Notes (Rev 1.1): Complex Sesqui-linear Tensor Conventions, Wedge Product Operations, Exterior, and other Derivative Conventions and Notations Used

Shane W. D.

10/3/2025

Abstract

Minimal notes regarding differential geometric definition of a complex sesqui-linear tensor product and wedge product operation, and definitions, and general formulas for exterior derivatives and other derivative definitions, including outcomes under under "Hermitian"' constraints.

Sesquilinear $\curlywedge$ and $\circledast$

We distinguish notations:

$$\omega^\mu \wedge \omega^\nu \quad \text{(standard wedge)}, \omega^\mu \curlywedge \omega^\nu \quad \text{(sesquilinear wedge)},$$

$$\omega^\mu \otimes \omega^\nu \quad \text{(standard tensor product)}, \qquad \omega^\mu \circledast \omega^\nu \quad \text{(sesquilinear tensor product)}.$$

Two-form translation .

$$\boxed{\omega^\mu \curlywedge \omega^\nu \quad \omega^\mu \wedge \omega^{\nu *}}$$

Two-legged object for the tensor product (analogous).

$$\boxed{\omega^\mu \circledast \omega^\nu \quad \omega^\mu \otimes \omega^{\nu *}}$$

Sesquilinearity on two-legged objects For scalars $a, b \in \mathbb{C}$ and one-forms α, β:

$$(a\,\alpha) \curlywedge (b\,\beta) = ab * (\alpha \curlywedge \beta), \qquad (a\,\alpha) \circledast (b\,\beta) = ab * (\alpha \circledast \beta).$$

The standard notation, in order to examine the operation that includes complexification explicitly, will be taken as "ignoring" the complex operations such as complex conjugation by default. It is not to say that it 'has to' ignore it i n general, but that we wish to examine the difference between 'ignoring' it and not. So we stick to that assumption within this context.

Three-legged $\curlywedge$ object

Define with alternating conjugation:

$$\alpha \curlywedge \beta \curlywedge \gamma \quad \leftrightarrow \quad \alpha \wedge \beta^* \wedge \gamma,$$

linear in slots 1 and 3, conjugate-linear in slot 2. Coefficient rule:

$$(a\alpha) \curlywedge (b\beta) \curlywedge (c\gamma) = (a\, b^*\, c)\, (\alpha \curlywedge \beta \curlywedge \gamma).$$

Associativity in the Sesquilinear Wedge

Issue. Binary-only $\curlywedge$ breaks associativity:

$$(\alpha \curlywedge \beta) \curlywedge \gamma \;\longleftrightarrow\; (\alpha \wedge \beta^*) \wedge \gamma, \qquad \alpha \curlywedge (\beta \curlywedge \gamma) \;\longleftrightarrow\; \alpha \wedge (\beta \wedge \gamma^*).$$

Rescue rule (alternating conjugation). Define the 3-ary operation by

$$\alpha \curlywedge \beta \curlywedge \gamma := \alpha \wedge \beta^* \wedge \gamma,$$

and re-bracket via

$$\alpha \curlywedge (\beta \curlywedge \gamma) \;\mapsto\; \alpha \wedge (\beta \curlywedge \gamma)^* \;\equiv\; \alpha \wedge \beta^* \wedge \gamma.$$

Result.

$$(\alpha \curlywedge \beta) \curlywedge \gamma \;\equiv\; \alpha \curlywedge (\beta \curlywedge \gamma) \;\equiv\; \alpha \curlywedge \beta \curlywedge \gamma,$$

so associativity holds *up to the alternating-slot conjugation convention.*

Sesquilinear Tensor Product $\circledast$

Binary translation.

$$\alpha \circledast \beta \;\;\leftrightarrow\;\; \alpha \otimes \beta^*.$$

n-ary definition (even-slot conjugation). For one-forms $\alpha_1, \ldots, \alpha_n$,

$$\alpha_1 \circledast \cdots \circledast \alpha_n \;:=\; \alpha_1 \otimes \alpha_2^* \otimes \alpha_3 \otimes \alpha_4^* \otimes \cdots \quad \text{(stars on slots } 2, 4, 6, \ldots).$$

