LATTICES, CPOS AND FIXPOINT THEOREMS

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Contents

1. Background

Definition 1.1. A relation$\preceq$ is a partial order on a set S if $\forall\, a,\, b,\, c \in S$,

(1) $a \preceq a$ (reflexivity)
(2) $a \preceq b \,\&\, b \preceq a \Rightarrow a = b$ (anti-symmetry)
(3) $a \preceq b \,\&\, b \preceq a \Rightarrow a = b$ (transitivity)

And $(S, \preceq)$ is called a partially ordered set.

Definition 1.2. Let S be a partially ordered set, then the dual of S, denoted by S^∂, is a partially ordered set such that if $a \leq b$ in S then $a \geq b$ in S^∂.

We can use similar idea to define the dual statement of a statement Φ about partially ordered sets: given A statement Φ about partially ordered sets, then the dual statement Φ^∂ is obtained by replacing every occurrence of $\leq$ with $\geq$. For example, consider the statement Φ: let S be a partially ordered set, $a \in X$ and $A \subseteq S$, then $x \in A$ if $x \leq a$. then its dual statement would be "let S be a partially ordered set, $a \in X$ and $A \subseteq X$, then $x \in A$ if $x \geq a$.

The Dual Priciple 1.3. Given a statement Φ which is true in all partially ordered sets, then its dual statement Φ^∂ is also ture in all partially ordered sets

Let S be a partially ordered set and Q be a subset of S, then $a \in Q$ if a *maxima*l element of Q if $x \in Q$, $a \leq x (a \geq x)$ implies $a = x$. And a is called a *greatest* element if $\forall\, x \in Q$, $x \leq a (x \geq a)$. Dually, a is called a *minimal/least* element. Clearly the greatest element is always a maximal element, then the converse may not be true because there may exist some elements that are not related to a in the set. An *upper bound* $y \in S$ of Q is an element such that $\forall\, x \in Q$, $x \leq y$, dually y is called a lower bound of Q. And let Q^u be the set of upper bound of Q, then y is called the supremum of Q if y is the least element of Q^u, dually y is called the infimum of Q.

Definition 1.4. Let S be a partially ordered set, then S is a chain (or totally ordered set) if $\forall\, a,\, b \in S$, either $a \leqslant b$ *or* $b \leqslant a$. S is called an antichain if $a \leqslant b$ iff $a = b$.

Definition 1.5. Let S be a partially ordered set and $a\,,b \in S$, then a is said to be covered by b , denoted by $a \longleftarrow b$, if $a < b$ and $a \leqslant c < b \Longrightarrow a = c$

Definition 1.6. Let S be a partially ordered set, and $Q \subseteq S$, then Q is a down-set or order ideal if $x \in Q$, $y \in S$ and $y \leq x$ then $y \in Q$. Dually Q is called a up-set or order filter.

Definition 1.7. Let P and Q be partially ordered sets, then a map $\varphi : P \rightarrow Q$ is said to be

(1) an order-preserving if $x \leq y$ in P implies $\varphi(x) \leq \varphi(y)$ in Q.
(2) an order-embedding if $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q.
(3) an order-isomorphism if it is an order-embedding and bijective.

2. Lattices

Definition 2.1. Let P be a non-empty ordered set, if $\forall x, y \in P$, $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist in P, then P is called a lattice. P is a complete lattice if $\forall S \subseteq P$, $\bigvee S$ and $\bigwedge S$ exist in P.

Alternatively, a lattice can be considered to be a algebraic structure.

Alternative definition of lattices 2.2. Let S be a partially ordered set and $\vee$ and $\wedge$ be two binary operators defined on S such that:

(1) $(a \vee b) \vee c = a \vee (b \vee c)$
(2) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
(3) $a \vee b = b \vee a$
(4) $a \wedge b = b \wedge a$
(5) $a \vee a = a$
(6) $a \wedge a = a$
(7) $a \vee (a \wedge b) = a$
(8) $a \wedge (a \vee b) = a$

It is can be proved that the above two definitions are identical.

Connecting Lemma 2.3. Let L be a lattice and $a, b \in L$, then

(1) $a \leq b$
(2) $a \vee b = b$
(3) $a \wedge b = a$

are equivalent.

Proof. (1)$\Rightarrow$(2), $a \leq b$ and $b \leq b$ imply $b \in \{a, b\}^u$. And $b \leq x \;\; \forall x \in \{a, b\}^u \Rightarrow b = a \vee b$. (2)$\Rightarrow$(3), $a \vee b = b \Rightarrow a \leq b \Rightarrow a$ is a lower bound for $\{a, b\}$. But $y \leq a$ for all lower bound y of $\{a, b\}$. Hence $a = a \wedge b$. (3)$\Rightarrow$ (1) is trivial. $\square$

Lemma 2.4. *Let P be a lattice, $a, b, c, d \in P$, then*

1. $a \leq b$ *implies* $a \vee c \leq b \vee c$ *and* $a \wedge c \leq b \wedge c$
2. $a \leq b$ *and* $c \leq d$ *implies* $a \vee c \leq b \vee d$ *and* $a \wedge c \leq b \wedge d$

Proof. (1) consider $b \vee c$, $b\,, c \leq b \vee c \Rightarrow a \leq b \leq b \vee c$ and $c \leq b \vee c$. Hence $b \vee c \in \{a, c\}^u$, then $a \vee c \leq b \vee c$. Now consider $a \wedge c$, a, $c \geq a \wedge c \Rightarrow b\,, c \geq a \wedge c$. Therefore $a \wedge c$ is a lower bound for $\{b\,, c\}$. Hence, $a \wedge c \leq b \wedge c$.

(2) Consider $b \vee d$, we have $a \leq b \leq b \vee d$ and $c \leq d \leq b \vee d$, therefore, $b \vee d$ is a upper bound for $\{a, c\}$, hence $a \vee c \leq b \vee d$. Now consider $a \wedge c \leq a$, c, then $a \wedge c \leq b$, d, then $a \wedge c$ is a lower bound for $\{b, d\}$. Hence $a \wedge c \leq b \wedge d$. $\square$

Definition 2.5. Let L be a lattice, then L has a ONE if $\exists 1 \in L$ such that $\forall a \in L$, $a \wedge 1 = a$ and L has a ZERO if $\exists 0 \in L$ such that $a \vee 0 = a$.

