times 70 = 840$$

Sum them up and then divide by the total number of runners

$$\frac{170 + 807.5 + 1,210 + 840}{55}$$

Mean time of runners = **50 mins**

Representing inequalities graphically

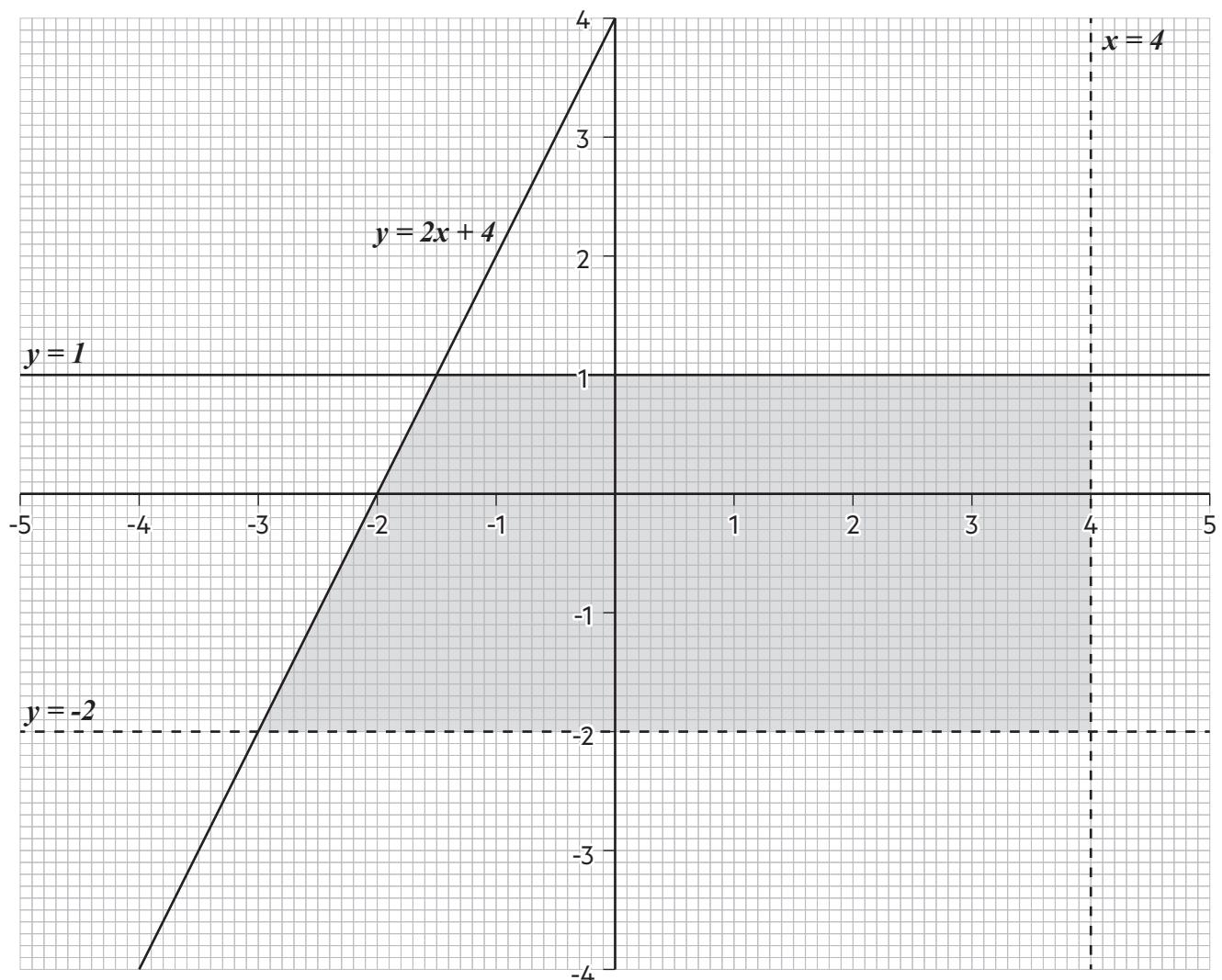
When drawing graphs of inequality, ignore the greater than ($>$, \geq) and less than ($<$, \leq) symbols and draw the lines as if they were = signs

eg. Shade the region that satisfies the inequalities: $x < 4$, $y \leq 2x + 4$, $y > -2$, $y \leq 1$

So when drawing the lines change the inequalities to $x = 4$, $y = 2x + 4$, $y = -2$, $y = 1$

You use solid lines when the values are included and dotted lines when they are not
ie

$x < 5$ – dotted $y \leq 2x + 4$ – solid $y > -2$ – dotted $y \leq 1$ – solid



Quadratic sequences

Example 1

Find the n th term of the non-linear (quadratic) sequence

5, 12, 23, 38, 57 ...

	5	12	23	38	57
First difference	\ 7	\ 11	\ 15	\ 38	
Second difference	\ \ 4	\ \ 4	\ \ 4	\ \ 4	

Using the equation $an^2 + bn + c$

The second differences of a quadratic sequence are constant and equal to $2a$

$$\text{So } a = 4 \div 2 = 2$$

The formula has a $2n^2$ term in it, next note the difference between the sequence $2n^2$ and the original sequence

$2n^2$	2	8	18	32	50
Sequence	5	12	23	38	57
Change	+3	+4	+5	+6	+7

If the change is **not constant**, then find the difference to find the value of b

The formula has a $2n^2 + n$ in it, to find the constant c ...

$$2n^2 + n + c = 5, \text{ when } n = 1$$

$$2 + 1 + c = 5$$

$$c + 3 = 5$$

$$c = 5 - 3$$

$$c = 2$$

So the formula for the n th term is $2n^2 + n + 2$

Example 2

Find the n th term of the non-linear (quadratic) sequence

3, 9, 19, 33, 51 ...

	3	9	19	33	51
First difference	\ 6	\ 10	\ 14	\ 18	
Second difference	\ \ 4	\ \ 4	\ \ 4	\ \ 4	

Using the equation $an^2 + bn + c$

The second differences of a quadratic sequence are constant and equal to $2a$

$$\text{So } a = 4 \div 2 = 2$$

The formula has a $2n^2$ term in it, next note the difference between the sequence $2n^2$ and the original sequence

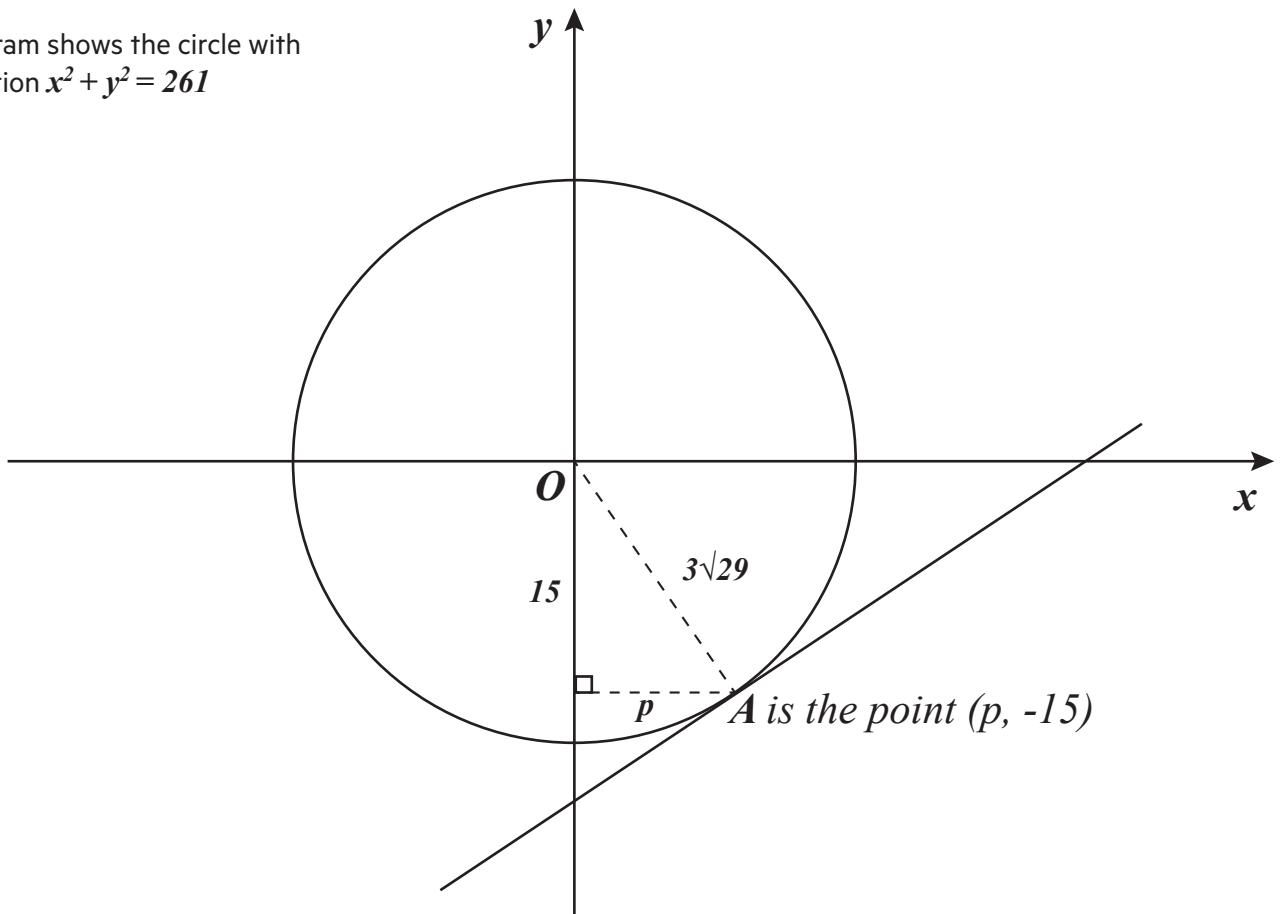
$2n^2$	2	8	18	32	50
Sequence	3	9	19	33	51
Change	+1	+1	+1	+1	+1

If the change is **constant** then $b = 0$, and c is equal to the change

So the formula for the n th term is $2n^2 + 1$

Finding the equation of the tangent to a circle

The diagram shows the circle with the equation $x^2 + y^2 = 261$



A tangent is drawn at point A with coordinates of $(p, -15)$, where $p > 0$
Find an equation of the tangent at A

Step 1

Calculate the radius of the circle

$$x^2 + y^2 = r^2$$

$$r^2 = 261$$

$$r = \sqrt{261}$$

$$r = 3\sqrt{29}$$

Step 2

Calculate p using Pythagoras theorem

$$(3\sqrt{29})^2 = 15^2 + p^2$$

$$261 = 225 + p^2$$

$$p^2 = 261 - 225$$

$$p^2 = 36$$

$$p = 6$$

Step 3

Calculate the gradient of the radius OA

$$\frac{\text{Change in } y}{\text{Change in } x} = \frac{-15}{6}$$

Step 4

The gradient of the tangent is the negative reciprocal of the gradient of the radius OA

$$= \frac{6}{15} = \frac{2}{5}$$

Step 5

Use $y = mx + c$ to calculate the equation of tangent

$$\begin{aligned} -15 &= \frac{2}{5}(6) + c & c &= \frac{-87}{5} \\ -15 &= \frac{12}{5} + c & y &= \frac{2}{5}x - \frac{87}{5} \\ -75 &= 12 + 5c & 5y &= 2x - 87 \\ 5c &= -75 - 12 = -87 & \end{aligned}$$

Similarity and congruence

Congruence

Two triangles are congruent when one of these conditions is true

SSS (all three sides are equal)

