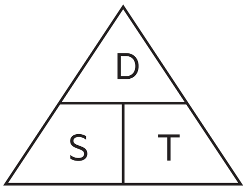
# Formulae

## Compound measures

### Distance, speed, time

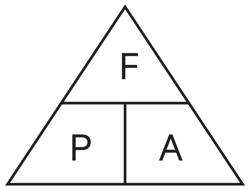



Distance = speed x time

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

### Force, pressure, area

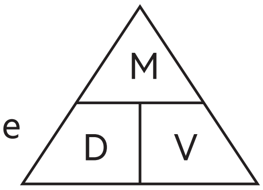



Force = pressure x area

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

$$\text{Area} = \frac{\text{Force}}{\text{Pressure}}$$

### Mass, density, volume

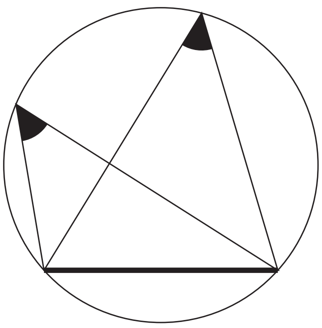



Mass = density x volume

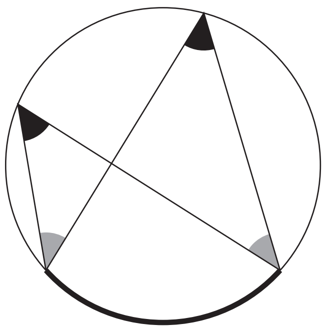
$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

$$\text{Volume} = \frac{\text{Mass}}{\text{Density}}$$

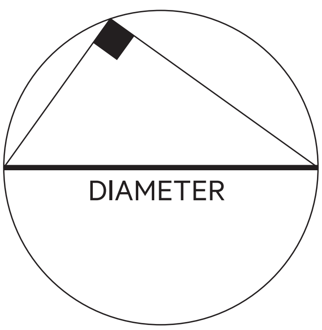
## Circle theorem



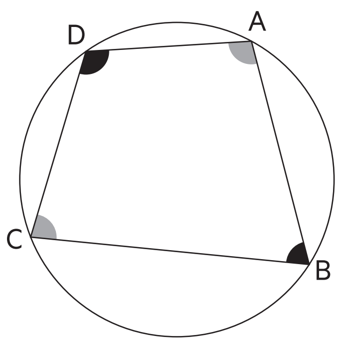
Angles in the same segment and standing on the same chord are **always equal**



Angles subtended from the same arc in the same segment are **always equal**

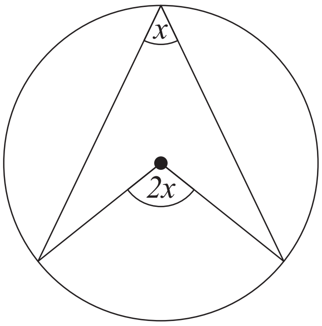



The angle in a semicircle is always **right angle**

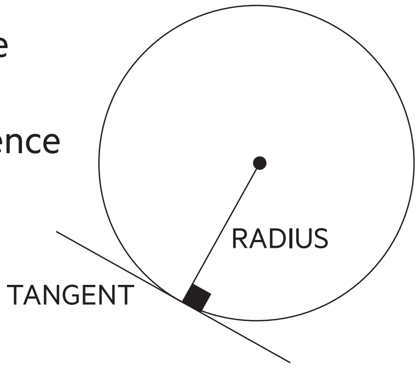



ABCD is a cyclical quadrilateral with all vertices on the circumference of the circle. **Diagonally opposite** angles add up to **180°**

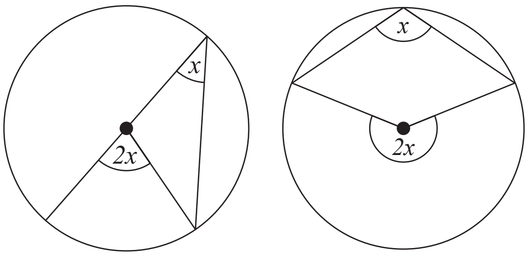
## Circle theorem continued

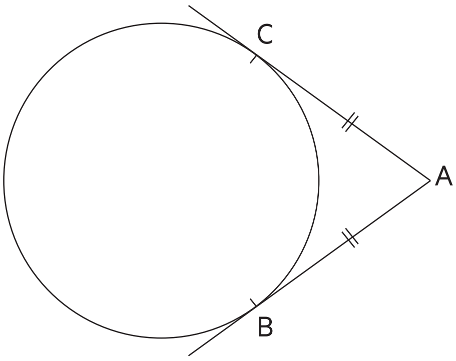



The angle at the centre of a circle is **twice** the angle at the circumference

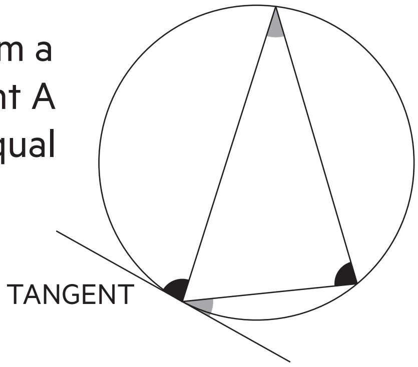



The angle between the tangent and the radius is always **90°**




Tangents from a common point A are always equal in length
**AB = AC**




The angle between a tangent and a chord is equal to the angle in the alternate segment (angle opposite the chord)

## Pythagoras theorem

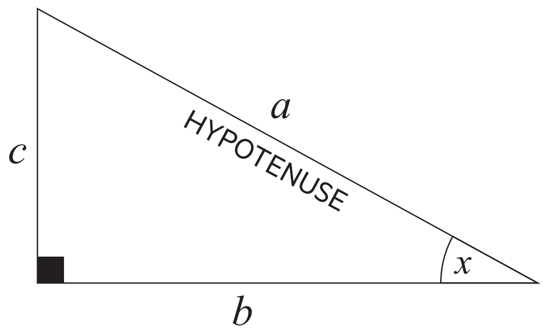
In a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides

$a^2 = b^2 + c^2$

$a = \sqrt{b^2 + c^2}$

$b = \sqrt{a^2 - c^2}$

$c = \sqrt{a^2 - b^2}$

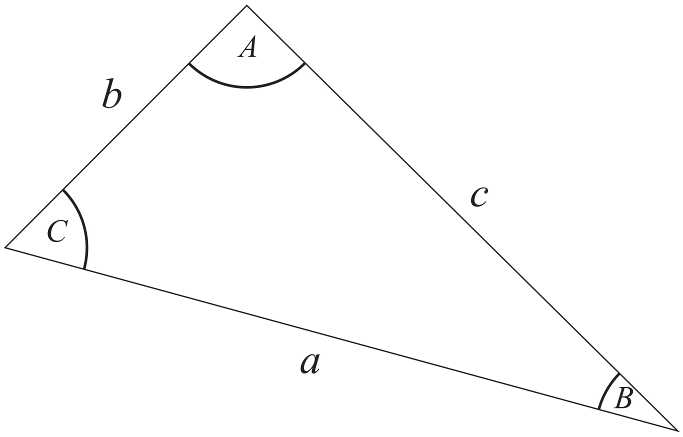



## SOH CAH TOA

$sin(x) = \dfrac{\textbf{O}\text{pposite}}{\textbf{H}\text{ypotenuse}}$ $\quad cos(x) = \dfrac{\textbf{A}\text{djacent}}{\textbf{H}\text{ypotenuse}}$ $\quad tan(x) = \dfrac{\textbf{O}\text{pposite}}{\textbf{A}\text{djacent}}$

In a right-angled triangle the **hypotenuse is always opposite the right angle**

## Trigonometry rules




### Sine rule

$$\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)}$$

### Cosine rule

$$a^2 = b^2 + c^2 - 2bcCos(A)$$

$$Cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$Cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$$

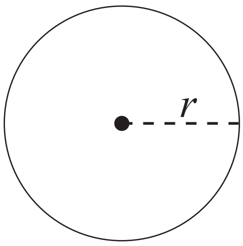
$$Cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

### Area of triangle

$$\frac{abSin(C)}{2}$$

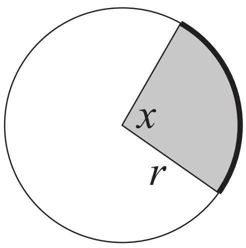
## Area and volumes

**Circle**




Area = $\pi r^2$

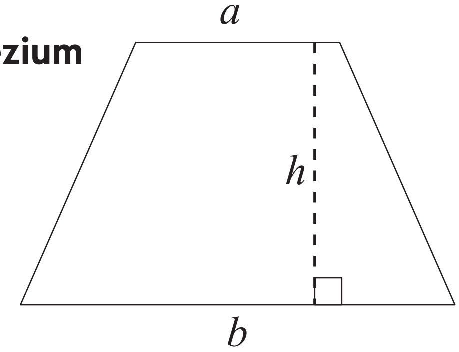
**Sector**




Area = $\pi r^2 \times \frac{x°}{360}$

Arc length = $2\pi r \times \frac{x°}{360}$

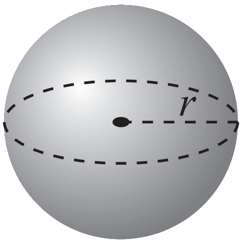
**Trapezium**




Area = $\frac{(a + b)h}{2}$

## Area and volumes continued

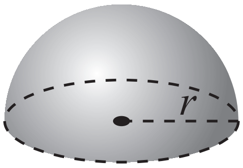
**Sphere**




Surface area = $4\pi r^2$

Volume= $\dfrac{4\pi r^3}{3}$

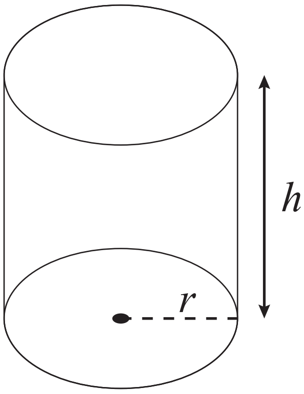
**Hemisphere**




Surface area = $2\pi r^2$

Volume= $\dfrac{2\pi r^3}{3}$

**Cylinder**

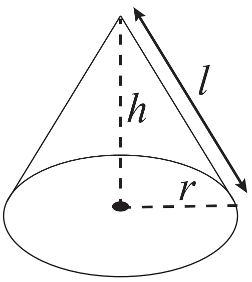



Curved surface area = $2\pi rh$

Total surface area = $2\pi rh + 2\pi r^2$

Volume= $\pi r^2 h$ (base area x height)

**Cone**

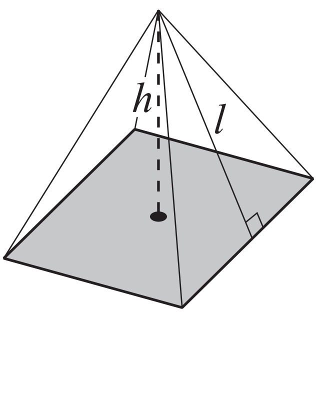



Curved surface area = $\pi rl$ (where $l$ is the slant length)

Total surface area = $\pi r^2 + \pi rl$

Volume= $\dfrac{\pi r^2 h}{3}$

**Pyramid**



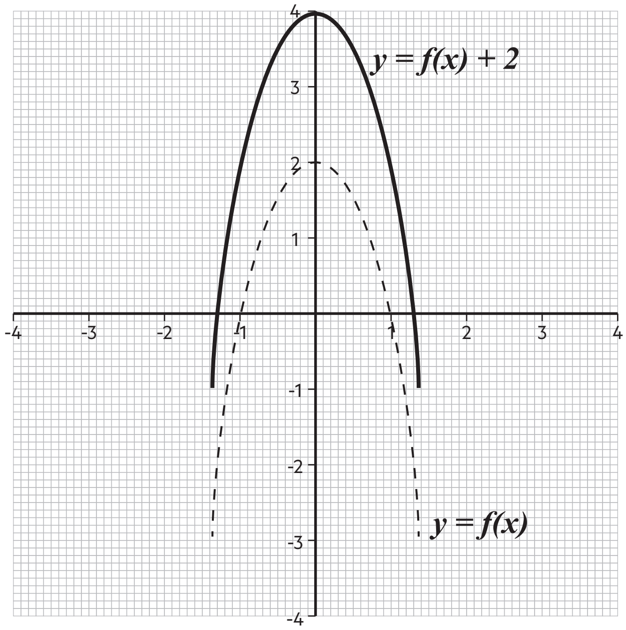

Volume= $\dfrac{base\ area \times h}{3}$

# Transformations of functions

$f(x) \pm a \longrightarrow$ Translation by $\begin{pmatrix} 0 \\ \pm a \end{pmatrix}$

same sign up or down;
**$y$ changes**

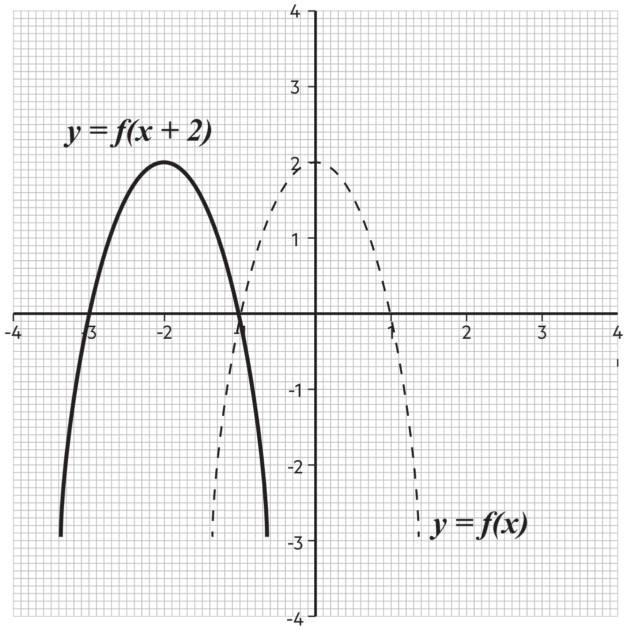
eg $y = f(x) + 2$




$f(x + a) \longrightarrow$ Translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$

left or right; **$x$ changes**
+5 $\longrightarrow$ 5 units **left**
-5 $\longrightarrow$ 5 units **right**

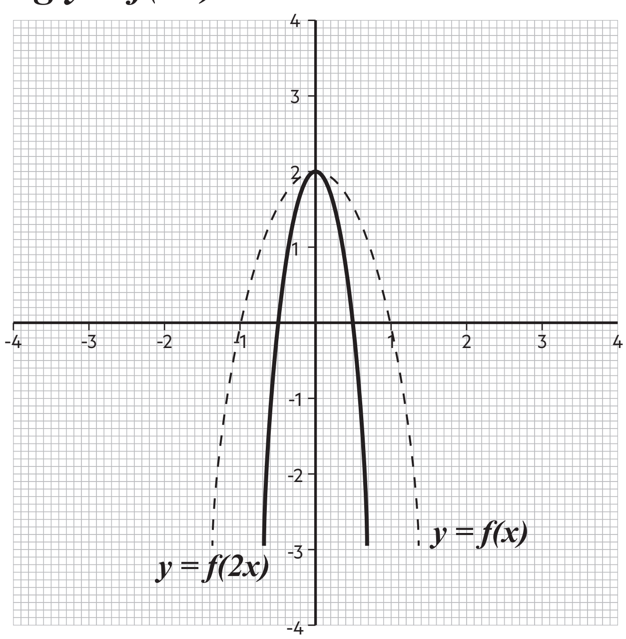
eg $y = f(x + 2)$




$f(ax) \longrightarrow$ Shrink by a scale factor of $\frac{1}{a}$; **$x$ changes**

