THE SUM OF THE RECIPROCALS OF THE SQUARES
A Proof by Leonhard Euler

As if one needed further evidence for the genius of Leonhard Euler, here is one of his solutions to the summation of a famous series. The sum of the reciprocals of the squares of the natural numbers was a question first posed in 1644 by Pietro Mengoli, and left unsolved until Leonhard Euler 1734 [1]. The original method that Euler used was not what follows, but an expansion of the series of the sine and cosine functions. What makes this particular method appealing is a reliance on multivariate calculus techniques [2]. It was well-known at the time that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ to some finite value; finding that specific value, however, is a far greater challenge. The object of this paper is to find, and prove, the exact value that this series converges to.

$$\lim_{n\to\infty}(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2}) = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Euler accomplished this by showing first that this series is equal to the following integrated region, and then finding the exact value of the definite integral.

$$\int_0^1\int_0^1 \frac{1}{1-xy}dydx$$

This double integral over the region $D = \{x, y|0 \leq x \leq 1, 0 \leq y \leq 1\}$, which is represented graphically below, at first appears to have nothing in common with this series; however, the integrand can be rewritten. In particular, $\frac{1}{1-xy}$ is of the form $\frac{1}{1-p}$ for $|p| < 1$ can be expanded as an infinite series.

$$S_n = 1 + p + p^2 + p^3 + \ldots + p^n$$

$$p * S_n = p + p^2 + p^3 + \ldots + p^{n+1}$$

$$s_n - p * s_n = 1 - p^{n+1} = s_n(1-p) => s_n = \frac{1-p^{n+1}}{1-p}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{1-p^{n+1}}{1-p} = \frac{1}{1-p}$$

$$p = xy \Rightarrow \frac{1}{1-xy} = xy + x^2y^2 + x^3y^3 + \ldots$$

$$\int_0^1 \int_0^1 \frac{1}{1-xy} dydx = \int_0^1 \int_0^1 (xy + x^2y^2 + x^3y^3 + \ldots) dydx = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

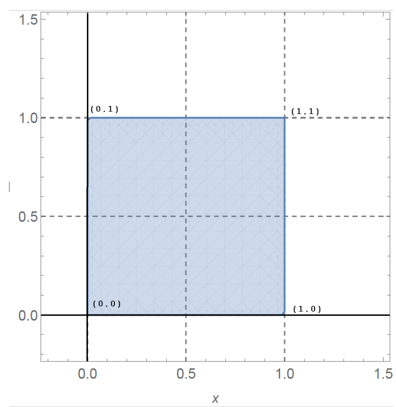
So we know that this integral is equal to the sum of the reciprocals of the squares. Euler performed a transformation of variables here to find the exact value of the double integral.

Let $x = \frac{u+v}{\sqrt{2}}$, and let $y = \frac{u-v}{\sqrt{2}}$.

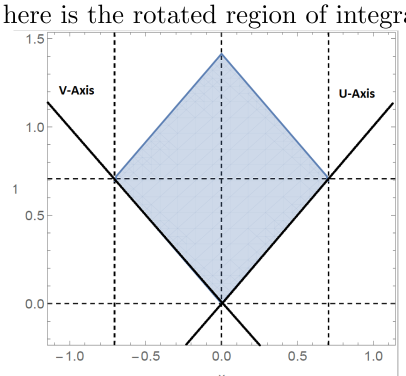
Because the determinant of the Jacobian matrix is 1, or equivalently because rotation is a linear transformation, $dydx = dudv$.

Intuitively this makes sense as the area of the transformed region is $1 * 1 = 1$, as the region D is rotated counterclockwise by 90 degrees, now changing the limits of integration accordingly to the bounds on the U-V axis.

Here is an image of the unit square representing the region D.



Now here is the rotated region of integration.



Here we substitute in (u, v) for (x, y) and evaluate.

$$\int_0^1\int_0^1 \frac{1}{1-xy}dydx = \int_0^{\frac{\sqrt{2}}{2}}\int_{-u}^{u} \frac{2}{(2-u^2)+v^2}dvdu + \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}}\int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{2}{(2-u^2)+v^2}dvdu = \boldsymbol{A} + \boldsymbol{B}$$

1 Evaluating the inside integral of $\boldsymbol{A}$:

Let $\alpha = \sqrt{2-u^2}$,

$$\Rightarrow \int_{-u}^{u} \frac{2}{(2-u^2)+v^2}dv = 2\int_{-u}^{u} \frac{1}{\alpha^2+v^2}dv = \frac{2}{\alpha}\arctan\frac{v}{\alpha}\Big|_{-u}^{u} = \frac{4}{\alpha}\arctan\frac{u}{\alpha}$$

$$= \frac{4}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}} = \int_0^{\frac{\sqrt{2}}{2}} \frac{4}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}}du$$