Coefficient pull-out. For scalars $a_k \in \mathbb{C}$,

$$(a_1\alpha_1) \circledast \cdots \circledast (a_n\alpha_n) \;=\; \big(a_1 a_2^* a_3 a_4^* \cdots\big)\; \alpha_1 \circledast \cdots \circledast \alpha_n.$$

Re-bracketing (associativity convention). Any parenthesized product is reinterpreted via the n-ary rule above:

$$(\cdots((\alpha_1 \circledast \alpha_2) \circledast \alpha_3) \cdots \circledast \alpha_n) \;\equiv\; \alpha_1 \circledast \cdots \circledast \alpha_n.$$

Conjugation toggle.

$$(\alpha_1 \circledast \cdots \circledast \alpha_n)^* \;\leftrightarrow\; \alpha_1^* \otimes \alpha_2 \otimes \alpha_3^* \otimes \alpha_4 \otimes \cdots \quad \text{(stars swap to the opposite parity slots).}$$

Basis example.

$$\omega^\mu \circledast \omega^\nu \circledast \omega^\rho \;\leftrightarrow\; \omega^\mu \otimes (\omega^\nu)^* \otimes \omega^\rho.$$

Derivatives of a Dyadic One-Form

We consider the sesquilinear dyad

$$\omega^\alpha \omega^{\dagger\beta} \;:=\; \omega^\alpha \circledast \omega^\beta,$$

and compare two rules for computing its exterior derivative.

Rule A (plain Leibniz for $\circledast$). By the dyadic analogy $X \circledast Y \sim XY^\dagger$, we take

$$\nabla_\gamma (X \circledast Y) \;\leftrightarrow\; (\nabla_\gamma X) Y \;+\; X \nabla_\gamma Y^\dagger.$$

Applying to $\omega^\alpha \circledast \omega^\beta$ gives

$$\nabla_\gamma(\omega^\alpha \circledast \omega^\beta) \leftrightarrow (\nabla_\gamma \omega^\alpha)\omega^{\beta\dagger} + \omega^\alpha \nabla_\gamma \omega^{\beta\dagger}.$$

By convention we will use the folowing definitions of the derivatives of basis one-form objects

$$\nabla_\gamma \omega^\mu = -\,\Lambda^{*\mu}{}_{\rho\gamma}\,\omega^\rho$$

$$\nabla_\gamma \omega^{\mu\dagger} = -\,\Lambda^{\mu}{}_{\rho\gamma}\,\omega^{\rho\dagger}$$

We next briefly evaluate the constructions of several items, including the covariant derivative of the dyadic object

$$\nabla_\gamma(\omega^\alpha\omega^{\dagger\beta}) \leftrightarrow -\,\Lambda^{*\alpha}{}_{\rho\gamma}\,\omega^\rho \circledast \omega^\beta \;-\; \Lambda^{\beta}{}_{\rho\gamma}\,\omega^\alpha \circledast \omega^\rho$$

a similar derivative on a symmetric portion of this object

$$\nabla_\gamma(\omega^\alpha \odot \omega^\beta)$$

with

$$\omega^\alpha \odot \omega^\beta := 2\omega^{(\alpha} \circledast \omega^{\beta)}$$

and derivatives of the exterior product

$$\nabla_\gamma(\omega^\alpha \curlywedge \omega^\beta)$$

and some other relevant objects.

0.1 Exterior Derivatives

Supposing we define the exterior derivative by a left acting rule $\mathbf{d}^L_\curlywedge \alpha \to \omega^\gamma \curlywedge \nabla_\gamma \alpha$, and assume α to be a 1-form and β to be a 1 form the exterior derivative of the exterior product of the two forms is

