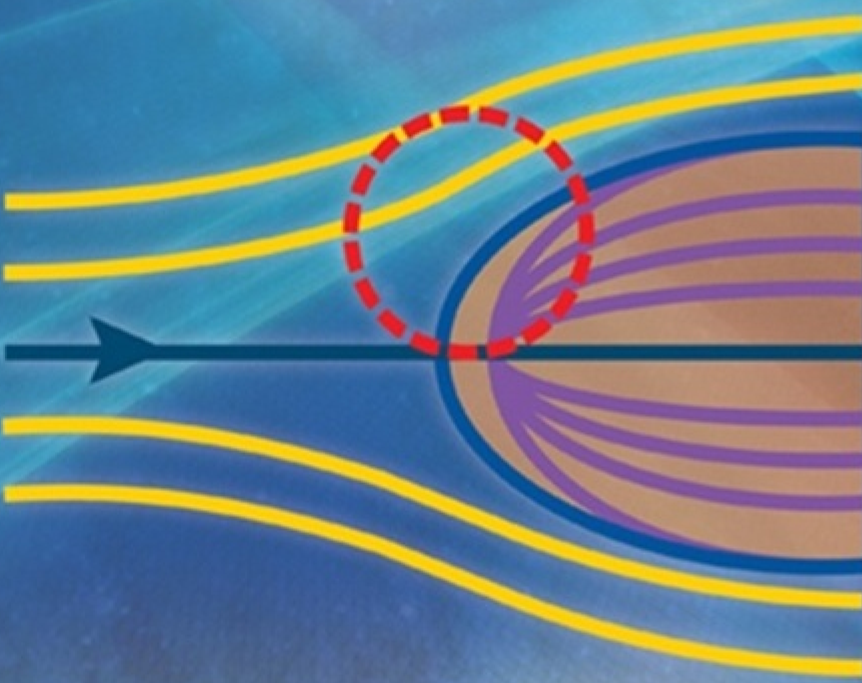
INTRODUCTION TO THEORETICAL AERODYNAMICS AND HYDRODYNAMICS

William R. Sears

Edited by Demetri P. Telionis



American Institute of Aeronautics and Astronautics

AIAA EDUCATION SERIES
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Editor-In-Chief

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Aerodynamics and Hydrodynamics

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Virginia Polytechnic Institute and State University

EDUCATION SERIES

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Virginia Polytechnic Institute and State University
Blacksburg, Virginia

Published by

AMERICAN INSTITUTE OF AERONAUTICS AND ASTRONAUTICS, INC.

1801 ALEXANDER BELL DRIVE, RESTON, VA 20191-4344

American Institute of Aeronautics and Astronautics, Inc., Reston, Virginia

1 2 3 4 5

Library of Congress Cataloging-in-Publication Data

Sears, William Rees, 1913-2002.

Introduction to theoretical aerodynamics and hydrodynamics / William R.Sears.

p. cm.

Includes bibliographical references and index.

ISBN 978-1-60086-773-6

1. Aerodynamics. 2. Hydrodynamics. I. Title.

TL570.S38 2011

533′.62—dc22

2010052541

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PREFACE

Professors William Sears was one of the greatest aerodynamicists of the twentieth century, the century that saw, during a fifty-year span, aeronautics and aviation start from airplanes built with bicycle parts to become supersonic aircraft and jumbo jets. Sears actively participated in all aspects of aeronautics and aviation development. Sears could claim a distinguished intellectual ancestry. He was a student of Theodore von Kármán, who was a student of Ludwig Prandtl. As Northrop's Chief of Aerodynamics and Flight Testing, Sears led the team that designed two new aircraft: the Northrop P-61 Black Widow and the Northrop YB-35 Flying Wing. He then founded the Graduate School of Aeronautical Engineering at Cornell University and served as its director while also serving as the editor of the *Journal of the Aeronautical Sciences*.

Sears' excitement for mathematics and aerodynamics was contagious. It "infected" most of his students and is reflected in *Introduction to Theoretical Aerodynamics and Hydrodynamics*. Sears was a master of connecting mathematical concepts with physical aerodynamic quantities. Readers of this book will find direct connections of fundamental concepts with practical applications, which are presented in the numerous examples worked out in the book, as well as in the problems. I was privileged to take many classes taught by Professor Sears, and I used my class notes and this book in the classes that I then taught.

Many of his students and associates, including myself, find the presentation of the material concise and to the point, presenting complex ideas succinctly, and thus covering adequately the most basic topics with just a few pages. As a result, even though this material appeared at first as a set of class notes prepared for Sears' classes at Cornell, many instructors from other universities chose to use them in their classes. Instructors today will find that this book can serve as an excellent textbook for a first course in ideal aerodynamics. Finally, I would like to acknowledge the very careful and competent help of Dr. Shereef Sadek, who reviewed the entire book.

Demetri P. Telionis
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June 2011

PREFACE (1970)

When these notes and problems were first prepared and published in 1948, they were intended to be only the first of four parts. The other three parts have never been prepared, however. "Part I" has, in the meantime, served as a textbook at Cornell and at several other institutions; its warm reception outside of Ithaca has both surprised and pleased me. It has inspired me to revise the material again, adding some new text and improving some of the Problems and Exercises. Some of the users of the book have kindly assisted by sending me their corrections and suggestions.

The following paragraphs from the Preface of 1948 are still pertinent:

The student is presumed to be familiar with the most common mathematical methods of engineering, including a course in Differential Equations. Preferably, the student should also have studied Vector Analysis and Functions of Complex Variables, although these subjects are both reviewed here. The student who is unprepared in these matters must expect to study them intensively as the course progresses.

Little of the material presented here is original. Most of it is already available in Lamb's *Hydrodynamics,* Glauert's *Aerofoil and Airscrew Theory,* Volume I and II of Durand's *Aerodynamic Theory*, and elsewhere. I therefore claim credit only for the presentation, and even this undoubtedly reflects the influence of my former teachers, Theodore von Kármán and Clark B. Millikan.

One further word before we get underway. A feature of this presentation is that new concepts and techniques are deliberately introduced at many points where they could have been avoided. My aim is not to treat a variety of subjects by the easiest methods available, but rather to educate the student in a broad way, to teach the student things that will be of value to the student in original work and study in a much wider field than I can possibly cover here.

William R. Sears

Kinematics of Fluid Flows

- Kinematics of fluid flow
- Functional representation of fluid motion
- Eulerian equations
- Continuity
- Stream function
- Vorticity and circulation

1.1 Introduction

The fundamental property of a fluid, which distinguishes it from other forms of matter, is that it cannot be in stationary equilibrium under oblique stresses, that is, stresses oblique to the surface separating any two parts. We can easily recognize that such stresses do exist in fluids that are in motion—consider, for example, how the fluid in a rotating circular vessel takes on the motion of the vessel.

We assume that the fluid is *continuous* and *homogeneous* in structure, that is, the properties of the smallest subdivisions are the same as those of large samples. We know that this is not so, because matter is ultimately made up of molecules and atoms, but in many applications the dimensions we are concerned with are large compared with the molecular structure, and the smallest sample of fluid that concerns us contains a great number of molecules. In such cases, the properties of any sample are the average values over many molecules, and the approximation of a continuum is found to be acceptable and useful. Nevertheless, it must always be borne in mind that results obtained on the assumption of a continuum may be erroneous whenever the molecular structure dimensions are relatively large. For example, at very high altitudes (low pressures), the molecular spacing is so great that air is not even approximately a continuum in its contact with a body with the size of an airplane wing.

Here, we have already made the first of a series of simplifying approximations, which we hope will enable us to arrive at a useful and tractable theory of aerodynamics. This emphasizes the inescapable fact that all physical sciences are approximate. It is the job of the scientist—and especially the engineer—to develop the ingenuity and judgment that enable him or her to make simplifying approximations and still retain the fundamental features of the phenomena being investigated.

1.2 Functional Representation of Fluid Motion

Let us now fix our attention on a fluid that is a continuum and does not consist of discrete particles. Nevertheless, we often speak of "fluid particles," such as "the velocity of a particle." This simply means an infinitesimal portion or sample of the fluid that might be made identifiable by coloring it, for example.

There are two common ways of writing equations to describe a fluid flow; both were derived by L. Euler, who lived 1707–1783:

1. *Lagrangian*. Let a, b, and c denote the coordinates of any fluid particle at the time $t = 0$. Let x, y, and z denote the coordinates of the same particle at time t. Then, the flow geometry is completely specified if we know $x = x(a, b, c, t)$, $y = y(a, b, c, t)$, and $z = z(a, b, c, t)$. These give the *trajectories* of various particles. The velocity components are $u = u(a, b, c, t)$, etc., and the temperature or any other property would be given by a function $\theta = \theta(a, b, c, t)$.
2. *Eulerian*. Instead of following individual particles, in this system we fix our attention on a point in space, x, y, z, and consider the velocity components, temperatures, and so on of the fluid particles that pass that point:

$$
\begin{aligned}
u &= u(x, y, z, t) \\
v &= v(x, y, z, t) \\
w &= w(x, y, z, t) \\
\theta &= \theta(x, y, z, t)
\end{aligned}
$$

These equations become especially simple if the flow is *steady*, that is, it does not vary with time. Hence, flow variables are functions of spacial coordinates *only*, that is, $u = u(x, y, z)$.

Let the fluid velocity be denoted by the vector $\boldsymbol{q}$, and then $\boldsymbol{q} = \boldsymbol{q}(x, y, z, t) = (u, v, w)$. A *streamline* is defined as a line everywhere tangent to $\boldsymbol{q}$; hence, its differential equations are

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w} \tag{1.1}$$

In a steady flow, a streamline is also a trajectory.

1.3 Eulerian Equations

We shall adopt the Eulerian equations for our study. We will have a need for a sort of "Lagrangian" derivative in the Eulerian system, that is, a rate of change of a property for a particle.

Consider any property of the fluid, for example, the density $\rho = \rho(x, y, z, t)$, and calculate its differential:

$$\mathrm{d}\rho = \frac{\partial \rho}{\partial x}\mathrm{d}x + \frac{\partial \rho}{\partial y}\mathrm{d}y + \frac{\partial \rho}{\partial z}\mathrm{d}z + \frac{\partial \rho}{\partial t}\mathrm{d}t = \mathrm{d}\boldsymbol{r} \cdot \mathrm{grad}\ \rho + \frac{\partial \rho}{\partial t}\mathrm{d}t$$

But for any given particle as it moves along, dx, dy, and dz are not independent; in fact, $\mathrm{d}x = u\,\mathrm{d}t, \mathrm{d}y = v\,\mathrm{d}t$, and $\mathrm{d}z = w\,\mathrm{d}t$, or briefly $\mathrm{d}\boldsymbol{r} = \boldsymbol{q}\,\mathrm{d}t$. Thus, the rate of change of the density of a particle is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial t} + \boldsymbol{q} \cdot \mathrm{grad}\ \rho$$

We use the symbol $\mathrm{D}/\mathrm{D}t$ for this type of derivative, sometimes called the *convective derivative*:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{q} \cdot \nabla \tag{1.2}$$

This can be applied to any fluid property, including vector properties such as velocity and acceleration. The acceleration of a particle, for example, is

$$\frac{\mathrm{D}\boldsymbol{q}}{\mathrm{D}t} = \frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{q} \cdot \nabla \boldsymbol{q} \tag{1.3}$$

1.4 Continuity

This is the expression of conservation of mass of the fluid. Consider an arbitrary fixed volume V enclosed in a surface S. Let $\boldsymbol{n}$ be the unit normal vector, outwardly drawn. The mass of fluid in V is

$$\int_V \rho\ \mathrm{d}\tau = m$$

say. If m increases, it means that fluid has entered through S:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\int_S \rho \boldsymbol{n} \cdot \boldsymbol{q} \, \mathrm{d}\sigma$$

and by the *divergence theorem*, this surface integral is equal to

$$-\int_V \mathrm{div}\,(\rho \boldsymbol{q}) \, \mathrm{d}\tau$$

But also, V being a fixed volume, we can write

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}\tau$$

Hence, for arbitrary choice of V, we have

$$\int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}\tau = -\int_V \mathrm{div}\,(\rho \boldsymbol{q}) \, \mathrm{d}\tau$$

The only way that these integrals can be equal for any and every choice of V is that their integrands be equal; thus, we obtain the general equation of continuity:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \boldsymbol{q}) = 0 \tag{1.4}$$

or noting that $\mathrm{div}\,(\rho \boldsymbol{q}) = \boldsymbol{q} \cdot \mathrm{grad}\,\rho + \rho\,\mathrm{div}\,\boldsymbol{q}$,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\mathrm{div}\,\boldsymbol{q} = 0 \tag{1.5}$$

There are two important special cases:

1. *Steady motion.* Because, for steady motion, $\partial \rho / \partial t = 0$ all partial derivatives $\partial(\ )/\partial t$ vanish, Eq. (1.4) becomes

$$\mathrm{div}\,(\rho \boldsymbol{q}) = 0 \tag{1.6}$$

2. *Incompressible flow.* If the density of every particle is constant, $\rho =$ constant, and Eq. (1.5) gives us

$$\mathrm{div}\,\boldsymbol{q} = 0 \tag{1.7}$$

Note that this is correct irrespective of whether the fluid is steady or not, and moreover it applies to the case of an inhomogeneous fluid, such as a stratified liquid, in which ρ varies throughout the fluid, provided each particle is incompressible.

Sample Problem 1.1

Consider a one-dimensional flow wherein each particle begins motion in the x direction at time $t = 0$ with velocity equal to K times its distance x from the origin and continues at that speed.

1. Write the Eulerian and Lagrangian formulas describing the motion.
2. How must the density vary in this flow in order that the equation of continuity will be satisfied? You may assume that at $t = 0$ the density has a constant value throughout the fluid given by $\rho = \rho_0$. Hint: Write out the equation of continuity [Eq. (1.5)] in terms of the Lagrangian variables. After integrating to find $\rho(a, t)$, check the Eulerian formulas as well.

Solution:

1. Lagrangian: $u = Ka = \partial x / \partial t$; $x(a, t) = a + Kat$.
   Eulerian: $u(x, t) = Ka(x, t) = Kx / (1 + Kt)$.
2. a. Continuity (in Eulerian variables): $\mathrm{D}\rho / \mathrm{D}t + \rho(\partial u / \partial x) = 0$.
   b. To rewrite this with Lagrangian variables:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} \text{ is exactly } \frac{\partial \rho(a,t)}{\partial t}$$

$$\frac{\partial u}{\partial x} = \lim \left(\frac{\Delta u}{\Delta x} \right)_{t=\text{const}}$$

which is

$$\frac{[\partial u(a,t)/\partial a]\Delta a}{[\partial x(a,t)/\partial a]\Delta a} = \frac{u_a}{x_a}$$

so that the equation becomes

$$\frac{\partial \rho}{\partial t} + \rho \frac{u_a}{x_a} = 0$$

and from Solution 1

$$\frac{\partial \rho}{\partial t} + \rho \frac{K}{1+Kt} = 0 \quad \text{or} \quad \frac{\partial \rho / \partial t}{\rho} = -\frac{K}{1+Kt}$$

Integrating with respect to t

$$\ell n\, \rho = -\ell n(1 + Kt) + f(a)$$

$$\rho = \frac{\text{function of } a}{1 + Kt}$$

But if $\rho = \rho_0 =$ constant at $t = 0$, the "function of a" is only the constant ρ_0:

$$\frac{\rho}{\rho_0} = \frac{1}{1+Kt}$$

1.5 Plane Flow

The term *plane flow* is used to describe situations where the flow is identical in every plane parallel to a given plane, and the velocity vector is everywhere parallel to this plane. Here this plane is called the xy plane; then, all of the fluid properties are functions of x, y, and t. When we draw a diagram of such a flow, we must remember that a point really denotes a line, a curve denotes a cylindrical surface, and so on.

For such flows the customary vector operators take on especially simple forms; for example in Cartesian coordinates, for any scalar Θ or any vector v,

$$\text{grad}\,\Theta = \boldsymbol{i}\frac{\partial\Theta}{\partial x} + \boldsymbol{j}\frac{\partial\Theta}{\partial y} \qquad \text{div}\,\boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$$

$$\text{curl}\,\boldsymbol{v} = \boldsymbol{k}\left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}\right) \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.8}$$

1.6 Axisymmetric Flow

We are sometimes concerned with another sort of "two-dimensional" flow, namely, that which is identical in every plane that contains a certain axis—that is, in every "meridional plane"—and the velocity vector lies

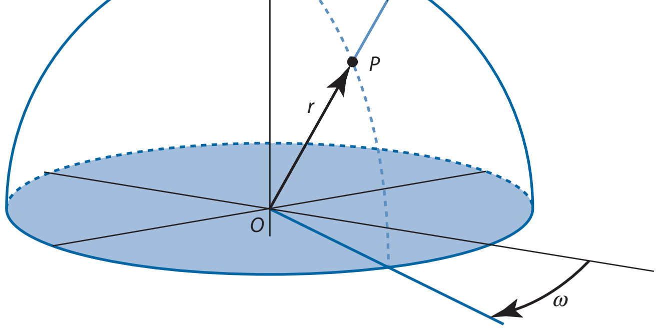


Fig. 1.1 Spherical coordinates r, θ, ω.

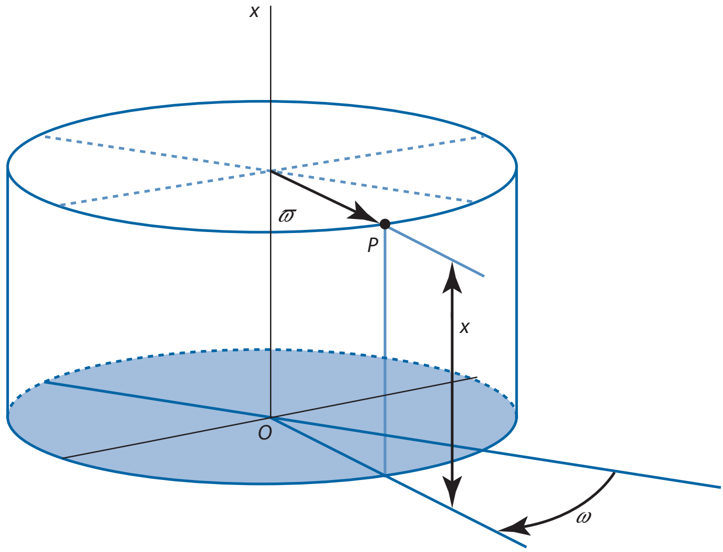


Fig. 1.2 Cylindrical coordinates x, ϖ, ω.

everywhere in these planes. To describe this more adequately, let us define *spherical* and *cylindrical coordinates*.

Figures 1.1 and 1.2 show how a given point P is located in spherical and cylindrical coordinates, respectively. Now if the flow is the same in all planes containing the x axis, that is, if the fluid properties are described by functions of r, θ or x, ϖ, we say that there is *axial symmetry*.

Here, the useful vector operators can best be worked out from the general-coordinate formulas (see appendix); again they will be simplified by virtue of $\partial(\ )/\partial\omega \equiv 0$.

1.7 Stream Function for Plane Flow

For incompressible flow, whether steady or unsteady, the equation of continuity becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.9}$$

This is satisfied *identically* if there is a function $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{1.10}$$

We can show that there is always such a function, and we can demonstrate its physical significance. It is called the *stream function* and was used as early as 1781 by Lagrange.

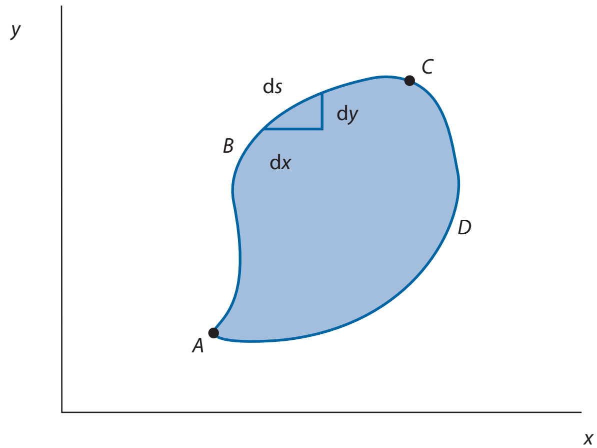


Fig. 1.3 Incompressible flow across line segments.

Consider the quantity of flow across *ABC* in Fig. 1.3. The flow across an element d*s* from left to right is $u\,dy - v\,dx$. Thus the flow across *ABC* is given by the line integral

$$\int_{ABC} (u\,dy - v\,dx)$$

But for homogeneous incompressible flow, the flow across *ADC* must be the same. In fact, the line integral $\int (u\,dy - v\,dx)$ must be independent of the path. Thus, if we choose a fixed starting point, say *A*, and for any path $c(x, y)$, we have

$$\int_{A}^{C(x,y)} (u\,dy - v\,dx) = \text{a function of } x, y$$
$$= \psi(x, y), \text{ say}$$

and its differential is

$$d\psi = u\,dy - v\,dx$$

Therefore, $u = \partial\psi / \partial y$ and $v = -\partial\psi / \partial x$, and the quantity of flow across any line drawn to a fixed point is exactly the function that we seek, that is, the stream function for plane incompressible flow.

The differential equation of a streamline is now seen to be $d\psi = 0$, according to this definition. Compare this with Eq. (1.1).

The great value of the stream function is that it replaces two velocity functions $u(x, y, t)$ and $v(x, y, t)$. Or, in other words, the scalar function ψ replaces the vector function $\boldsymbol{q}$. An analogous stream function can be defined for the case of *plane*, *steady*, and *compressible* flow, as shown in Problem 1.3.

1.8 Stream Function for Axisymmetric Flow

Sir G. G. Stokes (1842) pointed out that an analogous stream function can be defined for axisymmetric flow. Suppose the fluid is incompressible; the equation of continuity in terms of cylindrical coordinates is

$$\operatorname{div} \boldsymbol{q} = \frac{\partial v_x}{\partial x} + \frac{1}{\varpi}\frac{\partial \varpi v_\varpi}{\partial \varpi} = 0 \tag{1.11}$$

where the components of $\boldsymbol{q}$ are v_x and v_ϖ.

In this case, let us calculate the quantity of flow from left to right across the surface formed by rotating the curve OAP in Fig. 1.4 around the x axis:

$$\int_{OAP} (v_x 2\pi\varpi \, \mathrm{d}\varpi - v_\varpi 2\pi\varpi \, \mathrm{d}x)$$

Now, by virtue of the axial symmetry, no fluid goes in or out through the meridional plane nor crosses the x axis (Fig. 1.4). Thus, the flow through every surface OAP is the same, and

$$\int_{OAP} (v_x \varpi \, \mathrm{d}\varpi - v_\varpi \varpi \, \mathrm{d}x) = \int_{O}^{P(x,\varpi)} (v_x \varpi \, \mathrm{d}\varpi - v_\varpi \varpi \, \mathrm{d}x)$$

$$= \text{function of } x, \varpi = \psi(x, \varpi), \text{ say}$$

Then, $\mathrm{d}\psi = v_x \varpi \, \mathrm{d}\varpi - v_\varpi \varpi \, \mathrm{d}x$, which implies that

$$v_x = \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} \quad \text{and} \quad v_\varpi = -\frac{1}{\varpi}\frac{\partial \psi}{\partial x} \tag{1.12}$$

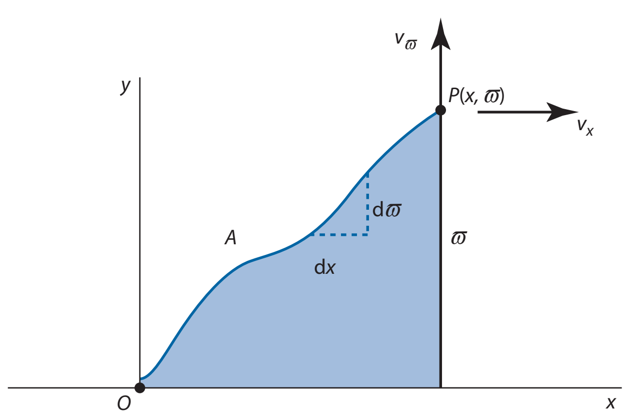


Fig. 1.4 Representation of axisymmetric flow across surface segments.

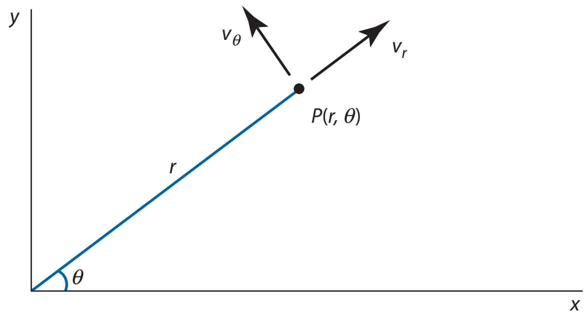


Fig. 1.5 Representation of axisymmetric flow with spherical coordinates.

Inspection will show that these relations satisfy Eq. (1.11) identically. Thus, ψ is a stream function.

The student can carry out an analogous calculation for the case when spherical coordinates are used. The results are

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{1.13}$$

where v_r and v_θ are the velocity components in the directions of increasing r and θ, respectively (Fig. 1.5). This calculation should include the working out of div $\boldsymbol{q}$ for this case and the verification that it vanishes identically as the result of Eq. (1.13).

Just as in the case of the plane stream function, Stokes' stream function can be defined for *steady*, *axisymmetric*, and *compressible* flow, by the method used in Problem 1.3.

1.9 Vorticity

The vector function Ω = curl $\boldsymbol{q}$, where $\boldsymbol{q}(x, y, z, t)$ is the velocity of the fluid, is called the *vorticity*. Its components are occasionally represented by the symbols ξ, η, and ζ, namely, in rectangular Cartesian coordinates

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

To give a physical picture of the meaning of vorticity, it is often said that curl $\boldsymbol{q}$ is twice the angular-velocity vector of the fluid particle. Or, because the particle is being deformed continually, perhaps we should say the *average* angular velocity at a point. This is easy to see

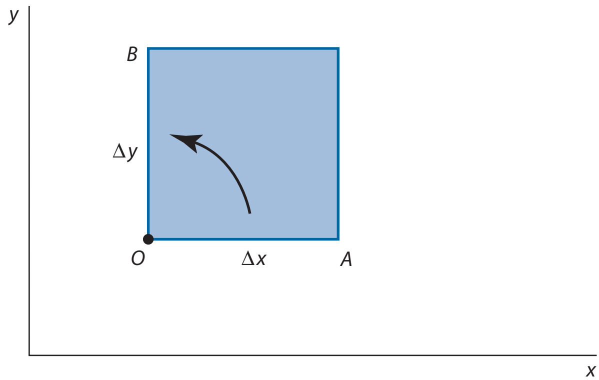


Fig. 1.6 Representation to calculate angular velocity.

in the case of plane flow. In this case $|\mathrm{curl}\,\boldsymbol{q}| = \zeta$. The angular velocity of side OA (Fig. 1.6) is

$$\frac{v(A) - v(O)}{\Delta x} = \frac{v + (\partial v / \partial x)\Delta x - v}{\Delta x}$$
$$= \frac{\partial v}{\partial x}$$

The angular velocity of OB is, similarly,

$$-\frac{u(B) - u(O)}{\Delta y} = -\frac{\partial u}{\partial y}$$

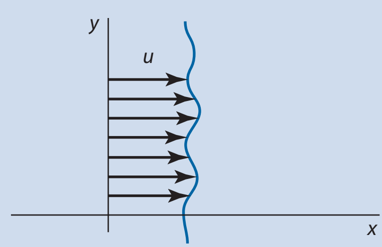
Thus, the average angular velocity ω, say, at the point O, is $\frac{1}{2}\zeta$.

The three-dimensional case is a little harder to visualize, to be sure, but yields an analogous result.

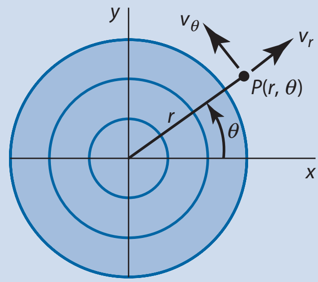
Example 1.1 Plane Flow in Parallel Planes

This is an exceedingly simple flow, described by $v = w = 0$; $u = u(x, y)$. The vorticity is just

$$\zeta = \frac{\partial u}{\partial y}$$



Example 1.2 Plane Flow in Concentric Circles



In plane polar coordinates,

$$\theta = \tan^{-1}\frac{y}{x}, \quad r = \sqrt{x^2 + y^2}$$
$$v = v_r \sin\theta + v_\theta \cos\theta$$
$$u = v_r \cos\theta - v_\theta \sin\theta$$

Here, we need an expression for ζ in these coordinates. Instead of using our general vector formulas, let us try the alternative method of direct transformation of variables.

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

By differentiating and substituting, one finds

$$\zeta = \frac{\partial v_\theta}{\partial r} - \frac{\partial v_r}{r\partial\theta} + \frac{v_\theta}{r} = \frac{1}{r}\left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta}\right)$$

Hence, in the present example of flow in concentric circles,

$$v_r = w = 0 \quad v_\theta = v_\theta(r, \theta)$$

$$\zeta = \frac{1}{r}\frac{\partial r v_\theta}{\partial r}$$

We note that the vorticity can be zero, even for this kind of flow, if v_θ varies as $1/r$. (We make an exception of the point $r = 0$, where $\zeta = \infty$. This is a *singularity* of the flow.) We also see that for solid-body rotation ($v_\theta = \omega r$) our previous result is confirmed, namely, $\zeta = 2\omega$.

We classify flows as *irrotational* and *rotational* depending on whether curl $\boldsymbol{q}$ is zero or not everywhere. The irrotational type will be found to be rather common, for sound physical reasons, and will occupy a considerable portion of our time.

1.10 Circulation

The line integral

$$\Gamma = \oint_C \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r} \tag{1.14}$$

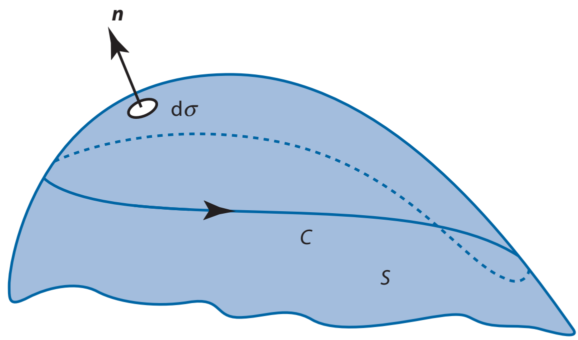


Fig. 1.7 Definition of symbols for calculating circulation and vorticity.

where $\boldsymbol{q}$ is the fluid velocity, taken about any closed curve C in space, is called the *circulation* about the contour C.

By Stokes' theorem, it is clear that the circulation and vorticity are related, for

$$\Gamma = \oint_C \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r} = \int_S \boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{q} \, \mathrm{d}\sigma$$

where the symbols are as defined in Fig. 1.7. The transformation is only permissible, of course, when $\boldsymbol{q}$ is finite and has continuous partial derivatives at each point of S; we may encounter some cases where certain singularities have to be excluded from such processes.

Obviously, if the flow is wholly irrotational, Γ will be zero for every contour. In any case, Γ is zero if C encloses only irrotational portions of the flow.

1.11 Velocity Potential

In regions where the flow is *irrotational*, the line integral $\int \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r}$ is independent of the path. For example, consider any two paths connecting A and B, as in Fig. 1.8. The line integral around the entire, closed path is the circulation and is zero because the flow is irrotational:

$$\Gamma = 0 = \int_{ADB} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r} + \int_{BCA} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r}$$

Thus,

$$\int_{ADB} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r} = \int_{ACB} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r}$$

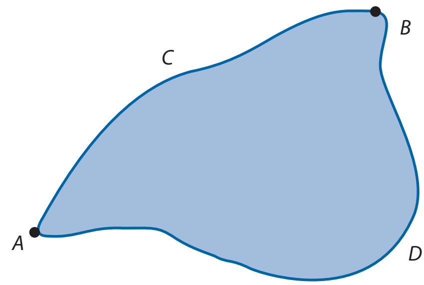


Fig. 1.8 Alternate paths from A to B.

and similarly for all other paths from A to B. Therefore, choosing A as a fixed point,

$$\int_A^{B(x,y,z)} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r} = \phi(x, y, z)$$

and

$$\mathrm{d}\phi = \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r}$$

Now we see that $\mathrm{d}\boldsymbol{r} \cdot \operatorname{grad} \phi = \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r}$, for arbitrary choice of $\mathrm{d}\boldsymbol{r}$. This means that

$$\boldsymbol{q} = \operatorname{grad} \phi \tag{1.15}$$

By retracing these steps, one will see immediately that this result has nothing to do with the physical meaning of $\boldsymbol{q}$. That is, the result, Eq. (1.15), will follow for every vector function $\boldsymbol{q}$ whose curl is zero.

Moreover, the condition curl $\boldsymbol{q} = 0$ is *necessary*, as well as *sufficient*, for the result $\boldsymbol{q} = \operatorname{grad} \phi$ because the curl of every gradient is identically zero.

In the case considered here, where $\boldsymbol{q}(x, y, z, t)$ is the fluid velocity, $\phi(x, y, z, t)$ is called the *velocity potential*. The surfaces $\phi = \text{constant}$ are called *equipotential* surfaces; thus, $\boldsymbol{q}$ is the vector perpendicular to these surfaces at every point, and its magnitude is that of the derivative $\partial\phi / \partial n$ in the direction normal to the surface. (The student should verify these statements, using the relation $\mathrm{d}\phi = \mathrm{d}\boldsymbol{r} \cdot \operatorname{grad} \phi$.)

Example 1.3 Irrotational Plane Parallel Flow

If this type of flow is to be irrotational, $\partial u / \partial y$ must be zero and $u = u(x)$. Then Eq. (1.15) reduces to $u = \mathrm{d}\phi / \mathrm{d}x$ and $\phi = \phi(x)$.

Example 1.4 Irrotational Plane Concentric Flow

We have seen in Example 1.2 that this is irrotational if $v_\theta = f(\theta)/r$, where $f(\theta)$ is any function of θ. In plane polar coordinates,

$$\text{grad}\,\phi = \boldsymbol{i}_r \frac{\partial \phi}{\partial r} + \boldsymbol{i}_\theta \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

hence, in this case

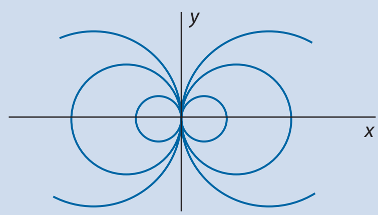
$$v_r = 0 = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{f(\theta)}{r} = \frac{\mathrm{d}\phi}{r\,\mathrm{d}\theta}, \quad \phi = \int f(\theta)\,\mathrm{d}\theta = \text{function of } \theta$$

Example 1.5 Plane Flow by Differentiation

Consider the plane flow given by

$$\phi = \frac{x}{(x^2+y^2)}$$

The equipotential lines (surfaces) are as shown in figure. By differentiation of ψ,



$$u = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}, \quad v = -\frac{2xy}{\left(x^2+y^2\right)^2}$$

The student should verify again by differentiation that this flow is irrotational.

Sample Problem 1.2

1. Show that the flow in Example 1.5 satisfies the equation of continuity for incompressible flow. Determine ψ. Sketch some streamlines.
2. Find the relations between v_r, v_θ, and $\psi(r,\theta)$ for plane incompressible flow.

Solution:

1. $$\psi = \int u\,\mathrm{d}y + f(x) = -\frac{y}{2(x^2+y^2)} + \frac{1}{2x}\tan^{-1}\frac{y}{x}\frac{y}{2(x^2+y^2)}$$

$$-\frac{x^2}{2x^3}\tan^{-1}\frac{y}{x} + f(x) = \frac{-y}{(x^2-y^2)} + f(x)$$

and

$$\psi = -\int v\,\mathrm{d}x + g(y) = -\frac{\cancel{2}y}{\cancel{2}(x^2 + y^2)} + g(y)$$

Comparing these forms, we conclude

$$\psi = -\frac{y}{(x^2 + y^2)} + \text{const}$$

To sketch streamlines, put $\psi =$ various constants; the result should look like the figure given in Example 1.5.

2. Flow across element in the figure below $= v_r r\,\mathrm{d}\theta - v_\theta\,\mathrm{d}r$. Putting this equal to $\mathrm{d}\psi$, we see that

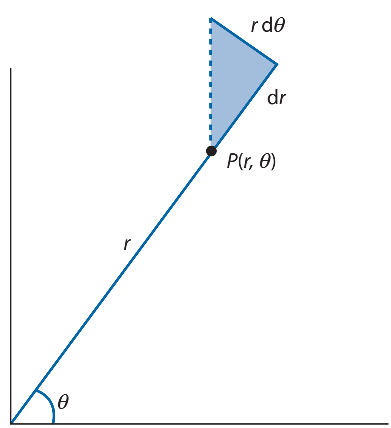
$$\frac{\partial \psi}{\partial \theta} = r v_r \quad \text{and} \quad \frac{\partial \psi}{\partial r} = -v_\theta$$

Alternatively, we can get this by direct calculation:

$$\left.\begin{aligned} v_r &= u\cos\theta + v\sin\theta \\ v_\theta &= v\cos\theta - u\sin\theta \end{aligned}\right\}$$

Thus,

$$\begin{cases} u = \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \psi}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial \psi}{\partial \theta}\dfrac{\partial \theta}{\partial y} \\[2ex] v = -\dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \psi}{\partial r}\dfrac{\partial r}{\partial x} - \dfrac{\partial \psi}{\partial \theta}\dfrac{\partial \theta}{\partial x} \end{cases}$$



1.12 Kinematic Properties of Rotational Flows

Let us adopt the symbol $\mathbf{\Omega}$ to represent the vorticity. Because the divergence of every curl is zero,

$$\operatorname{div} \mathbf{\Omega} = 0$$

The Cartesian components of $\mathbf{\Omega}$ are ξ, η, and ζ; we call the space curves that are parallel to $\mathbf{\Omega}$ *vortex lines*. Their differential equations are $\mathrm{d}x / \xi = \mathrm{d}y / \eta = \mathrm{d}z / \zeta$. The condition $\operatorname{div} \mathbf{\Omega} = 0$ can be thought of as meaning that vortex lines do not begin nor end in the fluid. We call a tube whose walls are made up of vortex lines a *vortex tube*. (The analogous tube made up of streamlines would be called a *stream tube*.)

The circulation about a vortex tube is clearly the circulation about a contour encircling the tube; it is constant along the length of the tube. Consider, for example, the contour in Fig. 1.9. For the entire contour shown,

$$\oint \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r} = \oint_{C_1} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r} - \oint_{C_2} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r}$$

$$= \int_S \boldsymbol{n} \cdot \mathbf{\Omega} \, \mathrm{d}\sigma$$

by Stokes' theorem. But, by definition, $\boldsymbol{n} \cdot \mathbf{\Omega} = 0$ everywhere on S; thus,

$$\oint_{C_1} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r} = \oint_{C_2} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{r} \quad \text{or} \quad \Gamma_1 = \Gamma_2$$

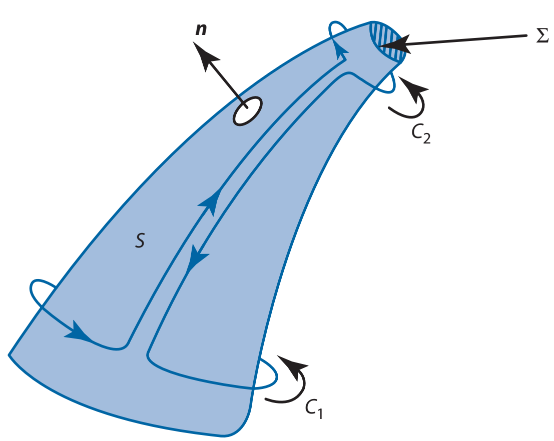


Fig. 1.9 Circulation about a vortex tube.

On the other hand, consider the application of Stokes' theorem to a cross-section of the tube:

$$\int_{\Sigma} \boldsymbol{n} \cdot \boldsymbol{\Omega} \, d\sigma = \Gamma = \text{constant along tube}$$

Thus, the average vorticity in the cross-section varies inversely as the cross-sectional area. The vorticity becomes very small if the tube spreads out. Suppose, however, that the tube is necked down; this makes the vorticity large. In the extreme case, we imagine that the tube is contracted to a line. Then the vorticity at this line becomes infinite, but the circulation is still the same, Γ. This is called a *vortex filament*, or briefly a "vortex," and Γ is its *strength*. It is a kind of mathematical approximation to the case where all of the vorticity is confined to a tube of relatively small cross-section, as often occurs in nature—for example, in a tornado. Outside the core of a tornado, the air is in practically irrotational motion. The irrotational concentric flow considered in Secs. 1.10 and 1.11 represents the case of a long, straight vortex filament; the singularity at the center is the filament, and there the vorticity is infinite, as predicted.

Clearly, a vortex tube or filament, consisting of vortex lines, cannot begin nor end in the fluid. But it can double back on itself in a ring or terminate at a boundary of the fluid.

1.13 Velocity Field of a Vortex in Incompressible Flow

To determine the velocity field of a vortex in an incompressible fluid, we begin with a more general case of rotational flow and later specialize for a vortex filament. Consider incompressible rotational flow in general. The velocity potential does not exist, but, as will be seen later, it may be possible to determine a *vector-potential function* $\boldsymbol{A}(x, y, z, t)$, such that

$$\boldsymbol{q} = \text{curl } \boldsymbol{A} \tag{1.16}$$

This form has the advantage of satisfying the incompressible equation of continuity identically, for the divergence of every curl is zero. Thus, $\boldsymbol{A}$ is related to a stream function; we shall explore this relationship in Problem 1.3.

Now we shall try to determine $\boldsymbol{A}(x, y, z, t)$ for any given distribution of vorticity $\boldsymbol{\Omega}(x, y, z, t)$, for then we shall have $\boldsymbol{q}(x, y, z, t)$ in terms of the vorticity—a sort of inversion of the relation $\boldsymbol{\Omega} = \text{curl } \boldsymbol{q}$. The relation between $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ is

$$\boldsymbol{\Omega} = \text{curl}^2 \boldsymbol{A} = \text{grad div } \boldsymbol{A} - \nabla^2 \boldsymbol{A} \tag{1.17}$$

This is a differential equation for $\boldsymbol{A}$, for given $\boldsymbol{\Omega}$, and our aim is to obtain a particular integral. We can now assume that div $\boldsymbol{A}=0$; this does not sacrifice any generality, for we are trying to calculate $\boldsymbol{A}$ for given $\boldsymbol{\Omega}$. If we can succeed in calculating it with this restriction, the problem will be solved. However, we shall have to check our result to verify that the divergence of A vanishes. With this assumption,

$$\boldsymbol{\Omega}=-\nabla^2 A \tag{1.18}$$

Equation (1.18) is Poisson's equation, and its solution is

$$\boldsymbol{A}(x, y, z, t)=\frac{1}{4\pi}\int\frac{\boldsymbol{\Omega}}{r}\,\mathrm{d}\tau \tag{1.19}$$

where r is the distance from the element $\mathrm{d}\tau$ to the point x, y, z, and the integration is carried throughout the entire fluid.

To prove that Eq. (1.19) is a solution of Eq. (1.18), consider the integral of $\nabla^2\boldsymbol{A}$, where $\boldsymbol{A}$ is given by Eq. (1.19), through an arbitrary volume V enclosed in a surface S:

$$\int_V \nabla^2\boldsymbol{A}\,\mathrm{d}\tau' = \int_S \boldsymbol{n}\cdot\nabla\boldsymbol{A}\,\mathrm{d}\sigma = \int_S \boldsymbol{n}\cdot\nabla\left(\frac{1}{4\pi}\int\frac{\boldsymbol{\Omega}}{r}\,\mathrm{d}\tau\right)\mathrm{d}\sigma$$

Because $\boldsymbol{\Omega}$ is the value at the element $\mathrm{d}\tau$ and is therefore independent of the integration over S, the contribution of the element $\mathrm{d}\tau$ to this integral is

$$\frac{1}{4\pi}\boldsymbol{\Omega}\,\mathrm{d}\tau\int_S \boldsymbol{n}\cdot\nabla\frac{1}{r}\,\mathrm{d}\sigma = -\frac{1}{4\pi}\boldsymbol{\Omega}\,\mathrm{d}\tau\int_S\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{r^3}\,\mathrm{d}\sigma$$

The last integral is either zero or 4π, depending on whether $\mathrm{d}\tau$ is inside or outside S. (This is sometimes called "Gauss' Theorem." See Wills, *Vector and Tensor Analysis*, p. 101, or any book on vector analysis.) Thus, the contribution of $\mathrm{d}\tau$ is zero if $\mathrm{d}\tau$ is outside V and is $-\boldsymbol{\Omega}\,\mathrm{d}\tau$ if $\mathrm{d}\tau$ is within V. Consequently, when the integration is taken throughout the fluid, the result is

$$\int_V \nabla^2\boldsymbol{A}\,\mathrm{d}\tau' = -\int_V \boldsymbol{\Omega}\,\mathrm{d}\tau$$

and as V is arbitrary, the integrands must be equal.

Equation (1.19) makes $\boldsymbol{A}$ a solution of Eq. (1.18), but not a solution of the differential equation we are trying to solve, Eq. (1.17), unless div $\boldsymbol{A}$ is zero, as has already been mentioned. Therefore, take the divergence of Eq. (1.19):

$$\operatorname{div} \boldsymbol{A} = \frac{1}{4\pi} \int \boldsymbol{\Omega} \cdot \nabla \frac{1}{r} \, \mathrm{d}\tau \tag{1.20}$$

But, using an obvious notation,

$$r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

we see that $\nabla(1/r) = -\nabla'(1/r)$, where ∇ denotes $\boldsymbol{i}\partial/\partial x + \cdots$ and ∇' denotes $\boldsymbol{i}\partial/\partial x' + \cdots$. Moreover,

$$\boldsymbol{\Omega} \cdot \nabla' \frac{1}{r} = \nabla' \cdot \frac{\boldsymbol{\Omega}}{r} - \frac{1}{r} \nabla' \cdot \boldsymbol{\Omega} = \nabla' \cdot \frac{\boldsymbol{\Omega}}{r}$$

Thus, the integral in Eq. (1.20) can be changed to a surface integral of $\boldsymbol{n} \cdot \boldsymbol{\Omega}/r$, by the divergence theorem. The surface it is integrated over is the surface enclosing all of the areas of rotational flow. But this will be the outer walls of vortex tubes, and on these $\boldsymbol{n} \cdot \boldsymbol{\Omega} = 0$. Hence, the divergence is zero as required [1].

1.14 Velocity Induced by a Vortex Filament

To illustrate the use of this result, Eq. (1.19), let us calculate the velocity in the field of a vortex filament. That is, let us assume that the vorticity is concentrated in a tube of very small cross-sectional area σ and circulation Γ as shown in Fig. 1.10. Then

$$\boldsymbol{A} = \frac{1}{4\pi} \int \frac{\boldsymbol{\Omega}}{r} \sigma \, \mathrm{d}\ell = \frac{1}{4\pi} \int \frac{\Gamma \boldsymbol{\ell}_1}{r} \, \mathrm{d}\ell = \frac{\Gamma}{4\pi} \int \frac{\mathrm{d}\boldsymbol{\ell}}{r}$$

where $\mathrm{d}\ell$ is an element of length along the filament, $\boldsymbol{\ell}_1$ is a unit vector in the direction of the filament, and $\mathrm{d}\boldsymbol{\ell}$ denotes $\boldsymbol{\ell}_1 \, \mathrm{d}\ell$ (Fig. 1.10).

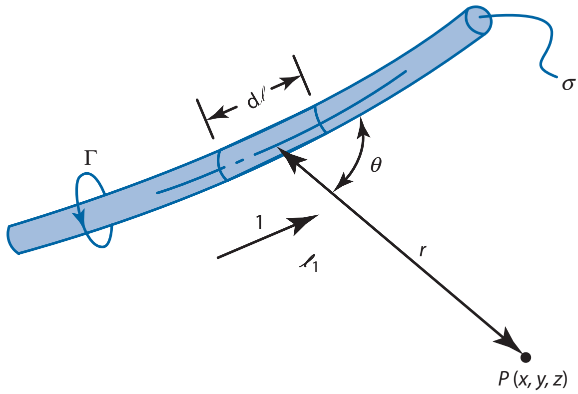


Fig. 1.10 Vorticity in a tube with very small cross-sectional area.

The velocity at $P(x, y, z)$, due to the particular element $\mathrm{d}\boldsymbol{\ell}$, is

$$\mathrm{d}\boldsymbol{q} = \frac{\Gamma}{4\pi}\operatorname{curl}\left(\frac{\mathrm{d}\boldsymbol{\ell}}{r}\right) = \frac{\Gamma}{4\pi}\operatorname{grad}\left(\frac{1}{r}\right)\times \mathrm{d}\boldsymbol{\ell} = -\frac{\Gamma}{4\pi}\frac{\boldsymbol{r}\times \mathrm{d}\boldsymbol{\ell}}{r^3} \tag{1.21}$$

Or, in other words, the velocity due to $\mathrm{d}\boldsymbol{\ell}$ is directed normal to the plane of $\mathrm{d}\boldsymbol{\ell}$ and $\boldsymbol{r}$, and its magnitude is

$$\mathrm{d}q = \frac{\Gamma}{4\pi r^2}\sin\theta\, \mathrm{d}\ell \tag{1.22}$$

where θ is defined in Fig. 1.10.

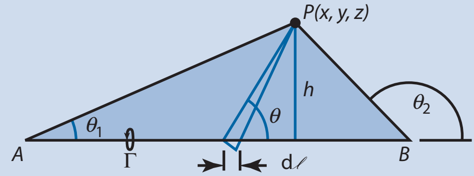
This is called the *formula of Biot and Savart*, by analogy with the expression for the magnetic flux due to a conductor carrying a current. This analogy also leads to the name *induced velocity* for $\mathrm{d}\boldsymbol{q}$.

The student may wonder about the physical mechanism by which a vortex "induces" velocities at a distance. In this chapter, of course, we are considering only kinematics, so we are not prepared to discuss any such mechanism (for example, viscosity, which might impart motion to the air surrounding a tornado). Actually, the Biot–Savart formula results essentially from the assumption of irrotational flow outside the tube and does not specify anything about the mechanism that produced this situation.

Example 1.6 Velocity Due to Rectilinear Vortex Segment

Consider the vortex segment *AB*. The velocity induced at *P* is normal to the plane of *AB* and *P* and has the magnitude

$$\Delta q = \frac{\Gamma}{4\pi}\int_A^B \frac{\sin\theta \, d\ell}{r^2}$$



Put $r = h/\sin\theta$, $d\ell = r\, d\theta/\sin\theta$, etc. Then

$$\Delta q = \frac{\Gamma}{4\pi}\int_{\theta_1}^{\theta_2} \sin\theta \frac{d\theta}{h} = \frac{\Gamma}{4\pi h}(\cos\theta_1 - \cos\theta_2) \tag{1.23}$$

Reference

[1] Lamb, H., *Hydrodynamics*, Cambridge University Press, London, 1945.

Problems

1.1 a. Consider a one-dimensional case where Lagrangian equations are used; $x = x(a,t), \rho = \rho(a,t)$, etc. Calculate the rate of change of ρ at a fixed point.

b. Next, consider a three-dimensional case, and show that it would be possible to derive an analogous formula.

1.2 A test missile carrying an ideal thermometer (no corrections or lag) is shot through the air along the path $\boldsymbol{r} = \boldsymbol{R}(t)$. If the temperature of the atmosphere is $\theta(x,y,z,t)$, what is the vector formula for $d\theta/dt$ recorded by the thermometer?

1.3 a. Discuss the relationship between the vector potential and the stream function for plane flow.

b. Show that a stream function can be defined for steady plane compressible flow. What is its physical meaning?

