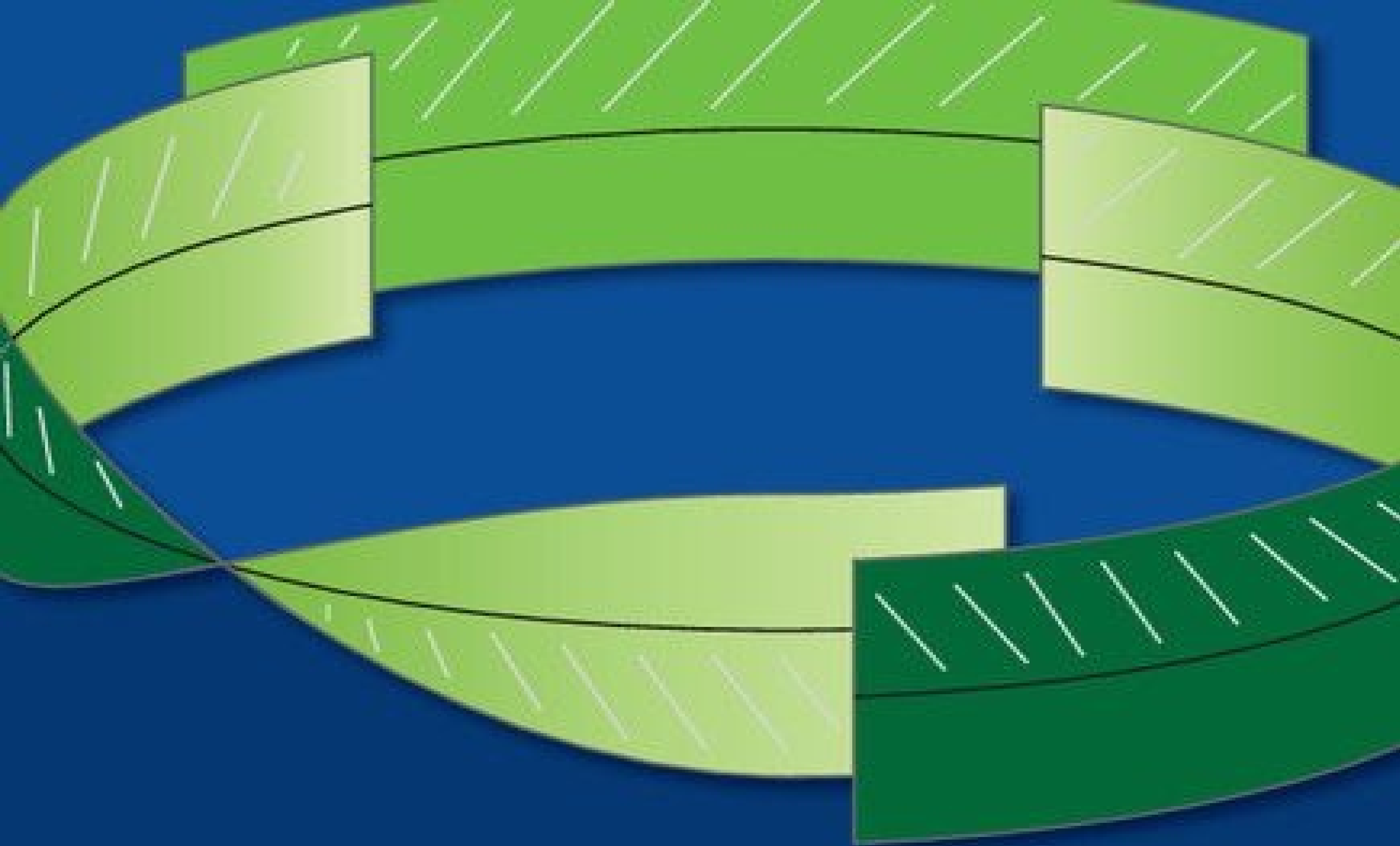
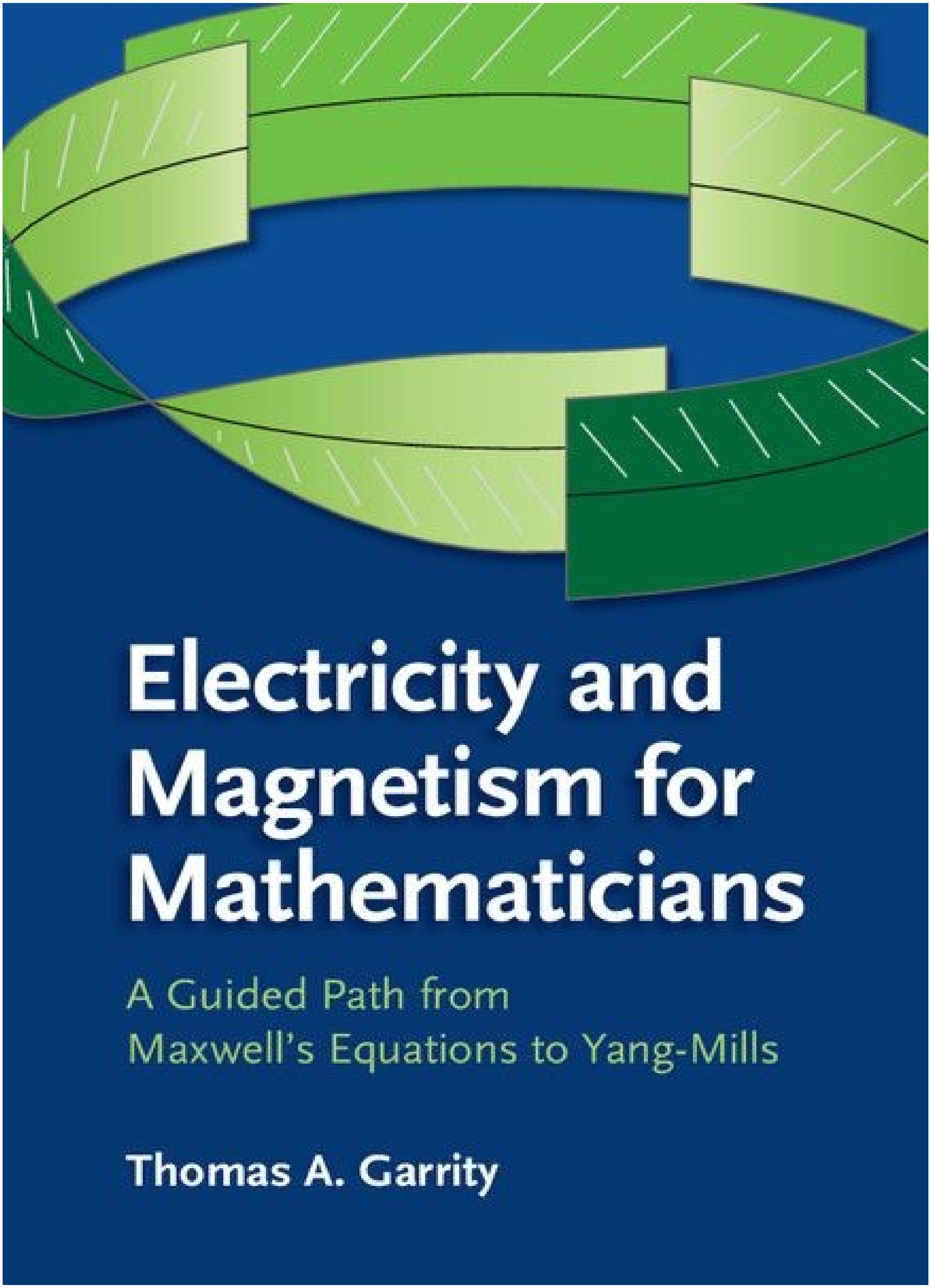


# Electricity and Magnetism for Mathematicians

## A Guided Path from Maxwell's Equations to Yang-Mills

**Thomas A. Garrity**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## Electricity and Magnetism for Mathematicians

This text is an introduction to some of the mathematical wonders of Maxwell's equations. These equations led to the prediction of radio waves, the realization that light is a type of electromagnetic wave, and the discovery of the special theory of relativity. In fact, almost all current descriptions of the fundamental laws of the universe can be viewed as deep generalizations of Maxwell's equations. Even more surprising is that these equations and their generalizations have led to some of the most important mathematical discoveries of the past thirty years. It seems that the mathematics behind Maxwell's equations is endless.