Period changes, becomes smaller

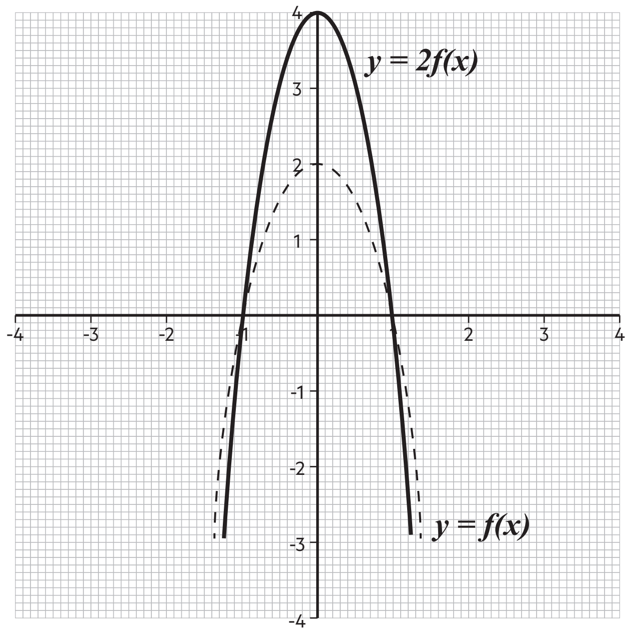
eg $y = f(2x)$



# Transformations of functions continued

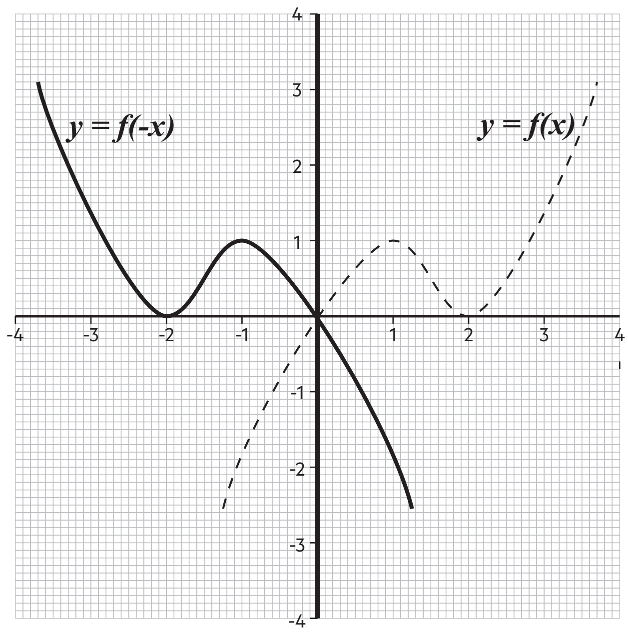
*af(x)* ⟶ Enlargement by scale factor **a**; ***y* changes**

**eg $y = 2f(x)$**



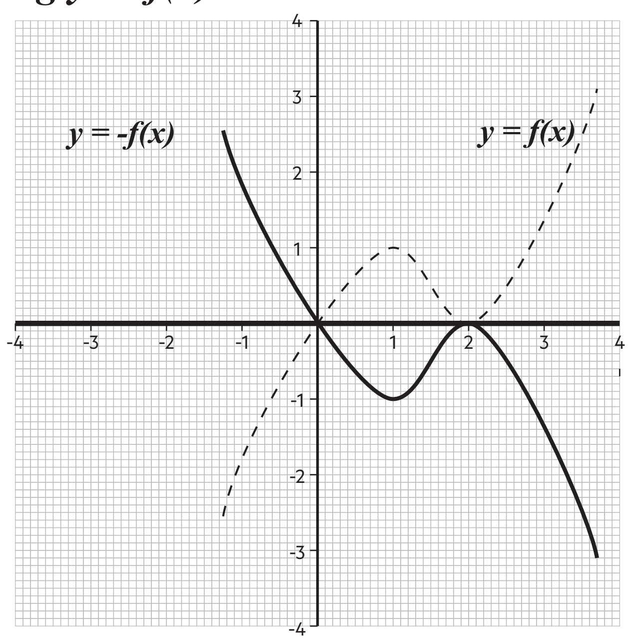

*f(-x)* ⟶ Reflection in the **y-axis; *x* changes**

**eg $y = f(-x)$**



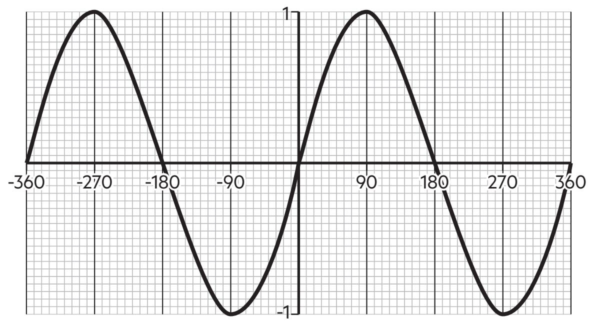

*-f(x)* ⟶ Reflection in the **x-axis; *y* changes**

**eg $y = -f(x)$**



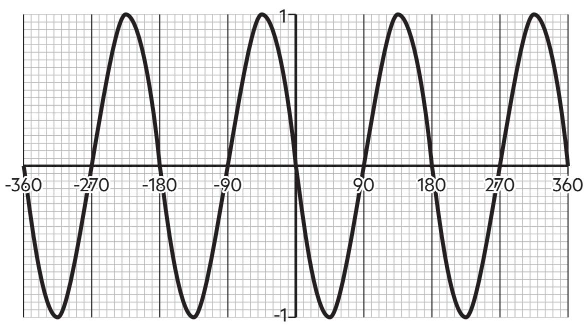
# Sin, cos and tan graphs

## *Sin graph*

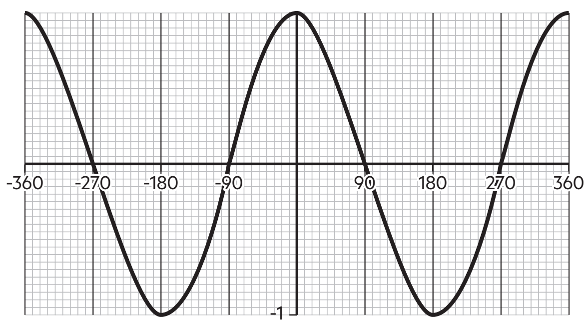



***y = sin(2x)*** ⟶ Shrink by a scale factor of $\frac{1}{2}$; ***x* changes**

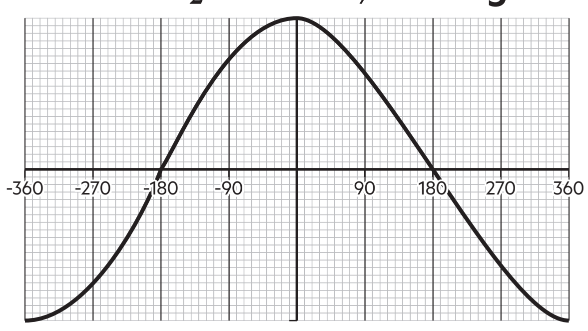
Period changes, becomes smaller



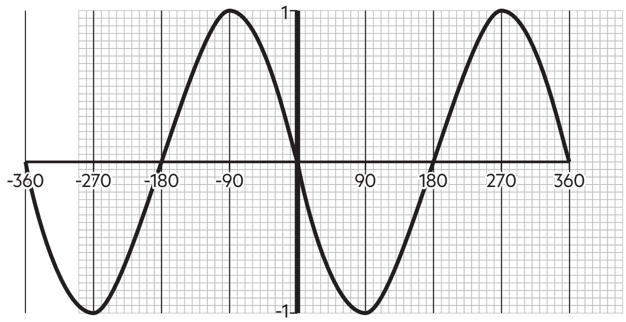

## *Cos graph*



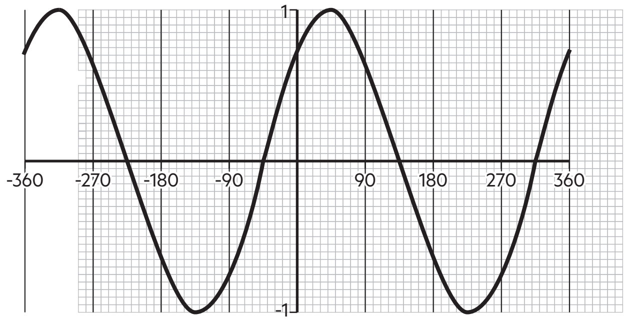

***y = Cos*** $(\frac{1}{2}x)$ Scale by a scale factor of 2; ***x* changes**



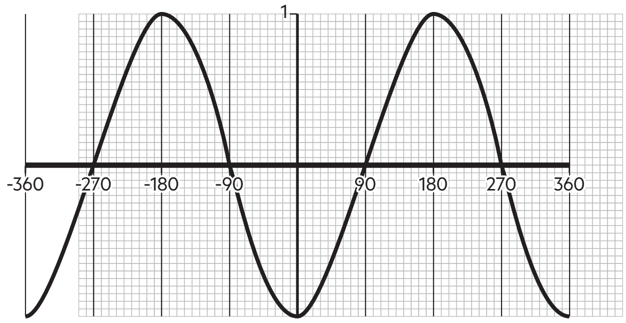

***y = sin(-x)*** ⟶ Reflection in the **y-axis; *x* changes**



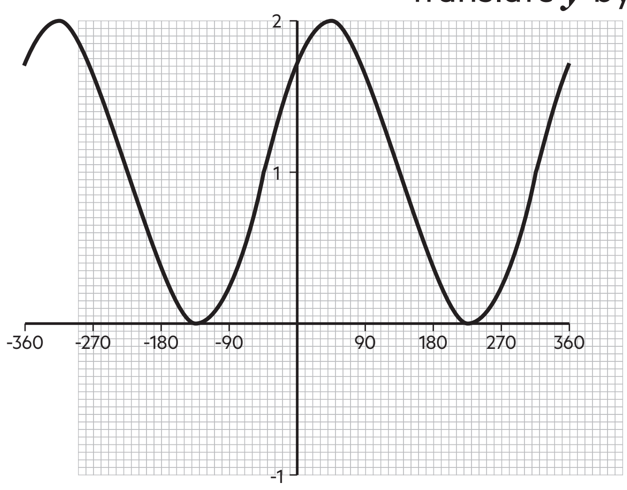

***y = sin(x + 45°)*** ⟶ Translate by -45º; ***x* changes**




***y = -Cos(x)*** ⟶ Reflection in the **x-axis; *y* changes**

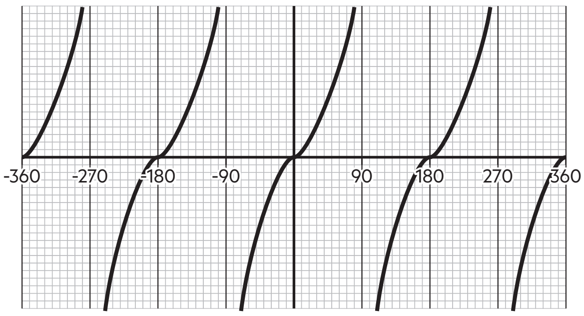



***y = Cos(x - 45°) + 1*** ⟶ translate ***x*** by + 45º; translate ***y*** by + 1



# Sin, cos and tan graphs continued

## *Tan graph*



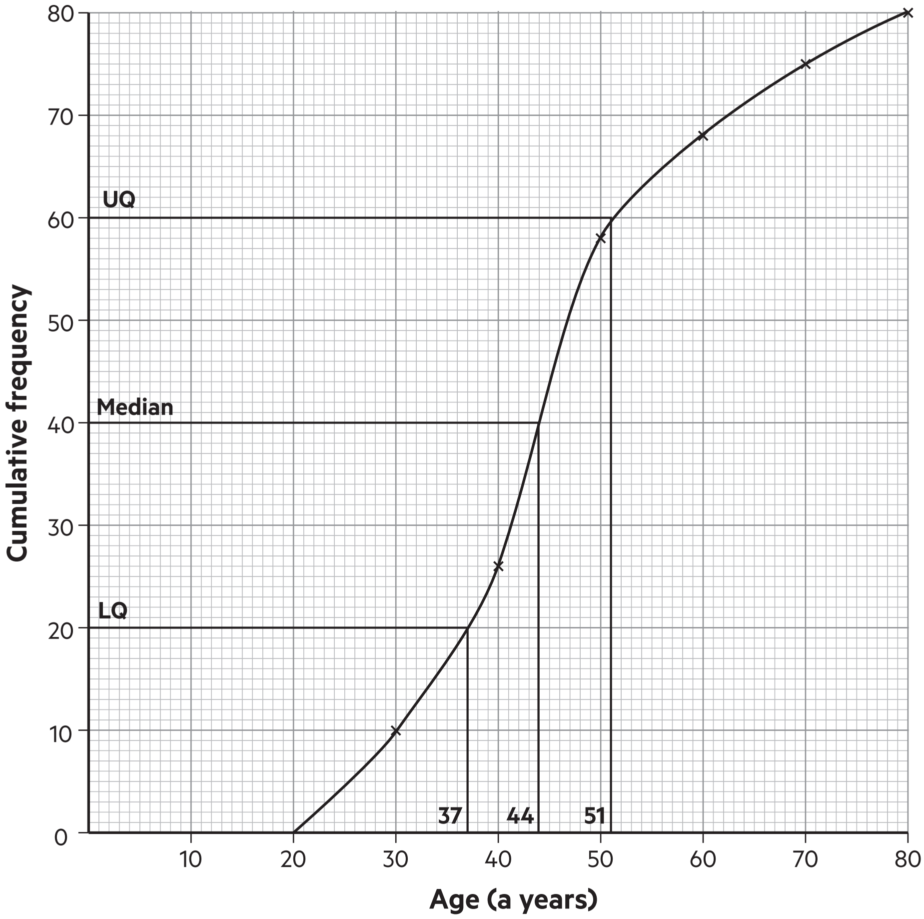

# Cumulative frequency and box plots

| Age (a years) | Frequency |
|---|---|
| *20 < a ≤ 30* | *10* |
| *30 < a ≤ 40* | *16* |
| *40 < a ≤ 50* | *32* |
| *50 < a ≤ 60* | *10* |
| *60 < a ≤ 70* | *7* |
| *70 < a ≤ 80* | *5* |

Cumulative frequency

| Age (a years) | Cumulative frequency |
|---|---|
| *20 < a ≤* **30** | *10* |
| *20 < a ≤* **40** | *26 (10+16)* |
| *20 < a ≤* **50** | *58 (26+32)* |
| *20 < a ≤* **60** | *68 (58+10)* |
| *20 < a ≤* **70** | *75 (68+7)* |
| *20 < a ≤* **80** | *80 (75+5)* |

When plotting a cumulative frequency graph, plot the marks at the **upper bound** of each range. Then join the points in a smooth curve




To find the **median value** divide the total cumulative frequency (80) by 2, in this case 40. Draw a line across from the y-axis and down to the x-axis to get the median age ⟶ **44 years**

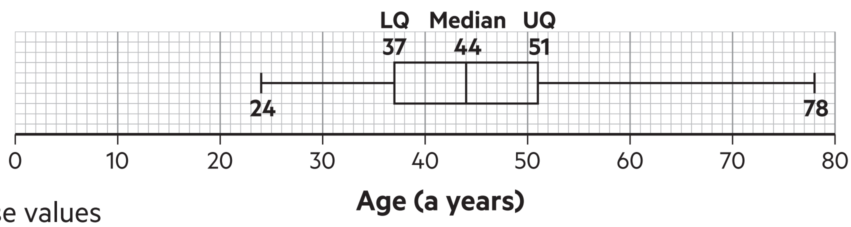
To draw a **box plot** of this graph we need to know the **Interquartile range (IQR)** – which is the range between the **lower quartile (LQ)** and the **upper quartile (UQ)** – the **maximum** and **minimum** values and the **median**

The minimum age was **24** and the maximum was **78**

To find the lower quartile divide the total cumulative frequency (80) by 4 ⟶ **20**

To find the upper quartile divide the total cumulative frequency (80) by 4, then multiply by 3 ⟶ **60**

Draw lines from y-axis and down to x-axis to get these values



# Histograms

Bar charts and frequency diagrams show data grouped in **equal** class intervals.
For data grouped **unequal** class intevals, you need a **histogram**

In a histogram the area of the bar represents the frequency. The height of each bar is the frequency density

$$\text{Frequency density} = \frac{\text{Frequency}}{\text{Class width}}$$

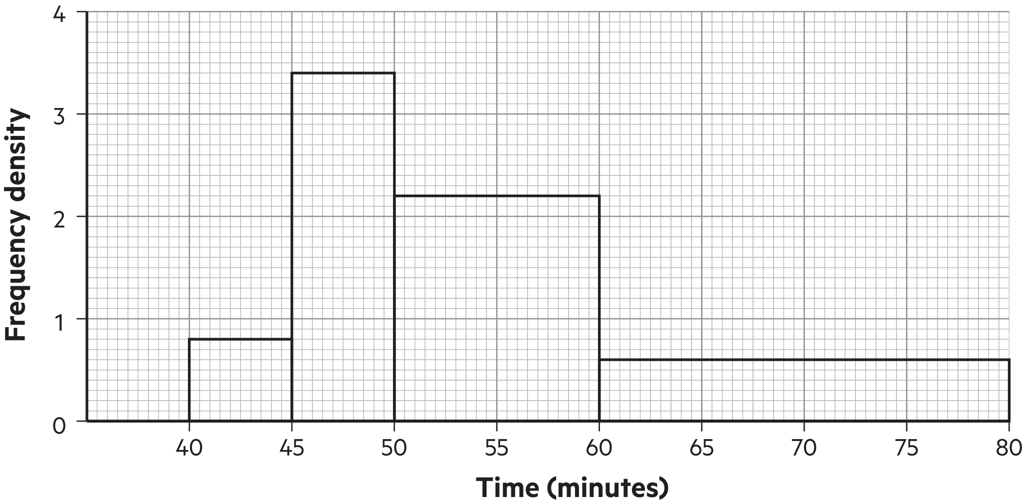
**Example**

This table shows the times taken for **55** runners to complete a fun run. Draw a histogram for this data