Lemma 2.6. *Complete lattices have both 1 and 0.*

Proof. Let P be a complete lattice, then $\forall S \subseteq P$, $\bigvee S$ exists in P. Now consider S=$\emptyset \subseteq P$ and let $a \in P$ be any element. Then we can say that $\forall, x \in \emptyset, x \leq a$, because there is no element x in $\emptyset$ such that $x > a$ or x is not related to a. Hence, we have $P = \emptyset^u$. since $\bigvee \emptyset = \sup\{\emptyset\}$ exists in P, P has a bottom element 0.

Similarly, we can prove that P has a top element 1. $\square$

Definition 2.7. Let L be a lattice and $\emptyset \neq M \subseteq L$, them M is a sublattice of L if $\forall\, a, b\, \in M$, $a \vee b \in M$ and $a \wedge b \in M$.

Remark 2.8. It is possible that M itself is a lattice but is not a sublattice of L.

Definition 2.9. Let L, K be lattices, a map $f\,:\, L \to K$ is said to be a homomorphism if $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)\ \ \forall a, b \in L$.

f is an isomorphism if f is bijective.

Lemma 2.10. *f(L) is a sublattice of K*

Proof. Let $x, y \in f(L)$, then $\exists a, b \in L$ such that $f(a) = x$ and $f(b) = y$.

Now consider $x \vee y = f(a) \vee f(b)$, since f is a homomorphism, we have $f(a) \vee f(b) = f(a \vee b) \in f(L)$, therefore $x \vee y \in f(L)$. Similarly, we can prove that $x \wedge y \in f(L)$. Hence $f(L)$ is a sublattice of K. $\square$

Lemma 2.11. *Let L, K be lattices and let $f\,:\, L \to K$ be a map. Then the following statements are equivalent.*

(1) *f is order-preserving.*
(2) *$f(a) \vee f(b) \leq f(a \vee b)$.*
(3) *$f(a \wedge b) \leq f(a) \wedge f(b)$.*

Proof. (1)$\Rightarrow$(2). $a\,, b \leq a \vee b$ and f is order-preserving $\Rightarrow$ $f(a)\,, f(b) \leq f(a \vee b)$. Therefore $f(a \vee b)$ is an upper bound for $\{f(a)\,, f(b)\}$. Hence $f(a) \vee f(b) \leq f(a \vee b)$.

(2)$\Rightarrow$(3). It holds because of the Dual Principal.

(3) $\Rightarrow$(1). Let $a\,, b \in L$ with $a \leq b$, then $a = a \wedge b$, by hypothesis, $f(a) = f(a \wedge b) \leq f(a) \wedge f(b) \leq f(b)$. $\square$

Definition 2.12. Let L be a lattice. A non-empty subset J of L is called an ideal if

(1) $\forall a, b \in J, a \vee b \in J$.
(2) $a \in L, b \in J$ and $a \leq b$ imply $a \in J$.

Definition 2.13. Let L a lattice. A non-empty subset G of L is called a filter if

(1) $\forall a, b \in G, a \wedge b \in G$.
(2) $a \in L, b \in G$ and $a \geq b$ imply $a \in G$.

Lemma 2.14. *Let P be a lattice, let S, $T \subseteq P$. Suppose that $\vee S$, $\vee T$, $\wedge S$, $\wedge T$ exist in P then*

(1) $\vee(S \cup T) = (\vee S) \vee (\vee T)$.
(2) $\wedge(S \cup T) = (\wedge S) \wedge (\wedge T)$

Definition 2.15. Let P and Q be partially ordered sets and $\varphi : P \to Q$ be a map such that $\varphi(\vee S) = \vee\varphi(S)$ for $\vee S$ exists in P. Then φ is said to preserve existing joins. Dually, φ preserves the existing meets.

Lemma 2.16. *Let P and Q be partially ordered sets, and $\varphi : P \to Q$ be an order-preserving map. Then*

(1) *Suppose that $S \subseteq P$ such that $\vee S$ exists in P and $\vee\varphi(\vee S)$ exists in Q, then $\varphi(\vee S) \geq \vee\varphi(S)$. Dually, $\varphi(\wedge S) \leq \wedge\varphi(S)$.*
(2) *if φ is an order-isomorphism, then φ preserves all existing joins and meets.*

Lemma 2.17. *Let P be a partially ordered set, $Q \subseteq P$ with the same order, let $S \subseteq Q$, then if $\vee_P S$ exists and is in Q then $\vee_Q S$ exists and $\vee_P S = \vee_Q S$. Dually it is true for $\wedge_P S$ and $\wedge_Q S$.*

Proof. Suppose $\vee_p S$ exists and it is in Q, then for all $x \in S$, $x \leq \vee_P S$. Since $\vee_p S$ is in Q, it is an upper bound for S in Q. Let y be any upper bound for S in Q. Then it is an upper bound for S in P. Then $y \leq \vee_p S$. Hence $\vee_p S = \vee_Q S$. □

Lemma 2.18. *Let P be an partially ordered set such that $\forall\, S \subseteq P$ and $S \neq \phi$, $\bigwedge S$ exists in P for every non-empty subset S, then $\bigvee S$ exists in P for every subset S of P which has an upper bound in P.*

Proof. Let $S \subseteq P$ and S has an upper bound in P, therefore $S^u \neq \emptyset$. By hypothesis, $\alpha = \bigwedge S^u$ exists in P. Consider $x \in S$, then $x \leq y \quad \forall y \in S^u$, therefore, x is a lower bound for S^u, then $x \leq \alpha$ since α is the greatest lower bound. Hence, α is an upper bound for S. Finally, $\bigvee S = \alpha$. □

Theorem 2.19. *Let P be a non-empty partially ordered set, then the following statements are equivalent.*

(1) *P is a complete lattice.*
(2) *$\wedge S$ exists in P, $\forall S \subseteq P$.*
(3) *P has a top element, and $\wedge S$ exists in P $\forall S \subseteq P$, $S \neq \varphi$*

Proof. (1) $\Rightarrow$ (2) is trivial, it follows the definition of complete lattices.