1.4 a. Find the relations between v_r, v_θ, and $\phi(r,\theta)$ for plane irrotational flow.

b. Find the relations between v_x, v_ϖ, v_ω, and $\phi(x,\varpi,\omega)$.

c. Find the relations between v_r, v_θ, v_ω, and $\phi(r,\theta,\omega)$.

1.5 What is the differential equation satisfied by Stokes' stream function if the flow is irrotational? Use cylindrical coordinates.

1.6 Consider again the one-dimensional flow of Sample Problem 1.1, wherein each particle begins motion in the x direction at time $t = 0$, with velocity K times its distance x from the origin, and continues at that speed.

a. Calculate the acceleration of a particle in this flow, using both Lagrangian and Eulerian formulas.

b. Calculate the time rate of change of velocity at a point for this flow, using both Lagrangian and Eulerian formulas.

1.7 Consider a vortex, in an incompressible fluid, in the shape of an infinitely long helix, whose equations are $x = k\theta$, $y = a\cos\theta$, and $z = a\sin\theta$. Calculate the axial velocity induced at a point on the axis.

Dynamics of Frictionless Fluids

- Frictionless fluids
- Eulerian equations of motion
- Dynamical equations
- Properties of barotropic motion
- Properties of irrotational and incompressible flow
- Physical interpretation of impulsive pressure

2.1 Frictionless Fluids

It has already been mentioned that the distinguishing feature of fluids is the absence of shearing stresses at rest. In many fluids, such as water and air, the shearing stresses are small relative to the normal stresses (pressures) even in motion. For some applications, it is permissible to make the approximation that they do not exist at all—then the fluid is called *frictionless* or *inviscid*, or it is described as an "ideal fluid."

The theory of such inviscid fluids has been developed in detail, especially for incompressible fluids, and is found to yield useful approximations to the flow characteristics of real fluids in many practical cases. Nevertheless, such an extreme simplification must be regarded with some suspicion, and we must be careful not to extend this approximate theory beyond its limits. The limitations of ideal-fluid theory in representing real-fluid flows can best be stated by mentioning some results of the theory of fluids of small (but not zero) viscosity:

1. The shearing stress in a viscous fluid is proportional to velocity derivatives such as $\partial u / \partial y$ and $\partial v / \partial x$. Hence, the ideal-fluid approximation breaks down where there are large velocity gradients.
2. It is found that the flow of a fluid of small viscosity around a "streamlined" body is practically the same as the inviscid-fluid flow, except in a thin layer near the surface of the body. This is

Prandtl's *boundary-layer* approximation. Its limitations are beyond the scope of this book; it must suffice to say that it applies to many practical cases in aerodynamics, both at low and high speeds. Thus, the ideal-fluid results are expected to be valid outside the boundary layer and other areas of large velocity gradients.

2.2 Pressure

In a frictionless fluid, the only stresses between neighboring fluid particles are the normal ones or pressures. In the absence of oblique stresses, the normal stresses must be equal in all directions, and the pressure is a scalar magnitude at any point of the flow (for proof, see [1], p. 2).

2.3 Eulerian Equations of Motion

Adopting again the Eulerian system of equations, we proceed to set up the differential equations of motion of a frictionless fluid. According to Newton's laws, the rate of change of momentum of a body equals the resultant force acting on the body. Let us consider a small element of fluid in x-, y-, and z-coordinate system (Fig. 2.1). The forces acting are 1) the pressure p and 2) possibly a certain "body force," that is, an external force such as gravity, represented by $\rho \boldsymbol{F}$ (ρ denotes the mass density, and $\boldsymbol{F}$ is the body force per unit mass). Consider first the x-momentum of the particle. The rate of change of this component Δm_1, say, is given by the "convective" type of derivative:

$$\frac{\mathrm{D}\,\Delta m_1}{\mathrm{D}t} = -\frac{\partial p}{\partial x}\Delta x \Delta y \Delta z + \rho \Delta x \Delta y \Delta z F_1$$

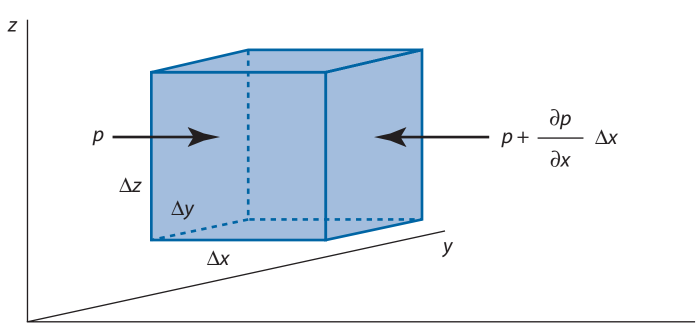


Fig. 2.1 Differential equations of motion of a frictionless fluid.

where $\Delta m_1 = \rho \Delta x \Delta y \Delta z u$. Also,

$$\frac{\mathrm{D}\Delta m_1}{\mathrm{D}t} = \rho \Delta x \Delta y \Delta z \frac{\mathrm{D}u}{\mathrm{D}t}$$

because the mass of the particle does not vary as it moves. Hence,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_1$$

or in vector form,

$$\frac{\mathrm{D}\boldsymbol{q}}{\mathrm{D}t} = -\frac{1}{\rho}\,\mathrm{grad}\; p + \boldsymbol{F} \tag{2.1}$$

Equation (2.1) gives us the Eulerian partial differential equations of motion for any frictionless fluid. The equations are nonlinear in general, as the term $\mathrm{D}\boldsymbol{q}/\mathrm{D}t$ contains $\boldsymbol{q}\cdot\nabla\boldsymbol{q}$.

It may be of interest to run through an alternative derivation of Eq. (2.1), which uses vector principles and is more rigorous. Consider any arbitrary sample of the fluid of volume V and surface S, and let S move with the flow so as to enclose always the same fluid particles. Then, if $\boldsymbol{m}$ denotes the momentum of this body of fluid,

$$\boldsymbol{m} = \int_V \rho\boldsymbol{q}\,\mathrm{d}\tau \quad \text{and} \quad \frac{\mathrm{D}\boldsymbol{m}}{\mathrm{D}t} = -\int_S p\boldsymbol{n}\,\mathrm{d}\sigma + \int_V \rho\boldsymbol{F}\,\mathrm{d}\tau$$

This equation does not use the the notation defined in Eq. (1.2), as $\boldsymbol{m}$ is a function of t only. We use it by analogy; no confusion seems likely to occur.

To calculate $\mathrm{D}\boldsymbol{m}/\mathrm{D}t$, we note that this derivative consists of two contributions: first, the part due to the rate of change of fluid momentum in the fixed volume $V(t_1)$ and second, the increment due to the distortion of S as it moves with the flow (Fig. 2.2):

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V (\rho\boldsymbol{q})\,\mathrm{d}\tau = \int_V \frac{\partial(\rho\boldsymbol{q})}{\partial t}\,\mathrm{d}\tau + \int_S \boldsymbol{n}\cdot\boldsymbol{q}(\rho\boldsymbol{q})\,\mathrm{d}\sigma$$

The second integral can be transformed by the divergence theorem for a dyadic product:

$$\begin{aligned}\int_S \boldsymbol{n}\cdot\boldsymbol{q}\rho\boldsymbol{q}\,\mathrm{d}\sigma &= \int_V \nabla\cdot(\boldsymbol{q}\rho\boldsymbol{q})\,\mathrm{d}\tau \\ &= \int_V \left\{\rho\boldsymbol{q}\,\mathrm{div}\,\boldsymbol{q} + \boldsymbol{q}\cdot\nabla(\rho\boldsymbol{q})\right\}\mathrm{d}\tau\end{aligned}$$

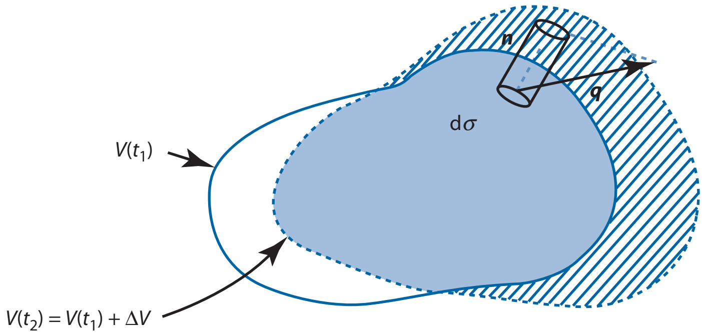


Fig. 2.2 Volume V at times t_1 and $t_1 + \Delta t$.

and finally, making use of the general equation of continuity, Eq. (1.5),

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\boldsymbol{q}\,\mathrm{d}\tau = \int_V \left\{\frac{\mathrm{D}\rho\boldsymbol{q}}{\mathrm{D}t} + \rho\boldsymbol{q}\,\mathrm{div}\,\boldsymbol{q}\right\}\mathrm{d}\tau = \int_V \rho\frac{\mathrm{D}\boldsymbol{q}}{\mathrm{D}t}\,\mathrm{d}\tau$$

Now equating the two expressions for $\mathrm{D}\boldsymbol{m}/\mathrm{D}t$ and transforming the pressure term to a volume integral, we have

$$-\int_V \mathrm{grad}\;p\,\mathrm{d}\tau + \int_V \rho\boldsymbol{F}\,\mathrm{d}\tau = \int_V \rho\frac{\mathrm{D}\boldsymbol{q}}{\mathrm{D}t}\,\mathrm{d}\tau$$

Because V is arbitrary, the integrands must be equal, and results in Eq. (2.1).

2.4 Other Forms of the Equations of Motion

The term $\boldsymbol{q}\cdot\nabla\boldsymbol{q}$ that occurs in the dynamical equations, Eq. (2.1), can be transformed by the vector identity

$$\mathrm{grad}\;q^2 = \mathrm{grad}\;\boldsymbol{q}\cdot\boldsymbol{q} = 2\boldsymbol{q}\cdot\nabla\boldsymbol{q} + 2\boldsymbol{q}\times\mathrm{curl}\;\boldsymbol{q}$$

Thus,

$$\boldsymbol{q}\cdot\nabla\boldsymbol{q} = \frac{1}{2}\mathrm{grad}\;q^2 - \boldsymbol{q}\times\boldsymbol{\Omega} \tag{2.2}$$

Also, it is often assumed that the body force $\boldsymbol{F}$ is derivable from a potential, that is, a conservative force, such as gravity. Then, we can write $\boldsymbol{F} = -\mathrm{grad}\;\mho$, and the equations appear in what is known as *Lamb's form*:

$$\frac{\partial\boldsymbol{q}}{\partial t} - \boldsymbol{q}\times\boldsymbol{\Omega} = -\mathrm{grad}\left(\frac{q^2}{2} + \mho\right) - \frac{1}{\rho}\mathrm{grad}\;p \tag{2.3}$$

This is about as far as we can go with complete generality, but the equations can be simplified still further if the fluid is *barotropic*; that is, if the density ρ depends on the pressure p only: $\rho = \rho(p)$. Examples of this state of affairs are compressible fluids flowing adiabatically ($p \sim \rho^{\kappa}$) (the adiabatic case is usually consistent with the inviscid approximation, for heat is transferred through the fluid by the same mechanism that produces viscosity) or isothermally ($p \sim \rho$) or, of course, incompressible fluids ($\rho = \text{constant}$). In these cases, the term $(1/\rho)\,\text{grad}\,p$ can also be expressed as the gradient of a function; consider

$$\mathrm{d}\boldsymbol{r}\cdot\frac{\text{grad}\ p}{\rho(p)} = \frac{\mathrm{d}p}{\rho(p)} = \mathrm{d}\int\frac{\mathrm{d}p}{\rho(p)} = \mathrm{d}\boldsymbol{r}\cdot\text{grad}\int\frac{\mathrm{d}p}{\rho(p)}$$

Because $\mathrm{d}\boldsymbol{r}$ is arbitrary,

$$\frac{\text{grad}\ p}{\rho(p)} = \text{grad}\int\frac{\mathrm{d}p}{\rho(p)}$$

and Lamb's form is reduced to

$$\frac{\partial \boldsymbol{q}}{\partial t} - \boldsymbol{q}\times\boldsymbol{\Omega} = -\text{grad}\left(\frac{q^2}{2} + \mho + \int\frac{\mathrm{d}p}{\rho}\right) \tag{2.4}$$

2.5 Integration of the Dynamical Equations in Special Cases

There are several important cases in which the Eulerian equations of motion can be integrated directly.

2.5.1 Irrotational Barotropic Flow

In this type of flow, $\boldsymbol{\Omega}$ is zero, and $\boldsymbol{q}$ is $\text{grad}\,\phi$. The left-hand side of Eq. (2.4) becomes simply $(\partial/\partial t)\,\text{grad}\,\phi$, and as the time and space derivatives are independent and can be exchanged in order, this is equal to $\text{grad}\,(\partial\phi/\partial t)$. Thus,

$$\text{grad}\left(\frac{\partial\phi}{\partial t} + \frac{q^2}{2} + \int\frac{\mathrm{d}p}{\rho} + \mho\right) = 0 \tag{2.5}$$

But when the gradient of a function is zero throughout a region, the function must certainly be constant throughout the region—or rather, as the gradient involves space derivatives only, the function must be constant throughout the region at any instant, but may vary with time. The integrated form is therefore

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+\int\frac{\mathrm{d}p}{\rho}+\mho=F(t) \tag{2.6}$$

Remember that the term $\int \mathrm{d}p/\rho$ is just a function of ρ (or p) whose form is known as soon as the particular barotropic law $\rho=\rho(p)$ is specified. For example, the simplest law is that of the incompressible fluid: $\rho=$ constant. Hence, the integrated equation for *incompressible*, frictionless, irrotational, nonsteady flow is

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+\frac{p}{\rho}+\mho=F(t) \tag{2.7}$$

2.5.2 Steady Barotropic Motion

For this case, we need not assume irrotational flow, and therefore, we return to the form Eq. (2.1) of the dynamical equation, but again assume $\rho=\rho(p)$. The equations then read, for steady flow,

$$\boldsymbol{q}\cdot\nabla\boldsymbol{q}=-\mathrm{grad}\left(\int\frac{\mathrm{d}p}{\rho}+\mho\right) \tag{2.8}$$

We shall now show that this can be integrated along individual streamlines; that is, we shall obtain an integral that will tell how the quantities behave along a streamline, but not how they change from streamline to streamline. Let an orthogonal curvilinear coordinate system be defined so that s is measured along a streamline, and n and ℓ are normal to it (Fig. 2.3). Then, $\boldsymbol{q}=(q,0,0)$ and $\boldsymbol{q}\cdot\nabla\boldsymbol{q}=[q(\partial q/\partial s),\ \ldots]$ (as may be verified by reference to the formulas in general curvilinear orthogonal coordinates).

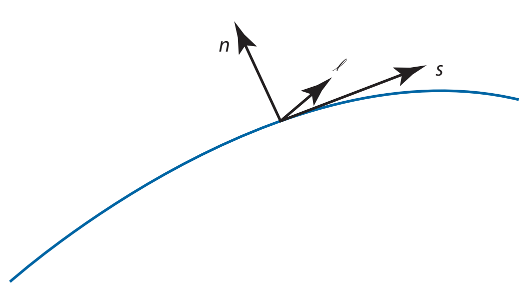


Fig. 2.3 Orthogonal curvilinear coordinate system.

Let us substitute this in Eq. (2.8) and then multiply both sides by d**s**:

$$q\frac{\partial q}{\partial s}\,\mathrm{d}\boldsymbol{s} = -\frac{\partial}{\partial s}\left(\int\frac{\mathrm{d}p}{\rho}+\mho\right)\mathrm{d}\boldsymbol{s}$$

and integrate along the streamline

$$\frac{1}{2}q^2+\int\frac{\mathrm{d}p}{\rho}+\mho = C_s \tag{2.9}$$

where C_s is the constant of integration, and we give it the subscript s to emphasize that the constant may vary from streamline to streamline.

Because Eqs. (2.6) and (2.9) must yield the same result in cases of steady, irrotational barotropic flow, we see that the *irrotational* assumption is equivalent to the condition of taking the same constant C_s for all streamlines:

$$\frac{1}{2}q^2+\int\frac{\mathrm{d}p}{\rho}+\mho = \text{constant} \tag{2.10}$$

and finally, if this flow is also *incompressible*,

$$\frac{1}{2}\rho q^2+p+\rho\mho = \text{constant} \tag{2.11}$$

This will be recognized as *Bernoulli's equation*, which is an energy equation, although we obtained it by integration of momentum equations. In fact, Eqs. (2.6), (2.7), and (2.9) are also sometimes called generalized forms of Bernoulli's equation. More general relations can be derived from the First Law of Thermodynamics, which reduce to forms like these in the appropriate cases; however, we do not need these more general forms in this book and therefore do not pursue the matter further.

Sample Problem 2.1

Consider "small-perturbation flow" wherein $u = U + u', v = v', w = w'$, and u', v', and w' are all small compared with U:

1. For steady incompressible flow without body forces, show that $p - p_\infty \approx -\rho U u'$, where p_∞ is the pressure in the undisturbed stream of speed U.

2. Show that the approximate equation for Θ (see Problem 2.2 at the end of the chapter) for this case is $\nabla^2\Theta \approx 0$. (The flow need not be steady or irrotational.)

Solution:

1. $$p+\frac{\rho}{2}q^2 = p+\frac{\rho}{2}(U^2+2U'u'+u'^2+v'^2+w'^2) = \text{constant}$$

Evaluating the constant in the undisturbed stream, we have

$$\text{constant} = p_\infty + \frac{\rho}{2}U^2$$

Now, if u', v', w' are small compared with U, their squares may be negligible in the pressure formula. Then,

$$p - p_\infty \approx \rho U u'$$

2. $$\text{grad}\,\Theta = \frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{q}\cdot\nabla\boldsymbol{q} \approx \frac{\partial \boldsymbol{q}}{\partial t} + U\frac{\partial \boldsymbol{q}}{\partial x}$$

$$\nabla^2\Theta \approx \frac{\partial}{\partial t}\cancel{\text{div}\,\boldsymbol{q}} + U\frac{\partial}{\partial x}\cancel{\text{div}\,\boldsymbol{q}} = 0$$

Prandtl called Θ the *acceleration potential*. We see that it is also (with a factor $-\rho$) the potential of the pressure perturbations.

2.6 Some Dynamical Properties of Rotational Flow

2.6.1 Constancy of Circulation

In any flow of a barotropic inviscid fluid, the circulation about any closed path does not vary with time if the contour is imagined to move with the fluid, that is, always to be made up of the same particles.

There is a proof in [1] for this remarkable theorem, which was given by Lord Kelvin (Sir W. Thomson) in 1869. We give here a different proof, which offers more generality. We begin by considering the contour integral

$$\oint_C \boldsymbol{V}\cdot d\boldsymbol{r}$$

where V is any vector quantity and C is carried by the fluid.

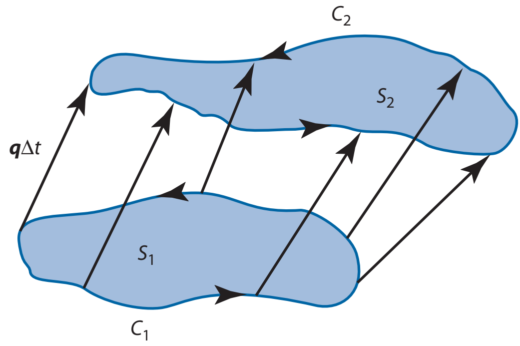


Fig. 2.4 Volume formed by surfaces S_1 and S_2 with the surface generated by $\boldsymbol{q}\Delta t$.

Let C_1 denote the contour C at a given time t_1; let C_2 be the new contour at time $t_1 + \Delta t$. The enclosed surfaces S_1 and S_2, together with the vectors $\boldsymbol{q}\Delta t$, make up a closed volume, as shown in Fig. 2.4. By Stokes' theorem,

$$\oint_C \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{r} = \int_S \boldsymbol{n}\cdot \mathrm{curl}\,\boldsymbol{V}\,\mathrm{d}\sigma$$

The change of

$$\oint_C \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{r}$$

during the interval Δt consists of two parts: 1) the surface integral

$$\Delta t \oint_{S_1} \boldsymbol{n}\cdot \mathrm{curl}\,\frac{\partial \boldsymbol{V}}{\partial t}\,\mathrm{d}\sigma$$

which arises because $\boldsymbol{V}$ changes with time at the original location, and 2) the difference

$$\int_{S_2} \boldsymbol{n}\cdot \mathrm{curl}\,\boldsymbol{V}\,\mathrm{d}\sigma - \int_{S_1} \boldsymbol{n}\cdot \mathrm{curl}\,\boldsymbol{V}\,\mathrm{d}\sigma$$

But the latter is equal to the flux of curl $\boldsymbol{V}$ through the sidewalls of the volume shown in Fig. 2.4 because curl $\boldsymbol{V}$ is always divergence-free.

Remembering the meaning of a scalar triple product (see Sec. A.4.3), we can calculate this influx, for infinitesimal Δt, by means of a contour integral around C_1, namely,

$$-\int_{C_1} \mathrm{d}\boldsymbol{r}\cdot \boldsymbol{q}\Delta t \times \mathrm{curl}\,\boldsymbol{V}$$

We now have the total rate of change in the form

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\oint_C \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{r} &= \oint_{C_1}\frac{\partial \boldsymbol{V}}{\partial t}\cdot \mathrm{d}\boldsymbol{r} - \oint_{C_1}\boldsymbol{q}\times \mathrm{curl}\,\boldsymbol{V}\cdot \mathrm{d}\boldsymbol{r}\\ &= \oint_{C_1}\left\{\frac{\partial \boldsymbol{V}}{\partial t} - \boldsymbol{q}\times \mathrm{curl}\,\boldsymbol{V}\right\}\cdot \mathrm{d}\boldsymbol{r}\end{aligned} \tag{2.12}$$

Considering the special case $\boldsymbol{V} = \boldsymbol{q}$, we see from Eq. (2.4) that the integrand in Eq. (2.12) is a gradient for any barotropic flow with conservative body force, and therefore the right-hand integral is equal to zero. For such flows,

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0 \tag{2.13}$$

as the theorem states.

Applying this to the contours that enclose vortex filaments, we see immediately that such vortices do not vary in strength as they move about in any barotropic inviscid fluid.

2.6.2 Zero Vorticity

From the theorem stated earlier, it becomes clear that if a fluid particle in this type of fluid once has zero vorticity it will always have zero vorticity. Consider a contour surrounding a very small sample of fluid; if $\boldsymbol{\Omega}$ is zero for this sample, Γ is also zero, and according to the theorem it must remain so. But this certainly implies that $\boldsymbol{\Omega}$ remains zero because the statement is true for every contour that surrounds any part of the sample. This result was first given by Helmholtz in 1858.

Thus, every flow that can be produced (without friction) in a barotropic fluid initially at rest, initially in a uniform stream or initially in any irrotational state, must be an irrotational flow. This would seem to give a strong reason for confining our attention to irrotational flows, but actually our occasional preoccupation with such flows has better, but more subtle, justification in the results of the boundary-layer theory mentioned in Sec. 2.1. In fact, the Kelvin and Helmholtz theorems, when carried to their ultimate conclusions, lead to physical absurdities, such as no force is exerted on any body submerged in a stream, and so forth. Nevertheless, the theorems are often of great usefulness in theoretical work, when they are used with discretion. As long as the fluid does not pass through any regions where viscosity is important (as in a boundary layer or a wake) or where barotropy is absent (as in a shock wave), we can expect it to remain irrotational.

2.7 Irrotational and Incompressible Flow

Throughout the rest of this book, we devote considerable attention to the study of irrotational motions of incompressible fluids. As we have seen, the irrotational approximation is likely to be valid throughout much of the flow.

The assumption of incompressibility is obviously permissible for most liquids but is harder to justify for gases. Nevertheless, as is well known, the importance of compressibility depends on the Mach number, that is, the ratio of q to the local speed of sound, which is a function of temperature. For many aeronautical cases, this ratio is not so great that compressibility can be ignored completely. Moreover, the theory of compressible fluids tells us that when the Mach numbers are somewhat greater, but still less than unity everywhere, the flow characteristics are very similar to those of incompressible fluids and in fact can usually be calculated by applying minor corrections to the latter.

Finally, there are strong pedagogical reasons for studying irrotational incompressible flows. They are the simplest kind of fluid flows, and yet they exhibit many of the mathematical and physical characteristics of more complicated cases. They will afford the student an introduction to the mathematical techniques of theoretical aerodynamics without the rather great difficulties that sometimes appear in the advanced subjects.

2.8 Physical Interpretation of Velocity Potential

Before we proceed to the treatment of these flows in detail, we will use the dynamical equations to obtain a physical interpretation of the velocity potential ϕ, which will be of considerable utility as we go on.

In the study of dynamics, it is common to define an *impulsive force* as the result of an infinite force acting through an infinitesimal time interval, so as to produce finite momentum of a solid body:

$$\boldsymbol{m} = \lim_{\substack{F \to \infty \\ \tau \to 0}} \int_0^\tau \boldsymbol{F}\, \mathrm{d}t$$

This may be thought of as the mathematical approximation to a hammer blow.

Impulsive pressure is defined in an analogous manner, namely, impulsive force per unit area.

Consider now our general Eulerian equations, Eq. (2.1), multiplied by dt and integrated over a time interval:

$$\int_0^\tau \frac{\partial \boldsymbol{q}}{\partial t}\,\mathrm{d}t + \int_0^\tau \boldsymbol{q}\cdot\nabla\boldsymbol{q}\,\mathrm{d}t = -\int_0^\tau \frac{1}{\rho}\,\mathrm{grad}\,p\,\mathrm{d}t + \int_0^\tau \boldsymbol{F}\,\mathrm{d}t$$

Assume, for simplicity, that $\rho = \text{constant}$. The first term can be integrated and the pressure term simplified:

$$\boldsymbol{q}(\tau) - \boldsymbol{q}(0) + \int_0^\tau \boldsymbol{q}\cdot\nabla\boldsymbol{q}\,\mathrm{d}t = -\frac{1}{\rho}\,\mathrm{grad}\int_0^\tau p\,\mathrm{d}t + \int_0^\tau \boldsymbol{F}\,\mathrm{d}t$$

Now suppose that $\tau \to 0$ while both p and $\boldsymbol{F}$ are increased indefinitely so as to result in impulsive pressures ϖ and force $\boldsymbol{J}$, respectively. Just as in the solid body mentioned earlier, finite velocities will result, and consequently the integral

$$\int_0^\tau \boldsymbol{q}\cdot\nabla\boldsymbol{q}\,\mathrm{d}t$$

must disappear:

$$\boldsymbol{q}(\tau) - \boldsymbol{q}(0) = -\frac{1}{\rho}\,\mathrm{grad}\,\varpi + \boldsymbol{J} \tag{2.14}$$

For example, if there are no impulsive body forces and the fluid was initially at rest,

$$\boldsymbol{q} = -\frac{1}{\rho}\,\mathrm{grad}\,\varpi \tag{2.15}$$

But for any *irrotational* flow, $\boldsymbol{q} = \mathrm{grad}\,\phi$. Thus, $\phi(x, y, z, t)$ can be interpreted as [minus $(1/\rho)$ times] that distribution of impulsive pressures that would be required to generate impulsively the flow existing at time t. Or alternately, $\rho\phi(x, y, z, t)$ is the impulsive-pressure distribution required to stop impulsively the flow observed at time t.

It is now clear that rotational flows cannot be started by impulsive pressures alone, for any flow so produced will have $\mathrm{curl}\,\boldsymbol{q} = 0$.

Moreover, the process of starting or stopping an irrotational flow impulsively should not be thought of as requiring the application of impulsive pressures throughout the field of flow by some mechanism. It will only be necessary to supply the *boundary* values of ϖ, for the pressures at interior points are internal forces exerted between adjacent fluid particles. In fact, from Eq. (2.15), the differential equation satisfied by ϖ is the same as for ϕ, namely,

$$\operatorname{div} \boldsymbol{q} = -\frac{1}{\rho}\nabla^2\varpi = 0; \quad \nabla^2\varpi = 0$$

As we shall see later, the solution of this equation is determined by its boundary values, and so ϖ will be distributed everywhere like ϕ when the boundary-impulsive pressures are supplied.

2.9 Equations in a Moving Coordinate System

Let $\boldsymbol{q}'(x', y', z', t)$ describe the flowfield in a coordinate system that is in motion relative to a fixed system x, y, and z. In general, the relations between coordinates, velocities, and accelerations are those of Eqs. (A.53–A.55), that is,

$$\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{R} \tag{2.16}$$

$$\boldsymbol{q} = \boldsymbol{q}' + \dot{\boldsymbol{R}} + \boldsymbol{\omega}\times\boldsymbol{r}' \tag{2.17}$$

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\mathrm{d}'\boldsymbol{q}'}{\mathrm{d}t} + 2\boldsymbol{\omega}\times\boldsymbol{q}' + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}') + \ddot{\boldsymbol{R}} \tag{2.18}$$

Thus, the left-hand side of Eq. (2.1) must be augmented by the addition of four new terms, in general, to constitute a differential equation for $\boldsymbol{q}'(x', y', z', t)$.

However, space derivatives such as grad, div, and curl are unaffected in form by the transformation of axes. The only changes in such terms therefore arise from the process of carrying out these operations on quantities, such as $\boldsymbol{q}$, which have additional terms. There are no such terms in the scalar quantity ρ; hence, Eq. (2.1) is altered only by the addition of the four new left-hand terms mentioned.

Before writing down the new equation of motion, let us consider the equation of continuity. Consider the term div $\boldsymbol{q}$, and let div′ denote the divergence operator in the primed system:

$$\begin{aligned}\operatorname{div} \boldsymbol{q} &= \operatorname{div}' \boldsymbol{q} = \operatorname{div}' \left\{\boldsymbol{q}' + \dot{\boldsymbol{R}} + \boldsymbol{\omega}\times\boldsymbol{r}'\right\}\\ &= \operatorname{div}' \boldsymbol{q}'\end{aligned} \tag{2.19}$$

because the divergences of the last two terms are zero [see Eq. (A.27)]. The physical meaning of this result should be clear to the reader.

Thus, the equations for the general case of inviscid fluid motion described in a moving coordinate system are (omitting the primes)

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\,\mathrm{div}\,\boldsymbol{q}=0 \tag{2.20}$$

and

$$\frac{\mathrm{D}\boldsymbol{q}}{\mathrm{D}t}+2\boldsymbol{\omega}\times\boldsymbol{q}+\dot{\boldsymbol{\omega}}\times\boldsymbol{r}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})+\ddot{\boldsymbol{R}}=-\frac{1}{\rho}\,\mathrm{grad}\;p+\boldsymbol{F} \tag{2.21}$$

The vorticity is not the same in the two systems. Using the primes as before, we see that

$$\begin{aligned}\mathrm{curl}\;\boldsymbol{q}=\mathrm{curl}'\;\boldsymbol{q}&=\mathrm{curl}'\left\{\boldsymbol{q}'+\dot{\boldsymbol{R}}+\boldsymbol{\omega}\times\boldsymbol{r}'\right\}\\&=\mathrm{curl}'\;\boldsymbol{q}'+2\boldsymbol{\omega}\end{aligned} \tag{2.22}$$

[see Eq. (A.28)]. Again, the physical meaning of this result should be clear to the reader. An important example is the case of flow that is irrotational but is viewed in a rotating coordinate system; it appears to be rotational.

Sample Problem 2.2

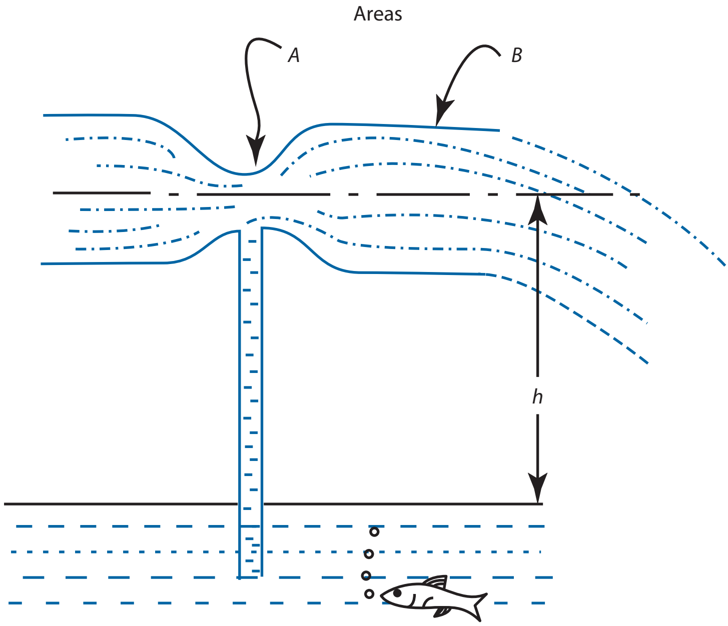
A stream in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A, is delivered at atmospheric pressure at a place where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth $(Q^2/2g)(A^{-2}-B^{-2})$ below the pipe, where Q is the volume flow rate.

Solution:

$$p_A+\frac{1}{2}\rho v_A^2=p_{\mathrm{atm}}+\frac{1}{2}\rho v_B^2$$

$$\begin{aligned}p_{\mathrm{atm}}-p_A&=\frac{1}{2}\rho v_A^2\left\{1-\frac{A^2}{B^2}\right\}\\&=\rho g h\end{aligned}$$

$$h=\frac{v_A^2}{2g}\left\{1-\frac{A^2}{B^2}\right\}=\frac{1}{2g}\frac{Q^2}{A^2}\left\{1-\frac{A^2}{B^2}\right\}$$



Reference

[1] Lamb, H. *Hydrodynamics*, Cambridge University Press, London, 1945. Sec. 1–11, 17–24, 30–36, 145, and 146.

Problems

2.1 a. By taking the curl of both sides of the Eulerian momentum equation for incompressible inviscid flow, show that the vorticity is constant for each fluid particle, in plane flow.

b. Make the same calculation for axisymmetric flow, evaluating $D\boldsymbol{\Omega}/Dt$. Can you explain these results? (Hint: Consider Kelvin's theorem).

2.2 a. For what class of flows can the fluid-acceleration vector be expressed as the gradient of a scalar Θ?

b. Write out the pressure–velocity relation that you would use for a case of unsteady, irrotational, isentropic gas flow. What is Θ for this case?

2.3 Write the expression for the kinetic energy of a fluid flow occupying a volume V bounded by a surface S, by integration over V. Now assume that the flow is incompressible and irrotational, and prove that the kinetic energy is given by

$$\frac{\rho}{2}\int_S \phi \frac{\partial \phi}{\partial n}\,\mathrm{d}\sigma$$

provided ϕ and its first derivatives are single-valued and continuous within V.

2.4 A straight tube of small bore, ABC, is bent so as to make the angle ABC a right angle, and AB equal to BC. The end C is closed, and the tube is placed with the end A upwards and AB vertical and is filled with liquid. If the end C is opened, prove that the pressure at any point of the vertical tube is instantaneously diminished one-half, and find the instantaneous change of pressure at any point of the horizontal tube, the pressure of the atmosphere being neglected [2].

2.5 Consider plane or axisymmetric, steady, barotropic flow. Show that the vorticity $\mathbf{\Omega}$ is related to the *Bernoulli constant* C_S as follows: $\mathbf{\Omega} = (1/q)(\partial C_S/\partial n)$, where $\partial/\partial n$ denotes the derivative taken normal to a streamline.

2.6 Suppose that a flow is irrotational and two-dimensional, but is unsteady. It becomes steady when viewed in a coordinate system rotating with constant angular velocity ω about an axis normal to the plane of the flow.

a. Give an example of such a flow.

b. Show that in this case the equations of motion can be integrated to give a new Bernoulli-type pressure formula. Assume barotropic flow and no body forces.

Irrotational Motion of an Incompressible Fluid: Laplace's Equation

- Laplace's equation
- Membrane analogy
- Plane irrotational incompressible flow
- Elementary plane and axisymmetric flows

3.1 Introduction

As the equation of continuity is given by div $\boldsymbol{q} = 0$ for homogeneous incompressible flow, where $\boldsymbol{q}$ is the gradient of the velocity potential ϕ, the differential equation satisfied by ϕ is called Laplace's equation:

$$\text{div grad } \phi = \nabla^2 \phi = 0 \tag{3.1}$$

The pressure–velocity relation is given by the integrated dynamical equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + \frac{p}{\rho} + \mho = F(t) \tag{3.2}$$

In a more general case, even if we restrict ourselves to barotropic fluids, we have five dependent variables—u, v, w, p, and ρ—and five equations to solve for them:

- Three equations of motion
- One equation of continuity
- One equation of state, $\rho = \rho(p)$

Now, we see that an extreme simplification has been achieved in the irrotational incompressible case, for the equation of state has degenerated to $\rho =$ constant, and we have replaced u, v, and w by the velocity potential ϕ, leaving only two unknowns (ϕ and p) and two equations, Eqs. (3.1) and (3.2).

Moreover, Eq. (3.2) has been integrated and constitutes a formula for calculating p when Eq. (3.1) has been solved. The only mathematical problem that remains is the solution of Laplace's equation (3.1), with the appropriate boundary conditions. The most surprising result is that the dynamical equations do not impose any restrictions on the flow! Any solution of the equation of continuity (3.1) is a possible flow pattern for some set of boundary conditions. Another statement of this situation is that *every kinematically possible flow is dynamically possible.*

It is also important to note that Eq. (3.1) does not involve t. In the case of nonsteady flow, the boundary conditions will vary with time. All that is required is that we solve Laplace's equation with the instantaneous boundary conditions. Another statement of this is that *every nonsteady flow pattern is a possible steady flow pattern* (and vice versa). Of course, the corresponding pressures will depend on whether the flow is steady or not.

3.2 Laplace's Equation

If f satisfies Laplace's equation in a region, then f has no maxima or minima in that region. Consider for any volume V, enclosed in a surface S, lying entirely inside the region,

$$O = \int_V \nabla^2 f \, \mathrm{d}\tau = \int_V \operatorname{div} \operatorname{grad} f \, \mathrm{d}\tau = \int_S \boldsymbol{n} \cdot \operatorname{grad} f \, \mathrm{d}\sigma = \int_S \frac{\partial f}{\partial n} \, \mathrm{d}\sigma$$

But if f has a maximum at any point P, we can surely obtain a negative value of

$$\int_S \frac{\partial f}{\partial n} \, \mathrm{d}\sigma$$

by taking V to enclose P and making it small enough. Similarly, we can obtain a positive value by integrating $\partial f / \partial n$ around a minimum. Consequently, there can be neither.

In a plane case, this property is even easier to see, especially when $\nabla^2 f$ is written in its approximate form in terms of differences (Fig. 3.1):

$$\frac{\partial^2 f}{\partial x^2} \approx \frac{[(f_2 - f_0)/\Delta] - [(f_0 - f_1)/\Delta]}{\Delta} = \frac{f_2 + f_1 - 2 f_0}{\Delta^2}$$

$$\frac{\partial^2 f}{\partial y^2} \approx \frac{[(f_4 - f_0)/\Delta] - [(f_0 - f_3)/\Delta]}{\Delta} = \frac{f_4 + f_3 - 2 f_0}{\Delta^2}$$

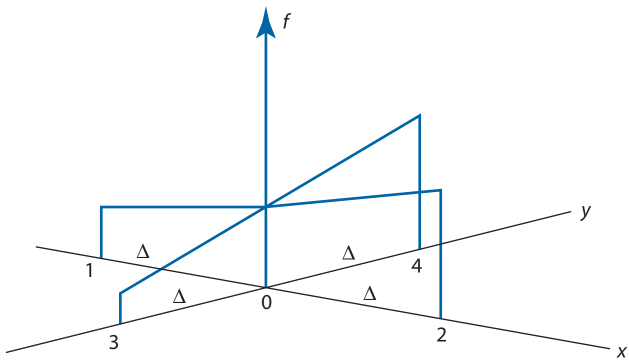


Fig. 3.1 Representation of the value of *f* at five points, namely points 0, 1, 2, 3, and 4.

Thus, if $\nabla^2 f = 0$,

$$f_0 \approx \frac{1}{4}(f_1 + f_2 + f_3 + f_4)$$

and the value of f at every point is equal to the mean of the four adjacent points. Obviously, this allows no maxima or minima.

Laplace's equation has many important applications in physics. It is the equation satisfied by the electrical, magnetic, and gravitational potentials in empty space and by other important functions. It also occurs in several places in the theory of elasticity. There is a great literature concerning the problem of constructing solutions to this equation, constituting the subject of Potential Theory.

The Laplacian is sometimes denoted by the symbol Δ instead of ∇^2; hence, one might find the equation written as $\Delta f = 0$. The solutions are sometimes called *harmonics*.

Perhaps the most important thing to note about Laplace's equation is that it is *linear*. It means, specifically, that the unknown function and/or its derivatives appear in the equation only in a linear fashion. As a result of this, any linear combination of solutions is also a solution:

$$\left.\begin{array}{c} \text{If} \quad \nabla^2 f = 0 \quad \text{and} \quad \nabla^2 g = 0, \\ \text{then} \quad \nabla^2 (f + g) = 0, \text{etc.} \end{array}\right\} \tag{3.3}$$

This property makes it possible for us to construct solutions by *superposition* of elementary solutions to fit our prescribed boundary conditions. This is an essential simplification.

3.3 Membrane Analogy

There is a simple case in physics that leads to Laplace's equation and constitutes an analogy that is helpful in hydrodynamics. Consider a thin flexible membrane under uniform tension T (per unit width or length), such as a soap film. Let the pressure difference between top and bottom be p and the height of the membrane over the xy plane be w. The equilibrium of vertical force components requires (assuming small slopes $\partial w / \partial x$ and $\partial w / \partial y$) (Fig. 3.2):

$$p \Delta x \Delta y = T \Delta y \frac{\partial^2 w}{\partial x^2} \Delta x + T \Delta x \frac{\partial^2 w}{\partial y^2} \Delta y \tag{3.4}$$

or

$$\nabla^2 w = \frac{p}{T}$$

Thus, the elevation of the membrane satisfies Poisson's equation in general and Laplace's equation when there is no pressure difference. By analogy, then, any plane potential problem, $\nabla^2 f = 0$, can be visualized as a question of the shape assumed by a soap film under given boundary conditions and no pressure difference. If boundary values of f are given, they correspond to a soap film stretched over a framework of given height. If boundary values of $\partial f / \partial n$ are given, as often occurs, they require raising and lowering of the framework to obtain the required slopes of the film at the boundary.

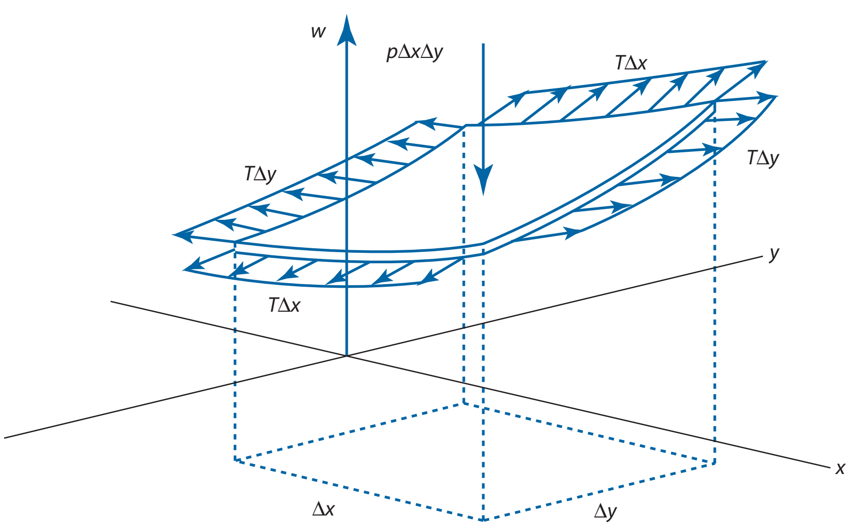


Fig. 3.2 Thin flexible membrane under tension.

This analogy has actually been employed in certain research studies, using a stretched rubber sheet for the membrane and measuring the altitude above a plane table with a suitable height gauge. Electrical analogies have been used even more commonly, usually with electrolytic tanks. In one recent investigation, the magnetic field (in the air) due to an arrangement of wires carrying alternating currents was explored by means of a small coil to determine the field of velocities induced by the analogous arrangement of vortex filaments.

3.4 Plane Irrotational Incompressible Flow

From our previous work, we collect the formulas appropriate to this case:

$$\boldsymbol{q} = u\boldsymbol{i} + v\boldsymbol{j} = \text{grad}\,\phi = \boldsymbol{i}\frac{\partial \phi}{\partial x} + \boldsymbol{j}\frac{\partial \phi}{\partial y} \tag{3.5}$$

$$\Omega = \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{3.6}$$

$$\text{div}\,\boldsymbol{q} = \text{div grad}\,\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0 \tag{3.7}$$

Also, $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$, and from Eq. (3.6),

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0 \tag{3.8}$$

Thus, both the velocity potential and the stream function satisfy Laplace's equation in this type of flow. Naturally, their boundary conditions are different, but it is interesting to note that ϕ and ψ for any such flow, upon being interchanged (a change of one sign is required), can be taken as ψ and ϕ for some other flow.

Equipotential lines are given by $\mathrm{d}\phi = 0$ or

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\,\mathrm{d}x + \frac{\partial \phi}{\partial y}\,\mathrm{d}y = u\,\mathrm{d}x + v\,\mathrm{d}y = 0$$

Thus,

$$\frac{\mathrm{d}x}{v} = \frac{\mathrm{d}y}{-u}$$

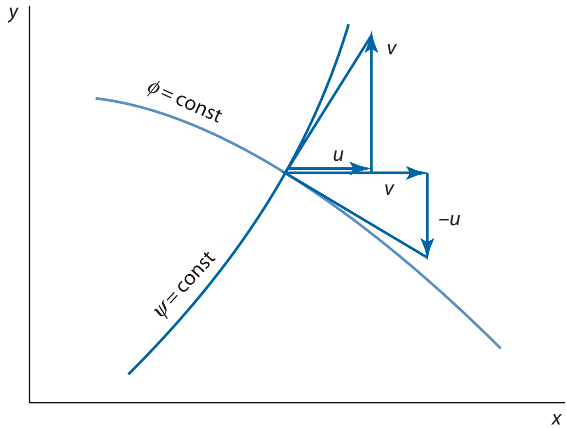


Fig. 3.3 Streamlines and equipotentials are orthogonal to each other.

Which means, of course, that the direction cosines of equipotentials are v and $-u$, relative to the x and y axes. But streamlines are given by $(dx/u)=(dy/v)$ (direction cosines u and v). Therefore, the streamlines and equipotentials are perpendicular, for

$$uv - vu = 0$$

This is not surprising, after all, because it agrees with our general interpretation of grad ϕ as a vector normal to the equipotentials, and with our physical interpretation of ϕ in terms of impulsive pressures (Fig. 3.3).

The network of streamlines and equipotential lines, everywhere perpendicular to each other, is an example of an *orthogonal net*.

Sample Problem 3.1

1. Suppose that the following flows are incompressible and irrotational, determine the velocity components and thence ϕ and ψ by integration, and verify by direct substitution that both ϕ and ψ satisfy Laplace's equation:

   a. Plane parallel flow shown in Example 1.1

   b. Plane concentric flow shown in Example 1.2

   c. Plane radial flow, that is, $v_\theta = 0$ and $v_r \neq 0$

2. Sketch some of the equipotentials and streamlines for these three flows, and also for a flow obtained by superimposing b) and c).
3. Assuming that these are all steady flows, calculate the pressure distributions in these four cases, and sketch the curves showing the results.

Solution:

1. a. With $v = 0$

$$\operatorname{div} \boldsymbol{q} = \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} = 0$$

$$\Omega = \frac{\partial u}{\partial y} - \cancel{\frac{\partial v}{\partial x}} = 0\text{; therefore, } u = \text{const} = U$$

Integrating the equation,

$$u = u(y) = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = 0 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\phi = Ux + \text{const}, \quad \psi = Uy + \text{const}$$

b. With $v_r = 0$

$$\operatorname{div} \boldsymbol{q} = \frac{1}{r}\left\{\cancel{\frac{\partial r v_r}{\partial r}} + \frac{\partial v_\theta}{\partial \theta}\right\} = 0$$

$$\Omega = \frac{1}{r}\left\{\frac{\partial r v_\theta}{\partial r} - \cancel{\frac{\partial v_r}{\partial \theta}}\right\} = 0\text{; therefore, } r v_\theta = \text{const} = C$$

Integrating the equation,

$$v_\theta = v_\theta(r) = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

$$v_r = 0 = \frac{\partial \theta}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$v_\theta = \frac{C}{r}$$

$$\phi = C\theta + \text{const}, \quad \psi = -C \ln r + \text{const}$$

c. With $v_\theta = 0$

$$\text{div}\, \boldsymbol{q} = \frac{1}{r}\left\{\frac{\partial r v_r}{\partial r} + \cancel{\frac{\partial v_\theta}{\partial \theta}}\right\} = 0$$

Integrating the equation,

$$r v_r = f(\theta) = r\frac{\partial \phi}{\partial r} = \frac{\partial \psi}{\partial \theta}$$

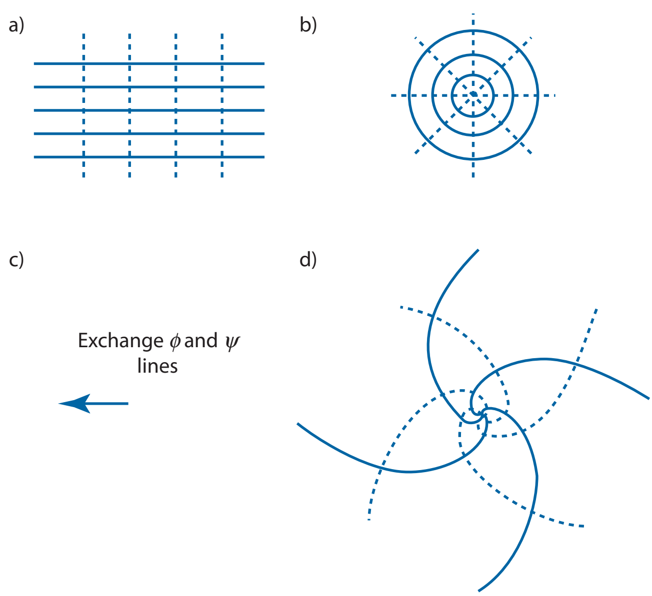
$$v_\theta = 0 = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

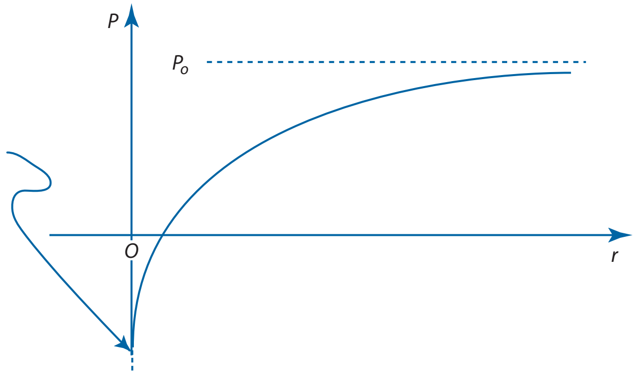
$$\Omega = \frac{1}{r}\left\{\cancel{\frac{\partial r v_\theta}{\partial r}} - \frac{\partial v_r}{\partial \theta}\right\} = 0;\ \text{therefore,}\ \ r v_r = \text{const} = K$$

$$v_r = \frac{K}{r}$$

$$\phi = K \ln r + \text{const}, \quad \psi = K\theta + \text{const}$$

2. Solid lines are streamlines, and dash lines are equipotentials:





3. a. $p+\dfrac{\rho}{2}q^2 = p+\dfrac{\rho}{2}U^2 = C;\ p = C-\dfrac{\rho}{2}U^2 = \text{constant}$

b. $p+\dfrac{\rho}{2}q^2 = p+\dfrac{\rho}{2}v_\theta^2 = p+\dfrac{\rho}{2}\left(\dfrac{C}{r}\right)^2 = \text{const}$

For example, we might evaluate the constant at $r=\infty$, where $v_\theta = 0$ and $p = p_0$, say:

$$p = p_0 = \frac{\rho}{2}\left(\frac{C}{r}\right)^2$$

c. $p+\dfrac{\rho}{2}q^2 = p+\dfrac{\rho}{2}v_r^2 = p+\dfrac{\rho}{2}\left(\dfrac{K}{r}\right)^2 = \text{const}$

$$p = p_0 - \frac{\rho}{2}\left(\frac{K}{r}\right)^2 \quad \text{where } p_0 = p(\infty)$$

d. $p+\dfrac{\rho}{2}q^2 = p+\dfrac{\rho}{2}(v_r^2+v_\theta^2) = p+\dfrac{\rho}{2}\dfrac{C^2+K^2}{r^2} = p_0$

This result indicates that for small enough r, we have unbounded negative pressures in the fluid. Clearly, some of our approximations have been violated, as demonstrated by the above figure.