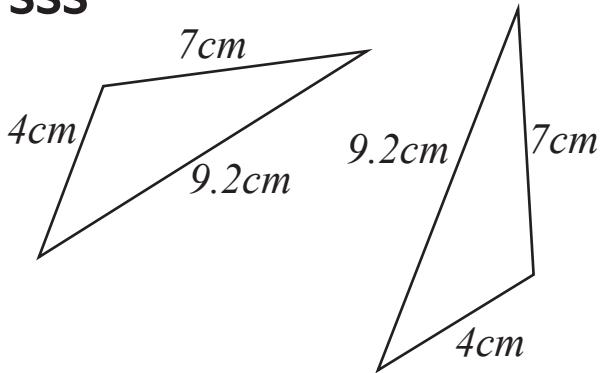
SAS (two sides and the included angle are equal)

AAS (two angles and a corresponding side are equal)

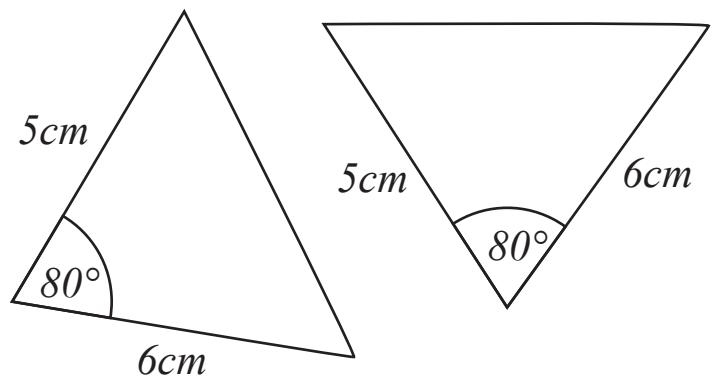
RHS (right angle, hypotenuse and one other side are equal)

Examples

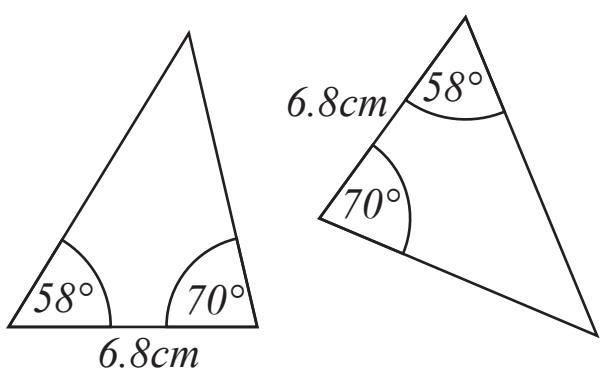
SSS



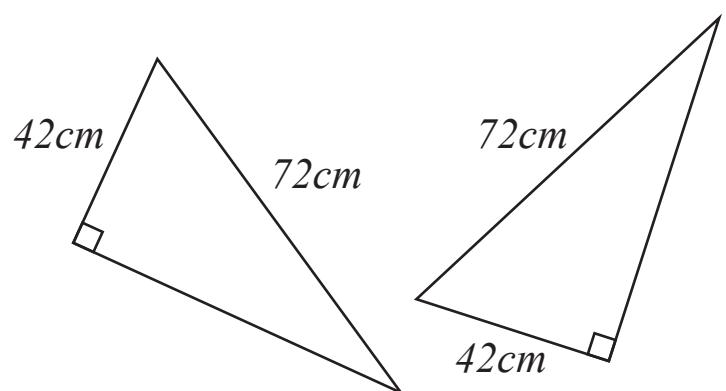
SAS



AAS

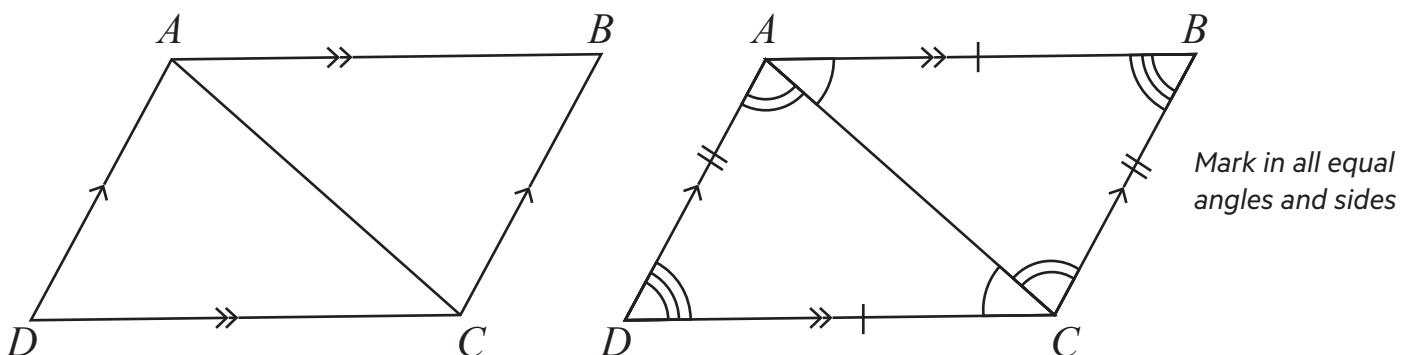


RHS



Example

$ABCD$ is a parallelogram. Prove that triangle ABC is congruent to triangle ADC



$AB = CD$ because opposite sides of a parallelogram are equal
 $BC = AD$ because opposite sides of a parallelogram are equal
 AC is common to both triangles
Therefore triangle ABC is congruent to triangle ADC (**SSS**)

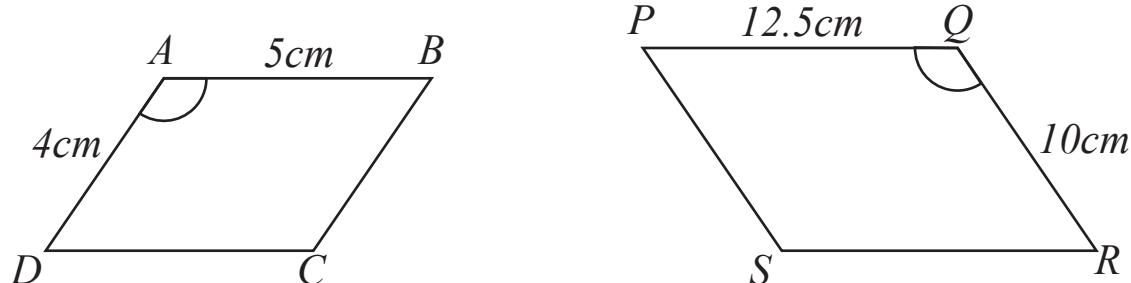
Similarity and congruence continued

Similarity

Shapes are **similar** when one shape is an enlargement of the other. **Corresponding angles are equal** and **corresponding sides are all in the same ratio**

Example 1

Here are two parallelograms $ABCD$ and $PQRS$. Angles DAB and PQR are the same. Prove that they are similar



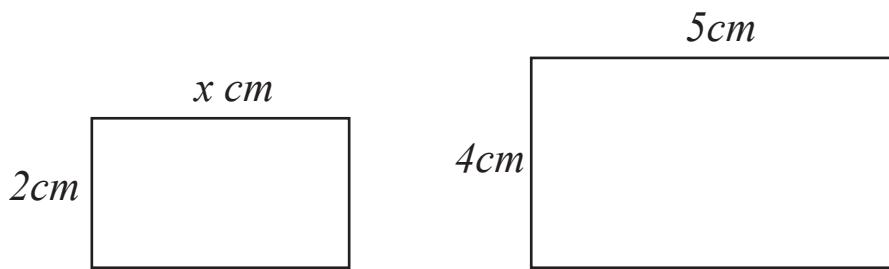
Find out the ratios of the corresponding sides, if they are the same the parallelograms are similar

$$\frac{AD}{QR} = \frac{4}{10} = \frac{2}{5} \quad \frac{AB}{PQ} = \frac{5}{12.5} = \frac{2}{5}$$

The ratios of the corresponding sides are the same therefore the shapes are similar

Example 2

These two rectangles are similar. Find the missing length x in the smaller rectangle

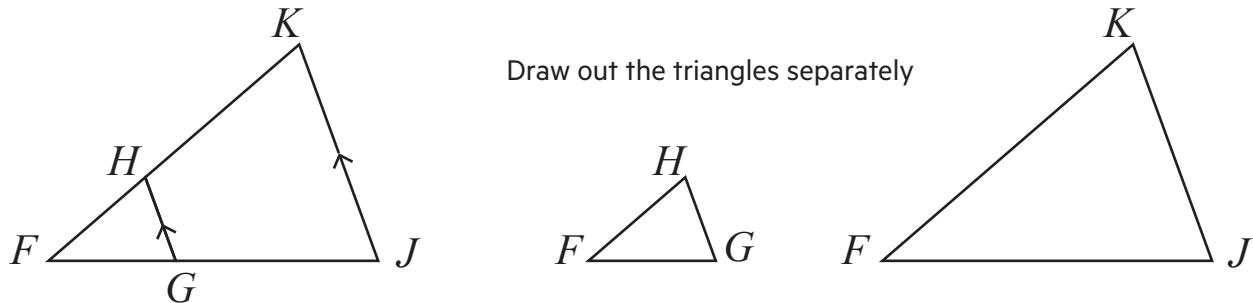


If the rectangles are similar the ratios of the corresponding sides will be the same, therefore

$$\frac{2}{4} = \frac{x}{5} \rightarrow \frac{1}{2} = \frac{x}{5} \rightarrow \frac{5}{10} = \frac{2x}{10} \rightarrow 2x = 5 \rightarrow x = 2.5$$

Example 3

Explain why triangles FGH and FJK are similar



Draw out the triangles separately

\widehat{HFG} and \widehat{KJF} are the same (common angle)

\widehat{FGH} and \widehat{FJK} are the same (corresponding angles)

\widehat{FHG} and \widehat{FKJ} are the same (corresponding angles)

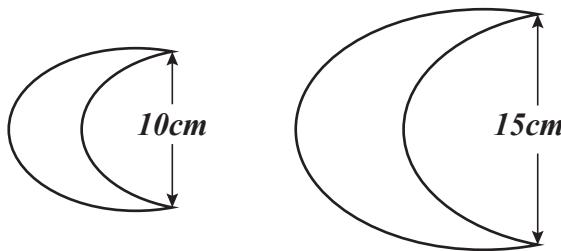
All corresponding angles are the equal therefore the triangles are similar

Similarity in 2D and 3D shapes

When a shape is enlarged by a **scale factor k** , any areas are enlarged by a **scale factor k^2** and volumes are enlarged by a **scale factor k^3**

Example 1

These two shapes are mathematically similar
The area of the smaller shape is **24cm^2**
Work out the area of the larger shape



First calculate the **linear scale factor (LSF)** of the larger shape compared to the smaller shape

$$LSF = \frac{15}{10} = \frac{3}{2}$$

Next calculate the **area scale factor (ASF)**, remember the **ASF** is the **square** of the **LSF**

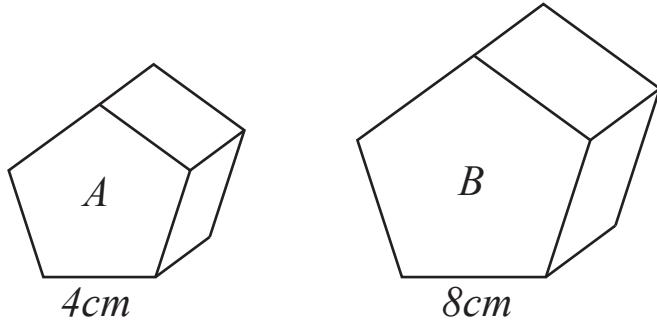
$$ASF = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2.25$$

