Partial orders and Posets

A relation $\leq$ is called a partial order on a set S if

1. $x \leq x\ , \forall x \in S$.
2. $x \leq y$ and $y \leq x$ imply $x = y$.
3. $x \leq y$ and $y \leq z$ imply $x \leq z$.

$(S, \leq)$ is called a partially ordered set or a poset.

S is called a chain if $\forall x\ , y \in S$ we have either $x \leq y$ or $y \leq x$.

S is called an antichain if $x \leq y \Rightarrow x = y \ \ \forall x, y \in S$.

Lattices and Complete lattices

Let P be a partially ordered set, then P is a lattice if $\forall x, y \in P$, $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist in P.

P is a complete lattice if $\forall S \subseteq P$, $\bigvee S$ and $\bigwedge S$ exist in P.

$\phi \neq M \subseteq P$ is a sublattice of P if $\forall a, b \in M$, $a \vee b$ and $a \wedge b$ are also in M.

$\top$ is the top element of P if $\forall x \in P,\ x \leq \top$. $\bot$ is the bottom of P if $\forall x \in P,\ x \geq \bot$.

Order-preserving maps

Let $P,\ Q$ be partially ordered sets, a map $\varphi : P \to Q$ is order-preserving if

$$x \leq y \in P \Rightarrow \varphi(x) \leq \varphi(y) \in Q$$

If φ is from P to itself then φ is called a self-map on P.

Fixpoint: given a partially ordered set P and F is a self-map on P, then $x \in P$ is a fixpoint for F if $F(x) = x$.

Knaster-Tarski Theorem

Let L be a complete lattice and $F : L \longrightarrow L$ an order-preserving self-map on L. Then

$$\alpha = \bigvee \{x \in L | x \leq F(x)\}$$

is a fixpoint of F. Futher it is the greatest fixpoint of F.

$$\beta = \bigwedge \{x \in L | x \geq F(x)\}$$

is the least fixpoint of F.

CPOs

Directed sets: Let P be a partially ordered set, $\phi \neq S \subseteq P$, then S is directed if $\forall x, y \in S$, $\exists z \in S$ s.t. $z \in \{x, y\}^u$.

CPOs: Let P be a partially ordered set, then P is a CPO if

1. P has a bottom element $\perp$.
2. $\bigsqcup D$ exists for each directed subset D of P.

SubCPOs: Let P be a CPO and $Q \subseteq P$, then Q is a subCPO of P if

1. $\perp \in Q$.
2. for all directed subset D of Q, $\bigsqcup_P D \in Q$.

Continuous maps

Definition : Let P, Q be CPOs, then the map $\varphi : P \longrightarrow Q$ is continuous if for all directed subset D of P, $\varphi(D)$ is directed in Q and $\varphi(\bigsqcup D) = \bigsqcup \varphi(D)$.

Lemma: Let P, Q be CPOs and $\varphi : P \longrightarrow Q$, then φ is order-preserving iff for all directed set D in P, we have $\bigsqcup \varphi(D) \leq \varphi(\bigsqcup D)$.

Corollary: If φ is continuous then it is order-preserving.

Fixpoint Theorems

1. Let P be a CPO, F an order-preserving self-map on P, define $\alpha = \bigsqcup_{n \geq 0} F^n(\bot)$. Then
   (a) if α is a fixpoint for F, then it is the least fixpoint.
   (b) if F is continuous, then it has the least fixpoint which equals to α.
2. Let P be a CPO and F an order-preserving self-map on P. Then F has a least fixpoint.

Fixpoint Theorem Cont.

Increasing maps: Let P be a CPO, then $F : P \longrightarrow P$ is increasing if

$$\forall x \in P, x \leq F(x)$$

Theorem: Let P be a CPO and F an increasing self-map on P, then F has a fixpoint.

Note: Here we can not say that F has a least fixpoint, in fact not even a minimal fixpoint.

Interesting results

Theorem: Let P be an partially ordered set. Then

1. If P is a lattice and every order-preserving map $F : P \longrightarrow P$ has a fixpoint, then P is complete.
2. If every order-preserving map $F : P \longrightarrow P$ has a least fixpoint, then P is a CPO.

Scott Topology

Let P be a CPO, let $\mathcal{F}$ be the collection of sets $U \in O(P)$ such that $\bigsqcup D \in U$ whenever D is a directed subset of U. Then $\mathcal{F}$ is a topology on P and $U \in \mathcal{F}$ are closed sets.

Theorem: Let P, Q be CPOs and topplogized as above, then the map $\varphi : P \longrightarrow Q$ is topologically continuous iff it is continuous in the CPO sense.