Now we show that (2) $\Rightarrow$ (3). Consider $\emptyset \subseteq P$, then $\wedge\emptyset$ exists in P. Let $p \in P$, then there is no element x in $\emptyset$ such that $x \nleq p$, hence every element $p \in P$ is a lower bound for $\emptyset$. Therefore $\wedge\emptyset$ exists in P and $\forall\, p \in P$, $p \leq \wedge\emptyset$, hence $\top = \wedge\emptyset$.

Now we show that (3) $\Rightarrow$ (1). Let $P \subseteq P$ and $P \neq \emptyset$, then $\wedge P$ exists in P. Let $\bot = \wedge P$, clearly for $\emptyset \subseteq P$, $\vee\emptyset = \top$ and since $\bot$exists in P, $\wedge\emptyset = \bot$.

Let $S \subseteq P$ and $S \neq \emptyset$, by hypothesis $\wedge S \in P$. Consider $\vee S$. since $\top \in P$, S has a upper bound, $\therefore$ $S^u \neq \emptyset$, hence $\wedge S^u$exists in P and $\vee S = \wedge S^u$.

To sum up, P is a complete lattice. □

Definition 2.20. Let P be an ordered set. If $C = \{c_0, c_1, \ldots, c_n\}$ is a finite chain in P with $|C| = n + 1$, then C is said to have length n. P have length n if the length of the longest chain in P is n. P is of finite length if it has a length $n \in \mathbb{N}$, and it has no infinite chains if every chain in P is finite.

P satisfies the Ascending Chain Condition if given any sequence $x_1 \leq x_2 \cdots \leq x_n \leq \cdots$ in P, $\exists\, k \in \mathbb{N}$ such that $x_i = x_k$ $\forall i = k, k{+}1, \ldots$. Dually, P satisfies the Descending Chain Condition.

3. Complete partially ordered sets

Definition 3.1. Let P be a partially ordered set and $\emptyset \neq S \subseteq P$, then S is said to be directed if $\forall\, x,\, y \in S$, $\exists\, z \in S$ such that $z \in \{x,\, y\}^u$.

If D is a directed subset of P, we write $\bigvee D$ as $\bigsqcup D$ if it exists.

Corollary 3.2. *S is directed in P if and only if $\forall F \subseteq S$, $|F| < \infty$, $\exists\, z \in S$ such that $z \in F^U$.*

Proof. Say S is directed, then $\forall x, y \in S$, $\exists z \in S$ such that $z \in \{x, y\}^u$. Suppose this is true for $F \subseteq S$ such that $|F| = k$, consider $F \bigcup \{s\}$ where $s \in S$. Suppose $a, b \in F \bigcup \{s\}$. If $a, b \in F$ then the proof is done. If $a = s$, by induction hypothesis, $\exists z \in S$ such that $z \in F^u$, and since S is directed, $\exists z_1 \in S$ such that $z_1 \in \{s, z\}^u$. Hence $z_1 \in \{F \bigcup \{s\}\}^u$.

Conversely, say $\forall F \subseteq S$, F is finite, $\exists z \in S$ such that $z \in F^u$. Then in particular it is true for $\{x, y | x, y \in S\}$. □

Definition 3.3. Let P be a partially ordered set, then P is a complete partially ordered set(CPO) if

(1) P has bottom element $\perp$.
(2) $\bigsqcup D$ exists for each directed subset D of P.

P is a pre-CPO if it satisfies the second property.

Definition 3.4. Let P be a CPO, $Q \subseteq P$, then Q is a subCPO of P if

(1) $\perp \in Q$
(2) if $D \subseteq Q$ is a directed in Q, then $\bigsqcup_Q D$ exists and $\bigsqcup_Q D = \bigsqcup_P D$

Definition 3.5. Let $P$and Q be pre-CPOs, let $\varphi : P \to Q$ be a map from P to Q, then φ is continuous if $\forall\, D \subseteq P$ directed in P, $\varphi(D)$ is directed in Q and $\varphi(\bigsqcup D) = \bigsqcup_Q \varphi(D)$. If P and Q are CPOs and $\varphi(\perp) = \perp$, then φ is said to be strict.

Lemma 3.6. *Let P and Q be CPOs and $\varphi : P \to Q$, then φ is order-preserving iff $\forall\, D \subseteq P$ directed in P, $\varphi(D)$ is directed in Q and $\sqcup \varphi(D) \leq \varphi(\sqcup D)$.*

Proof. Let $x, y \in \varphi(D)$, then $\exists\, a, b \in D$ such that $\varphi(a) = x$ and $\varphi(b) = y$. $a, b \in D \Rightarrow \exists\, c \in D$ such that $a \leq c$ and $b \leq c$. Now, consider $\varphi(c) \in \varphi(D)$,since φ is order-preserving, $\therefore a \leq c$ and $b \leq c \Rightarrow \varphi(a), \varphi(b) \leq \varphi(c)$, hence $\varphi(D)$ is directed in Q and $\forall\, x \in \varphi(D)$, $x \leq \varphi(\bigsqcup D)$. Therefore, $\bigsqcup \varphi(D) \leq \varphi(\bigsqcup D)$.