Next multiply the area of the smaller shape by the **ASF** to find the area of the larger shape

$$24 \times 2.25 = 54\text{cm}^2$$

Example 2

These two solids *A* and *B* are similar



Solid *A* has a volume of **80cm^3**

a) Work out the volume of solid *B*

Solid *B* has a total surface area of **160cm^2**
b) Work out the total surface area of solid *A*

a) First calculate the **linear scale factor (LSF)** of the larger shape compared to the smaller shape

$$LSF = \frac{8}{4} = 2$$

Next calculate the **volume scale factor (VSF)**, remember the **VSF** is the **cube** of the **LSF**

$$VSF = 2^3 = 8$$

Next multiply the volume of the smaller shape by the **VSF** to find the volume of the larger shape

$$80 \times 8 = 640\text{cm}^3$$

b) Calculate the **area scale factor (ASF)**, remember the **ASF** is the **square** of the **LSF**

$$ASF = \left(\frac{4}{8}\right)^2 = \frac{16}{64} = 0.25$$

Next multiply the total surface area (TSA) of the larger shape by the **ASF** to find the TSA of the smaller shape

$$160 \times 0.25 = 40\text{cm}^2$$

Rules of indices, roots and surds

Indices

Use these **rules of indices** to multiply, divide and work out the **power of a power**

$$x^m \times x^n = x^{m+n} \quad \frac{x^m}{x^n} = x^{m-n} \quad (x^m)^n = x^{m \times n}$$

Use these **rules of indices** to work out zero, negative and fractional powers

$$x^0 = 1 \quad x^{-n} = \frac{1}{x^n} \quad x^{\frac{1}{n}} = \sqrt[n]{x} \quad x^{\frac{n}{m}} = (\sqrt[m]{x})^n \quad x^{\frac{-n}{m}} = \frac{1}{(\sqrt[m]{x})^n}$$

Examples

$$2^{-5} \times 2^{12} = 2^{(-5+12)} = 2^7 = 128 \quad (3^{-7})^0 = 3^{(-7 \times 0)} = 3^0 = 1$$

$$\sqrt[8]{4^{32}} = (4^{32})^{\frac{1}{8}} = 4^{(32 \times \frac{1}{8})} = 4^4 = 256 \quad 3^4 \div 3^{-1} = \frac{3^4}{3^{-1}} = 3^{4 - (-1)} = 3^5 = 243$$

Surds

Use these rules to multiply and divide **surds**

$$\sqrt{mn} = \sqrt{m} \times \sqrt{n} \quad \sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}$$

Use these methods to **rationalise a denominator** to give the denominator as an integer

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b} \quad \frac{1}{a\sqrt{b}} = \frac{1}{a\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{b}}{ab}$$

$$\frac{c}{d + \sqrt{e}} = \frac{c}{d + \sqrt{e}} \times \frac{d - \sqrt{e}}{d - \sqrt{e}} = \frac{c(d - \sqrt{e})}{d^2 - e}$$

Examples

$$\sqrt{12} + \sqrt{27} = \sqrt{4 \times 3} + \sqrt{9 \times 3} = 2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$$

$$\sqrt{12} \times \sqrt{3} = \sqrt{4 \times 3} \times \sqrt{3} = 2\sqrt{3} \times \sqrt{3} = 2 \times 3 = 6$$

$$\sqrt{45} \div \sqrt{5} = \frac{\sqrt{45}}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{5}} = 3$$

$$\frac{4\sqrt{30}}{2\sqrt{10}} = \frac{4\sqrt{30}}{2\sqrt{10}} \times \frac{2\sqrt{10}}{2\sqrt{10}} = \frac{8\sqrt{300}}{4\sqrt{100}} = \frac{80\sqrt{3}}{40} = 2\sqrt{3}$$

$$(8 + 4\sqrt{3})(8 - 4\sqrt{3}) = 64 - 32\sqrt{3} + 32\sqrt{3} - 48 = 64 - 48 = 16$$

$$(\sqrt{20} - 6)(\sqrt{5} + 3) = (2\sqrt{5})\sqrt{5} + 6\sqrt{5} - 6\sqrt{5} - 18 = 10 - 18 = -8$$

Simultaneous equations

Simple simultaneous equations

When there are two unknowns you need two equations to find their values. These are called **simultaneous equations**

Example 1

$2x - y = 4$ The terms in y have **opposite signs**, so **add** the equations to eliminate the terms in y

$$3x + y = 11$$

$$\underline{5x + 0 = 15}$$

$$5x = 15$$

$$x = 3$$

When you have solved x , put x back into either equation to find y

$$(2 \times 3) - y = 4$$

$$6 - y = 4$$

$$y = 6 - 4$$

$$y = 2$$

Example 2

$3x + 2y = 13$ Multiply the first equation by 2 to make the coefficients of y the same

$$4x + 4y = 20$$

$$6x + 4y = 26$$

Subtract the equations to eliminate the terms in y

$$\underline{4x + 4y = 20}$$

$$2x + 0 = 6$$

$$2x = 6$$

$$x = 3$$

When you have solved x , put x back into either equation to find y

$$(3 \times 3) + 2y = 13$$

$$9 + 2y = 13$$

$$2y = 13 - 9$$

$$2y = 4$$

$$y = 2$$

Quadratic simultaneous equations

$2x - y = 7$ Rearrange the first equation to make y the subject

$$x^2 - 15 = y$$

$y = 2x - 7$ Substitute the value for y into the second equation

$x^2 - 15 = 2x - 7$ Rearrange so the right hand side is 0

$x^2 - 2x - 8 = 0$ Solve the quadratic equation

$(x - 4)(x + 2) \rightarrow x = 4 \text{ or } x = -2$ Substitute the values of x back into the first equation to find y values

$(2 \times 4) - y = 7 \quad (2 \times (-2)) - y = 7$ Solutions are $x = 4, y = 1$ and $x = -2, y = -11$

$$y = 1$$

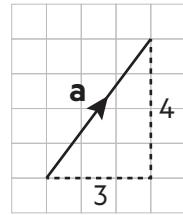
$$y = -11$$

Vectors

A **vector** is a quantity that has **magnitude** and **direction**

The **magnitude** of a vector is its **size**

Displacement is a change in position. A displacement can be written as $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ where 3 is the **x** component and 4 is the **y** component. This is called a **column vector**



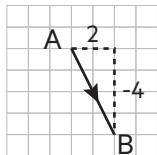
The displacement vector between two points A and B is written \vec{AB}

Example

Point A has the coordinates (3, 5) and point B has coordinates (5, 1)

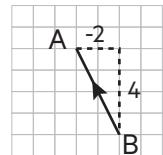
Write \vec{AB} as a column vector

$$\vec{AB} = \begin{pmatrix} \text{change in } x \\ \text{change in } y \end{pmatrix} = \begin{pmatrix} 5 - 3 \\ 1 - 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$



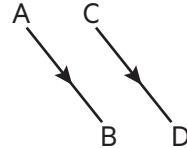
Write \vec{BA} as a column vector

$$\vec{BA} = \begin{pmatrix} \text{change in } x \\ \text{change in } y \end{pmatrix} = \begin{pmatrix} 3 - 5 \\ 5 - 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

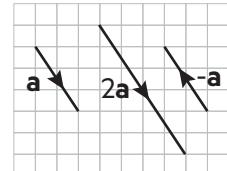


The **magnitude** of a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is its length ie $\sqrt{x^2 + y^2}$

If $\vec{AB} = \vec{CD}$ then the line segments AB and CD are **equal in length** and are **parallel**



$2\mathbf{a}$ is twice as long as \mathbf{a} and in the same direction
 $-\mathbf{a}$ is the same length as \mathbf{a} but the opposite direction



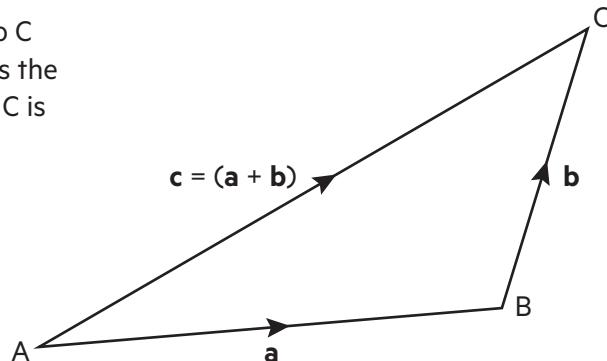
The two-stage journey from A to B and then from B to C has the same starting point and the same end point as the single journey from A to C. So A to B followed by B to C is equivalent to A to C

$$\vec{AB} + \vec{BC} = \vec{AC}$$

Triangle law for vector addition

Let $\vec{AB} = \mathbf{a}$, $\vec{BC} = \mathbf{b}$ and $\vec{AC} = \mathbf{c}$

Then $\mathbf{a} + \mathbf{b} = \mathbf{c}$ forms a triangle

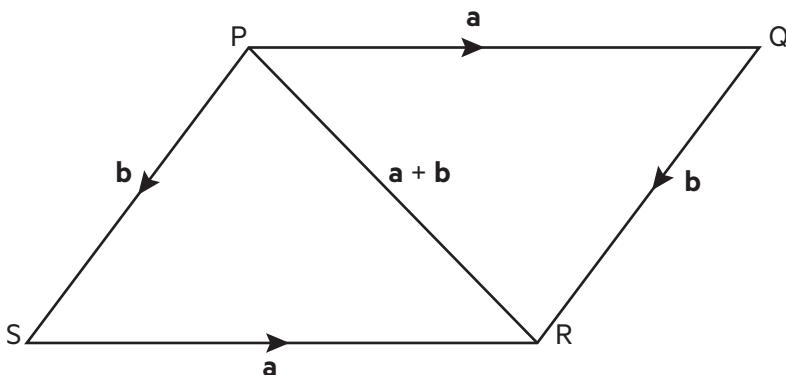


Example

If $\vec{AB} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\vec{BC} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$ find \vec{AC}

From the diagram above you know $\vec{AB} + \vec{BC} = \vec{AC}$, therefore $\vec{AC} = \begin{pmatrix} 2 + 7 \\ 5 - 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$

In a parallelogram PQRS where \vec{PQ} is \mathbf{a} and \vec{PS} is \mathbf{b} , the diagonal \vec{PR} of the parallelogram is $\mathbf{a} + \mathbf{b}$
 This is called the **parallelogram law for vector addition**



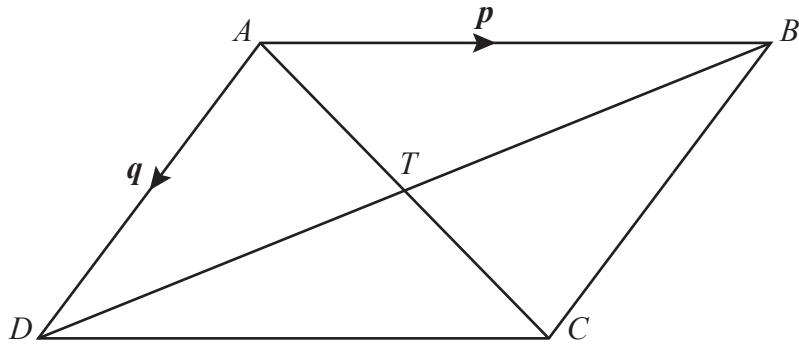
$\vec{PQ} = \vec{SR}$, therefore $\vec{SR} = \mathbf{a}$

$\vec{PS} = \vec{QR}$, therefore $\vec{QR} = \mathbf{b}$

Vectors continued

Example

AC and BD are diagonals of parallelogram $ABCD$, AC and BD intersect at T . Express \vec{AT} in terms of p and q



Step 1
 $\vec{AT} = \frac{\vec{AC}}{2}$

Step 2
 $\vec{AC} = p + q$

Step 3
 $\vec{AT} = \frac{p + q}{2} = \frac{1}{2}(p + q)$

With the origin O, the vectors OA and OB are called the position vectors of the points A and B