3.5 Elementary Plane Flows

3.5.1 Source or Sink

The radial flow of Sample Problem 3.1(1c) is called *source* flow if $v_r > 0$ and *sink* flow if $v_r < 0$. The point $r = 0$ is a singularity because v_r and its derivatives are infinite and continuity is not satisfied there. Naturally, the quantity of flow is the same across any closed curve C enclosing the singularity. Let us calculate it.

Quantity per unit time (per unit depth):

$$Q = \oint_C v_r r \, \mathrm{d}\theta$$

For simplicity, take C to be a circle of radius a:

$$Q = \int_0^{2\pi} v_r a \, \mathrm{d}\theta = \frac{B}{a} \int_0^{2\pi} a \, \mathrm{d}\theta = 2\pi B \tag{3.9}$$

where B is the constant in the expression $v_r = B/r$. Thus, the constant can be replaced in terms of the quantity Q, if desired.

There is another phenomenon that should be discussed: we see that ψ for this case is multivalued, namely [from Sample Problem 3.1(1c)] $\psi = \kappa\theta + \text{constant}$, and θ is a multivalued function at any point of the plane. Recalling the original meaning of ψ—the quantity flow across a curve drawn from the origin—we can see that this multivaluedness is inevitable in this case where fluid is being created at the origin at the rate Q.

To see this clearly, suppose ψ is determined by integration from a fixed point A (Fig. 3.4) not at the singularity. (Remember that a change in the starting point for ψ means simply the addition of a constant.) If any value is obtained for ψ at P by integrating along path (1), the value for path (2) must clearly be increased by the amount Q. The multivaluedness of the formula $\psi = B\theta + \text{constant}$ takes care of this, for at any point P we have

$$\begin{aligned} \psi &= B(\theta + 2\pi n) + \text{const} \\ &= B\theta + nQ + \text{const} \end{aligned}$$

where $0 \leqq \theta < 2\pi$ and n is any integer. The integer n can be interpreted as the number of counterclockwise circuits made around the singularity.

To avoid this multivaluedness, we ordinarily restrict θ to the range 0 to 2π. This permits us to apply theorems that postulate single-valued functions, but introduces some compensating complications, as will be seen later.

In summary, the flow due to a plane source of strength Q (quantity per unit time) is described by

$$\left.\begin{aligned} &\phi = \frac{Q}{2\pi} \ln r + \text{const}; \quad v_r = \frac{Q}{2\pi r}; \quad v_\theta = 0 \\ &\psi = \frac{Q}{2\pi}\theta + \text{const} \ \ (0 \leqq \theta < 2\pi) \end{aligned}\right\} \tag{3.10}$$

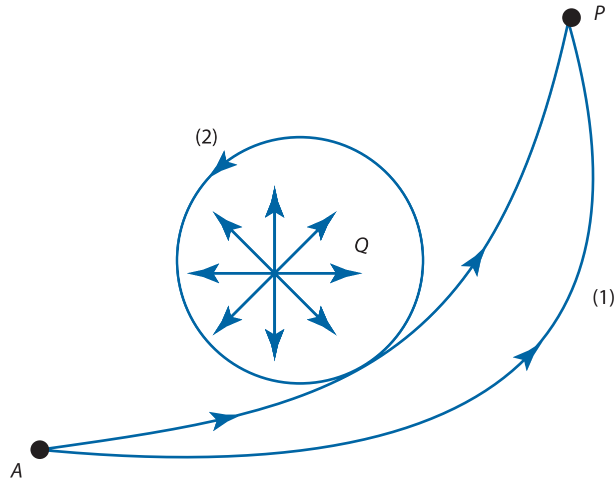


Fig. 3.4 Path of integration indicates the multivaluedness of ψ.

3.5.2 Vortex

The concentric flow of Sample Problem 3.1(1b) is called *vortex* flow. As we shall see, it is the flow induced by a straight vortex filament of infinite length according to the Biot–Savart formula.

Naturally, the circulation Γ is a convenient parameter to describe the strength of this flow. It is easily found to be equal to $2\pi A$, where A is the constant in the formula $v_\theta = A/r$.

In this case, ϕ is multivalued; again this is to be expected because ϕ was defined as $\int \boldsymbol{q}\cdot \mathrm{d}\boldsymbol{r}$ and Γ as $\oint \boldsymbol{q}\cdot \mathrm{d}\boldsymbol{r}$. Every complete counterclockwise circuit of the singularity at $r = 0$ increases the value of ϕ by the amount Γ; hence, we usually restrict θ as in the earlier case. In summary,

$$\left.\begin{aligned} &\phi = \frac{\Gamma}{2\pi}\theta + \text{const}\,(0 \leqq \theta < 2\pi) \\ &\psi = \frac{-\Gamma}{2\pi}\ln r + \text{const}; \quad v_\theta = \frac{\Gamma}{2\pi r}; \quad v_r = 0 \end{aligned}\right\} \tag{3.11}$$

3.5.3 Doublet (Dipole)

Consider the flow due to an equal and opposite source and sink arranged as in Fig. 3.5. By the superposition principle, the potential at $P(r,\theta)$ is

$$\phi = \frac{Q}{2\pi}\ln r - \frac{Q}{2\pi}\ln(r+\Delta r) + \text{const} = -\frac{Q}{2\pi}\ln\left(1+\frac{\Delta r}{r}\right) + \text{const}$$

Now, for $(\Delta r / r) < 1$, it is permissible to expand the logarithm in powers of $\Delta r / r$ as follows:

$$\phi = -\frac{Q}{2\pi}\left\{\frac{\Delta r}{r} - \frac{1}{2}\left(\frac{\Delta r}{r}\right)^2 + \cdots\right\} + \text{const}$$

Let us consider the singularity produced by letting the source and sink join while we increase their strengths so as to keep the product $Q\Delta\xi$ finite, that is, let

$$\lim_{\substack{Q\to\infty \\ \Delta\xi\to 0}} (Q\Delta\xi) \equiv \mu \neq \infty$$

$$\neq 0$$

Then, with $\Delta r \approx \Delta\xi \cos\theta$, we have

$$\phi = -\frac{\mu}{2\pi}\frac{\cos\theta}{r} + \text{const} \tag{3.12}$$

This singularity is called a *doublet* or *dipole*, and the flow is called doublet flow or dipole flow. Note that a doublet has direction; defining this direction as that from the sink to the source during the limiting process, we would say that Eq. (3.12) is applied to a *doublet at the origin, directed along the +x axis*. The corresponding velocities and stream function are easily found to be

$$v_r = \frac{\mu}{2\pi}\frac{\cos\theta}{r^2}, \quad v_\theta = \frac{\mu}{2\pi}\frac{\sin\theta}{r^2}, \quad \psi = \frac{\mu}{2\pi}\frac{\sin\theta}{r} + \text{const} \tag{3.13}$$

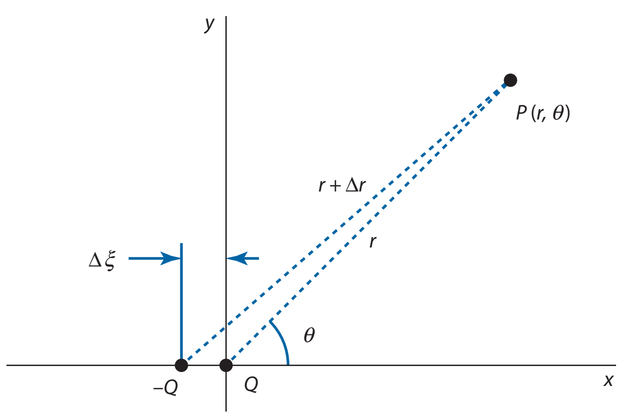


Fig. 3.5 Doublet (dipole) arrangement.

Sample Problem 3.2

Show that the streamlines and equipotentials of doublet flow are circles through the origin. Sketch a few of both.

Solution:
Streamlines are

$$\frac{\sin\theta}{r} = \text{const} \; = C, \text{ say}$$

or

$$\sin\theta = \frac{y}{r} = Cr; \quad y = Cr^2 = C(x^2 + y^2)$$

This should be recognized as the equation of circle of radius $1/2C$ with center on the y axis and passing through the origin. Equipotentials are $\cos\theta/r = \text{const}$; that is, circles with centers on the x axis are passing through the origin.

For the figure, take Fig. 1.9 and turn it 90 deg to see the streamlines.

Sample Problem 3.3

Determine ϕ for the flow due to two equal and opposite vortices, one at (0,0) and the other at (0, $\Delta\eta$). Let $\Gamma \to \infty$ and $\Delta\eta \to 0$ so that $\Gamma\Delta\eta \to \nu$. How does the result compare with a doublet?

Solution:

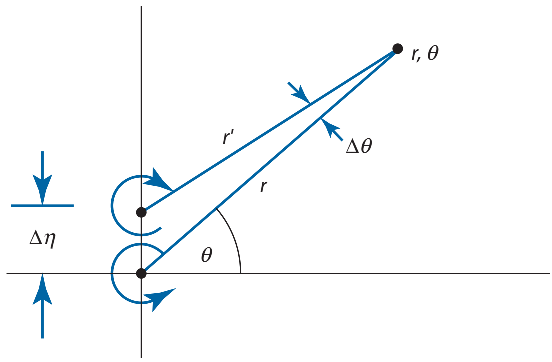
$$\phi = \frac{\Gamma}{2\pi}\theta - \frac{\Gamma}{2\pi}(\theta - \Delta\theta) = \frac{\Gamma}{2\pi}\Delta\theta$$

$$= \frac{\Gamma}{2\pi}\frac{\Delta\eta\cos\theta}{r} \to \frac{\nu}{2\pi}\frac{\cos\theta}{r}, \text{ say (+const)}$$

It is exactly the same as a doublet! So a doublet is either a source and sink *coalesced* (viewed from afar) or equal and opposite vortices. This is an important observation!

3.6 Other Elementary Plane Flows

The plane flow due to a doublet could easily be found by differentiating the potential (or stream function) for source flow. Upon consideration, it will be seen that differentiation with respect to a certain coordinate is exactly equivalent to the process of superposition and limit taking that we performed in Sec. 3.5.



Let ϕ at r,θ(or x, y) be caused by the combination of a sink $-Q$ at ξ,η and a source Q at $\xi+\Delta\xi,\eta$ (Fig. 3.6). Let r' be the distance from ξ,η to x, y, then

$$\phi = \frac{Q}{2\pi}\left\{-\ln r' + \ln(r'+\Delta r')\right\} + \text{const}$$

$$\approx \frac{Q}{2\pi}\frac{\partial}{\partial\xi}(\ln r')\Delta\xi + \text{const}$$

But

$$\frac{\partial}{\partial\xi}(\ln r') = -\frac{\partial}{\partial x}(\ln r')$$

hence,

$$\phi \approx -\frac{Q\Delta\xi}{2\pi}\frac{\partial}{\partial x}(\ln r')$$

and in the limit

$$\phi = -\frac{\mu}{2\pi}\frac{\partial}{\partial x}(\ln r') = -\frac{\mu}{2\pi}\frac{x-\xi}{r'^2} = -\frac{\mu}{2\pi}\frac{\cos\theta'}{r'}$$

as before.

Similarly, the potential due to a doublet at (ξ,η), directed in the $+y$ direction, is

$$-\frac{\mu}{2\pi}\frac{\partial}{\partial y}(\ln r')$$

The process can be continued, leading to singularities of higher and higher order, having increasingly large powers of r' in their denominators. Thus, their velocities diminish more and more rapidly as r' increases.

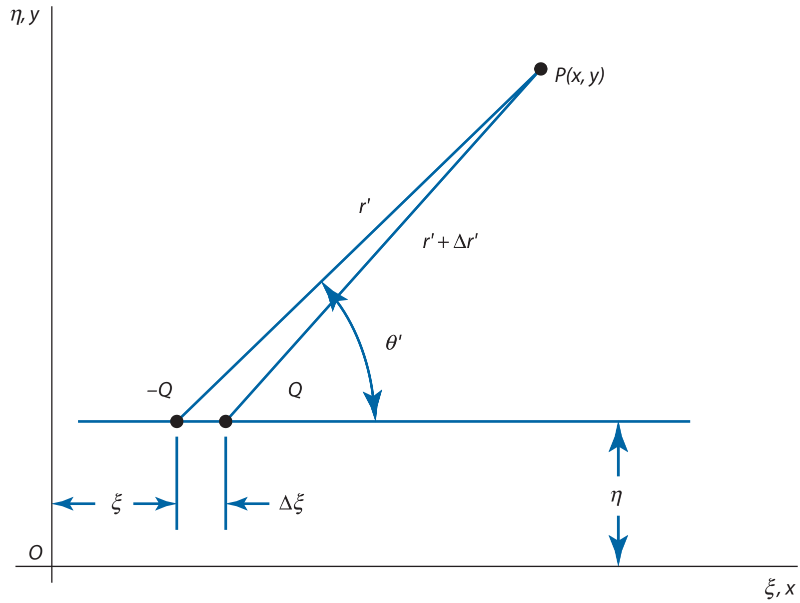


Fig. 3.6 Elementary plane flow with combined sink and source.

In problems where a stream function exists (plane or axisymmetric flow), the principle of superposition applies to ψ and ϕ. (Why?) Thus, the procedure of *coalescing* singularities (by differentiation, for example) is applicable to ψ and ϕ.

Sample Problem 3.4

Verify the result of Sample Problem 3.3 using the method of superposition by differentiation.

Solution:

Start with the potential of a vortex:

$$\phi_{\text{vortex}} = \frac{\Gamma}{2\pi}\theta + \text{const}$$

Take the derivative with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\Gamma}{2\pi}\theta\right) = \frac{\Gamma}{2\pi}\frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right) = \frac{\Gamma}{2\pi}\frac{1/x}{1-y^2/x^2} = \frac{\Gamma}{2\pi}\frac{x}{r^2}$$

So ϕ for this doublet is

$$\phi_{\text{doublet}} = C\frac{x^2}{r} + \text{const} = C\frac{\cos\theta}{r} + \text{const}$$

Now start with the stream function of a vortex:

$$\psi_{\text{vortex}} = \frac{\Gamma}{2\pi}\ln r + \text{const}$$

Take the derivative with respect to y:

$$\psi_{\text{doublet}} = \frac{\partial}{\partial y}\left(\frac{\Gamma}{2\pi r}\ln r\right) = \frac{\Gamma}{2\pi r}\frac{y}{r}$$

So ψ for this doublet is

$$\psi_{\text{doublet}} = C\frac{y}{r^2} + \text{const} = C\frac{\sin\theta}{r} + \text{const}$$

Sample Problem 3.5

Use the method of separation of variables to find the following classes of solutions for incompressible irrotational flow:

1. $\phi(x, y)$ sinusoidal in x (or in y)
2. $\phi(r,\theta)$ varying as a power of r in the plane flow
3. $\phi(x,\varpi) = \text{const}\ e^{\pm kx} J_0(k\varpi)$, where J_0 is the Bessel function of order zero
4. $\phi(x,\varpi)$ sinusoidal in x

In each case, find the corresponding stream function.

Solution:

1. $$\phi(x, y) = X(x)Y(y) + \text{const}$$

   Which implies

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

therefore, each side of this equation should be independent of both x and y.

If

$$\frac{X''}{X} = -k^2 = \text{const}, \quad X'' + k^2 X = 0$$

Therefore,

$$X \propto \begin{pmatrix} \sin \\ \text{or} \\ \cos \end{pmatrix} kx$$

Similarly

$$\frac{Y''}{Y} = +k^2, \quad Y'' - k^2 Y = 0$$

Therefore,

$$Y \propto e^{\pm ky}$$

and

$$\phi(x, y) = \text{const } e^{\pm ky} \begin{pmatrix} \sin \\ \text{or} \\ \cos \end{pmatrix} kx + \text{const}$$

If ϕ is sinusoidal in y, simply exchange x and y in this solution, and find that ϕ must be exponential in x.

2. $$\phi(r,\theta) = R(r)\Theta(\theta) + \text{const}$$

$$\nabla^2\phi = \frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta} = \frac{1}{r}(rR')'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Therefore,

$$\frac{r(rR')'}{R} = -\frac{\Theta''}{\Theta}$$

which implies that each side must be independent of both r and θ. If

$$\frac{r(rR')'}{R} = k^2 = \text{const}, \quad r^2 R'' + rR' - k^2 R = 0$$

and

$$R \propto r^{\pm k}; \quad \text{then } \Theta'' + k^2\Theta = 0$$

Therefore,

$$\Theta \propto \begin{pmatrix} \sin \\ \cos \end{pmatrix} k\theta$$

and finally

$$\phi(r,\theta) = \text{const}\, r^{\pm k} \begin{pmatrix} \sin \\ \cos \end{pmatrix} k\theta + \text{const}$$

3. $$\phi(x,\varpi) = X(x)\Pi(\varpi) + \text{const}$$

$$\nabla^2\phi = \phi_{xx} + \frac{1}{\varpi}(\varpi\phi_{\varpi})_{\varpi} = X''\Pi + \frac{1}{\varpi}X(\varpi\Pi')' = 0$$

Therefore,

$$\frac{X''}{X} = -\frac{1}{\varpi}\frac{(\varpi\Pi')'}{\Pi}$$

which implies that each side must be independent of both r and ϖ. If

$$\frac{X''}{X} = k^2 = \text{const}, \quad X \propto e^{\pm kx}; \quad \text{then}\, \frac{1}{\varpi}\frac{(\varpi\Pi')'}{\Pi} = -k^2$$

or

$$\Pi'' + \frac{1}{\varpi}\Pi' + k^2\Pi = 0$$

Bessel's famous equation is $f'' + (1/x)f' + [1-(n^2/X^2)]f = 0$, for which a solution is $f(x) = J_n(x)$. [See, for example, Dwight, *Tables of Integrals & Other Math. Data.* Equation (800).] Thus, a solution of our differential equation for Π, stated earlier, is $\Pi \propto J_0(k\varpi)$, and

$$\phi(x,\varpi) \propto e^{\pm kx} J_0(k\varpi) + \text{const}$$

There is another solution of Bessel's equation: namely, $f(x) = Y_n(x)$, sometimes called $N_n(x)$, so that a more general solution sinusoidal in x would involve both $J_0(k,\varpi)$ and $Y_0(k,\varpi)$.

4. Take

$$\frac{X''}{X} = -K^2 = \text{const} \quad \text{then}\ X \propto \begin{pmatrix} \sin \\ \cos \end{pmatrix} kx$$

and the equation for Π becomes $\Pi'' + (1/\varpi)\Pi' - k^2\Pi = 0$. This is the modified Bessel equation

$$f'' + \frac{1}{x}f' - \left(1 + \frac{n^2}{x^2}\right)f = 0$$

with $n=0$. The solution is $f(x)=I_n(x)$ or $K_n(x)$, where the I_n are functions that blow up at $x=\infty$ and the K_n at $x=0$. So in our problem, $\Pi \propto I_0(k\varpi)$ or $K_0(k\varpi)$, and

$$\phi(\mathrm{x},\varpi) \propto I_0(k\varpi)\begin{pmatrix}\sin\\ \cos\end{pmatrix}kx + \text{const}$$

$$\text{or} \propto K_0(k\varpi)\begin{pmatrix}\sin\\ \cos\end{pmatrix}kx + \text{const}$$

The solutions found in (3) and (4) are the analogs of the solutions found in (1).

3.7 Elementary Axisymmetric Flows

We now turn our attention to certain incompressible irrotational flows having axial symmetry. It will be good for us to collect, for reference, some of the formulas that pertain to this case, in both spherical and cylindrical coordinates.

3.7.1 Uniform Stream

The plane parallel flow that has been discussed in Sample Problem 3.1(1a) has axial symmetry as well as being a plane flow, provided it is aligned with the axis in question. It is given by

$$u = U = \text{const},\ v = w = 0, \quad \text{and} \quad \phi = Ux + \text{const} \tag{3.14}$$

Also, from Eq. (1.12), it is clear that Stokes' stream function for this flow is

$$\psi = \frac{1}{2}U\varpi^2 + \text{const} \tag{3.15}$$

(Naturally, this is not the same as the plane stream function for the same flow, $\psi = Uy + \text{const}$. Note the difference in dimensions of these two stream functions, which follows from their respective physical meanings.)

3.7.2 Source

The three-dimensional analog of our plane source flow has both axial and spherical symmetry, as we shall see. Consider a purely radial flow out of a point. Using spherical components, $v_r \neq 0$, and $v_\theta = v_\omega = 0$. Now the conditions for continuity and irrotational flow in this special case become, respectively,

$$\frac{\partial (r^2 v_r)}{\partial r} = 0 \quad \text{and} \quad \frac{\partial v_r}{\partial \theta} = 0 = \frac{\partial v_r}{\partial \omega}$$

Thus, $v_r = \text{const}/r^2$. The quantity flow through any spherical surface is obviously constant; if it is called Q, then we can write

$$v_r = \frac{Q}{4\pi r^2} \tag{3.16}$$

The relations between v_r and ϕ and ψ are very simple in this case; it can be easily verified by integrating

$$\phi = -\frac{Q}{4\pi r} + \text{const} \quad \text{and} \quad \psi = -\frac{Q}{4\pi} \cos\theta + \text{const} \tag{3.17}$$

3.7.3 Doublet

If a source and a sink exist in a body of fluid, the flow they produce has axial symmetry about the line joining them. If their strengths are equal and opposite, Q and $-Q$, and the distance between them along the x axis is $\Delta\xi$ (source lying in $+x$ direction relative to sink), and if a limiting process is performed so that they coalesce with $Q\Delta\xi \to \mu$, the resulting singularity is a *three-dimensional doublet,* and the resulting axisymmetric flow is called doublet flow. It is described by

$$\left.\begin{aligned} &\phi = -\frac{\mu}{4\pi r^2}\cos\theta + \text{const}; \quad \psi = \frac{\mu}{4\pi r}\sin^2\theta + \text{const} \\ &v_r = \frac{\mu}{2\pi r^3}\cos\theta; \quad v_\theta = \frac{\mu}{4\pi r^3}\sin\theta \end{aligned}\right\} \tag{3.18}$$

Sample Problem 3.6

Verify formulas of Eq. (3.18).

Solution:

$$\phi_{\text{source}} = \frac{-Q}{4\pi r} + \text{const}$$

$$\frac{\partial}{\partial x}\left(\frac{-Q}{4\pi r}\right) = -\frac{Q}{4\pi}\left(-\frac{x}{r^2}\right) = \frac{Q}{4\pi}\frac{\cos\theta}{r^2}$$

So ϕ for this doublet is proportional to $\cos\theta / r^2$.

$$\psi_{\text{source}} = -\frac{Q}{4\pi}\cos\theta + \text{const} = -\frac{Q}{4\pi}\frac{x}{r} + \text{const}$$

$$\frac{\partial}{\partial x}\left(-\frac{Q}{4\pi}\frac{x}{r}\right) = -\frac{Q}{4\pi}\left(\frac{1}{r} - \frac{x}{r^2}\frac{x}{r}\right) = -\frac{Q}{4\pi}\frac{r^2 - x^2}{r^3}$$

$$= -\frac{Q}{4\pi}\frac{\varpi^2}{r^3} = -\frac{Q}{4\pi}\frac{\sin^2\theta}{r}$$

So ψ for this doublet is proportional to $\sin^2\theta / r + \text{const}$.

3.8 Other Elementary Axisymmetric Flows

We shall now illustrate another technique of finding elementary solutions of partial differential equations. The equation in this case is Laplace's equation:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0 \tag{3.19}$$

We are trying to find solutions $\phi(r,\phi)$. A standard procedure is to assume that there is a solution of the form

$$\phi(r,\theta) = R(r)\Theta(\theta) \tag{3.20}$$

that is, a solution that consists of the product of two independent functions, r and θ. If this is the case, Eq. (3.19) becomes

$$\Theta\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{R}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = 0$$

or

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) \tag{3.21}$$

But in Eq. (3.21), we have functions of r and θ in the left and right sides, respectively. If they are to be equal for all choices of r and θ, both must be equal to a constant, say k. We now have two ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)-kR=0 \tag{3.22}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+k\Theta\sin\theta=0 \tag{3.23}$$

Equation (3.22) is satisfied by both $R=\text{const}\; r^n$ and $R=\text{const}/r^{n+1}$, provided $k=n(n+1)$. Equation (3.23) is a form of Legendre's equation, as can be seen by making the substitution $u=\cos\theta$; the result is

$$\frac{\mathrm{d}}{\mathrm{d}u}\left\{(1-u^2)\frac{\mathrm{d}\Theta}{\mathrm{d}u}\right\}+n(n+1)\Theta=0 \tag{3.24}$$

which is Legendre's equation [1]. A solution of this equation for any integral value of n is $P_n(u)$, where P_n denotes the "Legendre polynomial of the first kind." Thus, two particular integrals of Laplace's equation, having axial symmetry, are

$$\phi=\text{const}\; r^n P_n(\cos\theta) \quad \text{and} \quad \phi=\text{const}\frac{P_n(\cos\theta)}{r^{n+1}} \tag{3.25}$$

By superposition, any function of the following form is also a solution:

$$\phi=\sum_{n=0}^{\infty}\left\{A_n r^n P_n(\cos\theta)+B_n\frac{P_n(\cos\theta)}{r^{n+1}}\right\} \tag{3.26}$$

where A_n and B_n are constants.

There is another series of particular solutions, not included in the form [Eq. (3.26)], involving the "Legendre polynomials of the second kind," Q_n, which are also solutions of Eq. (3.24). However, these solutions have logarithmic singularities at $\theta=0$ and π, for all r, and therefore are not often useful in satisfying the boundary conditions encountered in hydrodynamics. If we restrict ourselves to solutions that do not have this type of singularity, then Eq. (3.26) can be shown to be a *complete* general solution of Eq. (3.19), that is, every solution of Eq. (3.19) can be expressed in this form. This knowledge is of tremendous value, as we can write down Eq. (3.26) and proceed to determine the A_n and B_n to fit our boundary conditions, knowing that the flow we are seeking is included in the form assumed.

Let us see what some of these solutions look like. The following tabulation shows the forms of $P_n(u)$ for a few values of n:

$$\left.\begin{aligned} &P_0(u)=1 \quad P_2(u)=\frac{1}{2}(3u^2-1) \\ &P_1(u)=u \quad P_3(u)=\frac{1}{2}(5u^3-3u), \text{ etc.} \end{aligned}\right\} \tag{3.27}$$

Thus, P_0 gives rise to the following solutions

$$\phi = C_1 \quad \text{and} \quad \phi = \frac{C_2}{r}$$

of which the first is trivial, whereas the second is our three-dimensional source or sink, as in Eq. (3.17). Proceeding to P_1, we find

$$\phi = C_1\, r\cos\theta \quad \text{and} \quad \phi = C_2\frac{\cos\theta}{r^2}$$

The first of these is of a type that disappears at the origin and grows indefinitely as r increases; the second is our three-dimensional doublet [Eq.(3.18)].

If the Laplace equation is written in cylindrical, instead of spherical, coordinates, and the same technique of finding solutions by separation of variables is employed, a different series of particular solutions is found, as will be illustrated by a problem at the end of this chapter.

Sample Problem 3.7

Investigate the type of flow given by the term $P_n(\cos\theta)/r^{n+1}$ for $n=2$. Compare this with the axisymmetric flow obtained by superimposing equal and opposite doublets.

Solution:

$$\phi = A_2\frac{1}{r^3}P_2(\cos\theta)+\text{const}$$

$$P_2(u)=\frac{1}{2}(3u^2-1); \quad P_2(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1)$$

so that

$$\phi = \frac{1}{2} A_2 \frac{1}{r^3}(3\cos^2\theta - 1) + \text{const}$$

By superposition of doublets (in the x direction, to retain axial symmetry),

$$\phi_{\text{doublet}} = C\frac{\cos\theta}{r^2} + \text{const} = C\frac{x}{r^3} + \text{const}$$

$$\frac{\partial}{\partial x}\frac{x}{r^3} = \frac{1}{r^3} - \frac{3x^2}{r^5} = \frac{1-3\cos^2\theta}{r^3}$$

so that

$$\phi \propto \frac{1-3\cos^2\theta}{r^2}$$

the flows are the same.

Problems

3.1 By direct substitution, verify that ϕ and ψ in Eq. (3.18) satisfy their respective differential equations (see Problem 1.5).

3.2 Sketch some streamlines and equipotentials (in a meridional plane) for three-dimensional doublet flow.

3.3 a. Plane vortex flow should be the same as the flow in any normal plane due to a doubly infinite straight vortex filament. Verify this by calculating the velocity induced by such a vortex filament.

b. Plane source flow should be the same as the flow in any normal plane due to a doubly infinite uniform distribution of three-dimensional sources along a straight line. Verify this by calculating the velocity v_ϖ for such a distribution. Hint: Calculate $\mathrm{d}v\varpi$ at x,ϖ due to a source of strength $Q'\mathrm{d}\xi$ at $\xi,0$; then, integrate from $\xi = -\infty$ to $\xi = +\infty$. [It is best to replace the variables in the integrand in terms of ϖ and a variable angle α (say), then to integrate over $0 \leq \alpha \leq \pi$.]

3.4 Let two equal and opposite plane doublets coalesce to produce the next higher-order singularity

a. with their directions opposed like this $\rightarrow\leftarrow$
b. with their directions opposed like this $\uparrow\downarrow$

Sketch some streamlines for each case.

3.5 In a case of irrotational incompressible flow, the velocity potential equals $A\cos^2\theta$ on the sphere $r=1$ and vanishes at $r=\infty$. What is $\phi(r,\theta)$? What is $\psi(r,\theta)$:

3.6 Consider the flow due to superposition of a uniform stream and a source. Write the expressions for ϕ and ψ. Sketch some typical streamlines, including the ones that go through a stagnation point ($q=0$):

a. Plane case
b. Three-dimensional case

3.7 Consider the flow due to superposition of a uniform stream and a doublet directed upstream. Write ϕ and ψ; sketch some streamlines.

a. Plane case: Show that a certain circle, whose center is at the doublet, is a streamline.
b. Three-dimensional case: Show that a certain sphere, whose center is at the doublet, is a stream surface.

3.8 Show that the Cartesian velocity components $u, v,$ and w satisfy Laplace's equation, whereas the components referred to other (curvilinear) systems generally do not. Verify for some of the elementary flows that are considered in this chapter.

3.9 If the flow considered in Problem 2.6 is incompressible, show the following:

a. that a stream function exists for the flow in rotating coordinates
b. that it satisfies Poisson's equation

Motion of Bodies in an Incompressible Frictionless Fluid

- Flow past a blunt-end rod, sphere, and circular cylinder
- Dirichlet's theorem
- Distribution of singularities and sources
- Elementary plane and axisymmetric flows
- Impulse and momentum

4.1 Introduction

The only restriction imposed on the flow of an inviscid fluid by the presence of a solid body is that there should be no flow through the surface of the body. This means that the normal (to the body surface) component of the fluid velocity at every point of the surface must be equal to the same component of the velocity of that part of the body. In particular, in steady flow the surface of the body must be a stream surface.

For steady flow, any stream surface of any of the flows set up in Chapter 3 could be replaced by solid surfaces without changing the exterior fluid flow. But there are only a few special stream surfaces that have any practical interest in this regard.

Consider, first, the flow resulting from the superposition of uniform stream and source, in Problem 3.6(a) of Chapter 3. In this flow, there is one streamline (stream surface) that presents a rounded "leading edge" to the stream and extends infinitely far downstream (Fig. 4.1). If the stream function is written as

$$\psi = Uy + \frac{Q}{2\pi}\theta + C \qquad (0 \leqq \theta < 2\pi) \tag{4.1}$$

where C is an arbitrary constant, then the streamline in question is the one defined by $\psi = (Q/2) + C$. It is one of the streamlines that

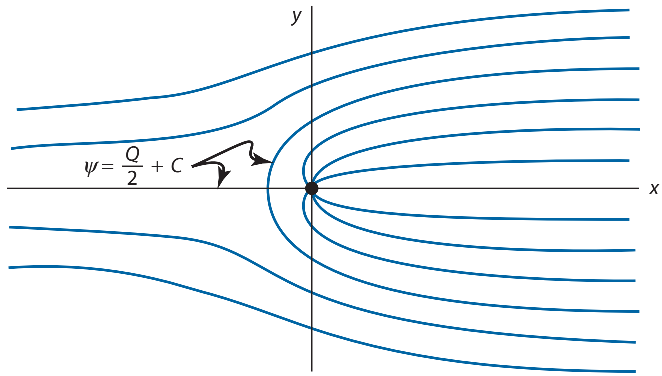


Fig. 4.1 Streamlines generated by a free stream and a line source.

passes through the *stagnation point* (point of zero velocity) at $x = -(Q/2\pi)U$ and $y = 0$.

Now this stream surface could be replaced, in a steady inviscid flow, by a solid body having the same shape, namely, a board having a rounded leading edge and extending indefinitely downstream. This body would then replace all of the fluid that exudes from the source. All of the characteristics of the flow outside the streamline $\psi = (Q/2) + C$ in Eq. (4.1) would be unchanged by this substitution! Thus, we have discovered a method of calculating the characteristics of steady flows about solid bodies, that is, to set up the flow due to appropriate singularities in a body of unlimited fluid, such that one stream surface has the shape of the desired solid. Then the flow exterior to this surface is the same as the desired flow exterior to the solid.

Let us consider the shape of the stream surface of Fig. 4.1 in detail. Its equation is from Eq. (4.1)

$$y = \frac{Q}{2U}\left(1 - \frac{\theta}{\pi}\right) \qquad (0 \leqq \theta < 2\pi) \qquad (4.2)$$

Far downstream ($\theta \to 0, 2\pi$), it approaches the asymptotic ordinate $\pm Q/2U$, which we will call $\pm h/2$. Hence, in a physical problem involving a board of thickness h, we would eliminate the source strength Q in favor of this physically significant dimension and the stream speed. Then, for the flow around the board we would write

$$\psi = U\left(y + \frac{h}{2}\frac{\theta}{\pi}\right) + C \qquad (0 \leqq \theta < 2\pi)$$

$$\phi = U\left(x + \frac{h}{2\pi}\ln r\right) + C' \qquad (4.3)$$

Sample Problem 4.1

In the case of flow past a blunt-edged plate (stream plus source), show that the curve of constant flow inclination is a circle through the nose and the origin (source). Show that its radius is

$$\frac{h}{4\pi}\sqrt{1+\frac{1}{k^2}}$$

where h is the asymptotic thickness of plate and k the tangent of angle of flow inclination.

Solution:

$$u=\phi_x=U\left\{1+\frac{h}{2\pi r}\frac{x}{r}\right\};\quad v=\phi_y=U\frac{h}{2\pi r}\frac{y}{r}$$

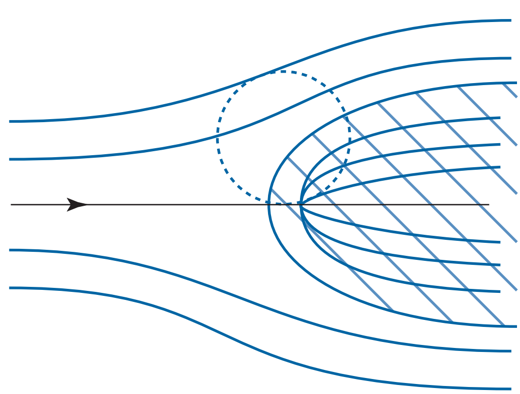
$$\text{Flow inclination}\equiv\frac{v}{u}=\frac{y}{2\pi(r^2/h)+x}=k$$

or

$$x^2+y^2+\frac{h}{2\pi}x-\frac{h}{2\pi k}y=0$$

A simple calculation (or an analytical geometry textbook) confirms that this is a circle of radius R,

$$R=\left\{\left(\frac{h}{4\pi}\right)^2+\left(\frac{h}{4\pi k}\right)^2\right\}^{1/2}=\frac{h}{4\pi}\sqrt{1+\frac{1}{k^2}}$$



It passes through the origin and the nose (stagnation point), where $x = -Q/2\pi U = -h/2\pi$ and $y = 0$.

4.2 Flow Past a Blunt-Ended Rod

By exactly the same reasoning, the symmetrical steady flow around the nose of a blunt-ended rod that extends indefinitely downstream can be determined from the superposition of a uniform stream and a three-dimensional source carried out in Problem 3.6(b). In this case, Stokes' stream function is

$$\psi = \frac{1}{2}U\varpi^2 - \frac{Q}{4\pi}\cos\theta + C \tag{4.4}$$

The streamline that passes through the stagnation point is defined by $\psi = (Q/4\pi) + C$ or

$$\varpi^2 = \frac{Q}{2\pi U}(1 + \cos\theta) \tag{4.5}$$

The asymptotic radius is therefore $\sqrt{Q/(\pi U)}$. Let this be called R, and then

$$\psi = \frac{1}{2}U\left\{\varpi^2 - \frac{1}{2}R^2\cos\theta\right\} + C \tag{4.6}$$

Some of the streamlines in a meridional plane are plotted in Fig. 4.2. We calculate the pressure distribution over such a nose shape in a problem at the end of this chapter. Such a calculation might be valuable in various practical problems, such as estimating the pressures on or near the nose of a dirigible or nacelle (although the shape is somewhat too blunt for most of these cases). Prandtl used this calculation to design the well-known type of pitot-static tube that bears his name.

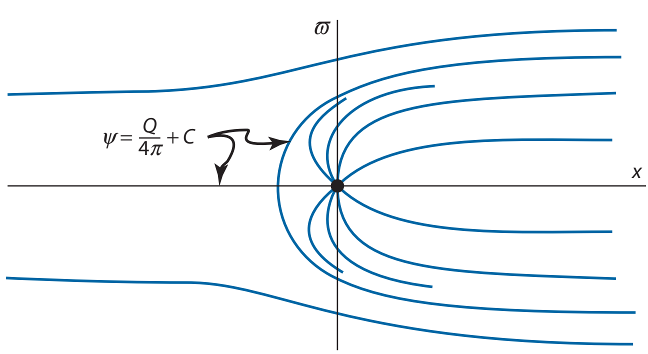


Fig. 4.2 Streamlines generated by a free stream and a point source.

Sample Problem 4.2

How far directly in front of the blunt-ended rod, measured in asymptotic diameters, is the stream speed reduced 1%?

Solution:
Along the streamline $\theta = -\pi$, the velocity is

$$v_x = U - \frac{Q}{4\pi x^2} = U\left\{1 - \frac{R^2}{4x^2}\right\}$$

At the point where $v_x / U = 0.99$,

$$\frac{R^2}{4x^2} = \frac{1}{100} \quad \text{or} \quad \frac{x}{2R} = \frac{5}{2}$$

The stagnation point is at $(R^2/4x^2) = 1$ or $(x/2R) = (1/4)$. Thus, the 1% point is 2.25 diameters upstream of the nose.

4.3 Flow Past a Sphere

The flow in Problem 3.7(b) can be used to calculate the steady irrotational flow past a sphere. After the doublet strength μ is replaced in terms of the stream speed U and the radius R of the sphere, as we can easily verify, the flow is described by

$$\psi = \frac{1}{2}U\left\{r^2 - \frac{R^3}{r}\right\}\sin^2\theta + C; \quad \phi = \frac{1}{2}U\left\{2r + \frac{R^3}{r^2}\right\}\cos\theta + C'$$

$$v_r = U\left\{1 - \frac{R^3}{r^3}\right\}\cos\theta; \quad v_\theta = -\frac{1}{2}U\left\{2 + \frac{R^3}{r^3}\right\}\sin\theta \tag{4.7}$$

On the sphere, the velocity is given by

$$v_\theta = -\frac{3}{2}U\sin\theta \tag{4.8}$$

so that there are two stagnation points, at $\theta = 0$ and π, and the maximum local speed is 50% greater than the stream speed. The pressure is given by

$$\frac{p - p_0}{\rho U^2/2} = 1 - \frac{9}{4}\sin^2\theta \tag{4.9}$$

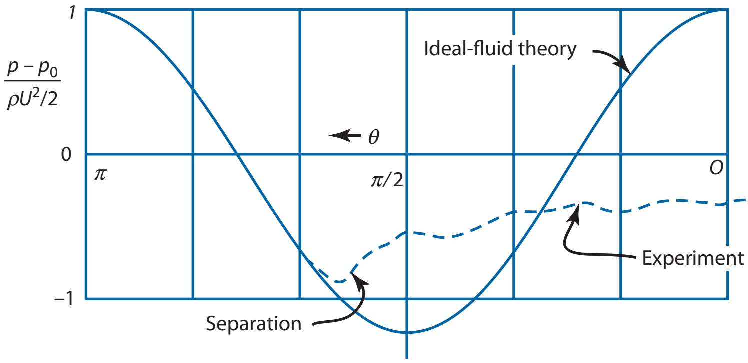


Fig. 4.3 Rough comparison of the pressure distributions.

Now, in addition to the axial symmetry, we see that the velocities and pressure are distributed symmetrically with respect to the equatorial plane $\theta = \pi/2$. Consequently, there can be no force on the sphere! This is surely in contradiction with our experiences and serves as a fresh reminder of the limitations of ideal-fluid theory in describing real-fluid flows. The trouble is that a sphere is not a "streamlined" body, and viscous effects are important in determining its flow pattern. Actually, the boundary layer separates from the surface just above the equator $\theta = \pi/2$ (in agreement with viscous-fluid theory), and from there the flow loses its resemblance to ideal-fluid flow. Figure 4.3 shows a rough comparison of the pressure distributions; it is clear that the boundary-layer separation is ultimately responsible for the appreciable drag of the sphere.

4.4 Flow Around a Circular Cylinder

In spite of the shortcomings of our inviscid-fluid theory, we shall proceed to the consideration of the plane steady irrotational flow around a circular cylinder, which was set up in Problem 3.7(a). In terms of the radius of the cylinder a, formulas for this case are

$$\psi = U\left\{r - \frac{a^2}{r}\right\}\sin\theta + C; \quad \phi = U\left\{r + \frac{a^2}{r}\right\}\cos\theta + C'$$

$$v_r = U\left\{1 - \frac{a^2}{r^2}\right\}\cos\theta; \quad v_\theta = -U\left\{1 + \frac{a^2}{r^2}\right\}\sin\theta \tag{4.10}$$

Thus, the maximum surface speed is $2U$ in this case, and the local pressures are correspondingly lower than on the sphere. Again, the pressure is distributed symmetrically fore and aft, and there is no force on the cylinder.

In this case, however, we encounter a new and important phenomenon. The boundary conditions satisfied by Eq. (4.10) are 1) $v_r = 0$ when $r = a$ and 2) $\boldsymbol{q} \to U\boldsymbol{i}$, as $r \to \infty$. But these would be just as well satisfied if we were to superimpose a plane vortex flow about the origin, of any desired strength, namely,

$$\psi = U\left\{r - \frac{a^2}{r}\right\}\sin\theta - \frac{\Gamma}{2\pi}\ln r + C$$

$$\phi = U\left\{r + \frac{a^2}{r}\right\}\cos\theta + \frac{\Gamma}{2\pi}\theta + C' \qquad (0 \leqq \theta < 2\pi)$$

$$v_r = U\left\{1 - \frac{a^2}{r^2}\right\}\cos\theta; \quad v_\theta = -U\left\{1 + \frac{a^2}{r^2}\right\}\sin\theta + \frac{\Gamma}{2\pi r} \tag{4.11}$$

With circulation, the local velocity at the surface becomes $-2U\sin\theta + \Gamma/2\pi a$. Thus, the stagnation points have moved to (Fig. 4.4)

$$\theta_0 = \sin^{-1}\frac{\Gamma}{4\pi U a} \tag{4.12}$$

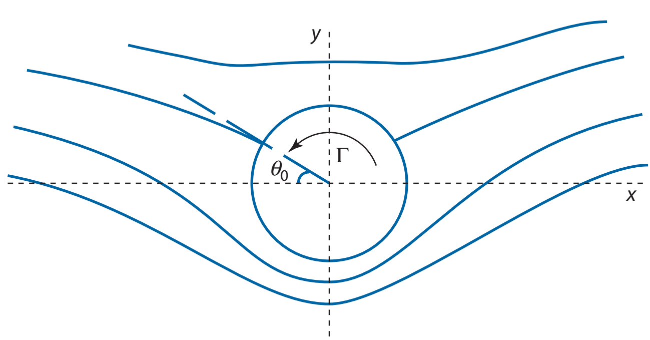


Fig. 4.4 Flow around a circular cylinder with circulation Γ.

provided $|\Gamma| \leqq 4\pi Ua$. (If $|\Gamma|$ has a greater value, the stagnation points merge and occur in the flow outside the cylinder; this case is of academic interest only.)

The fluid pressure on the cylinder is now

$$p = p_0 + \frac{\rho}{2}U^2 - \frac{\rho}{2}\left(2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2 \tag{4.13}$$

and again, by symmetry, we see that there is no force component in the x direction. The force component in the y direction is easily computed; it is, per unit length,

$$Y = -\int_0^{2\pi} pa\sin\theta\,\mathrm{d}\theta = -\rho U\Gamma \tag{4.14}$$

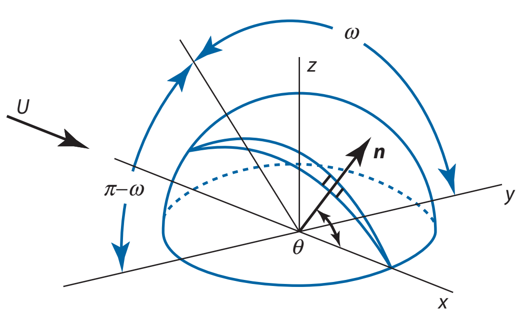
We see that, although there is no drag, there is a lift on a circular cylinder with circulation. This is obviously related to the so-called *Magnus effect*, which produces lift on a rotating cylinder. In a real fluid, the circulation is produced by the action of viscosity near the spinning cylinder. In an ideal fluid, however, we are at a loss to know what value to take for Γ.

Sample Problem 4.3

Calculate the total lift on a hemisphere of radius R fixed to the flat bottom of a channel carrying a steady stream of velocity U and density ρ.

Solution:
The pressure is given by Eq. (4.9) for $0 \leqq \omega \leqq \pi$ (see figure). We must calculate

$$F_z = -\int_S pn_z\,\mathrm{d}\sigma = -\int_0^\pi\int_0^\pi pn_z R\,\mathrm{d}\theta R\sin\theta\,\mathrm{d}\omega$$



The component n_z can be seen as $\sin\theta\sin\omega$. Thus,

$$F_Z = -\int_0^{\pi}\int_0^{\pi}\left\{p_0 + \rho\frac{U^2}{2}\left(1-\frac{9}{4}\sin^2\theta\right)\right\}R^2\sin\omega\sin^2\theta\,\mathrm{d}\omega\,\mathrm{d}\theta$$

$$= -2R^2\int_0^{\pi}\left\{p_0 + \rho\frac{U^2}{2}\left(1-\frac{9}{4}\sin^2\theta\right)\right\}\sin^2\theta\,\mathrm{d}\theta$$

$$= -\pi R^2\left\{p_0 + \rho\frac{U^2}{2}\right\} + \frac{9}{2}R^2\rho\frac{U^2}{2}\int_0^{\pi}\frac{3}{8}\pi\sin^4\theta\,\mathrm{d}\theta$$

$$= -\pi R^2 p_0 + \rho\frac{U^2}{2}\left(-1+\frac{27}{16}\right)\pi R^2$$

The first term is, of course, the hydrostatic load (downward) on the body. The second, with the coefficient $11\pi/16$, is the hydrodynamic lift, which will clearly overcome the hydrostatic force at a certain value of $\rho U^2/2$.

4.5 Multiply-Connected Regions and Cyclic Constants

The phenomenon encountered earlier, namely, that the flow was not uniquely determined by the differential equation and boundary conditions, is actually a very general situation. It occurs when the region containing fluid is not simply connected (as will be defined presently). Moreover, it is certainly not a trivial phenomenon, for, as we have seen, the indeterminacy lies in the value of Γ that determines the magnitude of the force on the body!

A *connected* region is one in which it is possible to draw a path from any point to any other without leaving the region. In bodies of fluid, it means that a path can be drawn that does not leave the fluid.

Two paths are *reconcilable* if they can be deformed into coincidence with one another without leaving the fluid. Also, a closed circuit is called *reducible* if it can be shrunk to a point without leaving the fluid.

A *simply-connected* region is one in which all paths between any two points are reconcilable; thus, every circuit that can be drawn is reducible.

In a *doubly-connected* region, two and only two irreconcilable paths can be drawn; thus, one irreducible circuit can be drawn, and all others are reducible or can be made to coincide with it.

In an *n-ply-connected* region, n and only n irreconcilable paths can be drawn, and this means that there are $n-1$ possible irreducible circuits.

For example, the region of the plane flow about our circular cylinder, shown earlier, is doubly connected, as, in fact, the region containing any infinite cylinder must be (see Fig. 4.5). A region containing two such cylinders

Fig. 4.5 Multiply-connected regions.

would be triply connected and so forth. The region containing a sphere or any other solid three-dimensional body (without holes) is simply connected, as is a region containing any number of such bodies. However, a region containing a ring is certainly doubly connected and so forth.

In determining the connectivity of a region, a singularity must be counted as a body so that a vortex ring makes a region doubly connected and so forth.

Now, the relation of all this to hydrodynamics is indicated by the following theorems, which are proved in the following section. In an n-ply-connected region, the values of the circulation about $n-1$ irreducible circuits must be specified, in addition to the usual boundary conditions to determine the flow. [Modern studies show that in certain complicated topological configurations there can occur irreducible circuits for which (by application of Stokes' theorem) the cyclic constant must be zero. Thus, the correspondence between connectivity (as defined earlier) and cyclic constants is not as simple as suggested here (correction made by Prof. W.Hayeb, 1960.)] These values of Γ are called *cyclic constants* because they are the values by which the velocity potential ϕ increases (or decreases) per circuit around the contours. Any flow in a multiply-connected region where all of the cyclic constants are not zero is called *cyclic flow*.

In the light of this theorem, we see why the flow about the circular cylinder was indeterminate. The region was doubly connected, and we needed, to determine the flow, some specifications for the one cyclic constant involved. In more complicated problems, the determination of the flow and the forces on bodies would require some sort of specification of a greater number of cyclic constants.

As stated at the beginning of this chapter, the boundary condition at an impermeable surface in frictionless flow is only that the normal (to the surface) components of fluid and surface velocities must be equal. There is a general way of writing this condition, which will be found to be extremely useful.