Conversely, let $a, b \in P$ and $a \leq b$, then the set $\{a, b\}$ is directed in P. $\therefore \varphi(\{a, b\})$ is directed in Q and $\bigsqcup \varphi(\{a, b\}) \leq \varphi(\bigsqcup \{a, b\}) = \varphi(b)$. Hence $\varphi(a) \leq \varphi(b) \Rightarrow \varphi$ is order-preserving. □

Corollary 3.7. *Let P and Q be CPOs and $\varphi : P \to Q$ be a map, if φ is continuous then it is order-preserving.*

Remark. Not every order-preserving map between CPOs is continuous. For example the map $\varphi : \wp(\mathbb{N}) \to \wp(\mathbb{N})$

Lemma 3.8. *Let P and Q be CPOs and $\varphi : P \to Q$ be a map, then φ is continuous if P satisfies (ACC).*

Proof. Let D be a directed subset in P, since P satisfies (ACC), D has a greatest element $\alpha = \bigsqcup D$. And φ is order-preserving implies $\varphi(D)$ is directed in Q and $\bigsqcup \varphi(D) \leq \varphi(\bigsqcup D) = \varphi(\alpha)$. Since $\alpha \in D$, $\varphi(\alpha) \leq \bigsqcup \varphi(D)$. Hence $\bigsqcup \varphi(D) = \varphi(\bigsqcup D)$. Thus φ is continuous. $\square$

Theorem 3.9. *Let P be an partially ordered set. Then P is a CPO if and only if each chain has a least upper bound in P.*

Proof. Say P is a CPO, and let $C \subseteq P$ be a chain , then $\forall\, x, y \in C$, either $x \leq y$ or $y \leq x$, therefore either $x \in \{x, y\}^u$ or $y \in \{x, y\}^u$. which implies C is directed in P. Then by definition, $\bigsqcup C$ exists in P.

To prove the converse is very complicated, I just do it for the case that P is a countable partially ordered set.

Consider $\emptyset \subseteq P$, then $\emptyset$ is a chain in P. By hypothesis, $\emptyset$ has a least upper bound in P, but $\emptyset^u = P$, therefore $\exists \perp \in P$ such that $\forall\, x \in P$, $x \geq \perp$.

Now we shall prove that for every directed subset D of P, $\bigsqcup D$ exists in P.

Suppose $D = \{x_0, x_1, \ldots, x_n, \ldots\}$ is a directed subset of P, then for each finite subset F of D, we fix an upper bound u_F of F in D.

Define sets D_i as follows:

$$D_0 = \{x_0\}, \quad D_{i+1} = D_i \bigcup \{y_{i+1},\, u_{D_i \bigcup \{y_{i+1}\}}\}$$

where y_{i+1} is the element x_n in $D \setminus D_i$, with the subscript n chosen as small as possible.

Claim 1. D_i has at least i elements

Proof. $D_0 = x_0$ implies D_0 has at least 0 elements. Suppose that D_k has at least k elements, consider $D_{k+1} = D_k \bigcup \{y_{k+1}, u_{\{D_k \bigcup \{y_{k+1}\}\}}\}$. Since D is an infinite set and D_k is finite, $D \setminus D_k \neq \emptyset$, therefore $y_{k+1} \in D \setminus D_k$. Hence, D_{k+1} has at least $k+1$ elements. By induction, D_i has at least i elements. □

Claim 2. D_i is directed

Proof. Consider $D_0 = x_0$, it is directed. Now suppose that D_k is directed, consider D_{k+1}.
Let $x, y \in D_{k+1}$.
Case 1: $x, y \in D_k$, then $\exists z \in \{x, y\}^u$ such that $z \in D_k \subseteq D_{k+1}$.
Case 2: $y = y_{k+1}$, then $u_{D_k \bigcup \{y_{k+1}\}}$ is an upper bound for D_{k+1}.
Case 3: $y = u_{D_k \bigcup \{y_{k+1}\}}$.
To sum up, D_i is directed. □

Claim 3. $\bigvee D_i$ exists in P

Proof. Consider D_{k+1} for $k \geq 0$, then $D_{k+1} = D_k \bigcup \{y_{k+1}, u_{D_k \bigcup \{y_{k+1}\}}\}$. Hence, $u_{D_k \bigcup \{y_{k+1}\}}$ is an upper bound for $D_k \bigcup \{y_{k+1}\}$ and $u_{D_k \bigcup \{y_{k+1}\}} \leq u_{D_k \bigcup \{y_{k+1}\}}$.
Therefore $u_{D_k \bigcup \{y_{k+1}\}} = \bigvee D_i$ □

Claim 4. $\{\bigvee D_i\}_{i \geq 1}$ form a chain in P.

Proof. $D_m \subseteq D_n$ for all $m \leq n$, therefore, $\bigvee D_m \leq \bigvee D_n$. □

Claim 5. $\sqcup \{\vee D_i\}_{i \geq 1}$ is an upper bound for D.

Proof. We first show that $x_i \in D_N, \forall i \leq N$.
$x_0 \in D_0$, hence this is true for D_0.
Now, suppose it is ture for $n = k$. Consider $n = k + 1$
$D_{k+1} = D_k \bigcup \{y_{k+1}, u_{D_k \bigcup \{y_{k+1}\}}\}$, but $x_i = y_{k+1} \in D \setminus D_k$ and x_i has the smallest index. Therefore $i \geq k + 1$.
Case 1: $i \geq k + 1$, then $x_{k+1} \in D_k \subseteq D_{k+1}$.
Case 2: $i = k + 1$, then $x_i \in D_{k+1}$
Therefore, by induction $x_i \in D_N$ for all $i \leq N$.
Now, let $x_n \in D$, then $x_n \in D_i \subseteq D, \forall i \geq n$

$\therefore x_n \leq \vee D_i \leq \bigsqcup \{\bigvee D_i\}_{i \geq 1} \forall n \in \mathbb{N}_0$.
But, $\vee D_i \in D$ by definition, therefore $\bigvee D_i \subseteq D$ implies $\bigsqcup \{\bigvee D_i\} \leq x, \forall x \in D^u$
$\therefore \bigsqcup \{\bigvee D_i\}_{i \geq 1} = \bigsqcup D$

□

Hence, P is a CPO. □

Definition 3.10. Let P be a partially ordered set, $F : P \to P$ be a self-map on P. Then $x \in P$ is a fixpoint of F if $F(x) = x$. It is a pre-fixpoint if $F(x) \leq x$, a post-fixpoint $x \leq F(x)$.

We use $\mu(F)$ to denote the least fixpoint of F and $\nu(F)$ to denote the greatest fixpoint.