In general, a point with coordinates (p, q) has position vector $\begin{pmatrix} p \\ q \end{pmatrix}$

Example

The points A, B, C, and D have coordinates $(1, 3)$, $(2, 7)$, $(-6, -10)$ and $(-1, 10)$ respectively

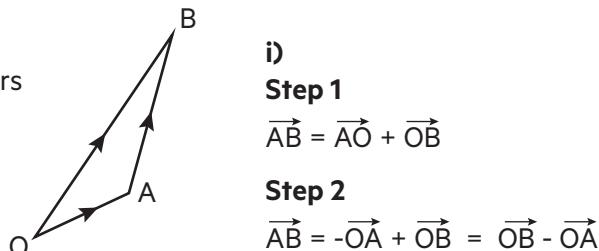
O is the origin

a) Write down the position vectors \vec{OA} and \vec{OB}

$$\vec{OA} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \vec{OB} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

b) Work out as column vectors

i) \vec{AB} ii) \vec{CD}



i)

Step 1
 $\vec{AB} = \vec{AO} + \vec{OB}$

Step 2

$$\vec{AB} = -\vec{OA} + \vec{OB} = \vec{OB} - \vec{OA}$$

Step 3

$$\vec{AB} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

ii)

Step 1

$$\vec{CD} = \vec{CO} + \vec{OD}$$

Step 2

$$\vec{CD} = -\vec{OC} + \vec{OD} = \vec{OD} - \vec{OC}$$

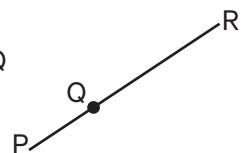
Step 3

$$\vec{CD} = \begin{pmatrix} -1 \\ 10 \end{pmatrix} - \begin{pmatrix} -6 \\ -10 \end{pmatrix} = \begin{pmatrix} 5 \\ 20 \end{pmatrix}$$

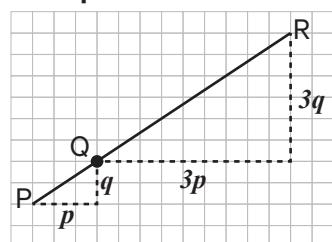
c) What do these results show about AB and CD ?

$$\vec{AB} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \vec{CD} = \begin{pmatrix} 5 \\ 20 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \vec{CD} = 5\vec{AB}$$

$\vec{PQ} = k\vec{QR}$ shows that the lines PQ and QR are parallel. Also they both pass through the point Q so PQ and QR are part of the same straight line. P, Q and R are said to be **colinear**



Example



$$\vec{PQ} = p + q \quad \vec{QR} = 3(p + q)$$

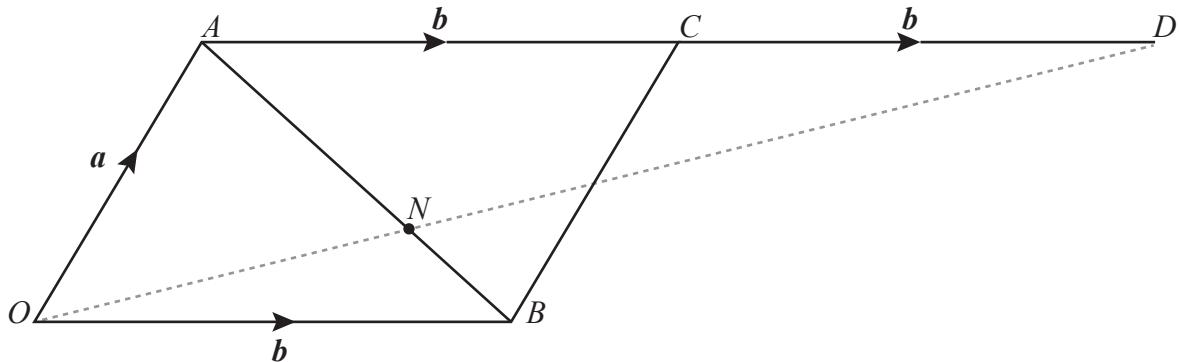
Same vector

\vec{PQ} and \vec{QR} have the **same vector** (parallel) and have a **point in common** (Q) therefore P, Q and R lie on the same straight line.

Vectors continued

Example

$OACB$ is a parallelogram



$$\overrightarrow{OA} = \mathbf{a} \text{ and } \overrightarrow{OB} = \mathbf{b}$$

$$D \text{ is a point such that } \overrightarrow{AC} = \overrightarrow{CD}$$

The point N divides AB in the ratio $2:1$

a) Write an expression of \overrightarrow{ON} in terms of \mathbf{a} and \mathbf{b}

$$\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN}$$

$$\overrightarrow{BN} = \frac{1}{3} \overrightarrow{BA}$$

$$\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$$

$$\overrightarrow{BN} = \frac{1}{3} (\mathbf{a} - \mathbf{b})$$

$$\overrightarrow{ON} = \mathbf{b} + \frac{1}{3} (\mathbf{a} - \mathbf{b}) = \frac{1}{3} (\mathbf{a} + 2\mathbf{b})$$

b) Prove that OND is a straight line

$$\overrightarrow{ND} = \overrightarrow{NA} + \overrightarrow{AD}$$

$$\overrightarrow{NA} = \frac{2}{3} \overrightarrow{BA} = \frac{2}{3} (\mathbf{a} - \mathbf{b})$$

$$\overrightarrow{ND} = \frac{2}{3} (\mathbf{a} - \mathbf{b}) + 2\mathbf{b} = \frac{2}{3} (\mathbf{a} + 2\mathbf{b})$$

\overrightarrow{ON} and \overrightarrow{ND} have the **same vector** and **share a common point** N therefore the points O , N and D are **colinear** and OND is a **straight line**

Functions

A function is a rule for working out values of y and given values of x

For example, $y = 3x$ and $y = x^2$ are functions. The notation $f(x)$ is read as 'f of x '. f is the function
 $f(x) = 3x$ means the function of x is $3x$

Examples

$$f(x) = \frac{10}{x} \quad \text{Work out } f(5)$$

$$\text{Substitute } x = 5 \text{ into } \frac{10}{x} \longrightarrow \frac{10}{5} = 2$$

$$g(x) = 2x^3 \quad \text{Work out } g(-5)$$

$$\text{Substitute } x = -5 \text{ into } 2x^3 \longrightarrow 2(-5)^3 = -250$$

$$f(x) = x + x^3 \quad g(x) = 3x^2 \quad \text{Work out } f(4) - g(2)$$
$$(4 + 4^3) - 3(2)^2 = 68 - 12 = 56$$

$$f(x) = 5x - 3 \quad \text{Work out the value of } a \text{ when } f(a) = 12$$
$$5a - 3 = 12$$
$$5a = 15$$
$$a = 3$$

Composite functions

fg is a composite function. To work out $fg(x)$, first work out $g(x)$ and then substitute your answer into $f(x)$

Examples

$$f(x) = 6 - 2x, g(x) = x^2 + 7 \quad \text{Work out } gf(7)$$

$$f(x) = 6 - (2 \times 7) = -8$$

$$g(-8) = (-8)^2 + 7 = 64 + 7 = 71$$

$$f(x) = 4x - 3, g(x) = 10 - x \quad \text{Work out } fg(x)$$

$$g(x) = 10 - x$$

$$f(10 - x) = 4(10 - x) - 3 = 40 - 4x - 3 = 37 - 4x$$

Inverse functions

An inverse function **reverses** the effect of the original function, this is usually written $f^{-1}(x)$

Examples

$$f(x) = 3 - 4x \quad \text{Find the inverse of } f(x)$$

Form an equation by making $y = f(x) \longrightarrow y = 3 - 4x$

Make x the subject

$$4x = 3 - y \longrightarrow x = \frac{3 - y}{4}$$

Finally, re-write the expression that is equal to x , replacing the y with an x

So the inverse of $f(x) = 3 - 4x$ is $\frac{3 - x}{4}$

$$f(x) = 16 - 3(x + 2) \quad \text{Find the inverse of } f(x)$$

As before, form an equation by making $y = f(x) \longrightarrow y = 16 - 3(x + 2)$

Make x the subject

$$y = 16 - 3x - 6 \longrightarrow 3x = 10 - y \longrightarrow x = \frac{10 - y}{3}$$

Finally, re-write the expression that is equal to x , replacing the y with an x

So the inverse of $f(x) = 16 - 3(x + 2)$ is $\frac{10 - x}{3}$

Probability

$$\text{Probability} = \frac{\text{number of successful outcomes}}{\text{total number of possible outcomes}}$$

Examples

What is the probability of rolling a **6** with a 6-sided die?

$$\text{Probability} = \frac{1}{6}$$

What is the probability of rolling an **even number** with a 6-sided die? ie if you roll a 2, 4 or 6

$$\text{Probability} = \frac{3}{6} = \frac{1}{2}$$

Independent events

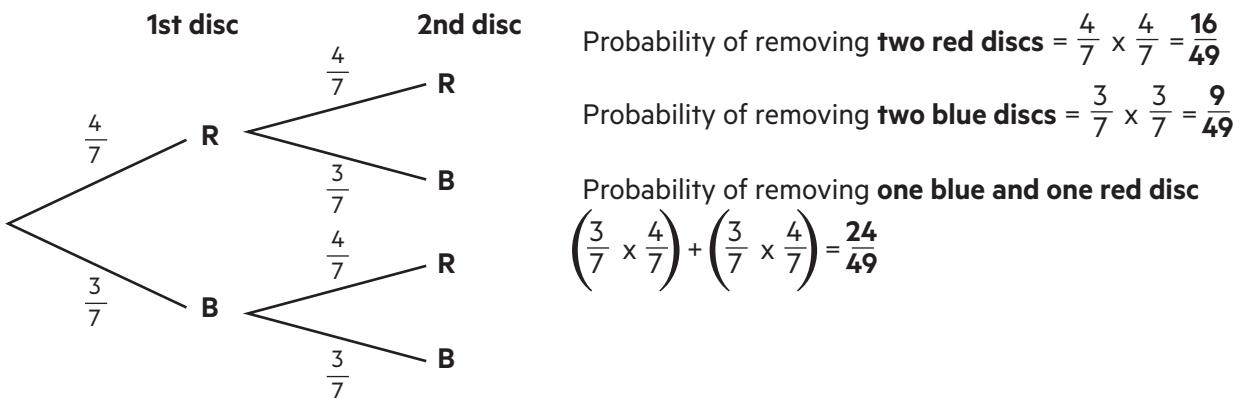
Events are **independent** when the outcome of one does not affect the outcome of the other. To find the combined probability of two independent events, multiply the probabilities

$$P(A \text{ and } B) = P(A) \times P(B)$$

A **tree diagram** shows the possible outcomes of two or more combined events

Example

A bag contains 4 red (R) discs and 3 blue (B) discs. One disc is taken at random, its colour is noted and then it is replaced. A second disc is then taken



$$\text{Probability of removing two red discs} = \frac{4}{7} \times \frac{4}{7} = \frac{16}{49}$$

$$\text{Probability of removing two blue discs} = \frac{3}{7} \times \frac{3}{7} = \frac{9}{49}$$

Probability of removing **one blue and one red disc**

$$\left(\frac{3}{7} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{4}{7}\right) = \frac{24}{49}$$

A sample space diagram shows all possible outcomes of two independent events

Example

Two six-sided dice are rolled and the results are added together

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Total number of possible outcomes = **36**

$$\text{Probability of rolling an **even** number} = \frac{18}{36} = \frac{1}{2}$$

$$\text{Probability of rolling a **7**} = \frac{6}{36} = \frac{1}{6}$$

$$\text{Probability of rolling **higher than a 7**} = \frac{15}{36} = \frac{5}{12}$$

$$\text{Probability of rolling a **double 1**} = \frac{1}{36}$$

Mutually exclusive events

Two events are **mutually exclusive** if they cannot happen at the same time. eg, when you roll an ordinary die you cannot get a 3 and an even number at the same time.