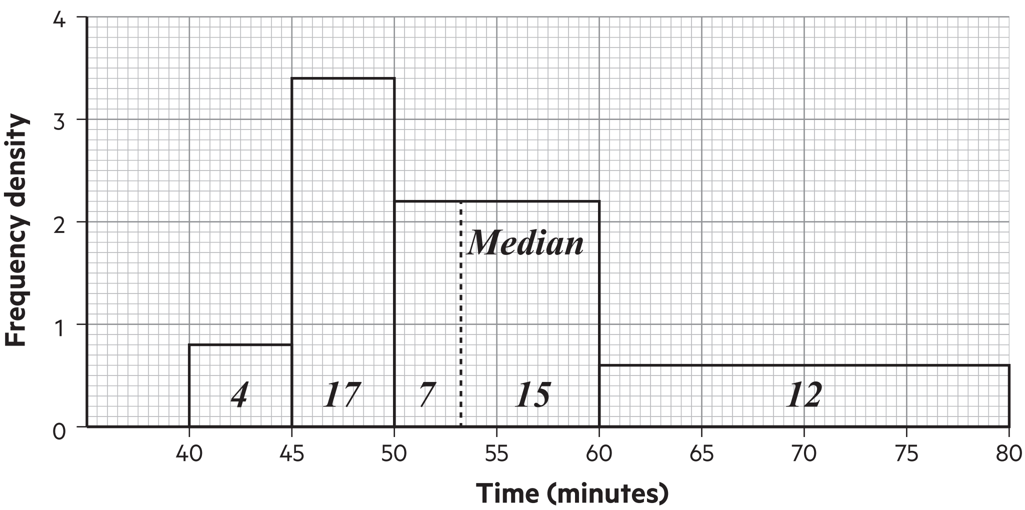
| *Time, t (minutes)* | $40 < t \leq 45$ | $45 < t \leq 50$ | $50 < t \leq 60$ | $60 < t \leq 80$ |
|---|---|---|---|---|
| ***Frequency*** | 4 | 17 | 22 | 12 |

Firstly you need to calculate the **frequency density**

| *Time, t (minutes)* | $40 < t \leq 45$ | $45 < t \leq 50$ | $50 < t \leq 60$ | $60 < t \leq 80$ |
|---|---|---|---|---|
| ***Frequency*** | 4 | 17 | 22 | 12 |
| ***Frequuncy density*** | 0.8 | 3.4 | 2.2 | 0.6 |




**Estimating the median value of a histogram**




**Step 1**

Find the median. The median is the 28th value...

$$\frac{55 + 1}{2} = 28$$

**Step 2**

You now know the median lies in the class $50 < t \leq 60$

**Step 3**

Use the frequency density formula to find the width of the class from 50 to the median.

Frequency = $28 - (17 + 4) = 7$

Frequency density = $2.2$

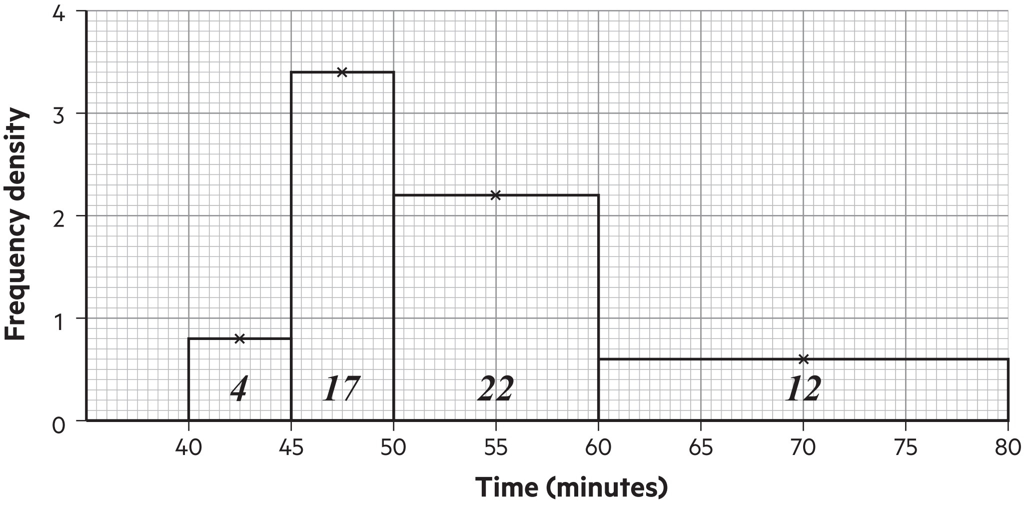
Class width = $7 \div 2.2 = 3.18$

**Step 4**

An estimate for the median is $50 + 3.18 = 53.2$ *minutes (1dp)*

# Histograms continued

### Estimating the mean value of a histogram




To find the mean of a histogram use the midpoint of each class interval and multiply by its frequency

$4 \times 42.5 = 170$

$17 \times 47.5 = 807.5$

$22 \times 55 = 1{,}210$

$12 \times 70 = 840$

Sum them up and then divide by the total number of runners

$$\frac{170 + 807.5 + 1{,}210 + 840}{55}$$

Mean time of runners = ***50 mins***

# Representing inequalities graphically

When drawing graphs of inequality, ignore the greater than (>, ≥) and less than (<,≤) symbols and draw the lines as if they were = signs

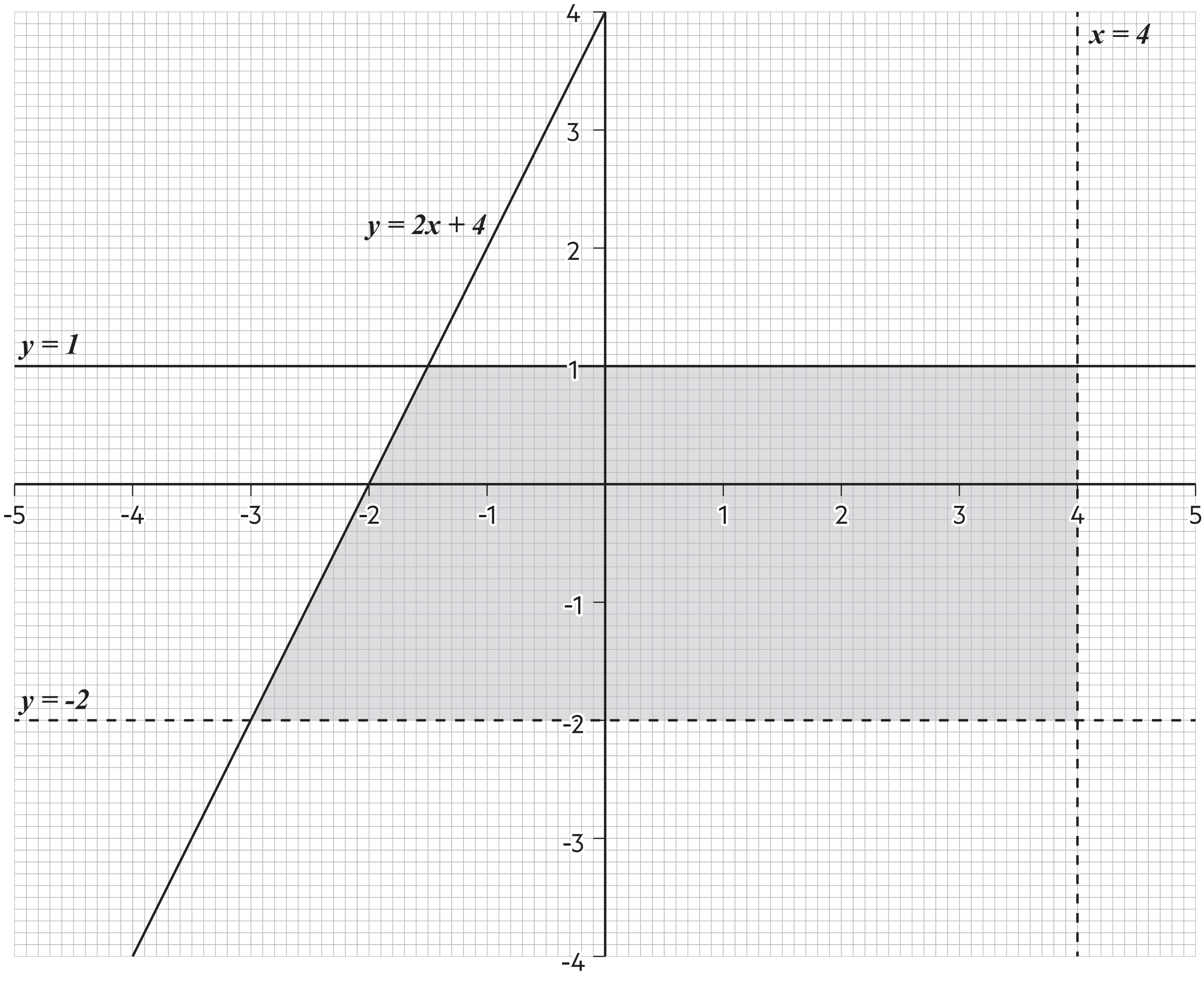
eg. Shade the region that saitisfies the inequalites: $x < 4, y \leq 2x + 4, y > -2, y \leq 1$

So when drawing the lines change the inequalities to $x = 4, y = 2x + 4, y = -2, y = -1$

You use solid lines when the values are included and dotted lines when they are not

ie

$x < 5$ – *dotted* $y \leq 2x + 4$ – *solid* $y > -2$ – *dotted* $y \leq 1$– *solid*



# Quadratic sequences

**Example 1**
Find the *nth* term of the non-linear (quadratic) sequence
5, 12, 23, 38, 57 ...

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 5 | | 12 | | 23 | | 38 | | 57 |
| **First difference** | | 7 | | 11 | | 15 | | 38 | |
| **Second difference** | | | 4 | | 4 | | 4 | | |

Using the equation $an^2 + bn + c$

The second differences of a quadratic sequence are constant and equal to $2a$

So $a = 4 \div 2 = 2$

The formula has a $2n^2$ term in it, next note the difference between the sequence $2n^2$ and the original sequence

| | | | | | |
|---|---|---|---|---|---|
| $2n^2$ | 2 | 8 | 18 | 32 | 50 |
| **Sequence** | 5 | 12 | 23 | 38 | 57 |
| **Change** | +3 | +4 | +5 | +6 | +7 |
| | 1 | 1 | 1 | 1 | |

If the change is **not constant,** then find the difference to find the value of $b$

The formula has a $2n^2 + n$ in it, to find the constant $c$...

$2n^2 + n + c = 5,$ when $n = 1$
$2 + 1 + c = 5$
$c + 3 = 5$
$c = 5 - 3$
$c = 2$

So the formula for the *nth* term is $2n^2 + n + 2$

**Example 2**
Find the *nth* term of the non-linear (quadratic) sequence
3, 9, 19, 33, 51 ...

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 3 | | 9 | | 19 | | 33 | | 51 |
| **First difference** | | 6 | | 10 | | 14 | | 18 | |
| **Second difference** | | | 4 | | 4 | | 4 | | |

Using the equation $an^2 + bn + c$

The second differences of a quadratic sequence are constant and equal to $2a$

So $a = 4 \div 2 = 2$

The formula has a $2n^2$ term in it, next note the difference between the sequence $2n^2$ and the original sequence

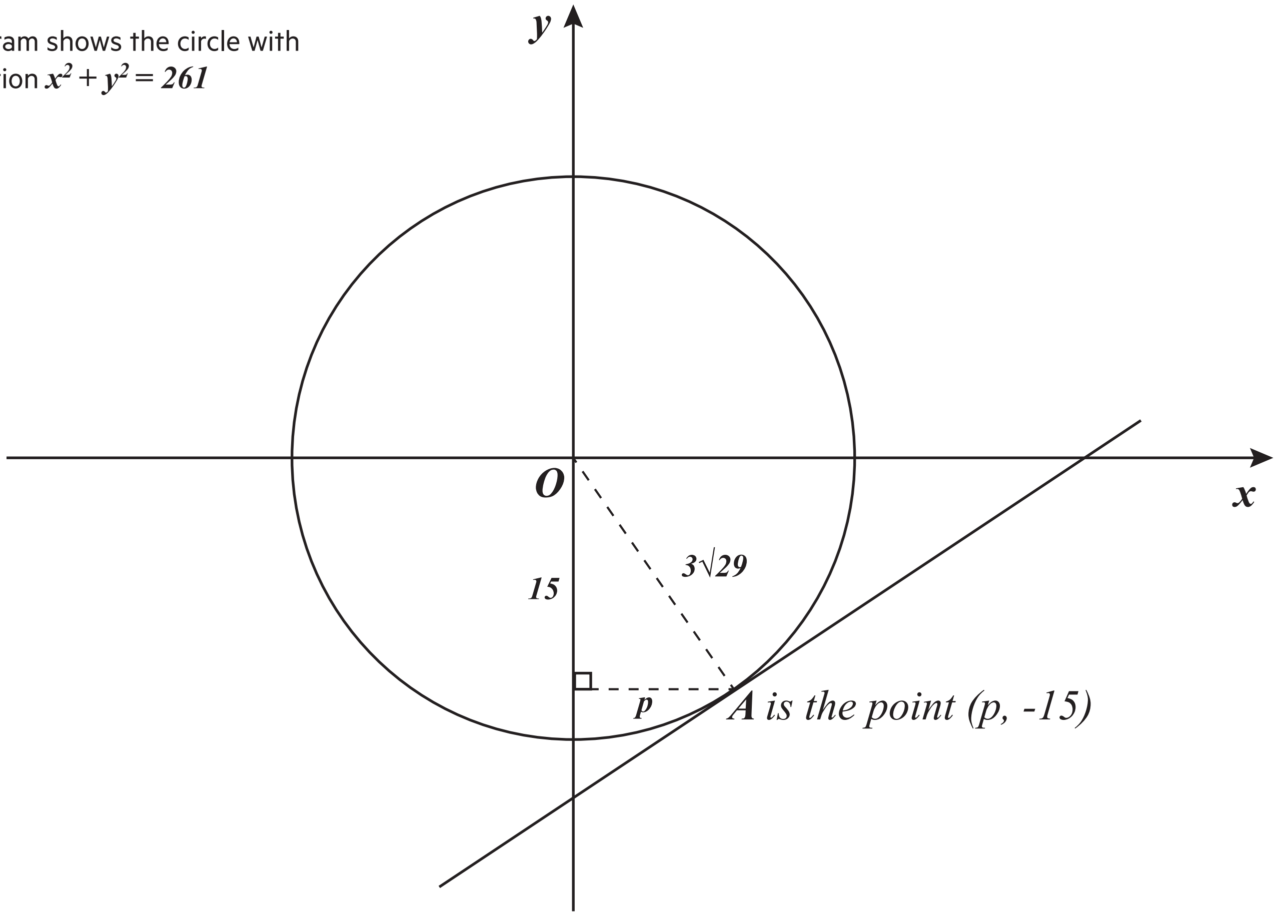
| | | | | | |
|---|---|---|---|---|---|
| $2n^2$ | 2 | 8 | 18 | 32 | 50 |
| **Sequence** | 3 | 9 | 19 | 33 | 51 |
| **Change** | +1 | +1 | +1 | +1 | +1 |

If the change is **constant** then $b = 0$, and $c$ is equal to the change

So the formula for the *nth* term is $2n^2 + 1$

# Finding the equation of the tangent to a circle

The diagram shows the circle with the equation $x^2 + y^2 = 261$




A tangent is drawn at point $A$ with coordinates of $(p, -15)$, where $p > 0$
Find an equation of the tangent at $A$