Definition 3.11. Let P be a CPO, Y be a subset of P and let $F : P \to P$ be a self-map on P. Then F is said to be increasing if for all $x \in P$, $x \leq F(x)$. Y is F- invariant if $F(Y) \subseteq Y$.

Lemma 3.12. *Let P be a CPO, $F : P \to P$ be self-map on P, then there exists a $F-$ invariant subCPO P_0 of P such that for all $F-$ invariant subCPO P_α of P, we have $P_0 \subseteq P_\alpha$.*

Proof. Let C be the collection of all $F-invariant$ subCPO of P, since P itself is a $F-invariant$ subCPO of P, $P \in C$, hence $C \neq \emptyset$.

Let $P_\alpha \in C$, then consider $P_0 = \bigcap_{\alpha \in \Lambda} P_\alpha$ where Λ is an index set. We shall prove that it is a $F-invariant$ subCPO of P.

P_α is a subCPO of P, $\forall \alpha \in \Lambda$, then $\perp \in P_\alpha$ for all α, hence $\perp \in P_0$. Let D be a directed subset in P_0, then D is directed in P_α for all α and it is directed in P. Therefore $\sqcup_p D$ exists. And since P_α is a subCPO, we have $\sqcup_p D \in P_\alpha$ for all $\alpha \in \Lambda$. Therefore $\sqcup_p D \in P_0$. Hence P_0 is an $F-invariant$ subCPO of P. And $P_0 \subseteq P_\alpha$ for all $\alpha \in \Lambda$ by definition. □

Fixpoint theorem One 3.13. Let P be a CPO, let F be an order-preserving self-map on P and let $\alpha = \sqcup_{n \geq 0} F^n(\perp)$.

(1) If $\alpha \in fix(F)$, then $\alpha = \mu(F)$.
(2) If F is continuous, then $\mu(F)$ exists and equals α.

Proof. (1) Since F is order-preserving self-map on P, $\perp \leq F(\perp)$. Applying F^n to $\perp$, we have $F^n(\perp) \leq F^{n+1}(\perp)$ for all $n \in \mathbb{N}$. Then we get a chain

$$\perp \leq F(\perp) \leq F^2(\perp) \ldots \leq F^n(\perp) \ldots$$

in P. Since P is a CPO, $\alpha = \bigsqcup_{n \geq 0} F^n(\perp)$ exists in P. Suppose x_0 be any fixpoint of F, then $F^n(X_0) = x_0$ for all $n \in \mathbb{N}_0$. And $\perp \leq x_0 \Rightarrow F^n(\perp) \leq F^n(x_0)$ since F is order-preserving. Therefore $\alpha \leq F^n(x_0) = x_0 \; \forall n \in \mathbb{N}_0$. Hence, if α is a fixpoint then it is the least fixpoint.

(2) Consider $F(\alpha) = F(\bigsqcup_{n \geq 0} F^n(\perp))$. Since F is continuous, $F(\alpha) = \bigsqcup_{n \geq 0} F(F^n(\perp)) = \bigsqcup_{n \geq 1} F^n(\perp)$. But $\perp \leq F^n(\perp) \;\; \forall n \in \mathbb{N}_0$, therefore $\bigsqcup_{n \geq 1} F^n(\perp) = \bigsqcup_{n \geq 0} F^n(\perp) = \alpha$. Hence α is a fixpoint, then by (1), it is the least fixpoint. $\square$

Theorem 3.14. *Let P be a partially ordered set and F be an order-preserving self-map on P.*

(1) *Suppose F has a least pre-fixpoint $\mu_*(F)$. Then F has a least fixpoint, and $F(x) \leq x$ implies $\mu(F) \leq x$. Also $\mu(F) = \mu_*(F)$.*
(2) *Suppose P is a complete lattice, then $\mu_*(F)$ exists.*

Proof. (1) Suppose that $\mu_*(F)$ exists. Now consider $F(\mu_*(F))$. Since $\mu_*(F)$ is a pre-fixpoint, $F(\mu_*(F)) \leq \mu_*(F)$. And F is order-preserving implies $F(F(\mu_*(F))) \leq F(\mu_*(\mathrm{F}))$. Therefore, $F(\mu_*(F))$ is also a pre-fixpoint, hence we have $\mu_*(F) \leq F(\mu_*(F))$. Hence, $\mu_*(F) = F(\mu_*(F)) \Rightarrow \mu_*(F)$ is a fixpoint. $fix(F) \subseteq pre(F)$, we have $\mu(F) = \mu_*(F)$.

(2) Suppose P is a complete lattice, then consider $\wedge pre(F)$, since F is order-preserving, $F(\wedge pre(F)) \leq F(y) \leq y \;\; \forall y \in pre(F)$. Therefore, $F(\wedge pre(F)) \leq \wedge pre(F)$, hence $\wedge pre(F) \in pre(F)$ and it is the least pre-fixpoint. Then by (1), $\wedge pre(F) = \mu(F)$. $\square$

Lemma 3.15. *Let P be a CPO, then the increasing order-preserving self-maps on P have a common fixpoint.*

Proof. Let $I(P)$ be the set of all increasing order-preserving self-map on P. Since $id_p \in I(P)$, $I(P) \neq \emptyset$. Let $F, G \in I(P)$ and $x \in P$. Then

$F(x) \leq F(G(x))$ since G is increasing and F is order-preserving. And $G(x) \leq F(G(x))$ since F is increasing. Therefore $F \circ G$ is an upper bound for $\{F, G\}$ in $I(P)$. Hence $I(P)$ is a directed subset of the CPO $< P \rightarrow P >$ of all order-preserving self-maps on P.

Let $H = \sqcup I(P)$ in $< P \rightarrow P >$. Then H is an order-preserving self-map on P. Let $x \in P$, consider $H(x)$. For all $F \in I(P)$, we have $x \leq F(x)$. And $F \leq H$ implies $F(x) \leq H(x)$. Hence, $x \leq H(x)$, therefore $H \in I(P)$. Also $F \circ H$ is in $I(P)$, therefore $F \circ H \leq H$, but F is increasing implies $H \leq F \circ H$. Hence $H = F \circ H$.

