1. Showing A/J is a complete intersection

We work over an algebraically closed field k of characteristic $p > 0$. Fix an element $c \in k$, and define the polynomial

$$g(z) = \prod_{i=0}^{n-1} \left(\sum_{k=0}^{p-1} (-1)^k \binom{c}{k} x_i^k z^k \right).$$

Consider the elements

$$f_i = [z^p] \frac{1}{1 - x_i z} g(z)$$

in $k[x_0, \dots, x_{n-1}]$, where we consider $\frac{1}{1-x_i z} g(z)$ as a formal power series in z and denote the coefficient of z^p in r by $[z^p] r(z, x_0, \dots, x_{n-1})$.

Lemma 1.1. Let $y_0, \dots, y_{r-1}$ be distinct elements of k. Then the polynomials $b_i(z) = \prod_{j \neq i} (1 - y_j z)$ are linearly independent.

Proof. Assume that for some $\lambda_i \in k$ we have $\sum_i \lambda_i b_i(z) = 0$. We can consider $b_i(z)$ as power series in z. Then $\left(\prod_j (1 - y_j z) \right) \left(\sum_i \frac{\lambda_i}{1 - y_i z} \right) = 0$ as power series. Therefore since the ring of power series is an integral domain, $\sum_i \frac{\lambda_i}{1 - y_i z} = 0$. Then in particular for $\ell = 0, \dots, r-1$ we have $\sum_i \lambda_i y_i^\ell = 0$. This is just the product of the λ vector with the Vandermonde matrix for the y_i, which will have nonzero determinant since they are distinct. Therefore the λ vector must be 0. $\square$

Proposition 1.2. For c generic, if $x_0, \dots, x_{n-i} \in k$ satisfy $f_i(x_0, \dots, x_{n-1}) = 0$ for all i, then $x_0 = \dots = x_{n-1} = 0$.

Proof. Suppose that the distinct elements of $\{x_0, \dots, x_{n-1}\}$ are $\{y_0, \dots, y_{r-1}\}$ and that y_i occurs with multiplicity m_i so that

$$\sum_i m_i y_i = 0.$$

Then we see that

$$g(z) = \prod_{i=0}^{r-1} \left(\sum_{k=0}^{p-1} (-1)^k \binom{c}{k} y_i^k z^k \right)^{m_i}.$$

If $r = 1$, then $g(z) = \left(\sum_{k=0}^{p-1} (-1)^k \binom{c}{k} y_0^k z^k \right)^n$ and $f_0(z) = \frac{1}{1-y_0 z} \left(\sum_{k=0}^{p-1} (-1)^k \binom{c}{k} y_0^k z^k \right)^n$. We note that $[z^p] f_0(z) = \sum_{k=0}^{p} y_0^{p-k} [z^k] g(z)$. For $k = 1, \dots, p-1$ we note that the term $[z^k] g(z)$ is a linear combination of terms of the form $\binom{n}{\ell_1} \binom{n-\ell_1}{\ell_2} \dots \binom{n-\ell_1-\dots-\ell_{s-1}}{\ell_s} (-1)^k \binom{c}{k_1}^{\ell_1} \dots \binom{c}{k_s}^{\ell_s} y_0^k$ where $\ell_1, \dots, \ell_s$ are positive integers and $k_1, \dots, k_s$ are distinct positive integers with $\sum \ell_i k_i = k$ - this represents the product of ℓ_i terms $(-1)^{k_i} \binom{c}{k_i} y_0^{k_i} z^{k_i}$.

. In particular this means that $0 < \ell_1 < p$, so $\binom{n}{\ell_1}$ is 0 in characteristic p, so all of these terms are 0. This means that $[z^p] f_0(z) = y_0^p [z^0] g(z) + [z^p] g(z) = y_0^p + [z^p] g(z)$. Note that $[z^p] g(z)$ is the sum of terms of the form $\binom{n}{\ell_1} \binom{n-\ell_1}{\ell_2} \dots \binom{n-\ell_{s-1}}{\ell_s} (-1)^k \binom{c}{k_1} \dots \binom{c}{k_s} y_0^k$ where $\ell_1, \dots, \ell_s$ are positive integers and $k_1, \dots, k_s$ are distinct positive integers with $\sum \ell_i k_i = p$; this term is 0 unless $\ell_1 = p$, so $[z^p] g(z) = \binom{n}{p} (-1)^p \binom{c}{1}^p y_0^p = \binom{n}{p} (-c)^p y_0^p$. Therefore $[z^p] f_0(z) = \left(1 + \binom{n}{p} (-c)^p \right) y_0^p$; if this is 0 but $y_0 \neq 0$, then $1 + \binom{n}{p} (-c)^p = 0$; this is false since c is generic. Therefore $y_0 = 0$. This completes the case $r = 1$. We now assume $r > 1$.