**Step 1**
Calculate the radius of the circle

$x^2 + y^2 = r^2$

$r^2 = 261$

$r = \sqrt{261}$

$r = 3\sqrt{29}$

**Step 2**
Calculate $p$ using Pythagoras theorem

$(3\sqrt{29})^2 = 15^2 + p^2$

$261 = 225 + p^2$

$p^2 = 261 - 225$

$p^2 = 36$

$p = 6$

**Step 3**
Calculate the gradient of the radius $OA$

$$\frac{\text{Change in } y}{\text{Change in } x} = \frac{-15}{6}$$

**Step 4**
The gradient of the tangent is the negative reciprocal of the gradient of the radius $OA$

$$= \frac{6}{15} = \frac{2}{5}$$

**Step 5**
Use $y = mx + c$ to calculate the equation of tangent

$-15 = \frac{2}{5}(6) + c$

$-15 = \frac{12}{5} + c$

$-75 = 12 + 5c$

$5c = -75 - 12 = -87$

$c = \frac{-87}{5}$

$y = \frac{2}{5}x - \frac{87}{5}$

$5y = 2x - 87$

# Similarity and congruence

**Congruence**

Two triangles are congruent when one of these conditions is true

**SSS** (all three sides are equal)

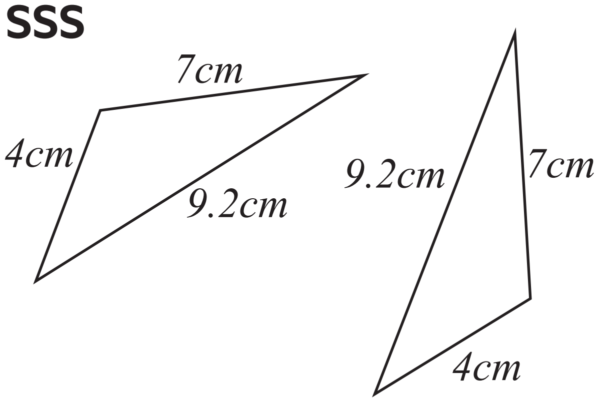
**SAS** (two sides and the included angle are equal)

**AAS** (two angles and a corresponding side are equal)

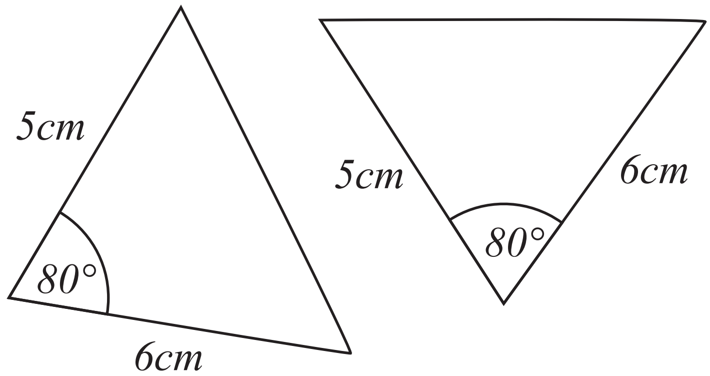
**RHS** (right angle, hypotenuse and one other side are equal)

**Examples**

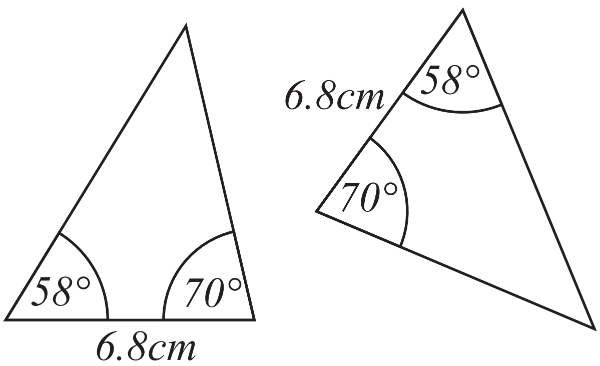
## SSS



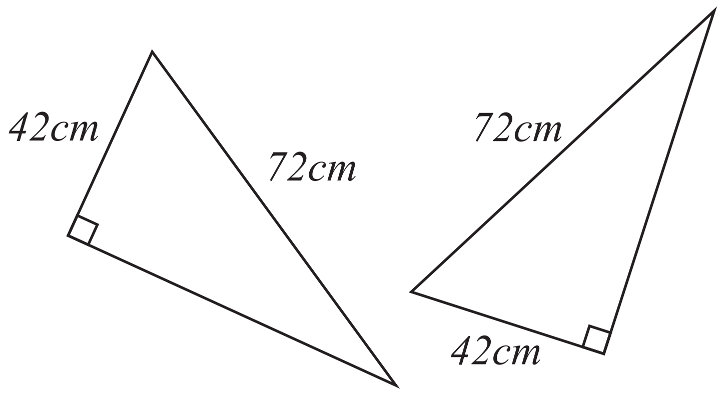

## SAS




## AAS

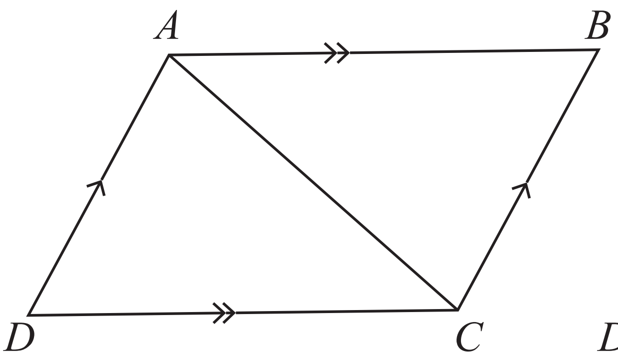


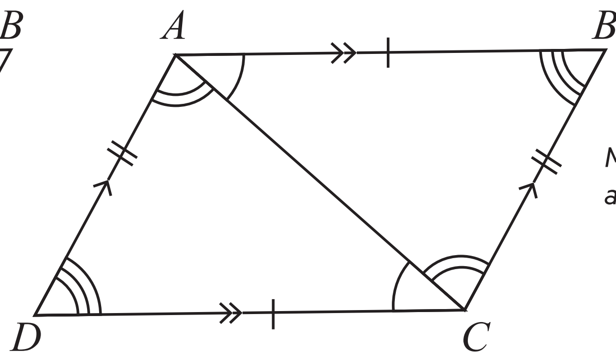

## RHS




**Example**

$ABCD$ is a parallelogram. Prove that triangle $ABC$ is congruent to $ADC$




*Mark in all equal angles and sides*

$AB = CD$ because opposite sides of a parallelogram are equal

$BC = AD$ because opposite sides of a parallelogram are equal

$AC$ is common to both triangles

Therefore triangle $ABC$ is congruent to triangle $ACD$ **(SSS)**

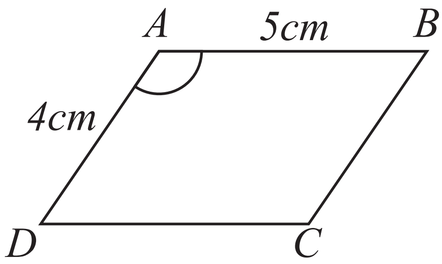
# Similarity and congruence continued

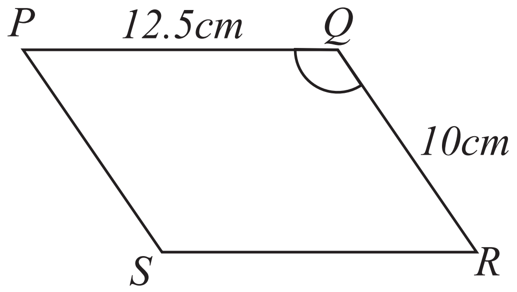
**Similarity**

Shapes are **similar** when one shape is an enlargement of the other. **Corresponding angles are equal** and **corresponding sides are all in the same ratio**

**Example 1**

Here are two parallelograms $ABCD$ and $PQRS$. Angles $DAB$ and $PQR$ are the same. Prove that they are similar



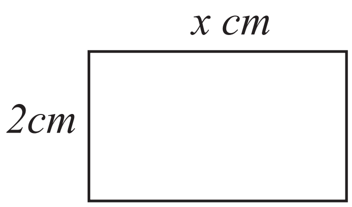

Find out the ratios of the corresponding sides, if they are the same the parallelograms are similar

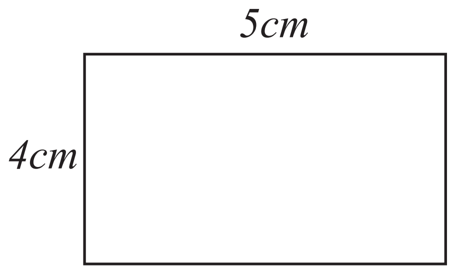
$$\frac{AD}{QR} = \frac{4}{10} = \frac{2}{5} \qquad \frac{AB}{PQ} = \frac{5}{12.5} = \frac{2}{5}$$

The ratios of the corresponding sides are the same therefore the shapes are similar

**Example 2**

These two rectangles are similar. Find the missing length $x$ in the smaller rectangle



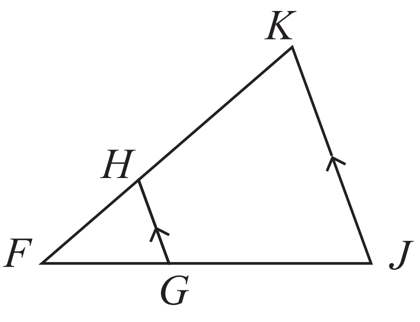




If the rectangles are similar the ratios of the corresponding sides will be the same, therefore

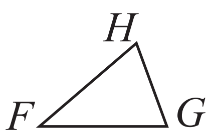
$$\frac{2}{4} = \frac{x}{5} \longrightarrow \frac{1}{2} = \frac{x}{5} \longrightarrow \frac{5}{10} = \frac{2x}{10} \longrightarrow 2x = 5 \longrightarrow x = 2.5$$

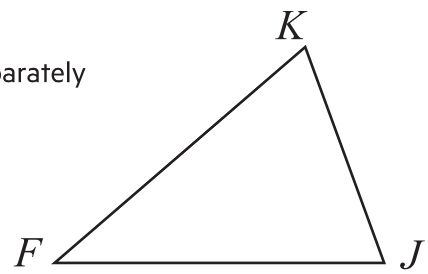
**Example 3**

Explain why triangles $FGH$ and $FJK$ are similar




Draw out the triangles separately




$H\hat{F}G$ and $K\hat{F}J$ are the same (common angle)

$F\hat{G}H$ and $F\hat{J}K$ are the same (corresponding angles)

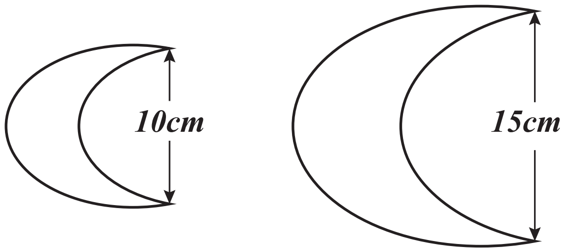
$F\hat{H}G$ and $F\hat{K}J$ are the same (corresponding angles)

All corresponding angles are the equal therefore the triangles are similar

# Similarity in 2D and 3D shapes

When a shape is enlarged by a ***scale factor k,*** any areas are enlarged by a ***scale factor*** $k^2$ and volumes are enlarged by a ***scale factor*** $k^3$

**Example 1**
These two shapes are mathematically similar
The area of the smaller shape is $24cm^2$
Work out the area of the larger shape




First calculate the **linear scale factor (LSF)** of the larger shape compared to the smaller shape

$$LSF = \frac{15}{10} = \frac{3}{2}$$

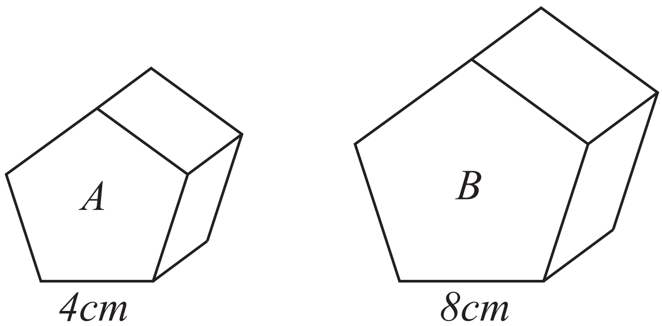
Next calculate the **area scale factor (ASF),** remember the **ASF** is the **square** of the **LSF**

$$ASF = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = \mathbf{2.25}$$

Next multiply the area of the smaller shape by the **ASF** to find the area of the larger shape

$$\mathbf{24 \times 2.25 = 54cm^2}$$

**Example 2**
These two solids $A$ and $B$ are similar




Solid $A$ has a volume of $80cm^3$
a) Work out the volume of solid $B$

Solid $B$ has a total surface area of $160cm^2$
b) Work out the total surface area of solid $A$

a) First calculate the **linear scale factor (LSF)** of the larger shape compared to the smaller shape

$$LSF = \frac{8}{4} = \mathbf{2}$$

Next calculate the **volume scale factor (VSF),** remember the **VSF** is the **cube** of the **LSF**

$$VSF = 2^3 = \mathbf{8}$$

Next multiply the volume of the smaller shape by the **VSF** to find the volume of the larger shape

$$\mathbf{80 \times 8 = 640cm^3}$$

b) Calculate the **area scale factor (ASF),** remember the **ASF** is the **square** of the **LSF**

$$ASF = \left(\frac{4}{8}\right)^2 = \frac{16}{64} = \mathbf{0.25}$$

Next multiply the total surface area (TSA) of the larger shape by the **ASF** to find the TSA of the smaller shape

$$\mathbf{160 \times 0.25 = 40cm^2}$$

# Rules of indices, roots and surds

## Indices

Use these **rules of indices** to **multiply, divide** and work out the **power of a power**

$$x^m \times x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n} \qquad (x^m)^n = x^{m \times n}$$

Use these **rules of indices** to work out zero, negative and fractional powers

$$x^0 = 1 \qquad x^{-n} = \frac{1}{x^n} \qquad x^{\frac{1}{n}} = \sqrt[n]{x} \qquad x^{\frac{n}{m}} = (\sqrt[m]{x})^n \qquad x^{\frac{-n}{m}} = \frac{1}{(\sqrt[m]{x})^n}$$

**Examples**

$$2^{-5} \times 2^{12} = 2^{(-5+12)} = 2^7 = 128 \qquad (3^{-7})^0 = 3^{(-7 \times 0)} = 3^0 = 1$$

$$\sqrt[8]{4^{32}} = (4^{32})^{\frac{1}{8}} = 4^{(32 \times \frac{1}{8})} = 4^4 = 256 \qquad 3^4 \div 3^{-1} = \frac{3^4}{3^{-1}} = 3^{4-(-1)} = 3^5 = 243$$

## Surds

Use these rules to multiply and divide **surds**

$$\sqrt{mn} = \sqrt{m} \times \sqrt{n} \qquad \sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}$$

Use these methods to **rationalise a denominator** to give the denominator as an integer

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b} \qquad \frac{1}{a\sqrt{b}} = \frac{1}{a\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{b}}{ab}$$

$$\frac{c}{d+\sqrt{e}} = \frac{c}{d+\sqrt{e}} \times \frac{d-\sqrt{e}}{d-\sqrt{e}} = \frac{c(d-\sqrt{e})}{d^2-e}$$

**Examples**

$$\sqrt{12} + \sqrt{27} = \sqrt{4 \times 3} + \sqrt{9 \times 3} = 2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$$

$$\sqrt{12} \times \sqrt{3} = \sqrt{4 \times 3} \times \sqrt{3} = 2\sqrt{3} \times \sqrt{3} = 2 \times 3 = 6$$

$$\sqrt{45} \div \sqrt{5} = \frac{\sqrt{45}}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{5}} = 3$$

$$\frac{4\sqrt{30}}{2\sqrt{10}} = \frac{4\sqrt{30}}{2\sqrt{10}} \times \frac{2\sqrt{10}}{2\sqrt{10}} = \frac{8\sqrt{300}}{4\sqrt{100}} = \frac{80\sqrt{3}}{40} = 2\sqrt{3}$$

$$(8 + 4\sqrt{3})(8 - 4\sqrt{3}) = 64 - 32\sqrt{3} + 32\sqrt{3} - 48 = 64 - 48 = 16$$

$$(\sqrt{20} - 6)(\sqrt{5} + 3) = (2\sqrt{5})\sqrt{5} + 6\sqrt{5} - 6\sqrt{5} - 18 = 10 - 18 = -8$$