Now, consider $x \in P$, then $H(x) = F(H(x))$, therefore $H(x) \in P$ is a fixpoint for F for all $x \in P$. $\square$

Fixpoint theorem Two 3.16. Let P be a CPO and let $F : P \rightarrow P$ be order-preserving. Then F has a least fixpoint.

Proof. Define $\Phi : \wp(P) \rightarrow \wp(P)$ such that $\Phi(X) = \{\perp\} \bigcup F(X)\{\bigsqcup D | D \subseteq X \textit{ and } D \textit{ is directed}\}$ for all $X \subseteq P$.

Now, consider $Y \subseteq X \subseteq P$, then $F(Y) \subseteq F(X)$ and $\{\bigsqcup D \mid D \subseteq Y \textit{ and } D \textit{ is directed}\} \subseteq \{\bigsqcup D \mid D \subseteq X \textit{ and } D \textit{ is directed}\}$. Therefore, Φ is order-preserving, hence by Knaster-Tarski fixpoint theorem, it has a least fixpoint given by $P_0 = \bigcap\{X \in \wp(P) | \Phi(X) \subseteq X\}$. By definition, this is the smallest $F - invariant$ subCPO of P.

Setp 1: show that $P_0 \subseteq post(F)$.

Let $Q = post(F)$. Clearly, $\perp \in Q$. And since F is order-preserving, $F(Q) \subseteq Q$. Let D be a directed subset of Q, then $\bigsqcup F(D) \leq F(\bigsqcup D)$, since $x \leq F(x)$ for all $x \in D$, we have $\bigsqcup D \leq \bigsqcup F(D)$, hence $\bigsqcup D \leq F(\bigsqcup D)$. Therefore $\bigsqcup D \in post(P)$. Then we have $\Phi(Q) \subseteq Q$.

Step 2: If $x \in P$ is a fixpoint, then $P_0 \subseteq \downarrow x$.

Let x be a fixpoint, consider $\downarrow x$, let $y \in \downarrow x$, then $y \leq x \Rightarrow F(y) \leq F(x) = x$, then $F(y) \in \downarrow x$, hence $\downarrow x$ is $F - invariant$. And it is trivial that $\downarrow x$ is a subCPO of P. Thus $P_0 \subseteq \downarrow x$.

Define $G = F|_{P_0} : P_0 \rightarrow P_0$, Since P_0 is also a CPO, then $\exists a \in P_0$ such that $G(a) = a$, hence $F(a) = a$. We need to show that a is both the top of P_0 and the least element of F.

Suppose that $x \in P$ is a fixpoint, then $\Phi(\downarrow x) \subseteq \downarrow x$. Then we have $P_0 = \mu(\Phi) \subseteq \downarrow x$. Since $a \in P_0$ and $P_0 \subseteq \downarrow x$, we have $a \leq x$. □

Theorem(Fixpoint Theorem Three) 3.17. Let P be a CPO and let F be an increasing self-map on P. Then F has a fixpoint.

Proof. Let P_0 be the smallest $F - invariant$ subCPO of P, a element x_0 in P_0 is called a roof of F if for all $y \in P_0, y \leq x$, $F(y) \leq x$.
Define $Z_x = \{y \in P_0 | y \leq x \text{ } or \text{ } F(x) \leq y\}$.

Claim 1. $Z_x = P_0$

Proof. First we show that Z_x is a subCPO of P.
Consider $\perp \in P$, since P_0 is a subCPO, $\perp \in P_0$ and for all $x \in P$, $\perp \leq x$, hence $\perp \in Z_x$.
Let D be a directed subset in Z_x, then D is directed in P and P_0. Therefore $\bigcup_p D$ exists and it is in P_0 since P_0 is a subCPO.
Case 1: $\forall d \in D, d \leq x$, then x is an upper bound for D, therefore $\bigsqcup_p D \leq x$ since $\bigsqcup_p D = \sup(D)$.
Case 2: $\exists d_0 \in D$ such that $d_0 \nleq x$, then $d_0 \in Z_x$ implies $F(x) \leq d_0 \leq \bigsqcup_p D$ and $\bigsqcup_p D \in P_0$, therefore $\bigsqcup_p D \in Z_x$.
To sum up, $\bigsqcup_p D \in Z_x$ and $\perp \in Z_x$ imply Z_x is a subCPO of P. □

Claim 2. Z_x is $F - invariant$.

Proof. Let $y \in Z_x$, consider $F(y)$.
Since P_0 is $F - invariant$, $F(x)$ is in P_0 for all $x \in P_0$, especially for x being a roof element. Then
Case 1: $x = y$, then $F(x) \leq F(y)$, therefore $F(y) \in Z_x$.
Case 2: $x \neq y$, then $y \in Z_x$ implies $y < x$ or $F(x) \leq y$. If $y < x$, then since F is increasing, we have $y \leq F(y)$, therefore $F(x) \leq y \leq F(y)$, hence $F(y) \in Z_x$.
Therefore, Z_x is $F - invariant$.
We proved that Z_x is an $F - invariant$ subCPO of P and $Z_x \subseteq P_0$, but P_0 is the smallest $F - invariant$ subCPO of P, therefore $Z_x = P_0$. □

Claim 3. Define $Z = \{x \in P_0 | x \text{ } is \text{ } a \text{ } roof\}$. Since $\perp \in Z$, Z is not empty. Then Z is $F - invariant$.