The goal of this book is to explain to mathematicians the underlying physics behind electricity and magnetism and to show their connections to mathematics. Starting with Maxwell's equations, the reader is led to such topics as the special theory of relativity, differential forms, quantum mechanics, manifolds, tangent bundles, connections, and curvature.

THOMAS A. GARRITY is the William R. Kenan, Jr. Professor of Mathematics at Williams, where he was the director of the Williams Project for Effective Teaching for many years. In addition to a number of research papers, he has authored or coauthored two other books, *All the Mathematics You Missed [But Need to Know for Graduate School]* and *Algebraic Geometry: A Problem Solving Approach*. Among his awards and honors is the MAA Deborah and Franklin Tepper Haimo Award for outstanding college or university teaching.

# ELECTRICITY AND MAGNETISM FOR MATHEMATICIANS

## A Guided Path from Maxwell's Equations to Yang-Mills

THOMAS A. GARRITY

*Williams College, Williamstown, Massachusetts*
with illustrations by Nicholas Neumann-Chun

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# Contents

# List of Symbols

| Symbol | Name |
|---|---|
| $\nabla$ | nabla |
| $\triangle$ | Laplacian |
| $^T$ | transpose |
| $\in$ | element of |
| $\mathrm{O}(3,\mathbb{R})$ | orthogonal group |
| $\mathbb{R}$ | real numbers |
| $\rho(\cdot,\cdot)$ | Minkowski metric |
| $\wedge^k(\mathbb{R}^n)$ | $k$-forms on $\mathbb{R}^n$ |
| $\wedge$ | wedge |
| $\circ$ | composed with |
| $\star$ | star operator |
| $\mathcal{H}$ | Hilbert space |
| $\langle\cdot,\cdot\rangle$ | inner product |
| $\mathbb{C}$ | complex numbers |
| $L^2[0,1]$ | square integrable functions |
| $^*$ | adjoint |
| $\subset$ | subset of |
| $\mathcal{S}$ | Schwartz space |
| $h$ | Planck constant |
| $\cap$ | set intersection |
| $\cup$ | set union |
| $GL(k,\mathbb{R})$ | general linear group |
| $C_p^\infty$ | germ of the sheaf of differentiable functions |
| $\Gamma(E)$ | space of all sections of $E$ |
| $\nabla$ | connection |
| $\otimes$ | tensor product |
| $\odot$ | symmetric tensor product |

# Acknowledgments

There are many people who have helped in the preparing of this book. First off, an earlier draft was used as the text for a course at Williams College in the fall of 2009. In this class, Ben Atkinson, Ran Bi, Victoria Borish, Aaron Ford, Sarah Ginsberg, Charlotte Healy, Ana Inoa, Stephanie Jensen, Dan Keneflick, Murat Kologlu, Edgar Kosgey, Jackson Lu, Makisha Maier, Alex Massicotte, Merideth McClatchy, Nicholas Neumann-Chun, Ellen Ramsey, Margaret Robinson, Takuta Sato, Anders Schneider, Meghan Shea, Joshua Solis, Elly Tietsworth, Stephen Webster, and Qiao Zhang provided a lot of feedback. In particular Stephen Webster went through the entire manuscript again over the winter break of 2009–2010. I would like to thank Weng-Him Cheung, who went through the whole manuscript in the fall of 2013. I would also like to thank Julia Cline, Michael Mayer, Cesar Melendez, and Emily Wickstrom, all of whom took a course based on this text at Williams in the fall of 2013, for helpful comments.

Anyone who would like to teach a course based on this text, please let me know (tgarrity@williams.edu). In particular, there are write-ups of the solutions for many of the problems. I have used the text for three classes, so far. The first time the prerequisites were linear algebra and multivariable calculus. For the other classes, the perquisites included real analysis. The next time I teach this course, I will return to only requiring linear algebra and multivariable calculus. As Williams has fairly short semesters (about twelve to thirteen weeks), we covered only the first fifteen chapters, with a brief, rapid-fire overview of the remaining topics.

In the summer of 2010, Nicholas Neumann-Chun proofread the entire manuscript, created its diagrams, and worked a lot of the homework problems. He gave many excellent suggestions.

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I am grateful for all of their help.

# 1
# A Brief History

**Summary:** The unification of electricity, magnetism, and light by James Maxwell in the 1800s was a landmark in human history and has continued even today to influence technology, physics, and mathematics in profound and surprising ways. Its history (of which we give a brief overview in this chapter) has been and continues to be studied by historians of science.

## 1.1 Pre-1820: The Two Subjects of Electricity and Magnetism

Who knows when our ancestors first became aware of electricity and magnetism? I imagine a primitive cave person, wrapped up in mastodon fur, desperately trying to stay warm in the dead of winter, suddenly seeing a spark of static electricity. Maybe at the same time in our prehistory someone felt a small piece of iron jump out of their hand toward a lodestone. Certainly lightning must have been inspiring and frightening, as it still is.

But only recently (meaning in the last four hundred years) have these phenomena been at all understood. Around 1600, William Gilbert wrote his infuential *De Magnete*, in which he argued that the earth was just one big magnet. In the mid-1700s, Benjamin Franklin showed that lightning was indeed electricity. Also in the 1700s Coulomb's law was discovered, which states that the force $F$ between two stationary charges is

$$F = \frac{q_1 q_2}{r^2},$$

where $q_1$ and $q_2$ are the charges and $r$ is the distance between the charges (after choosing correct units). Further, in the 1740s, Leyden jars were invented to store electric charge. Finally, still in the 1700s, Galvani and Volta, independently, discovered how to generate electric charges, with the invention of galvanic, or voltaic, cells (batteries).

## 1.2 1820–1861: The Experimental Glory Days of Electricity and Magnetism

In 1820, possibly during a lecture, Hans Christian Oersted happened to move a compass near a wire that carried a current. He noticed that the compass's needle jumped. People knew that compasses worked via magnetism and at the same time realized that current was flowing electricity. Oersted found solid proof that electricity and magnetism were linked.

For the next forty or so years amazing progress was made finding out how these two forces were related. Most of this work was rooted in experiment. While many scientists threw themselves into this hunt, Faraday stands out as a truly profound experimental scientist. By the end of this era, most of the basic empirical connections between electricity and magnetism had been discovered.

## 1.3 Maxwell and His Four Equations

In the early 1860s, James Clerk Maxwell wrote down his four equations that linked the electric field with the magnetic field. (The real history is quite a bit more complicated.) These equations contain within them the prediction that there are electromagnetic waves, traveling at some speed $c$. Maxwell observed that this speed $c$ was close to the observed speed of light. This led him to make the spectacular conjecture that light is an electromagnetic wave. Suddenly, light, electricity, and magnetism were all part of the same fundamental phenomenon.

Within twenty years, Hertz had experimentally shown that light was indeed an electromagnetic wave. (As seen in Chapter 6 of [27], the actual history is not quite such a clean story.)

## 1.4 Einstein and the Special Theory of Relativity

All electromagnetic waves, which after Maxwell were known to include light waves, have a remarkable yet disturbing property: These waves travel at a fixed speed $c$. This fact was not controversial at all, until it was realized that this speed was independent of any frame of reference.