# Simultaneous equations

## Simple simultaneous equations

When there are two unknowns you need two equations to find their values. These are called **simultaneous equations**

**Example 1**

$2x - y = 4$ The terms in $y$ have **opposite signs**, so **add** the equations to eliminate the terms in $y$

$3x + y = 11$

$5x + 0 = 15$

$5x = 15$

$x = 3$ When you have solved $x$, put $x$ back into either equation to find $y$

$(2 \times 3) - y = 4$

$6 - y = 4$

$y = 6 - 4$

$y = 2$

**Example 2**

$3x + 2y = 13$ Multiply the first equation by $2$ to make the coe fficients of $y$ the same

$4x + 4y = 20$

$6x + 4y = 26$ Subtract the equations to eliminate the terms in $y$

$4x + 4y = 20$

$2x + 0 = 6$

$2x = 6$

$x = 3$ When you have solved $x$, put $x$ back into either equation to find $y$

$(3 \times 3) + 2y = 13$

$9 + 2y = 13$

$2y = 13 - 9$

$2y = 4$

$y = 2$

## Quadratic simultaneous equations

$2x - y = 7$ Rearrange the first equation to make $y$ the subject

$x^2 - 15 = y$

$y = 2x - 7$ Substitute the value for $y$ into the second equation

$x^2 - 15 = 2x - 7$ Rearrange so the right hand side is $0$

$x^2 - 2x - 8 = 0$ Solve the quadratic equation

$(x - 4)(x + 2) \longrightarrow x = 4 \text{ or } x = -2$ Substitute the values of $x$ back into the first equation to find $y$ values

$(2 \times 4) - y = 7$

$y = 1$

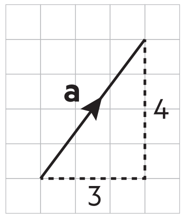
$(2 \times (-2)) - y = 7$

$y = -11$

Solutions are $x = 4, y = 1$ and $x = -2, y = -11$

# Vectors

A **vector** is a quantity that has **magnitude** and **direction**
The **magnitude** of a vector is its **size**
**Displacement** is a change in position. A displacement can be written as $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ where 3 is the $x$ component and 4 is the $y$ component. This is called a **column vector**

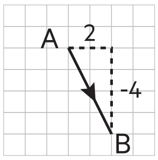



The displacement vector between two points A and B is written $\overrightarrow{AB}$

**Example**
Point A has the coordinates (3, 5) and point B has cordinates (5, 1)

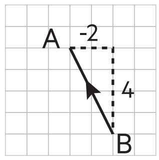
Write $\overrightarrow{AB}$ as a column vector

$$\overrightarrow{AB} = \begin{pmatrix} \text{change in } x \\ \text{change in } y \end{pmatrix} = \begin{pmatrix} 5 - 3 \\ 1 - 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$



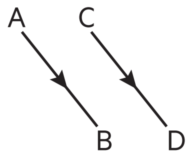

Write $\overrightarrow{BA}$ as a column vector

$$\overrightarrow{BA} = \begin{pmatrix} \text{change in } x \\ \text{change in } y \end{pmatrix} = \begin{pmatrix} 3 - 5 \\ 5 - 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

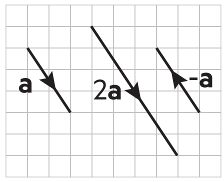



The **magnitude** of a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is its length ie $\sqrt{x^2 + y^2}$

If $\overrightarrow{AB} = \overrightarrow{CD}$ then the line segments AB and CD are **equal in length** and are **parallel**




2**a** is twice as long as **a** and in the same direction
-**a** is the same length as **a** but the opposite direction



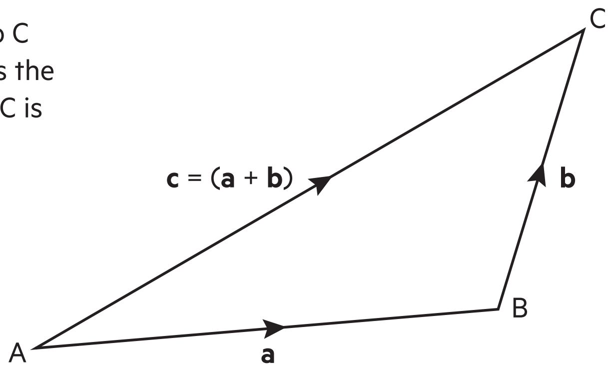

The two-stage journey from A to B and then from B to C has the same starting point and the same end point as the single journey from A to C. So A to B followed by B to C is equivalent to A to C

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

**Triangle law for vector addition**

Let $\overrightarrow{AB}$ = **a**, $\overrightarrow{BC}$ = **b** and $\overrightarrow{AC}$ = **c**

Then **a** + **b** = **c** forms a triangle

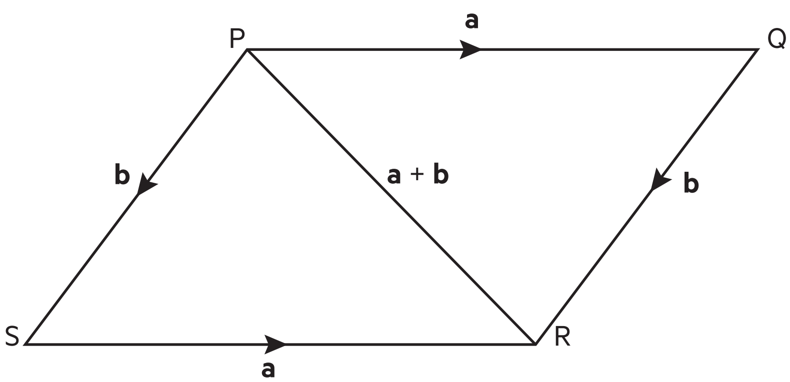



**Example**

If $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$ find $\overrightarrow{AC}$

From the diagram above you know $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, therefore $\overrightarrow{AC} = \begin{pmatrix} 2 + 7 \\ 5 - 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$

In a parallelogram PQRS where $\overrightarrow{PQ}$ is **a** and $\overrightarrow{PS}$ is **b**, the diagonal $\overrightarrow{PR}$ of the parallelogram is **a** + **b**
This is called the **parallelogram law for vector additon**



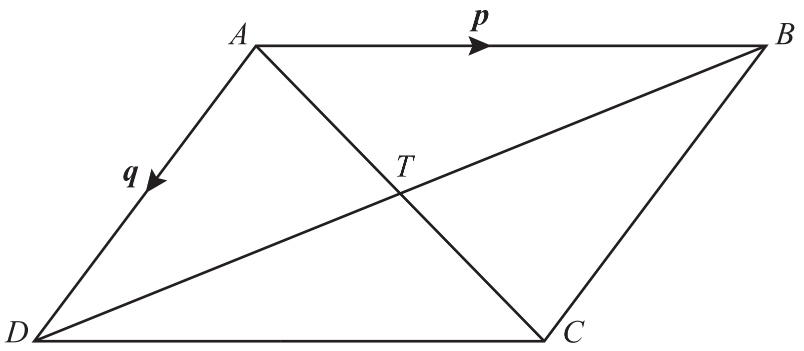

$\overrightarrow{PQ} = \overrightarrow{SR}$, therefore $\overrightarrow{SR}$ = **a**

$\overrightarrow{PS} = \overrightarrow{QR}$, therefore $\overrightarrow{QR}$ = **b**

# Vectors continued

**Example**

$AC$ and $BD$ are diagonals of parallelogram $ABCD$, $AC$ and $BD$ intersect at $T$. Express $\overrightarrow{AT}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$




**Step 1**

$\overrightarrow{AT} = \dfrac{\overrightarrow{AC}}{2}$

**Step 2**

$\overrightarrow{AC} = \boldsymbol{p} + \boldsymbol{q}$

**Step 3**

$\overrightarrow{AT} = \dfrac{\boldsymbol{p} + \boldsymbol{q}}{2} = \dfrac{1}{2}(\boldsymbol{p} + \boldsymbol{q})$

With the origin O, the vectors OA and OB are called the postion vectors of the points A and B

In general, a point with coordinates (p, q) has postion vector $\begin{pmatrix} p \\ q \end{pmatrix}$

**Example**

The points A, B, C, and D have coordinates (1, 3), (2, 7), (-6, -10) and (-1, 10) respectively

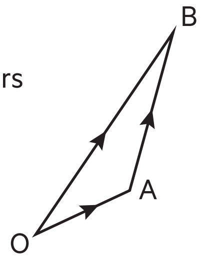
O is the origin

a) Write down the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

b) Work out as column vectors

i) $\overrightarrow{AB}$ ii) $\overrightarrow{CD}$




**i)**

**Step 1**

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

**Step 2**

$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$

**Step 3**

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

**ii)**

**Step 1**

$\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD}$

**Step 2**

$\overrightarrow{CD} = -\overrightarrow{OC} + \overrightarrow{OD} = \overrightarrow{OD} - \overrightarrow{OC}$

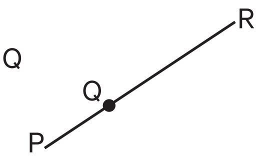
**Step 3**

$\overrightarrow{CD} = \begin{pmatrix} -1 \\ 10 \end{pmatrix} - \begin{pmatrix} -6 \\ -10 \end{pmatrix} = \begin{pmatrix} 5 \\ 20 \end{pmatrix}$

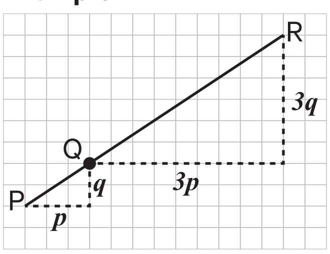
c) What do these results show about AB and CD?

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\overrightarrow{CD} = \begin{pmatrix} 5 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\overrightarrow{CD} = 5\overrightarrow{AB}$

$\overrightarrow{PQ} = \boldsymbol{k}\overrightarrow{QR}$ shows that the lines PQ and QR are parallel. Also they both pass through the point Q so PQ and QR are part of the same straight line. P, Q and R are said to be **colinear**




**Example**




$\overrightarrow{PQ} = \boldsymbol{p} + \boldsymbol{q}$ $\overrightarrow{QR} = \boldsymbol{3(p + q)}$

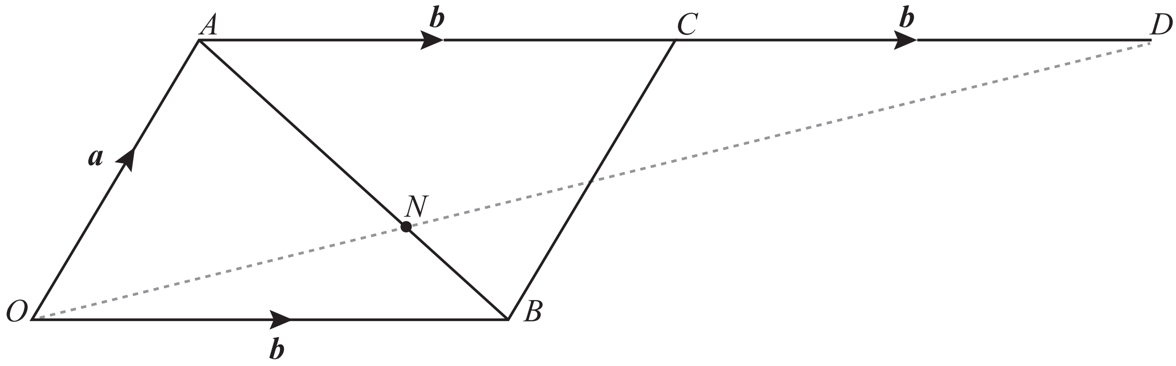
Same vector

$\overrightarrow{PQ}$ and $\overrightarrow{QR}$ have the **same vector** (parallel) and have a **point in common** (Q) therefore P, Q and R lie on the same straight line.

# Vectors continued

**Example**

$OACB$ is a parallelogram




$\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

$D$ is a point such that $\overrightarrow{AC} = \overrightarrow{CD}$

The point $N$ divides $AB$ in the ratio $2:1$

a) Write an expression of $\overrightarrow{ON}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

$\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN}$

$\overrightarrow{BN} = \frac{1}{3}\overrightarrow{BA}$

$\overrightarrow{BA} = \boldsymbol{a - b}$

$\overrightarrow{BN} = \frac{1}{3}\boldsymbol{(a - b)}$

$\overrightarrow{ON} = b + \frac{1}{3}\boldsymbol{(a - b)} = \frac{1}{3}\boldsymbol{(a + 2b)}$

b) Prove that $OND$ is a straight line

$\overrightarrow{ND} = \overrightarrow{NA} + \overrightarrow{AD}$

$\overrightarrow{NA} = \frac{2}{3}\overrightarrow{BA} = \frac{2}{3}\boldsymbol{(a - b)}$

$\overrightarrow{ND} = \frac{2}{3}\boldsymbol{(a - b)} + \boldsymbol{2b} = \frac{2}{3}\boldsymbol{(a + 2b)}$

$\overrightarrow{ON}$ and $\overrightarrow{ND}$ have the **same vector** and **share a common point** $N$ therefore the points $O$, $N$ and $D$ are **colinear** and $OND$ is a **straight line**

# Functions

A function is a rule for working out values of $y$ and given values of $x$
For example, $y = 3x$ and $y = x^2$ are functions. The notation f$(x)$ is read as 'f of $x$'. f is the function
f$(x) = 3x$ means the function of $x$ is $3x$

**Examples**

f$(x) = \frac{10}{x}$ Work out f(5)

Substitute $x = 5$ into $\frac{10}{x} \longrightarrow \frac{10}{5} = \mathbf{2}$

g$(x) = 2x^3$ Work out g(-5)

Substitute $x = -5$ into $2x^3 \longrightarrow 2(-5)^3 = \mathbf{-250}$

f$(x) = x + x^3$ g$(x) = 3x^2$ Work out f(4) - g(2)
$(4 + 4^3) - 3(2)^2 = 68 - 12 = \mathbf{56}$

f$(x) = 5x - 3$ Work out the value of $a$ when f$(a) = 12$
$5a - 3 = 12$
$5a = 15$
$\mathbf{a = 3}$

## Composite functions

fg is a composite function. To work out fg$(x)$, first work out g$(x)$ and then substitute your answer into f$(x)$

**Examples**

f$(x) = 6 - 2x$, g$(x) = x^2 + 7$ Work out gf(7)

f$(x) = 6 - (2 \times 7) = -8$

g$(-8) = (-8)^2 + 7 = 64 + 7 = \mathbf{71}$

f$(x) = 4x - 3$, g$(x) = 10 - x$ Work out fg$(x)$

g$(x) = 10 - x$

f$(10 - x) = 4(10 - x) - 3 = 40 - 4x - 3 = \mathbf{37 - 4x}$

## Inverse functions

An inverse function **reverses** the effect of the original function, this is usually written $f^{-1}(x)$

**Examples**

f$(x) = 3 - 4x$ Find the inverse of f$(x)$

Form an equation by making $y = f(x) \longrightarrow y = 3 - 4x$

Make $x$ the subject
$4x = 3 - y \longrightarrow x = \frac{3 - y}{4}$

Finally, re-write the expression that is equal to $x$, replacing the $y$ with an $x$

So the inverse of f$(x) = 3 - 4x$ is $\mathbf{\frac{3 - x}{4}}$

f$(x) = 16 - 3(x + 2)$ Find the inverse of f$(x)$

As before, form an equation by making $y = f(x) \longrightarrow y = 16 - 3(x + 2)$

Make $x$ the subject

$y = 16 - 3x - 6 \longrightarrow 3x = 10 - y \longrightarrow x = \frac{10 - y}{3}$

Finally, re-write the expression that is equal to $x$, replacing the $y$ with an $x$

So the inverse of f$(x) = 16 - 3(x + 2)$ is $\mathbf{\frac{10 - x}{3}}$

# Probability

Probability = $\dfrac{\text{number of successful outcomes}}{\text{total number of possible outcomes}}$

**Examples**

What is the probability of rolling a **6** with a 6-sided die?