The reader should recognize that the equation

$$F(x,y,z,t)=0 \tag{4.14a}$$

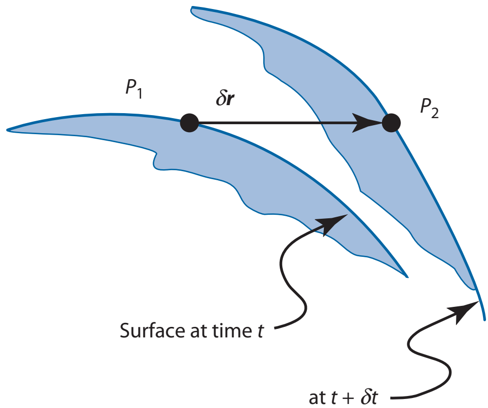


Fig. 4.6 Boundary condition at an impermeable surface in frictionless flow.

describes a surface moving in the x, y, and z space. Furthermore, the statement

$$\delta F = \frac{\partial F}{\partial t}\delta t + \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z = 0 \tag{4.14b}$$

means that the increments and δt, δx, δy, and δz are taken so that both P_1(at time t) and P_2(at time $t+\delta t$) lie on the surface, where $\delta \boldsymbol{r} = (\delta x, \delta y, \delta z)$ is a vector from P_1 to P_2 (see Fig. 4.6).

Our boundary condition requires that a fluid particle at P_1 move in such a way that at $t+\delta t$ it finds itself at a point P_2. By dividing this equation by δt and identifying $\delta \boldsymbol{r} / \delta t$ as the fluid velocity $\boldsymbol{q}$, we have this boundary condition in the form

$$\frac{\mathrm{D}F}{\mathrm{D}t} = \frac{\partial F}{\partial t} + \boldsymbol{q} \cdot \nabla F = 0$$

on

$$F(x, y, z, t) = 0 \tag{4.14c}$$

Sample Problem 4.4

Suppose an impermeable surface is described by $y = Y(x,t)$. Using Eq. (4.14c), show that the boundary condition for imcompressible frictionless flow over this surface is $v(x,Y) = (\partial Y / \partial t) + u(x,Y)(\partial Y / \partial x)$.

Solution:
If $y = Y(x,t)$, the function $F(x,y,z,t)$ is

$$F \equiv Y(x,t) - y = 0$$

The boundary condition reads

$$\frac{DF}{Dt} = \frac{\partial Y}{\partial t} + u\frac{\partial Y}{\partial x} - v = 0 \text{ on } F = 0$$

or

$$v = \frac{\partial Y}{\partial t} + u\frac{\partial Y}{\partial x} \text{ on } y = Y(x,t)$$

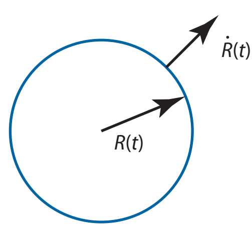
Sample Problem 4.5

An underwater explosion can be approximated by the case of an expanding impermeable sphere of radius $R = R(t)$ in an unlimited ocean of incompressible fluid otherwise at rest. If the density is ρ and the pressure far from the explosion is p_0, what is the pressure observed at a distance r from the center?

Solution:
To find the pressure, we have to find q and ϕ_t. At any instant, the boundary conditions are

$$v_n = v_r = \dot{R} \quad \text{at} \quad r = R(t)$$

$$q \to 0 \quad \text{as} \quad r \to \infty$$



Note the spherical symmetry. Clearly, the flow at this instant is purely radial. It is just three-dimensional source flow:

$$\phi(r,t) = \frac{-Q(t)}{4\pi r} + \text{const}$$

$$v_r(r,t) = \frac{Q(t)}{4\pi r^2}$$

To find $Q(t)$, we satisfy the boundary condition at $r = R(t)$:

$$\dot{R}(t) = \frac{Q(t)}{4\pi R^2} \quad \text{or} \quad Q(t) = 4\pi R^2 \dot{R}$$

Thus,

$$\phi(r,t) = -\frac{R^2\dot{R}}{r} + \text{const}$$

$$v_r(r,t) = \frac{R^2\dot{R}}{r^2}$$

Bernoulli's equation for unsteady flow is

$$\rho\phi_t + \rho\frac{q^2}{2} + p = F(t)$$

We evaluate $F(t)$ at $r = \infty$, where ϕ_t and q vanish and p becomes p_0; thus, $F(t) = p_0 = \text{const}$ and

$$\begin{aligned} p &= p_0 - \rho\phi_t - \rho\frac{q^2}{2} \\ &= p_0 - \rho\frac{\partial}{\partial t}\left(-\frac{R^2\dot{R}}{r} + \text{const}\right) - \frac{1}{2}\rho\left(\frac{R^2\dot{R}}{r^2}\right)^2 \\ &= p_0 + \rho\frac{2R\dot{R}^2}{r} + \rho\frac{R^2\ddot{R}}{r} - \frac{1}{2}\rho\left(\frac{R^2\dot{R}}{r^2}\right)^2 \end{aligned}$$

4.6 Dirichlet's Theorem

Most of the fluid-flow cases that concern us are included in a theorem, which is sometimes known as *Dirichlet's theorem*. The solution of Laplace's equation, $\nabla^2 f = 0$ in a region, is uniquely determined (up to an additive constant) by the boundary values of f, or by the boundary values of $\partial f / \partial n$ provided that the cyclic constants are also given.

To prove this, we consider various cases separately. Consider first a simply-connected region with f given all over the boundary. (This is called the *first boundary-value problem*.) Imagine that two solutions f_1 and f_2 have been found. Then, $f_1 - f_2$ is a third solution, and it has the boundary value zero all over the boundary. But no solutions of Laplace's equation can have a maximum or minimum within the region, and therefore $f_1 - f_2$ is zero everywhere in the region. Thus, f_1 and f_2 are, in fact, identical, and the theorem is proved.

Consider next the *second boundary-value problem*, where $(\partial f / \partial n)$ is given over the boundary, and again let the region be simply connected.

Again, imagine that there are two solutions f_1 and f_2, and therefore a third, $f_3 \equiv f_1 - f_2$. Now, it is clear that $\partial f_3 / \partial n = 0$ all over the boundary; but then

$$\int_S f_3 \frac{\partial f_3}{\partial n} \, \mathrm{d}\sigma = \int_V (\mathrm{grad}\ f_3)^2 \, \mathrm{d}\tau = 0 \tag{4.15}$$

where the volume integral is taken throughout the region. This can only mean that grad f_3 is zero everywhere, and f_3 is a constant. Thus, f_1 and f_2 differ only by a constant, and the theorem is again proved.

Let us now proceed to multiply-connected regions. In the "first" problem, the same proof applies for if f_1 and f_2 have the same boundary values, they must have the same cyclic constants. Our proof for the "second" problem is not valid if the cyclic constants are not given, but if they are given, as stated in the theorem, the third solution f_3 is not cyclic, having all of its cyclic constants equal to zero. Then, the same proof is valid, and the same result is obtained.

The theorem also applies to "mixed boundary-value problems," in which each f and $\partial f / \partial n$ are given over part of the boundary and can be extended to infinite regions and so forth. To proceed further in this subject, and also to see the present proofs worked out in somewhat more detail, the student is referred to [1], Secs. 40, 41, 47, 48, and 52.

The importance of Dirichlet's theorem in hydrodynamics should be obvious. Without this theorem, we could never be sure whether the flow pattern we calculated for a certain boundary-value situation was correct. Now, we see that any solution that we can construct must be the only possible one if it satisfies the differential equation and the boundary conditions and has the right values of circulation.

It may be advisable to add a few words here about the mysterious business of specifying the cyclic constants. Back in history, theoreticians were led by Helmholtz's theorems to specify the cyclic constants to be all zero; this led to physically ridiculous results, as we have already seen, and therefore to a complete rejection of theoretical hydrodynamics by practical engineers. In modern aerodynamics, the circulations are actually specified by considerations derived from viscous-fluid flow—even though ideal-fluid theory is being used. This was the great improvement introduced by Kutta, Joukowski, and later Prandtl, all of whom were interested in aerodynamics, which led to the rapprochement of classical hydrodynamics and practical aerodynamics, beginning in the first decade of the 1900s.

Sample Problem 4.6

Find the exact flaw in the proof of Sec. 4.6 for the "second" problem when applied to a multiply-connected region with undetermined cyclic constants.

Solution:
In a multiply-connected region, f_3 is multivalued unless all of its cyclic constants are equal to zero, that is, unless the cyclic constants of the problem are all determined, so that they are the same for f_1 and f_2. If f_3 is multivalued, the integral transformation (surface to volume) in Eq. (4.15) is invalid.

4.7 Barriers

Even when the cyclic constants are all specified by some methods or other, and the flow is uniquely determined, we are left with a certain degree of complication in multiply-connected regions, namely, a cyclic velocity potential is not single valued. Its value at any point of the region can be represented as

$$\phi(x,y,z)=\hat{\phi}(x,y,z)+\sum_{i=1}^{n-1}k_i\Gamma_i \tag{4.16}$$

where $\hat{\phi}$ is single valued, n denotes the connectivity of the region, Γ_i are the cyclic constants, that is, the values of the circulation, and the k_i are positive and negative integers. Each k_i can be increased or decreased by one by traversing an irreducible circuit once. This eliminates the possibility of applying many useful theorems that are restricted to single-valued functions.

To avoid this difficulty, as we have already done in Sec. 3.5.2, we simply restrict all paths, contours, ranges of integration, etc., so that no complete circuits of bodies or singularities with circulation can be made. In other words, we restrict all the k_i in Eq. (4.16) to zero. This is usually done by introducing imaginary *barriers* between bodies or between bodies and boundary, so as to render the region simply connected. Upon reflection, one will see that an n-ply-connected region requires exactly $n-1$ barriers; if fewer are introduced, the region will remain multiply connected; if more are introduced, the connectivity is destroyed.

Now the values of ϕ are different on the two sides of any barrier (if the barrier is needed). The barriers, although they are imaginary and do not affect the flow, must be considered as parts of the boundaries of the fluid. Thus, any integrations that are supposed to be carried over the boundaries of the region will have to be carried over the barriers as well.

For example, to make ϕ for the circular cylinder (Sec. 4.4) single valued, we can introduce a barrier running from any point of the cylinder out to infinity. Clearly, there is a jump in ϕ, of the amount Γ, across this barrier at any point (Fig. 4.7).

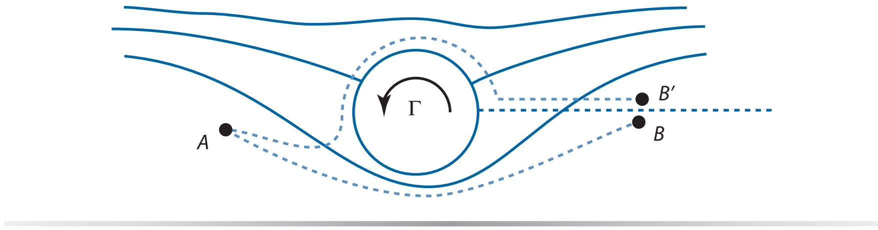


Fig. 4.7 Jump in ϕ across a barrier.

Sample Problem 4.7

Consider the irrotational concentric plane flow between two concentric circular cylinders of radii a and b. What is ϕ? What barrier is needed to make ϕ single valued? Calculate the kinetic energy of the flow (per unit depth) by surface integral and verify the result by direct volume integration.

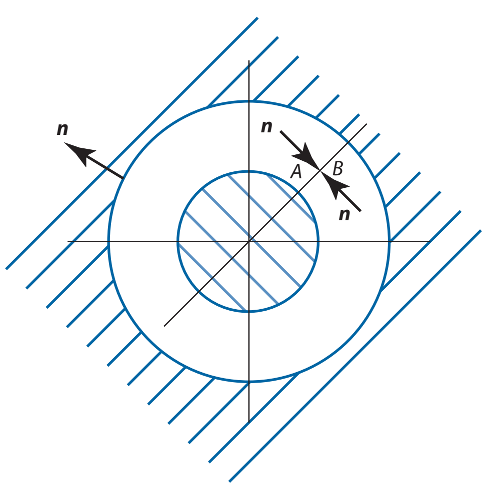
Solution:

$$\phi = \frac{\Gamma}{2\pi}\theta + \text{const}$$

The barrier required is any barrier running from the inner to the outer cylinder, so as to render θ single valued. The radial barrier AB, shown in the figure, would be a good one.

$$\text{K.E.} = \frac{\rho}{2}\int_S \phi \frac{\partial \phi}{\partial \boldsymbol{n}}\, \mathrm{d}\sigma$$

where $\boldsymbol{n}$ is directed out of the region.



The integrand vanishes at both inner and outer cylinders because $(\partial\phi/\partial \boldsymbol{n}) = 0$. Thus,

$$\begin{aligned}\text{K.E.} &= \frac{\rho}{2}\int_a^b \phi_{\text{top}}(-v_\theta)\,\mathrm{d}r + \frac{\rho}{2}\int_a^b \phi_{\text{bottom}} v_\theta\,\mathrm{d}r \\ &= \frac{\rho}{2}\int_a^b \{\phi_{\text{bottom}} - \phi_{\text{top}}\} v_\phi\,\mathrm{d}r = \frac{\rho}{2}\int_a^b \Gamma\frac{\Gamma}{2\pi r}\,\mathrm{d}r \\ &= \rho\frac{\Gamma^2}{4\pi}\ln\frac{b}{a}\end{aligned}$$

[Note: You must resist the temptation to think of $\int_S$ as a line integral, having a direction; it is just a *sum* of values $\phi(\partial\phi/\partial r)$ times values of area elements $\mathrm{d}\sigma = \mathrm{d}r$.]

To check the result by a direct volume integration,

$$\begin{aligned}\text{K.E.} &= \frac{\rho}{2}\int_V q^2\mathrm{d}\tau = \frac{\rho}{2}\int_a^b\int_0^{2\pi}\left(\frac{\Gamma}{2\pi r}\right)^2 r\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \frac{\rho}{2}\frac{\Gamma^2}{4\pi^2}2\pi\int_a^b\frac{\mathrm{d}r}{r}\end{aligned}$$

It is interesting that K.E. $\to\infty$ as either $a\to 0$ or $b\to\infty$; in the first case, $v_\theta\to\infty$. In the second case, much fluid is in motion. This often complicates analyses using energy principles.

4.8 Distribution of Singularities: Plane Case

In view of our success in representing the flow around bodies by the use of point singularities such as sources and doublets, we are led to the consideration of a line or sheet singularities, which we can produce by distributing point singularities nonuniformly. The practical engineering problem that we are approaching is the calculation of the flow around a *given* body. This investigation will bring us closer to the solution of this problem. We will take up the plane case first and then proceed to the axisymmetric.

4.8.1 Source–Sink Distributions

Consider a source distribution $f(\xi)$ along the x axis. The elementary stream function at x, y due to a source strength $f(\xi)\,\mathrm{d}\xi$ at ξ (Fig. 4.8) is

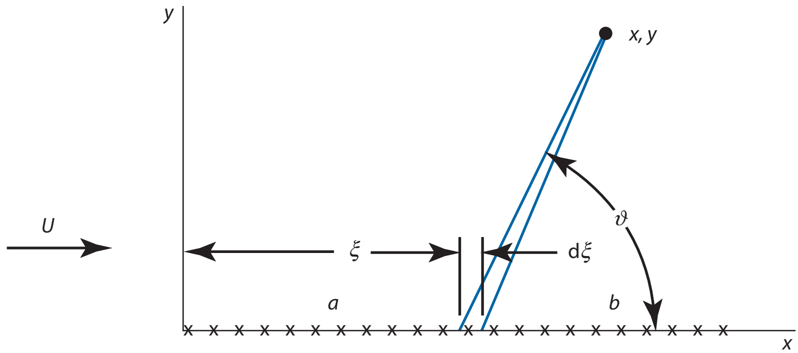


Fig. 4.8 Sources distributed along the x axis.

$$\mathrm{d}\psi = \frac{1}{2\pi} f(\xi)\,\mathrm{d}\xi\,\vartheta + C \qquad (0 \le \vartheta < 2\pi)$$

where ϑ denotes $\tan^{-1}[y/(x-\xi)]$. Hence, if the sources are distributed between $\xi = a$ and $\xi = b$ according to the distribution function $f(\xi)$, the total stream function, including a stream of speed U, is

$$\psi = Uy + \frac{1}{2\pi}\int_a^b f(\xi)\vartheta\,\mathrm{d}\xi + C \tag{4.17}$$

The velocity components are

$$u = U + \frac{1}{2\pi}\int_a^b f(\xi)\frac{\partial\vartheta}{\partial y}\,\mathrm{d}\xi$$

$$v = -\frac{1}{2\pi}\int_a^b f(\xi)\frac{\partial\vartheta}{\partial x}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_a^b f(\xi)\frac{\partial\vartheta}{\partial\xi}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{\vartheta(a)}^{\vartheta(b)} f(\xi)\,\mathrm{d}\vartheta \tag{4.18}$$

Consider these formulas in the limit $y \to 0$, that is, very close to the sheet singularity. First, calculating v for such a point at very small positive y, one finds a contribution only from the sources in the immediate neighborhood of the point. (The student should verify this.) The approximate formula for v is then,

$$v(x, y) \approx \frac{1}{2\pi} f(x)\int_0^\pi \mathrm{d}\vartheta = \frac{1}{2} f(x) \qquad \text{for} \qquad y \to 0+ \tag{4.19}$$

Repeating the calculation for a point just below the line, we find the same result but with the sign changed.

This says that the fluid leaves the line singularity normally, equally divided between the two sides of the line.

In the same approximation, the u component can be calculated. For small y,

$$\frac{\partial \vartheta}{\partial y} = \frac{x-\xi}{(x-\xi)^2 + y^2} \approx \frac{1}{x-\xi}$$

except in a small interval near $x = \xi$. But this time, the contribution from this small interval has cancelling positive and negative parts (please verify!), and the result is

$$u(x, y) \approx \frac{1}{2\pi} P\int_a^b f(\xi)\frac{\mathrm{d}\xi}{x-\xi} \qquad \text{for} \qquad y \to 0 \tag{4.20}$$

Here, the symbol P means that the Cauchy Principal Value of the improper integral is to be taken. This is defined as follows:

$$P\int_a^b f(\xi)\frac{\mathrm{d}\xi}{x-\xi} \equiv \lim_{\epsilon \to 0}\left\{\int_a^{x-\epsilon} f(\xi)\frac{\mathrm{d}\xi}{x-\xi} + \int_{x+\epsilon}^b f(\xi)\frac{\mathrm{d}\xi}{x-\xi}\right\} \tag{4.21}$$

provided that this limit exists. It does exist when the singularity of the integrand is of the type involved here, namely, $(x-\xi)^{-1}$ at $x = \xi$, for the positive and negative infinities cancel one another (see [2]).

We have succeeded in constructing an interesting line singularity (actually a sheet singularity because we are considering plane flow). Its properties are given by Eqs. (4.17–4.20); its flow is symmetrical about the x axis.

Let us investigate the use of this sheet singularity to reproduce the steady plane flow about a slender cylinder $y = Y(x)$. The boundary condition is

$$\frac{v}{u} = Y'(x) \quad \text{on} \quad y = Y(x)$$

or, approximately because we expect this to be a small-perturbation flow (as defined in Sample Problem 2.1),

$$\frac{v}{u} = \frac{v}{U + u'} \approx \frac{v}{U} = Y'(x) \quad \text{on} \quad y \approx 0 \tag{4.22}$$

The only other boundary condition is applied at great distances from the cylinder, namely, there the flow must be uniform and parallel at speed U. Now, the velocities associated with our sheet singularity vanish at large distances (please verify!) so that we can construct this flow by superimposing

1) a uniform stream and 2) a properly designed source–sink sheet. Then, $u = U + u'$ and $v = v'$, where u' and v' are given by Eq. (4.18), and, from Eq. (4.19), the source-strength function for a slender cylinder is given by its slope:

$$f(x) \approx 2UY'(x) \tag{4.23}$$

This makes the approximate calculation of the surface velocities and pressures for any given slender symmetrical body very easy.

The approximation that we have made is based on u' and v' being small and $f(x)$ being a relatively slow varying function. It is now clear that this means that $Y'(x)$ and the higher derivatives of $Y(x)$ must be small. This is what "slender body" really means; surely, the actual size of the body cannot matter, but only its slope, curvature, etc.

These assumptions are almost always violated at the leading and trailing edges. In fact, there are stagnation points (unless the profile has cusps at these edges), and this certainly implies gross violation of the assumption $u' \ll U$. The slender-body calculation must certainly be expected to give erroneous results near stagnation points. Actually, it is found to give good indications of the pressure and velocities elsewhere.

It should be emphasized that even though the slender-body approximations are used to determine $f(\xi)$, Eqs. (4.18) can be used (without further approximations) to calculate the flowfield at locations around the cylinder, not just on its surface.

When the approximations are not appropriate, or when more accuracy is required, the problem of determining $f(\xi)$ is difficult to solve. The profile $y = Y(x)$ must be a streamline; in fact, Eq. (4.17) indicates that, because the cylinder is symmetrical about the x axis, it must be given by the streamline

$$\psi = \frac{1}{2}\int_a^b f(\xi)\,\mathrm{d}\xi + C$$

(The student can verify.) (Note that the total source strength is zero if the profile is closed. However, we need not assume that this will always be the case.) Thus, the equation to be solved for $f(\xi)$ is an integral equation:

$$Y(x) = \frac{1}{2}\int_a^b \frac{f(\xi)}{U}\left(1 - \frac{\vartheta}{\pi}\right)\mathrm{d}\xi \tag{4.24}$$

Such equations can usually be solved only by approximate methods, such as approximating to $f(\xi)$ by a stepwise constant function, or direct numerical methods. This will be illustrated in the following section. It might also be mentioned that there are other methods, using complex variable, that are available in plane-flow problems, as we shall see later, and these are not restricted to symmetrical cases.

Sample Problem 4.8

Show that the pressure distribution on a very thin elliptical cylinder in symmetrical flow is approximately constant. Hint: Use a trigonometric substitution to eliminate the radical and note that

$$P\int_0^{\pi} \frac{\cos n\theta \, \mathrm{d}\theta}{\cos\theta - \cos\alpha} = \pi \frac{\sin n\alpha}{\sin\alpha} \qquad (n = 0, 1, 2, \quad)$$

Solution:
Let the cylinder be given by

$$\frac{x^2}{a^2} + \frac{y^2}{t^2} = 1 \quad \text{or} \quad Y(x) = \pm t\sqrt{1 - \frac{x^2}{a^2}}$$

Because the cylinder is thin ($t \ll a$), we will use Eqs. (4.19) and (4.22) and write the boundary condition for the top surface as

$$v(x,Y) \approx v(x,0+) = UY'(x) = -U\frac{t}{a}\frac{x}{\sqrt{a^2 - x^2}} = \frac{1}{2}f(x)$$

Then, according to Eq. (4.20),

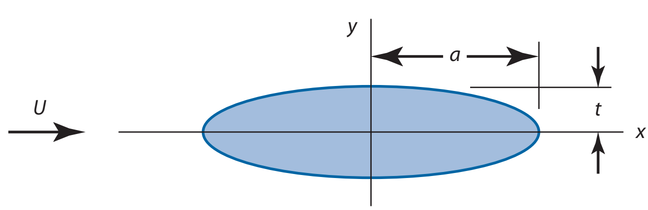
$$u'(x,Y) \approx u'(x,0) = \frac{-1}{2\pi} 2U \frac{t}{a} P\int_{-a}^{a} \frac{\xi}{\sqrt{a^2 - \xi^2}} \frac{\mathrm{d}\xi}{x - \xi}$$

$$= -\frac{Ut}{\pi a} P\int_0^{\pi} \frac{\cos\theta}{\sin\theta} \frac{-a\sin\theta \, \mathrm{d}\theta}{a\cos\alpha - a\cos\theta} = -\frac{Ut}{\pi a} P\int_0^{\pi} \frac{\cos\theta \, \mathrm{d}\theta}{\cos\alpha - \cos\theta}$$

where $x = a\cos\alpha$ and $\xi = a\cos\theta$.

According to the formula given in the problem, this is equal to

$$-\frac{Ut}{\pi a}\left(-\pi\frac{\sin\alpha}{\sin\alpha}\right) = \frac{Ut}{a}$$



and the pressure on the cylinder is approximately (see Sample Problem 2.1a)

$$p - p_\infty \approx -\rho U u' = -\rho U^2 \frac{t}{a}$$

We thus conclude that the pressure on the cylinder is practically constant.

4.8.2 Vortex Distributions

Now, if $f(\xi)$ were the distribution function for vortices, rather than sources, formula (4.17) would give us the velocity potential due to this distribution, which is called a *vortex sheet*. In Eq. (4.18), we would have formulas for v and $-u$, instead of u and v.

It is clear, therefore, that with these alterations, our formulas for source–sink distributions are also formulas for vortex sheets. Of particular interest are formulas (4.19), which now pertain to $u(x,0+)$, and Eq. (4.20), which pertains to $v(x,0)$. The new sheet singularity produces a flow *anti*symmetrical about the x axis and a discontinuous u' (rather than v') at the x axis.

Sample Problem 4.9

Suppose the vortex-strength distribution function is

$$f(x) = \sqrt{a^2 - x^2} \quad \text{in} \quad |x| \leq a$$

Then what are u and v near the vortex sheet?

Solution:
As $y \to 0+, |x| \leq a$, Eqs. (4.19) and (4.20), reinterpreted for the vortex sheet, give us

$$-u(x,y) \to \frac{1}{2} f(x) = \frac{1}{2}\sqrt{a^2 - x^2}$$

and

$$+v(x,y) \to \frac{1}{2\pi} P\int_{-a}^{a} \sqrt{a^2 - \xi^2}\,\frac{d\xi}{x - \xi}$$

$$= \frac{1}{2\pi} P\int_0^\pi \sin\theta \frac{\sin\theta\, d\theta}{\cos\alpha - \cos\theta} = \frac{1}{2\pi} P\int_0^\pi \frac{1-\cos 2\theta}{2} \frac{d\theta}{\cos\alpha - \cos\theta}$$

$$= \frac{1}{2\pi}\left\{0 + \frac{\pi}{2}\frac{\sin 2\alpha}{\sin\alpha}\right\} = \frac{1}{2}\cos\alpha = \frac{1}{2}\frac{\alpha}{a}$$

(using the definite-integral formula given in Sample Problem 4.8).

Sample Problem 4.10

Consider a distribution of doublets (dipoles), directed parallel to the y axis along the x axis. Set up the formula for $\phi(x, y)$ [or $\psi(x, y)$] and, by integration by parts, show that this sheet singularity is just a vortex sheet.

Solution:
The potential for a doublet $\uparrow$ is [see Problem 3.4(b)]

$$\phi = C\,\frac{y}{r^2} + \text{const} = C\,\frac{y}{x^2 + y^2} + \text{const}$$

Thus, for a distribution as specified,

$$\phi(x, y) = \int_a^b f(\xi)\frac{y}{(x-\xi)^2 + y^2}\,\mathrm{d}\xi + \text{const}$$

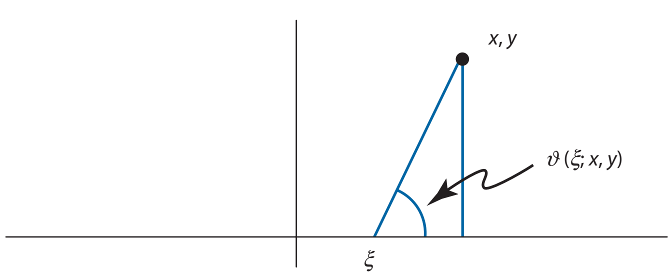
Integrate by parts: Let $\tilde{u} = f(\xi)$.

$$\mathrm{d}\tilde{u} = f'(\xi)\,\mathrm{d}\xi; \quad \mathrm{d}\tilde{v} = \frac{y}{(x-\xi)^2 + y^2}\,\mathrm{d}\xi$$

$$\tilde{v} = \tan^{-1}\frac{y}{x-\xi}$$

$$\phi(x, y) = \left[f(\xi)\tan^{-1}\frac{y}{x-\xi} \right]_a^b - \int_a^b f'(\xi)\tan^{-}\frac{y}{x-\xi}\,\mathrm{d}\xi + \text{const}$$

Because the $\tan^{-1}$ is equal to $\vartheta(\xi : x, y)$ (see figure), the second term is the same as a vortex distribution of strength $-2\pi f'(x)$ except for an additive constant.



What about the integrated part $[\ ]_a^b$? The identification of f' as the vortex strength provides the tip-off about this term, namely that it is f' that determines the flow. So f cannot have discontinuities unless we are willing to accept infinite vortex strength (concentrated vortices). Consider a vortex distribution of strength $-2\pi f'$ in the interval $a \leqq x \leqq b$. Write its ϕ and integrate by parts:

$$\phi(x,y) = -\int_a^b f'(\xi)\vartheta\,\mathrm{d}\xi = \left[-f(\xi)\vartheta\right]_a^b - \int_a^b f(\xi)\frac{y}{(x-\xi)^2+y^2}\,\mathrm{d}\xi + \text{const}$$

The second term is the one we started with, earlier. But if $f(x)$ is zero for $x \leqq a$, it will not be zero for $x \geqq b$, in general. So the case considered in the problem is a rather special one. Our conclusion is as follows: a doublet distribution $f(x)$ is equivalent to a vortex sheet of strength $f'(x)$, but usually the doublet distribution will extend infinitely to left or right or both. (Perhaps we should conclude we do not like doublet sheets as well as vortex sheets!)

4.9 Distributions of Sources: Axisymmetric Case

Repeating the calculation of the last section, using three-dimensional sources, one can compute Stokes' stream function for a distribution $f(\xi)$:

$$\psi(x,\varpi) = \frac{1}{2}U\varpi^2 - \frac{1}{4\pi}\int_a^b f(\xi)\cos\vartheta\,\mathrm{d}\xi + C; \quad \text{where } \vartheta = \tan^{-1}\frac{\varpi}{x-\xi} \tag{4.25}$$

The velocity components v_x and v_ϖ are found by differentiating this expression:

$$v_\varpi = -\frac{1}{\varpi}\frac{\partial\psi}{\partial x} = \frac{\varpi}{4\pi}\int_a^b f(\xi)\frac{\mathrm{d}\xi}{\left[(x-\xi)^2+\varpi^2\right]^{3/2}} \tag{4.26}$$

$$v_x = \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = U + \frac{1}{4\pi}\int_a^b f(\xi)\frac{(x-\xi)\,\mathrm{d}\xi}{\left[(x-\xi)^2+\varpi^2\right]^{3/2}} \tag{4.27}$$

These equations can be used to calculate the symmetrical flow about bodies of revolution.

If the *slender-body* approximations are valid, we can again obtain a simplified formula from Eq. (4.26). Here, the flow very close to the axis is found to be the same as near a uniform line source [Problem 3.3(b)], that is,

$$v_{\varpi}(x,\varpi) \approx \frac{f(x)}{2\pi\varpi} \qquad \text{for} \quad \varpi \to 0 \tag{4.28}$$

If the expression (4.27) for v_x is taken to the limit $\varpi \to 0$, it is found to be logarithmically infinite:

$$v_x = U + \frac{1}{2\pi} f'(x) \ln \varpi + \cdots \tag{4.28a}$$

Now, if the body shape $\varpi = R(x)$ is given, and the slender-body approximations are permissible, the calculation is simple, for $f(\xi)$ is calculated from Eq. (4.28) and the approximate boundary condition:

$$R'(x) = \frac{v_{\varpi}}{v_x} \approx \frac{v_{\varpi}}{U} \approx \frac{f(x)}{2\pi R(x) U}$$

$$f(x) \approx 2\pi URR' = U \frac{\mathrm{d}}{\mathrm{d}x}(\pi R^2) \tag{4.29}$$

That is, the source distribution is proportional to the slope of the curve of cross-sectional area of the body of revolution.

Unfortunately, formula (4.28a) is not a good approximation in practical cases. To be sure, the term retained, $f'(x) \ln \varpi$, is $0(\in^2 \ln \in)$ if $R'(x)$ is $0(\in)$ [see Eq. (4.29)], whereas the next (neglected) term is $0(\in^2)$, which is a higher-order term. Nevertheless, in practical cases these terms are not (numerically) very different, and it is advisable (and consistent) to retain both orders in approximating to the pressure distribution, for example. Therefore, Eq. (4.27) is actually used instead of Eq. (4.28a), and the v_{ϖ}^2 term is retained in Bernoulli's equation, even for slender bodies.

However, if slender-body approximations are not adequate at all, one must attack the integral equation given by

$$\psi(x,R) = \frac{1}{4\pi} \int_a^b f(\xi)\,\mathrm{d}\xi + C$$

namely,

$$UR^2(x) = \frac{1}{2\pi}\int_a^b f(\xi)(1+\cos\vartheta)\,\mathrm{d}\xi \tag{4.30}$$

In this case, a procedure due to von Kármán [3] is used. Here $f(\xi)$ is taken to be a stepwise constant function, and values of $R^2(x)$ are imposed at the same number of stations, N. The integral becomes a summation, and the integral equation becomes an $N\times N$ matrix equation. Such an equation can easily be solved for the N values of f, using computer equipment and well-known routines for matrix inversion. Because such equipment was not available to von Kármán, he proposed a different successive approximation procedure that now seems out of date.

Sample Problem 4.11

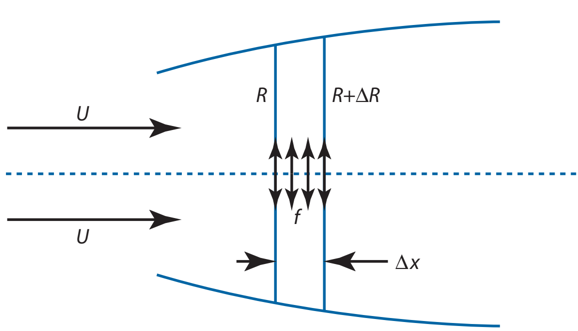
Derive Eq. (4.29) directly by assuming only that $v_x \approx U$ and applying the principle of continuity.

Solution:
All of the fluid exuded at the axis in the element Δx must flow through the area $\pi(R+\Delta R)^2$ at (approximately) the speed U. Thus,

$$\pi(R+\Delta R)^2 U - \pi R^2 U = f(x)\Delta x$$

or

$$f(x) = 2\pi R U \frac{\Delta R}{\Delta x} + \cdots$$



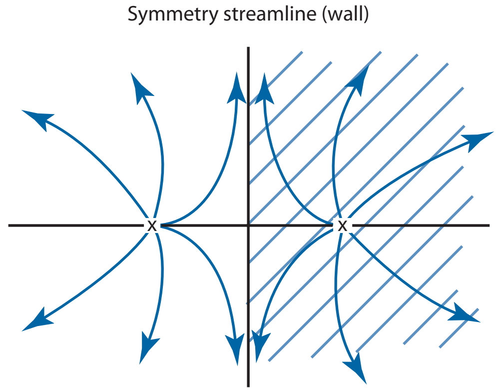


Fig. 4.9 Construction of a solid wall using the method of images.

4.10 Method of Images

Consider two sources of equal strength in a fluid otherwise at rest. By any one of several methods, one can easily verify that the plane of symmetry between them is a stream surface. Consequently, it could be replaced by an impermeable smooth wall (Fig. 4.9). Therefore, the superposition of the flows due to a source and its "mirror image" in the plane gives the flow due to a source placed near a plane wall. This is an example of the "method of images."

In the same way, the flows due to a doublet near a plane, a vortex (not necessarily straight) near a plane and so forth, are found by the simple matter of calculating for an unbounded fluid containing the singularity and its image. Naturally, the flow of a stream parallel to the plane past any such singularity is found just as easily. But is not particularly useful because the body shape resulting is distorted, as indicated in Fig. 4.10.

It is more interesting to determine the "images" of simple singularities in circular cylinders and spheres, that is, to determine the systems of singularities that produce the same flow in an unlimited fluid.

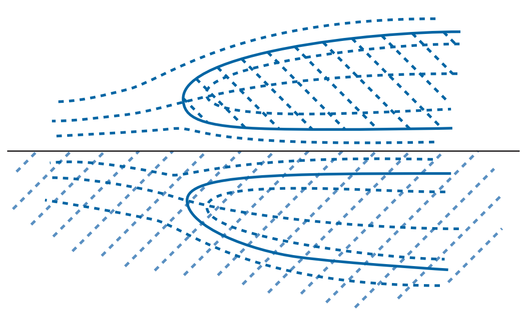
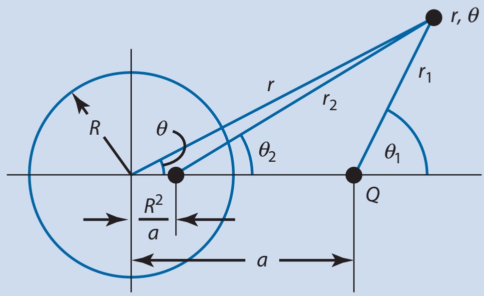


Fig. 4.10 Flow between two boards can be formed with a free stream and two sources.

Example 4.1 Plane Flow due to a Source near a Circular Cylinder



Consider a source of strength Q at a distance a from the center of the cylinder, as in the figure. Let us try the device of placing an equal source at the "image point" in the cylinder, that is, at the distance R^2/a from the center. Will this make ψ a constant on the circle, as required?

At any point,

$$\psi = \frac{Q}{2\pi}(\theta_1+\theta_2)+C \quad \left(0 \le \frac{\theta_1}{\theta_2} < 2\pi\right)$$

$$\theta_1 = \sin^{-1}\left(\frac{r\sin\theta}{r_1}\right) \quad \text{and} \quad \theta_2 = \sin^{-1}\left(\frac{r\sin\theta}{r_2}\right)$$

For points on the cylinder,

$$r_1^2 = (R\cos\theta - a)^2 + (R\sin\theta)^2; \quad r_2^2 = \left(R\cos\theta - \frac{R^2}{a}\right)^2 + (R\sin\theta)^2$$

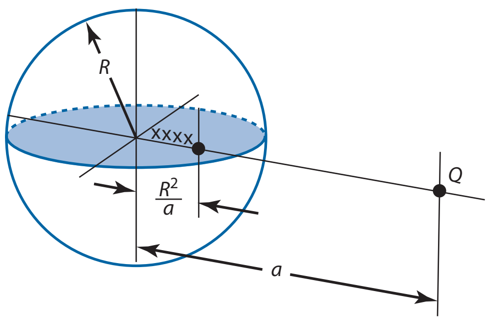
and thus, $r_2 = (R/a)r_1$. Now, to evaluate ψ on the cylinder, let us calculate $\theta_1+\theta_2$ in terms of its sine:

$$\begin{aligned}
\sin(\theta_1+\theta_2) &= \frac{R\sin\theta}{r_1}\sqrt{1-\left(\frac{R\sin\theta}{r_2}\right)^2} + \frac{R\sin\theta}{r_2}\sqrt{1-\left(\frac{R\sin\theta}{r_1}\right)} \\
&= \frac{R\sin\theta}{r_1 r_2}\left\{\sqrt{r_2^2-(R\sin\theta)^2}+\sqrt{r_1^2-(R\sin\theta)^2}\right\} \\
&= \frac{a\sin\theta}{r_1^2}\left\{\left(R\cos\theta-\frac{R^2}{a}\right)+(R\cos\theta-a)\right\} \\
&= -\sin\theta, \quad \text{and by inspection,} \quad \theta_1+\theta_2 = \pi+\theta
\end{aligned}$$

Thus, we find that ψ is not constant on the cylinder, but is equal to $(Q/2\pi)\theta + C'$. This is easy to correct by simply adding a sink at the center of the circle. This is the final system of singularities to represent the flow of a source near a cylinder: an equal source at the "image point" and an equal but opposite sink at the center. Now, we see that the total source strength within the circle of radius R has been made zero, as must be the case.

Sample Problem 4.12

Verify that the image system to represent a three-dimensional source Q near a sphere of radius R consists of a source of strength $(R/a)Q$ at the "image point" (see figure) and a uniform distribution of sinks, of strength $-Q/R$ per unit length, between the center and the "image point."



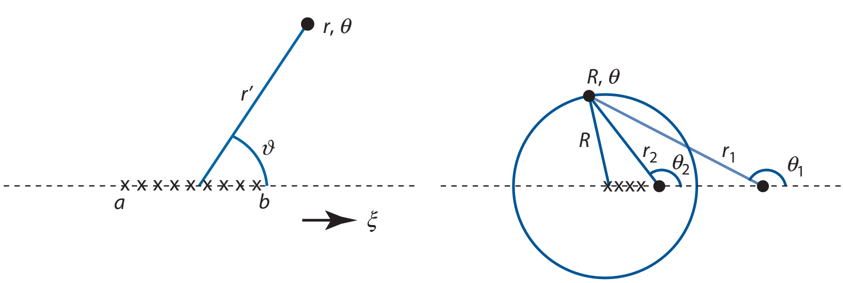
Solution:
The stream function of a source is given in Eq. (3.17). The contribution due to a line-source of constant strength k lying on the axis between a and b is

$$\Delta\psi(r,\theta) = -\frac{k}{4\pi}\int_a^b \cos\vartheta \, d\xi = \frac{k}{4\pi}\int_a^b dr'$$
$$= \frac{k}{-4\pi}\{r'_b - r'_a\}$$

Here r' and ϑ are measured from an element $d\xi$.

Thus,

$$\psi(R,\theta) = \frac{-Q/R}{4\pi}(r_2 - R) - \frac{Q}{4\pi}\frac{R}{a}\cos\theta_2 - \frac{Q}{4\pi}\cos\theta_1 + \text{const}$$



or

$$\frac{\psi(R,\theta)}{Q/R} = \frac{1}{R}(r_2 - R) + \frac{R}{a}\frac{\sqrt{r_2^2 - R^2\sin^2\theta}}{r_2} + \frac{\sqrt{r_1^2 - R^2\sin^2\theta}}{r_1} + \text{const}$$

$$= \frac{r_1}{a} + \frac{2R\cos\theta - (R^2/a) - a}{r_1} + \text{const}$$

(see Sec. 4.10)

$$= \cancel{\frac{r_1}{a}} - \cancel{\frac{r_1}{a}} + \text{const}$$

Thus, for $r = R$, the stream function is a constant, and the sphere is a stream surface.

Sample Problem 4.13

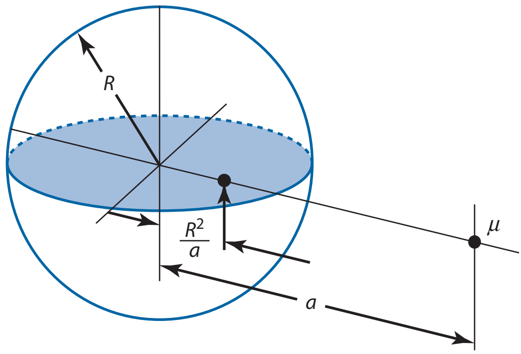
Verify that the image system to represent a doublet μ near a sphere, directed toward the sphere, consists of an opposed doublet $(R/a)^3\mu$ at the "image point" (see figure).

Solution:

$$\psi(R,\theta) = \frac{\mu}{4\pi}\left\{\frac{\sin^2\theta_1}{r_1} - \left(\frac{R}{a}\right)^3\frac{\sin^2\theta_2}{r_2}\right\} + \text{const}$$

$$= \frac{\mu}{4\pi}\left\{\frac{R^2\sin^2\theta}{r_1^3} - \left(\frac{R}{a}\right)^3\frac{R^2\sin^2\theta}{r_2^3}\right\} + \text{const}$$

$$= 0 \quad \text{because} \quad r_2 = \frac{R}{a}r_1$$



4.11 Nonsteady Flows

As we have already emphasized in Chapter 3, the process of finding the flow pattern in an unsteady case is fundamentally the same as in a steady case, but the calculation of pressures is different. Naturally, the device of making the body contour a stream surface is not applicable—at least not directly.

We shall illustrate two useful techniques by considering the problem of the flow produced by a sphere moving with nonuniform velocity through an infinite body of fluid otherwise at rest (see Fig. 4.11). Let the x be chosen in the direction of (instantaneous) motion of the sphere, and the value of this speed be U. The instantaneous boundary condition is

$$\frac{\partial\phi}{\partial r'} = U\cos\theta' \tag{4.31}$$

where r' is the radius measured from the center of the sphere at any instant and θ' the corresponding angle. (Note that primes (′) in this section do *not* represent derivatives.) The only other boundary condition is the vanishing of the velocities at an infinite distance from the sphere, which follows from physical considerations. The region is simply connected, and there are no cyclic constants to worry about. From Eq. (3.26), a complete general solution is

$$\phi = \sum_{n=0}^{\infty} B_n \frac{P_n(\cos\theta')}{r'^{n+1}} \tag{4.32}$$

which gives us

$$\frac{\partial\phi}{\partial r'} = -\sum_{n=0}^{\infty}(n+1)B_n\frac{P_n(\cos\theta')}{r'^{n+2}}$$

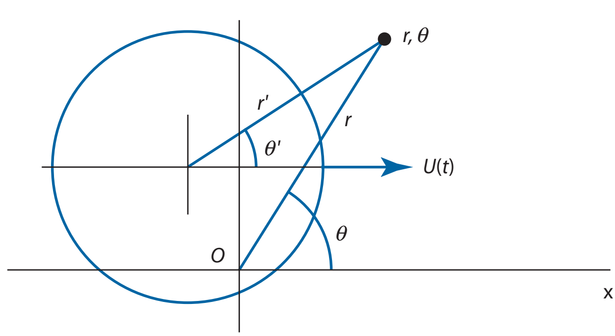


Fig. 4.11 Sphere moving with velocity $U(t)$ is a fluid at rest.

and on the sphere,

$$\frac{\partial \phi}{\partial r'} = -\sum_{n=0}^{\infty}(n+1)\frac{B_n}{R^{(n+2)}}P_n(\cos\theta') \tag{4.33}$$

But Eq. (4.31) is already in this form [see Eq. (3.27)], where the coefficients have the following values:

$$B_1 = -\frac{UR^3}{2}; \quad B_2 = B_3 = \cdots = 0$$

Thus,

$$\phi = -\frac{UR^3}{2}\frac{\cos\theta'}{r'^2} + C \tag{4.34}$$

By comparison with Eq. (3.18), we see that this is just the same (in the region exterior to the sphere) as the flow due to a stationary doublet, but this might be dismissed at this point as merely coincidence. Now, to express ϕ as a function of time, as it certainly is, we must recognize that U, θ', and r' can all vary with t. For example, if the motion is always in the same direction, but nonsteady, the pressure at the stagnation point ($\theta = 0$) is calculated as follows.

Static pressure at infinity: $p_0 = p + \rho\frac{\partial \phi}{\partial t} + \rho\frac{q^2}{2}$

$$p - p_0 = -\rho\frac{q^2}{2} - \rho\frac{\partial \phi}{\partial t} = -\rho\frac{U^2}{2} + \rho\frac{R^3}{2}\frac{\partial}{\partial t}\left(\frac{U}{r'^2}\right) \tag{4.35}$$

Now, for any point directly in front of the sphere and on its surface, $\partial r'/\partial t = -U$; hence, the second term is found to be

$$\rho\frac{R^3}{2}\frac{\partial}{\partial t}\left(\frac{U}{r'^2}\right) = \rho\frac{RU'}{2} + \rho U^2 \tag{4.36}$$

and the pressure formula for this particular point is

$$p - p_0 = \rho\frac{U^2}{2} + \rho\frac{RU'}{2} \tag{4.37}$$

Our notation here is somewhat unfortunate. As stated earlier, primes on r and θ denote polar coordinates centered on the center of the sphere. But a prime on U denotes differentiation with respect to time.

If the motion is curvilinear so that the other components of the sphere's velocity, say V and W, are instantaneously zero but their derivatives are not zero, there will, of course, be additional terms in the pressure formula. These would be found by including V and W terms in ϕ, which would then contribute to $\partial\phi/\partial t$.

Another procedure to find Eq. (4.34) is to begin with a steady flow of a fluid about a stationary sphere. From Eq. (4.7), the potential for this case is

$$\phi_1 = \frac{1}{2}U\left\{2r' + \frac{R^3}{r'^2}\right\}\cos\theta' + C \tag{4.38}$$

Now, it is easy to write the formula for this flow in terms of coordinates that move *with the stream* instead of being fixed in the sphere. We simply superimpose on the entire space a uniform flow of velocity $-U$:

$$\phi = \phi_1 - Ur'\cos\theta' = \frac{1}{2}U\frac{R^3}{r'^2}\cos\theta' + C \tag{4.39}$$

This, then, is the potential for a sphere moving (from right to left) with velocity U through a fluid at rest. Therefore, it is (after change of sign) the potential we are seeking for the nonsteady case, for the instant at which the velocities of the sphere coincide. Remember that the nonsteady character of a flow does not affect the pattern, but only the pressures.

We now see why the flow is doublet flow. We originally set up the flow about a sphere by superposition of doublet and stream; in the nonsteady case, we have canceled the stream.

Sample Problem 4.14

A sphere moves in a straight line with speed $U(t)$ through an unbounded fluid otherwise at rest. By integration of pressures, determine the force on the sphere.

Solution:
At any instant, for the sphere moving from left to right.

$$\phi(r,\theta,t) = -\frac{1}{2}U(t)\frac{R}{r'^2}\cos\theta' + \text{const}$$

$$\frac{\partial\phi}{\partial t} = -\frac{1}{2}U'(t)\frac{R^3}{r'^2}\cos\theta' + U(t)\frac{R^3}{r'^3}\cos\theta'\frac{\partial r'}{\partial t} + \frac{1}{2}U(t)\frac{R^3}{r'^2}\sin\theta'\frac{\partial\theta'}{\partial t}$$

Now, recall that r' and θ' are coordinates of a fixed point relative to the moving center so that (see figure)

$$\frac{\partial r'}{\partial t} = -U\cos\theta' \quad \text{and} \quad \frac{\partial \theta'}{\partial t} = \frac{U\sin\theta'}{r'}$$

For points on the sphere, therefore,

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}U'R\cos\theta' - U^2\cos^2\theta' + \frac{1}{2}U^2\sin^2\theta'$$

Also,

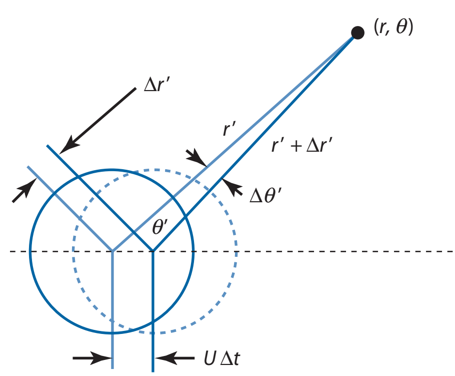
$$v_{r'} = U\cos\theta' \quad \text{and} \quad v_{\theta'} = \frac{U}{2}\sin\theta'$$

The Bernoulli equation for this case is Eq. (2.7) with $U = 0$. Far from the sphere, the pressure is p_∞, unaffected by the sphere's motion, and the velocity and ϕ_t vanish. Thus, $F(t) = 0$, and

$$\begin{aligned} p - p_\infty &= -\rho\left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}v_{r'}^2 + \frac{1}{2}v_{\theta'}^2\right\} \\ &= -\rho\left\{-\frac{1}{2}U'R\cos\theta' - \frac{1}{2}U^2\cos^2\theta + \frac{5}{8}U^2\sin^2\theta'\right\} \end{aligned}$$

The axial force on the body is

$$F_x = -\int_S p n_x \,\mathrm{d}\sigma = -\int_0^\pi p(R,\theta')\cos\theta' 2\pi R\sin\theta' R\,\mathrm{d}\theta'$$



Clearly, the symmetrical terms in the pressure formula stated earlier will not contribute to F_x. The first term contributes

$$\rho\int_0^{\pi} 1U'R^3 \cancel{2}\pi\cos^2\theta\sin\theta' d\theta = \pi\rho U'R^3 \left.\frac{\cos^3\theta'}{3}\right|_0^{\pi}$$

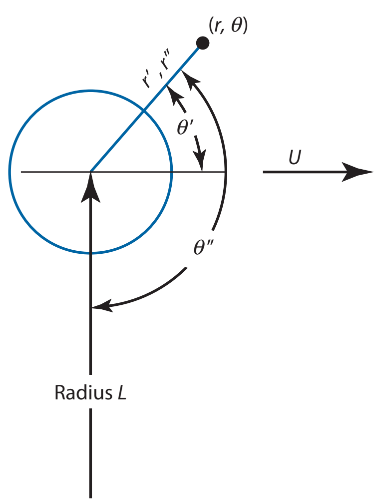
$$= -\frac{2}{3}\pi\rho U'R^3 = -\frac{1}{2}\rho U' \forall$$

where $\forall$ is the volume of the sphere. It is also clear, by symmetry, that this is the only nonzero force component.