When events are mutually exclusive you can add their probabilities

$$\text{For mutually exclusive events } P(A \text{ or } B) = P(A) + P(B)$$

Probability continued

Examples

A standard pack of 52 playing cards is shuffled and a card is chosen at random

Find the probability of choosing a diamond or a spade

$$P(\text{diamond}) = \frac{13}{52} \quad P(\text{spade}) = \frac{13}{52} \longrightarrow P(\text{diamond or spade}) = \frac{13}{52} + \frac{13}{52} = \frac{26}{52} = \frac{1}{2}$$

Find the probability of choosing a black ace or a heart

$$P(\text{black ace}) = \frac{2}{52} \quad P(\text{heart}) = \frac{13}{52} \longrightarrow P(\text{black ace or heart}) = \frac{2}{52} + \frac{13}{52} = \frac{15}{52}$$

Conditional probability

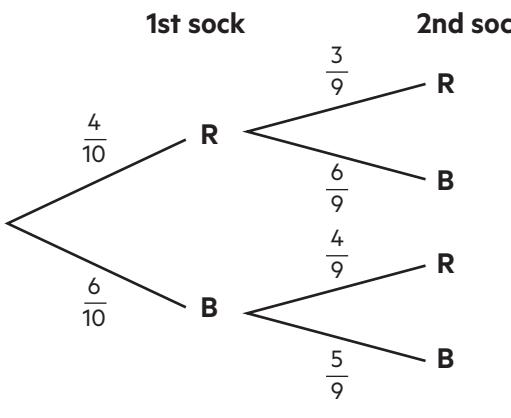
When events are **dependent** the outcome of the first event influences the outcome of the second event.

Conditional probabilities are probabilities associated with **dependent** events

In a tree diagram showing conditional probabilities, the probabilities on the branches for the second outcome depend on what happened on the first outcome

Examples

Matt has 4 red (R) and 6 black (B) socks in a drawer. He takes out two socks at random.



$$\text{Probability of removing two red socks} = \frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$$

Probability of a **matching pair of socks...**

$$\text{Probability of removing two red socks} = \frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$$

PLUS

$$\text{Probability of removing two black socks} = \frac{6}{10} \times \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$$

$$\frac{1}{3} + \frac{2}{15} = \frac{5}{15} + \frac{2}{15} = \frac{7}{15}$$

Exam style question

There are n counters in a bag. 8 of the counters are red and the rest are blue

Adam takes a counter from the bag at random and does not replace it

He then takes another counter at random from the bag

The probability that Adam takes two blue counters is $\frac{1}{5}$

a) Show that $n^2 - 21n + 90 = 0$

b) Find the value of n

Step 1

Begin by defining the number of blue counters $\rightarrow (n - 8)$

Step 2

Set up an equation based on what you know

$$\frac{n-8}{n} \times \frac{n-8-1}{n-1} = \frac{1}{5} \longrightarrow \frac{(n-8)(n-9)}{n(n-1)} = \frac{1}{5} \longrightarrow \frac{n^2 - 17n + 72}{n^2 - n} - \frac{1}{5} = 0$$

$$\longrightarrow 5n^2 - 85n + 360 - n^2 + n = 0 \longrightarrow 4n^2 - 84n + 360 = 0 \longrightarrow n^2 - 21n + 90 = 0 \longrightarrow (n - 15)(n - 6)$$

$$n = 15 \text{ or } 6$$

n must be greater than 8 therefore $n = 15$

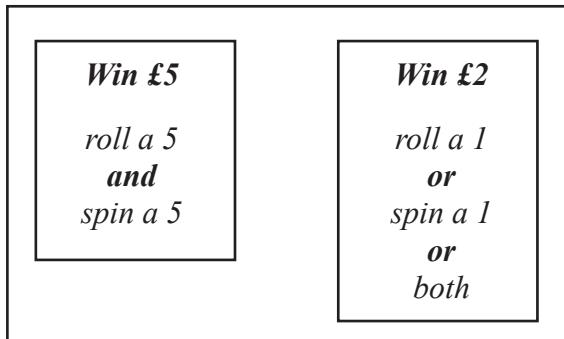
Probability continued

Exam style question

David has designed a game. He uses a fair 6-sided die and a fair 5-sided spinner

The dice is numbered **1 to 6**, the spinner is numbered **1 to 5**

Each player rolls the die **once** and spins the spinner **once**. A player can win either **£5** or **£2**



David expects **30 people** will play his game

Each person will pay David **£1 to play the game**

a) Work out how much **profit** David can expect to make

b) Give a reason why David's actual profit may be different to the profit he expects to make

Step 1

Find the probability of someone winning £5

$$\frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$$

Step 2

Find the probability of someone winning £2

Probability of someone rolling a **1 and not** spinning a **1**

$$\frac{1}{6} \times \frac{4}{5} = \frac{4}{30}$$

Probability of someone spinning a **1 and not** rolling a **1**

$$\frac{1}{5} \times \frac{5}{6} = \frac{5}{30}$$

Probability of someone spinning a **1 and** rolling a **1**

$$\frac{1}{5} \times \frac{1}{6} = \frac{1}{30}$$

Probability of any of these 3 events happening

$$\frac{4}{30} + \frac{5}{30} + \frac{1}{30} = \frac{10}{30} = \frac{1}{3}$$

Step 3

David receives £30 from the players

Number of players he would expect to win £5

$$\frac{1}{30} \times 30 = 1$$

Number of players he would expect to win £2

$$\frac{1}{3} \times 30 = 10$$

Amount of money he would have to pay out

$$(1 \times 5) + (10 \times 2) = 5 + 20 = \text{£25}$$

Therefore he would expect to make **£5 profit** ($30 - 25$)

A reason why David's actual profit may be different to the profit he expects to make is that his calculations are **theoretical**. In reality they could be different. Also more or less than 30 people could play the game.

Venn diagrams

$A \cap B$, means '**A intersection B**'

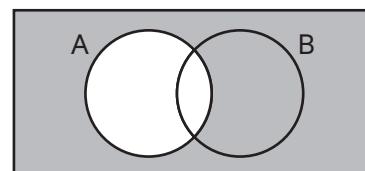
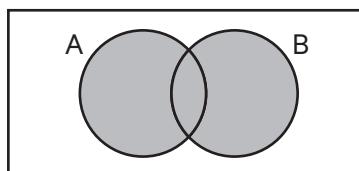
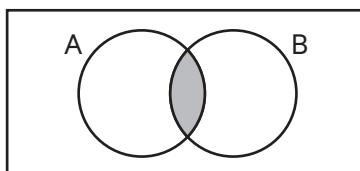
All the elements in **A and in B**

$A \cup B$, means '**A union B**'

All the elements in **A or B or both**

A' means all the elements

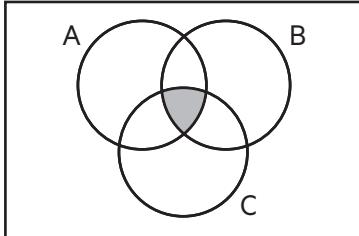
not in A



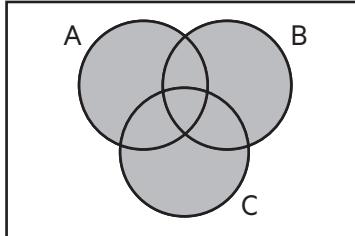
ξ means the universal set – all elements being considered

Venn diagrams continued

$A \cap B \cap C$, means the **intersection** of A, B and C



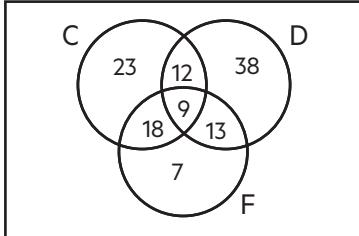
$A \cup B \cup C$, means the **union** of A, B and C



$P(A \cap B | B)$ means the probability of A and B **given** B

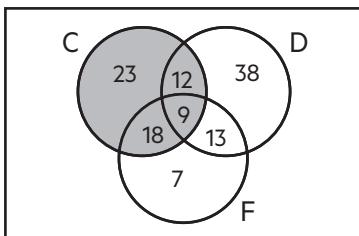
Example

Caitlin did a survey of pet owners owning cats (C), dogs (D) and fish (F)



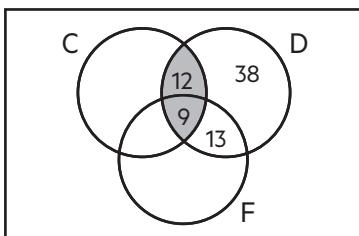
How many took part in the survey?

$$23 + 12 + 38 + 18 + 9 + 13 + 7 = 120$$



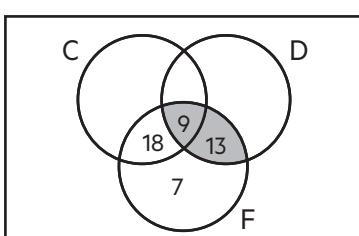
Work out $P(C)$

$$P(C) = \frac{23 + 12 + 9 + 18}{120} = \frac{62}{120} = \frac{31}{60}$$



Work out $P(C \cap D | D)$. This means find the intersection of C and D, given D
This essentially means ignore everything outside of D

$$P(C \cap D | D) = \frac{12 + 9}{38 + 12 + 9 + 13} = \frac{21}{72} = \frac{7}{24}$$



Work out $P(D \cap F | F)$. This means find the intersection of D and F, given F
As before ignore everything outside of F

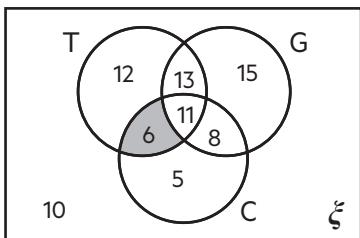
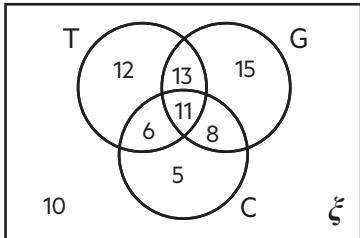
$$P(D \cap F | F) = \frac{13 + 9}{9 + 13 + 18 + 7} = \frac{22}{47}$$

Venn diagrams continued

Further example

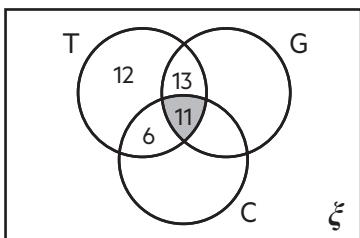
This Venn diagram shows the sports played by 80 students

The three most popular sports were tennis (T), golf (G) and cricket (C). A student is picked at random



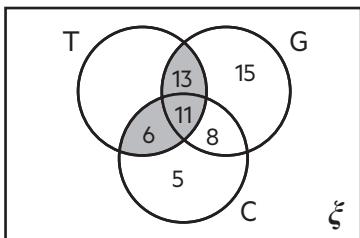
Work out $P(T \cap C \cap G')$. This means find the intersection of T and C and everything outside of G

$$P(T \cap C \cap G') = \frac{6}{80} = \frac{3}{40}$$



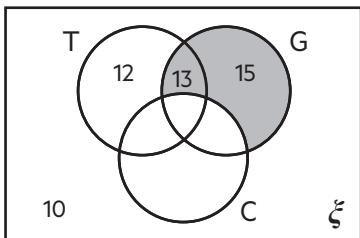
Work out $P(T \cap G \cap C | T)$. This means find the intersection of T, G and C given T. As before ignore everything outside of T

$$P(T \cap G \cap C | T) = \frac{11}{12 + 13 + 11 + 6} = \frac{11}{42}$$



Work out $P(T | G \cup C)$. This means given the union of G and C, find T As before ignore everything outside of $G \cup C$

$$P(T | G \cup C) = \frac{11 + 13 + 6}{8 + 5 + 13 + 11 + 6 + 15} = \frac{30}{58} = \frac{15}{29}$$