Probability = $\mathbf{\dfrac{1}{6}}$

What is the probability of rolling an **even number** with a 6-sided die? ie if you roll a 2, 4 or 6

Probability = $\dfrac{3}{6} = \mathbf{\dfrac{1}{2}}$

## Independent events

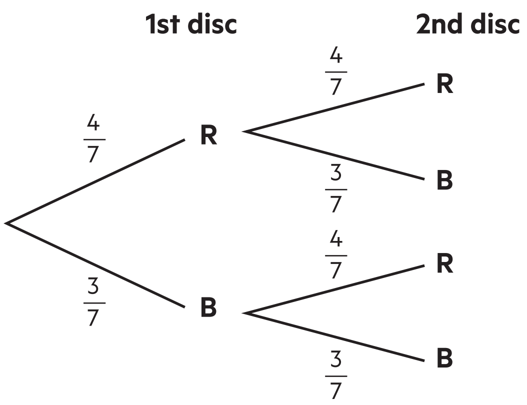
Events are **independent** when the outcome of one does not affect the outcome of the other. To find the combined probability of two independent events, multiply the probablilities

P(A and B) = P(A) x P(B)

A **tree diagram** shows the possible outcomes of two or more combined events

**Example**

A bag contains 4 red (R) discs and 3 blue (B) discs. One disc is taken at random, its colour is noted and then it is replaced. A second disc is then taken




Probability of removing **two red discs** = $\dfrac{4}{7} \times \dfrac{4}{7} = \mathbf{\dfrac{16}{49}}$

Probability of removing **two blue discs** = $\dfrac{3}{7} \times \dfrac{3}{7} = \mathbf{\dfrac{9}{49}}$

Probability of removing **one blue and one red disc**

$$\left(\frac{3}{7} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{4}{7}\right) = \mathbf{\frac{24}{49}}$$

A sample space diagram shows all possible outcomes of two independent events

**Example**

Two six-sided dice are rolled and the results are added together

| | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| **1** | 2 | 3 | 4 | 5 | 6 | 7 |
| **2** | 3 | 4 | 5 | 6 | 7 | 8 |
| **3** | 4 | 5 | 6 | 7 | 8 | 9 |
| **4** | 5 | 6 | 7 | 8 | 9 | 10 |
| **5** | 6 | 7 | 8 | 9 | 10 | 11 |
| **6** | 7 | 8 | 9 | 10 | 11 | 12 |

Total number of possible outcomes = **36**

Probability of rolling an **even** number = $\dfrac{18}{36} = \mathbf{\dfrac{1}{2}}$

Probability of rolling a **7** = $\dfrac{6}{36} = \mathbf{\dfrac{1}{6}}$

Probability of rolling **higher than a 7** = $\dfrac{15}{36} = \mathbf{\dfrac{5}{12}}$

Probability of rolling a **double 1 =** $\mathbf{\dfrac{1}{36}}$

## Mutually exclusive events

Two events are **mutually exclusive** if they cannot happen at the same time. eg, when you roll an ordinary die you cannot get a 3 and an even number at the same time.

When events are mutually exclusive you can add their probabilities

For mutually exclusive events P(A or B) = P(A) + P(B)

# Probability continued

**Examples**
A standard pack of 52 playing cards is shuffled and a card is chosen at random
Find the probability of choosing a diamond or a spade

P(diamond) = $\frac{13}{52}$ P(spade) = $\frac{13}{52}$ $\longrightarrow$ P(diamond or spade) = $\frac{13}{52} + \frac{13}{52} = \frac{26}{52} = \mathbf{\frac{1}{2}}$

Find the probability of choosing a black ace or a heart

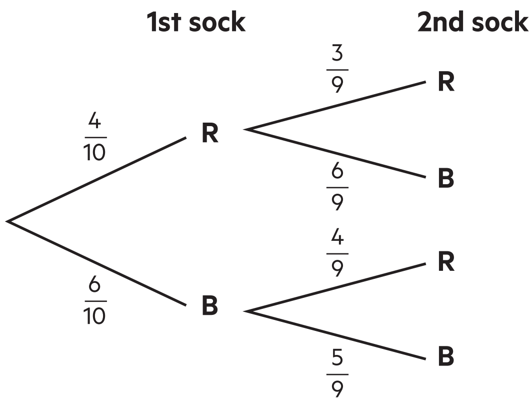
P(black ace) = $\frac{2}{52}$ P(heart) = $\frac{13}{52}$ $\longrightarrow$ P(black ace or heart) = $\frac{2}{52} + \frac{13}{52} = \mathbf{\frac{15}{52}}$

## Conditional probability

When events are **dependent** the outcome of the first event influences the outcome of the second event.
**Conditional probabilities** are probabilities associated with **dependent** events

In a tree diagram showing conditional probabilities, the probabilities on the branches for the second outcome depend on what happened on the first outcome

**Examples**
Matt has 4 red (R) and 6 black (B) socks in a drawer. He takes out two socks at random.




Probability of removing **two red socks** = $\frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \mathbf{\frac{2}{15}}$

Probability of a **matching pair of socks...**

Probability of removing **two red socks** = $\frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \mathbf{\frac{2}{15}}$

**PLUS**

Probability of removing **two black socks** = $\frac{6}{10} \times \frac{5}{9} = \frac{30}{90} = \mathbf{\frac{1}{3}}$

$\frac{1}{3} + \frac{2}{15} = \frac{5}{15} + \frac{2}{15} = \mathbf{\frac{7}{15}}$

## Exam style question

There are $\boldsymbol{n}$ counters in a bag. 8 of the counters are red and the rest are blue
Adam takes a counter from the bag at random and does not replace it
He then takes another counter at random from the bag

The probability that Adam takes two blue counters is $\frac{1}{5}$
a) Show that $n^2 - 21n + 90 = 0$
b) Find the value of $n$

**Step 1**
Begin by defining the number of blue counters $\longrightarrow (n - 8)$

**Step 2**
Set up an equation based on what you know

$$\frac{n-8}{n} \times \frac{n-8-1}{n-1} = \frac{1}{5} \longrightarrow \frac{(n-8)(n-9)}{n(n-1)} = \frac{1}{5} \longrightarrow \frac{n^2 - 17n + 72}{n^2 - n} - \frac{1}{5} = 0$$

$$\longrightarrow 5n^2 - 85n + 360 - n^2 + n = 0 \longrightarrow 4n^2 - 84n + 360 = 0 \longrightarrow \mathbf{n^2 - 21n + 90 = 0} \longrightarrow (n-15)(n-6)$$

$$n = 15 \text{ or } 6$$

$n$ must be greater than $8$ therefore $\boldsymbol{n = 15}$

# Probability continued

**Exam style question**
David has designed a game. He uses a fair 6-sided die and a fair 5-sided spinner
The dice is numbered **1 to 6**, the spinner is numbered **1 to 5**
Each player rolls the die **once** and spins the spinner **once**. A player can win either **£5** or **£2**

| ***Win £5*** | ***Win £2*** |
|---|---|
| *roll a 5* ***and*** *spin a 5* | *roll a 1* ***or*** *spin a 1* ***or*** *both* |

David expects **30 people** will play his game
Each person will pay David **£1 to play the game**

a) Work out how much **profit** David can expect to make

b) Give a reason why David's actual profit may be different to the profit he expects to make

**Step 1**
Find the probabilty of someone winning £5

$$\frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$$

**Step 2**
Find the probabilty of someone winning £2
Probability of someone rolling a $1$ **and not** spinning a $1$

$$\frac{1}{6} \times \frac{4}{5} = \frac{4}{30}$$

Probability of someone spinning a $1$ **and not** rolling a $1$

$$\frac{1}{5} \times \frac{5}{6} = \frac{5}{30}$$

Probability of someone spinning a $1$ **and** rolling a $1$

$$\frac{1}{5} \times \frac{1}{6} = \frac{1}{30}$$

Probability of any of these 3 events happening

$$\frac{4}{30} + \frac{5}{30} + \frac{1}{30} = \frac{10}{30} = \frac{1}{3}$$

**Step 3**
David receives £30 from the players

Number of players he would expect to win £5

$$\frac{1}{30} \times 30 = 1$$

Number of players he would expect to win £2

$$\frac{1}{3} \times 30 = 10$$

Amount of money he would have to pay out

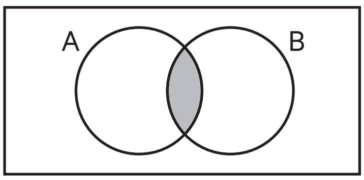
$$(1 \times 5) + (10 \times 2) = 5 + 20 = \boldsymbol{£25}$$

Therefore he would expect to make **£5 profit** (30 - 25)

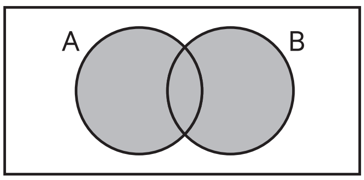
A reason why David's actual profit may be different to the profit he expects to make is that his calculations are **theoretical.** In reality they could be different. Also more or less than 30 people could play the game.

# Venn diagrams

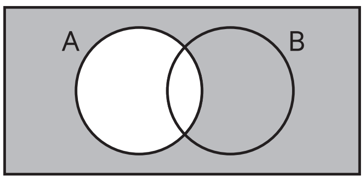
A ∩ B, means 'A **intersection** B'
All the elements in A **and in** B




A ∪ B, means 'A **union** B'
All the elements in A **or** B or **both**



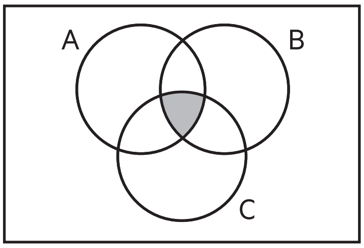

A' means all the elements **not** in A



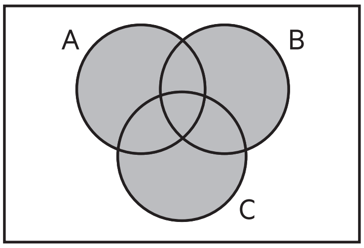

$\xi$ means the universal set – all elements being considered

# Venn diagrams continued

A ∩ B ∩ C, means the **intersection** of A, B and C



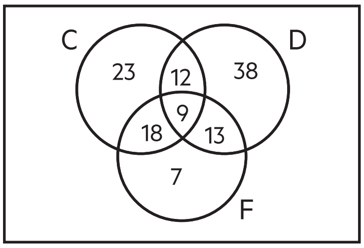

A ∪ B ∪ C, means the **union** of A, B and C




P(A ∩ B | B) means the probability of A and B **given** B

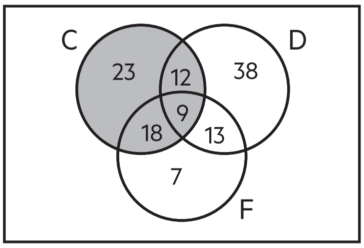
**Example**

Caitlin did a survey of pet owners owning cats (C), dogs (D) and fish (F)



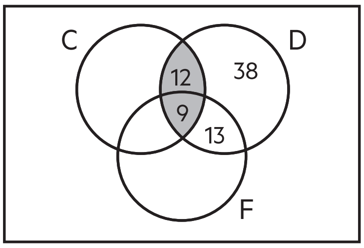

How many took part in the survey?

23 + 12 + 38 + 18 +9 + 13 + 7 = **120**




Work out P(C)

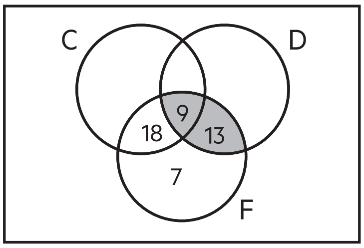
$$P(C) = \frac{23 + 12 + 9 + 18}{120} = \frac{62}{120} = \mathbf{\frac{31}{60}}$$




Work out P(C ∩ D | D). This means find the intersection of C and D, given D

This essentially means ignore everything outside of D

$$P(C \cap D \mid D) = \frac{12 + 9}{38 + 12 + 9 + 13} = \frac{21}{72} = \mathbf{\frac{7}{24}}$$




Work out P(D ∩ F | F). This means find the intersection of D and F, given F

As before ignore everything outside of F

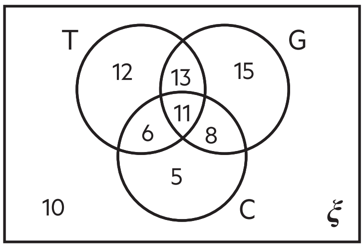
$$P(D \cap F \mid F) = \frac{13 + 9}{9 + 13 + 18 + 7} = \mathbf{\frac{22}{47}}$$

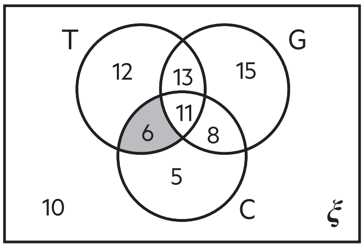
# Venn diagrams continued

**Further example**

This Venn diagram shows the sports played by 80 students

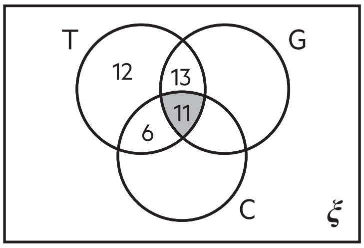
The three most popular sports were tennis (T), golf (G) and cricket (C). A student is picked at random



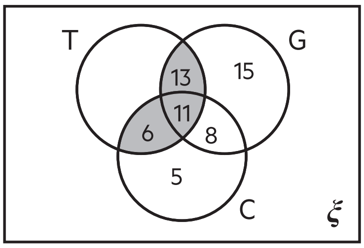

Work out P(T ∩ C ∩ G'). This means find the intersection of T and C and everything outside of G

$$P(T \cap C \cap G') = \frac{6}{80} = \mathbf{\frac{3}{40}}$$




Work out P(T ∩ G ∩ C | T). This means find the intersection of T, G and C given T. As before ignore everything outside of T

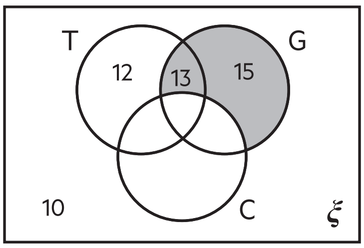
$$P(T \cap G \cap C \mid T) = \frac{11}{12 + 13 + 11 + 6} = \mathbf{\frac{11}{42}}$$




Work out P(T | G ∪ C). This means given the union of G and C, find T

As before ignore everything outside of G ∪ C

$$P(T \mid G \cup C) = \frac{11 + 13 + 6}{8 + 5 + 13 + 11 + 6 + 15} = \frac{30}{58} = \mathbf{\frac{15}{29}}$$

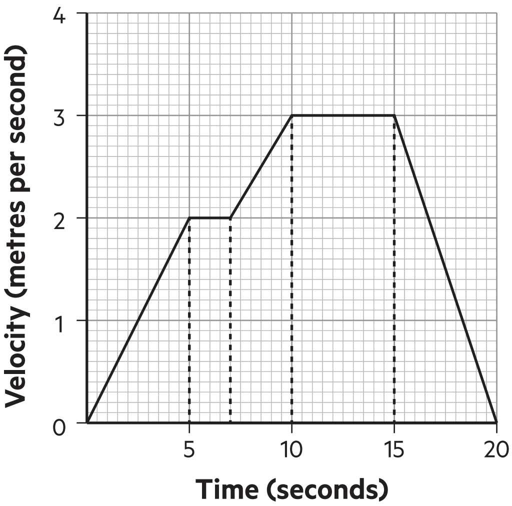



Work out P(G | C'). This means find G given everything outside of C

$$P(G \mid C') = \frac{13 + 15}{12 + 13 + 15 + 10} = \frac{28}{50} = \mathbf{\frac{14}{25}}$$

# Distance, speed, time graphs

On a velocity-time graph, the gradient is the rate of **change of velocity over time**, or the **acceleration** $a = \frac{v}{t}$
The **area** under a velocity time graph represents the **distance travelled**




Acceleration over the first 5 seconds

$$a = \frac{v}{t} \longrightarrow a = \frac{2}{5} \longrightarrow a = 0.4m/s^2$$

Total distance travelled is the area under the graph

$$\frac{1}{2}(5\times2) + (2\times2) + \frac{1}{2}((2 + 3) \times 3) + (5\times3) + \frac{1}{2}(5\times3)$$

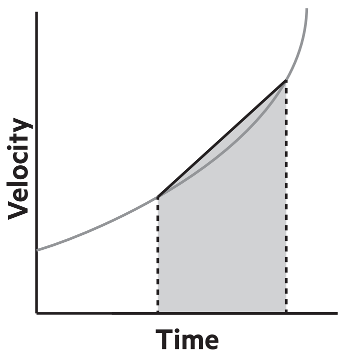
$$\downarrow$$

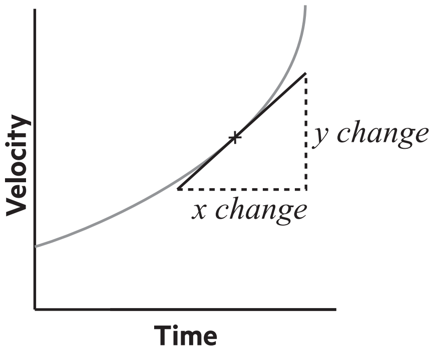
$$10 + 4 + 7.5 + 15 + 7.5 = \mathbf{44m}$$