Proof. Take $y \in P_0$ and $x \in Z$. Then $P_0 = Z_x$, hence we have either $y \leq x$ or $F(x) \leq y$.
Suppose $y < x$, then $F(y) \leq x \leq F(x)$, hence $F(x)$ is a roof. Therefore, Z is $F-invariant$. ☐

Claim 4. Z is a subCPO of P

Proof. Let $x, y \in Z$, then x, y are both roof elements, therefore in particular, $Z_x = P_0$, since $y \in P_0$, we have either $y \leq x$ or $x \leq F(x) \leq F(y)$, therefore Z is a chain. Hence $\bigsqcup_p Z$ exists in P_0.
Let D be a directed subset in Z, we need to show that $\bigsqcup_P D$ is in Z. Since P is a CPO and D is directed in P, we have $\bigsqcup_P D$ exists, moreover, P_0 is a subCPO $\Rightarrow \bigsqcup_P D$ is in P_0.
So suppose D is non-empty, and let $y \in P_0$ such that $y < \bigsqcup_P D$, then for all $x \in Z$, we have either $y \leq x$ or $F(x) \leq y$.
Case 1: there exists $x \in Z$ such that $y \leq x$, then $F(y) \leq x \leq \bigsqcup_P D$.
Case 2: for all $x \in Z$, we have $F(x) \leq y$, then $x \leq F(x) \leq y$ implies y is an upper bound for D, hence $\bigsqcup_P D \leq y$ ⋇.
Therefore, $\bigsqcup_P D$ is a roof element, then it is in Z.
Thus, Z is a subCPO of P. Therefore, $Z = P_0$. Hence P_0 is also a chain. ☐

Claim 5. P_0 has a top element.

Proof. P is a CPO and P_0 is a chain imply $\bigsqcup_p P_0$ exists in P. Also P_0 is a directed subset of P_0, then $\exists \top_{p_0} \in P_0$. And $\top_{p_0} = \bigsqcup_p P_0$. ☐

Now, P_0 is $F-invariant$
$\Rightarrow F(P_0) \subseteq P_0$
$\Rightarrow F(\top_{P_0}) \in P_0$
But $\top_{P_0} \leq F(\top_{P_0}) \leq \top_{P_0}$
$\therefore \top_{P_0} = F(\top_{P_0})$, hence it is a fixpoint of F. ☐

Note that: in this case we can not claim that F has a least fixpoint, in fact, not even a minimal fixpoint. For example, consider the set $P = 1 \oplus \mathbb{N}^{\partial}$ and $F : P \to P$ such that $F(\perp) = 0$, and $F(x) = x \ \forall x \in \mathbb{N}$.

The Knaster-Tarski Theorem 3.18. Let L be a complete lattice and $F : L \to L$ be an order-preserving self-map on P. Then $\alpha = \bigvee\{x \in L | x \le F(x)\}$ is the greatest fixpoint of F. Dually, F has a least fixpoint given by $\bigwedge\{x \in L | F(x) \le x\}$.

Proof. Let $H = \{x \in L |\ x \le F(x)\}$. Then for all $x \in H$, we have $x \le F(x) \le F(\alpha)$, thus $F(\alpha)$ is a upper bound for H, and $\alpha \le F(\alpha)$ since α is the least upper bound.

Now, F is order-preserving, then $F(\alpha) \le F(F(\alpha))$, which implies that $F(\alpha) \in H$, hence $F(\alpha) \le \alpha$. Therefore $F(\alpha) = \alpha$, then α is a fixpoint. Suppose x_0 is a fixpoint of F, then $x_0 \le F(x_0) = x_0$, therefore x_0 is in H. Hence $x_0 \le \alpha$.

The proof for the Dual statement is similar. □

Theorem 3.19. *Let P be an partially ordered set.*

(1) *If P is a lattice and every order-preserving map $F : P \to P$ has a fixpoint, then P is a complete lattice.*
(2) *If every order-preserving map $F : P \to P$ has a least fixpoint, then P is a CPO.*

The proof for this theorem is quite complicated, it can be found in [2].

4. CPOs and Topology

Definition 4.1. Let X be a set and τ be a collection of subsets of X such that

(1) $X, \phi \in \tau$.
(2) arbitrary union of elements of τ is in τ.
(3) finite intersection of elements of τ is in τ.

Then τ is called a topology on X and X is a topological space. A subset S of X in τ is said to be open, a subset F of X is said to be closed if $\exists S \subseteq X$ and $S \in \tau$ such that $F = X \setminus S$.

Definition 4.2. Let X, Y be topological spaces, and let $f : X \to Y$ be a map, then f is said to be continuous if $f^{-1}(F)$ is closed in X for all closed subsets F of Y.

Theorem 4.3. *Let P be a CPO. Let $\mathcal{F}$ be a collection of subsets U of P such that $U \in O(P)$ and $\bigsqcup D \in U$ whenever D is a directed subset of U. Then*

(1) *$\mathcal{F}$ is a topology of X.*
(2) *Let P and Q be topologized as above. Then the map $\varphi : P \to Q$ is topologically continuous if and only if it is continuous in the CPO sense.*

Proof. Clearly, $\emptyset \in \mathcal{F}$. Consider $\emptyset \neq D \subseteq U$, let D be directed in U.

$P \in O(P)$ is trivial. Let $D \subseteq P$ be directed, since P is a CPO, $\sqcup D$ exists in P, hence $P \in \mathcal{F}$.

Let Λ be an index set and let $U_\alpha \in \mathcal{F}$ for all $\alpha \in \Lambda$. Consider $\bigcap = \bigcap_{\alpha\in\Lambda} U_\alpha$.

Let $x \in \bigcap$, $y \in P$ and $y < x$. Then

$x \in \bigcap \Rightarrow \forall\alpha \in \Lambda\,, x \in U_\alpha$

$\Rightarrow y \in U_\alpha$ $\forall\alpha \in \Lambda$ since U_α are down-sets.

Hence, $y \in \bigcap$. Therefore $\bigcap \in O(P)$.

Let $D \subseteq \bigcap$ be directed, then

$D \subseteq U_\alpha \ \forall\alpha \in \Lambda \Rightarrow \bigsqcup D \in U_\alpha \ \forall\alpha \ \Rightarrow \bigsqcup D \in \bigcap$.

Hence, arbitrary intersections of U_α is still in $\mathcal{F}$. Now we shall prove that finite union of U_n is still in $\mathcal{F}$.

Let $x \in \bigcup_{n=1}^{N}$ and $y \leq x$, where $N \in \mathbb{N}$.

$\Rightarrow \exists k \in \{1, 2, \ldots, N\}$ such that $x \in U_k \Rightarrow \ y \in U_k \ \Rightarrow \ y \in \bigcup_{n=1}^{N} U_n$.