To make this surprise more concrete, we turn to Einstein's example of shining lights on trains. (No doubt today the example would be framed in terms of airplanes or rocket ships.) Imagine you are on a train traveling at 60 miles per hour. You turn on a flashlight and point it in the same direction as the train is moving. To you, the light moves at a speed of $c$ (you think your speed is zero miles per hour). To someone on the side of the road, the light should move at a speed of 60 miles per hour $+ c$. But according to Maxwell's equations, it does not. The observer off the train will actually see the light move at the same speed $c$, which is no different from your observation on the train. This is wacky and suggests that Maxwell's equations must be wrong.

In actual experiments, though, it is our common sense (codified in Newtonian mechanics) that is wrong. This led Albert Einstein, in 1905, to propose an entirely new theory of mechanics, the special theory. In large part, Einstein discovered the special theory because he took Maxwell's equations seriously as a statement about the fundamental nature of reality.

## 1.5 Quantum Mechanics and Photons

What is light? For many years scientists debated whether light was made up of particles or of waves. After Maxwell (and especially after Hertz's experiments showing that light is indeed a type of electromagnetic wave), it seemed that the debate had been settled. But in the late nineteenth century, a weird new phenomenon was observed. When light was shone on certain metals, electrons were ejected from the metal. Something in light carried enough energy to forcibly eject electrons from the metal. This phenomenon is called the *photoelectric effect*. This alone is not shocking, as it was well known that traditional waves carried energy. (Many of us have been knocked over by ocean waves at the beach.) In classical physics, though, the energy carried by a traditional wave is proportional to the wave's amplitude (how high it gets). But in the photoelectric effect, the energy of the ejected electrons is proportional not to the amplitude of the light wave but instead to the light's frequency. This is a decidedly non-classical effect, jeopardizing a wave interpretation for light.

In 1905, in the same year that he developed the Special Theory of Relativity, Einstein gave an interpretation to light that seemed to explain the photoelectric effect. Instead of thinking of light as a wave (in which case, the energy would have to be proportional to the light's amplitude), Einstein assumed that light is made of particles, each of which has energy proportional to the frequency, and showed that this assumption leads to the correct experimental predictions.

In the context of other seemingly strange experimental results, people started to investigate what is now called quantum mechanics, amassing a number of partial explanations. Suddenly, over the course of a few years in the mid-1920s, Born, Dirac, Heisenberg, Jordan, Schrdinger, von Neumann, and others worked out the complete theory, finishing the first quantum revolution. We will see that this theory indeed leads to the prediction that light must have properties of both waves and particles.

## 1.6 Gauge Theories for Physicists: The Standard Model

At the end of the 1920s, gravity and electromagnetism were the only two known forces. By the end of the 1930s, both the strong force and the weak force had been discovered.

In the nucleus of an atom, protons and neutrons are crammed together. All of the protons have positive charge. The rules of electromagnetism would predict that these protons would want to explode away from each other, but this does not happen. It is the strong force that holds the protons and neutrons together in the nucleus, and it is called such since it must be strong enough to overcome the repelling force of electromagnetism.

The weak force can be seen in the decay of the neutron. If a neutron is just sitting around, after ten or fifteen minutes it will decay into a proton, an electron, and another elementary particle (the electron anti-neutrino, to be precise). This could not be explained by the other forces, leading to the discovery of this new force.

Since both of these forces were basically described in the 1930s, their theories were quantum mechanical. But in the 1960s, a common framework for the weak force and the electromagnetic force was worked out (resulting in Nobel Prizes for Abdus Salam, Sheldon Glashow, and Steven Weinberg in 1979). In fact, this framework can be extended to include the strong force. This common framework goes by the name of the *standard model*. (It does not include gravity.)

Much earlier, in the 1920s, the mathematician Herman Weyl attempted to unite gravity and electromagnetism, by developing what he called a *gauge theory*. While it quickly was shown not to be physically realistic, the underlying idea was sufficiently intriguing that it resurfaced in the early 1950s in the work of Yang and Mills, who were studying the strong force. The underlying mathematics of their work is what led to the unified electro-weak force and the standard model.

Weyl's gauge theory was motivated by symmetry. He used the word "gauge" to suggest different gauges for railroad tracks. His work was motivated by the desire to shift from global symmetries to local symmetries. We will start with global symmetries. Think of the room you are sitting in. Choose a corner and label this the origin. Assume one of the edges is the $x$-axis, another the $y$-axis, and the third the $z$-axis. Put some unit of length on these edges. You can now uniquely label any point in the room by three coordinate values.

Of course, someone else might have chosen a different corner as the origin, different coordinate axes, or different units of length. In fact, any point in the room (or, for that matter, any point in space) could be used as the origin, and so on. There are an amazing number of different choices.

Now imagine a bird flying in the room. With your coordinate system, you could describe the path of the bird's flight by a curve ( $x(t)$,$y(t)$,$z(t)$). Someone else, with a different coordinate system, will describe the flight of the bird by three totally different functions. The flight is the same (after all, the bird does not care what coordinate system you are using), but the description is different. By changing coordinates, we can translate from one coordinate system into the other. This is a global change of coordinates. Part of the deep insight of the theory of relativity, as we will see, is that which coordinate changes are allowed has profound effects on the description of reality.

Weyl took this one step further. Instead of choosing one global coordinate system, he proposed that we could choose different coordinate systems at each point of space but that all of these local coordinate systems must be capable of being suitably patched together. Weyl called this patching "choosing a gauge."

## 1.7 Four-Manifolds

During the 1950s, 1960s, and early 1970s, when physicists were developing what they called gauge theory, leading to the standard model, mathematicians were developing the foundations of differential geometry. (Actually this work on differential geometry went back quite a bit further than the 1950s.) This mainly involved understanding the correct nature of curvature, which, in turn, as we will see, involves understanding the nature of connections. But sometime in the 1960s or 1970s, people must have begun to notice uncanny similarities between the physicists' gauges and the mathematicians' connections. Finally, in 1975, Wu and Yang [69] wrote out the dictionary between the two languages (this is the same Yang who was part of Yang-Mills). This alone was amazing. Here the foundations of much of modern physics were shown to be the same as the foundations of much of differential geometry.

Through most of the twentieth century, when math and physics interacted, overwhelmingly it was the case that math shaped physics:

$$\text{Mathematics} \Rightarrow \text{Physics}$$

Come the early 1980s, the arrow was reversed. Among all possible gauges, physicists pick out those that are Yang-Mills, which are in turn deep generalizations of Maxwell's equations. By the preceding dictionary, connections that satisfy Yang-Mills should be special.

This leads us to the revolutionary work of Simon Donaldson. He was interested in four-dimensional manifolds. On a four-manifold, there is the space of all possible connections. (We are ignoring some significant facts.) This space is infinite dimensional and has little structure. But then Donaldson decided to take physicists seriously. He looked at those connections that were Yang-Mills. (Another common term used is "instantons.") At the time, there was no compelling mathematical reason to do this. Also, his four-manifolds were not physical objects and had no apparent link with physics. Still, he looked at Yang-Mills connections and discovered amazing, deeply surprising structure, such as that these special Yang-Mills connections form a five-dimensional space, which has the original four-manifold as part of its boundary. (Here we are coming close to almost criminal simplification, but the underlying idea that the Yang-Mills connections are linked to a five-manifold that has the four-manifold as part of its boundary is correct.) This work shocked much of the mathematical world and transformed four-manifold theory from a perfectly respectable area of mathematics into one of its hottest branches. In awarding Donaldson a Field's Medal in 1986, Atiyah [1] wrote:

> The surprise produced by Donaldson's result was accentuated by the fact that his methods were

> completely new and were borrowed from theoretical physics, in the form of Yang-Mills equations. … Several mathematicians (including myself) worked on instantons and felt very pleased that they were able to assist physics in this way. Donaldson, on the other hand, conceived the daring idea of reversing this process and of using instantons on a general 4-manifold as a new geometrical tool.

