Notes (Rev 1): Complex sesqui-linear tensor and wedge product operations

Shane W. D.

10/3/2025

Abstract

Notes regarding differential geometric definition of a complex sesquilinear tensor product and wedge product operation, and definitions, and equivalency of exterior derivative outcome on a dyadic object between the two systems under hermitian constraints.

Sesquilinear $\curlywedge$ and $\circledast$

We distinguish notations:

$$\omega^\mu \wedge \omega^\nu \quad \text{(standard wedge)}, \omega^\mu \curlywedge \omega^\nu \quad \text{(sesquilinear wedge)},$$

$$\omega^\mu \otimes \omega^\nu \quad \text{(standard tensor product)}, \qquad \omega^\mu \circledast \omega^\nu \quad \text{(sesquilinear tensor product)}.$$

Two-form translation .

$$\boxed{\omega^\mu \curlywedge \omega^\nu \quad \omega^\mu \wedge \omega^{\nu *}}$$

Two-legged object for the tensor product (analogous).

$$\boxed{\omega^\mu \circledast \omega^\nu \quad \omega^\mu \otimes \omega^{\nu *}}$$

Sesquilinearity on two-legged objects For scalars $a, b \in \mathbb{C}$ and one-forms α, β:

$$(a\,\alpha) \curlywedge (b\,\beta) = ab\,(\alpha \curlywedge \beta), \qquad (a\,\alpha) \circledast (b\,\beta) = ab\,(\alpha \circledast \beta).$$

Three-legged $\curlywedge$ object

Define with alternating conjugation:

$$\alpha \curlywedge \beta \curlywedge \gamma \;\;\leftrightarrow\;\; \alpha \wedge \beta^* \wedge \gamma,$$

linear in slots 1 and 3, conjugate-linear in slot 2. Coefficient rule:

$$(a\alpha) \curlywedge (b\beta) \curlywedge (c\gamma) = (a\, b^*\, c)\, (\alpha \curlywedge \beta \curlywedge \gamma).$$

Associativity in the Sesquilinear Wedge

Issue. Binary-only $\curlywedge$ breaks associativity:

$$(\alpha \curlywedge \beta) \curlywedge \gamma \;\longleftrightarrow\; (\alpha \wedge \beta^*) \wedge \gamma, \qquad \alpha \curlywedge (\beta \curlywedge \gamma) \;\longleftrightarrow\; \alpha \wedge (\beta \wedge \gamma^*).$$

Rescue rule (alternating conjugation). Define the 3-ary operation by

$$\alpha \curlywedge \beta \curlywedge \gamma := \alpha \wedge \beta^* \wedge \gamma,$$

and rebracket via

$$\alpha \curlywedge (\beta \curlywedge \gamma) \;\mapsto\; \alpha \wedge (\beta \curlywedge \gamma)^* \;\equiv\; \alpha \wedge \beta^* \wedge \gamma.$$

Result.

$$(\alpha \curlywedge \beta) \curlywedge \gamma \;\equiv\; \alpha \curlywedge (\beta \curlywedge \gamma) \;\equiv\; \alpha \curlywedge \beta \curlywedge \gamma,$$

so associativity holds *up to the alternating-slot conjugation convention.*

Sesquilinear Tensor Product $\circledast$

Binary translation.

$$\alpha \circledast \beta \;\;\leftrightarrow\;\; \alpha \otimes \beta^*.$$

n-ary definition (even-slot conjugation). For one-forms $\alpha_1, \ldots, \alpha_n$,

$$\alpha_1 \circledast \cdots \circledast \alpha_n \;:=\; \alpha_1 \otimes \alpha_2^* \otimes \alpha_3 \otimes \alpha_4^* \otimes \cdots \quad \text{(stars on slots } 2, 4, 6, \ldots\text{)}.$$

Coefficient pull-out. For scalars $a_k \in \mathbb{C}$,

$$(a_1\alpha_1) \circledast \cdots \circledast (a_n\alpha_n) \;=\; \big(a_1 a_2^* a_3 a_4^* \cdots\big)\; \alpha_1 \circledast \cdots \circledast \alpha_n.$$

Rebracketing (associativity convention). Any parenthesized product is reinterpreted via the n-ary rule above:

$$(\cdots((\alpha_1 \circledast \alpha_2) \circledast \alpha_3)\cdots \circledast \alpha_n) \;\equiv\; \alpha_1 \circledast \cdots \circledast \alpha_n.$$

Conjugation toggle.