This can be evaluated with the substitution $u = \sqrt{2}\sin\theta, du = \sqrt{2}\cos\theta d\theta$

$$\Rightarrow \int_a^b \frac{4}{\sqrt{2-\sqrt{2}\sin\theta^2}}\arctan\frac{\sqrt{2}\sin\theta}{\sqrt{2-\sqrt{2}\sin\theta^2}} * \sqrt{2}\cos\theta d\theta = 2*\theta^2\Big|_a^b$$

where $u = \sqrt{2}\sin\theta \Rightarrow \theta = \arcsin\frac{u}{\sqrt{2}}$

This means that the total of $\boldsymbol{A}$ is just

$$2*\arcsin(\frac{u}{\sqrt{2}})^2\Big|_0^{\frac{\sqrt{2}}{2}} = \frac{\pi^2}{18}$$

2 Evaluating the Integral $\boldsymbol{B}$:

$$\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}}\int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{2}{(2-u^2)+v^2}dvdu$$

From our earlier result, it was shown that the inside integral stays the same, and only the limits of integration change.

$$\frac{4}{\sqrt{2-u^2}}\arctan\frac{v}{\sqrt{2-u^2}}\Big|_{u-\sqrt{2}}^{-u+\sqrt{2}} = \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}[\arctan(\frac{u-\sqrt{2}}{\sqrt{2-u^2}}) - \arctan(\frac{-u+\sqrt{2}}{\sqrt{2-u^2}})]du$$

Just as last time let $u = \sqrt{2}\sin\theta, du = \sqrt{2}\cos\theta$ and the integral simplifies to

$$2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \arctan\left(\frac{-\sqrt{2}\sin\theta + \sqrt{2}}{\sqrt{2\cos^2\theta}}\right) - \arctan\left(\frac{\sqrt{2}\sin\theta - \sqrt{2}}{\sqrt{2\cos^2\theta}}\right) d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \arctan\left(\frac{-\sin\theta + 1}{\cos\theta}\right) d\theta - 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \arctan\left(\frac{\sin\theta - 1}{cos\theta}\right) d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \arctan\left(\frac{-\sin\theta + 1}{\cos\theta}\right) d\theta = 4\theta\arctan\left(\frac{-\sin\theta + 1}{\cos\theta}\right) d\theta - 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \theta\frac{1}{1+\Delta^2}\frac{d\Delta}{d\theta} d\theta$$

For simplicity,

$\Delta = \frac{1-sin\theta}{\cos\theta}$

$\frac{d\Delta}{d\theta} = \frac{\sin\theta - 1}{\cos^2\theta}$

$\Delta^2 = \frac{1-2\sin\theta+\sin^2\theta}{\cos^2\theta}$

$1+\Delta^2 = \frac{1-2\sin\theta+\sin^2\theta+\cos^2\theta}{\cos^2\theta} = \frac{2-2\sin\theta}{\cos^2\theta}$

$\frac{1}{1+\Delta^2} = \frac{1}{2}\frac{\cos^2\theta}{1-1\sin\theta}$

By substituting these back in, it can be seen that the integral miraculously simplifies.

$$4\theta\arctan\left(\frac{-\sin\theta + 1}{\cos\theta}\right) d\theta - 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \theta\frac{1}{1+\Delta^2}\frac{d\Delta}{d\theta} d\theta$$

$$\theta\arctan\left(\frac{-\sin\theta + 1}{\cos\theta}\right) d\theta - 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \theta\frac{1}{2}\frac{\cos^2\theta}{(1-\sin\theta)}\frac{(-1)(1-\sin\theta)}{\cos^2\theta} d\theta$$

$$= \theta\arctan\left(\frac{-\sin\theta + 1}{\cos\theta}\right) d\theta - 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \theta d\theta = 4\frac{\pi^2}{36} = \frac{\pi^2}{12} = \boldsymbol{B}$$

Adding $\boldsymbol{B}$ to $\boldsymbol{A}$ and equating it to the original geometric series, we find that the summation of the infinite series of the reciprocals of the squares of all positive integers,

$$4\frac{\pi^2}{36} + \frac{\pi^2}{18} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \blacksquare$$

References

[1] Ayoub, Raymond, *Euler and the zeta function*, Amer. Math. Monthly 81: 106786 1974.

[2] Stewart, James, *Calculus: Early Transcendentals*, Cengage Learning, Boston, 7th edition, 2013.