Many of the finest mathematicians of the 1980s started working on developing this theory, people such as Atiyah, Bott, Uhlenbeck, Taubes, Yau, Kobayashi, and others.

Not only did this work produce some beautiful mathematics, it changed how math could be done. Now we have

$$\text{Physics} \Rightarrow \text{Mathematics}$$

an approach that should be called *physical mathematics* (a term first coined by Kishore Marathe, according to [70]: This text by Zeidler is an excellent place to begin to see the power behind the idea of physical mathematics).

Physical mathematics involves taking some part of the real world that is physically important (such as Maxwell's equations), identifying the underlying mathematics, and then taking that mathematics seriously, even in contexts far removed from the natural world. This has been a major theme of mathematics since the 1980s, led primarily by the brilliant work of Edward Witten. When Witten won his Field's Medal in 1990, Atiyah [2] wrote:

> Although (Witten) is definitely a physicist his command of mathematics is rivaled by few mathematicians, and his ability to interpret physical ideas in mathematical form is quite unique. Time and again he has surprised the mathematical community by a brilliant application of physical insight leading to new and deep mathematical theorems.

The punchline is that mathematicians should take seriously underlying mathematical structure of the real world, even in non-real world situations. In essence, nature is a superb mathematician.

## 1.8 This Book

There is a problem with this revolution of physical mathematics. How can any mere mortal master both physics and mathematics? The answer, of course, is you cannot. This book is a compromise. We concentrate on the key underlying mathematical concepts behind the physics, trying at the same time to explain just enough of the real world to justify the use of the mathematics. By the end of this book, I hope the reader will be able to start understanding the work needed to understand Yang-Mills.

## 1.9 Some Sources

One way to learn a subject is to study its history. That is not the approach we are taking. There are a number of good, accessible books, though. Stephen J. Blundell's *Magnetism: A Very Short Introduction* [4] is excellent for a popular general overview. For more technical histories of the early days of electromagnetism, I would recommend Steinle's article "Electromagnetism and Field Physics" [62] and Buchwald's article "Electrodynamics from Thomson and Maxwell to Hertz" [7].

Later in his career, Abraham Pais wrote three excellent books covering much of the history of twentieth century physics. His *Subtle Is the Lord: The Science and the Life of Albert Einstein* [51] is a beautiful scientific biography of Einstein, which means that it is also a history of much of what was important in physics in the first third of the 1900s. His *Niels Bohr's Times: In Physics, Philosophy, and Polity* [52] is a scientific biography of Bohr, and hence a good overview of the history of early quantum mechanics. His *Inward Bound* [53] is a further good reference for the development of quantum theory and particle physics.

It appears that the ideas of special relativity were "in the air" around 1905. For some of the original papers by Einstein, Lorentz, Minkowski, and Weyl, there is the collection [19]. Poincar was also actively involved in the early days of special relativity. Recently two biographies of Poincar have been written: Gray's *Henri Poincar: A Scientific Biography* [27] and Verhulst's *Henri Poincar: Impatient Genius* [67]. There is also the still interesting paper of Poincar that he gave at the World's Fair in Saint Louis in 1904, which has recently been reprinted [54].

At the end of this book, we reach the beginnings of gauge theory. In [50], O'Raifeartaigh has collected some of the seminal papers in the development of gauge theory. We encourage the reader to look at the web page of Edward Witten for inspiration. I would also encourage people to look at many of the expository papers on the relationship between mathematics and physics in volume 6 of the collected works of Atiyah [3] and at those in volume 4 of the collected works of Bott [5]. (In fact, perusing all six volumes of Atiyah's collected works and all four volumes of Bott's is an excellent way to be exposed to many of the main themes of mathematics of the last half of the twentieth century.) Finally, there is the wonderful best seller *The Elegant Universe* by Brian Greene [28].

# 2
# Maxwell's Equations

**Summary:** The primary goal of this chapter is to state Maxwell's equations. We will then see some of their implications, which will allow us to give alternative descriptions for Maxwell's equations, providing us in turn with a review of some of the basic formulas in multivariable calculus.

## 2.1 A Statement of Maxwell's Equations

Maxwell's equations link together three vector fields and a real-valued function. Let

$$E = E(x,y,z,t) = (E_1(x,y,z,t), E_2(x,y,z,t), E_3(x,y,z,t))$$

and

$$B = B(x,y,z,t) = (B_1(x,y,z,t), B_2(x,y,z,t), B_3(x,y,z,t))$$

be two vector fields with spacial coordinates $(x,y,z)$ and time coordinate $t$. Here $E$ represents the electric field while $B$ represents the magnetic field. The third vector field is

$$j(x,y,z,t) = (j_1(x,y,z,t), j_2(x,y,z,t), j_3(x,y,z,t)),$$

which represents the current (the direction and the magnitude of the flow of electric charge). Finally, let

$$\rho(x,y,z,t)$$

be a function representing the charge density. Let *c* be a constant. (Here *c* is the speed of light in a vacuum.) Then these three vector fields and this function satisfy Maxwell's equations if

$$\begin{aligned}
\operatorname{div}(E) &= \rho \\
\operatorname{curl}(E) &= -\frac{\partial B}{\partial t} \\
\operatorname{div}(B) &= 0 \\
c^2 \operatorname{curl}(B) &= j + \frac{\partial E}{\partial t}.
\end{aligned}$$

(Review of the curl, the divergence, and other formulas from multivariable calculus is in the next section.)

We can reinterpret these equations in terms of integrals via various Stokes-type theorems. For example, if *V* is a compact region in space with smooth boundary surface *S*, as in Figure 2.1, then for any vector field *F* we know from the Divergence Theorem that

$$\int\int_S F \cdot n \mathrm{d}A = \int\int\int_V \operatorname{div}(F) \mathrm{d}x \mathrm{d}y \mathrm{d}z,$$

where $n$ is the unit outward normal of the surface $S$.

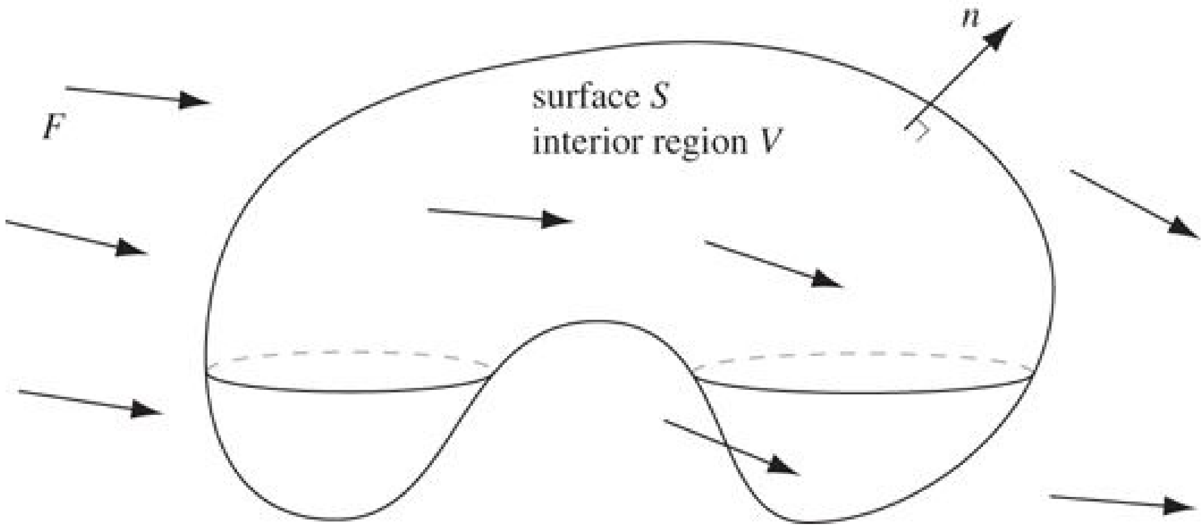


Figure 2.1

In words, this theorem says that the divergence of a vector field measures how much of the field is flowing out of a region.