$(\alpha_1 \circledast \cdots \circledast \alpha_n)^* \;\leftrightarrow\; \alpha_1^* \otimes \alpha_2 \otimes \alpha_3^* \otimes \alpha_4 \otimes \cdots$ (stars swap to the opposite parity slots).

Basis example.

$$\omega^\mu \circledast \omega^\nu \circledast \omega^\rho \;\leftrightarrow\; \omega^\mu \otimes (\omega^\nu)^* \otimes \omega^\rho.$$

Exterior Derivative of a Dyadic One-Form

We consider the sesquilinear dyad

$$\omega^\alpha \omega^{\dagger\beta} \;:=\; \omega^\alpha \circledast \omega^\beta,$$

and compare two rules for computing its exterior derivative.

Rule A (plain Leibniz for $\circledast$). By the dyadic analogy $X \circledast Y \sim XY^\dagger$, we take

$$d(X \circledast Y) \;=\; (dX) \circledast Y \;+\; X \circledast dY.$$

Applying to $\omega^\alpha \circledast \omega^\beta$ gives

$$d(\omega^\alpha \omega^{\dagger\beta}) = (d\omega^\alpha) \circledast \omega^\beta + \omega^\alpha \circledast d(\omega^\beta)^\dagger.$$

Using

$$d\omega^\mu = -\,\Lambda^{*\mu}{}_{\rho\gamma}\, \omega^\rho \curlywedge \omega^\gamma,$$

we obtain

$$\boxed{d(\omega^\alpha \omega^{\dagger\beta}) = -\,\Lambda^{*\alpha}{}_{\rho\gamma}\, (\omega^\rho \curlywedge \omega^\gamma) \circledast \omega^\beta \;-\; \Lambda^{\beta}{}_{\rho\gamma}\, \omega^\alpha \circledast (\omega^\rho \curlywedge \omega^\gamma).} \quad \text{(A)}$$

Rule B (graded Leibniz for $\circledast$). Alternatively, one may impose the graded sign rule for degree 1 objects:

$$d(X \circledast Y) \;=\; (dX) \circledast Y \;-\; X \circledast dY.$$

Applying to $\omega^\alpha \circledast \omega^\beta$ gives

$$\boxed{d(\omega^\alpha \omega^{\dagger\beta}) = -\,\Lambda^{*\alpha}{}_{\rho\gamma}\, (\omega^\rho \curlywedge \omega^\gamma) \circledast \omega^\beta \;+\; \Lambda^{*\beta}{}_{\rho\gamma}\, \omega^\alpha \circledast (\omega^\rho \curlywedge \omega^\gamma).} \quad \text{(B)}$$

Compatibility. Equations (A) and (B) differ only in the second term. Rule A has $-\Lambda^{\beta}{}_{\rho\gamma}$, while Rule B has $+\Lambda^{*\beta}{}_{\rho\gamma}$. However:

- the antisymmetry of $\curlywedge$ enforces contraction with $\Lambda_{[\rho\gamma]}$, and
- in this framework the connection is Hermitian (purely imaginary), so $\Lambda^{*\beta}{}_{[\rho\gamma]} = -\,\Lambda^{\beta}{}_{[\rho\gamma]}$.

Thus the two results coincide, and both rules give the same final expression:

$$\boxed{d(\omega^{\alpha}\omega^{\dagger\beta}) = -\,\Lambda^{*\alpha}{}_{\rho\gamma}\,(\omega^{\rho}\curlywedge\omega^{\gamma})\circledast\omega^{\beta} \;-\; \Lambda^{\beta}{}_{\rho\gamma}\,\omega^{\alpha}\circledast(\omega^{\rho}\curlywedge\omega^{\gamma}).}$$

Conclusion. For dyads of one-forms, the plain and graded Leibniz rules for $\circledast$ are *compatible* under the Hermitian connection assumption.