Sample Problem 4.15

A sphere describes a circular path of radius L, with constant speed U in an unbounded fluid otherwise at rest. Determine the pressure on it, and thence the force on the sphere.

Solution:
The sphere is instantaneously moving horizontally with speed U and accelerating radially toward the center with acceleration U^2/L. The horizontal motion affects the pressure distribution but not (see Sample Problem 4.14) the force on the sphere because U' is (instantaneously) zero. The radial acceleration affects the pressure distribution through the $\partial\phi/\partial t$ term, but produces no velocity components because the radial velocity is zero.



Let the horizontal motion at speed U be described in the same terms as in Sample Problem 4.14:

$$\phi_{\text{Hor.}} = -\frac{1}{2}U\frac{R^3}{r'^2}\cos\theta' + \text{const}$$

and first the radial motion at speed V (say) be described by

$$\phi_{\text{Rad.}} = -\frac{1}{2}V\frac{R^3}{r'^2}\cos\theta'' + \text{const}$$

where θ', θ'', r', and r'' are defined in the figure (primes and double primes denote polar coordinates—not derivatives). The total ϕ is found by superposition:

$$\phi = \phi_{\text{Hor.}} + \phi_{\text{Rad.}}$$

The pressure distribution on the sphere is, therefore,

$$\begin{aligned} p - p_\infty &= -\rho\left\{\frac{\partial\phi_{\text{Rad.}}}{\partial t} + \frac{1}{2}v_{r'}^2 + \frac{1}{2}v_{\theta'}^2\right\} \\ &= -\rho\left\{-\frac{1}{2}V'R\cos\theta'' - \frac{1}{2}U^2\cos^2\theta' + \frac{5}{8}U^2\sin^2\theta'\right\} \end{aligned}$$

The U^2 terms are symmetrical in all respects and yield no force on the sphere. The V term, by comparison with Sample Problem 4.14, yields the radial (outward) force

$$\frac{1}{2}\rho V'\mathcal{V} = \frac{1}{2}\rho\frac{U^2}{L}\mathcal{V}$$

where $\mathcal{V}$ is the volume of the sphere.

4.12 Impulse and Momentum

There is an easy way to calculate the forces acting on bodies moving through fluids. It is based on a concept very closely analogous to *momentum*, but modified to avoid certain difficulties in infinite regions.

For the time being, let us consider noncyclic flows only. Consider a body moving through a fluid, and calculate the momentum of the fluid contained between the body and an impermeable envelope Σ (Fig. 4.12):

$$\rho\int_V \operatorname{grad}\phi\,\mathrm{d}\tau = \rho\int_S \boldsymbol{n}\phi\,\mathrm{d}\sigma + \rho\int_\Sigma \boldsymbol{n}\phi\,\mathrm{d}\sigma \tag{4.40}$$

where V is the volume enclosed between the body and the envelope Σ, S is the body surface, and n is outwardly drawn in each case (that is, out of the fluid). Now in plane flow, if Σ is taken to be very large, ϕ has terms of order r^{-1} (doublet), and dσ varies as r. Hence, the product ϕ dσ does not vanish for very large r, and the result of the integration over Σ will depend on the shape of Σ. Similarly, in a three-dimensional case, $\phi = O(r^{-2})$ and $\mathrm{d}\sigma = O(r^2)$, and the same indeterminacy appears if the approach to an unbounded fluid is attempted.

This indeterminacy in the momentum of an infinite body of fluid—even though the flow is produced by a finite body—is troublesome and can even lead to erroneous results. It has been proposed by some writers to avoid the difficulty by requiring that Σ always have a spherical form, as a convention. Another method was invented by Lord Kelvin, however, who showed that the indeterminacy has nothing to do with the forces on the body, and therefore the integration over Σ can be eliminated entirely.

In fact, Kelvin showed that the external force $\boldsymbol{F}(t)$ acting on the body is given by the following formula:

$$\boldsymbol{F}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\boldsymbol{P} + \rho\int_S \boldsymbol{n}\phi\,\mathrm{d}\sigma\right\} \tag{4.41}$$

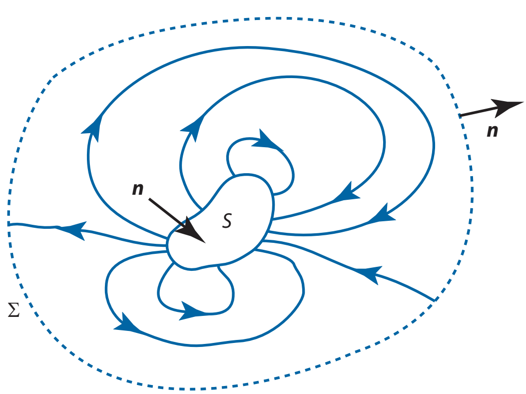


Fig. 4.12 Momentum of fluid contained between the body and an impermeable envelope Σ.

where $\boldsymbol{P}$ is the momentum of the body itself. Now, the force $\boldsymbol{F}(t)$ could be calculated by differentiating the expression for momentum, accounting for the pressures on the large envelope Σ:

$$\boldsymbol{F}(t) - \int_\Sigma \boldsymbol{n}p \, \mathrm{d}\sigma = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\boldsymbol{P} + \rho\int_V \operatorname{grad}\phi \, \mathrm{d}\tau\right\}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left\{\boldsymbol{P} + \rho\int_S \boldsymbol{n}\phi \, \mathrm{d}\sigma\right\} + \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \boldsymbol{n}\phi \, \mathrm{d}\sigma \tag{4.42}$$

Thus, Kelvin's result [Eq. (4.41)] tells us that

$$\int_\Sigma \boldsymbol{n}p \, \mathrm{d}\sigma = \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \boldsymbol{n}\phi \, \mathrm{d}\sigma \tag{4.43}$$

In other words, although the integral

$$\int_\Sigma \boldsymbol{n}\phi \, \mathrm{d}\sigma$$

is indeterminate, its time derivative is determinate, and in fact equal to the total force exerted (by an external structure, say) on the large envelope enclosing the fluid.

The proof of Kelvin's theorem will appear in the next section; it will consist of proving Eq. (4.43). The quantity in brackets in Eq. (4.41) is called the *impulse* of the motion. We denote it by $\boldsymbol{I}(t)$; hence,

$$\boldsymbol{I}(t) \equiv \boldsymbol{P} + \rho\int_S \boldsymbol{n}\phi \, \mathrm{d}\sigma \tag{4.44}$$

and the theorem is

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} \tag{4.45}$$

In an analogous manner, the moment $\boldsymbol{M}(t)$ on the body, referred to fixed origin, was also calculated by Kelvin. It is found to be

$$\boldsymbol{M} = \frac{\mathrm{d}\boldsymbol{I}_m}{\mathrm{d}t} \tag{4.46}$$

where $\boldsymbol{I}_m$ might be called the *moment of impulse*:

$$\boldsymbol{I}_m(t) \equiv \boldsymbol{P}_m + \rho \int_S \boldsymbol{r} \times \boldsymbol{n}\phi \, \mathrm{d}\sigma \tag{4.47}$$

$\boldsymbol{r}$ being the radius vector from the specified origin, and $\boldsymbol{P_m}$ being the moment of momentum of the body about the same origin.

It might be advisable to point out explicitly that the force $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ are not the resultants of fluid pressures that we usually calculate. They are the external force and moment required to overcome both the inertia of the body and that of the fluid. The latter, of course, manifests itself in fluid pressures on the body so that the resultant of the pressures, with sign reversed, is included in $\boldsymbol{F}$ and the moment $\boldsymbol{M}$.

4.13 Proof of Kelvin's Result: Force Equals Rate of Change of Impulse

To prove formula (4.43), we simply calculate p on the surface Σ according to Bernoulli's theorem:

$$\int_\Sigma \boldsymbol{n}p \, \mathrm{d}\sigma = \int_\Sigma \boldsymbol{n} \left\{ p_0 - \frac{\rho}{2} q^2 - \rho \frac{\partial \phi}{\partial t} \right\} \mathrm{d}\sigma \tag{4.48}$$

Now,

$$\int_\Sigma \boldsymbol{n}p_0 \, \mathrm{d}\sigma = 0$$

Also, the term involving q^2 can be made to vanish if Σ is made indefinitely large, for $q^2 = O(r^{-4})$ in a plane case or $O(r^{-6})$ in three dimensions. [It might be noted, for future use, that $q^2 = O(r^{-2})$ even in a plane case with circulation, whereas $\mathrm{d}\sigma = O(r)$.]

Finally, in the term

$$\rho \int_\Sigma \boldsymbol{n}(\partial \phi / \partial t) \mathrm{d}\sigma$$

the order of integration and differentiation can be reversed as long as the limit has not yet been taken. Consequently, formula (4.43) is approximately true for all finite Σ, the error being made as small as desired by taking Σ sufficiently large.

This completes our proof of the theorem regarding the force. The extension to the moment is not essentially different.

Sample Problem 4.16

Verify the result of Sample Problem 4.14 for the force on the sphere, using the formula involving impulse.

Solution:

$$\boldsymbol{I} = \rho \int_S \boldsymbol{n}\phi \, \mathrm{d}\sigma; \qquad \phi = -\frac{1}{2} U R^3 \frac{\cos\theta'}{r'^2} + \text{const}$$

[See Eq. (4.34)]. The x component is

$$\begin{aligned} \boldsymbol{I}_x &= \rho \int_S n_x \phi \, \mathrm{d}\sigma \qquad\qquad (n_x = -\cos\theta') \\ &= -\frac{1}{2}\rho U R^3 \int_0^\pi -\frac{\cos^2\theta'}{R^2} 2\pi R \sin\theta' R \, \mathrm{d}\theta' \\ &= -\pi\rho U R^3 \left. \frac{\cos^3\theta'}{3} \right|_0^\pi = \frac{2}{3}\pi\rho U R^3 \end{aligned}$$

The external force p_x that must be applied to the body to produce this motion is, therefore,

$$p_x = \frac{\mathrm{d}\boldsymbol{I}_x}{\mathrm{d}t} = \frac{2}{3}\pi\rho U' R^3$$

in agreement with Sample Problem 4.14.

Sample Problem 4.17

What initial acceleration results when a sphere of density ρ', immersed in an unbounded liquid of density ρ, is allowed to fall under the influence of gravity?

Solution:
Let $\mho$ denote the volume of the sphere $(4/3)\pi R^3$. The buoyant force tending to raise the sphere is $\rho g \mho$, and the sphere's weight is $\rho' g \mho$. The net upward force is therefore $(\rho - \rho') g \mho$.

To calculate the acceleration, we equate this to the rate of change of vertical impulse, including the momentum of the sphere:

$$\begin{aligned} (\rho - \rho') g \mho &= \frac{\mathrm{d}}{\mathrm{d}t}\{m_x + I_x\} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\rho + \rho'\right\}\mho U = \left\{\frac{1}{2}\rho + \rho'\right\}\mho U' \end{aligned}$$

or

$$U' = \frac{\rho - \rho'}{(1/2)\rho + \rho'} g$$

For example, an evacuated sphere $n = \infty$ has an initial upward acceleration of $2g$. Does this suggest an experiment with a light bulb in water—or are you content simply to watch bubbles rise in beer?

4.14 Impulse Components in Terms of Kinetic Energy

The analogy between Kelvin impulse and ordinary momentum extends to the relationship between impulse and kinetic energy. Suppose the components of $\boldsymbol{I}$ are ξ, η, and ζ, referred to an xyz-coordinate system, and suppose that the velocity components of the body are U, V, and W. Let X, Y, and Z be the force components acting on the body, and consider the work done by them during a small time interval δt. The work done by X is

$$\int_0^{\delta t} XU \,\mathrm{d}t$$

or with finite forces and for a very short interval,

$$U \int_0^{\delta t} X \,\mathrm{d}t$$

But

$$\int_0^{\delta t} X \,\mathrm{d}t$$

is $\delta\xi$. The total work increment during the interval is

$$U\delta\xi + V\delta\eta + W\delta\zeta \tag{4.49}$$

We shall see (Sample Problem 4.19) that ξ has the form $m_{11}U + m_{12}V + m_{13}W$, etc., where the m_{ij} are constants, and $m_{ij} = m_{ji}$. Thus, the total work increment

Eq. (4.49) is equal to

$$U\left\{m_{11}\delta U + m_{12}\delta V + m_{13}\delta W\right\} + V\left\{m_{21}\delta U + m_{22}\delta V + m_{23}\delta W\right\}$$
$$+W\left\{m_{31}\delta U + m_{32}\delta V + m_{33}\delta W\right\}$$

which, upon rearrangement and in view of the symmetry of the m_{ij}, is the same as

$$\delta U\left\{m_{11}U + m_{12}V + m_{13}W\right\} + \delta V\left\{m_{21}U + m_{22}V + m_{23}W\right\}$$
$$+\delta W\left\{m_{31}U + m_{32}V + m_{33}W\right\} \tag{4.50}$$

Hence, expression (4.49) for the total work increment can be written as

$$\xi\delta U + \eta\delta V + \varsigma\delta W \tag{4.51}$$

Now, let us equate this to the increment of kinetic energy during the same interval, which is

$$\delta T = \frac{\partial T}{\partial U}\delta U + \frac{\partial T}{\partial V}\delta V + \frac{\partial T}{\partial W}\delta W \tag{4.52}$$

Because all of the increments are arbitrary, their coefficients can be equated, and we have new expressions for the impulse components:

$$\xi = \frac{\partial T}{\partial U}, \quad \eta = \frac{\partial T}{\partial V}, \quad \zeta = \frac{\partial T}{\partial W} \tag{4.53}$$

Actually, we might have easily carried along the three components of $\boldsymbol{I}_m$ say λ, μ, υ, together with the angular-velocity components of the body, P, Q, and R, in the preceding derivation. It is suggested that the student carry out this extension, noting that ϕ is also linear in P, Q, and R. The results are

$$\lambda = \frac{\partial T}{\partial P}, \quad \mu = \frac{\partial T}{\partial Q}, \quad \nu = \frac{\partial T}{\partial R} \tag{4.54}$$

These formulas are sometimes useful. Note that the indeterminacy encountered in the momentum does not occur in the kinetic energy, for

$$T = \frac{\rho}{2}\left\{\int_S \phi\frac{\partial\phi}{\partial n}\,\mathrm{d}\sigma + \int_\Sigma \phi\frac{\partial\phi}{\partial n}\,\mathrm{d}\sigma\right\} \tag{4.55}$$

and the integrand in Eq. (4.55) vanishes satisfactorily as $\Sigma \to \infty$.

Sample Problem 4.18

Calculate the kinetic energy of the sphere in a liquid, and verify by differentiation the formula for impulse.

Solution:
According to Eq. (4.55), the kinetic energy is

$$T = \frac{\rho}{2}\int_S \phi \frac{\partial \phi}{\partial n}\, \mathrm{d}\sigma$$

with

$$\phi = -\frac{1}{2}U\frac{R^3}{r^2}\cos\theta + \text{const}$$

$$T = \frac{\rho}{2}\int_0^{\pi} -\frac{1}{2}UR\cos\theta \cdot (-U\cos\theta)\cdot 2\pi R\sin\theta R\,\mathrm{d}\theta$$

(***n*** being directed *out* of the fluid)

$$= -\frac{\pi}{2}\rho U^2 R^3 \left.\frac{\cos^3\theta}{3}\right|_0^{\pi} = \frac{\pi}{3}\rho U^2 R^3$$

Thus, the impulse component in the U direction is

$$\frac{\partial T}{\partial U} = \frac{2\pi}{3}\rho U R^3$$

as in Sample Problem 4.16.

4.15 Case of Pure Translation: Apparent Mass

As we have already noted, the velocity potential for the acyclic flow for a rigid body moving through an unbounded fluid otherwise at rest has the form

$$\phi = U\phi_1 + V\phi_2 + W\phi_3 + P\phi_4 + Q\phi_5 + R\phi_6 + C \tag{4.56}$$

where $\phi_1, \phi_2, \ldots$ are the velocity potentials for unit velocity (or angular velocity) along (or about) the three coordinate axes. Let us consider here the case of motion without rotation: $P = Q = R = 0$. We see that (omitting, for the time being, the body's momentum, which can always be added later)

$$\boldsymbol{I} = \rho\int_S \boldsymbol{n}\phi\,\mathrm{d}\sigma = U\left\{\rho\int_S \boldsymbol{n}\phi_1\,\mathrm{d}\sigma\right\} + V\left\{\rho\int_S \boldsymbol{n}\phi_2\,\mathrm{d}\sigma\right\} + W\left\{\rho\int_S \boldsymbol{n}\phi_3\,\mathrm{d}\sigma\right\} \tag{4.57}$$

If the components of $\boldsymbol{n}$ are l, m, and n,

$$\xi = U\left\{\rho\int_S \ell\phi_1\,\mathrm{d}\sigma\right\} + V\left\{\rho\int_S \ell\phi_2\,\mathrm{d}\sigma\right\} + W\left\{\rho\int_S \ell\phi_3\,\mathrm{d}\sigma\right\}$$

$$\eta = U\left\{\rho\int_S m\phi_1\,\mathrm{d}\sigma\right\} + \cdots \qquad \text{etc.}$$

and, in general, adopting a self-evident notation, the ith component of $\boldsymbol{I}$ is

$$\boldsymbol{I}_i = \sum_{j=1}^{3} U_j \rho \int_S \ell_i \phi_j\,\mathrm{d}\sigma \qquad (i = 1, 2, 3)$$

$$= \sum_{j=1}^{3} m_{ij} U_j \tag{4.58}$$

The potential $\phi_i(\boldsymbol{r}, t)$ for motion at unit velocity in the i direction is always of the form $f(\boldsymbol{r}')$, where $\boldsymbol{r}'$ is measured from axes moving with the body. Thus, each ϕ_i is a function depending only on the shape of the body, and the m_{ij} are a set of nine constants for any given body (Fig. 4.13).

It should be clear at this point that, ordinarily, the impulse vector is *not* parallel to the body-velocity vector. This, of course, is strikingly different from ordinary momentum of a body in space. It can be shown, however, that there are three mutually perpendicular directions, for every body, such that, when these are selected as axes,

$$m_{ij} = 0 \qquad \text{for } i \neq j \tag{4.59}$$

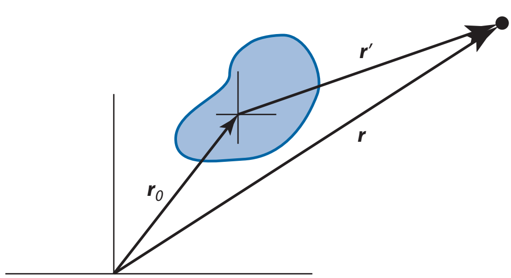


Fig. 4.13 Absolute and relative position vectors of a point in a flow generated by the motion of a body.

Lamb called these *axes of permanent translation* for a reason that will appear next. For motion along these particular directions,

$$I_i = m_{ii} U_i \qquad (i = 1, 2, 3) \tag{4.60}$$

or, including the momentum of the body,

$$I_i = (m + m_{ii}) U_i \tag{4.61}$$

where m denotes the mass of the body.

When it moves in one of these three directions, the body seems to have simply an increased momentum. The sum $m + m_{ii}$ is called the *apparent mass* for motion in the i direction, and m_{ii} is called the *additional apparent mass* for this direction. Note that, in general, each body has *three* different apparent masses; for bodies of revolution two are equal; only for a sphere are all three the same!

The process of finding the axes of permanent translation for a given body is analogous to finding the principal axes in dynamics. We shall not go into it here, but will simply point out that these axes can often be found by considerations of symmetry. For ellipsoids, they must coincide with the principal axes—this statement takes care of bodies of revolution and elliptical cylinders.

When these special directions are known, the simplest method of calculating the impulse, for motion in any arbitrary direction, is to resolve the velocity into components along the axes of permanent translation. The impulse components are then given by Eq. (4.61).

Sample Problem 4.19

Prove that the coefficients of Eq. (4.58) satisfy the equation $m_{ij} = m_{ji}$.

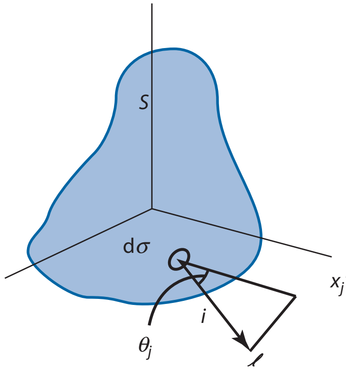
Solution:

$$m_{ij} \equiv \rho \int_S \ell_i \phi_j \, \mathrm{d}\sigma$$

where $\boldsymbol{\ell} = (\ell_1, \ell_2$, and $\ell_3)$ is the unit normal to S, directed out of the fluid, and ϕ_j is the potential for motion of the body at unit speed in the j direction.

The boundary condition for this motion is

$$\frac{\partial \phi}{\partial \ell} j = \cos\theta_j = \ell_j$$



Thus,

$$m_{ij} = \rho \int_S \frac{\partial \phi_i}{\partial \ell} \phi_j \cdot \mathrm{d}\sigma$$

But by Green's Theorem [Problem A.6(b)],

$$\int_S \left\{ \phi_j \frac{\partial \phi_i}{\partial \ell} - \phi_i \frac{\partial \phi_j}{\partial \ell} \right\} \mathrm{d}\sigma = \int_V \left\{ \phi_j \nabla^2 \phi_i - \phi_i \nabla^2 \phi_j \right\} \mathrm{d}\tau = 0$$

where the volume integral is carried through the whole volume of fluid outside the body. Hence,

$$\int_S \phi_j \frac{\partial \phi_i}{\partial \ell} \mathrm{d}\sigma = \int_S \phi_i \frac{\partial \phi_j}{\partial \ell} \mathrm{d}\sigma$$

and therefore $m_{ij} = m_{ji}$.

4.16 Steady Translation: D'Alembert's Paradox

It is clear from Eq. (4.58) that there is no force on any rigid body in steady translatory motion, for the impulse components are given by the formula

$$I_i = mU_i + \sum_1^3 m_{ij} U_j \qquad (i = 1,2,3) \tag{4.62}$$

and all of the quantities in this formula are constant. This is *D'Alembert's Paradox*. It results from the fact that we are assuming acyclic flow as well as neglecting viscosity.

The moment is easily calculated in this case:

$$\boldsymbol{M} = \frac{\mathrm{d}\boldsymbol{I}_m}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{ \boldsymbol{P}_m + \rho \int_S \boldsymbol{r} \times \boldsymbol{n}\phi \, \mathrm{d}\sigma \right\} = \rho \frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{r} \times \boldsymbol{n}\phi \, \mathrm{d}\sigma \tag{4.63}$$

But, again $\phi(\boldsymbol{r}, t)$ has the form $f(\boldsymbol{r}')$ and is constant on S, whereas $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{r}'$ and $\mathrm{d}\boldsymbol{r}_0 / \mathrm{d}t = \boldsymbol{V}$, the vector velocity of the body. Thus,

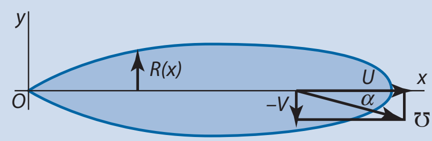
$$\boldsymbol{M} = \rho \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{r}' \times \boldsymbol{n}\phi \,\mathrm{d}\sigma + \boldsymbol{V} \times \int_S \boldsymbol{n}\phi \,\mathrm{d}\sigma \right\} = \boldsymbol{V} \times \boldsymbol{I} \tag{4.64}$$

This means that there is a moment on every body, even in steady translation, unless $\boldsymbol{I}$ is parallel to $\boldsymbol{V}$, which is not generally true, as we have seen. It is the case for motion along the axes of permanent translation, hence their name. If a body is moving in one of these directions, there is no moment tending to rotate it.

Example 4.2 Approximate Calculation of Airship Pitching Moment

The pitching moment on an airship flying at angle of attack α is

$$\boldsymbol{M} = (\boldsymbol{i}\mho \cos\alpha - \boldsymbol{j}\mho \sin\alpha) \times \boldsymbol{I}$$



Now, by symmetry, the axis of revolution and any transverse axis are *axes of permanent translation*, and moreover, the apparent mass for transverse motion is usually many times greater than that for longitudinal motion. Thus,

$$\boldsymbol{M} = \mho^2 \sin\alpha \cos\alpha \left\{ m_{xx} - m_{yy} \right\} \boldsymbol{k} \approx -\mho^2 \sin\alpha \cos\alpha \, m_{yy} \boldsymbol{k} \tag{4.65}$$

or for small α,

$$\boldsymbol{M} \approx -\mho^2 \alpha m_{yy} \boldsymbol{k} \tag{4.66}$$

Clearly, the fluid-pressure moment on the airship is unstable in sign, for the external moment calculated in Eq. (4.66) is required to maintain the angle of attack α, and it is nose down in sign. It is actually obtained from tail surfaces, which are therefore operated quite differently from those of a stable airplane.

Now, m_{yy} is difficult to evaluate exactly. For long, narrow shapes, it is often assumed that the flow at every station is nearly plane. This approximation will surely break down near the nose, and perhaps near the tail, but might be expected to work well over most of the length. If it is adopted, the additional apparent mass m_{yy} can be calculated from that of a circular cylinder in plane flow, which is $\pi\rho R^2$. (The student should verify this.) Thus,

$$m_{yy} \approx \pi\rho \int_0^L R^2(x) \,\mathrm{d}x \tag{4.67}$$

which is ρ times the volume of the airship.

4.17 Approximate Calculation of Airship Force Distribution

The concept of impulse can be used to obtain a very simple approximation to the loading on an airship, or other body elongated in the flight direction, flying at an angle of attack.

As we have mentioned, the transverse flow over such a body, except near the ends, is nearly plane. Consider a lamina of the fluid, of thickness Δx, taken perpendicular to the x direction as shown in Fig. 4.14. At time t, the motion in the lamina is practically the plane flow about a cylinder of radius $R(x)$ moving downward with velocity $-V = \mho \sin\alpha$. At time $t+\delta t$, however, the flow in the same lamina is that due to a cylinder of radius $R + R'U\ \delta t$ moving with the same velocity.

The vertical impulse component $-\eta$ has been increased by the amount

$$\delta\Delta(-\eta) = \pi\rho\left\{(R + R'U\,\delta t)^2 - R^2\right\}\Delta x\,\mho\,\sin\alpha$$

$$\approx \pi\rho\,\mho^2\,\frac{\mathrm{d}R^2}{\mathrm{d}x}\,\delta t\ \Delta x\ \alpha$$

Now, this increase in downward impulse can only arise from the action of forces on the fluid. The only possible forces are those applied by the body, which are equal and opposite to the fluid forces on the body. Thus, the approximate upward load on the body, per unit length, is given by

$$\pi\rho\mho^2\frac{\mathrm{d}R^2}{\mathrm{d}x}\alpha \quad \text{or} \quad \rho\mho^2 S'(x)\alpha \tag{4.68}$$

where $S(x)$ denotes the cross-sectional area and S' its derivative.

This simple result is found to agree remarkably well with more accurate calculations except, as we expected, near the nose. The same scheme, of considering an elongated shape to set up practically plane transverse flow, can be used in numerous other instances.

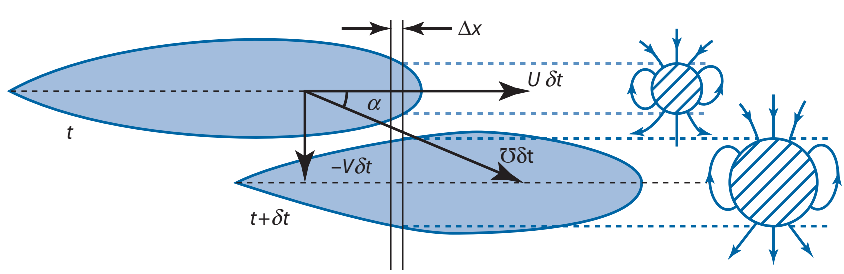


Fig. 4.14 Lamina of fluid taken perpendicular to x direction.

4.18 Extension to Cyclic Flows

We can extend Kelvin's concept of *impulse* to flows with circulation by using barriers. Remember that barriers are introduced in multiply-connected regions to prevent encircling any circuit having circulation, and thereby to render the velocity potential single valued.

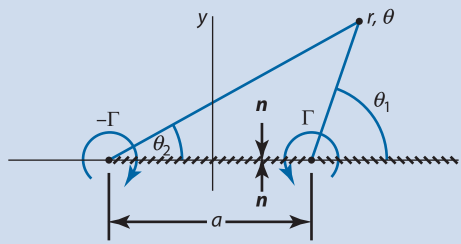
We have already noted that ϕ is discontinuous across a barrier. Thus, in carrying out surface integrals over the boundaries of the region, we shall have to include integrals taken over both sides of the barriers. Using this device, we can extend to cyclic flows all of the useful transformations of volume to surface integrals.

In particular, we can extend the definition of *impulse.* If the flow is to be produced impulsively from rest, impulsive pressures corresponding to the discontinuity in ϕ will have to be provided. Lamb imagines that the barrier is a soluble membrane, and that it dissolves immediately after the flow starts, but is capable of transmitting the impulsive pressures before that.

Example 4.3 Impulse of a Vortex Pair

Let us calculate the impulse of the plane flow produced by two equal and opposite line vortices a distance a apart. The potential is

$$\phi = \frac{\Gamma}{2\pi}(\theta_1 - \theta_2) + C \qquad (4.69)$$



The region is triply connected, and the two cyclic constants have been specified. We introduce two barriers, placing them (arbitrarily) between the two singularities and from the right-hand one out to infinity, as shown in the figure. The jump in ϕ across the first barrier is $-\Gamma$, that is,

$$\phi(x,+0) - \phi(x,-0) = \Gamma \qquad \left(\frac{-a}{2} \leqq x \leqq \frac{a}{2}\right) \qquad (4.70)$$

The jump across the second is seen to be zero, so that actually this second barrier is not needed. The contribution of the first barrier to the impulse component η is (per unit depth)

$$\begin{aligned} \rho \int_S m\phi \, \mathrm{d}\sigma &= \rho \int_{-a/2}^{a/2} (-1)\phi(x,+0)\,\mathrm{d}x + \rho \int_{-a/2}^{a/2} (+1)\phi(x,-0)\,\mathrm{d}x \\ &= -\rho\Gamma a \end{aligned} \qquad (4.71)$$

It may be desirable for the student to investigate the possibility of additional impulse contributions at the singularities, by considering them to be small circular cylinders of radius $\in$. If $\in$ is very small, the formula for ϕ is substantially unchanged. It will be found that the impulse contributions of the integrals taken around the small cylinders vanish as $\in \rightarrow 0$. By symmetry, the component ξ is zero. Thus, the quantity given in Eq. (4.71) is the total impulse of the vortex pair.

Sample Problem 4.20

The vortices making up a vortex pair are moved directly away from one another at the rate $\dot{a}$. What is the direction and magnitude of the force that must be exerted on each vortex?

Solution:
The impulse at any instant is $I_y = -\rho\Gamma a$, that is, $\rho\Gamma a(t)$ directed in the direction normal to the barrier. The force on the system is therefore $\rho\Gamma \mathrm{d}a / \mathrm{d}t$ in the same direction. By symmetry, half of this force must be applied to each vortex.

References

[1] Lamb, *Hydrodynamics*: parts of Chap. 5, 40, 41, 47, 48, 52, 64, 92, 96, 97, 117–120, 122.

[2] Jeffreys, and Jeffreys, *Methods of Mathematical Physics*, Cambridge, 1944, p. 348.

[3] NACA TM 574.

Problems

Assume incompressible irrotational flow for these problems.

4.1 Calculate and plot the surface-pressure distribution

$$\frac{p - p_0}{(1/2)\rho U^2}$$

for the two-dimensional leading edge treated in Sec. 4.1.

4.2 Show that the pressure distribution on the blunt rod produced by a stream and a source is given by

$$p - p_0 = \rho \frac{U^2}{2}(1 - 4t^2 + 3t^4)$$

where t is defined as the (cylindrical) radius at any point of the rod divided by the ultimate radius.

4.3 Consider an infinite circular cylinder of radius R placed in a steady stream of velocity U and density ρ, at an angle of yaw β.

a. Discuss the possibility of obtaining this flow from that of Sec. 4.4 by superimposing a velocity component in the direction of the cylindrical axis.

b. Sketch a typical streamline at the cylinder (top view).

c. Calculate the pressure distribution on the cylinder.

d. Calculate the lift if the circulation about the cylinder is Γ.

4.4 a. The additional apparent masses (per unit length) of an infinite elliptical cylinder with semiminor axis a and semimajor axis b, in plane flow, are $\pi\rho a^2$ for motion parallel to b and $\pi\rho b^2$ for motion parallel to a. (These will be verified later.) What are the direction and magnitude of the initial acceleration if a force F acts at an angle γ with the major axis? Let the density of cylinder material be $k\rho$ and that of the fluid ρ.

b. What moment acts on this cylinder if it is in steady motion at angle of attack α?

4.5 For an infinite flat plate of width $2a$ moving normal to itself with unit speed in fluid otherwise at rest, the surface values of velocity potential are $\sqrt{a^2 - x^2}$ (on top) and $-\sqrt{a^2 - x^2}$ (on bottom), x being measured along the plate from the center. (This will be verified later.)

a. Calculate the additional apparent mass.

b. What "lift" and "drag" result if the plate is moving through a fluid at a constant angle of attack α and velocity $V(t)$? What pitching moment? (Assume no circulation.)

4.6 A sphere moves through a fluid at constant speed U and in constant direction while its radius increases at the constant rate $\dot{R}$. (Assume mass of sphere is constant.)

a. What force must be provided?

b. Calculate the rate at which the kinetic energy of the flow is increasing. Discuss the energy balance.

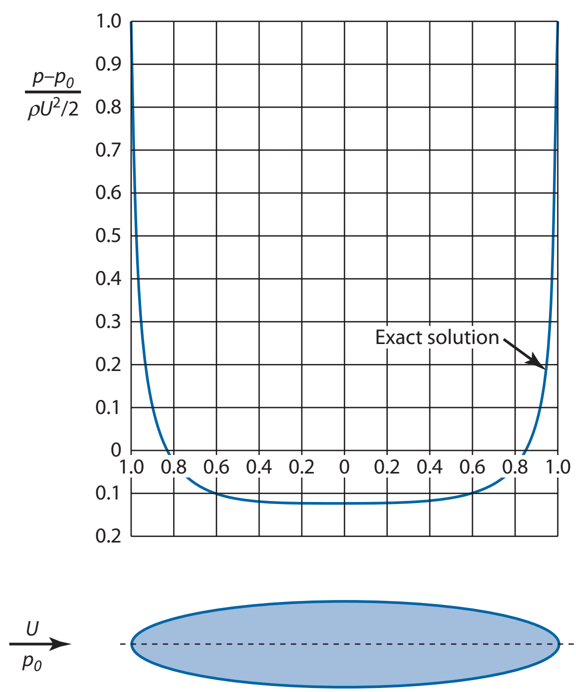
4.7 An engineer requires a formula for the maximum velocities produced by nacelles, fuselages, etc., of various slenderness in symmetrical flow.

As a typical family, consider the oval bodies of revolution each of which is represented by a source and a sink located at a distance $2a$ apart in a uniform stream of velocity U.

a. Determine the maximum velocity on the body (that is, the velocity at its midsection) in terms of U and R/a, where R is the maximum (midsection) radius of the body.

b. Show that your maximum velocity formula gives the correct results in the limiting cases of 1) $R \to 0$ and 2) when the body becomes a sphere.

4.8 Work out the approximate surface-pressure distribution $(p - p_0)/(\rho U^2/2)$ for an ellipsoid of revolution of fineness ratio 5, using the slender-body approximations. Plot your results on the below chart for comparison with the exact solution. (The latter is taken from [1], Sec. 105.) *Note:* Make your approximate calculation correct to $O(\in)^2$ where $R'(x) = O(\in)$.



4.9 If $\psi_0(r,\theta)$ is Stokes' stream function for unbounded axisymmetric flow with no singularities within $r \leqq a$, show that after introduction of a solid sphere of radius a at the origin the stream function is

$$\psi(r,\theta)=\psi_0(r,\theta)-\frac{r}{a}\psi_0\left(\frac{a^2}{r},\theta\right)$$

4.10 a. Show that the flow about an infinitely long cylindrical blade rotating with angular velocity ω about an axis normal to the cylindrical axis in an infinite body of fluid otherwise at rest is given by

$$u=\omega y\frac{\partial\varphi_1}{\partial x}$$

$$v=\omega\left[\varphi_1-2x\right]$$

$$w=\omega y\frac{\partial\varphi_1}{\partial z}$$

where x, y, and z are Cartesian coordinates, with y lying parallel to the cylindrical axis and z parallel to the axis of rotation. Also, $\varphi_1(x,z)$ denotes the potential for plane flow at unit stream speed about the same cylinder.

b. Describe this flow pattern in words.

Plane Irrotational Incompressible Flow: Complex Variables

- Regular functions and complex variables
- Conformal mapping
- General expressions for the force on a cylinder
- Force and moment in a transformed plane

5.1 Introduction

In this chapter, by introducing complex variables and the concept of regular functions, we shall arrive at powerful methods for the solution of plane-flow problems. In fact, it becomes clear that most of the plane problems solved heretofore in this book might have been solved by simpler methods if these devices had been used.

Let us review our equations for this type of flow:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad &&\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0\\
&u=\frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y} \qquad &&v=\frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x}\\
&\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0 \qquad &&\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=0
\end{aligned}\right\} \tag{5.1}$$

We now define x to be the real part and y the imaginary part of a complex variable z, that is, $z \equiv x+iy$, where $i^2=-1$. The following notation and nomenclature can also be used:

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

Modulus of z: $\quad r=\sqrt{x^2+y^2}\equiv |z|$

Argument of z: $\quad \theta = \tan^{-1}\frac{y}{x} \equiv \arg z$

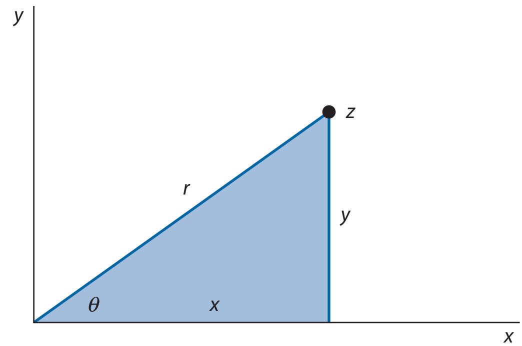


Fig. 5.1 Argand diagram.

Figure 5.1, called an Argand diagram, will be used continually; it is just a rectangular Cartesian coordinate diagram in which the real part of a complex number is plotted as the abscissa and the imaginary part as the ordinate so that the complex number becomes a sort of plane vector.

The value of this device in our work comes from constructing a complex number F, whose real part is the potential ϕ and imaginary part is the stream function, represented by ψ:

$$F \equiv \phi(x, y) + i\psi(x, y) \tag{5.2}$$

This combination is called the *complex potential*.

Now, F is a function of z, for if z is specified, x and y are known, and ϕ and ψ can be determined and combined to form F. Graphically, as shown in Fig. 5.2, a point selected in the z plane defines a point in the F plane, where the latter has ϕ for its abscissa and ψ for its ordinate. Similarly, a succession of points in the z plane, forming a curve, determines a curve in the F plane, provided that ϕ and ψ are suitable continuous functions.

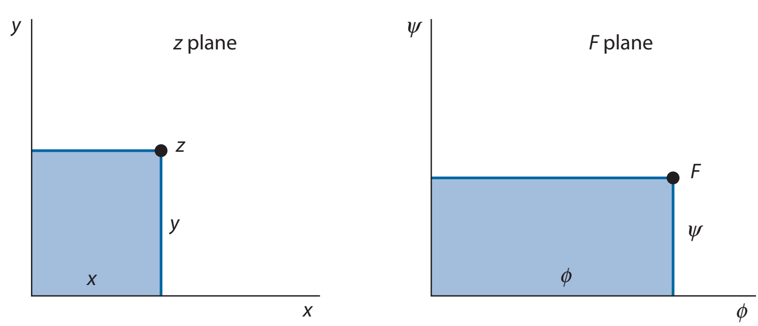


Fig. 5.2 Graphical representation of a point in z and F plane.

5.2 Regular Functions

There is a certain class of functions $f(z)$, of the complex variable z, that is especially useful; these are the functions that can be said to have a *derivative* at each point. Let us calculate the derivative by extending the usual definition in a manner that seems appropriate in this case. Suppose

$$f(z) = U(x,y) + iV(x,y)$$

Then,

$$\Delta f \equiv U(x+\Delta x, y+\Delta y) + iV(x+\Delta x, y+\Delta y) - U(x,y) - iV(x,y) \tag{5.3}$$

and we define

$$\frac{\mathrm{d}f}{\mathrm{d}z} \equiv \lim_{\Delta z \to 0}\left\{\frac{\Delta f}{\Delta z}\right\} \tag{5.4}$$

This is illustrated in the Argand diagrams in Fig. 5.3, where the corresponding complex increments Δz and Δf are shown. Now, it is clear that, for any given Δz, Δf has a certain direction and magnitude, and the limit of $(\Delta f / \Delta z)$ will, in general, depend on these directions. Hence, $(\mathrm{d}f / \mathrm{d}z)$ will not be a function of z alone, but will also depend on the direction of Δz. It is easy to show this by calculation. From Eqs. (5.3) and (5.4), assuming U and V to be *differentiable* functions,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}z} &= \lim_{\left.\begin{smallmatrix}\Delta x\\ \Delta y\end{smallmatrix}\right\}\to 0}\left\{\frac{U(x+\Delta x, y+\Delta y) - U(x,y) + i[V(x+\Delta x, y+\Delta y) - V(x,y)]}{\Delta x + i\Delta y}\right\}\\ &= \lim_{\left.\begin{smallmatrix}\Delta x\\ \Delta y\end{smallmatrix}\right\}\to 0}\left\{\frac{(\partial U/\partial x)\Delta x + (\partial U/\partial y)\Delta y + i[(\partial V/\partial x)\Delta x + (\partial V/\partial y)\Delta y]}{\Delta x + i\Delta y}\right\}\end{aligned}$$

or, if we let Δ denote $\Delta y/\Delta x$,

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{\left.\begin{smallmatrix}\Delta x\\ \Delta y\end{smallmatrix}\right\}\to 0}\left\{\frac{(\partial U/\partial x) + i(\partial V/\partial x) + [(\partial U/\partial y) + i(\partial V/\partial y)]\Delta}{1 + i\Delta}\right\} \tag{5.5}$$

By adding and subtracting terms, this can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} + \lim_{\left.\begin{smallmatrix}\Delta x\\ \Delta y\end{smallmatrix}\right\}\to 0}\left(\left\{\frac{(\partial U/\partial y) + (\partial V/\partial x) + i[(\partial V/\partial y) - (\partial U/\partial x)]}{1 + i\Delta}\right\}\Delta\right) \tag{5.6}$$

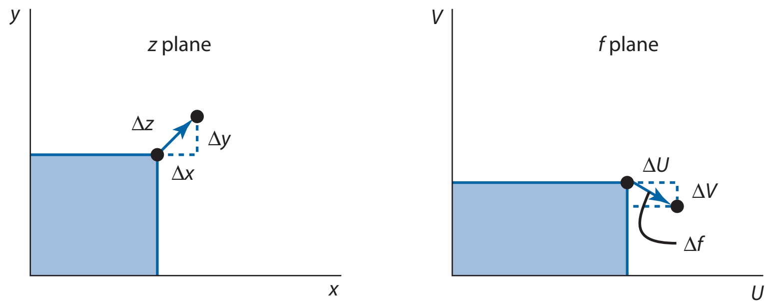


Fig. 5.3 Argand diagrams of increments Δz and Δf.

In this form, the part of $(\mathrm{d}f/\mathrm{d}z)$ that depends on the direction Δ is collected in one additive term. This part vanishes, and the *derivative is independent of direction*, if

$$\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}=0, \quad \frac{\partial V}{\partial y}-\frac{\partial U}{\partial x}=0 \tag{5.7}$$

It is only in cases where $(\mathrm{d}f/\mathrm{d}z)$ is independent of direction, and therefore is a *function of z*, say $f'(z)$. It can be said that $f(z)$ possesses a derivative at z. At such a point, $f(z)$ is said to be *regular*. The most useful functions are those that are regular at all points within a region or are regular everywhere except at a few singularities and so forth. Conditions (5.7), together with the differentiability of U and V, are the *Cauchy–Riemann* (CR) *conditions* for regularity of a function. They are both necessary and sufficient.

If, in addition to being regular in a region, a function is also single valued, it is often termed *analytic*.

Sample Problem 5.1

Show that Eq. (5.5) can be written $(\partial V/\partial y)-i(\partial U/\partial y)$ plus a term depending on Δ and that the elimination of this term leads again to Eq. (5.7). Explain.

Solution:
The expression preceding Eq. (5.5), upon dividing numerator and denominator by $i\Delta y$, reads

$$\frac{\mathrm{d}f}{\mathrm{d}z}=\lim_{\substack{\Delta x\\ \Delta y}\}\to 0}\left\{\frac{(\partial V/\partial y)-i(\partial U/\partial y)+[(\partial V/\partial x)-i(\partial U/\partial x)]\Delta^{-1}}{1-i\Delta^{-1}}\right\}$$

[equivalent to dividing numerator and denominator of Eq. (5.5) by $i\Delta$]

$$= \frac{\partial V}{\partial y} - i\frac{\partial U}{\partial y} + \lim_{\substack{\Delta x \\ \Delta y}\Big\}\to 0} \left[\frac{(\partial V/\partial x) - i(\partial U/\partial x) + i(\partial V/\partial y) + (\partial U/\partial y)}{1 - i\Delta^{-1}} \right] \Delta^{-1}$$

The meaning is that both $(\partial U/\partial x) + i(\partial V/\partial x)$ and $(\partial v/\partial y) - i(\partial u/\partial y)$ are correct expressions for $\mathrm{d}f/\mathrm{d}x$ for a regular function. The first is the derivative taken in the real direction ($\Delta z = \Delta x$), and the second is the derivative in the imaginary direction ($\Delta z = i\Delta y$); they are equal.

Sample Problem 5.2

Are the following regular functions?

1. z^n
2. $\cos z$

$$\left.\begin{array}{ll} 3. & \mathcal{R}[g(z)] \equiv \text{“the real part of”}\, g(z) \\ 4. & I[g(z)] \equiv \text{“the imaginary part of”}\, g(z) \\ 5. & |g(z)| \\ 6. & \bar{g}(z) \equiv \text{the complex conjugate of } g(z) \end{array}\right\} \begin{array}{l} \text{where } g(z) \text{ is known to be} \\ \text{a regular function of } z \end{array}$$

7. $h[g(z)]$, where both $h(z)$ and $g(z)$ are regular functions of z

Note that in part 6), if the real and imaginary parts of g are ξ and η, the complex conjugate $\bar{g}$ denotes $\xi - i\eta$.

Solution:
(At this point, the student is supposed to work out these seven cases using the CR conditions. Later, we learn that this tedious process is not usually necessary.)

1. $z^n = r^n e^{ni\theta} = r^n \cos n\theta + ir^n \sin n\theta$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(r^n \cos n\theta) = nr^{n-1}\frac{\partial r}{\partial y}\cos n\theta - n \sin n\theta r^n \frac{\partial \theta}{\partial y}$$

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(r^n \sin n\theta) = nr^{n-1}\frac{\partial r}{\partial x}\sin n\theta - n \cos n\theta r^n \frac{\partial \theta}{\partial x}$$

$$\frac{\partial V}{\partial y}=\frac{\partial}{\partial y}(r^n \sin n\theta)=nr^{n-1}\frac{\partial r}{\partial y}\sin n\theta - n\cos n\theta r^n\frac{\partial \theta}{\partial y}$$

$$\frac{\partial U}{\partial x}=\frac{\partial}{\partial x}(r^n \cos n\theta)=nr^{n-1}\frac{\partial r}{\partial x}\cos n\theta - n\sin n\theta r^n\frac{\partial \theta}{\partial x}$$

Now,

$$\frac{\partial r}{\partial y}=\frac{y}{r},\quad \frac{\partial r}{\partial x}=\frac{x}{r},\quad \frac{\partial \theta}{\partial x}=\frac{-y}{r^2},\quad \frac{\partial \theta}{\partial y}=\frac{x}{r^2}$$

and substitution in these four expressions will confirm that the CR conditions [Eq. (5.7)] are satisfied. Yes, z^n is a regular function of z.

2. $\cos z=\cos(x+iy)=\frac{1}{2}(\mathrm{e}^{ix-y}+\mathrm{e}^{-ix+y})$

$$=\frac{1}{2}(\mathrm{e}^{-y}+\mathrm{e}^{y})\cos x+\frac{i}{2}(\mathrm{e}^{-y}-\mathrm{e}^{y})\sin x$$

$$\frac{\partial U}{\partial y}=\frac{1}{2}(\mathrm{e}^{y}-\mathrm{e}^{-y})\cos x=-\frac{\partial V}{\partial x}$$

$$\frac{\partial V}{\partial y}=\frac{1}{2}(-\mathrm{e}^{-y}-\mathrm{e}^{y})\sin x=\frac{\partial U}{\partial x}$$

Yes, $\cos z$ is a regular function of z.

3. $U=R[g(x)],\ V=0$

In general, unless $g(z)$ is trivial,

$$\frac{\partial U}{\partial y}\neq 0=\frac{\partial V}{\partial x}\quad\text{and}\quad -\frac{\partial U}{\partial x}\neq 0=\frac{\partial V}{\partial y}$$

The real part of a regular function of z is *not* a regular function of z.

4. This is similar to part 3): *not* a regular function of z.
5. Again, $V=0$ whereas $U\neq 0$. This is the same as part 3) and 4): the absolute value of a regular function of z is *not* a regular function of z.
6. $\overline{g}(z)\equiv \xi(x,y)-i\eta(x)$

Now, we know that

$$\frac{\partial \xi}{\partial y}+\frac{\partial \eta}{\partial x}=0 \quad \text{and} \quad \frac{\partial \eta}{\partial y}-\frac{\partial \xi}{\partial x}=0$$

But,

$$\frac{\partial U}{\partial y}=\frac{\partial \xi}{\partial y} \quad \text{and} \quad \frac{\partial V}{\partial x}=-\frac{\partial \eta}{\partial x}$$

$$\frac{\partial V}{\partial y}=-\frac{\partial \eta}{\partial y} \quad \text{and} \quad \frac{\partial U}{\partial x}=\frac{\partial \xi}{\partial x}$$

$$\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}=\frac{\partial \xi}{\partial y}-\frac{\partial \eta}{\partial x}\neq 0$$

$$\frac{\partial V}{\partial y}-\frac{\partial U}{\partial x}=-\frac{\partial \eta}{\partial y}-\frac{\partial \xi}{\partial x}\neq 0$$

The complex conjugate of a regular function of z is *not* a regular function of z.

7. Let $h(z)=M(x,y)+iN(x,y)$ and $g(z)=P(x,y)+iQ(x,y)$. Then, $h(g)=M(P,Q)+iN(P,Q)$. We know that

$$\frac{\partial M}{\partial Q}+\frac{\partial N}{\partial P}=0 \quad \text{and} \quad \frac{\partial N}{\partial Q}-\frac{\partial M}{\partial P}=0$$

(How do we know this?)