Then the first of Maxwell's equations can be restated as

$$\begin{aligned}\int\int_S E \cdot n \mathrm{d}A &= \int\int\int \mathrm{div}(E)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int\int\int \rho(x, y, z, t)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \text{total charge inside the region } V.\end{aligned}$$

Likewise, the third of Maxwell's equations is:

$$
\begin{aligned}
\int\int_S B \cdot n \mathrm{d}A &= \int\int\int \operatorname{div}(B)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int\int\int 0\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= 0 \\
&= \text{There is no magnetic charge inside the region } V.
\end{aligned}
$$

This is frequently stated as "There are no magnetic monopoles," meaning there is no real physical notion of magnetic density.

The second and fourth of Maxwell's equations have similar integral interpretations. Let $C$ be a smooth curve in space that is the boundary of a smooth surface $S$, as in Figure 2.2. Let $T$ be a unit tangent vector of $C$. Choose a normal vector field $n$ for $S$ so that the cross product $T \times n$ points into the surface $S$.

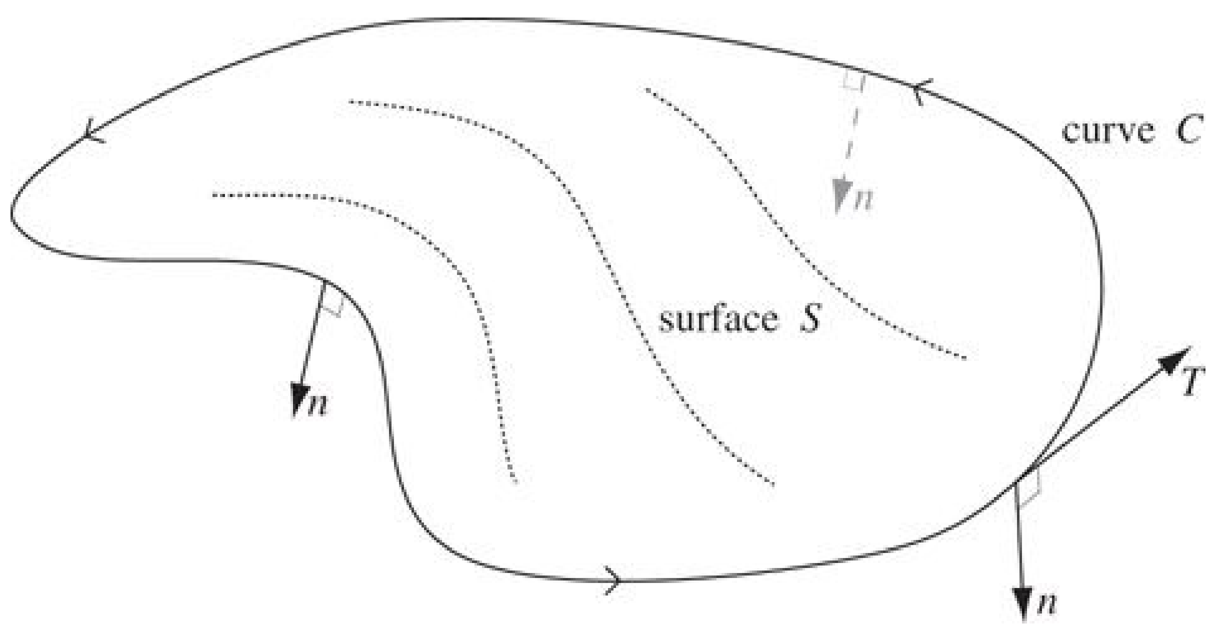

Figure 2.2

Then the classical Stokes Theorem states that for any vector field $F$ , we have

$$\int_C F \cdot T \mathrm{d}s = \int\int_S \mathrm{curl}(F) \cdot n \mathrm{d}A.$$

This justifies the intuition that the curl of a vector field measures how much the vector field $F$ wants to twirl.

Then the second of Maxwell's equations is equivalent to

$$\int_C E \cdot T \mathrm{d}s = -\int\int_S \frac{\partial B}{\partial t} \cdot n \mathrm{d}A.$$

Thus the magnetic field $B$ is changing in time if and only if the electric field $E$ is curling.

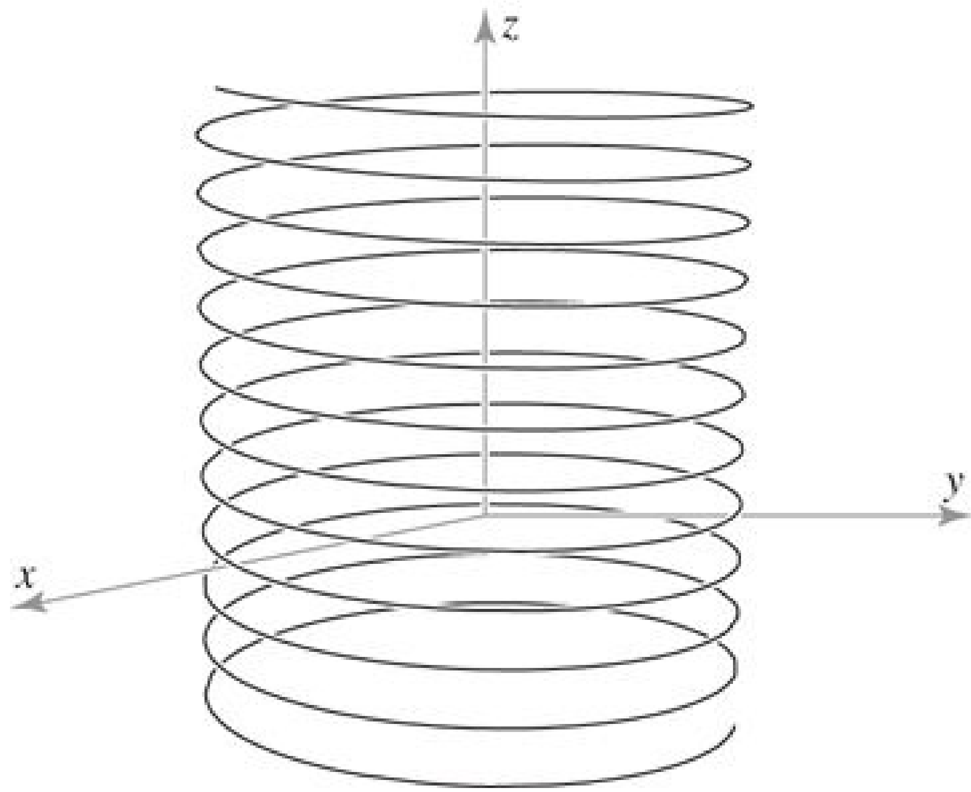


Figure 2.3

This is the mathematics underlying how to create current in a wire by moving a magnet. Consider a coil of wire, centered along the $z$-axis (i.e., along the vector $k = (0,0,1)$).

The wire is coiled (almost) in the $xy$-plane. Move a magnet through the middle of this coil. This means that the magnetic field $B$ is changing in time in the direction $k$. Thanks to Maxwell, this means that the curl of the electric field $E$ will be non-zero and will point in the direction $k$. But this means that the actual vector field $E$ will be "twirling" in the $xy$-plane, making the electrons in the coil move, creating a current.

This is in essence how a hydroelectric dam works. Water from a river is used to move a magnet through a coil of wire, creating a current and eventually lighting some light bulb in a city far away.

The fourth Maxwell equation gives

$$c^2 \int_C B \cdot T \mathrm{d}s = \int \int_S \left( j + \frac{\partial E}{\partial t} \right) \cdot n \mathrm{d}A.$$

Here current and a changing electric field are linked to the curl of the magnetic field.