Average speed is the total distance travelled divided by the total time taken

$$s = \frac{d}{t} \longrightarrow s = \frac{44}{20} \longrightarrow s = \mathbf{2.2m/s}$$

If the velocity-time graph is **not linear** you can still estimate the distance travelled by drawing a **chord** between two points on the graph and drawing straight lines down to the x-axis to create a **trapezium**. The area of the trapezium is an estimate for the area under this part of the graph

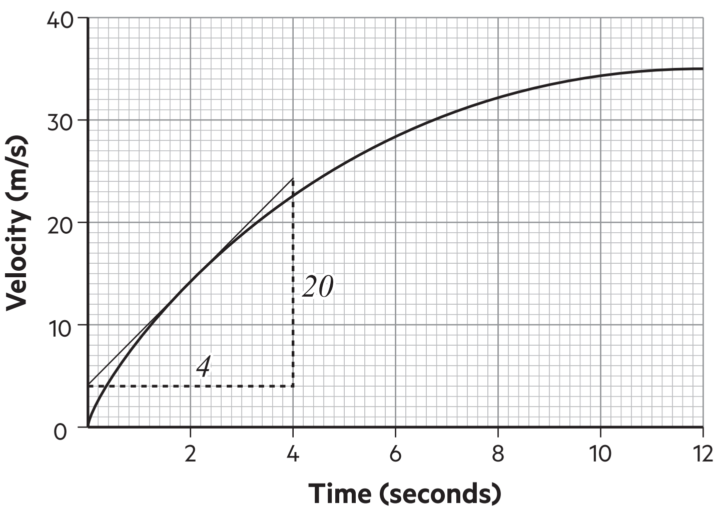



To find the **acceleration** at any given point, draw a **tangent** to the graph and the **gradient of the tangent** will be the **acceleration at that point**

$$a = \frac{y\ change}{x\ change}$$

**Example**

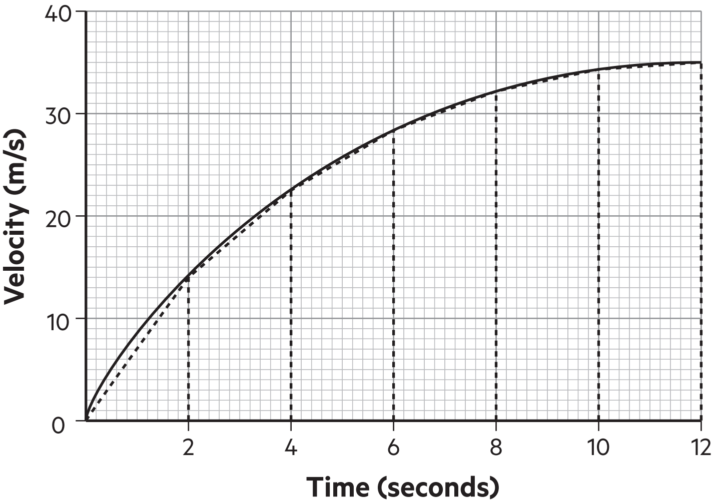



Estimate the acceleration at t = 2 seconds
Draw a tangent to the graph at t = 2 and calculate the gradient

$$a = \frac{20}{4} = 5m/s^2$$

# Distance, speed, time graphs continued

**Example**




Estimate the distance travelled over the 12 seconds
Divide the graph into sections, drawing a chord to the graph at each interval to create trapezia (first one here is a triangle) then sum up the areas to find the total

$$\frac{1}{2}(14\times2) + \frac{1}{2}((14 + 22.5) \times 2) + \frac{1}{2}((22.5 + 28.5) \times 2) + \frac{1}{2}((28.5 + 32) \times 2) + \frac{1}{2}((32 + 34.5) \times 2) + \frac{1}{2}((34.5 + 35) \times 2)$$

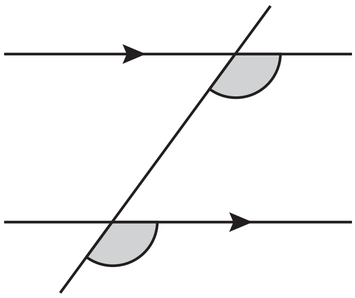
$$= 28 + 36.5 + 51 + 60.5 + 66.5 + 69.5 = \mathbf{312m}$$

# Angle properties

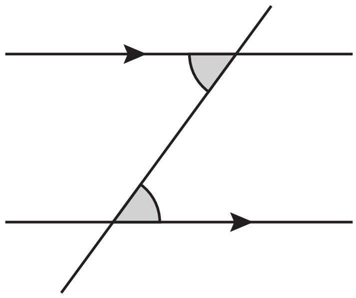
## Parallel lines

There are three properties of angles in parallel lines – **alternate, corresponding** and **interior**

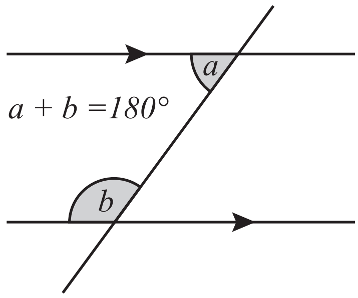
**Corresponding** angles are the same



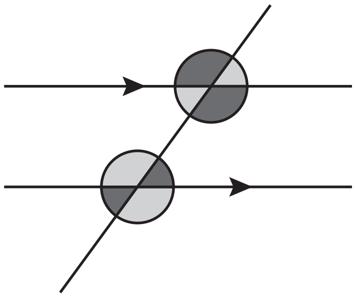
**Alternate** angles are the same



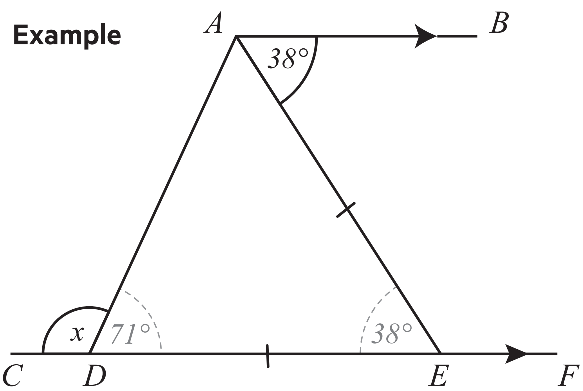
**Interior** angles add up to 180°




**Vertically opposite** angles are the same



**Example**




$CDEF$ is a straight line. $AB$ is parallel to $CF$. $DE = AE$, $B\widehat{A}E = 38°$
Calculate the size of the angle marked $x$
Give reasons for your answer

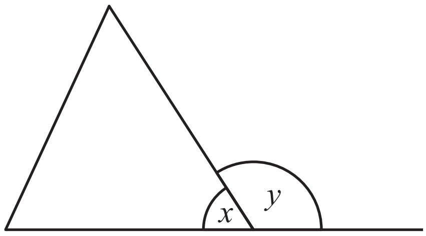
$B\widehat{A}E$ and $A\widehat{E}D$ are the same (alternate angles) = $38°$

$E\widehat{A}D$ and $E\widehat{D}A$ are the same (isoceles triangle and angles in a triangle sum to 180) = $\frac{180 - 38}{2} = \mathbf{71°}$

$x + 71 = 180$ (angles in a straight line add up to 180)

$x = 180 - 71 = \mathbf{109°}$

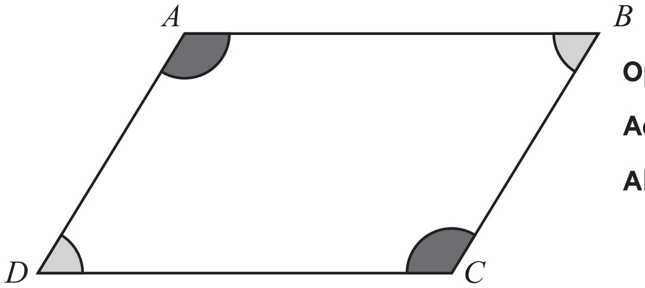
# Angle properties continued




When one side of a triangle is extended at the vertex:
The angle marked $x$ is called the **interior angle**
The angle marked $y$ is called the **exterior angle**

$\boldsymbol{x + y = 180°}$ (angles on a straight line)

### Parallelograms




**Opposite angles** are **equal** – $\widehat{BAD} = \widehat{BCD}$ and $\widehat{ABC} = \widehat{CDA}$

**Adjacent angles** add up to **180°** – $\widehat{ABC} + \widehat{BCD} = 180^o$

**All interior angles** add up to **360°**

### Interior angles of polygons

The sum of the interior angles of a polygon with $\boldsymbol{n}$ sides = $\boldsymbol{(n -2) \times 180}$
eg a pentagon has $\boldsymbol{5}$ sides, therefore sum of interior angles = $\boldsymbol{(5 - 2) \times 180 = 540°}$
A regular polygon with $\boldsymbol{15}$ sides, the sum of interior angles = $\boldsymbol{(15 - 2) \times 180 = 2{,}340°}$

To find the number of sides when you know the sum of the interior angles
eg. the sum of the interior angles of a polygon is 1,620°

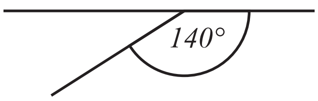
$(n - 2) \times 180 = 1{,}620$

$n - 2 = \frac{1620}{180}$

$n - 2 = 9$

$\boldsymbol{n = 11}$

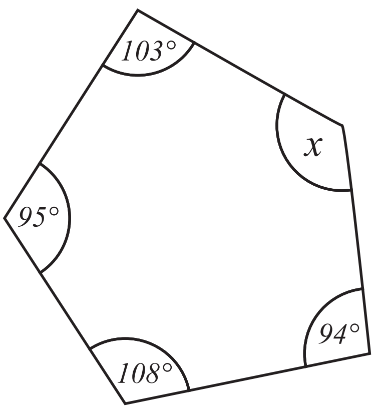
**Example 1**
The interior angle of a regular polygon is $140^o$, how many sides does the polygon have?




**Interior + exterior angle = 180°,** therefore exterior angle = $180 - 140 = \boldsymbol{40^o}$

For a regular polygon the **number of sides** = $\frac{360}{\text{exterior angle}} = \frac{360}{40} = \boldsymbol{9}$

**Example 2**
Work out the size of angle $x$




**Step 1**
The sum of the interior angles of a polygon with $\boldsymbol{n}$ sides = $\boldsymbol{(n -2) \times 180}$
$(5 - 2) \times 180 = \boldsymbol{540}$

**Step 2**
$x = 540 - 103 - 95 - 108 - 94 = \boldsymbol{140°}$

# Ratio and proportion

**Example 1**
Share £126 between Lu and Katie in the ratio **2 : 5**

**Step 1**
Find out how many parts there are in total: 5 + 2 = **7**

**Step 2**
Find out how much each part is worth: £126 ÷ 7 = **£18**

**Step 3**
Find out how much 2 parts and 5 parts is
Lu gets 2 x 18 = **£36**
Katie gets 5 x 18 = **£90**

**Example 2**
Margaret is in Switzerland, the local supermarket sells Reblochon cheese
Each box of Reblochon cheese costs **3.10** Swiss francs. It weighs **160g**
In England, a box of Reblochon cheese cost **£13.55 per kg**
The exchange rate is **£1 = 1.65** francs
Work out whether Reblochon cheese is better value for money in Switzerland or England

**Step 1**
Find out how much the cheese costs in Switzerland **per kg**

$\frac{1000}{160} \times 3.10 =$ **19.375 Swiss francs**

**Step 2**
Convert the cost of the cheese in Switzerland from Swiss francs into pounds

$\frac{19.375}{1.65} =$ **£11.74**

**Step 3**
Compare the two prices
Switzerland = £11.74/kg
England = £13.55/kg

Therefore the cheese is better value in **Switzerland by £1.81/kg**

**Example 3**
There are only blue pens, green pens and red pens in a box
The ratio of blue pens to green pens is 2 : 5
The ratio of green pens to red pens is 4 : 1
There are less than 100 pens in the box
What is the greatest number of red pens in the box?

**Step 1**
Green is the common colour in the two ratios, so make them comparable by multiplying the two ratios to make them the same
2 : 5 x 4 = **8 : 20**
4 : 1 x 5 = **20 : 5**
So the ratio of blue : green : red pens is **8 : 20 : 5**

**Step 2**
Sum the ratios to find the number of pens: 8 + 20 + 5 = **33**

**Step 3**
We know there are less than 100 pens in box so $\frac{99}{33} =$ **3**

**Step 4**
Multiply the ratio of red pens by 3: 5 x 3 = **15 red pens in the box**

# Percentages

**Simple interest** is the interest calculated only on the original amount. It is the **same** each year

**Example**
John invests £14,500 for 3 years at simple interest of 6.75%. What is the value of the investment after 3 years?

**Step 1**
Find the amount of interest for **1 year**

$14500 \times \frac{6.75}{100} = \mathbf{£978.75}$

**Step 2**
As simple interest is the same each year, multiply the interest for one year by 3

$978.75 \times 3 = \mathbf{£2936.25}$

**Step 3**
Add the interest to the original investment

$14500 + 2936.25 = \mathbf{£17,436.25}$

**Compound interest** is the interest calculated on the original amount **plus** the interest added each year

**Example**
Calculate the interest on borrowing £40 for 3 years if the compound interest rate is 5% per year
Year 1: £40 x 1.05 = £42
Year 2: £42 x 1.05 = £44.10
Year 3: £44.10 x 1.05 = £46.31

This is the same as £40 x 1.05 x 1.05 x 1.05
This can also be written as £40 x $1.05^3$

A generic formula for compound interest can therefore be written as
Investment value after $n$ years = Original investment x multiplier$^n$

## Percentage change

You can calculate a percentage change using the formula

$$\text{Percentage change} = \frac{\text{new amount - original amount}}{\text{original amount}} \times 100$$

**Example**
Charlie invests £3,200, when his investment matures, he receives £3,328
What is the percentage increase in his investment?

Using the formula above

$$\frac{3328 - 3200}{3200} \times 100 = \mathbf{4\%}$$

## Profit and loss

**Profit** is when you buy something and then sell it for **more** than you originally paid for it. The amount of profit is the sale price **minus** the original cost

**Loss** is when you buy something and then sell it for **less** than you originally paid for it. The size of the loss is the sale price **minus** the original cost

**Example**
Lucy spent **£11.40** buying ingredients to make cupcakes. She sold all the cakes for a total of **£39.90**
What percentage profit did she make?

**Step 1**
Calculate the amount of **profit** Lucy made. Remember **profit** is the sale price **minus** the original cost
£39.90 - £11.40 = **£28.50**

**Step 2**
Use the percentage change formula to calculate the percentage profit

$$\text{Percentage profit} = \frac{28.50}{11.40} \times 100 = \mathbf{250\%\ profit}$$

# Percentages continued

**Inverse operations** can be used to find the original amount after a percentage increase or decrease

**Example**
In one year, the value of a car dropped by 12% to £9,240
How much was the car worth at the start of the year?