Now,

$$\frac{\partial M(P,Q)}{\partial y}=\frac{\partial M}{\partial P}\frac{\partial P}{\partial Y}+\frac{\partial M}{\partial Q}\frac{\partial Q}{\partial y}$$

whereas

$$\frac{\partial N(P,Q)}{\partial x}=\frac{\partial N}{\partial P}\frac{\partial P}{\partial x}+\frac{\partial N}{\partial Q}\frac{\partial Q}{\partial x}$$

$$=-\frac{\partial M}{\partial Q}\frac{\partial Q}{\partial y}-\frac{\partial M}{\partial P}\frac{\partial P}{\partial y}=-\frac{\partial M(P,Q)}{\partial y}$$

An analogous calculation discloses that

$$\frac{\partial N(P,Q)}{\partial y} = \frac{\partial M(P,Q)}{\partial x}$$

The conclusion is that a regular function of a regular function of z is, itself, a regular function of z.

5.3 Complex Potential Is a Regular Function

It is clear that $F(z)$, as defined in Eq. (5.2), is a regular function of z, for the CR conditions are exactly the relations between the first derivatives in Eq. (5.1). Consequently, every plane irrotational flow defines a regular function of a complex variable. Moreover, the converse is true, namely, that every such function defines an irrotational flow (or rather two different irrotational flows), for the real and imaginary parts must satisfy the CR conditions and must therefore satisfy Laplace's equation (which should be verified).

Because the CR conditions are linear, the sums of regular functions are regular, and flows can be built up by adding regular functions, that is, by superposition.

The singularities of flow patterns are points where regularity is absent, either because Eq. (5.7) is not satisfied or simply because the partial derivatives do not exist at such points.

Moreover, it is seldom necessary even to check the CR equations for any given function to see whether it is regular, for we can prove that if $f(t)$ is any analytic function of a real variable t, $f(z)$ is regular. Here, we use "analytic" in a different sense, as in real-variable theory, namely, to designate functions that can be expanded in power series of t. This includes, of course, all polynomial expressions and rational combinations thereof, and also the transcendental functions such as $\sin t$, e^t, $\ln t$, $J_n(t)$, all of which can be defined by means of their power series. The theorem is that if one replaces t by z in these functions, one will obtain a function that is regular in a corresponding region.

To prove this, note that for these functions,

$$\frac{\partial f(z)}{\partial x} = \frac{\partial}{\partial x} f(x+iy) = f'(x+iy) \tag{5.8}$$

whereas

$$\frac{\partial f(z)}{\partial y} = \frac{\partial}{\partial y} f(x+iy) = if'(x+iy) \tag{5.9}$$

Hence, if f is written as $f_1 + if_2$, it appears that

$$f' = \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x} = -i\left\{\frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y}\right\} \tag{5.10}$$

which is same as Eq. (5.7).

The converse of this is also true, for the CR conditions imply $\nabla^2 f = 0$, and it can be proved that the most general solution of this equation in two dimensions is of the form $f = F_1(x+iy) + F_2(x-iy)$. A check will disclose that only the first of these satisfies the CR conditions, whereas the second satisfies slightly different conditions and is therefore rejected.

Sample Problem 5.3

Determine the complex potentials for plane source, vortex, doublet, flow past leading edge, and flow past cylinder and also for the flow of Example 1.5.

Solution:

Source: From Sec. 3.1,

$$\begin{aligned}\phi + i\psi &= \frac{Q}{2\pi}(\ln r + i\theta) + \text{const}\\ &= \frac{Q}{2\pi}\ln(re^{i\theta}) + \text{const}\\ &= \frac{Q}{2\pi}\ln z + \text{const}\end{aligned}$$

where Q is real.

Vortex: From Sec. 5.5.2,

$$\begin{aligned}\phi + i\psi &= \frac{\Gamma}{2\pi}(\theta - i\ln r) + \text{const}\\ &= \frac{-i\Gamma}{2\pi}\ln(re^{i\theta}) + \text{const}\\ &= \frac{-i\Gamma}{2\pi}\ln z + \text{const}\end{aligned}$$

where Γ is real.

Doublet: From Sec. 5.5.3,

$$\phi + i\psi = -\frac{\mu}{2\pi r}(\cos\theta - i\sin\theta) + \text{const}$$
$$= -\frac{\mu}{2\pi r}\mathrm{e}^{-i\theta} + \text{const}$$
$$= -\frac{\mu}{2\pi z} + \text{const}$$

where μ is real. Imaginary and complex μs should also be considered.

Flow of Eqs. (4.3):

$$\phi + i\psi = U\left\{z + \frac{h}{2\pi}\ln z\right\} + \text{const}$$

Flow past circular cylinder: From Eqs. (4.11),

$$\phi + i\psi = U\left\{r\mathrm{e}^{i\theta} + \frac{a^2}{r}\mathrm{e}^{-i\theta}\right\} - \frac{i\Gamma}{2\pi}\ln z + \text{const}$$
$$= U\left\{z + \frac{a^2}{z}\right\} - \frac{i\Gamma}{2\pi}\ln z + \text{const}$$

Doublet: Using a result of Sample Problem 1.2,

$$\phi + i\psi = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2} + \text{const}$$
$$= \frac{r\cos\theta - ir\sin\theta}{r^2} + \text{const}$$
$$= \frac{1}{r}\mathrm{e}^{-i\theta} + \text{const}$$
$$= \frac{1}{z} + \text{const}$$

5.4 Complex Velocity

It is a great simplification to have found that every regular function of a complex variable has, for its real and imaginary parts, plane harmonics and the converse (that is, any plane harmonic is the real or imaginary part of a regular function of a complex variable). But the advantages of this technique do not end here.

Consider the derivative of the complex potential:

$$F'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \quad \text{or} \quad -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}$$

$$= u - iv \tag{5.11}$$

This is called the *complex velocity*; it is the complex conjugate of the velocity vector plotted as a complex number in the Argand diagram. Its modulus is the flow speed q. It is another regular function, for it is easy to verify that the derivatives and integrals of regular functions are also regular. We sometimes denote it by the symbol $w(z)$.

Sample Problem 5.4

Prove that $w = \mathrm{e}^{-i\theta}(v_r - iv_\theta)$.

Solution:
There are several ways to prove this:

1. Because $F(z)$ is a regular function of z, the derivative $w = F'(z)$ can be taken in any direction. Let us try the r direction, that is,

$$F'(z) = \lim \frac{\Delta F}{\Delta r \mathrm{e}^{i\theta}} = \mathrm{e}^{-i\theta}\left(\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r}\right) = \mathrm{e}^{-i\theta}(v_r - iv_\theta)$$

2. Start with $w = u - iv$, and note that $u = v_r \cos\theta - v_\theta \sin\theta$ and $v = v_\theta \cos\theta + v_r \sin\theta$; thus,

$$w = v_r \cos\theta - v_\theta \sin\theta - i(v_\theta \cos\theta + v_r \sin\theta)$$

$$= v_r(\cos\theta - i\sin\theta) - iv_\theta(\cos\theta - i\sin\theta)$$

$$= \mathrm{e}^{-i\theta}(v_r - iv_\theta)$$

3. Start with $w = u - iv = \dfrac{\partial \phi}{\partial x} - i\dfrac{\partial \phi}{\partial y}$

$$= \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial x} - i\left\{\frac{\partial \phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial y}\right\}$$

$$= \frac{\partial \phi}{\partial r}\left(\frac{x}{r} - i\frac{y}{r}\right) + \frac{\partial \phi}{\partial \theta}\left(\frac{-y}{r^2} - i\frac{x}{r^2}\right)$$

and with

$$r=\sqrt{x^2+y^2},\quad \theta=\tan^{-1}\frac{y}{x}$$
$$w=v_r(\cos\theta-i\sin\theta)-iv_\theta(\cos\theta-i\sin\theta)$$

Sample Problem 5.5

Prove that, if C is any closed contour in the z plane,

$$\oint_C w(z)\,\mathrm{d}z=\Gamma+iQ \tag{5.12}$$

where Γ is the circulation about C and Q is the quantity of flow across C, that is, the total source strength enclosed by C.

Solution:

$$\begin{aligned}\oint_C w(z)\mathrm{d}z&=\oint_C(u-iv)(\mathrm{d}x+i\,\mathrm{d}y)\\&=\oint_C(u\,\mathrm{d}x+v\,\mathrm{d}y)+i\oint_C(u\,\mathrm{d}y-v\,\mathrm{d}x)\\&=\Gamma+iQ\end{aligned}$$

(see Secs. 1.8 and 1.10).

5.5 Complex Potentials of Simple Flows

Now we know that every regular function represents a plane flow, we can try the same scheme we used in Sec. 5.2, namely, to write down some simple functions and see what flows they give us.

5.5.1 $F=Cz$

Let C be complex, equal to $A+iB$, say, where A and B are real numbers.

$$w\equiv F'=u-iv=A+iB \tag{5.13}$$

Thus, $u=A$, and $v=-B$. This is the complex potential for a parallel stream, inclined to the x axis. There are no singularities in this flow pattern, unless we designate the "point at infinity," where F becomes infinite but the flow is still the same (Fig. 5.4).

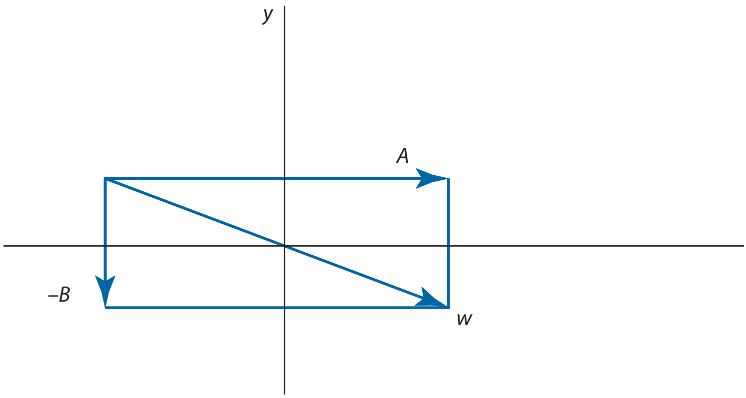


Fig. 5.4 Velocity of $F = Cz$.

5.5.2 $F = Az^2$

Let A be a real number. Then,

$$\phi = A\{x^2 - y^2\}, \quad \psi = 2Axy \tag{5.14}$$

and we see that the streamlines and equipotentials form two orthogonal families of hyperbolas, as shown in Fig. 5.5. This complex potential can be used, for example, to study the impinging of fluid against a plane wall, or the flow in a right-angle corner (Fig. 5.6).

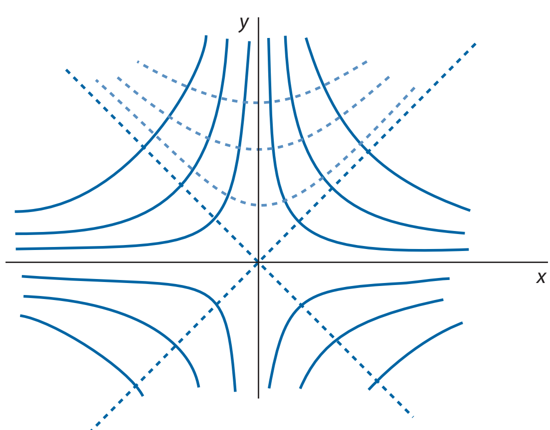


Fig. 5.5 Streamlines and equipotentials of $F = Az^2$.

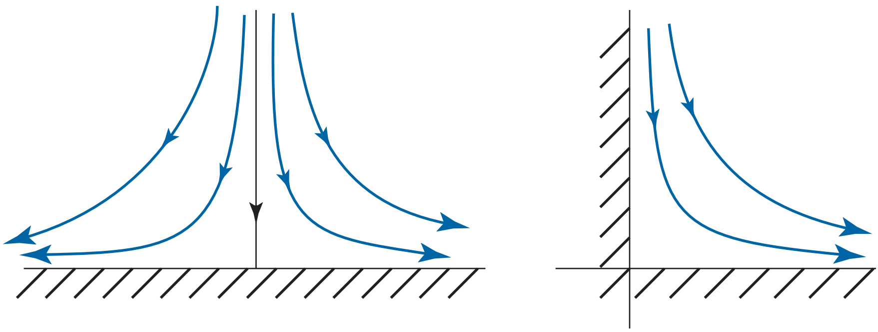


Fig. 5.6 Flow in a right-angle corner, given by $F = Az^2$.

The complex velocity is $w = 2Az$; thus, $u = 2Ax$ and $v = -2Ay$. There is a stagnation point at the origin.

The flow in the immediate neighborhood of any stagnation point produced by flow impinging on a smooth body has this character, and the expression for F in powers of $z - z_0$, where z_0 is the stagnation point, begins with a term $(z - z_0)^2$.

Again, there are no singularities in the finite region of the flow. At infinity, both F and velocities become infinite.

5.5.3 $F = Az^n$

Assume, for the time being, that A is real and n is real and positive.

$$\phi = Ar^n \cos n\theta \quad \text{and} \quad \psi = Ar^n \sin n\theta$$

Note that ψ is constant when $\theta = 0, \pi/n, 2\pi/n, \ldots,$ for all values of r. Upon investigation, it becomes clear that this gives us the flow in a reentrant corner when $n > 1$ [as, for example, $n = 2$, case in Sec. 5.5.2, or $n = 4$ (Fig. 5.7a)], and the flow around an exterior corner when $n < 1$ (Fig. 5.7b).

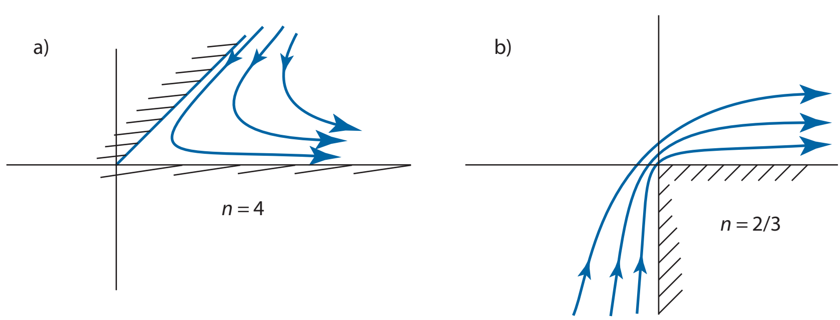


Fig. 5.7 a) Flow in a reentrant corner and b) flow around exterior corner.

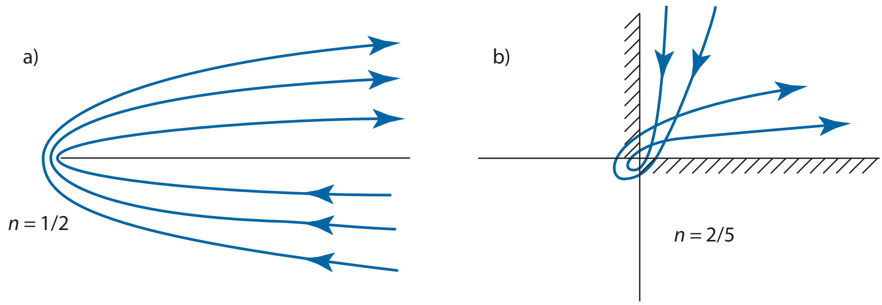


Fig. 5.8 Flows in $F = Az^2$ when a) $n = \frac{1}{2}$ and b) $n = \frac{2}{5}$.

When $n = \frac{1}{2}$, we obtain the interesting flow pattern shown in Fig. 5.8. It gives us the behavior of an inviscid incompressible fluid passing around the edge of a plate and will be met again in several practical problems. However, for any smaller values of *n,* the flow seems to lose its physical significance, for the streamlines cross over each other (Fig. 5.8). Hence, we will have to restrict *n* to values equal to or greater than $\frac{1}{2}$.

When $n > 1$, there is a stagnation point at the origin; this is typical of every flow in an interior corner. When $n < 1$, there occurs a singularity at the origin, for the velocity and all of its derivatives become infinite. This is characteristic of every exterior sharp corner in inviscid incompressible flow; naturally, we must expect significant viscous effects at any such point in a real fluid. The intermediate case, $n = 1$, is, of course, our parallel stream (case in Sec. 5.5.1).

5.5.4 $F = C / z$

This case has already been encountered by the student in Sample Problem 5.3. It represents doublet flow. If C is real, the doublet is directed along the x axis; if imaginary, it is directed along the y axis. The origin is a singularity.

5.5.5 $F = C \ln z$

This case was also discussed in Sample Problem 5.3 because it represents plane source or vortex flow, depending on whether C is real or imaginary. If C is complex, we get the superposed flow due to source and vortex; this is an interesting flow, which we might have to look at.

We see that multivaluedness of these flows arises from the ambiguity of the logarithm of a complex variable.

Sample Problem 5.6

Determine the character of the flow given by $F = e^z$. Sketch some streamlines. Suggest, if you can, a physical interpretation.

Solution:

$$F(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

$$\phi = e^x \cos y \qquad \psi = e^x \sin y$$

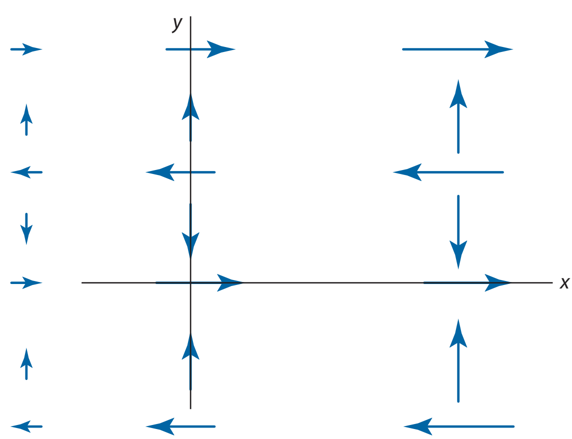
$$u = e^x \cos y \qquad v = -e^x \sin y$$

A clue as to the flow pattern is given by these velocities: note that they vary sinusoidally with y and exponentially with x.

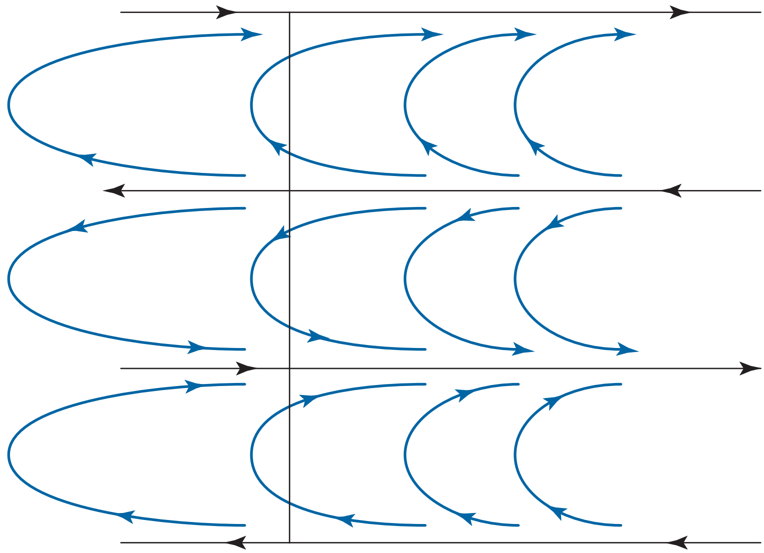
The streamlines are given by $\psi = \text{const}$

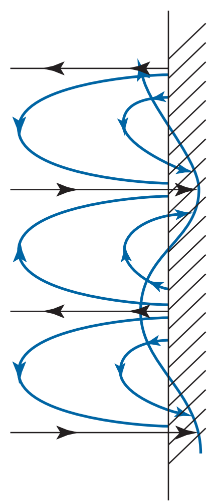
$$x = \ln \frac{\text{const}}{\sin y} = -\ln \frac{\sin y}{\text{const}}$$

which confirms that the flow looks something like this:



A physical interpretation: Suppose a vertical wall is deforming sinusoidally as drawn. As it passes through the plane shape with normal velocities u as drawn, the flow of a fluid lying to its left is given by the formula $F(s) \propto e^z$.





5.6 Conformal Mapping

The most powerful method of finding solutions to plane boundary-value problems is the use of conformal mapping, which is a method of relating complicated problems to simpler ones. We have seen that any function $\zeta(z)$ of a complex variable z represents a *mapping* of the z plane into a ζ plane. Now, if the function is *regular*, the mapping has the property of preserving the geometrical relationships of points and elements in any small region, which is described by the word *conformal*. In effect, a conformal transformation is one that does not distort the map in any small neighborhood so

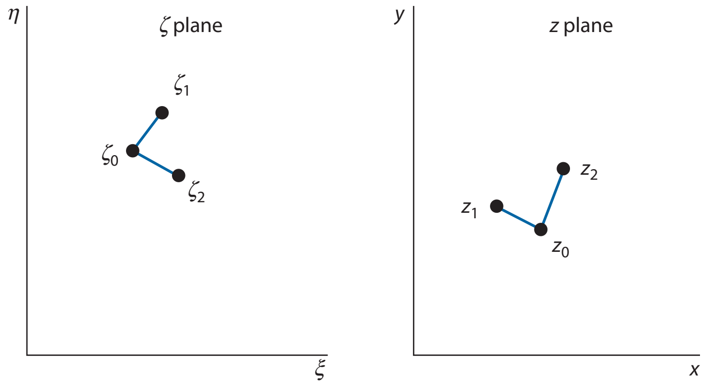


Fig. 5.9 Three points in ζ and z plane.

that the angle between any two line elements is preserved. This follows from the existence of a derivative at each point, for the derivative is a complex number that tells the magnification and orientation of the transformed line element, and it is the same for every element at any given point.

Let us write this out: consider three points ζ_0, ζ_1, and ζ_2, corresponding to z_0, z_1, and z_2 (Fig. 5.9). The existence of the derivatives of ξ and η ensures that ζ_1 and ζ_2 will be close to ζ_0 if z_1 and z_2 are close to z_0. Let $z - z_0 = re^{i\theta}$ and $\zeta - \zeta_0 = se^{i\vartheta}$. Then, except for second-order terms in r and s,

$$\zeta - \zeta_0 \approx (z - z_0)\zeta'(z_0)$$

or

$$se^{i\vartheta} \approx re^{i\theta}\zeta'(z_0)$$

where, if $\zeta(z)$ is regular at z_0, the derivative $\zeta'(z_0)$ is a fixed complex number. In particular, $s_1e^{i\vartheta_1} = r_1e^{i\theta_1}\zeta'(z_0)$ and $s_2e^{i\vartheta_2} = r_2e^{i\theta_2}\zeta'(z_0)$ so that

$$\frac{s_1}{s_2}e^{i(\vartheta_1 - \vartheta_2)} = \frac{r_1}{r_2}e^{i(\theta_1 - \theta_2)} \tag{5.15}$$

Hence,

$$\frac{s_1}{s_2} = \frac{r_1}{r_2} \quad \text{and} \quad \vartheta_1 - \vartheta_2 = \theta_1 - \theta_2 \tag{5.16}$$

Thus, the angle between the elements is preserved, as predicted, and the small triangle formed by the three points is transformed into a similar triangle. The ratio of the sides of the triangles

$$\frac{s_1}{r_1} = \frac{s_2}{r_2} = \cdots = |\zeta'(z_0)|$$

is called the *magnification* of the transformation at z_0.

We see that a conformal transformation transforms any smooth curve into another smooth curve. The most useful transformations are those that are conformal at all but a few points, called singular points or singularities, where the property breaks down. A curve that passes through such a point, of course, might undergo a sharp distortion in the transformation.

5.7 Singularities of a Transformation

The analysis stated in Sec. 5.6, regarding conformality, breaks down if $\zeta'(z_0)$ is either 0 or ∞ [See Eq. (5.15)]. Any point where the derivative assumes either of these values is called a *singularity* of the transformation. Suppose, for example, that $\zeta'(z)$ has a "zero of nth order" at $z = z_0$. It means that $\zeta'(z)$ can be written in the form $(z - z_0)^n f(z)$, where $f(z)$ is a function that is neither 0 nor ∞ at z_0. Or, for points near z_0, it means

$$\zeta - \zeta_0 \approx (z - z_0)^{n+1} f(z_0) \tag{5.17}$$

If we repeat this argument, for this case, we find that

$$\frac{s_1}{s_2} = \left(\frac{r_1}{r_2}\right)^{n+1} \quad \text{and} \quad \vartheta_1 - \vartheta_2 = (n+1)(\theta_1 - \theta_2) \tag{5.18}$$

Thus, angles are multiplied by the factor $n+1$ while being transformed, at such a point.

For example, if $\zeta'(z)$ has a "simple zero" ($n = 1$) at z_0, angles are doubled. If a smooth curve passes through such a point in the z plane, a cusp will appear in the ζ plane (Fig. 5.10) and so forth.

If $\zeta'(z)$ becomes infinite at a point $z = z_0$, then the derivative $dz/d\zeta$ of the reverse transformation has a zero. We see that there is no essential difference between these two cases.

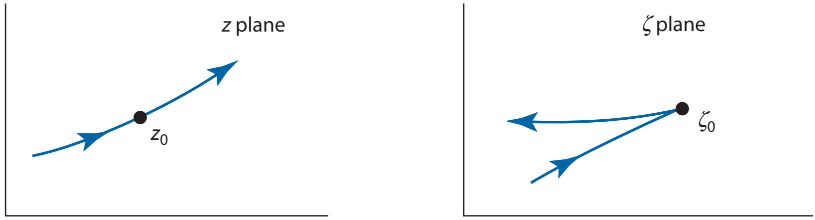


Fig. 5.10 Cusp formation in ζ plane.

5.8 Conformal Mapping in Hydrodynamics

Suppose that $f(z)=\phi(x,y)+i\psi(x,y)$ is the complex potential of a flow in the z plane and $\zeta(z)$ is a conformal transformation, mapping the z plane upon the ζ plane. By writing the transformation in the form $z=z(\zeta)$ and substituting for z in $f(z)$, we obtain a new function:

$$f(z)=f[z(\zeta)]\equiv F(\zeta) \tag{5.19}$$

Now, $F(\zeta)$ is a regular function of ζ (this is left for the student to prove), and therefore, $F(\zeta)$ defines a flow in the ζ plane. In other words, the conformal transformation of a flow pattern is another flow pattern.

Moreover, the streamlines in the z plane are transformed into the streamlines in the ζ plane, and the same for equipotentials, for if $F=\Phi+i\psi$, it is clear that

$$f(z)=f(\zeta) \quad \text{implies} \quad \begin{cases}\phi(x,y)=\Phi(\xi,\eta)\\ \psi(x,y)=\Psi(\xi,\eta)\end{cases} \tag{5.20}$$

Consequently, for steady flows, the contour of a solid body in the flow is mapped onto the contour of another solid body in the ζ plane. In any case, steady or unsteady, a boundary-value problem involving certain boundaries in the z plane, upon transformation, becomes another boundary-value problem in the ζ plane, involving the transformed boundaries. Obviously, the steady-flow solid-body case is only a special case. The great value of conformal mapping is that the transformed boundaries may be much simpler than the originals; for example, an airfoil profile can be transformed into a circle. The problem of finding the transformation to map one given curve into another given curve and the region exterior (or interior) to the first curve into that exterior (or interior) to the second does not have an explicit solution in general. Often, it can be accomplished by approximate, numerical methods only.

Before going further, we point out another important property of conformal transformations of irrotational flows, namely, the relationship between the transformed complex velocities. If $\zeta = \zeta(z)$ and $f(z) = F(\zeta)$, as stated earlier, then

$$F'(\zeta) = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{w(z)}{\zeta'(z)} \tag{5.21}$$

Thus, the complex velocities at corresponding points are in the (complex) ratio ζ'. The flow speeds are in the ratio $|\zeta'|$; the flow directions differ by the angle $\arg \zeta'$. In particular, at any singularity of the transformation, the velocities are in the ratio 0 or ∞.

Sample Problem 5.7

Show that the circulation about a closed curve C_z in the z plane, and also the total flow across it, is equal to the circulation about the transformed contour C_ζ, and the flow across it.

Solution:

$$\oint_{C_z} w(x)\,\mathrm{d}z = \Gamma_z + iQ_z \text{ in the } z \text{ plane}$$

Now, suppose a transformation is introduced, $z = z(\zeta)$, mapping the z plane on the ζ plane and the closed contour C_z on the contour C_ζ (also closed). The preceding formula reads

$$\Gamma_z + iQ_z = \oint_{C_z} w(z)\,\mathrm{d}z \equiv \oint_{C_z} \frac{\mathrm{d}F(z)}{\mathrm{d}z}\mathrm{d}z = \oint_{C_\zeta} \frac{\mathrm{d}\mathcal{F}(\zeta)}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\mathrm{d}\zeta$$

(This is just a substitution of variable in a definite integral.) But this is nothing but

$$\oint_{C_\zeta} \frac{\mathrm{d}\mathcal{F}(\zeta)}{\mathrm{d}\zeta}\mathrm{d}\zeta = \Gamma_\zeta + iQ_\zeta$$

5.9 Some Examples of Conformal Transformations

In this section, we shall have a look at some typical conformal transformations. The procedure here shows what is meant by "discussing" a transformation, namely, one determines how certain coordinate lines are transformed, where certain typical points or regions are mapped, whether there are any singularities, and so forth.

5.9.1 $\zeta = C_z$

This is the simplest transformation. If C is real, it is simply a change of scale. If C is complex, say equal to $ae^{i\alpha}$, then (with $\zeta = se^{i\vartheta}$ and $z = re^{i\theta}$ as before):

$$se^{i\vartheta} = are^{i(\theta+\alpha)} \tag{5.22}$$

This is a rotation of the z plane through an angle α, together with a change of scale if $a \neq 1$ (Fig. 5.11).

In view of this result, it is permissible for us to omit any multiplicative constant, real or complex, if we wish, in considering subsequent transformations.

After we have determined the characteristics of a transformation, we can always multiply by a constant to rotate the plane and/or change its scale.

5.9.2 $\zeta = z^2$

Let $\zeta = \xi + i\eta$. Then,

$$\xi = x^2 - y^2 \quad \text{and} \quad \eta = 2xy \tag{5.23}$$

Thus, the coordinates $\xi = \text{constant}$ and $\eta = \text{constant}$ in the ζ plane are transformed into orthogonal families of hyperbolas (see Fig. 5.5). These equations can also be written in the form

$$\xi = \left(\frac{\eta}{2y}\right)^2 - y^2 \quad \text{and} \quad \xi = x^2 - \left(\frac{\eta}{2x}\right)^2 \tag{5.24}$$

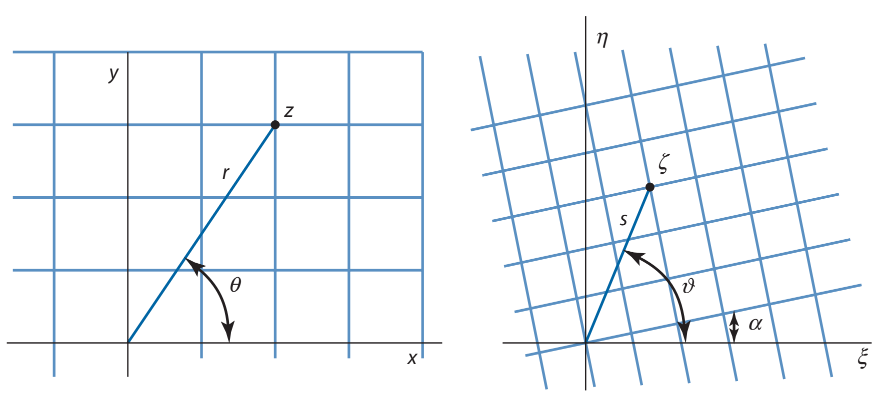


Fig. 5.11 Mapping of the transformation $\xi = C_z$.

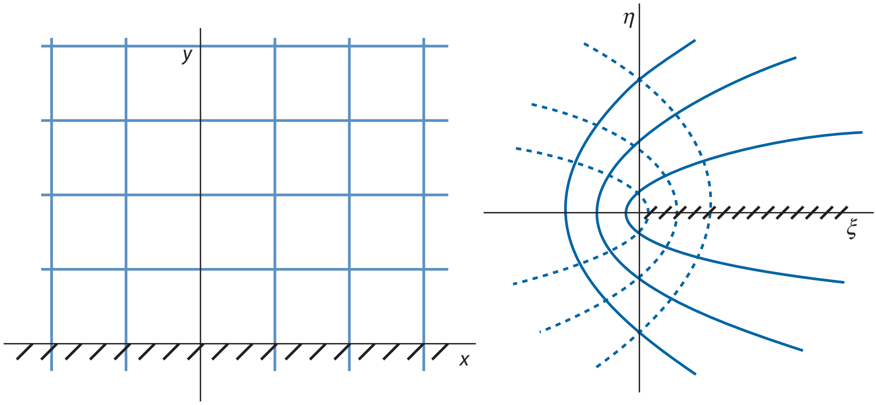


Fig. 5.12 Straight lines in z plane transform to parabolas in ζ plane.

We see that the coordinate lines in the z plane are transformed into orthogonal families of parabolas (Fig. 5.12).

Now a certain complication appears in this example, namely, the mapping is not one-to-one. We get two points in the z plane corresponding to every single point in the ζ plane (except the origin). This is easily seen to be inevitable because every number has two square roots, differing in sign, that is, differing in a factor $e^{\pi x i}$. This complication is not serious. We can easily make the mapping one-to-one by confining our attention to the upper (or the lower) half of the z plane. Working with either half, we get the entire ζ plane.

The German mathematician B. Riemann (1826–1866) had a more ingenious idea. He imagined, in a case like this, that the ζ plane consists not of a single sheet, but of two connecting sheets, as shown in Fig. 5.13.

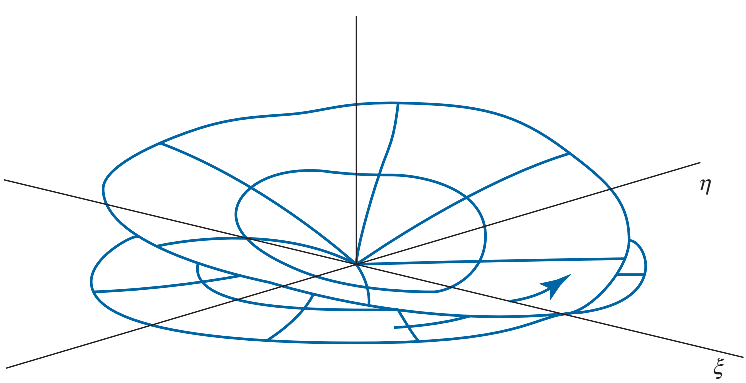


Fig. 5.13 Connecting sheets of ζ plane.

One of these comes from the upper half-plane in z, the other from the lower. They are joined along the positive ξ axis.

There is a singularity at $z=0$, for $d\zeta/dz=0$. Because $d\zeta/dz$ is equal to $2z$, it is clear that it is a simple zero; hence, angles are doubled at this point.

5.9.3 $\zeta=(a^2/z)$

In these transformations a is a real number. The inverse transformation $z=a^2/\zeta$ is the same transformation. Note that $\zeta'(z)=-a^2/z^2$ and $z'(\zeta)=-a^2/\zeta^2$. Thus, the origin in each plane is a singularity. It is not useful to inquire about the index n at these singularities because, in either case, the origin is transformed into the "point" at infinity.

According to this transformation,

$$se^{i\vartheta}=\frac{a^2}{r}e^{-i\theta};\quad s=\frac{a^2}{r},\quad \vartheta=-\theta \tag{5.25}$$

Hence, each point in ζ plane is mapped in the complex conjugate of its image in the circle $r=a$ (Fig. 5.14). Any circle about the origin, of radius r, is mapped on a circle about the origin of radius a^2/r and vice versa. Thus, the entire region *exterior* to the circle of radius a is mapped on the region *interior* and vice versa.

This transformation is called an *inversion* in the circle of radius a. It has the property of transforming any circle into a circle, providing we recognize straight lines as circles of infinite radius passing through the "point" at infinity. In fact, it is a special case of a family of transformations that have this property. These will be discussed in Sec. 5.9.6.

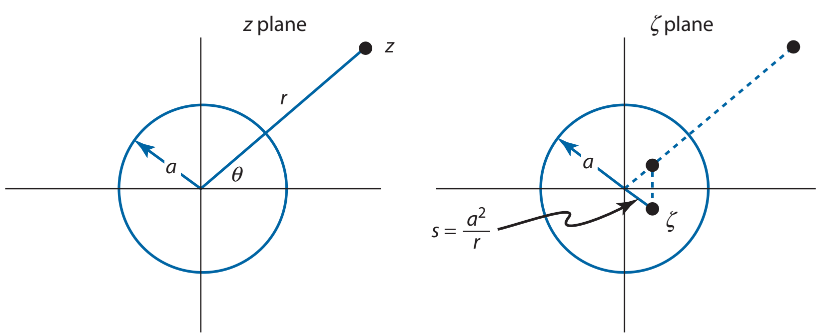


Fig. 5.14 Example for inverse transformation.

Sample Problem 5.8

Verify that, in an inversion, circles are transformed into circles and straight lines into circles through the origin.

Solution:
Here are three proofs: Let the transformation be $z = c^2/\zeta$.

1. Consider the circle $(x-a)^2 + (y-b)^2 = R^2$. Note that

$$x + iy = \frac{c^2}{\xi + i\eta} = c^2 \frac{\xi - i\eta}{\xi^2 + \eta^2} \quad \text{so that} \quad x = \frac{c^2\xi}{\rho^2}, \quad y = -\frac{c^2\eta}{\rho^2}$$

where $\rho^2 \equiv \xi^2 + \eta^2$. The circle becomes

$$\left(\xi - \frac{a}{c^2}\rho^2\right)^2 + \left(-\eta - \frac{b}{c^2}\rho^2\right)^2 = \frac{R^2}{c^4}\rho^4$$

or

$$\xi^2 - 2\frac{a}{c^2}\xi\rho^2 + \frac{a^2}{c^4}\rho^4 + \eta^2 + 2\frac{b}{c^2}\eta\rho^2 + \frac{b^2}{c^4}\rho^4 = \frac{R^2}{c^4}\rho^4$$

or dividing by ρ^2,

$$1 - 2\frac{a}{c^2}\xi + 2\frac{b}{c^2}\eta + \left(\frac{a^2 + b^2 - R^2}{c^4}\right)\rho^2 = 0$$

This is the equation for a circle in the (ξ, η) plane.

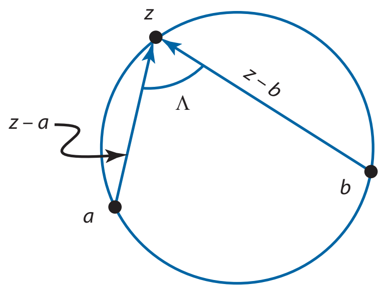
2. This proof by Dr. H.-H. Chu is essentially the same as proof 1 but carried out very neatly in vector notation:

A circle in the z plane has the form

$$z\bar{z} + \bar{b}z + b\bar{z} = R^2 = \text{real} \quad \text{(see vector-analysis books)}$$

which becomes

$$\frac{c^2}{\zeta}\frac{c^2}{\bar{\zeta}} + \bar{b}\frac{c^2}{\zeta} + b\frac{c^2}{\bar{\zeta}} = R^2$$



or

$$\zeta\bar{\zeta} - \frac{\bar{b}c^2}{R^2}\bar{\zeta} - \frac{bc^2}{R^2}\zeta = \frac{c^4}{R^2}$$

which is a circle in the ζ plane.

3. A third proof that will teach us something interesting about geometry in the complex plane is as follows: The equation of any circle through points $z = a$ and $z = b$ in the z plane is

$$\arg\left(\frac{z-a}{z-b}\right) = \text{const} = \Lambda$$

Now, let $z = c^2 / \zeta$:

$$\arg\frac{(c^2/\zeta) - a}{(c^2/\zeta) - b} = \arg\frac{(c^2/a) - \zeta}{(c^2/b) - \zeta} = \Lambda$$

This is the equation of a circle through $\zeta = c^2/a$ and $\zeta = c^2/b$ in the ζ plane. It is interesting that the included angle Λ is the same in both planes: this is because the straight lines in our diagram transform into straight lines (Why?) and angles are preserved.

5.9.4 $\zeta = z + (a^2/z)$

In these transformations a is a real number. This transformation is especially useful in airfoil theory. The derivative is

$$\zeta'(z) = 1 - \frac{a^2}{z^2} = \left[\frac{(z-a)(z+a)}{z^2}\right] \tag{5.26}$$

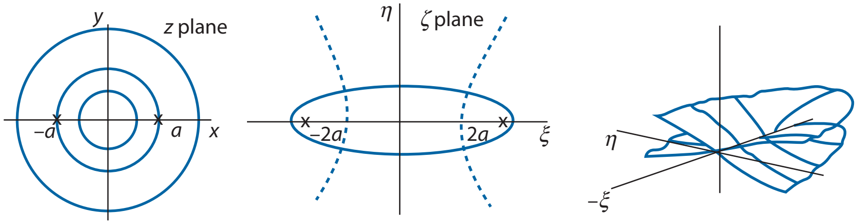


Fig. 5.15 Example of two-to-one transformation.

Thus, there are singularities at $z = a$ and $z = -a$, which are transformed into $\zeta = 2a$ and $-2a$. Angles are doubled at these points, in going from z to ζ. Now, write the transformation in exponential form:

$$se^{i\vartheta} = re^{i\theta} + \frac{a^2}{r}e^{-i\theta} \tag{5.27}$$

Note that the circle $r = a$ is mapped on the ξ axis between $\pm 2a$, for

$$ae^{i\theta} + \frac{a^2}{a}e^{-i\theta} = 2a\cos\theta \tag{5.28}$$

Moreover, the point z and the point (a^2/z) are mapped on the same point in the ζ plane. But one of these is inside and one outside the circle of radius a. Hence, this is another two-to-one transformation, or the ζ plane has two Riemann sheets, one corresponding to the exterior and the other to the interior of the circle of radius a (Fig. 5.15). They are connected only along the "cut" $-2a \leqq \xi \leqq 2a$, $\eta = 0$.

An important property of this transformation is that it leaves the region at infinity unchanged. Some additional properties of this transformation will be investigated in a problem at the end of this chapter.

5.9.5 $\zeta = \ln z$

The derivative gives $\zeta'(z) = 1/z$; hence, $z = 0$ is a singularity; it goes to $-\infty$ in the mapping. Take real and imaginary parts:

$$\xi = \ln \eta, \quad \eta = \theta \tag{5.29}$$

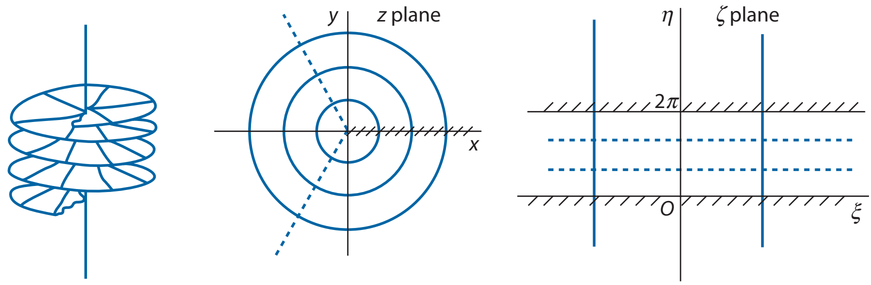


Fig. 5.16 Mapping of z plane on a strip in ζ plane.

The entire z plane is mapped on a strip of width 2π in the ζ plane (Fig. 5.16). Alternatively, if the entire ζ plane is considered, one gets infinitely many sheets in the z plane, one for each strip of width 2π.

The utility of this type of transformation appears when a repetitious (that is, periodic) type of boundary problem is encountered. Suppose a flow is known in the z plane of a type that does not involve velocity discontinuities across the positive real axis. Now, the transformed flow in the ζ plane will be identical along $\eta = 0$ and $\eta = 2\pi$. This is exactly what is needed in a physical problem where the flow situation is repeated without end in the η direction (Fig. 5.17).

5.9.6 $\zeta = [(az + b)/(cz + d)]$

In these transformations a, b, c, and d are complex constants. This is called the *linear fractional transformation* or the *bilinear transformation*. It is the transformation, which is discussed in Sec. 5.9.3, that transforms circles into circles. The transformation can be written in the form

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(\zeta-\zeta_1)(\zeta_2-\zeta_3)}{(\zeta-\zeta_3)(\zeta_2-\zeta_1)} \tag{5.30}$$

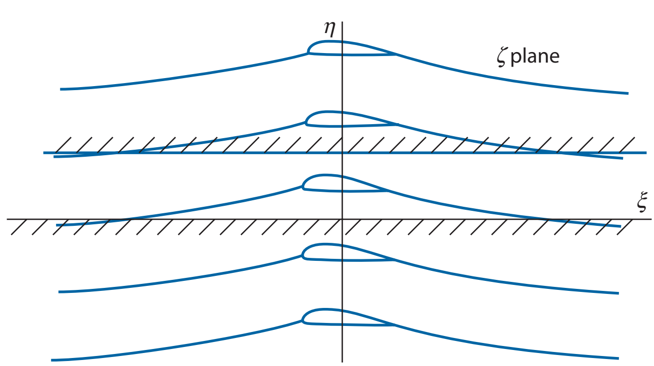


Fig. 5.17 Physical problem with endless flow situation in η direction.

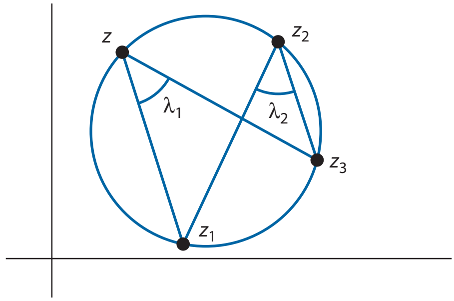


Fig. 5.18 Point z lies on a circle passing through z_1, z_2, and z_3.

To verify this, one needs only to solve Eq. (5.30) for ζ; it will be found in the form of the transformation stated earlier.

This is a relatively powerful device because it will map any three given points z_1, z_2, and z_3 into any other three selected points ζ_1, ζ_2, and ζ_3. Moreover, because three points determine a circle, the circle passing through z_1, z_2, and z_3 will be mapped on the circle through ζ_1, ζ_2, and ζ_3. The transformation is often used to map three given points onto the unit circle, or two points onto a straight line (the third is then mapped into the "point" at infinity) and so forth.

Most of these statements concerning the linear fractional transformation are easy to prove if we notice the following regarding the ratios equated in Eq. (5.30):

$$\arg\left\{\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right\} = \arg\left\{\frac{z-z_1}{z-z_3}\right\} - \arg\left\{\frac{z_2-z_1}{z_2-z_3}\right\}$$
$$= \lambda_1 - \lambda_2 \tag{5.31}$$

But if z lies on the circle through z_1, z_2, and z_3, as shown, $\lambda_1 = \lambda_2$, and the left-hand side of Eq. (5.30) is real. This requires that ζ also lies on a circle through ζ_1, ζ_2, and ζ_3 (Fig. 5.18).

Sample Problem 5.9

Show that, for any four points z_1, z_2, z_3, and z_4, the ratio

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

is invariant in any linear fractional transformation, that is, it is equal to the corresponding ratio computed for the transformed points ζ_1,

ζ_2, ζ_3, and ζ_4. What does this imply regarding the transformation of circles?

Solution:
Suppose the linear fractional transformation is

$$z=\frac{a\zeta+b}{c\zeta+d}$$

Then, the "anharmonic ratio" specified in this problem becomes

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}=\frac{\left(\frac{a\zeta_1+b}{c\zeta_1+d}-\frac{a\zeta_2+b}{c\zeta_2+d}\right)\left(\frac{a\zeta_3+b}{c\zeta_3+d}-\frac{a\zeta_4+b}{c\zeta_4+d}\right)}{\left(\frac{a\zeta_1+b}{c\zeta_1+d}-\frac{a\zeta_4+b}{c\zeta_4+d}\right)\left(\frac{a\zeta_3+b}{c\zeta_3+d}-\frac{a\zeta_2+b}{c\zeta_2+d}\right)}$$

$$=\frac{\left[(a\zeta_1+b)(c\zeta_2+d)-(a\zeta_2+b)(c\zeta_1+d)\right]}{\left[(a\zeta_1+b)(c\zeta_4+d)-(a\zeta_4+b)(c\zeta_1+d)\right]}$$

$$\times\frac{\left[(a\zeta_3+b)(c\zeta_4+d)-(a\zeta_4+b)(c\zeta_3+d)\right]}{\left[(a\zeta_3+b)(c\zeta_2+d)-(a\zeta_2+b)(c\zeta_3+d)\right]}$$

which becomes, after simplification,

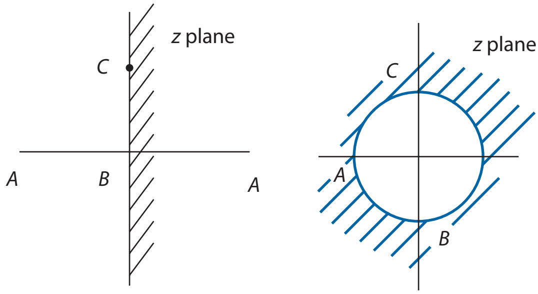
$$\frac{(\zeta_1-\zeta_2)(\zeta_3-\zeta_4)}{(\zeta_1-\zeta_4)(\zeta_3-\zeta_2)}$$

Sample Problem 5.10

Determine the transformation that maps the y axis into the unit circle, the area to its left being mapped inside the circle, the point at infinity into $\zeta=-1$, the point $z=0$ into $\zeta=-i$, and the point $z=i$ into itself.

Solution:
We can use the linear fractional transformation in the form of Eq. (5.30):

$$\frac{(z-z_A)(z_B-z_C)}{(z-z_C)(z_B-z_A)}=\frac{(\zeta-\zeta_A)(\zeta_B-\zeta_C)}{(\zeta-\zeta_C)(\zeta_B-\zeta_A)}$$



namely,

$$\frac{\cancel{(z-\infty)}(0-i)}{(z-i)\cancel{(0-\infty)}}=\frac{(\zeta+1)(-i-i)}{(\zeta-i)(-i+1)}$$

or

$$\zeta=\frac{1-i+2z}{1+i-2z}$$

The student should check his/her result: does it transform the three points as required?

The similarity between some of these transformations and the complex potentials of some of our elementary flow patterns has probably been noticed. (Compare Secs. 5.5.1 and 5.5.2 with Secs. 5.9.1 and 5.9.2, etc.) Any complex potential function $F(z)$ can be thought of as a conformal transformation that transforms the streamlines and equipotentials of the z plane into the rectangular Cartesian coordinates of an F plane. This point of view is sometimes useful.

5.10 Example of the Use of Conformal Mapping

Let us attempt to calculate the plane steady flow about a plate of width $4a$ placed in a stream (Fig. 5.19). The boundary conditions require that u be zero on front and back of the plate and that $u=U$, $v=0$ at infinity. The first boundary condition can also be stated by requiring that the plate be a streamline.

Now if we apply the appropriate transformation, we can map this region into the region outside of a circular cylinder, retaining the same conditions

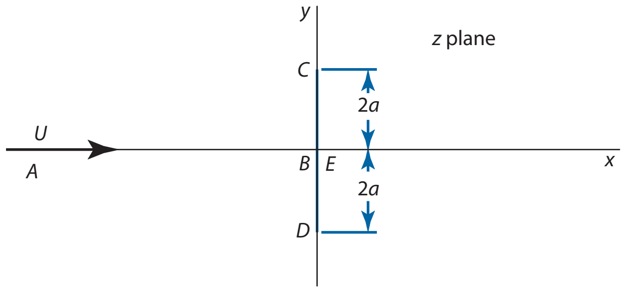


Fig. 5.19 Points represented in z plane.

at infinity. This will permit us to use our known results for the steady flow around a circular cylinder. It is easily verified using the results of Secs. 5.9.1 and 5.9.4 that the correct transformation is

$$z = -iz_1 \quad \text{where} \quad z_1 = it + \frac{a^2}{it}$$

or

$$z = t - \frac{a^2}{t} \tag{5.32}$$

The student should verify that the points of the z plane (Fig. 5.19) are mapped as shown in Fig. 5.20.