# 2.2 Other Versions of Maxwell's Equations

## 2.2.1 Some Background in Nabla

This section is meant to be both a review and a listing of some of the standard notations that people use. The symbol $\triangledown$ is pronounced "nabla" (sometimes $\nabla$ is called "del"). Let

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$
$$= i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

where $i = (1,0,0)$, $j = (0,1,0)$, and $k = (0,0,1)$. Then for any function $f(x,y,z)$, we set the gradient to be

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$
$$= i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}.$$

For a vector field

$$F = F(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$$
$$= (F_1, F_2, F_3)$$
$$= F_1 \cdot i + F_2 \cdot j + F_3 \cdot k,$$

define the divergence to be:

$$\nabla \cdot F = \mathrm{div}(F)$$
$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

The curl of a vector field in this notation is

$$
\begin{aligned}
\nabla \times F &= \operatorname{curl}(F) \\
&= \det \begin{pmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} \\
&= \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right), \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).
\end{aligned}
$$

### 2.2.2 Nabla and Maxwell

Using the nabla notation, Maxwell's equations have the form

$$
\begin{aligned}
\nabla \cdot E &= \rho \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \cdot B &= 0 \\
c^2 \nabla \times B &= j + \frac{\partial E}{\partial t}
\end{aligned}
$$

Though these look like four equations, when written out they actually form eight equations. This is one of the exercises in the following section.

## 2.3 Exercises

The first few problems are exercises in the nabla machinery and in the basics of vector fields.

**Exercise 2.3.1.** *For the function* $f(x,y,z) = x^2 + y^3 + xy^2 + 4z$, *compute grad*$(f)$, *which is the same as computing* $\nabla(f)$.

**Exercise 2.3.2.**

*a. Sketch, in the xy-plane, some representative vectors making up the vector field*

$$F(x,y,z) = (F_1, F_2, F_3) = (x,y,z),$$

*at the points*

$$(1,0,0), (1,1,0), (0,1,0), (-1,1,0), (-1,0,0), (-1,-1,0), (0,-1,0), (1,-1,0).$$

*b. Find div*$(F) = \nabla \cdot F$.

*c. Find curl*$(F) = \nabla \times F$.

**Comment:** Geometrically the vector field $F(x,y,z) = (x,y,z)$ is spreading out but not "twirling" or "curling" at all, as is reflected in the calculations of its divergence and curl.

**Exercise 2.3.3.**

*a. Sketch, in the xy-plane, some representative vectors* *making up the vector field*

$$F(x,y,z) = (F_1, F_2, F_3) = (-y, x, 0),$$

*at the points*

$$(1,0,0), (1,1,0), (0,1,0), (-1,1,0), (-1,0,0), (-1,-1,0), (0,-1,0), (1,-1,0).$$

*b. Find div(F) = $\nabla \cdot F$.*

*c. Find curl(F) = $\nabla \times F$.*

**Comment:** As compared to the vector field in the previous exercise, this vector field $F(x,y,z) = (-y,x,0)$ is not spreading out at all but does "twirl" in the *xy*-plane. Again, this is reflected in the divergence and curl.

**Exercise 2.3.4.**

*Write out Maxwell's equations in local coordinates (meaning not in vector notation). You will get eight equations. For example, one of them will be*

$$\frac{\partial}{\partial y} E_3 - \frac{\partial}{\partial z} E_2 = -\frac{\partial}{\partial t} B_1.$$

**Exercise 2.3.5.**

*Let* $c = 1$. *Show that*

$$\begin{aligned} E &= (y - z, -2zt, -x - z^2) \\ B &= (-1 - t^2, 0, 1 + t) \\ \rho &= -2z \\ j &= (0, 2z, 0) \end{aligned}$$

*satisfy Maxwell's equations.*

**Comment:** In the real world, the function $\rho$ and the vector fields $E$, $B$, and $j$ are determined from experiment. That is not how I chose the function and vector fields in problem 5. In Chapter 7, we will see that given any function $\phi(x,y,z,t)$ and vector field $A(x,y,z,t) = (A_1,A_2,A_3)$, if we set

$$\begin{aligned} E &= -\nabla(\phi) - \frac{\partial A}{\partial t} \\ &= \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) - \left(\frac{\partial A_1}{\partial t}, \frac{\partial A_2}{\partial t}, \frac{\partial A_3}{\partial t}\right) \\ B &= \nabla \times A \end{aligned}$$

and, using these particular $E$ and $B$, set

$$\begin{aligned} \rho &= \nabla \cdot E \\ j &= \nabla \times B - \frac{\partial B}{\partial t}, \end{aligned}$$

we will have that $\rho$, $E$, $B$, and $j$ satisfy Maxwell's equations. For this last problem, I simply chose, almost at random, $\phi\,(x,y,z,t) = xz$ and

$$A = (-yt + x^2, x + zt^2, -y + z^2 t).$$

The punchline of Chapter 7 is that the converse holds, meaning that if the function $\rho$ and the vector fields $E$, $B$, and $j$ satisfy Maxwell's equations, then there must be a function $\phi\,(x,y,z,t)$ and a vector field $A$ such that $E = -\nabla(\phi) - \frac{\partial A}{\partial t}$ and $B = \nabla \times A$. The $\phi\,(x,y,z,t)$ and $A$ are called the *potentials*.

# 3
# Electromagnetic Waves

**Summary:** When the current $j$ and the density $\rho$ are zero, both the electric field and the magnetic field satisfy the wave equation, meaning that both fields can be viewed as waves. In the first section, we will review the wave equation. In the second section, we will see why Maxwell's equations yield these electromagnetic waves, each having speed $c$.

## 3.1 The Wave Equation

Waves permeate the world. Luckily, there is a class of partial differential equations (PDEs) whose solutions describe many actual waves. We will not justify why these PDEs describe waves but instead will just state their form. (There are many places to see heuristically why these PDEs have anything at all to do with waves; for example, see [26].)

The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0.$$

Here the goal is to find a function $y = y(x,t)$, where $x$ is position and $t$ is time, that satisfies the preceding equation. Thus the “unknown” is the function $y(x,t)$. For a fixed $t$, this can describe a function that looks like Figure 3.1.

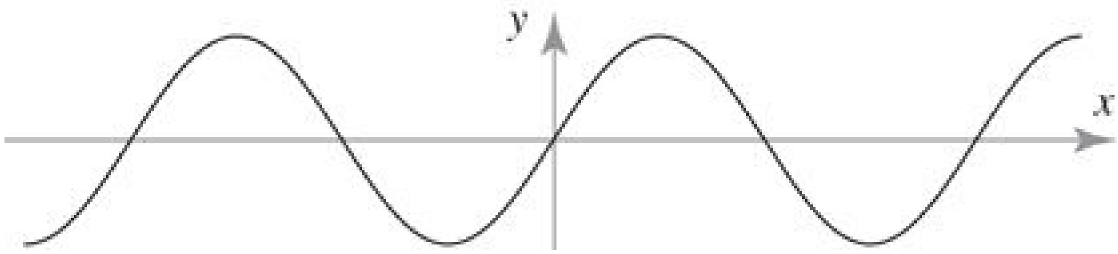


Figure 3.1

From the heuristics of the derivation of this equation, the speed of this wave is $v$.

The two-dimensional wave equation is

$$\frac{\partial^2 z}{\partial t^2} - v^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 0.$$

Here, the function $z(x,y,t)$ is the unknown, where $x$ and $y$ describe position and $t$ is again time. This could model the motion of a wave over the $(x,y)$-plane. This wave also has speed $v$.