It is very important to remember that a **12% drop** means the car is worth **88% of its original value**
Therefore
Original value x 0.88 = new value
Original value = $\frac{9240}{0.88}$ = **£10,500**

# Fractions, decimals and percentages

When making calculations that are a mix of fractions, decimals and percentages, put them all into a single type

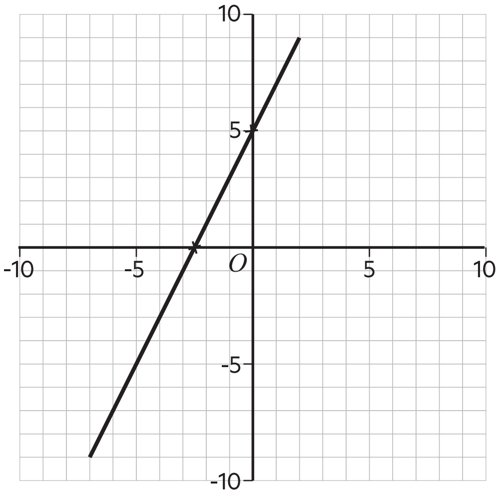
**Example**
Mr Mason asks 240 year 11 students what they want to do next year
15% of the students want to go to college
$\frac{3}{4}$ of the students want to stay at school
The rest of the students do not know. Work out the number of students who do not know

**Step 1**
Convert the percentage and fraction to decimals
15% = 0.15
$\frac{3}{4}$ = 0.75

**Step 2**
Add the two decimals together
0.15 + 0.75 = 0.9

**Step 3**
Rest of students = 1 - 0.9 = 0.1 = 10%
So 10% of students do not know
240 x 0.1 = **24** students do not know what they want to do next year

# Linear graphs



Plotting a linear (straight line) graph

**Example**

Draw the graph of $y = 2x + 5$

**Step 1**

Find the $y$ intercept, by making $x = 0$

$y = (2 \times 0) + 5$

$y = \mathbf{5}$

$x = \mathbf{0}$

**Step 2**

Find the $x$ intercept, by making $y = 0$

$0 = 2x + 5$

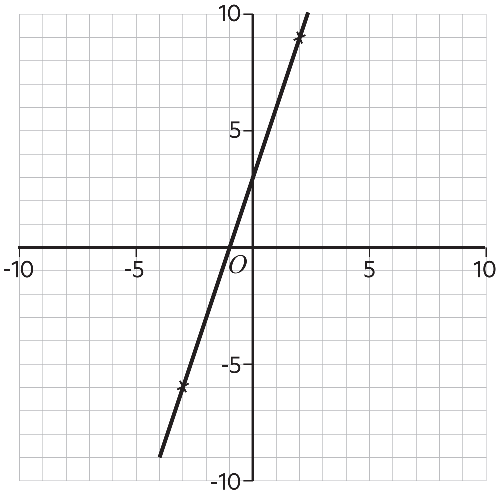
$2x = -5$

$x = \mathbf{-2.5}$

$y = \mathbf{0}$

**Step 3**

Join the two points together



You don't just have to use the $x$ and $y$ intercepts you can substitute any values for $x$ or $y$ to find two points. Putting in two different values for $x$ however, is easier

**Example**

Draw the graph of $y = 3x + 3$

**Step 1**

Let $x = 2$

$y = (3 \times 2) + 3$

$y = \mathbf{9}$

$x = \mathbf{2}$

**Step 2**

Let $x = -3$

$y = (3 \times -3) + 3$

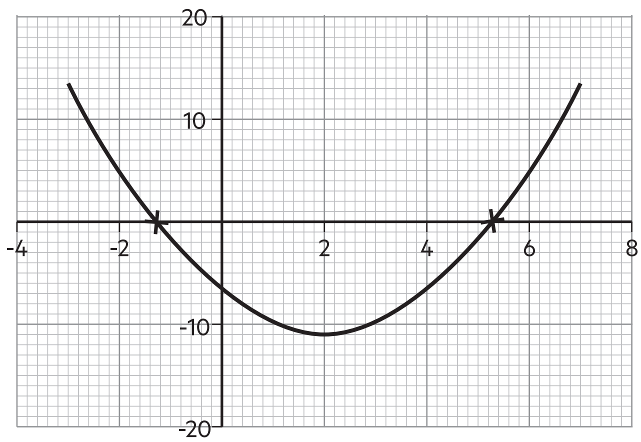
$y = \mathbf{-6}$

$x = \mathbf{-3}$

**Step 3**

Join the two points together

# Solving quadratic equations graphically

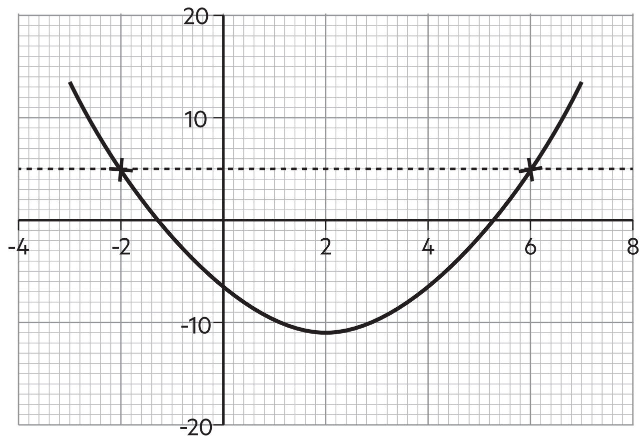


This is a graph of $y = x^2 - 4x - 7$

Estimate the solutions to the equation $x^2 - 4x - 7 = 0$

This basically means where does the graph $x^2 - 4x - 7$ cross the $x$-axis. Because when $y = 0$, the line crosses the $x$-axis

So an estimate the solutions to the equation $x^2 - 4x - 7 = 0$ are $\mathbf{x = 5.3}$ and $\mathbf{x = -1.3}$



Using the same graph $y = x^2 - 4x - 7$

Estimate the solutions to the equation $x^2 - 4x - 7 = 5$

This basically means where does the graph $x^2 - 4x - 7$ cross the line $y = 5$

So an estimate the solutions to the equation $x^2 - 4x - 7 = 5$ are $\mathbf{x = 6}$ and $\mathbf{x = -2}$

# Quadratic equations

A **quadratic expression** contains a term in $x^2$ but no higher power of $x$
It is usually written in the form
$ax^2 + bx + c$
where $a, b$ and $c$ are constants and $a \neq 0$

The **quadratic formula** can be used to find the solution to a quadratic equation $ax^2 + bx + c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

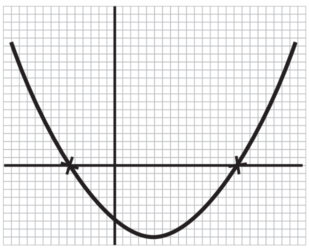
**Example**

$3x^2 - 7x - 1 = 0$

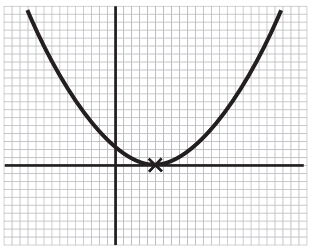
$$x = \frac{7 \pm \sqrt{-7^2 - 4(3 \times -1)}}{2 \times 3} \longrightarrow \frac{7 \pm \sqrt{49 + 12}}{6} \longrightarrow \frac{7 \pm \sqrt{61}}{6} \longrightarrow \frac{14.81}{6} \text{ or } \frac{-0.81}{6} \longrightarrow \begin{array}{l} x = \mathbf{-0.14} \\ x = \mathbf{2.47} \end{array}$$

In the quadratic formula, the expression under the square root sign $(b^2 - 4ac)$ is called the **discriminant**. It tells you whether the equation has 0, 1 or 2 real roots

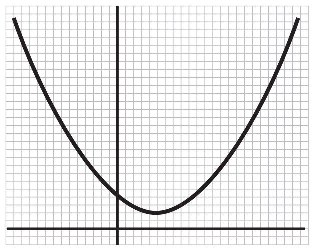
- When $b^2 - 4ac > 0$ there are two distinct solutions to the quadratic equation (two real roots)
- When $b^2 - 4ac = 0$ there is one solution to the quadratic equation (one real roots)
- When $b^2 - 4ac < 0$ there are no real solutions to the quadratic equation (no real roots)



**Two real roots**



**One real root**



**No real roots**

**Completing the square** is a technique used to identify the line of symmetry and the turning point of a quadratic graph

$ax^2 + bx + c$ can be written in the form $a\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

**Example**

A quadratic graph has the formula $y = x^2 - x - 4$ find the turning point by completing the square

Using the formula above $\left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 4 \longrightarrow \left(x - 0.5\right)^2 - 0.25 - 4 \longrightarrow \left(x - 0.5\right)^2 - 4.25$

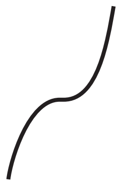
$x = 0.5, y = -4.25$

The $x$ value is also the **line of symmetry** of the graph

# Graphs of cubic equations

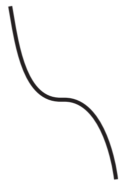
A **cubic function** is one whose highest power of $x$ is $x^3$
It is written in the form $y = ax^3 + bx^2 + cx + d$

When $a > 0$ the function looks like

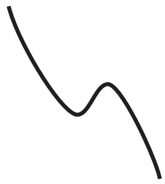


or

When $a < 0$ the function looks like

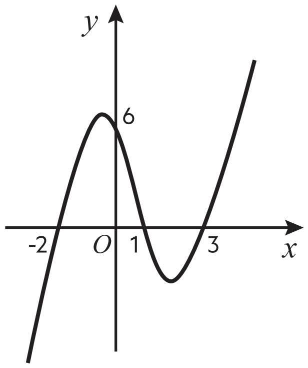


or



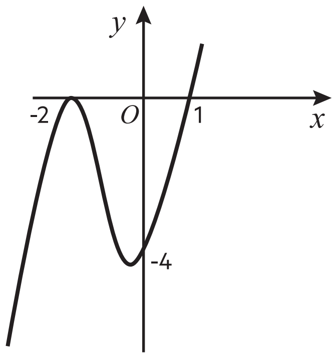
When the graph of a cubic function $y$ crosses the $x$ axis three times, the equation $y = 0$ has **three** solutions

For example $y = (x + 2)(x - 1)(x - 3)$
$y$ intercept, when $x = 0$ is $(0 + 2) \times (0 - 1) \times (0 - 3) = 2 \times (-1) \times (-3) = \mathbf{6}$



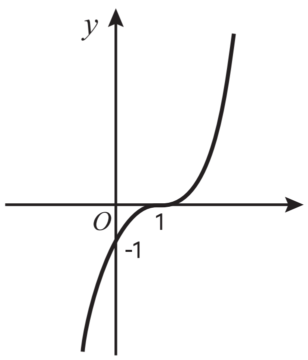

When the graph of a cubic function $y$ crosses the $x$ axis once and touches $x$ axis once, the equation $y = 0$ has **three** solutions but one of them is repeated

For example $y = (x - 1)(x + 2)^2$
$y$ intercept, when $x = 0$ is $(0 - 1) \times (0 + 2)^2 = -1 \times 4 = \mathbf{-4}$

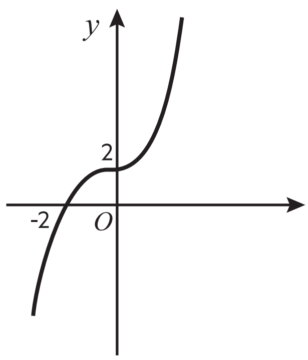



When the graph of a cubic function $y$ crosses the $x$ axis once, the equation $y = 0$ can have

One distinct repeated solution,
for example $y = (x - 1)^3$
$y$ intercept, when $x = 0$ is $(0 - 1)^3 = -1^3 = \mathbf{-1}$



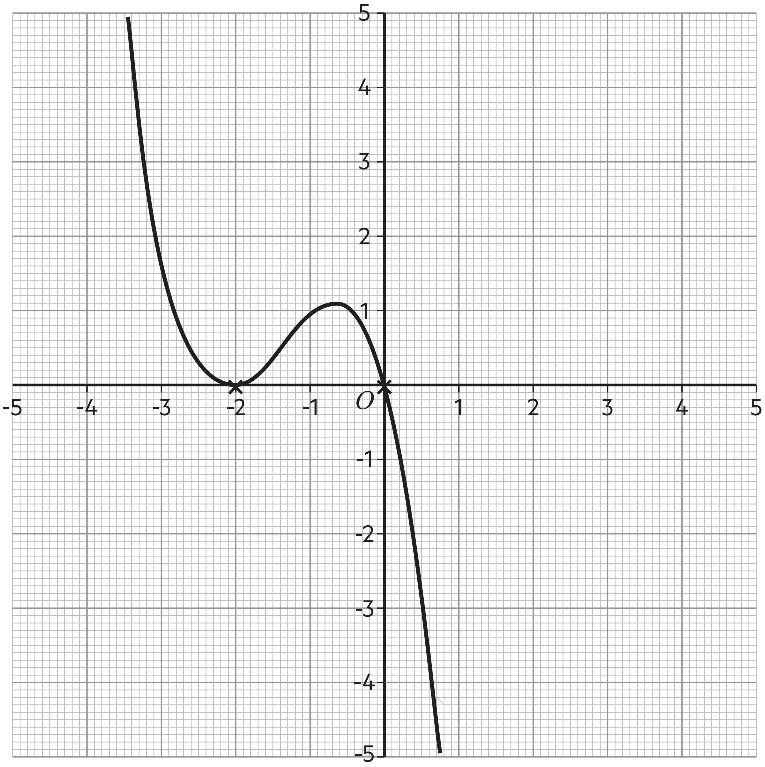

or only one real solution,
for example $y = (x + 2)(x^2 + x + 1)$
The quadratic $(x + 2)(x^2 + x + 1)$ has no real solutions
$y$ intercept, when $x = 0$ is $(0 + 2) \times (0 + 0 + 1) = 2 \times 1 = \mathbf{2}$



# Graphs of cubic equations continued

**Example**

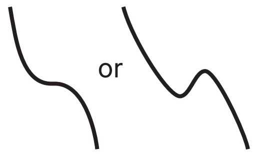
Sketch the graph of $y = -x(x + 2)^2$ marking clearly the points of intersection with the axes




**Step 1**

Work out what shape your graph is going to be
Remember the formula $y = ax^3 + bx^2 + cx + d$

When $a < 0$ the function looks like




**Step 2**

Work out where the graph crosses the axes
The graph has **three real solutions** but one of them is repeated $x = 0$ and $x = -2$ (twice)
$y$ intercept = $-0(0 + 2)^2 = 0 \times 2^2 = \mathbf{0}$

**Step 3**

Mark these points on your graph and draw the line to fit these points

**Showing that a cubic graph has a solution between two values**

**Example**

Show that the equation $x^3 + x = 7$ has a solution between $1$ and $2$

$x^3 + x - 7 = 0$

Let $x = 2$

$2^3 + 2 - 7 = 8 + 2 - 7 = \mathbf{3}$

Let $x = 1$

$1^3 + 1 - 7 = \mathbf{-5}$

There is a change of sign in y, therefore the equation has a solution between $2$ and $1$

**Using iterative formula to find a root of a cubic graph**

Using the equation $x^3 + x = 7$ from the above example, find the root to 4 decimal places

Set up your iterative formula

$x^3 = 7 - x$

$x = \sqrt[3]{7 - x}$

$x = \sqrt[3]{7 - x_n}$

You know the root is between $1$ and $2$ so start the iterative process with $2$

$x_0 = 2$

$x_1 = \sqrt[3]{7 - 2} = 1.70998$

$x_2 = \sqrt[3]{7 - 1.70998} = 1.7424$

$x_3 = \sqrt[3]{7 - 1.74242} = 1.7389$

$x_4 = \sqrt[3]{7 - 1.73885} = 1.7392$

$x_5 = \sqrt[3]{7 - 1.73924} = \mathbf{1.7392}$

Continue with the iterative process until the 4th decimal place stays the same when rounded to 4 decimal places

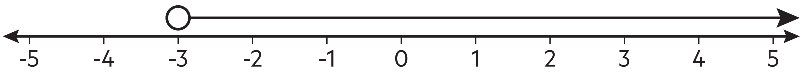
# Linear inequalities

Linear inequalities usually illustrate their solutions using a number line.
When using a number line, a small solid circle is used for ≤ or ≥ and a hollow circle is used for > or <

**Example**
Solve the inequality $x + 6 > 3$ and illustrate the solution on a number line.

**Step 1**
You want to get $x$ on its own on one side of the inequality
$x > 3 - 6$ subtract 6 from both sides
$x > -3$

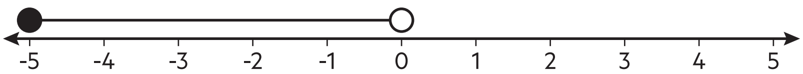
**Step 2**
Illustrate the inequality on a number line, remember as the sign is > you want to use a hollow circle




**Example 2**
Show the inequality $-3 \leq x + 2 < 2$ on a number line.

**Step 1**
You want to get $x$ on its own on in the middle of the inequality
$-3 - 2 \leq (x + 2) - 2 < 2 - 2$ subtract 2 from each part
$-5 \leq x < 0$

**Step 2**
Illustrate the inequality on a number line, remember where the sign is ≤ you want to use a solid circle




# Common values of Sin Cos Tan

| | *Angle* | | | | |
|---|---|---|---|---|---|
| | $0°$ | $30°$ | $45°$ | $60°$ | $90°$ |
| *Sin* | $0$ | $0.5$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ |
| *Cos* | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $0.5$ | $0$ |
| *Tan* | $0$ | $\frac{\sqrt{3}}{3}$ | $1$ | $\sqrt{3}$ | *inf* |