Hence, $\bigcup_{n=1}^{N} U_n$ is a down-set.

Let $D \subseteq \bigcup_{n=1}^{N} U_n$.

Case 1: $\exists k \in \{1, 2, \ldots, N\}$ such that $D \subseteq U_k$, then $\bigsqcup D \in \bigcup_{n=1}^{N} U_n$.

Case 2: $\exists k_1, k_2$ such that $U_{k_1} \bigcap D \neq \emptyset$ and $U_{k_2} \bigcap D \neq \emptyset$.

We shall prove that the second case is impossible.

Consider $D \subseteq U_1 \bigcup U_2 = U_1 \bigcup (U_2 \setminus (U_1 \bigcap U_2))$, $x, y \in U_1 \bigcup U_2$ and $z \in \{x, y\}^u$.

Consider $x, y \in U_1$

Case 1: $z \in U_1$.

Case 2: $z \in U_2 \ \Rightarrow x, y \in U_1 \bigcap U_2 \ \Rightarrow x, y \in U_2$.

Consider $x, y \in U_2 \setminus (U_1 \bigcap U_2)$.

Case 1: $z \in U_1 \Rightarrow x, y \in U_1$ ⁕.

Case 2: $z \in U_2 \Rightarrow x, y, z \in U_2$.

If $x \in U_1$ and $y \in U_2 \setminus (U_1 \bigcap U_2)$.

Case 1: $z \in U_1 \Rightarrow y \in U_1$ ⁕.

Case 2: $z \in U_2 \Rightarrow x \in U_2 \Rightarrow x \in U_1 \bigcap U_2$.

To sum up, $\forall x, y \in D$, we have either $x, y \in U_1$ or $x, y \in U_2$.

$\therefore$ either $D \subseteq U_1$ or $D \subseteq U_2$.

$\therefore$ $U_1 \bigcup U_2 \in \mathcal{F}$.

Suppose the above statement is true for $\bigcup_{n=1}^{k} U_n$ for any $U_n \in \mathcal{F}$, consider $\bigcup_{n=1}^{k+1} U_n$.

Let D be a directed subset of $\bigcup_{n=1}^{k+1} U_n$, since $\bigcup_{n=1}^{k} U_n$ and U_{k+1} are both in $\mathcal{F}$, by induction hypothesis, we have either $D \subseteq \bigcup_{n=1}^{k} U_n$ or $D \subseteq U_{k+1}$, therefore $\bigsqcup D \in \bigcup_{n=1}^{k+1} U_n$, which implies it is in $\mathcal{F}$.

Hence, finite unions of U_n is still in $\mathcal{F}$.

Therefore, $\mathcal{F}$ is a topology on P. In fact, it is called the Scott topology.

Now we prove the second theorem.

Say $\varphi : P \to Q$ is topologically continuous.

Let $x, y \in P$ such that $x \leq y$. And $\varphi(x), \varphi(y) \in Q$.

Consider $\downarrow \varphi(y) \subseteq Q$. It is a down-set and let D be a directed subset of it, then by definition $\varphi(y)$ is an upper bound for D, hence $\bigsqcup D \leq \varphi(y) \Rightarrow \bigsqcup D \in \downarrow \varphi(y)$. Therefore, it is in $\mathcal{F}_Q$.

Since φ is topologically continuous, $\varphi^{-1}(\varphi(y)) \in \mathcal{F}_{\mathcal{P}}$.

$y \in \varphi^{-1}(\varphi(y)) \Rightarrow x \leq y \in \varphi^{-1}(\varphi(y))$.

$\Rightarrow \varphi(x) \in \downarrow \varphi(y)$

$\Rightarrow \varphi(x) \leq \varphi(y)$

$\therefore \varphi$ is order-preserving.

Let $S \subseteq P$ be directed, since φ is order-preserving, we have $\varphi(S)$ is directed in Q and $\bigsqcup \varphi(S) \leq \varphi(\bigsqcup S)$.

Now consider $S \subseteq F = \varphi^{-1}(\downarrow \bigsqcup \varphi(S)) \subseteq P$, if F is empty, then S is also empty, hence $\bigsqcup \varphi(\emptyset) = \varphi(\bigsqcup \emptyset) = \perp_Q$.

If F is non-empty, then F is in $\mathcal{F}_{\mathcal{P}}$ since φ is topologically continuous, then $\bigsqcup S$ exists in F. Hence, $\varphi(\bigsqcup S) \in \downarrow \bigsqcup \varphi(S)$.

Therefore $\varphi(\bigsqcup S) \leq \bigsqcup \varphi(S) \Rightarrow \varphi(\bigsqcup S) = \bigsqcup \varphi(S)$.

Hence φ is continuous in the CPO sense.

Conversely, say φ is continuous in the CPO sense, then φ is order-preserving.

Let $U \in \mathcal{F}_Q$, consider $\varphi^{-1}(U) \subseteq P$.

Let $x \in \varphi^{-1}(U)$, $y \in P$ and $y \leq x$. Since φ is order-preserving, we have $\varphi(y) \leq \varphi(x)$. But $\varphi(x) \in U \in O(Q)$, therefore $\varphi(y) \in U$, hence $y \in \varphi^{-1}(U)$.

Therefore $\varphi^{-1}(U)$ is a down-set.

Let D be a directed subset of $\varphi^{-1}(U)$. Consider $\varphi(\bigsqcup D) = \bigsqcup \varphi(D)$.

Since $D \subseteq \varphi^{-1}(U)$ and D is directed, we have $\varphi(D) \subseteq U$ and $\varphi(D)$ is directed. Therefore $\bigsqcup \varphi(D)$ exists in U. Hence, $\bigsqcup D$ exists in $\varphi^{-1}(U)$.

Therefore, $\varphi^{-1}(U) \in \mathcal{F}_{\mathcal{P}}$. So φ is topologically continuous. $\square$

References

[1] Introduction to lattices and order, 2nd Edition, B. A. Davey and H. A. Priestley.

[2] Chain-complete posets and directed sets with applications, Algebra Universalis 6 (1976), 53-68. G. Markowsky.