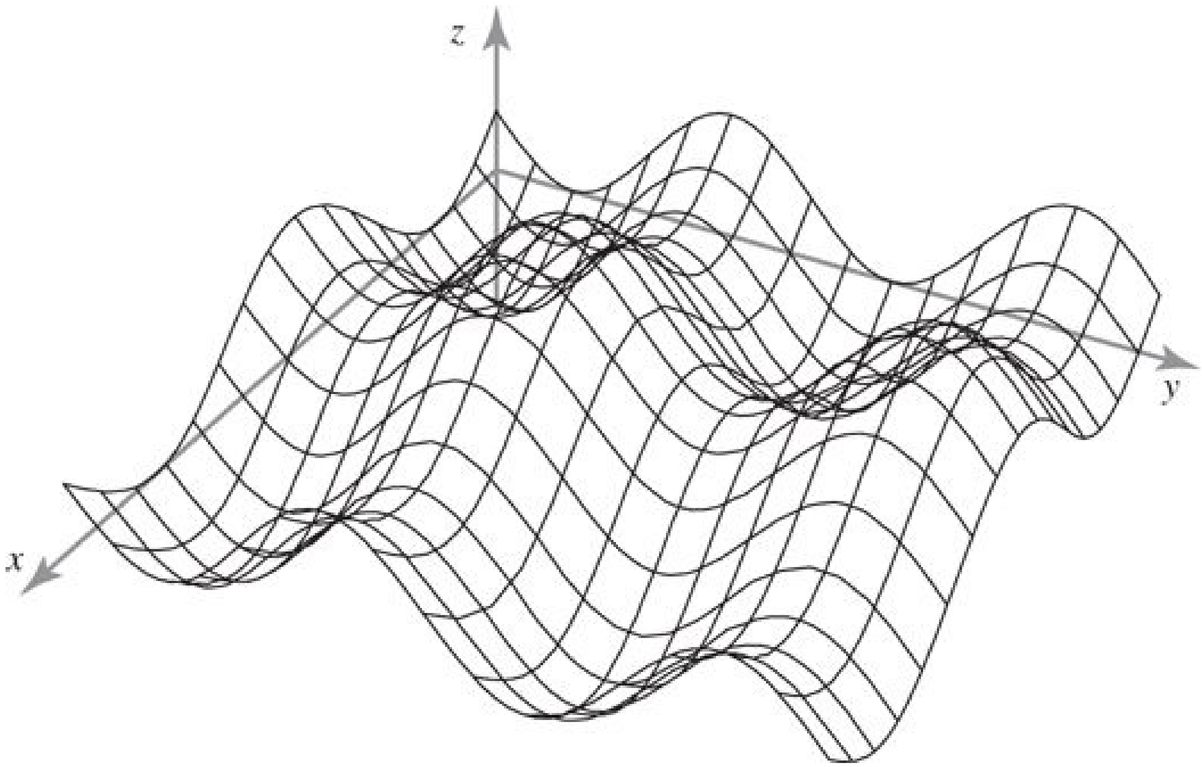


Figure 3.2

The three-dimensional wave equation is

$$\frac{\partial^2 f}{\partial t^2} - v^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0.$$

Once again, the speed is *v*.

Any function that satisfies such a PDE is said to satisfy the wave equation. We expect such functions to have wave-like properties.

The sum of second derivatives $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ occurs often enough to justify its own notation and its own name, the *Laplacian*. We use the notation

$$\triangle(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

This has a convenient formulation using the nabla notation $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. Thinking of $\nabla$ as a vector, we interpret its dot product with itself as

$$\begin{aligned}
\nabla^2 &= \nabla \cdot \nabla \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \\
&= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},
\end{aligned}$$

and thus we have

$$\nabla^2 = \triangle.$$

Then we can write the three-dimensional wave equation in the following three ways:

$$\begin{aligned}
0 &= \frac{\partial^2 f}{\partial t^2} - v^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) \\
&= \frac{\partial^2 f}{\partial t^2} - v^2 \triangle(f) \\
&= \frac{\partial^2 f}{\partial t^2} - v^2 \nabla^2(f).
\end{aligned}$$

But this wave equation is for functions $f(x,y,z,t)$. What does it mean for a vector field $F = (F_1,F_2,F_3)$ to satisfy a wave equation? We will say that the vector field $F$ satisfies the wave equation

$$\frac{\partial^2 F}{\partial t^2} - v^2 \triangle(F) = 0,$$

if the following three partial differential equations

$$\frac{\partial^2 F_1}{\partial t^2} - v^2 \left( \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) = 0$$
$$\frac{\partial^2 F_2}{\partial t^2} - v^2 \left( \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2} \right) = 0$$
$$\frac{\partial^2 F_3}{\partial t^2} - v^2 \left( \frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right) = 0$$

hold.

## 3.2 Electromagnetic Waves

In the half-century before Maxwell wrote down his equations, an amazing amount of experimental work on the links between electricity and magnetism had been completed. To some extent, Maxwell put these empirical observations into a more precise mathematical form. These equations’s strength, though, is reflected in that they allowed Maxwell, for purely theoretical reasons, to make one of the most spectacular intellectual leaps ever: Namely, Maxwell showed that electromagnetic waves that move at the speed $c$ had to exist. Maxwell knew that this speed $c$ was the same as the speed of light, leading him to predict that light was just a special type of elecromagnetic wave. No one before Maxwell realized this. In the 1880s, Hertz proved experimentally that light was indeed an electromagnetic wave.

We will first see intuitively why Maxwell's equations lead to the existence of electromagnetic waves, and then we will rigorously prove this fact. Throughout this section, assume that there is no charge ( $\rho = 0$) and no current ($j = 0$) (i.e., we are working in a vacuum). Then Maxwell's equations become

$$\begin{aligned} \nabla \cdot E &= 0 \\ \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \cdot B &= 0 \\ c^2 \nabla \times B &= \frac{\partial E}{\partial t}. \end{aligned}$$

These vacuum equations are themselves remarkable since they show that the waves of the electromagnetic field move in space without any charge or current.

Suppose we change the magnetic field (possibly by moving a magnet). Then $\frac{\partial B}{\partial t} \neq 0$ and hence the electric vector field $E$ will have non-zero curl. Thus $E$ will have a change in a direction perpendicular to the change in $B$. But then $\frac{\partial E}{\partial t} \neq 0$, creating curl in $B$, which, in turn, will prevent $\frac{\partial B}{\partial t}$ from being zero, starting the whole process over again, never stopping.

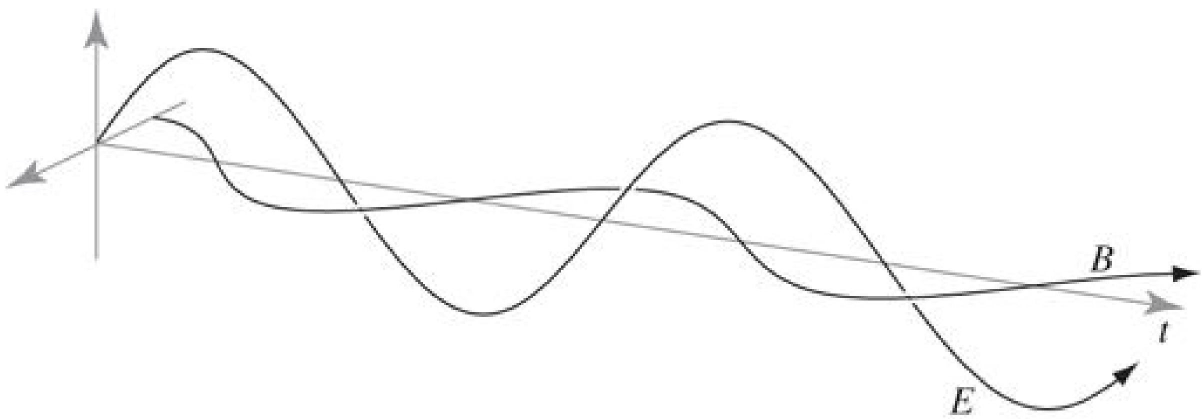


Figure 3.3

This is far from showing that we have an actual wave, though.