$$\begin{aligned}
\omega^\gamma \curlywedge \nabla_\gamma(\alpha \curlywedge \beta) &\leftrightarrow \omega^\gamma \curlywedge \nabla_\gamma(ab^*\omega^\alpha \curlywedge \omega^\beta) \\
&= \omega^\gamma \curlywedge a_{;\gamma}b^*\omega^\alpha \curlywedge \omega^\beta + \omega^\gamma \curlywedge ab^*_{;\gamma}\omega^\alpha \curlywedge \omega^\beta \\
&= \omega^\gamma \curlywedge a_{;\gamma}\alpha \curlywedge b\omega^\beta - a\omega^\alpha \curlywedge b_{;\gamma}\omega^\gamma \curlywedge \omega^\beta \\
&= \underbrace{\omega^\gamma \curlywedge a_{;\gamma}\omega^\alpha}_{\mathbf{d}\alpha} \curlywedge b\omega^\beta - a\omega^\alpha \curlywedge \underbrace{\omega^\gamma \curlywedge b_{;\gamma}\omega^\beta}_{\mathbf{d}\beta} \\
&\equiv \mathbf{d}_\curlywedge \alpha \curlywedge \beta - \alpha \curlywedge \mathbf{d}_\curlywedge \beta
\end{aligned}$$

In general as the degree of form for α and β are generalized

$$\boxed{\mathbf{d}_\curlywedge(\alpha_p \curlywedge \beta_q) = (\mathbf{d}_\curlywedge \alpha_p) \curlywedge \beta_q + (-1)^p\,\alpha_p \curlywedge (\mathbf{d}_\curlywedge \beta_q)} \qquad \text{with} \quad \mathbf{d}_\curlywedge := \omega^\gamma \curlywedge \nabla_\gamma.$$

In general we will always assume the left acting convention, and omit the "‘L"’ notation going forward.

0.1.1 Exterior Derivative of Basis One Form

$$\boxed{\mathbf{d}_L\omega^\alpha = \omega^\gamma \curlywedge \nabla_\gamma\omega^\alpha = -\,\Lambda^\alpha{}_{\rho\gamma}\,\omega^\gamma \curlywedge \omega^\rho = \,\Lambda^\alpha{}_{[\rho\gamma]}\,\omega^\rho \curlywedge \omega^\gamma}$$

0.1.2 A complementary type of derivative operation to the exterior derivative

We will define an additional derivative operation, which will be referred to informally as "co-terior" derivative

$$\mathbf{d}_\odot\alpha := \omega^\gamma \odot \nabla_\gamma\alpha$$

It acts analogously to the exterior derivative but extracts only the symmetric part of the sesquilinear derivative.

0.1.3 Exterior Derivative of $\alpha \odot \beta$

If we consider the case of α and β each to be one forms

$$\begin{aligned}
\omega^\gamma \curlywedge \nabla_\gamma(\alpha \odot \beta) &\leftrightarrow \omega^\gamma \curlywedge \nabla_\gamma(ab^*\omega^\alpha \odot \omega^\beta) \\
= \omega^\gamma \curlywedge a_{;\gamma}b^*\omega^\alpha \odot \omega^\beta &+ \omega^\gamma \curlywedge ab^*_{;\gamma}\omega^\alpha \odot \omega^\beta \\
= \omega^\gamma \curlywedge a_{;\gamma}\alpha \odot b\omega^\beta &- a\omega^\alpha \curlywedge b_{;\gamma}\omega^\gamma \odot \omega^\beta \\
= \underbrace{\omega^\gamma \curlywedge a_{;\gamma}\omega^\alpha}_{\mathbf{d}_\curlywedge\alpha} \odot b\omega^\beta &- a\omega^\alpha \curlywedge \underbrace{\omega^\gamma \odot b_{;\gamma}\omega^\beta}_{\mathbf{d}_\odot\beta} \\
&\equiv \mathbf{d}_\curlywedge\alpha \odot \beta - \alpha \curlywedge \mathbf{d}_\odot\beta
\end{aligned}$$

In general when α and β are extended to arbitrary p and q forms

$$\boxed{\mathbf{d}_\curlywedge(\alpha_p \odot \beta_q) = (\mathbf{d}_\curlywedge\alpha_p) \odot \beta_q \;+\; (-1)^p\,\alpha_p \curlywedge (\mathbf{d}_\odot\beta_q)}$$