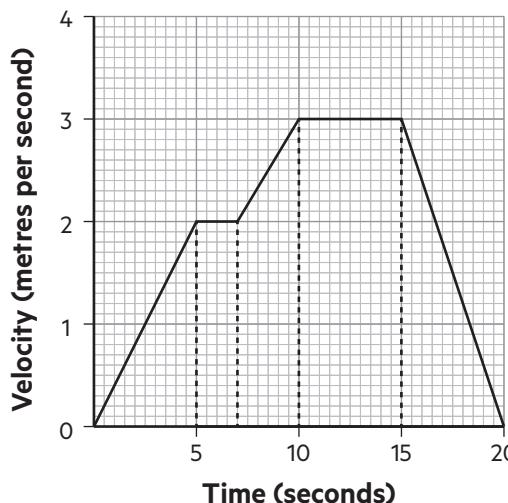


Work out $P(G | C')$. This means find G given everything outside of C

$$P(G | C') = \frac{13 + 15}{12 + 13 + 15 + 10} = \frac{28}{50} = \frac{14}{25}$$

Distance, speed, time graphs

On a velocity-time graph, the gradient is the rate of **change of velocity over time**, or the **acceleration** $a = \frac{v}{t}$
The **area** under a velocity time graph represents the **distance travelled**



Acceleration over the first 5 seconds

$$a = \frac{v}{t} \rightarrow a = \frac{2}{5} \rightarrow a = 0.4 \text{ m/s}^2$$

Total distance travelled is the area under the graph

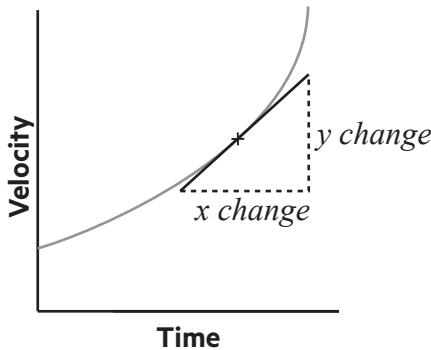
$$\frac{1}{2}(5 \times 2) + (2 \times 2) + \frac{1}{2}((2+3) \times 3) + (5 \times 3) + \frac{1}{2}(5 \times 3)$$

$$10 + 4 + 7.5 + 15 + 7.5 = 44 \text{ m}$$

Average speed is the total distance travelled divided by the total time taken

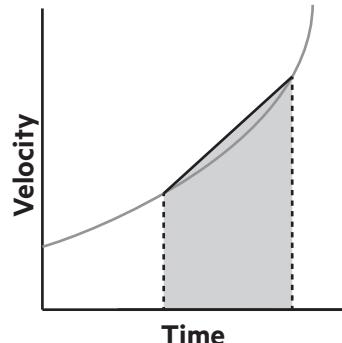
$$s = \frac{d}{t} \rightarrow s = \frac{44}{20} \rightarrow s = 2.2 \text{ m/s}$$

If the velocity-time graph is **not linear** you can still estimate the distance travelled by drawing a **chord** between two points on the graph and drawing straight lines down to the x-axis to create a **trapezium**. The area of the trapezium is an estimate for the area under this part of the graph

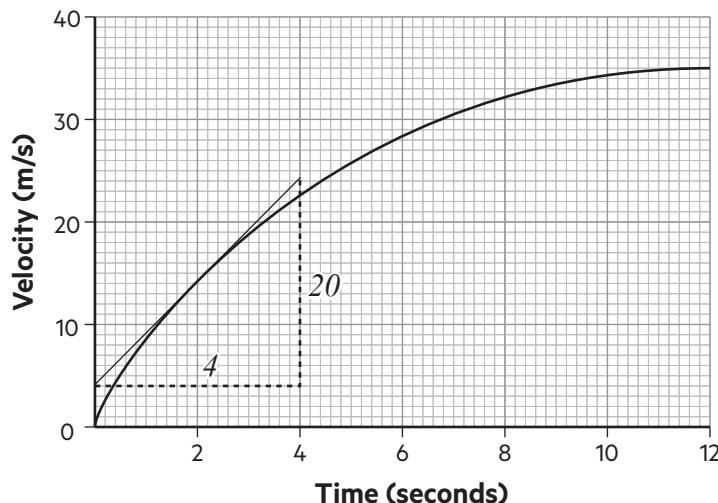


To find the **acceleration** at any given point, draw a **tangent** to the graph and the **gradient of the tangent** will be the **acceleration at that point**

$$a = \frac{y \text{ change}}{x \text{ change}}$$



Example

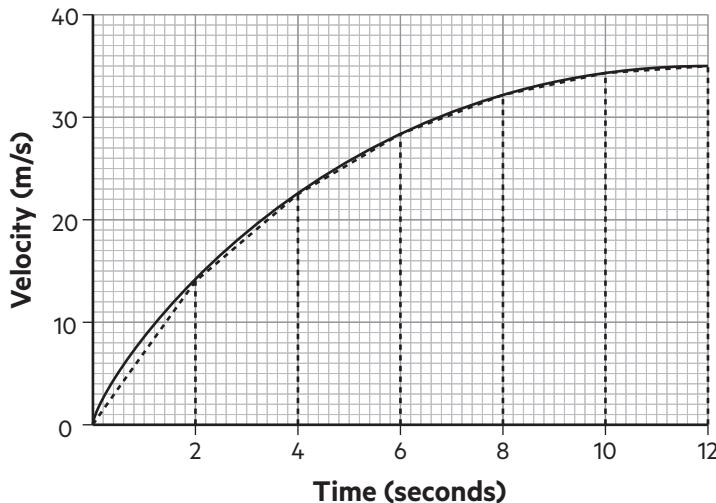


Estimate the acceleration at $t = 2$ seconds
Draw a tangent to the graph at $t = 2$ and calculate the gradient

$$a = \frac{20}{4} = 5 \text{ m/s}^2$$

Distance, speed, time graphs continued

Example



Estimate the distance travelled over the 12 seconds
Divide the graph into sections, drawing a chord to the graph at each interval to create trapezia (first one here is a triangle) then sum up the areas to find the total

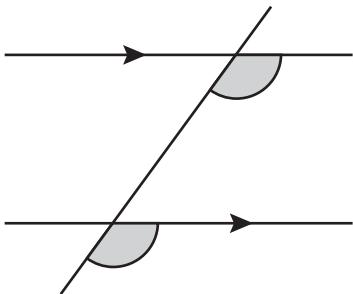
$$\begin{aligned} & \frac{1}{2}(14 \times 2) + \frac{1}{2}((14 + 22.5) \times 2) + \frac{1}{2}((22.5 + 28.5) \times 2) + \\ & \frac{1}{2}((28.5 + 32) \times 2) + \frac{1}{2}((32 + 34.5) \times 2) + \frac{1}{2}((34.5 + 35) \times 2) \\ & = 28 + 36.5 + 51 + 60.5 + 66.5 + 69.5 = 312\text{m} \end{aligned}$$

Angle properties

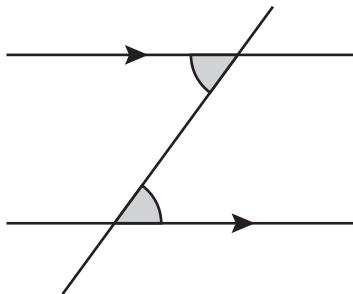
Parallel lines

There are three properties of angles in parallel lines – **alternate**, **corresponding** and **interior**

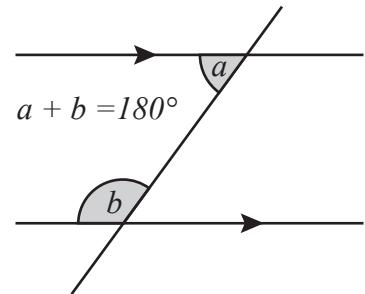
Corresponding angles are the same



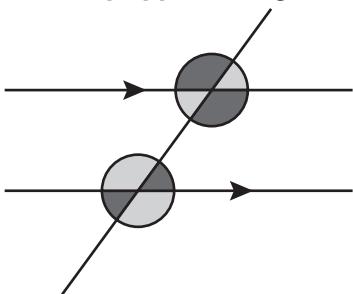
Alternate angles are the same



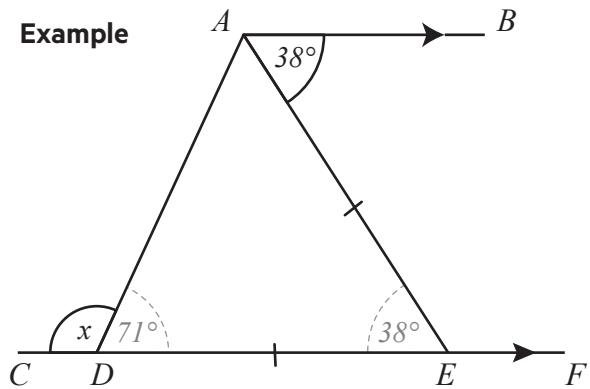
Interior angles add up to 180°



Vertically opposite angles are the same



Example



$CDEF$ is a straight line. AB is parallel to CF . $DE = AE$, $\hat{BAE} = 38^\circ$
Calculate the size of the angle marked x
Give reasons for your answer

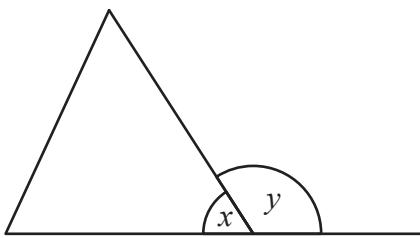
\hat{BAE} and \hat{AED} are the same (alternate angles) = 38°

\hat{EAD} and \hat{EDA} are the same (isosceles triangle and angles in a triangle sum to 180) = $\frac{180 - 38}{2} = 71^\circ$

$x + 71 = 180$ (angles in a straight line add up to 180)

$$x = 180 - 71 = 109^\circ$$

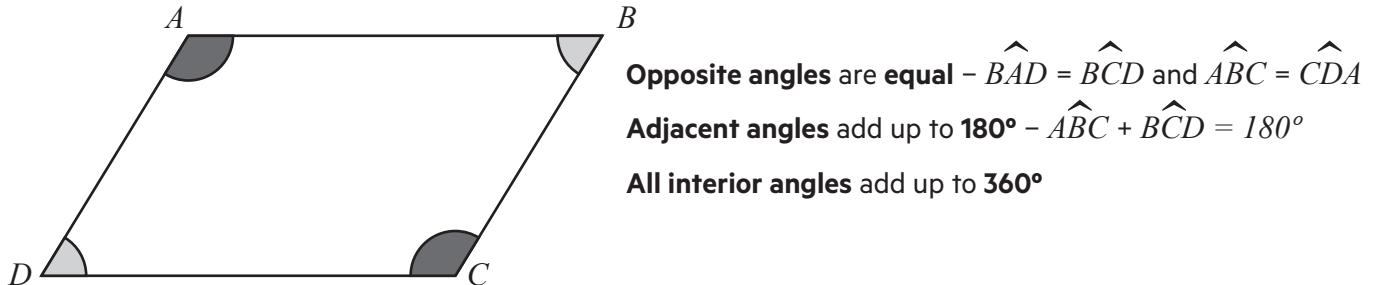
Angle properties continued



When one side of a triangle is extended at the vertex:
 The angle marked x is called the **interior angle**
 The angle marked y is called the **exterior angle**

$$x + y = 180^\circ \text{ (angles on a straight line)}$$

Parallelograms



Interior angles of polygons

The sum of the interior angles of a polygon with n sides = $(n - 2) \times 180$
 eg a pentagon has 5 sides, therefore sum of interior angles = $(5 - 2) \times 180 = 540^\circ$
 A regular polygon with 15 sides, the sum of interior angles = $(15 - 2) \times 180 = 2,340^\circ$

To find the number of sides when you know the sum of the interior angles

eg. the sum of the interior angles of a polygon is $1,620^\circ$

$$(n - 2) \times 180 = 1,620$$

$$n - 2 = \frac{1620}{180}$$

$$n - 2 = 9$$

$$n = 11$$

Example 1

The interior angle of a regular polygon is 140° , how many sides does the polygon have?