Now we show that the electric field $E$ satisfies the wave equation

$$\frac{\partial^2 E}{\partial t^2} - c^2 \Delta(E) = 0.$$

We have

$$\begin{aligned}\frac{\partial^2 E}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial E}{\partial t}\right)\\ &= \frac{\partial}{\partial t}(c^2\nabla \times B)\\ &= c^2\nabla \times \frac{\partial B}{\partial t}\\ &= -c^2\nabla \times \nabla \times E\\ &= c^2\left(\triangle(E_1), \triangle(E_2), \triangle(E_3)\right),\end{aligned}$$

which means that the electric field $E$ satisfies the wave equation. Note that the second and fourth lines result from Maxwell's equations. The fact that $\nabla \times \nabla \times E = -(\triangle(E_1), \triangle(E_2), \triangle(E_3))$ is a calculation coupled with Maxwell's first equation, which we leave for the exercises. The justification for the third equality, which we also leave for the exercises, stems from the fact that the order of taking partial derivatives is interchangeable. The corresponding proof for the magnetic field is similar and is also left as an exercise.

# 3.3 The Speed of Electromagnetic Waves Is Constant

### 3.3.1 Intuitive Meaning

We have just seen there are electromagnetic waves moving at speed $c$, when there is no charge and no current. In this section we want to start seeing that the existence of these waves, moving at that speed $c$, strikes a blow to our common-sense notions of physics, leading, in the next chapter, to the heart of the Special Theory of Relativity.

Consider a person A. She thinks she is standing still. A train passes by, going at the constant speed of 60 miles per hour. Let person B be on the train. B legitimately can think that he is at the origin of the important coordinate system, thus thinking of himself as standing still. On this train, B rolls a ball forward at, say, 3 miles per hour, with respect to the train. Observer A, though, would say that the ball is moving at $3 + 60$ miles per hour. So far, nothing controversial at all.

Let us now replace the ball with an electromagnetic wave. Suppose person B turns it on and observes it moving in the car. If you want, think of B as turning on a flashlight. B will measure its speed as some $c$ miles per hour. Observer A will also see the light. Common sense tells us, if not screams at us, that A will measure the light as traveling at $c + 60$ miles per hour.

But what do Maxwell's equations tell us? The speed of an electromagnetic wave is the constant $c$ that appears in Maxwell's equations. But the value of $c$ does not depend on the initial choice of coordinate system. The

$(x,y,z,t)$ for person $A$ and the $(x,y,z,t)$ for person $B$ have the same $c$ in Maxwell. Of course, the number $c$ in the equations is possibly only a "constant" once a coordinate system is chosen. If this were the case, then if person $A$ measured the speed of an electromagnetic wave to be some $c$, then the corresponding speed for person $B$ would be $c$- 60, with this number appearing in person B's version of Maxwell's equations. This is now an empirical question about the real world. Let $A$ and $B$ each measure the speed of an electromagnetic wave. What physicists find is that for both observers the speed is the same. Thus, in the preceding train, the speed of an electromagnetic wave for both $A$ and $B$ is the same $c$. This is truly bizarre.

### 3.3.2 Changing Coordinates for the Wave Equation

Suppose we again have two people, $A$ and $B$. Let person $B$ be traveling at a constant speed $\alpha$ with respect to $A$, with $A$'s and $B$'s coordinate systems exactly matching up at time $t = 0$.

To be more precise, we think of person $A$ as standing still, with coordinates $x$¢ for position and $t'$ for time, and of person $B$ as moving to the right at speed $\alpha$, with position coordinate $x$ and time coordinate $t$. If the two coordinate systems line up at time $t = t' = 0$, then classically we would expect

$$
\begin{aligned}
x' &= x + \alpha t \\
t' &= t,
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
x &= x' - \alpha t \\
t &= t'.
\end{aligned}
$$

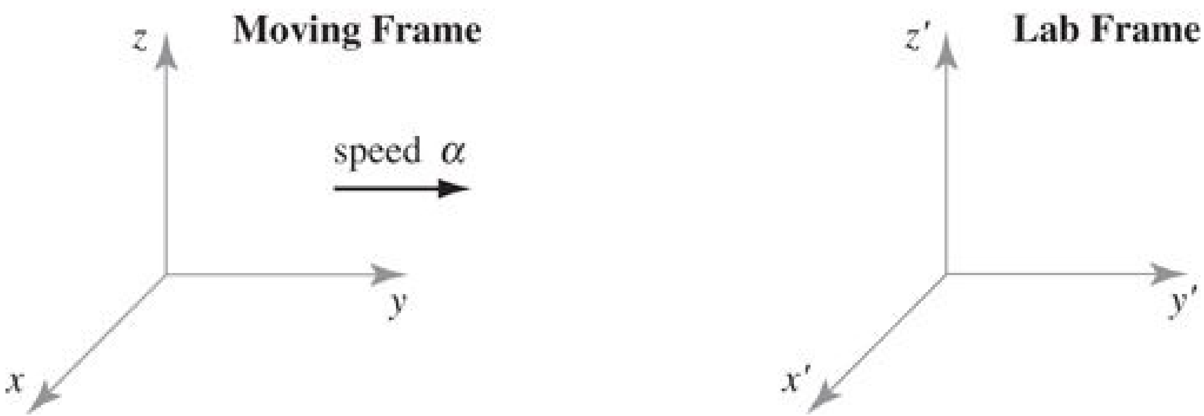


Figure 3.4

This reflects our belief that the way a stopwatch works should not be influenced by how fast it is moving. (This belief will also be shattered by the Special Theory of Relativity.)

Suppose in the reference frame for *B* we have a wave $y(x,t)$ satisfying

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0.$$

In $B$'s reference frame, the speed of the wave is $v$. From calculus, this speed $v$ must be equal to the rate of change of $x$ with respect to $t$, or in other words $v = \mathrm{d}x/\mathrm{d}t$. This in turn forces

$$\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x},$$

as we will see now. Fix a value of $y(x,t) = y_0$. Then we have some $x_0$ such that

$$y_0 = y(x_0, 0).$$

This means that for all $x$ and $t$ such that $x_0 = x-vt$, then $y_0 = y(x,t)$. The speed of the wave is how fast this point with $y$-coordinate $y_0$ moves along the $x$-axis with respect to $t$. Then

$$\begin{aligned} 0 &= \frac{\mathrm{d}y}{\mathrm{d}t} \\ &= \frac{\partial y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial y}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} \\ &= \frac{\partial y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial y}{\partial t} \\ &= v\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t}, \end{aligned}$$

giving us that $\partial y/\partial t = -v\,\partial y/\partial x$.

Person $A$ is looking at the same wave but measures the wave as having speed $v + \alpha$. We want to see explicitly that under the appropriate change of coordinates this indeed happens. This is an exercise in the chain rule, which is critically important in these arguments.

Our wave $y(x,t)$ can be written as function of $x'$ and $t'$, namely, as

$$y(x,t) = y(x' - \alpha t', t').$$

We want to show that this function satisfies

$$\frac{\partial^2 y}{\partial t'^2} - (v + \alpha)^2 \frac{\partial^2 y}{\partial x'^2} = 0.$$

The key will be that

$$\begin{aligned}
\frac{\partial}{\partial x'} &= \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} \\
&= \frac{\partial}{\partial x} \\
\frac{\partial}{\partial t'} &= \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} \\
&= -\alpha\frac{\partial}{\partial x} + \frac{\partial}{\partial t}
\end{aligned}$$

by the chain rule.

We start by showing that

$$\frac{\partial y}{\partial t'} = -\alpha \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t}$$

and

$$\frac{\partial y}{\partial x'} = \frac{\partial y}{\partial x},$$

whose proofs are left for the exercises.

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