0.1.4 Exterior derivative of basis two form

$$\begin{aligned}
\mathbf{d}_\curlywedge(\omega^\alpha \curlywedge \omega^\beta) &= \mathbf{d}_\curlywedge\omega^\alpha \curlywedge \omega^\beta - \omega^\alpha \curlywedge \mathbf{d}_\curlywedge\omega^\beta \\
&= \Lambda^\alpha{}_{[\rho\gamma]}\,\omega^\rho \curlywedge \omega^\gamma \curlywedge \omega^\beta - \omega^\alpha \curlywedge \Lambda^\beta{}_{[\rho\gamma]}\,\omega^\rho \curlywedge \omega^\gamma \\
&= \Lambda^\alpha{}_{[\rho\gamma]}\,\omega^\rho \curlywedge \omega^\gamma \curlywedge \omega^\beta - \Lambda^{*\beta}{}_{[\rho\gamma]}\,\omega^\alpha \curlywedge \omega^\rho \curlywedge \omega^\gamma \\
&= \Lambda^\alpha{}_{[\rho\gamma]}\,\omega^\beta \curlywedge \omega^\rho \curlywedge \omega^\gamma - \Lambda^{*\beta}{}_{[\rho\gamma]}\,\omega^\alpha \curlywedge \omega^\rho \curlywedge \omega^\gamma.
\end{aligned}$$

Strictly within the context the the connection is constrained to be "Hermitian on its lower two indices" the antisymmetric portion of the connection on these two indices considered to be purely imaginary.

Thus,

$$\boxed{\mathbf{d}^H_{\curlywedge} = (\omega^\alpha \curlywedge \omega^\beta) \to 2\Lambda^{[\alpha}{}_{[\rho\gamma]}\, \omega^{\beta]} \curlywedge \omega^\rho \curlywedge \omega^\gamma}$$

in this context.

0.1.5 "Co-terior" derivative of basis two forms

$$\begin{aligned}\mathbf{d}_\odot(\omega^\alpha \curlywedge \omega^\beta) &= \omega^\gamma \odot \nabla_\gamma(\omega^\alpha \circledast \omega^\beta - \omega^\beta \circledast \omega^\alpha) \\ &= \omega^\gamma \odot (\nabla_\gamma\omega^\alpha \circledast \omega^\beta + \omega^\alpha \circledast \nabla_\gamma\omega^\beta - \nabla_\gamma\omega^\beta \circledast \omega^\alpha - \omega^\beta \circledast \nabla_\gamma\omega^\alpha) \\ &= \omega^\gamma \odot (\nabla_\gamma\omega^\alpha \curlywedge \omega^\beta + \omega^\alpha \curlywedge \nabla_\gamma\omega^\beta) \\ &= \mathbf{d}_\odot\omega^\alpha \curlywedge \omega^\beta + \omega^\alpha \odot \omega^\gamma \curlywedge \nabla_\gamma\omega^\beta \\ &= \mathbf{d}_\odot\omega^\alpha \curlywedge \omega^\beta + \omega^\alpha \odot \mathbf{d}_\curlywedge\omega^\beta\end{aligned}$$

Explicitly,

$$\begin{aligned}\mathbf{d}_\odot(\omega^\alpha \curlywedge \omega^\beta) &= -\omega^\gamma \odot \Lambda^{*\alpha}_{\rho\gamma}\omega^\rho \curlywedge \omega^\beta - \omega^\alpha \odot \omega^\gamma \curlywedge \Lambda^{*\beta}_{\rho\gamma}\omega^\rho \\ &= -\Lambda^{\alpha}_{\rho\gamma}\omega_\gamma \odot \omega^\rho \curlywedge \omega^\beta - \Lambda^{*\beta}_{\rho\gamma}\omega^\alpha \odot \omega^\gamma \curlywedge \omega^\rho\end{aligned}$$

$$\boxed{\mathbf{d}_\odot(\omega^\alpha \curlywedge \omega^\beta) = +\Lambda^{\alpha}_{\rho\gamma}\omega^\beta \odot \omega^\rho \curlywedge \omega^\gamma + \Lambda^{*\beta}_{\rho\gamma}\omega^\alpha \odot \omega^\rho \curlywedge \omega^\gamma}$$