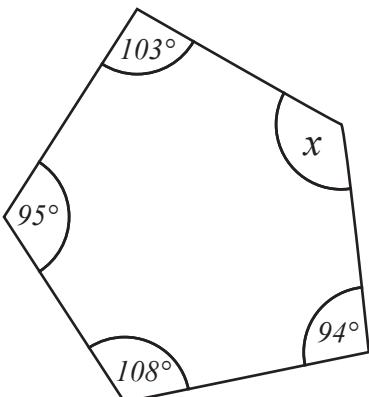


Interior + exterior angle = 180° , therefore exterior angle = $180 - 140 = 40^\circ$

For a regular polygon the **number of sides** = $\frac{360}{\text{exterior angle}} = \frac{360}{40} = 9$

Example 2

Work out the size of angle x



Step 1

The sum of the interior angles of a polygon with n sides = $(n - 2) \times 180$
 $(5 - 2) \times 180 = 540$

Step 2

$$x = 540 - 103 - 95 - 108 - 94 = 140^\circ$$

Ratio and proportion

Example 1

Share £126 between Lu and Katie in the ratio **2 : 5**

Step 1

Find out how many parts there are in total: $5 + 2 = \mathbf{7}$

Step 2

Find out how much each part is worth: $\text{£}126 \div 7 = \mathbf{\text{£}18}$

Step 3

Find out how much 2 parts and 5 parts is

Lu gets $2 \times 18 = \mathbf{\text{£}36}$

Katie gets $5 \times 18 = \mathbf{\text{£}90}$

Example 2

Margaret is in Switzerland, the local supermarket sells Reblochon cheese

Each box of Reblochon cheese costs **3.10** Swiss francs. It weighs **160g**

In England, a box of Reblochon cheese cost **£13.55 per kg**

The exchange rate is **£1 = 1.65** francs

Work out whether Reblochon cheese is better value for money in Switzerland or England

Step 1

Find out how much the cheese costs in Switzerland **per kg**

$$\frac{1000}{160} \times 3.10 = \mathbf{19.375 \text{ Swiss francs}}$$

Step 2

Convert the cost of the cheese in Switzerland from Swiss francs into pounds

$$\frac{19.375}{1.65} = \mathbf{\text{£}11.74}$$

Step 3

Compare the two prices

Switzerland = £11.74/kg

England = £13.55/kg

Therefore the cheese is better value in **Switzerland by £1.81/kg**

Example 3

There are only blue pens, green pens and red pens in a box

The ratio of blue pens to green pens is **2 : 5**

The ratio of green pens to red pens is **4 : 1**

There are less than 100 pens in the box

What is the greatest number of red pens in the box?

Step 1

Green is the common colour in the two ratios, so make them comparable by multiplying the two ratios to make them the same

$$2 : 5 \times 4 = \mathbf{8 : 20}$$

$$4 : 1 \times 5 = \mathbf{20 : 5}$$

So the ratio of blue : green : red pens is **8 : 20 : 5**

Step 2

Sum the ratios to find the number of pens: $8 + 20 + 5 = \mathbf{33}$

Step 3

We know there are less than 100 pens in box so $\frac{99}{33} = \mathbf{3}$

Step 4

Multiply the ratio of red pens by $3 : 5 \times 3 = \mathbf{15 \text{ red pens in the box}}$

Percentages

Simple interest is the interest calculated only on the original amount. It is the **same** each year

Example

John invests £14,500 for 3 years at simple interest of 6.75%. What is the value of the investment after 3 years?

Step 1

Find the amount of interest for **1 year**

$$14500 \times \frac{6.75}{100} = \text{£978.75}$$

Step 2

As simple interest is the same each year, multiply the interest for one year by 3

$$978.75 \times 3 = \text{£2936.25}$$

Step 3

Add the interest to the original investment

$$14500 + 2936.25 = \text{£17,436.25}$$

Compound interest is the interest calculated on the original amount **plus** the interest added each year

Example

Calculate the interest on borrowing £40 for 3 years if the compound interest rate is 5% per year

$$\text{Year 1: } £40 \times 1.05 = £42$$

$$\text{Year 2: } £42 \times 1.05 = £44.10$$

$$\text{Year 3: } £44.10 \times 1.05 = £46.31$$

This is the same as $£40 \times 1.05 \times 1.05 \times 1.05$

This can also be written as $£40 \times 1.05^3$

A generic formula for compound interest can therefore be written as

Investment value after **n** years = Original investment x multiplier^{**n**}

Percentage change

You can calculate a percentage change using the formula

$$\text{Percentage change} = \frac{\text{new amount} - \text{original amount}}{\text{original amount}} \times 100$$

Example

Charlie invests £3,200, when his investment matures, he receives £3,328

What is the percentage increase in his investment?

Using the formula above

$$\frac{3328 - 3200}{3200} \times 100 = 4\%$$

Profit and loss

Profit is when you buy something and then sell it for **more** than you originally paid for it. The amount of profit is the sale price **minus** the original cost

Loss is when you buy something and then sell it for **less** than you originally paid for it. The size of the loss is the sale price **minus** the original cost

Example

Lucy spent **£11.40** buying ingredients to make cupcakes. She sold all the cakes for a total of **£39.90**

What percentage profit did she make?

Step 1

Calculate the amount of **profit** Lucy made. Remember **profit** is the sale price **minus** the original cost

$$£39.90 - £11.40 = £28.50$$

Step 2

Use the percentage change formula to calculate the percentage profit

$$\text{Percentage profit} = \frac{28.50}{11.40} \times 100 = \mathbf{250\% \ profit}$$

Percentages continued

Inverse operations can be used to find the original amount after a percentage increase or decrease

Example

In one year, the value of a car dropped by 12% to £9,240

How much was the car worth at the start of the year?

It is very important to remember that a **12% drop** means the car is worth **88% of its original value**

Therefore

Original value $\times 0.88 = \text{new value}$

$$\text{Original value} = \frac{9240}{0.88} = \mathbf{\$10,500}$$

Fractions, decimals and percentages

When making calculations that are a mix of fractions, decimals and percentages, put them all into a single type

Example

Mr Mason asks 240 year 11 students what they want to do next year

15% of the students want to go to college

$\frac{3}{4}$ of the students want to stay at school

4

The rest of the students do not know. Work out the number of students who do not know

Step 1

Convert the percentage and fraction to decimals

$$15\% = 0.15$$

$$\frac{3}{4} = 0.75$$

Step 2

Add the two decimals together

$$0.15 + 0.75 = 0.9$$

Step 3

$$\text{Rest of students} = 1 - 0.9 = 0.1 = 10\%$$

So 10% of students do not know

$$240 \times 0.1 = \mathbf{24} \text{ students do not know what they want to do next year}$$

Linear graphs

Plotting a linear (straight line) graph

Example

Draw the graph of $y = 2x + 5$

Step 1

Find the y intercept, by making $x = 0$

$$y = (2 \times 0) + 5$$

$$y = 5$$

$$x = 0$$

Step 2

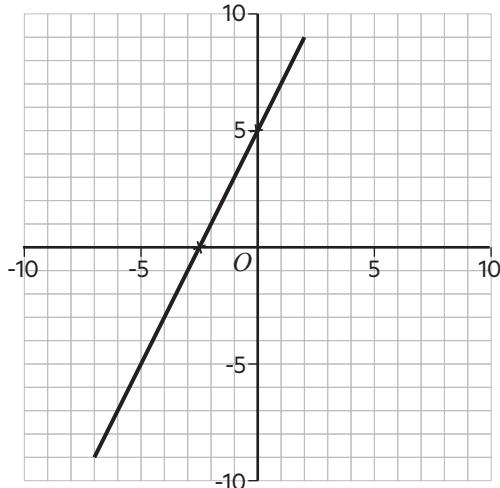
Find the x intercept, by making $y = 0$

$$0 = 2x + 5$$

$$2x = -5$$

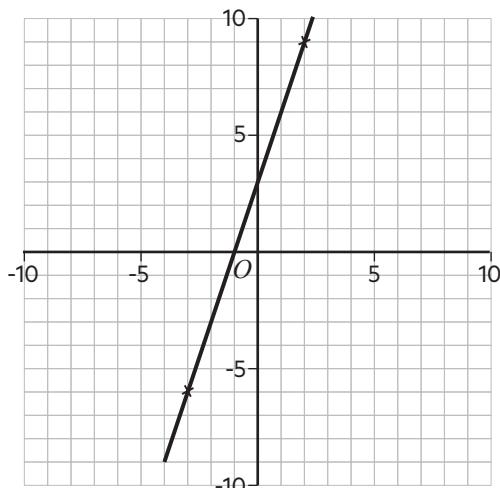
$$x = -2.5$$

$$y = 0$$



Step 3

Join the two points together



You don't just have to use the x and y intercepts you can substitute any values for x or y to find two points. Putting in two different values for x however, is easier

Example

Draw the graph of $y = 3x + 3$

Step 1

Let $x = 2$

$$y = (3 \times 2) + 3$$

$$y = 9$$

$$x = 2$$

Step 2

Let $x = -3$

$$y = (3 \times -3) + 3$$

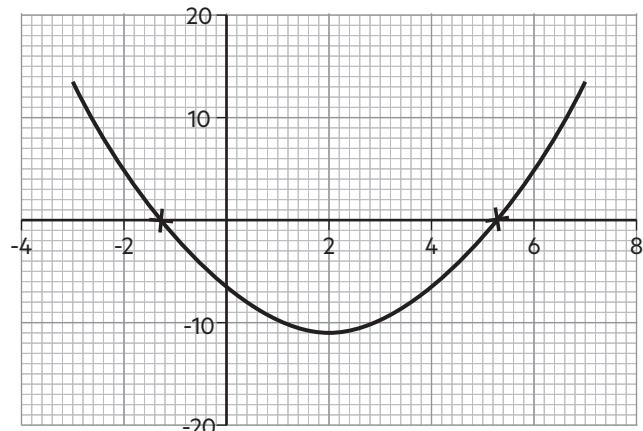
$$y = -6$$

$$x = -3$$

Step 3

Join the two points together

Solving quadratic equations graphically



This is a graph of $y = x^2 - 4x - 7$

Estimate the solutions to the equation $x^2 - 4x - 7 = 0$

This basically means where does the graph $x^2 - 4x - 7$ cross the x -axis. Because when $y = 0$, the line crosses the x -axis

So an estimate the solutions to the equation $x^2 - 4x - 7 = 0$ are $x = 5.3$ and $x = -1.3$

Using the same graph $y = x^2 - 4x - 7$

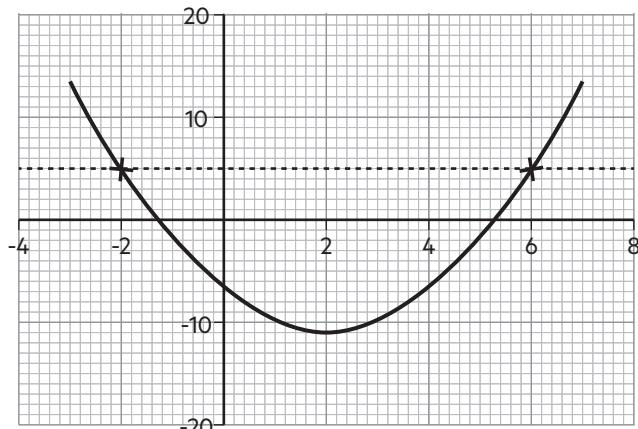
Estimate the solutions to the equation $x^2 - 4x - 7 = 5$