Now, the complex potential of this flow is (from Sample Problem 5.3)

$$\mathcal{F}(t) = U\left\{t + \frac{a^2}{t}\right\} - \frac{i\Gamma}{2\pi}\ln t + C \tag{5.33}$$

Let us assume, for simplicity, that there is no circulation.

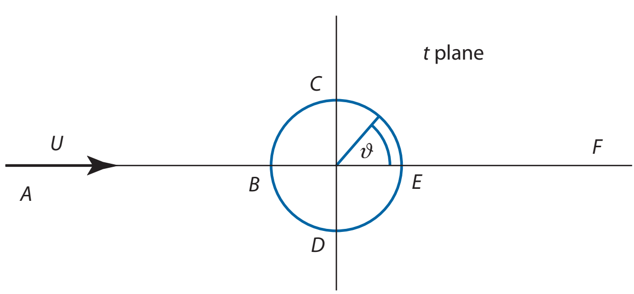


Fig. 5.20 Points mapped in t plane.

Theoretically, we should solve Eq. (5.32) for $t(z)$ and substitute in Eq. (5.33) to obtain the complex potential in the z plane, say $F(z)$. But, for practical use, it is probably easier to consider Eqs. (5.32) and (5.33) as giving the desired results in parametric form. Similarly, the velocity can be expressed as

$$\mathcal{F}'(t) = U\left\{1 - \frac{a^2}{t^2}\right\}; \quad F'(z) = \frac{I'(t)}{z'(t)} = \frac{I'(t)}{1 + (a^2/t^2)} \tag{5.34}$$

For example, on the plate, $x = 0$ and $y = 2a \sin\vartheta$, where ϑ is the angle indicated in Fig. 5.20, and

$$F' = U\frac{1 - e^{-2i\vartheta}}{1 + e^{-2i\vartheta}} = Ui\tan\vartheta = -iv \tag{5.35}$$

or

$$v(\pm 0, y) = \mp U\frac{y}{\sqrt{4a^2 - y^2}} \tag{5.36}$$

Suppose, instead, that it is desired to calculate the nonsteady flow due to the passage of this flat plate through a fluid otherwise at rest. As we have seen earlier, this flow can be obtained from the preceding steady flow, by superimposing the uniform flow $F_1(z) = -Uz$; then,

$$\begin{aligned}\mathcal{F}(t) = U\left\{t + \frac{a^2}{t}\right\} - Uz + C &= U\left\{t + \frac{a^2}{t}\right\} - U\left\{t - \frac{a^2}{t}\right\} + C \\ &= 2U\frac{a^2}{t} + C\end{aligned} \tag{5.37}$$

For example, let us calculate ϕ on the plate to check the statements made in Problem 4.7:

$$\begin{aligned}\mathcal{F} &= 2Uae^{-i\vartheta} + C \\ \phi &= 2Ua\cos\vartheta + C_1\end{aligned} \tag{5.38}$$

This is a convenient parametric form, together with $y = 2a\sin\vartheta$, from which one will have no difficulty in checking the desired formulas.

It is important to avoid the error of superimposing the stream of velocity $-U$ in the t plane, instead of the z plane. Remember that the given boundary conditions must be satisfied in the z plane.

Sample Problem 5.11

For the nonsteady plate problem of Sec. 5.10, determine the correct boundary condition at the circle $t = ae^{i\vartheta}$, and verify that the potential of Eq. (5.37) satisfies it.

Solution:
The boundary condition at the plate is $u = -U$ for $x = 0$, $-2a \leqq y \leqq 2a$.

The transformation of velocities is

$$\frac{d\mathcal{F}}{dt} = e^{-i\vartheta}(v_\rho - iv_\vartheta) = \frac{dF}{dz}\frac{dz}{dt}$$
$$= (u - iv)\left(1 + \frac{a^2}{t^2}\right)$$

where $t \equiv \rho e^{i\theta}$

On the circle $t = ae^{i\vartheta}$, then,

$$e^{-i\vartheta}(v_\rho - iv_\vartheta) = (u - iv)(1 + e^{-2i\vartheta})$$

or

$$v_\rho - iv_\vartheta = (u - iv)2\cos\vartheta$$

so that $v_\rho = 2u\cos\vartheta$ and $v_\vartheta = 2v\cos\vartheta$.

The boundary condition is therefore $v_\rho(a,\vartheta) = -2U\cos\vartheta$. Does Eq. (5.37) satisfy this?

$$\frac{d\mathcal{F}}{dt} = e^{-i\vartheta}(v_\rho - iv_\vartheta) = -2U\frac{a^2}{t^2}$$
$$= -2Ue^{-2i\vartheta} \quad \text{on} \quad \rho = a$$

$$v_\rho - iv_\vartheta = -2Ue^{-i\vartheta} = -2U(\cos\vartheta - i\sin\vartheta)$$

5.11 General Expressions for the Force on a Cylinder

Consider any *steady* plane flow about a solid cylinder, irrotational everywhere except at various singularities that might exist in the flow. Let X and Y be the components of the resultant fluid force on the cylinder. Clearly,

$$X = -\oint_{C_1} p\,dy \quad \text{and} \quad Y = \oint_{C_1} p\,dx \tag{5.39}$$

where C_1 denotes the contour of the cylinder (Fig. 5.21).

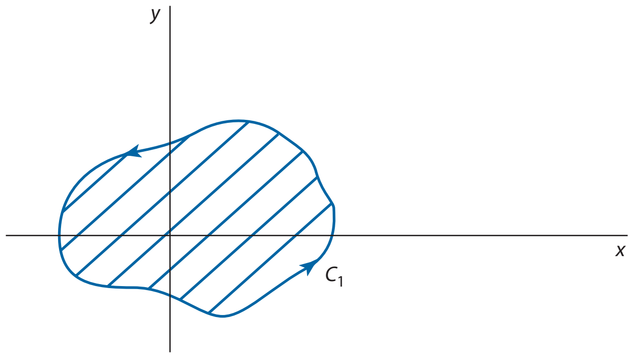


Fig. 5.21 Contour integrals over the cylinder surface yield force components.

Now, $p = p_0 - (1/2)\rho w\bar{w}$, where w denotes the complex velocity, $\bar{w}$ denotes its complex conjugate, and p_0 is a constant.

The integration of either $p_0\,dx$ or $p_0\,dy$ around a closed contour gives zero; hence, to calculate either X or Y, we are concerned only with $w\bar{w}$. It will be found convenient to combine these two integrations by adding X to $-iY$:

$$X - iY = \frac{1}{2} i\rho \oint_{C_1} w\bar{w}\{\mathrm{d}x - i\,\mathrm{d}y\} \tag{5.40}$$

But, $\bar{w}\{\mathrm{d}x - i\,\mathrm{d}y\}$ is equal to

$$(u + iv)(\mathrm{d}x - i\,\mathrm{d}y) = (u\,\mathrm{d}x + v\,\mathrm{d}y) + i(v\,\mathrm{d}x - u\,\mathrm{d}y) \tag{5.41}$$

and the second term of this is zero along any streamline in steady flow, including, of course, the contour C_1. Therefore, $\bar{w}\{\mathrm{d}x - i\,\mathrm{d}y\}$, on any streamline, is equal to $w\,\mathrm{d}z$, which has the same real part and the same (zero) imaginary part. Hence, the contour integral in Eq. (5.40) can be written as

$$X - iY = \frac{1}{2} i\rho \oint_{C_1} w^2\,\mathrm{d}z \tag{5.42}$$

The moment, about the origin of coordinates, of the fluid pressures on the cylinder can be calculated in an analogous way (see Problem 5.3). The result is

$$M_0 = \frac{-1}{2}\rho\mathcal{R}\left\{\oint_{C_1} w^2 z\,\mathrm{d}z\right\} \tag{5.43}$$

where M_0 is positive in the counterclockwise sense, and $\mathcal{R}\{\ \}$, as earlier, denotes the real part of $\{\ \}$.

5.12 Cauchy's Theorems Regarding Contour Integrals

The French mathematician A.-L. Cauchy (1789–1857) gave three fundamental theorems concerning contour integrals:

1. If $f(z)$ is analytic everywhere within and on a contour C, then

$$\oint_C f(z)\,\mathrm{d}z = 0 \tag{5.44}$$

This theorem is proved in many mathematics books. We shall not bother to prove it here because it seems to follow from our earlier work. Specifically, imagine $f(z)$ to be the complex velocity of some flow, as every regular function must be. The contour integral then gives us $\Gamma + iQ$, according to Eq. (5.12), and if there are no singularities on or within C, it is clear that $\Gamma = 0 = Q$.

2. If $f(z)$ is analytic everywhere within the region between C_1 and C_2, then

$$\oint_{C_1} f(z)\,\mathrm{d}z = \oint_{C_2} f(z)\,\mathrm{d}z \tag{5.45}$$

Consider the contour C, made up of C_1, C_2 (in reverse sense), and the connecting paths shown (Fig. 5.22). By theorem 1, the contour integral about this entire contour is zero. The contributions of the connecting paths cancel each other, and Eq. (5.45) follows.

3. If $f(z)$ is analytic everywhere within a region, then for any closed contour C in the region and any point Z inside C,

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \tag{5.46}$$

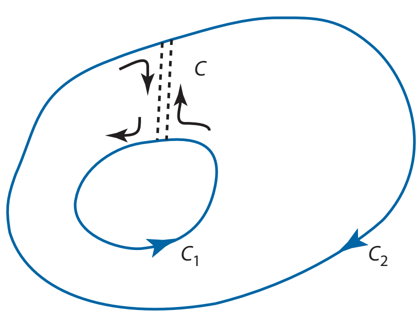


Fig. 5.22 Connecting paths between C_1 and C_2.

Take a very small contour C_1 about z, consisting of a circle of radius $\in$. By theorem 2,

$$\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \approx f(z)\int_0^{2\pi} \frac{i \in \mathrm{e}^{i\vartheta}\,\mathrm{d}\vartheta}{\in \mathrm{e}^{i\vartheta}} = 2\pi i f(z) \tag{5.47}$$

which proves the theorem.

Sample Problem 5.12

Show that

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta \qquad (n = 1, 2 \quad)$$

under the same conditions as stated in theorem 3. Here, $f^{(n)}$ denotes the nth derivative of $f(z)$.

Solution:

1. Differentiate both sides of Eq. (5.46) n times with respect to z, or
2. See [1].

5.13 Blasius's Formulas

Consider Eqs. (5.42) and (5.43) in the light of Cauchy's second theorem. It is clear that the contour integrations need not be carried out along C_1, the cylinder contour, but can be taken around any convenient contour C that encloses no singularities other than the cylinder itself:

$$X - iY = \frac{1}{2} i\rho \oint_C w^2\,\mathrm{d}z \tag{5.48}$$

$$M_0 = \frac{-1}{2}\rho R\left\{\oint_C w^2 z\,\mathrm{d}z\right\} \tag{5.49}$$

These were given by the German hydrodynamicist H. Blasius in 1910. They give us the forces and moment (counterclockwise about the origin) of the fluid pressures in steady flow. For example, if the flow is due to a uniform stream in which the cylinder is immersed, we can take C infinitely large.

5.14 Laurent's Expansion

To carry this matter still further, and to obtain finally a powerful method of evaluating the contour integrals in Eqs. (5.48) and (5.49)—or any other contour integrals—we require the following theorem:

If $f(z)$ is *analytic* everywhere within an annular space whose center is z_0, then, for every point in this ring, $f(z)$ can be expanded as follows:

$$f(z) = \sum_{n=-\infty}^{\infty} A_n \{z - z_0\}^n \tag{5.50}$$

and the series converges uniformly.

Furthermore, the coefficients A_n are given by

$$A_n = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \tag{5.51}$$

where C_1, the outer circle of the space, is used for $n = 0, 1, 2, \ldots$, and C_2, the inner circle, is used for $n = -1, -2, \ldots$.

The situation is shown in Fig. 5.23, where $f(z)$ has certain singularities inside C_2 and others outside C_1. It is clear that this is a generalization of Taylor's series. In fact, if there are no singularities inside C_2, that circle can be shrunk to a point, whereupon the ring-shaped region of convergence of Laurent's series becomes the ordinary circle of convergence of a Taylor series. Moreover, the Laurent series is not unique at a point, but depends entirely on the choice of z_0. The theorem is named after P.A. Laurent, who published it in 1843.

Laurent's theorem is proved by applying Cauchy's third theorem to the ring-shaped region: let C consist of C_1, C_2 (in reverse), and two connecting paths, as before.

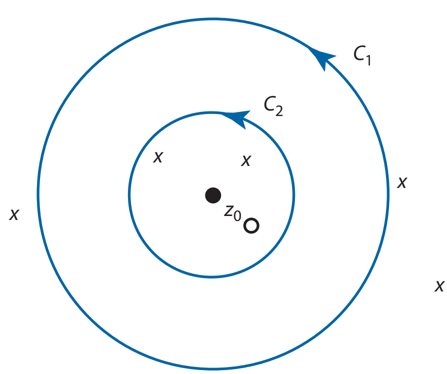


Fig. 5.23 Singularities inside C_2 and outside C_1.

Then, we find

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \tag{5.52}$$

Let $z = z_0 + h$; then on $C_1, |h| < |\zeta - z_0|$, whereas on $C_2, |h| > |\zeta - z_0|$. Thus, in the first integral, we can expand $(\zeta - z)^{-1}$ in powers of $h/(\zeta / z_0)$; in the second integral, we expand it in powers of the reciprocal. The student should carry out this calculation, which leads directly to the result stated in Eqs. (5.50) and (5.51). The convergence and uniformity of convergence remain to be proved, but will not be considered here (see [1], Sec. 5.6 and [2], Sec. 29).

In Laurent's expansion, we have a result of tremendous importance, which leads us to a complete general solution of Laplace's equation for plane flows. To be sure, the expansion was limited to *analytic* functions, whereas our complex potentials are often logarithmic. But the complex velocity must be analytic, for we admit only single-valued velocities. Hence, the most general form of $w(z)$ is a Laurent expansion. The Laurent expansion for any function for a given annular ring can be shown to be unique.

5.15 Residues

Consider the integral

$$\oint_C f(z)\,\mathrm{d}z$$

where C lies within the annulus of convergence of the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} A_n \{z - z_0\}^n \tag{5.53}$$

By Cauchy's second theorem, the contour integral has the same values as the integral around any circle having z_0 as a center of radius R, say, in the annular region. Let $z - z_0 = Re^{i\theta}$, and carry out the integration term-by-term. (It is permissible to do so in view of the uniform convergence.)

$$\oint_C f(z)\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} A_n \int_0^{2\pi} R^n \mathrm{e}^{ni\theta} Ri\mathrm{e}^{i\theta}\,\mathrm{d}\theta \tag{5.54}$$

But, this integral is zero for every term except $n = -1$, for which it has the value $2\pi i$. Hence,

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i A_{-1} \tag{5.55}$$

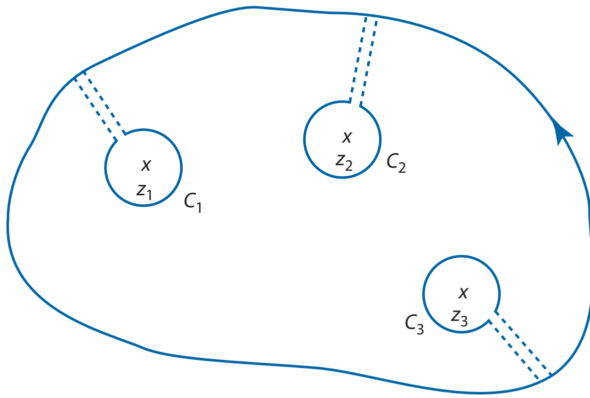


Fig. 5.24 Contour of integration made up of circles around singularities.

This important result means that, to evaluate any contour integral, one needs only to determine the Laurent expansion of the integrand for a region including the contour—or rather, the $(z-z_0)^{-1}$ term in such an expansion—whereupon the integral is $2\pi i$ times the coefficient of this term.

Now, for any isolated singularity z_1 of $f(z)$, there is a Laurent expansion in powers of $(z-z_1)$ that is valid in the neighborhood of z_1. The coefficient of $(z-z_1)^{-1}$ in this expansion is called the *residue* of $f(z)$ at z_1, or Res $f(z)|_{z_1}$. Obviously, the value of

$$\oint_{C_1} f(z)\,\mathrm{d}z$$

where C_1 encloses z_1 and no other singularity, is $2\pi i$ Res $f(z)|_{z_1}$. Moreover, the integral over a contour C that encloses a number of singularities can be made up of the sum of integrals taken each around a single singularity (Fig. 5.24), that is,

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &= 2\pi i\left\{\mathrm{Res}\ f(z)\Big|_{z_1} + \mathrm{Res}\ f(z)\Big|_{z_2} + \cdots\right\} \\ &= 2\pi i \sum_C \mathrm{Res}\ f(z) \end{aligned} \tag{5.56}$$

Equations (5.55) and (5.56) are alternative expressions. Sometimes, it will be more convenient to determine A_{-1}; in other cases, it will be simpler to evaluate the individual residues and add them.

5.16 Poles and Essential Singularities

If $f(z)$, in the immediate neighborhood of z_1, can be expressed in the form $g(z)\,(z-z_1)^{-n}$, where $g(z)$ is regular at z_1 and n is a positive integer, then $f(z)$ has a *pole of order n* at z_1. If $f(z)$ is not regular at z_1 but cannot be so expressed, then $f(z)$ has either an "essential singularity" or a "branch point" at z_1.

The student should verify that the Laurent expansion of $f(z)$ for the neighborhood of a pole of order n begins with a term $(z-z_1)^{-n}$. Near an essential singularity, the expansion has an infinite number of terms with negative exponents. There is no way to expand a multivalued function in a Laurent series in the neighborhood of a branch point, that is, a singularity such as $z=0$ in the case $f(z)=\ln z$, or $z=a$ in the case $f(z)=(z-a)^{1/2}$, etc. However, there might be Laurent expansions for regions not including the branch point (for example, [1], Sec. 5.6, Example 2), and also one can expand the function after making in single valued by means of a suitable transformation.

It is usually easy to evaluate the residue at a pole, for one need only put the function into the form $g(z)\,(z-z_1)^{-n}$ and calculate the Taylor expansion of $g(z)$ about z_1:

$$g(z)=g(z_1)+(z-z_1)g'(z_1)+\frac{1}{2!}(z-z_1)^2 g''(z_1)$$
$$+\cdots+\frac{1}{r!}(z-z_1)^r g^{(r)}(z_1)+\cdots \tag{5.57}$$

The residue is then seen to be the coefficient of $(z-z_1)^{n-1}$ in this expansion, that is,

$$\operatorname{Res} f(z)\Big|_{z_1}=\frac{1}{(n-1)!}g^{(n-1)}(z_1) \tag{5.58}$$

For example, at a pole of order one (a "simple pole"),

$$\operatorname{Res} f(z)\Big|_{z_1}=g(z_1) \tag{5.59}$$

Contour integration, employing residues, is often used to evaluate difficult definite integrals. In Example 5.1 we illustrate this process by means of the simple integral

$$\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta$$

The value of this integral, as we know, is π. This method can be used to evaluate any integral from 0 to 2π of a rational function of trigonometric functions.

Example 5.1 Evaluate $\int_0^{2\pi} \cos^2\theta d\theta$

Let $e^{i\theta} = z$. Then, $dz = ie^{i\theta} d\theta$, or $d\theta = dz/iz$. Also, $\cos\theta = (z + z^{-1})/2$, etc. The integral is easily seen to be the contour integral, around the unit circle, of a certain $f(z)$, where $z = re^{i\theta}$ in general:

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \oint_C \frac{z^2 + 2 + z^{-2}}{4} \frac{dz}{iz} = \frac{1}{4i} \oint_C (z + 2z^{-1} + z^{-3}) \, dz$$

Thus, $f(z) = z + 2z^{-1} + z^{-3}$ and

$$\int_0^{2\pi} \cos^2\theta d\theta = \frac{2\pi i}{4i} \sum_C \text{Res} f(z)$$

Now, the only singularity of $f(z)$ within the unit circle is at $z = 0$, where there is a pole of order 3. The residue is certainly 2, for the function itself is in the form of a Laurent expansion about this singularity. Hence,

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{\pi}{2} 2 = \pi$$

Sample Problem 5.13

Find the singular points of $z/[(z-a)(z+b)^2]$, and evaluate the residue at each.

Solution:
There is a simple pole at $z = a$ and a double pole at $z = -b$.

Near $z = a$, the function is of the form $(z-a)^{-1} g(z)$, where $g(z) \equiv [z/(z+b)^2]$. Hence, according to Eq. (5.59), the residue is simply $g(a) = [a/(a+b)^2]$.

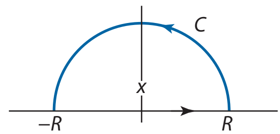
Near $z = -b$, the function is $(z+b)^{-2} g(z)$, where $g(z) \equiv [z/(z-a)]$. According to Eq. (5.58), the residue is

$$-\frac{1}{(n-1)!} g^{(n-1)}(-b) = g''(-b) = -\frac{a}{(a+b)^2}$$

Sample Problem 5.14

Show that

$$\int_0^{\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-a}$$



where a is positive. Hint: Consider the contour integral of $e^{iz}(z^2+a^2)^{-1}$ around the contour with $R > 1$ in the figure, and then let $R \to \infty$.

Solution:
Consider

$$\oint_C \frac{e^{ix}dz}{z^2+a^2}$$

The contour C (in the figure) encloses a simple pole at $z = ia$. The residue there is $(e^{-a}/2ia)$; thus, the value of the contour integral is $\pi e^{-a}/a$. Now, for large R, the contour integral consists of the integral along the x axis, namely,

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos x\,dx}{x^2+a^2} + i\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+a^2}$$

$$= 2\int_0^{\infty} \frac{\cos x\,dx}{x^2+a^2} + 0 \qquad \text{by symmetry}$$

plus a line integral from 0 to π along the arc $z = Re^{i\theta}$. The integrand along this arc is

$$\frac{\exp[iR(\cos\theta + i\sin\theta)]}{R^2e^{2i\theta}+a^2} = \frac{\exp(-R\sin\theta)}{R^2e^{2i\theta}+a^2}e^{iR\cos\theta}$$

As $R \to \infty$, this is bounded by $1/(R^2e^{2i\theta}+a^2)$, which can easily be shown to go toward zero like R^{-2}.

Thus, the contour integral finally consists solely of two times the given real integral and has the value $\pi e^{-a}/a$.

5.17 Force on a Body in a Parallel Stream

Consider a general case of a cylinder placed in a parallel stream. There are no singularities outside the body contour in the finite region; therefore, the Laurent expansion for $w(z)$ must hold between some finite radius and infinity.

At infinity, $w(z)$ must become $U - iV = A$, say, where U and V are the undisturbed stream components. It is clear that the expansion must have the form

$$w(z) = A + \frac{B}{z} - \frac{C}{z^2} - \frac{2D}{z^3} - \cdots \tag{5.60}$$

where B, C, D, … are constants, and the complex potential must be

$$F(z) = Az + B\,\ell n\, z + \frac{C}{z} + \frac{D}{z^2} + \cdots + \text{const} \tag{5.61}$$

Now, the formula for $X - iY$ requires a contour integration of w^2; in this case, it can be performed away from the body, where Eq. (5.60) holds. The term in z^{-1} in w^2 is easily seen to be $2ABz^{-1}$ so that

$$X - iY = \frac{1}{2} i\rho \oint_C w^2\, \mathrm{d}z = -2\pi\rho AB \tag{5.62}$$

But B, the coefficient of $\ell n\, z$ in $F(z)$, is equal to $(Q - i\Gamma)/2\pi$, where Q and Γ are the total source and vortex strength enclosed by the body contour. Presumably, the contour is closed and Q is zero, but actually the formulas are still valid for a finite distribution of sources or sinks not totaling zero (as an exercise, the student should repeat the derivation of Blasius's formulas for the case when there is flow into or out of the body), so we will carry the symbol Q:

$$X - iY = -\rho(U - iV)(Q - i\Gamma)$$

or

$$X = -\rho(UQ - V\Gamma); \; Y = -\rho(U\Gamma + VQ) \tag{5.63}$$

It will easily be verified that these formulas represent a force in the stream direction equal to $-\rho UQ$ and a force normal to the stream equal to $\rho U\Gamma$ (Fig. 5.25). We have already calculated one example of this situation, namely, the flow over a circular cylinder with circulation. The formula for the transverse force component $\rho U\Gamma$ is called the Kutta–Joukowski formula. It was obtained by Lanchester in 1894, but not analytically, and later by Kutta in 1902 and Joukowski in 1905.

The moment M_0 can be calculated in a similar fashion. It will easily be verified that the coefficient of z^{-1} in the expansion of $w^2 z$ is $B^2 - 2AC$; hence,

$$\begin{aligned} M_0 &= -\frac{1}{2}\rho\mathcal{R}\left\{2\pi i(B^2 - 2AC)\right\} \\ &= \pi\rho\mathcal{I}\{B^2 - 2AC\} \end{aligned} \tag{5.64}$$

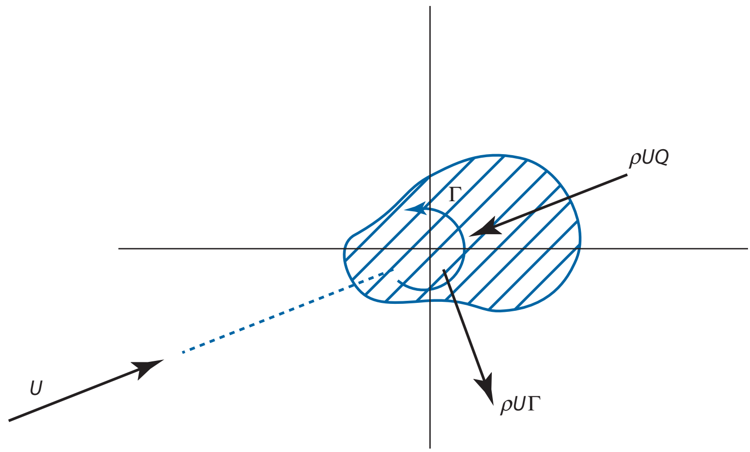


Fig. 5.25 Direction of force in a parallel stream.

But for any closed cylinder B^2 is real, and the formula reduces to

$$M_0 = -2\pi\rho\mathcal{I}\{AC\} \tag{5.65}$$

If the x direction has been selected in the stream direction, A is equal to U, the stream speed, and

$$M_0 = -2\pi\rho U\mathcal{I}\{C\} \tag{5.66}$$

Remember that M_0 is the counterclockwise moment of the fluid pressures.

Equation (5.66) is interesting; it shows that the moment is proportional to the coefficient of z^{-1} in $F(z)$. Investigation will disclose that contributions to this coefficient might come from doublets whose axes are not parallel to the stream, and from vortices that are not located at the origin. A singularity consisting of a doublet and a vortex is the simplest kind that produces a force and moment.

5.18 Calculation of Force and Moment in a Transformed Plane

In many cases, $F(z)$ is determined by conformal transformation from another plane. In aeronautical work, we frequently use transformations of the form

$$Z = \zeta + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \cdots \tag{5.67}$$

These have the property of maintaining the region at infinity unchanged; an example, of course, is the transformation that is shown in Sec. 5.9.4.

When a transformation of this form is used, the force and moment can be calculated without transforming the results to the z plane at all. Suppose $I(\zeta)$ has been determined; by the same theory of Laurent expansions as we used in Sec. 5.14, we see that $I(\zeta)$ must have the form

$$I(\zeta) = A\zeta + B\,\ln\zeta + \frac{C}{\zeta} + \cdots \tag{5.68}$$

where $A = U - iV$ and moreover $B = (Q - i\Gamma)/2\pi$, because Q and Γ for any contour are invariant in a conformal transformation (see Sample Problem 5.7). Now,

$$\begin{aligned} X - iY &= \frac{1}{2} i\rho \oint_C w^2(z)\,\mathrm{d}z = \frac{1}{2} i\rho \oint_{C'} \left[\frac{I'(\zeta)}{z'(\zeta)}\right]^2 z'(\zeta)\,\mathrm{d}\zeta \\ &= \frac{1}{2} i\rho \oint_{C'} \frac{[I'(\zeta)^2]}{z'(\zeta)}\,\mathrm{d}\zeta \end{aligned} \tag{5.69}$$

where C' denotes the transformed contour. The integrand can be evaluated from Eqs. (5.67) and (5.68), and the integration can be carried out at a large value of $|\zeta|$, where the Laurent expansion in descending powers of ζ is valid:

$$\frac{[I'(\zeta)]^2}{z'(\zeta)} = \frac{A^2 + (2AB/\zeta) + \cdots}{1 - (a_1/\zeta^2) - (2a_2/\zeta^3)}$$

$$= \left(A^2 + \frac{2AB}{\zeta} + \cdots\right)\left(1 + \frac{a_1}{\zeta^2} + \cdots\right)$$

$$X - iY = -2\pi\rho AB \tag{5.70}$$

In other words, it is not necessary to transform Eq. (5.68) to the z plane because the force can be calculated in terms of quantities describing the flow in the ζ plane. In fact, Eq. (5.70) can be interpreted as a statement that "the force is the same in the z and ζ planes"—a result that could have been anticipated by the invariance of Q and Γ in a transformation.

The moment can be calculated by a similar process (Problem 5.3) and does not turn out to be invariant. The result is

$$M_0 = \pi\rho\mathcal{I}\{2A^2 a_1 + B^2 - 2AC\} \tag{5.71}$$

where A, B, and C are defined in Eq. (5.68) and a_1 in Eq. (5.67).

References

[1] Whittaker, E.T., and Watson, G.N., *Modern Analysis*, Cambridge, 1935, p.89.

[2] Knopp, K., *Theory of Functions*, Part I, Dover, New York, 1935.

Problems

5.1 Consider the transformation $\zeta = z + (a^2/z)$, where a is real.

a. Show that along the strip $\eta = 0$, $-2a < \xi < 2a$, the fluid velocity components are related as follows:

$$v_\xi = -\frac{v_\theta}{2\sin\theta}, \quad v_\eta = \frac{v_r}{2\sin\theta}$$

where $\zeta = \xi + i\eta$, $z = re^{i\theta}$, and v_r, v_θ, v_ξ, and v_η represent the various velocity components in the directions indicated.

b. Show that radial lines and circles about the origin in the z plane are transformed into orthogonal families of hyperbolas and ellipses in the ζ plane.

c. Investigate the transformation of a circle that passes through both singular points in the z plane but whose center is not at the origin.

5.2 Verify the results stated in Problem 4.4 for the apparent masses of an elliptic cylinder. Hint: It is easier to perform the integrations in the transformed plane than in the plane of the ellipse.

5.3 a. Prove

$$M_0 = \frac{-1}{2}\rho\mathcal{R}\left\{\oint_{C_1} w^2 z\,\mathrm{d}z\right\} \tag{5.43}$$

b. Prove

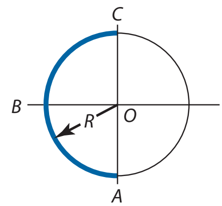
$$M_0 = \pi\rho\mathcal{I}\{2A^2 a_1 + B^2 - 2AC\} \tag{5.71}$$

5.4 A flat plate of width $4a$ rotates without circulation with angular velocity ω, about its center, in a fluid mass otherwise at rest.

a. Determine and sketch the velocity distribution on the plate.

b. Determine the additional apparent inertia.

5.5 Consider a flat sheet bent into the form of a semicircle of radius R. A source of strength Q is located at the center O. Determine the complex potential of the flow.



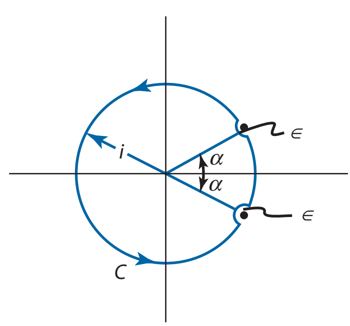
5.6 Two equal and opposite vortices are held a distance a apart in a fluid otherwise at rest (plane flow). Use Blasius's formula to calculate the force required to hold the vortices in place.

5.7 Show that

$$P\int_0^{\pi}\frac{\cos n\theta\,\mathrm{d}\theta}{\cos\theta-\cos\alpha}=\pi\frac{\sin n\alpha}{\sin\alpha}$$

where α is a constant; $n=1,2,\ \dots$; and P denotes Cauchy's principal value. Hint: using the same method as shown in example in Sec. 5.16, we put $\mathrm{e}^{i\theta}=z$; then,

$$\begin{aligned}P\int_0^{2\pi}\frac{\cos n\theta\,\mathrm{d}\theta}{\cos\theta-\cos\alpha}&=2P\int_0^{\pi}\frac{\cos n\theta\,\mathrm{d}\theta}{\cos\theta-\cos\alpha}\\&=\lim_{\epsilon\to 0}\oint_C\frac{z^n+z^{-n}}{z+z^{-1}-\mathrm{e}^{i\alpha}-\mathrm{e}^{-i\alpha}}\frac{\mathrm{d}z}{iz}\\&=-i\lim_{\epsilon\to 0}\oint_C\frac{z^n+z^{-n}}{(z-\mathrm{e}^{i\alpha})(z-\mathrm{e}^{-i\alpha})}\mathrm{d}z\end{aligned}$$



It will be necessary to evaluate the residues at any singularities inside C and also the contributions of the indentations at $e^{\pm i\alpha}$.

5.8 Investigate the transformation of hydrodynamical singularities, in general, at regular points of conformal transformation. [Assume $w(z) = \text{const}\,(z = z_0)^{-n}$ + (regular terms) near $z = z_0$, and determine the behavior (leading term) of the complex velocity in the ζ plane near $\zeta = \zeta(z_0)$.]

5.9 a. Show that the drift velocity of a vortex Γ at z_0 in a flow described by the complex potential $f(z)$ is $f'(z_0) + i\Gamma/2\pi(z - z_0)$.

b. If $z = z(\zeta)$ is a conformal mapping and $f(z) = F(\zeta)$, show that this drift velocity is

$$\left\{F'(\zeta_0) + \frac{i\Gamma}{2\pi(\zeta - \zeta_0)}\right\}\frac{1}{z'(\zeta_0)} - \frac{i\Gamma}{4\pi}\frac{z''(\zeta_0)}{[z'(\zeta_0)]^2}$$

where ζ_0 denotes $\zeta(z_0)$.

Vector Analysis

A.1 Introduction

Vector notation is a useful mathematical device, which not only provides a convenient shorthand system but, what is more important, also permits us to write relations involving vector quantities in a way that clarifies their meaning and emphasizes their independence of coordinate systems. Its use here is partly for pedagogical reasons, that is, it is a good thing for a student to learn. It is certainly not a necessity, for an entire course in aerodynamics can be presented without a single vector symbol.

Vector quantities in physics are those that possess both direction and magnitude, and in addition a special property regarding their transformations from one coordinate system to another, which is defined later. Force, velocity, acceleration, and angular velocity are familiar examples of vector quantities, as we shall see. Each of these has a physical aspect (its physical meaning) and a numerical aspect. The numerical aspect always involves three numbers related to a coordinate system, for example, the values of three components along Cartesian axes.

The simplest physical vector quantity is a *line vector*, that is, a linear displacement. Every vector quantity can be said to define a line vector, for a line vector can be produced having the same numerical aspect in the same coordinate system. We implicitly make use of this fact every time we draw a diagram representing a vector quantity as a line vector, as in Fig. A.1.

Strictly, the set of three numbers (referred to a coordinate system) that we have called the "numerical aspect" is a *vector*. Thus, a vector is a kind of "hyper-number," consisting of three scalar numbers, which define a line vector. But not every physical quantity having direction and magnitude is a vector quantity, for not every set of three numbers

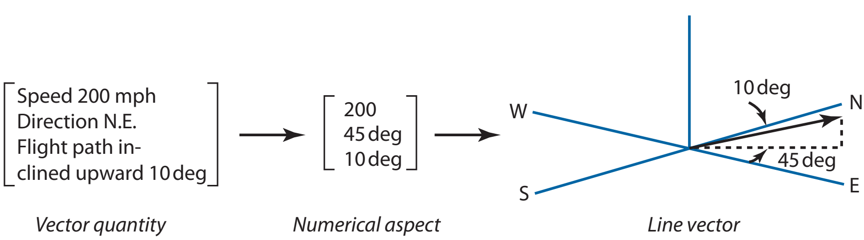


Fig. A.1 Representation of vector quantity as a line vector.

referring to a specified coordinate system defines a proper line vector, as we shall see. Our present definition is incomplete until we investigate how a line vector is transformed from one coordinate system to another. Vector quantities are those that transform the same way.

A.2 Methods of Representing Vectors in Symbols

Before making this investigation, let us consider some scheme for representing vectors in symbols.

A.2.1 Addition of Vectors

We may write the three numbers of the set in parentheses, separated by commas. For example, if a rectangular Cartesian coordinate system is used, the well-known rule for addition of vectors becomes

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Also, another well-known rule states that

$$(a_1, a_2, a_3)m = (ma_1, ma_2, ma_3)$$

where m is a scalar number.

A.2.2 Unit Vectors

We can denote vectors by underlined symbols (bold face, in printed work), and scalar numbers by ordinary symbols. In particular, let us define three *unit vectors,* $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ each having unit magnitude and being directed along the x, y, and z axes of a rectangular Cartesian coordinate system, respectively.

Then, using the second rule mentioned earlier, we can write an expression for any vector $\boldsymbol{a}$ as the sum of its components, that is,

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$$

For instance, the position vector $\boldsymbol{r}$, denoting the displacement of any point from the origin, is

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

In this system, we also write $a \equiv |\boldsymbol{a}| \equiv \sqrt{a_1^2 + a_2^2 + a_3^2}$ and

$$\cos(\boldsymbol{a}, \boldsymbol{i}) = \frac{a_1}{a}; \quad \cos(\boldsymbol{a}, \boldsymbol{j}) = \frac{a_2}{a}; \quad \cos(\boldsymbol{a}, \boldsymbol{k}) = \frac{a_3}{a}$$

Also, $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j} + ma_3\boldsymbol{k}$ (as used earlier).

The vector equation $\boldsymbol{a} = \boldsymbol{b}$ implies three scalar equations:

$$a_1 = b_1; \quad a_2 = b_2; \quad a_3 = a_3$$

We shall employ this notation in much of our study. Its great advantage is that it does not imply the use of any particular coordinate system. A statement such as $\boldsymbol{a} = \boldsymbol{b}$ is independent of coordinate systems, for if two vectors are equal, their three numbers are equal in any coordinate system. This is extremely important in physics, for we usually are concerned with relationships between physical quantities, true for any choice of coordinates.

A.2.3 Cartesian Tensor Notation

Assume that a rectangular Cartesian system is employed. Let the three coordinates be called x_1, x_2, and x_3. Then, for a vector whose corresponding components are a_1, a_2, and a_3, we write simply a_n, where the subscript is understood to take the values 1, 2, and 3.

Thus, a vector is recognized by the presence of a subscript and a scalar by the absence of a subscript.

The various rules mentioned earlier now take the forms

$$\text{(Sum of } u_n \text{ and } v_n) = u_n + v_n$$
$$\text{(Product of } m \text{ and } v_k) = mv_k$$
$$\text{Equality of two vectors: } a_p = b_p$$

(Note that this constitutes three scalar equations, as stated earlier, because the subscript, in this case p, must take the values 1, 2, and 3.)

This system has the advantage that we have not had to invent any new symbols or nomenclature—only the new concept of one symbol representing three scalar numbers in succession. This is the notation of Tensor Analysis, but it is simplified here by the fact that we restrict it to rectangular Cartesian coordinate systems.

It is the simplest notation to compute with. It requires the least memory work. We will sometimes use it, as, for example, in the following section.

A.3 Transformation of Vectors

Consider two rectangular Cartesian coordinate systems rotated with respect to one another, as shown in Fig. A.2.

Beginning with a familiar notation, let l_1, m_1, and n_1 denote the direction cosines of the x' axis, with respect to the x, y, and z axes, respectively. Let l_2, m_2, and n_2 denote those of y', and l_3, m_3, and n_3 those of z'.

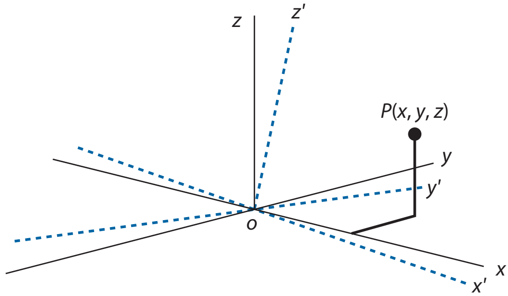


Fig. A.2 Two Cartesian coordinate systems rotating with respect to each other.

Then, to find the coordinates of a point $P\,(x, y, z)$ in the primed system, note that in moving a distance x along the x axis one moves $l_1 x$ along x', $l_2 x$ along y', and $l_3 x$ along z', etc. Hence, the new coordinates of P are

$$
\begin{aligned}
x' &= l_1 x + m_1 y + n_1 z \\
y' &= l_2 x + m_2 y + n_2 z \\
z' &= l_3 x + m_3 y + n_3 z
\end{aligned}
\tag{A.1}
$$

Also, if we transform from x', y', and z' to x, y, and z, by a similar calculation, we find

$$
\begin{aligned}
x &= l_1 x' + l_2 y' + l_3 z' \\
y &= m_1 x' + m_2 y' + m_3 z' \\
z &= n_1 x' + n_2 y' + n_3 z'
\end{aligned}
\tag{A.2}
$$

Now, suppose we had used a different notation, namely,

$$
\begin{aligned}
&x_1, x_2, x_3 \quad \text{instead of } x, y, z \\
&x_1', x_2', x_3' \quad \text{instead of } x', y', z'
\end{aligned}
\tag{A.3}
$$

$$
\left.
\begin{aligned}
&a_{11}, a_{21}, a_{31} \quad \text{instead of } l_1, m_1, n_1 \\
&a_{12}, a_{22}, a_{32} \quad \text{instead of } l_2, m_2, n_2 \\
&a_{13}, a_{23}, a_{33} \quad \text{instead of } l_3, m_3, n_3
\end{aligned}
\right\}
$$

(In summary, a_{ij} represents the cosine of the angle between the x_i and x_j' axes.)

Then Eqs. (A.1) and (A.2) become

$$x_i' = \sum_{j=1}^{3} a_{ji} x_j \qquad i = 1, 2, 3 \qquad \text{(A.4)}$$

$$x_i = \sum_{j=1}^{3} a_{ij} x_j' \qquad i = 1, 2, 3 \qquad \text{(A.5)}$$

A *vector* is defined as a set of three numbers u_1, u_2, and u_3 referred to a coordinate system x_1, x_2, and x_3 having the property that when transferred to the x_1', x_2', and x_3' system the corresponding quantities are given by

$$u_i' = \sum_{j=1}^{3} a_{ji} u_j \qquad i = 1, 2, 3 \qquad \text{(A.6)}$$

This is really the same as our earlier definition in terms of a line vector because Eqs. (A.4) and (A.5) are the transformation formulas for a line vector.

It is clear that, to test whether a physical quantity is a vector quantity, one must have a definition that permits examination of its transformation formula. Let us consider some simple examples.

Example A.1 Velocity of a Point

Show that the velocity of a point $P(x_1, x_2, x_3)$ is a vector. The components of this quantity along the three axes are $\mathrm{d}x_1/\mathrm{d}t$, $\mathrm{d}x_2/\mathrm{d}t$, and $\mathrm{d}x_3/\mathrm{d}t$. Calculating the velocity in the primed system, we find

$$\frac{\mathrm{d}x_1'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{3} a_{ji} x_j = \sum_{j=1}^{3} a_{ji} \frac{\mathrm{d}x_j}{\mathrm{d}t}$$

This has exactly the form required by Eq. (A.6). Hence, the velocity of a point is a vector quantity.

Example A.2 Set of Numbers $\partial u/\partial x$

Show that the set of numbers $\partial u/\partial x_i$, where u is a scalar function $u(x_1, x_2, x_3)$ is a vector. Let $w_1 \equiv \partial u/\partial x_1$ and $w_1' \equiv \partial u/\partial x_1$. Now note how w_1' is expressed in terms of w_1:

$$w_1' = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} \frac{\partial x_1}{\partial x_1'} = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} a_{ji} = \sum_{j=1}^{3} a_{ji} w_j \quad \text{[from Eq. (A.5)]}$$

Hence, $\partial u/\partial x_1$ is vector.

Sample Problem A.1

Is $(\partial u / \partial x_1)^{-1}$ a vector?

Solution:
The question is, can we form a vector whose components V_i, say, in an x_1, x_2, and x_3 system are, respectively,

$$\frac{1}{\partial u / \partial x_1}, \frac{1}{\partial u / \partial x_2}, \quad \text{and} \quad \frac{1}{\partial u / \partial x_3}$$

If $u(x_1, x_2, x_3)$ is properly defined, these three scalar functions exist: the proposed vector would surely have direction and magnitude.

The test is begun by writing the components of the proposed vector in another coordinate system, say the x_1', x_2', x_3' system, where

$$x_i' = \sum_{j=1}^{3} a_{ji} x_j \quad \text{and} \quad x_j = \sum_{p=1}^{3} a_{jp} x_p'$$

Let the function u, when described in terms of x_1', x_2', x_3' be called $U(x_1', x_2', x_3')$. Then, the proposed vector would have components v_i', say, as follows:

$$\frac{1}{\partial U / \partial x_1'}, \frac{1}{\partial U / \partial x_2'}, \quad \text{and} \quad \frac{1}{\partial U / \partial x_3'}$$

Are these related correctly to v_i? They are, respectively,

$$v_j' = 1 / \frac{\partial U}{\partial x_j'} = 1 / \sum_{p=1}^{3} \frac{\partial u}{\partial x_p} \frac{\partial x_p}{\partial x_j'} = 1 / \sum_{p=1}^{3} a_{pj} \frac{\partial u}{\partial x_p}$$

$(\text{for } i = 1, 2, 3)$. But this is certainly not equal to

$$\sum_{p=1}^{3} a_{pj} v_p$$

so the proposed vector is *not* a vector.

A.4 Multiplication of Vectors

A.4.1 Scalar Product

The *scalar product* of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as the scalar number given by the product of their scalar magnitudes and the cosine of the angle between them:

$$\boldsymbol{u} \cdot \boldsymbol{v} \equiv uv \cos(\boldsymbol{u}, \boldsymbol{v}) \tag{A.7}$$

From the definition, it is clear that this product is independent of any choice of coordinates. Furthermore, it is seen to be equal to the product of u and the projec·tion of $\boldsymbol{v}$ on $\boldsymbol{u}$, or vice versa.

Obviously, $\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{v}\cdot\boldsymbol{u}$. Moreover, it is easy to prove

$$\boldsymbol{u}\cdot(\boldsymbol{a}+\boldsymbol{b})=\boldsymbol{u}\cdot\boldsymbol{a}+\boldsymbol{u}\cdot\boldsymbol{b}$$

because the projection of $(\boldsymbol{a}+\boldsymbol{b})$ on $\boldsymbol{u}$ is the sum of the projections of $\boldsymbol{a}$ and of $\boldsymbol{b}$. Using this last rule, we obtain the expression of $\boldsymbol{u}\cdot\boldsymbol{v}$ in terms of Cartesian components:

If

$$\boldsymbol{u}\equiv u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k} \quad \text{and} \quad \boldsymbol{v}\equiv v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k}$$

then

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_1v_1+u_2v_2+u_3v_3 \tag{A.8}$$

Note that the rule for calculating the angle $(\boldsymbol{u},\boldsymbol{v})$ between two vectors is implied here, namely, by combination of Eqs. (A.7) and (A.8),

$$\cos(\boldsymbol{u},\boldsymbol{v})=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{uv}=\frac{u_1}{u}\frac{v_1}{v}+\frac{u_2}{u}\frac{v_2}{v}+\frac{u_3}{u}\frac{v_3}{v}$$

which is the sum of products of direction cosines. Because $\boldsymbol{i}\cdot\boldsymbol{i}=\boldsymbol{j}\cdot\boldsymbol{j}=\boldsymbol{k}\cdot\boldsymbol{k}=1$ whereas $\boldsymbol{i}\cdot\boldsymbol{j}=\boldsymbol{i}\cdot\boldsymbol{k}=\boldsymbol{j}\cdot\boldsymbol{k}=0$.

Note that if a scalar product vanishes, then one of the vectors has zero magnitude or the two vectors are at right angles.

Sample Problem A.2

Show from Eq. (A.8) that $\boldsymbol{u}\cdot\boldsymbol{v}$ is invariant with respect to transformation of (Cartesian) coordinates.

Solution:
According to Eq. (A.8),

$$\boldsymbol{u}\cdot\boldsymbol{v}=\sum_{i=1}^{3}u_iv_i$$

Because $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors,

$$\begin{aligned}\sum_{i=1}^{3}u_iv_i&=\sum_{i=1}^{3}\sum_{\ell=1}^{3}a_{i\ell}u'_\ell\sum_{m=1}^{3}a_{im}v'_m\\&=\sum_{\ell=1}^{3}u'_\ell\sum_{m=1}^{3}v'_m\sum_{i=1}^{3}a_{i\ell}a_{im}\,(a_{i\ell}=\delta_{\ell m})\\&=\sum_{\ell=1}^{3}u'_\ell v'_\ell\end{aligned}$$

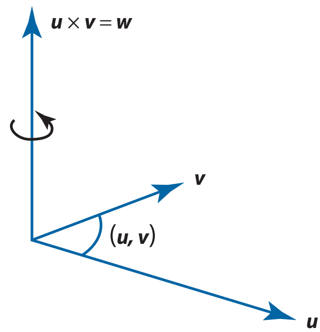


Fig. A.3 Intervening angle ***u***, ***v***.

A.4.2 Vector Products

The *vector product* of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as a vector whose direction is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$ and whose magnitude is the product of their magnitudes and the sine of the angle between them, that is,

$$\boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{v}; \quad w = uv \sin(\boldsymbol{u}, \boldsymbol{v}) \tag{A.9}$$

In this case, the direction of $\boldsymbol{w}$ needs further specification. By convention, $\boldsymbol{u} \cdot \boldsymbol{v}$ is given the direction of advance of a right-hand screw, the direction of rotation being from $\boldsymbol{u}$ toward $\boldsymbol{v}$, via the angle $\boldsymbol{u},\boldsymbol{v}$ intervening, as in Fig. A.3.

It is seen that in this case

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$$

It is useful to note that the magnitude of $\boldsymbol{u} \cdot \boldsymbol{v}$ is equal to twice the area of the triangle determined by $\boldsymbol{u}$ and $\boldsymbol{v}$ in their common plane, or the area of the parallelogram erected on $\boldsymbol{u}$ and $\boldsymbol{v}$. If the vector product $\boldsymbol{u} \cdot \boldsymbol{v}$ vanishes, it might be because $u = 0$, $v = 0$, or $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel.