Within the constraint that the connections are Hermitian in the lower two indices

$$\boxed{\mathbf{d}^H_\odot(\omega^\alpha \curlywedge \omega^\beta) \to +2\Lambda^{[\alpha}_{\rho\gamma}\omega^{\beta]} \odot \omega^\rho \curlywedge \omega^\gamma}$$

0.1.6 Exterior Derivative of $\omega^\alpha \odot \omega^\beta$

$$\mathbf{d}_\curlywedge(\omega^\alpha \odot \omega^\beta) = \mathbf{d}_\curlywedge\omega^\alpha \odot \omega^\beta - \omega^\alpha \curlywedge \mathbf{d}_\odot\omega^\beta$$

Explicitly,

$$\begin{aligned}\mathbf{d}_\curlywedge(\omega^\alpha \odot \omega^\beta) &= -\omega^\gamma \curlywedge \Lambda^{*\alpha}_{\rho\gamma}\omega^\rho \odot \omega^\beta + \omega^\alpha \curlywedge \omega^\gamma \odot \Lambda^{*\beta}_{\rho\gamma}\omega^\rho \\ &= -\Lambda^{\alpha}_{\rho\gamma}\omega^\gamma \curlywedge \omega^\rho \odot \omega^\beta + \Lambda^{*\beta}_{\rho\gamma}\omega^\alpha \curlywedge \omega^\gamma \odot \omega^\rho\end{aligned}$$

$$\boxed{\mathbf{d}_\curlywedge(\omega^\alpha \odot \omega^\beta) = -\Lambda^{\alpha}_{\rho\gamma}\omega^\beta \curlywedge \omega^\rho \odot \omega^\gamma + \Lambda^{*\beta}_{\rho\gamma}\omega^\alpha \curlywedge \omega^\rho \odot \omega^\gamma}$$

Within the constraint that the connections are Hermitian in lower two indices, the symmetric part is real so

$$\boxed{\mathbf{d}^H_\curlywedge\omega^\alpha \odot \omega^\beta = -2\Lambda^{[\alpha}_{(\rho\gamma)}\omega^{\beta]} \curlywedge \omega^\rho \odot \omega^\gamma}$$

0.1.7 Note on Permissible Index Swaps in Mixed Products

When manipulating mixed products such as $\omega^\gamma \curlywedge \omega^\rho \odot \omega^\beta$, the allowed transformations follow from the defining symmetries of the sesquilinear products:

$$\omega^\mu \curlywedge \omega^\nu = -\,\omega^\nu \curlywedge \omega^\mu, \qquad \omega^\mu \odot \omega^\nu = +\,\omega^\nu \odot \omega^\mu.$$

Hence one may exchange the one–forms *within each product* according to

$$\begin{aligned} \omega^\gamma \curlywedge \omega^\rho \odot \omega^\beta &\xrightarrow{(1)} -\,\omega^\rho \curlywedge \omega^\gamma \odot \omega^\beta \quad \text{(antisymmetry of } \curlywedge) \\ &\xrightarrow{(2)} -\,\omega^\rho \curlywedge \omega^\beta \odot \omega^\gamma \quad \text{(symmetry of } \odot) \\ &\xrightarrow{(3)} +\,\omega^\beta \curlywedge \omega^\rho \odot \omega^\gamma \quad \text{(second swap in } \curlywedge). \end{aligned}$$

The two sign reversals in steps (1) and (3) cancel, yielding an overall positive sign:

$$\boxed{\omega^\gamma \curlywedge \omega^\rho \odot \omega^\beta = +\,\omega^\beta \curlywedge \omega^\rho \odot \omega^\gamma.}$$

Remark. The sequence above represents a *permutation of indices*, not an algebraic re-association of the objects themselves. In this case all objects are basis-oneforms so the distinction is moot. But in general for objects it is only intended that indices are permutated to be safe.. For general composite or coefficient–weighted forms, additional rules must be specified before operations beyond pure index relabeling.