This basically means where does the graph

$x^2 - 4x - 7$ cross the line $y = 5$

So an estimate the solutions to the equation $x^2 - 4x - 7 = 5$

are $x = 6$ and $x = -2$



Quadratic equations

A **quadratic expression** contains a term in x^2 but no higher power of x

It is usually written in the form

$$ax^2 + bx + c$$

where a , b and c are constants and $a \neq 0$

The **quadratic formula** can be used to find the solution to a quadratic equation $ax^2 + bx + c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

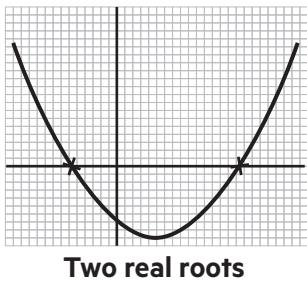
Example

$$3x^2 - 7x - 1 = 0$$

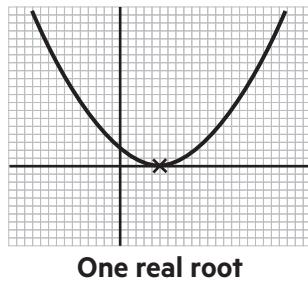
$$x = \frac{7 \pm \sqrt{-7^2 - 4(3 \times -1)}}{2 \times 3} \rightarrow \frac{7 \pm \sqrt{49 + 12}}{6} \rightarrow \frac{7 \pm \sqrt{61}}{6} \rightarrow \frac{14.81}{6} \text{ or } \frac{-0.81}{6} \rightarrow x = -0.14 \quad x = 2.47$$

In the quadratic formula, the expression under the square root sign ($b^2 - 4ac$) is called the **discriminant**. It tells you whether the equation has 0, 1 or 2 real roots

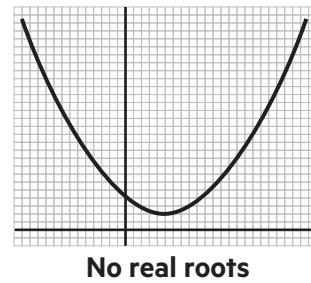
- When $b^2 - 4ac > 0$ there are two distinct solutions to the quadratic equation (two real roots)
- When $b^2 - 4ac = 0$ there is one solution to the quadratic equation (one real roots)
- When $b^2 - 4ac < 0$ there are no real solutions to the quadratic equation (no real roots)



Two real roots



One real root



No real roots

Completing the square is a technique used to identify the line of symmetry and the turning point of a quadratic graph

$$ax^2 + bx + c \text{ can be written in the form } a\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Example

A quadratic graph has the formula $y = x^2 - x - 4$ find the turning point by completing the square

$$\text{Using the formula above } \left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 4 \rightarrow \left(x - 0.5\right)^2 - 0.25 - 4 \rightarrow \left(x - 0.5\right)^2 - 4.25 \\ x = 0.5, y = -4.25$$

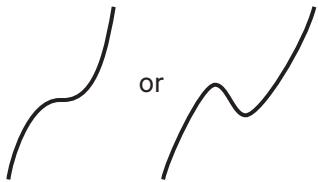
The x value is also the **line of symmetry** of the graph

Graphs of cubic equations

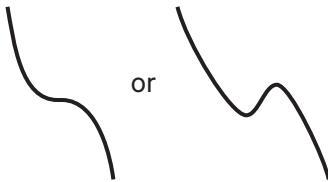
A **cubic function** is one whose highest power of x is x^3

It is written in the form $y = ax^3 + bx^2 + cx + d$

When $a > 0$ the function looks like



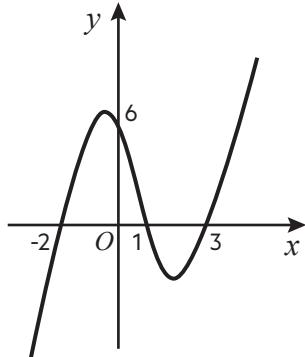
When $a < 0$ the function looks like



When the graph of a cubic function y crosses the x axis three times, the equation $y = 0$ has **three** solutions

For example $y = (x + 2)(x - 1)(x - 3)$

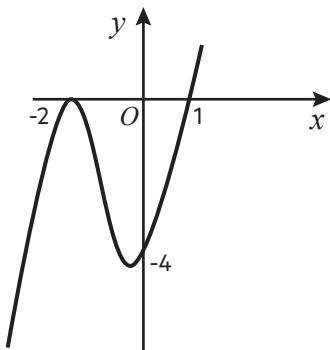
y intercept, when $x = 0$ is $(0 + 2) \times (0 - 1) \times (0 - 3) = 2 \times (-1) \times (-3) = 6$



When the graph of a cubic function y crosses the x axis once and touches x axis once, the equation $y = 0$ has **three** solutions but one of them is repeated

For example $y = (x - 1)(x + 2)^2$

y intercept, when $x = 0$ is $(0 - 1) \times (0 + 2)^2 = -1 \times 4 = -4$



When the graph of a cubic function y crosses the x axis once, the equation $y = 0$ can have

One distinct repeated solution,

for example $y = (x - 1)^3$

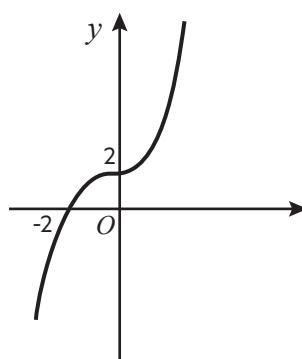
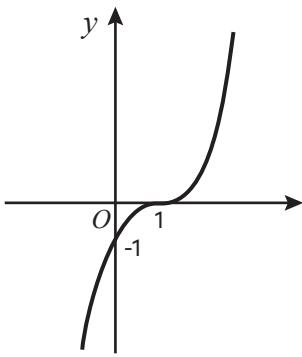
y intercept, when $x = 0$ is $(0 - 1)^3 = -1^3 = -1$

or only one real solution,

for example $y = (x + 2)(x^2 + x + 1)$

The quadratic $(x + 2)(x^2 + x + 1)$ has no real solutions

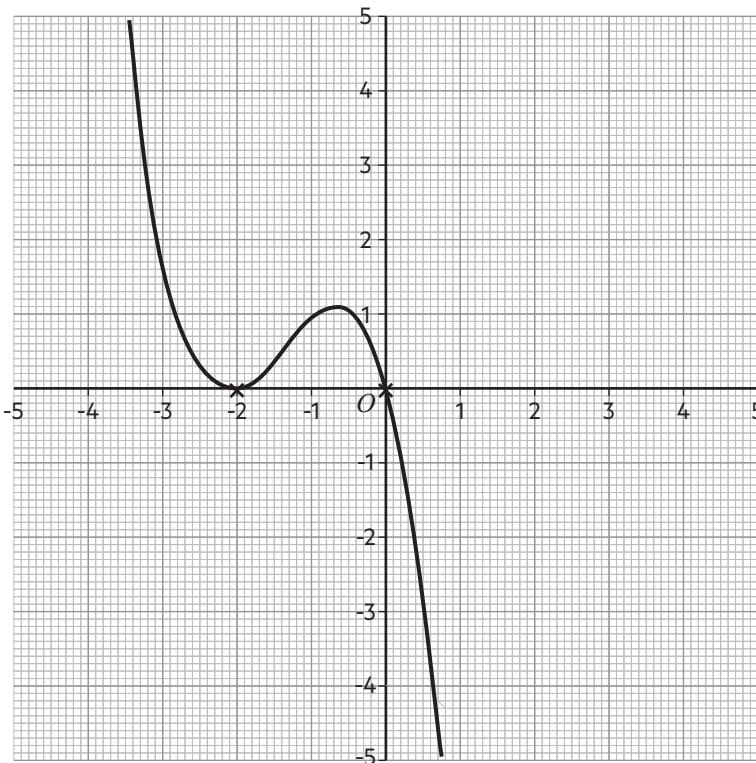
y intercept, when $x = 0$ is $(0 + 2) \times (0 + 0 + 1) = 2 \times 1 = 2$



Graphs of cubic equations continued

Example

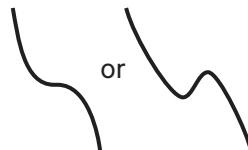
Sketch the graph of $y = -x(x + 2)^2$ marking clearly the points of intersection with the axes



Step 1

Work out what shape your graph is going to be
Remember the formula $y = ax^3 + bx^2 + cx + d$

When $a < 0$ the function looks like



Step 2

Work out where the graph crosses the axes
The graph has **three real solutions** but one of them is repeated $x = 0$ and $x = -2$ (twice)
 y intercept = $-0(0 + 2)^2 = 0 \times 2^2 = 0$

Step 3

Mark these points on your graph and draw the line to fit these points

Showing that a cubic graph has a solution between two values

Example

Show that the equation $x^3 + x = 7$ has a solution between 1 and 2

$$x^3 + x - 7 = 0$$

$$\text{Let } x = 2$$

$$2^3 + 2 - 7 = 8 + 2 - 7 = 3$$

$$\text{Let } x = 1$$

$$1^3 + 1 - 7 = -5$$

→ There is a change of sign in y , therefore the equation has a solution between 2 and 1

Using iterative formula to find a root of a cubic graph

Using the equation $x^3 + x = 7$ from the above example, find the root to 4 decimal places

Set up your iterative formula

$$x^3 = 7 - x$$

$$x = \sqrt[3]{7 - x}$$

$$x = \sqrt[3]{7 - x_n}$$

You know the root is between 1 and 2 so start the iterative process with 2

$$x_0 = 2$$

$$x_1 = \sqrt[3]{7 - 2} = 1.70998$$

$$x_2 = \sqrt[3]{7 - 1.70998} = 1.7424$$

$$x_3 = \sqrt[3]{7 - 1.7424} = 1.7389$$

$$x_4 = \sqrt[3]{7 - 1.7389} = 1.7392$$

$$x_5 = \sqrt[3]{7 - 1.7392} = 1.7392$$

Continue with the iterative process
until the 4th decimal place stays the same when rounded to 4 decimal places

Linear inequalities

Linear inequalities usually illustrate their solutions using a number line.

When using a number line, a small solid circle is used for \leq or \geq and a hollow circle is used for $>$ or $<$

Example

Solve the inequality $x + 6 > 3$ and illustrate the solution on a number line.

Step 1

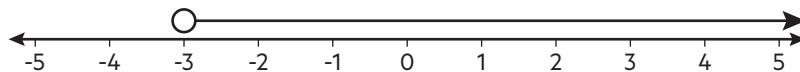
You want to get x on its own on one side of the inequality

$x > 3 - 6$ subtract 6 from both sides

$$x > -3$$

Step 2

Illustrate the inequality on a number line, remember as the sign is $>$ you want to use a hollow circle



Example 2

Show the inequality $-3 \leq x + 2 < 2$ on a number line.

Step 1

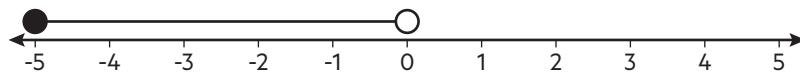
You want to get x on its own in the middle of the inequality

$-3 - 2 \leq (x + 2) - 2 < 2 - 2$ subtract 2 from each part

$$-5 \leq x < 0$$

Step 2

Illustrate the inequality on a number line, remember where the sign is \leq you want to use a solid circle



Common values of Sin Cos Tan

	Angle				
	0°	30°	45°	60°	90°
\sin	0	0.5	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
\cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	0.5	0
\tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	inf