To determine the expression for $\boldsymbol{u} \cdot \boldsymbol{v}$ in terms of Cartesian components, we can proceed as follows:

$$\boldsymbol{w} = \boldsymbol{\varepsilon}\, uv \sin(\boldsymbol{u}, \boldsymbol{v})$$

where $\boldsymbol{\varepsilon}$ is a unit vector in the direction perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$. Now $\boldsymbol{w} \cdot \boldsymbol{i}$ equals w_1, which we wish to compute, and therefore

$$w_1 = \boldsymbol{i} \cdot \boldsymbol{\varepsilon}\, uv \sin(\boldsymbol{u}, \boldsymbol{v})$$

But this equals twice the area of the projection onto the *yz* plane of the triangle determined by $\boldsymbol{u}$ and $\boldsymbol{v}$ (Fig. A.4). This area can be calculated easily in terms of u_2, v_2, u_3, and v_3:

$$w_1 = u_2 v_3 - u_3 v_2$$

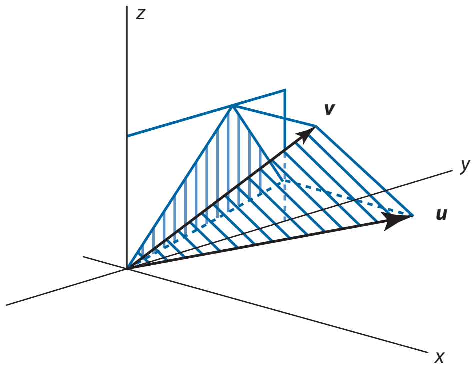


Fig. A.4 Projection on *yz* plane.

The formulas for w_2 and w_3 can be found by a similar calculation or can be determined by cyclic substitution of subscripts; finally,

$$\boldsymbol{u}\times\boldsymbol{v}=(u_2v_3-u_3v_2)\boldsymbol{i}+(u_3v_1-u_1v_3)\boldsymbol{j}+(u_1v_2-u_2v_1)\boldsymbol{k} \tag{A.10}$$

or, as an aid to memory,

$$\boldsymbol{u}\cdot\ \boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

We can now see that the distributive law is obeyed, that is,

$$\boldsymbol{u}\cdot\ (\boldsymbol{v}_1+\boldsymbol{v}_2)=\boldsymbol{u}\cdot\ \boldsymbol{v}_1+\boldsymbol{u}\cdot\ \boldsymbol{v}_2 \tag{A.11}$$

(The student can verify this.)

A.4.3 Scalar Triple Product

According to the definition stated earlier, the scalar triple product $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is a scalar number having a value equal to the volume of the parallelopiped erected on $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ (Fig. A.5). Moreover,

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

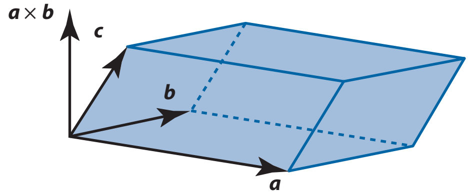


Fig. A.5 Parallelopiped erected on ***a***, ***b***, and ***c***.

Obviously, the parentheses used here are unnecessary, for you cannot form the vector product of a scalar $\boldsymbol{a}\cdot\boldsymbol{b}$ and a vector $\boldsymbol{c}$. Hence, we omit them; furthermore, from the determinant form we see

$$a\cdot b\times c=\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}=\boldsymbol{b}\cdot\boldsymbol{c}\times\boldsymbol{a},\ \text{etc.,}$$

$$=-\boldsymbol{b}\cdot\boldsymbol{a}\times\boldsymbol{c}=-\boldsymbol{a}\cdot\boldsymbol{c}\times\boldsymbol{b},\ \text{etc.}$$

A.4.4 Vector Triple Product

For the vector triple product $\boldsymbol{a}\cdot\ (\boldsymbol{b}\cdot\ \boldsymbol{c})$ the parentheses are needed. Note that $\boldsymbol{b}\cdot\ \boldsymbol{c}$ is perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{c}$; hence, $\boldsymbol{a}\cdot\ (\boldsymbol{b}\cdot\ \boldsymbol{c})$, being perpendicular to both $\boldsymbol{a}$ and $(\boldsymbol{b}\cdot\ \boldsymbol{c})$, lies in the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$. On the contrary, $(\boldsymbol{a}\cdot\ \boldsymbol{b})\cdot\ \boldsymbol{c}$ lies in the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$.

There is an important formula that expresses $\boldsymbol{a}\cdot\ (\boldsymbol{b}\cdot\ \boldsymbol{c})$ in terms of components in the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$:

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \tag{A.12}$$

(The student should verify this by writing out the terms.)

A.5 Derivatives Involving Vectors

A.5.1 Gradient

Consider a scalar function $u=u(x,y,z)$ that is differentiable and has continuous derivatives. Let us define the gradient of u at x, y, and z as the limiting value of a certain surface integral over a surface surrounding the point x, y, and z, as follows:

$$\text{grad}\, u\equiv\lim_{V\to o}\frac{1}{V}\int_S u\boldsymbol{n}\,\mathrm{d}\sigma \tag{A.13}$$

where S is the area enclosing the volume V, $\mathrm{d}\sigma$ is the element of area, and $\boldsymbol{n}$ is the unit vector normal to the surface at each point of the surface integration.

Now, by virtue of the continuity postulated, we can take V very small, in the form of a cube, say, with sides Δx, Δy, and Δz (Fig. A.6). Then, neglecting second-order quantities, $V = \Delta x \Delta y \Delta z$, and

$$\int_S u\boldsymbol{n}\,\mathrm{d}\sigma \approx -u\boldsymbol{i}\Delta y\Delta z - u\boldsymbol{j}\Delta x\Delta z - u\boldsymbol{k}\Delta x\Delta y + \left(u + \frac{\partial u}{\partial x}\Delta x\right)\boldsymbol{i}\Delta y\Delta z$$

$$+\left(u + \frac{\partial u}{\partial y}\Delta y\right)\boldsymbol{j}\Delta x\Delta z + \left(u + \frac{\partial u}{\partial z}\Delta z\right)\boldsymbol{k}\Delta x\Delta y$$

$$\approx \left\{\frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial u}{\partial y}\boldsymbol{j} + \frac{\partial u}{\partial z}\boldsymbol{k}\right\}V$$

Hence, in the limit,

$$\text{grad } u = \boldsymbol{i}\frac{\partial u}{\partial x} + \boldsymbol{j}\frac{\partial u}{\partial y} + \boldsymbol{k}\frac{\partial u}{\partial z} \tag{A.14}$$

(We recognize this as the vector involved in Example A.2) Another symbol often used for grad u is ∇u.

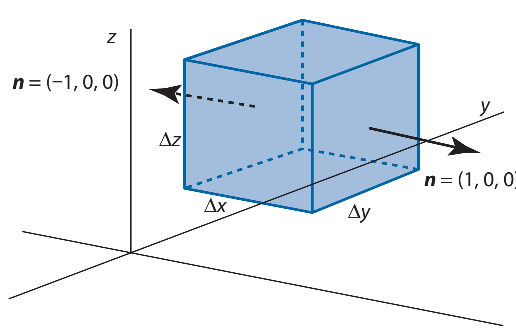


Fig. A.6 Form of a cube with sides Δx, Δy, and Δz.

A.5.2 Divergence

Consider now a *vector function*, $\boldsymbol{v} = \boldsymbol{v}(x, y, z) \equiv v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ where v_1, v_2, and v_3 are all scalar functions of x, y, and z having continuous derivatives. We define

$$\text{div } \boldsymbol{v} \equiv \lim_{V \to o} \frac{1}{V} \int_S \boldsymbol{n} \cdot \boldsymbol{v} \mathrm{d}\sigma \tag{A.15}$$

where V, S, etc., are defined as stated earlier.

Now, by calculating for a small cubical volume, one can easily confirm the following equality:

$$\text{div } \boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_2}{\partial z} \tag{A.16}$$

Another symbol used for div $\boldsymbol{v}$ is $\nabla \cdot \boldsymbol{v}$.

Sample Problem A.3

Show by direct transformation that div $\boldsymbol{v}$ is independent of the choice or (Cartesian) coordinate system, as of course it must be according to Eq. (A.15).

Solution:

$$\text{div } \boldsymbol{v}' = \sum_{i=1}^{3} \frac{\partial v_i'}{\partial x_i'}$$

where v_1', v_2', v_3' are the components of $\boldsymbol{V}$ along the primed axes, that is,

$$v_i' = \sum_{k=1}^{3} a_{ki} v_k$$

Thus,

$$\begin{aligned}
\text{div } \boldsymbol{v}' &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i'} \sum_{k=1}^{3} a_{ki} v_k = \sum_{i=1}^{3} \sum_{p=1}^{3} \frac{\partial}{\partial x_p} \sum_{k=1}^{3} a_{ki} v_k \\
&= \sum_{i=1}^{3} \sum_{p=1}^{3} \sum_{k=1}^{3} a_{ki} \frac{\partial v_k}{\partial x_p} a_{pi} = \sum_{p=1}^{3} \sum_{k=1}^{3} \delta_{kp} \\
&= \sum_{p=1}^{3} \frac{\partial v_p}{\partial x_p}
\end{aligned}$$

A.5.3 Curl

We define

$$\text{curl}\, \boldsymbol{v} \equiv \lim_{V \to o} \frac{1}{V} \int_S \boldsymbol{n} \cdot \boldsymbol{v}\, d\sigma \tag{A.17}$$

and find, by considering a small cube that

$$\text{curl}\, \boldsymbol{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \boldsymbol{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \boldsymbol{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \boldsymbol{k} \tag{A.18}$$

or, to assist the memory, purely symbolically we write

$$\text{curl}\, \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Another symbol used for curl $\boldsymbol{v}$ is $\nabla \times \boldsymbol{v}$.

A.5.4 Laplacian

The *Laplacian* of a scalar function $u(x, y, z)$ is

$$\begin{aligned} \nabla^2 u &\equiv \text{div grad}\, u \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \end{aligned} \tag{A.19}$$

By analogy, the Laplacian of a vector function is the vector whose rectangular Cartesian components are the Laplacians of the vector's corresponding components:

$$\nabla^2 \boldsymbol{v} = \boldsymbol{i} \nabla^2 v_1 + \boldsymbol{j} \nabla^2 v_2 + \boldsymbol{k} \nabla^2 v_3 \tag{A.20}$$

[See also Problem A.1 (b) at the end of this chapter.] (Here we have defined a vector, whose components in a rectangular Cartesian system are given. We can use this vector generally, but must do more work to find its expression in a non-Cartesian system. This limitation does not occur in other cases where we have defined our quantities in a way that is independent of coordinate systems [for example, in formulas (A.7), (A.9), (A.13), (A.15), (A.17), (A.21), (A.23), and the first form of formula (A.19)].

A.5.5 Differential

From the original definition of grad u, we can deduce that the differential du is given by the formula

$$\mathrm{d}u = \mathrm{d}\boldsymbol{r} \cdot \operatorname{grad} u \tag{A.21}$$

where $\boldsymbol{r}$ is the position vector and d$\boldsymbol{r}$ is any directed line element. This means that du is the increment of u corresponding to a position increment d$\boldsymbol{r}$ (Fig. A.7).

To prove this, note that

$$\mathrm{d}\boldsymbol{r} \cdot \operatorname{grad} u \approx \frac{1}{V} \int_S u\boldsymbol{n} \cdot \mathrm{d}\boldsymbol{r}\, \mathrm{d}\sigma$$

But $\boldsymbol{n} \cdot \mathrm{d}\boldsymbol{r}\, \mathrm{d}\sigma$ is a volume element erected on dσ, as in Fig. A.7, and the result of the integration is

$$\int_S u\boldsymbol{n} \cdot \mathrm{d}\boldsymbol{r}\, \mathrm{d}\sigma = \mathrm{d}\left(\int_V u\, \mathrm{d}r \right)$$

In the limit of vanishing V, this becomes $V\,\mathrm{d}u$, and the formula is proved.

In rectangular Cartesian coordinates, this result is immediately evident, for $\mathrm{d}\boldsymbol{r} = \boldsymbol{i}\,\mathrm{d}x + \boldsymbol{j}\,\mathrm{d}y + \boldsymbol{k}\,\mathrm{d}z$, and

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial z}\mathrm{d}z = \mathrm{d}\boldsymbol{r} \cdot \operatorname{grad} u$$

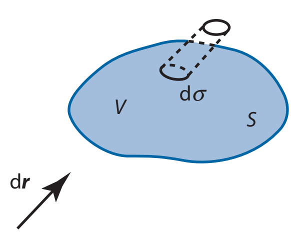


Fig. A.7 A volume element erected on dσ along the direction of dr.

Similarly, for a vector function $\boldsymbol{v}(x, y, z)$,

$$\begin{aligned} \mathrm{d}\boldsymbol{v} &\equiv \boldsymbol{i}\,\mathrm{d}v_1 + \boldsymbol{j}\,\mathrm{d}v_2 + \boldsymbol{k}\,\mathrm{d}v_3 \\ &= \left(\mathrm{d}x \frac{\partial}{\partial x} + \mathrm{d}y \frac{\partial}{\partial y} + \mathrm{d}z \frac{\partial}{\partial z} \right) (\boldsymbol{i}v_1 + \boldsymbol{j}v_2 + \boldsymbol{k}v_3) \\ &= \mathrm{d}\boldsymbol{r} \cdot \nabla \boldsymbol{v} \end{aligned} \tag{A.22}$$

We have defined a new differential operator $\mathrm{d}\boldsymbol{r} \cdot \nabla$ in Eq. (A.22). It is not difficult to prove that $\mathrm{d}\boldsymbol{r} \cdot \nabla \boldsymbol{v}$ satisfies our definition of a vector, and therefore Eq. (A.22) is independent of the choice of coordinate system (see note at end of Sec. A.5.4).

Equations (A.21) and (A.22) lead immediately to the formulas for the *directed derivative* in the direction of a given vector $\boldsymbol{s}$:

$$\frac{\partial u}{\partial s} = \boldsymbol{s} \cdot \operatorname{grad} u \tag{A.23}$$

$$\frac{\partial \boldsymbol{v}}{\partial s} = \boldsymbol{s} \cdot \nabla \boldsymbol{v} \tag{A.24}$$

Again, we have defined a new vector operator $\boldsymbol{s} \cdot \nabla$, equal to

$$\frac{s_1}{s} \frac{\partial}{\partial x} + \frac{s_2}{s} \frac{\partial}{\partial y} + \frac{s_3}{s} \frac{\partial}{\partial z}$$

in rectangular Cartesian coordinates (see note at end of Sec. A.5.4).

A.5.6 Note on the Symbol ∇

Inspection of all of the formulas described earlier will disclose that it is correct to consider the symbol ∇ as representing a *vector operator* $\boldsymbol{i}(\partial / \partial x) + \boldsymbol{j}(\partial / \partial y) + \boldsymbol{k}(\partial / \partial z)$. In all of the formulas, if one treats this operator as a vector, with the appropriate vector-multiplication signs, one gets the right result. This is a convenient aid to the memory, but it must be mentioned that this is true for rectangular Cartesian coordinate systems only. As will be seen later, the expressions for div, grad, curl, etc., in a more general, curvilinear system do not bear much resemblance to one another, and this device is not useful. Again, we wish to emphasize that the quantities obtained as the results of operations with these operators are independent of coordinate systems; only their expressions in terms of components change.

Example A.3 Evaluate grad *r*

Consider grad r, where $r = \sqrt{x^2 + y^2 + z^2}$. Then, $\partial r / \partial x = x / r$, etc., and finally grad $r = \boldsymbol{r} / r$.

Example A.4 Evaluate grad r^n

Here $\partial r^n / \partial x = n r^{n-1} (\partial r / \partial x)$, etc., so that grad $r^n = n r^{n-1}$ grad $r = n r^{n-2} \boldsymbol{r}$.

Example A.5 Evaluate $\nabla^2 r^n$

Now, $\nabla^2 r^n = \operatorname{div} \operatorname{grad} r^n = \operatorname{div} (n r^{n-2} \boldsymbol{r})$. Thus,

$$\nabla^2 r^n = \frac{\partial}{\partial x}(n r^{n-2} x) + \frac{\partial}{\partial y}(n r^{n-2} y) + \frac{\partial}{\partial z}(n r^{n-2} z),$$

which will easily be found equal to $n(n+1) r^{n-2}$.

Sample Problem A.4

Prove that grad u has the magnitude and direction of the maximum rate of change of u.

Solution:
For any given $\Delta \boldsymbol{r}$,

$$\begin{aligned} \Delta u &= \Delta \boldsymbol{r} \cdot \operatorname{grad} u \\ &= \Delta \boldsymbol{r} \left| \operatorname{grad} u \right| \cos (\Delta \boldsymbol{r}, \nabla u) \end{aligned}$$

where $\cos (\Delta \boldsymbol{r}, \nabla u)$ denotes the cosine of the angle between $\Delta \boldsymbol{r}$ and the (unknown) direction of the vector ∇u, Thus,

$$\begin{aligned} \left| \operatorname{grad} u \right| &= \frac{\Delta u / \Delta r}{\cos (\Delta \boldsymbol{r}, \nabla u)} \\ &= \frac{\Delta u}{\Delta r} \end{aligned}$$

when the cosine has its greatest value, namely, 1, that is, when $\Delta \boldsymbol{r} \,||\, \nabla u$ and Δr has its minimum value Δn. In other words, $\left| \operatorname{grad} u \right| = (\Delta u / \Delta n)$ and the direction of grad u is perpendicular to constant-u lines.

A.6 Expansion Formulas

The following formulas are of general utility. Let u denote any differentiable scalar function of x, y, and z, and $\boldsymbol{v}$ and $\boldsymbol{w}$ any such vector functions.

1. div $(u\boldsymbol{v})$

$$\frac{\partial u v_1}{\partial x} = \frac{\partial u}{\partial x} v_1 + u \frac{\partial v_1}{\partial x}$$

hence,

$$\text{div}(u\boldsymbol{v}) = \boldsymbol{v} \cdot \text{grad } u + u \text{ div } \boldsymbol{v} \tag{A.25}$$

2. curl $(u\boldsymbol{v})$

$$\frac{\partial u v_2}{\partial z} - \frac{\partial u v_3}{\partial y} = \frac{\partial u}{\partial z} v_2 - \frac{\partial u}{\partial y} v_3 + u\left(\frac{\partial v_2}{\partial z} - \frac{\partial v_3}{\partial y}\right)$$

hence,

$$\text{curl } (u\boldsymbol{v}) = (\text{grad } u) \cdot \ \boldsymbol{v} + u \text{ curl } \boldsymbol{v} \tag{A.26}$$

3. div $(\boldsymbol{v} \cdot \ \boldsymbol{w})$

$$\frac{\partial}{\partial x}(v_2 w_3 - v_3 w_2) = w_3 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial w_3}{\partial x} - \frac{\partial v_3}{\partial x} w_2 - v_3 \frac{\partial w_2}{\partial x}$$

$$\frac{\partial}{\partial y}(v_3 w_1 - v_1 w_3) = \frac{\partial v_3}{\partial y} w_1 + v_3 \frac{\partial w_1}{\partial y} - \frac{\partial v_1}{\partial y} w_3 - v_1 \frac{\partial w_3}{\partial y}$$

Adding and rearranging terms, we have

$$\text{div } (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{w} \cdot \text{curl } \boldsymbol{v} - \boldsymbol{v} \cdot \text{curl } \boldsymbol{w} \tag{A.27}$$

4. curl $(\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{w} \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \text{ div } \boldsymbol{w} - \boldsymbol{w} \text{ div } \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \boldsymbol{w}$ (A.28)
5. grad $(\boldsymbol{v} \cdot \boldsymbol{w}) = \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \times \text{curl } \boldsymbol{w} + \boldsymbol{w} \times \text{curl } \boldsymbol{v}$ (A.29)
6. div curl $\boldsymbol{v}$

$$\frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = 0 \tag{A.30}$$

Hence, *the divergence of any curl is zero.*

7. curl grad u

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial z}-\frac{\partial}{\partial z}\frac{\partial u}{\partial y}=0 \tag{A.31}$$

Hence, *the curl of any gradient is zero.*

8. curl curl $\boldsymbol{v}$ or curl$^2\boldsymbol{v}$

$$\text{curl}^2\boldsymbol{v}=\text{grad div }\boldsymbol{v}-\nabla^2\boldsymbol{v} \tag{A.32}$$

[Note that the verification of Eqs. (A.28), (A.29), and (A.32) is left to the student.]

A.7 Volume and Surface Integrals—Gauss's Theorem

We shall now prove three important theorems, which permit the transformation of certain types of volume integrals into surface integrals, and vice versa. Let u and v denote arbitrary scalar and vector functions of x, y, and z as earlier, and let us now stipulate that these are *defined, continuous, and single valued* in a certain region of space, and, moreover, *that their first derivatives with respect to x, y, and z satisfy the same requirements.*

Now consider the surface integral

$$\int_S u\boldsymbol{n}\,\mathrm{d}\sigma$$

carried over any closed surface S within the region, enclosing a volume V, $\boldsymbol{n}$ being the unit normal vector directed outward. It is clear that, if the volume V is subdivided into small volumes V_i, this integral equals the sum of all of the integrals

$$\int_{S_i} u\boldsymbol{n}\,\mathrm{d}\sigma$$

taken over the small surfaces S_i because integrations over neighboring elements will cancel one another, and only the integration over the outside will remain:

$$\int_{S_i} u\boldsymbol{n}\,\mathrm{d}\sigma=\sum\int_{S_i} u\boldsymbol{n}\,\mathrm{d}\sigma$$

But, in the limit, the surface integral over the small surface becomes grad $u\,\mathrm{d}\tau$, according to our definition of the gradient, Eq. (A.13), and the summation becomes a volume integration. Thus, we have the theorem:

$$\int_S u\boldsymbol{n}\, \mathrm{d}\sigma = \int_V \operatorname{grad} u\, \mathrm{d}\tau \tag{A.33}$$

By entirely analogous reasoning, using the definitions of the divergence and curl, one will verify that

$$\int_S \boldsymbol{n} \cdot \boldsymbol{v}\, \mathrm{d}\sigma = \int_V \operatorname{div} \boldsymbol{v}\, \mathrm{d}\tau \tag{A.34}$$

and

$$\int_S \boldsymbol{n} \times \boldsymbol{v}\, \mathrm{d}\sigma = \int_V \operatorname{curl} \boldsymbol{v}\, \mathrm{d}\tau \tag{A.35}$$

Equation (A.34) is known as the Divergence theorem or Gauss's theorem.

Example A.6 Directive Derivative

$$\int_V \nabla^2 u\, \mathrm{d}\tau = \int_V \operatorname{div} \operatorname{grad} u\, \mathrm{d}\tau = \int_S \boldsymbol{n} \cdot \operatorname{grad} u\, \mathrm{d}\sigma = \int_S \frac{\partial u}{\partial n}\, \mathrm{d}\sigma$$

where $\partial u / \partial n$ is the directed derivative in the outward normal direction.

A.8 Stokes's Theorem

From our definitions for grad $\boldsymbol{u}$ and curl $\boldsymbol{v}$, it is not difficult to prove that

$$\boldsymbol{n} \times \operatorname{grad} u \approx \frac{1}{S} \int_C u\, \mathrm{d}\boldsymbol{r} \tag{A.36}$$

and

$$\boldsymbol{n} \times \operatorname{curl} \boldsymbol{v} \approx \frac{1}{S} \int_C \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} \tag{A.37}$$

where S denotes a very small surface element in the field, C is the small contour that forms the boundary of S, and $\boldsymbol{n}$ is a unit vector normal to S. The line integrals in Eqs. (A.36) and (A.37) are taken in the direction that would advance a right-hand screw in the $\boldsymbol{n}$ direction. (For proof, see [1].)

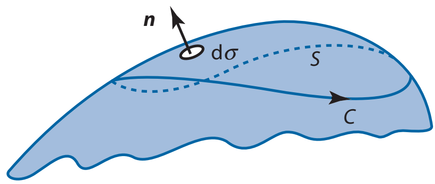


Fig. A.8 Conditions for Stokes's theorem.

With this knowledge, two more important transformation theorems follow; these relate certain surface integrals to contour integrals:

$$\int_S \boldsymbol{n} \times \text{grad}\, u \, d\sigma = \int_C u \, d\boldsymbol{r} \tag{A.38}$$

$$\int_S \boldsymbol{n} \cdot \text{curl}\, \boldsymbol{v} \, d\sigma = \int_C \boldsymbol{v} \cdot d\boldsymbol{r} \tag{A.39}$$

Equation (A.39) is known as Stokes's Theorem. The conditions on u and $\boldsymbol{v}$ are analogous to those imposed earlier, that is, the functions and their first derivations must be finite, continuous, and single valued in the region. The area S, enclosed by the contour C, need not be flat; $\boldsymbol{n}$ is normal to S at every point, and the direction of C is chosen as described earlier (Fig. A.8).

A.9 Vector Operators in General Curvilinear Orthogonal Coordinates

We will have need for the expressions of several vector differential operators in terms of curvilinear orthogonal coordinates, such as spherical and cylindrical. Therefore, we will work out some general expressions here that will apply to any such systems.

Suppose x_1, x_2, and x_3 are mutually orthogonal curvilinear coordinates, and that the line-element vector in this system has components

$$d\boldsymbol{s} = (h_1\, dx_1, h_2\, dx_2, h_3\, dx_3)$$

where $h_1 = h_1(x_1, x_2, x_3)$, etc.

For example, if x_1, x_2, and x_3 are spherical coordinates, as in Fig. A.9, we take $x_1 = r$, $x_2 = \omega$, and $x_3 = \theta$. The line element is $d\boldsymbol{s} = (dr, r \sin\theta\, d\omega, r\, d\theta)$; hence, $h_1 = 1$, $h_2 = r \sin\theta$, and $h_3 = r$.

In each case, we could actually transform our earlier formulas to the new system, but it is much easier to make use of relations (or definitions) that are independent of coordinates and express these in the new system. (There is, to be sure, a certain shortcoming in our treatment, which shows

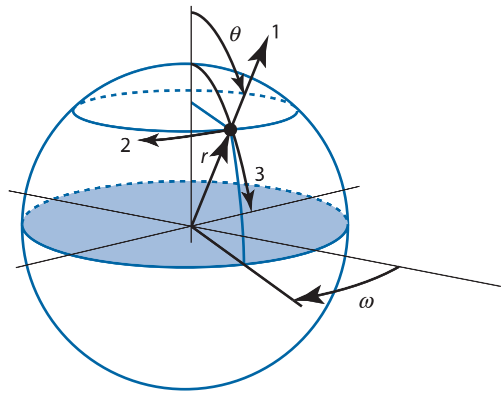


Fig. A.9 Spherical coordinates.

up at this point. We have said that certain quantities are "independent of coordinate system," but have proved this only for rectangular Cartesian systems. To rectify this would require considerable elaboration, as is furnished by the general theory of tensors (rather than Cartesian tensors.)

A.9.1 Grad *u*

We have the formula $\mathrm{d}u = \mathrm{d}\boldsymbol{s}\cdot \operatorname{grad} u$, which is completely general. But in any coordinate system

$$\mathrm{d}u = \frac{\partial u}{\partial x_1}\mathrm{d}x_1 + \frac{\partial u}{\partial x_2}\mathrm{d}x_2 + \frac{\partial u}{\partial x_3}\mathrm{d}x_3$$

and d$\boldsymbol{s}$ has been written as just shown; thus,

$$\frac{\partial u}{\partial x_1}\mathrm{d}x_1 + \frac{\partial u}{\partial x_2}\mathrm{d}x_2 + \frac{\partial u}{\partial x_3}\mathrm{d}x_3 = h_1\,\mathrm{d}x_1(\operatorname{grad} u)_1 + h_2\,\mathrm{d}x_2(\operatorname{grad} u)_2 + h_3\,\mathrm{d}x_3(\operatorname{grad} u)_3$$

Now $\mathrm{d}x_1$, $\mathrm{d}x_2$, and $\mathrm{d}x_3$ are completely arbitrary; hence, this equation can be true only if their coefficients are equal. Thus,

$$\operatorname{grad} u = \left(\frac{1}{h_1}\frac{\partial u}{\partial x_1}, \frac{1}{h_2}\frac{\partial u}{\partial x_2}, \frac{1}{h_3}\frac{\partial u}{\partial x_3}\right) \tag{A.40}$$

A.9.2 div *v*

For this operator, we return to the original definition; thus, denoting by v_1, v_2, and v_3, the components of $\boldsymbol{v}$ in the 1, 2, and 3 directions at any point,

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} \approx &(h_1 h_2 h_3 \Delta x_1 \Delta x_2 \Delta x_3)^{-1}\Big\{-v_1 h_2 h_3 \Delta x_2 \Delta x_3 \\
&-v_2 h_3 h_1 \Delta x_3 \Delta x_1 - v_3 h_1 h_2 \Delta x_1 \Delta x_2 \\
&+\left[v_1 h_2 h_3 + \frac{\partial}{\partial x_1}(v_1 h_2 h_3)\Delta x_1\right]\Delta x_2 \Delta x_3 \\
&+\left[v_2 h_3 h_1 + \frac{\partial}{\partial x_2}(v_2 h_3 h_1)\Delta x_2\right]\Delta x_3 \Delta x_1 \\
&+\left[v_3 h_1 h_2 + \frac{\partial}{\partial x_3}(v_3 h_1 h_2)\Delta x_3\right]\Delta x_1 \Delta x_2\Big\} \\
&= \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial x_1}(h_2 h_3 v_1) + \frac{\partial}{\partial x_2}(h_3 h_1 v_2) + \frac{\partial}{\partial x_3}(h_2 h_1 v_3)\right\}
\end{aligned}
\tag{A.41}
$$

A.9.3 curl *v*

Apply Stokes's theorem to one face of the element in Fig. A.10, say the 1–3 face:

$$
\begin{aligned}
\int_S \boldsymbol{n} \cdot \operatorname{curl} v \, d\sigma = \int_C \boldsymbol{v} \cdot d\boldsymbol{s} \\
&= v_1 h_1 \Delta x_1 - v_3 h_3 \Delta x_3 + \left[v_3 h_3 + \frac{\partial}{\partial x_1}(v_3 h_3)\Delta x_1\right]\Delta x_3 \\
&-\left[v_1 h_1 + \frac{\partial}{\partial x_3}(v_1 h_1)\Delta x_3\right]\Delta x_1 \\
&= \left[\frac{\partial}{\partial x_1}(h_3 v_3) - \frac{\partial}{\partial x_3}(h_1 v_1)\right]\Delta x_1 \Delta x_3
\end{aligned}
$$

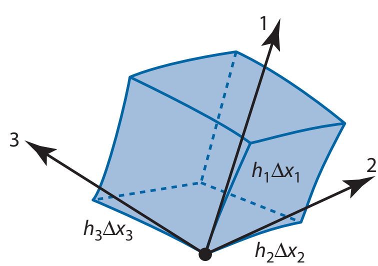


Fig. A.10 Stokes's theorem for one face of an element.

But also

$$\int_S \boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{v} \, d\sigma \approx -h_1 h_3 \Delta x_1 \Delta x_3 (\operatorname{curl} \boldsymbol{v})_2$$

Thus,

$$(\operatorname{curl} \boldsymbol{v})_2 = \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial x_3}(h_1 v_1) - \frac{\partial}{\partial x_1}(h_3 v_3)\right]$$

and by cyclic substitution,

$$\left.\begin{aligned}(\operatorname{curl} \boldsymbol{v})_3 &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}(h_2 v_2) - \frac{\partial}{\partial x_2}(h_1 v_1)\right] \\ (\operatorname{curl} \boldsymbol{v})_1 &= \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial x_2}(h_3 v_3) - \frac{\partial}{\partial x_3}(h_2 v_2)\right]\end{aligned}\right\} \quad \text{(A.42)}$$

or, symbolically,

$$\operatorname{curl} \boldsymbol{v} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \boldsymbol{i}_1 & h_2 \boldsymbol{i}_2 & h_3 \boldsymbol{i}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix} \quad \text{(A.43)}$$

A.9.4 Laplacian

For $\nabla^2 u$, we simply employ Eqs. (A.40) and (A.41):

$$\begin{aligned}\nabla^2 u &= \operatorname{div} \operatorname{grad} u \\ &= \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right\}\end{aligned} \quad \text{(A.44)}$$

The most convenient way to write out $\nabla^2 \boldsymbol{v}$ is by use of expansion formula (A.32):

$$\nabla^2 \boldsymbol{v} = \operatorname{grad} \operatorname{div} \boldsymbol{v} - \operatorname{curl}^2 \boldsymbol{v} \quad \text{(A.45)}$$

which can be expanded by use of formulas (A.40), (A.41), and (A.43).

A.9.5 $\boldsymbol{u}\cdot\nabla\boldsymbol{v}$

This useful vector can be calculated as follows: Suppose, for now, that $\boldsymbol{u}$ is a constant. Then, from Eq. (A.29),

$$\boldsymbol{u}\cdot\nabla\boldsymbol{v} = \mathrm{grad}(\boldsymbol{u}\cdot\boldsymbol{v}) - \boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{v}$$

and its components can be found by use of Eqs. (A.40) and (A.43):

$$\left.\begin{aligned}
(\boldsymbol{u}\cdot\nabla\boldsymbol{v})_1 &= \frac{1}{h_1}\left(u_1\frac{\partial v_1}{\partial x_1}+u_2\frac{\partial v_2}{\partial x_1}+u_3\frac{\partial v_3}{\partial x_1}\right)+\frac{1}{h_1}\left(v_1\frac{\partial u_1}{\partial x_1}+v_2\frac{\partial u_2}{\partial x_1}+v_3\frac{\partial u_3}{\partial x_1}\right)\\
&\quad-\frac{u_2}{h_1h_2}\left[\frac{\partial h_2v_2}{\partial x_1}-\frac{\partial h_1v_1}{\partial x_2}\right]+\frac{u_3}{h_3h_1}\left[\frac{\partial h_1v_1}{\partial x_3}-\frac{\partial h_3v_3}{\partial x_1}\right]\\
(\boldsymbol{u}\cdot\nabla\boldsymbol{v})_2 &= \frac{1}{h_2}\left(u_1\frac{\partial v_1}{\partial x_2}+u_2\frac{\partial v_2}{\partial x_2}+u_3\frac{\partial v_3}{\partial x_2}\right)+\frac{1}{h_2}\left(v_1\frac{\partial u_1}{\partial x_2}+v_2\frac{\partial u_2}{\partial x_2}+v_3\frac{\partial u_3}{\partial x_2}\right)\\
&\quad-\frac{u_3}{h_2h_3}\left[\frac{\partial h_3v_3}{\partial x_2}-\frac{\partial h_2v_2}{\partial x_3}\right]+\frac{u_1}{h_1h_2}\left[\frac{\partial h_2v_2}{\partial x_1}-\frac{\partial h_1v_1}{\partial x_2}\right]\\
(\boldsymbol{u}\cdot\nabla\boldsymbol{v})_3 &= \frac{1}{h_3}\left(u_1\frac{\partial v_1}{\partial x_3}+u_2\frac{\partial v_2}{\partial x_3}+u_3\frac{\partial v_3}{\partial x_3}\right)+\frac{1}{h_3}\left(v_1\frac{\partial u_1}{\partial x_3}+v_2\frac{\partial u_2}{\partial x_3}+v_3\frac{\partial u_3}{\partial x_3}\right)\\
&\quad-\frac{u_1}{h_3h_1}\left[\frac{\partial h_1v_1}{\partial x_3}-\frac{\partial h_3v_3}{\partial x_1}\right]+\frac{u_2}{h_2h_3}\left[\frac{\partial h_3v_3}{\partial x_2}-\frac{\partial h_2v_2}{\partial x_3}\right]
\end{aligned}\right\}\quad\text{(A.46)}$$

We must carry this analysis to the point where we have a form that is just the same whether $\boldsymbol{u}$ is constant or variable. We know this is possible because $\boldsymbol{u}$ is not differentiated in $\boldsymbol{u}\cdot\nabla\boldsymbol{v}$ so that variability of $\boldsymbol{u}$ cannot affect the result.

To do this, we must first calculate the derivatives $\partial u_i/\partial x_j$, which are not zero even if $\boldsymbol{u}$ is constant. We calculate the rotation of the coordinate axes due to increments Δx_1, Δx_2, and Δx_3, and then the projections of u_1, u_2, and u_3 on the new, rotated axes. The angle of rotation about the ith axis is found to be (using the summation convention)

$$\alpha_i = \varepsilon_{ijk}\frac{\partial(h\Delta x)_k}{(h\partial x)_j}$$

and the formula for the projections of $\boldsymbol{u}$ on the rotated axes is

$$u_i' = u_i + \varepsilon_{ijk} u_j \alpha_k$$

From these results, it can easily be verified that the derivatives $\partial u_i / \partial x_j$ are given by the following table:

j	$i = 1$	$i = 2$	$i = 3$
1	$-\frac{u_2}{h_2}\frac{\partial h_1}{\partial x_2} - \frac{u_3}{h_3}\frac{\partial h_1}{\partial x_3}$	$\frac{u_1}{h_2}\frac{\partial h_1}{\partial x_2}$	$\frac{u_1}{h_3}\frac{\partial h_1}{\partial x_3}$
2	$\frac{u_2}{h_1}\frac{\partial h_2}{\partial x_1}$	$-\frac{u_3}{h_3}\frac{\partial h_2}{\partial x_3} - \frac{u_1}{h_1}\frac{\partial h_2}{\partial x_1}$	$\frac{u_2}{h_3}\frac{\partial h_2}{\partial x_3}$
3	$\frac{u_3}{h_1}\frac{\partial h_3}{\partial x_1}$	$\frac{u_3}{h_2}\frac{\partial h_3}{\partial x_2}$	$-\frac{u_1}{h_1}\frac{\partial h_3}{\partial x_1} - \frac{u_2}{h_2}\frac{\partial h_3}{\partial x_2}$

Substituting these in Eq. (A.46a) and collecting terms, we arrive at the following result:

$$\left.\begin{aligned}
(\boldsymbol{u}\cdot\nabla\boldsymbol{v})_1 &= \frac{1}{h_1}\left[u_1\frac{\partial v_1}{\partial x_1} + u_2\frac{\partial v_2}{\partial x_1} + u_3\frac{\partial v_3}{\partial x_1} + \frac{1}{h_2}(u_1v_2 - u_2v_1)\frac{\partial h_1}{\partial x_2} + \frac{1}{h_3}(u_1v_3 - u_3v_1)\frac{\partial h_1}{\partial x_3}\right] \\
&\quad - \frac{u_2}{h_1h_2}\left(\frac{\partial h_2v_2}{\partial x_1} - \frac{\partial h_1v_1}{\partial x_2}\right) + \frac{u_3}{h_3h_1}\left(\frac{\partial h_1v_1}{\partial x_3} - \frac{\partial h_3v_3}{\partial x_1}\right) \\
(\boldsymbol{u}\cdot\nabla\boldsymbol{v})_2 &= \frac{1}{h_2}\left[u_1\frac{\partial v_1}{\partial x_2} + u_2\frac{\partial v_2}{\partial x_2} + u_3\frac{\partial v_3}{\partial x_2} + \frac{1}{h_3}(u_2v_3 - u_3v_2)\frac{\partial h_2}{\partial x_3} + \frac{1}{h_1}(u_2v_1 - u_1v_2)\frac{\partial h_2}{\partial x_1}\right] \\
&\quad - \frac{u_3}{h_2h_3}\left(\frac{\partial h_3v_3}{\partial x_2} - \frac{\partial h_2v_2}{\partial x_3}\right) + \frac{u_1}{h_1h_2}\left(\frac{\partial h_2v_2}{\partial x_1} - \frac{\partial h_1v_1}{\partial x_2}\right) \\
(\boldsymbol{u}\cdot\nabla\boldsymbol{v})_3 &= \frac{1}{h_3}\left[u_1\frac{\partial v_1}{\partial x_3} + u_2\frac{\partial v_2}{\partial x_3} + u_3\frac{\partial v_3}{\partial x_3} + \frac{1}{h_1}(u_3v_1 - u_1v_3)\frac{\partial h_3}{\partial x_1} + \frac{1}{h_2}(u_3v_2 - u_2v_3)\frac{\partial h_3}{\partial x_2}\right] \\
&\quad - \frac{u_1}{h_3h_1}\left(\frac{\partial h_1v_1}{\partial x_3} - \frac{\partial h_3v_3}{\partial x_1}\right) + \frac{u_2}{h_2h_3}\left(\frac{\partial h_3v_3}{\partial x_2} - \frac{\partial h_2v_2}{\partial x_3}\right)
\end{aligned}\right\} \tag{A.46a}$$

This result does not involve differentiation of $\boldsymbol{u}$ nor its components on either side of the equation. It must hold for either variable or constant $\boldsymbol{u}$.

A.10 Dyadic Products and Second-Order Tensors

Much of our work can be simplified if we extend our definitions of vector multiplication to include the *dyadic product* $\boldsymbol{uv}$ as described below. For our purpose, this need only be defined by the relations

$$(\boldsymbol{uv})\cdot\boldsymbol{w}\equiv\boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{w})$$
$$\boldsymbol{w}\cdot(\boldsymbol{uv})\equiv(\boldsymbol{w}\cdot\boldsymbol{u})\boldsymbol{v} \qquad \text{(A.47)}$$

Actually, the dyadic product $\boldsymbol{uv}$ is a special form of *second-order tensor*; it can easily be seen to satisfy the definition of such a tensor. This definition can be stated as follows, with reference to the x_i and x_i' coordinate systems (Sec. A.3).

A *second-order tensor* is a set of nine numbers p_{ij}, having the property that when transferred from the x_1, x_2, and x_3 system to the x_1', x_2', and x_3' system the corresponding quantities are given by

$$p_{ij}'=\sum_{k=1}^{3}\sum_{\ell=1}^{3}a_{ki}a_{\ell j}p_{k\ell} \qquad (i,j=1,2,3) \qquad \text{(A.48)}$$

In the case of $\boldsymbol{uv}$, of course, the nine numbers involved are the products u_iv_j $(i,j=1,2,3)$.

Now, the utility of dyadic products comes from the fact that they are tensors, and therefore relations involving them are independent of coordinate systems, and that many operations can be performed with them just as with vectors. (For example, see Problem A.8.)

There are other second-order tensors that are not dyadic products of vectors. To write them, we often have to give up ordinary vector notation and employ tensor notation; if we restrict ourselves to rectangular Cartesian coordinates, we can use Cartesian-tensor notation, as in Eq. (A.48). Let u_i be a vector, and consider the set of nine numbers $\partial u_i/\partial x_j$. This is easily shown to be a second-order tensor; it might be represented by the symbol grad $\boldsymbol{u}$ or $\nabla\boldsymbol{u}$.

Sample Problem A.5

If ϕ is any dyadic product, $\phi\cdot(\boldsymbol{a}\times\boldsymbol{b})=(\phi\times\boldsymbol{a})\cdot\boldsymbol{b}$. Prove it.

Solution:

If $\phi=\boldsymbol{uv}$, $\boldsymbol{uv}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ means $\boldsymbol{u}[\boldsymbol{v}\cdot(\boldsymbol{a}\times\boldsymbol{b})]$ according to Eq. (A.47). Now, the square bracket is a scalar triple product, and so this is equal to $\boldsymbol{u}[(\boldsymbol{v}\times\boldsymbol{a})\cdot\boldsymbol{b}]$. But, again referring to Eq. (A.47), we see that this can be written as $\{\boldsymbol{u}(\boldsymbol{v}\times\boldsymbol{a})\}\cdot\boldsymbol{b}$ where the curly bracket is a dyadic product. According to Example A.3, this can be written as $\{(\boldsymbol{uv})\times\boldsymbol{a}\}\cdot\boldsymbol{b}$.

The student should note that this holds for any second-order tensor (dyad), not just dyadic products of vectors. Simply write ϕ as $\phi_{11}\boldsymbol{ii}+\phi_{12}\boldsymbol{ij}+\cdots$ and apply our result term-by-term.

A.11 Rotating Coordinate Systems

Suppose that the primed coordinate system of Fig. A.2 is in motion relative to the unprimed system. For now, suppose that their origins coincide and that the relative motion consists only of rotation. Consider now a vector $\boldsymbol{v}$ that is a function of time t and whose time derivative is needed. Because of the relative motion, the derivative will appear different to observers in the two coordinate systems. For example, a vector that is constant in either system would seem to vary with time to an observer fixed in the other system. We will use the notation $\mathrm{d}/\mathrm{d}t$ to denote the derivative observed in the unprimed system and $\mathrm{d}'/\mathrm{d}t$ to denote the derivative observed in the primed system.

Let $\boldsymbol{\omega}$ be the vector angular velocity of the primed system relative to the unprimed, that is, a vector along the instantaneous axis of rotation, having the magnitude of the rate of rotation $\boldsymbol{\omega}$, and the direction given by the right-hand-screw rule. The reader can easily verify, by considering what happens during an infinitesimal time interval Δt, that

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=\frac{\mathrm{d}'\boldsymbol{v}}{\mathrm{d}t}+\boldsymbol{\omega}\times\boldsymbol{v} \tag{A.49}$$

If this formula is applied to the special case of the position vector $\boldsymbol{r}$, we have a familiar result, relating the velocity vectors in the two coordinate systems:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}=\frac{\mathrm{d}'\boldsymbol{r}}{\mathrm{d}t}+\boldsymbol{\omega}\times\boldsymbol{r} \tag{A.50}$$

We can obtain the relation between acceleration vectors by careful differentiation, making use of the general rule Eq. (A.49):

$$\begin{aligned}\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}&=\frac{\mathrm{d}'}{\mathrm{d}t}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)+\boldsymbol{\omega}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\\&=\frac{\mathrm{d}'}{\mathrm{d}t}\left(\frac{\mathrm{d}'\boldsymbol{r}}{\mathrm{d}t}+\boldsymbol{\omega}\times\boldsymbol{r}\right)+\boldsymbol{\omega}\times\left(\frac{\mathrm{d}'\boldsymbol{r}}{\mathrm{d}t}+\boldsymbol{\omega}\times\boldsymbol{r}\right)\\&=\frac{\mathrm{d}'^2\boldsymbol{r}}{\mathrm{d}t^2}+2\boldsymbol{\omega}\times\frac{\mathrm{d}'\boldsymbol{r}}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\boldsymbol{r}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\end{aligned} \tag{A.51}$$

Here, we have written $\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t$ instead of $\mathrm{d}'\boldsymbol{\omega}/\mathrm{d}t$ because $\boldsymbol{\omega}$ is a vector that is always the same in both systems.

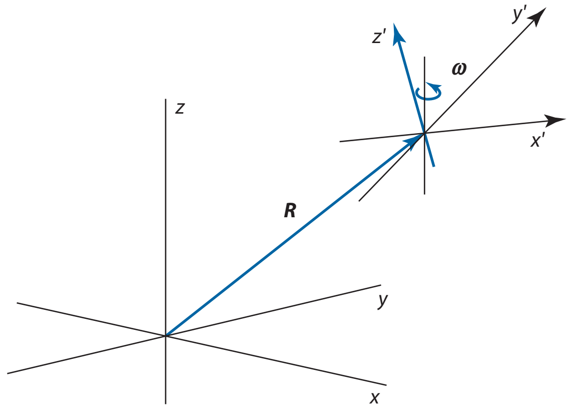


Fig. A.11 Movement of primed system.

The first term of Eq. (A.51) is the acceleration viewed in the primed system; the second is the "Coriolis acceleration," which depends on the velocity in the primed system; the meaning of the third term is clear; and because the last term is the generalized centripetal acceleration

$$\left|\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\right| = \omega^2 r \sin(\boldsymbol{\omega},\boldsymbol{r}) \tag{A.52}$$

Finally, for complete generality, let us suppose that the primed system is displaced from the unprimed and is moving relative to it in a quite general way (Fig. A.11). The generalization of our previous results is easily accomplished, and the results are

$$\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{R} \tag{A.53}$$

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}'\boldsymbol{r}'}{\mathrm{d}t} + \dot{\boldsymbol{R}} + \boldsymbol{\omega}\times\boldsymbol{r}' \tag{A.54}$$

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}'^2\boldsymbol{r}'}{\mathrm{d}t^2} + 2\boldsymbol{\omega}\times\frac{\mathrm{d}'\boldsymbol{r}'}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\boldsymbol{r}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}') + \ddot{\boldsymbol{R}} \tag{A.55}$$

Reference

[1] Wills, A. P., *Vector and Tensor Analysis*, Prentice-Hall, Inc., New York, 1931, pp. 86, 87.

Problems

A.1 a. Prove that the following are vector quantities: acceleration of a point, the product $m\boldsymbol{u}$, where m is a scalar and $\boldsymbol{u}$ a vector, and $\nabla^2\boldsymbol{v}$ as defined in Eq. (A.20).

b. Determine whether $\partial^2 u/\partial x_i^2$ (no sum) is a vector.

A.2 Prove that

a.

$$\nabla\times\boldsymbol{v}\cdot\boldsymbol{a}=\nabla\cdot\boldsymbol{v}\times\boldsymbol{a}\quad\text{if }\boldsymbol{a}=\text{const}$$

b.

$$\boldsymbol{a}\cdot\nabla(\boldsymbol{u}\cdot\boldsymbol{v})=(\boldsymbol{a}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{v}+(\boldsymbol{a}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{u}$$

A.3 Prove that

$$[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]\boldsymbol{r}=[\boldsymbol{r}\,\boldsymbol{b}\,\boldsymbol{c}]\boldsymbol{a}+[\boldsymbol{a}\,\boldsymbol{r}\,\boldsymbol{c}]\boldsymbol{b}+[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{r}]\boldsymbol{c}$$

where $[\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}]$ denotes the scalar triple product.
Hint: Expand $(\boldsymbol{a}\cdot\ \boldsymbol{b})\cdot\ (\boldsymbol{c}\cdot\ \boldsymbol{d})$.

A.4 A surface S has a plane boundary curve C. Show that $\int_S \boldsymbol{n}\,\mathrm{d}\sigma$ is a vector normal to the plane of C.

A.5 a. A scalar function $u(x,y,z)$ is known to depend only on r. Calculate grad u.

b. Give an example of this situation.

A.6 a. Convert to a volume integral:

$$\int_S (u\,\text{grad}\,v - v\,\text{grad}\,u)\cdot\boldsymbol{n}\,\mathrm{d}\sigma$$

b. Apply the result to calculate the integral when $v=1/r$ and the origin lies inside V. What is the value of the integral when the origin lies outside V?

A.7 By transforming a surface integral to a volume integral, derive Archimedes's principle (which states that a submerged body is buoyed up by a force equal to the weight of the displaced liquid) from the hydrostatic principle (which states that the pressure at depth z is ρgz where ρ is the (constant) mass density of the liquid). This is for a body of arbitrary shape.

A.8 Deduce the "divergence theorem" for the dyadic product of two vectors.

A.9 Calculate

a.
$$\int_S (\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{n}\, \mathrm{d}\sigma$$

b.
$$\int_S \boldsymbol{r}(\boldsymbol{a} \cdot \boldsymbol{n})\, \mathrm{d}\sigma$$

where $\boldsymbol{a}$ is a constant, $\boldsymbol{r}$ is the position vector, and $\boldsymbol{n}$ the unit normal vector on the surface S, which encloses a volume V.

A.10 Consider the vector operator $\boldsymbol{a} \times \nabla$: Prove

$$(\boldsymbol{a} \times \nabla)u = \boldsymbol{a} \times \text{grad}\, u$$

$$(\boldsymbol{a} \times \nabla) \cdot \boldsymbol{v} \quad (\text{or} \quad \boldsymbol{a} \times \nabla \cdot \boldsymbol{v}) = \boldsymbol{a} \cdot \nabla \times \boldsymbol{v} = \boldsymbol{a} \cdot \text{curl}\, \boldsymbol{v}$$

$$(\boldsymbol{a} \times \nabla) \times \boldsymbol{v} = (\text{grad}\, \boldsymbol{v}) \cdot \boldsymbol{a} - \boldsymbol{a}\, \text{div}\, \boldsymbol{v}$$

A.11 Prove

$$\int_S (\boldsymbol{n} \times \nabla) \times \boldsymbol{v}\, \mathrm{d}\sigma = \oint \mathrm{d}\boldsymbol{r} \times \boldsymbol{v}$$

A.12 What is the total acceleration of a missile flying in a northerly direction at constant altitude and constant speed U at latitude 40°N?

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SUPPORTING MATERIALS

Many of the topics introduced in this book are discussed in more detail in other AIAA publications. For a complete listing of titles in the AIAA Education Series, as well as other AIAA publications, please visit www.aiaa.org.

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