Define the associated function

$$a(z) = \prod_{i=0}^{r-1} \left(\sum_{k=0}^{p-1} (-1)^k \binom{c}{k} y_i^k z^k \right)^{m_i - 1} \left(\sum_{k=0}^{p-1} (-1)^k \binom{c-1}{k} y_i^k z^k \right).$$

Then for $i = 0, \dots, r-1$ we let $a_i(z) = \left(\sum_{k=0}^{p-1} (-1)^k \binom{c}{k} y_i^k z^k \right)^{m_i - 1} \left(\sum_{k=0}^{p-1} (-1)^k \binom{c-1}{k} y_i^k z^k \right)$ so $a(z) = \prod_{i=0}^{r-1} a_i(z)$.

$$
\begin{aligned}
a_i'(z) &= \frac{\partial}{\partial z}\left(\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-1}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^k z^k\right)\right)\\
&= \left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^k z^k\right)\frac{\partial}{\partial z}\left(\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-1}\right)+\\
&\quad \left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-1}\frac{\partial}{\partial z}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^k z^k\right)\\
&= \left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^k z^k\right)(m_i-1)\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-2}\left(\sum_{k=1}^{p-1}k(-1)^k\binom{c}{k}y_i^k z^{k-1}\right)+\\
&\quad \left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-1}\left(\sum_{k=1}^{p-1}k(-1)^k\binom{c-1}{k}y_i^k z^{k-1}\right)\\
&= (m_i-1)\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^k z^k\right)\left(\sum_{k=1}^{p-1}c(-1)^k\binom{c-1}{k-1}y_i^k z^{k-1}\right)+\\
&\quad \left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)\left(\sum_{k=1}^{p-1}(c-1)(-1)^k\binom{c-2}{k-1}y_i^k z^{k-1}\right)\\
&= (m_i-1)\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^k z^k\right)\left(\sum_{k=0}^{p-2}c(-1)^{k+1}\binom{c-1}{k}y_i^{k+1} z^{k}\right)+\\
&\quad \left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)\left(\sum_{k=0}^{p-2}(c-1)(-1)^{k+1}\binom{c-2}{k}y_i^{k+1} z^{k}\right)\\
&= \frac{-(m_i-1)y_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^k z^k\right)\left(\sum_{k=0}^{p-2}c(-1)^{k}\binom{c-1}{k}y_i^{k} z^{k}\right)(1-y_iz)+\\
&\quad \frac{-y_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)\left(\sum_{k=0}^{p-2}(c-1)(-1)^{k}\binom{c-2}{k}y_i^{k} z^{k}\right)(1-y_iz)\\
&= \frac{-(m_i-1)y_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^k z^k\right)\\
&\left(\sum_{k=0}^{p-2}c(-1)^{k+1}\binom{c-1}{k}y_i^{k+1}z^{k+1}+\sum_{k=0}^{p-2}c(-1)^k\binom{c-1}{k}y_i^k z^k\right)+\\
&\quad \frac{-y_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^k z^k\right)\\
&\left(\sum_{k=0}^{p-2}(c-1)(-1)^{k+1}\binom{c-2}{k}y_i^{k+1}z^{k+1}+\sum_{k=0}^{p-2}(c-1)(-1)^k\binom{c-2}{k}y_i^k z^k\right)
\end{aligned}
$$

$$= \frac{-(m_i-1)y_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^kz^k\right)$$

$$\left(\sum_{k=1}^{p-1}c(-1)^k\binom{c-1}{k-1}y_i^kz^k+\sum_{k=0}^{p-2}c(-1)^k\binom{c-1}{k}y_i^kz^k\right)+$$

$$\frac{-y_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)$$

$$\left(\sum_{k=1}^{p-1}(c-1)(-1)^k\binom{c-2}{k-1}y_i^kz^k+\sum_{k=0}^{p-2}(c-1)(-1)^k\binom{c-2}{k}y_i^kz^k\right)$$

$$= \frac{-(m_i-1)y_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^kz^k\right)$$

$$\left(c(-1)^{p-1}\binom{c-1}{p-1}y_i^{p-1}z^{p-1}+\sum_{k=0}^{p-1}c(-1)^k\binom{c-1}{k}y_i^kz^k\right)+$$

$$\frac{-y_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)$$

$$\left(-(c-1)(-1)^{p-1}\binom{c-2}{p-1}y_i^{p-1}z^{p-1}+\sum_{k=0}^{p-1}(c-1)(-1)^k\binom{c-1}{k}y_i^kz^k\right)$$

$$= \frac{-(m_i-1)cy_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^kz^k\right)$$

$$\left(-(-1)^{p-1}\binom{c-1}{p-1}y_i^{p-1}z^{p-1}+\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^kz^k\right)+$$

$$\frac{-y_i(c-1)}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)$$

$$\left(-(-1)^{p-1}\binom{c-2}{p-1}y_i^{p-1}z^{p-1}+\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^kz^k\right)$$

$$= \frac{-(m_i-1)cy_i-y_i(c-1)}{1-y_iz}a_i(z)$$

$$+\frac{-(m_i-1)cy_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^kz^k\right)\left(-(-1)^{p-1}\binom{c-1}{p-1}y_i^{p-1}z^{p-1}\right)$$

$$+\frac{-y_i(c-1)}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)\left(-(-1)^{p-1}\binom{c-2}{p-1}y_i^{p-1}z^{p-1}\right)$$

$$= \frac{y_i(1-m_ic)}{1-y_iz}a_i(z)+v_i(z)$$

where $v_i(z) = \frac{-(m_i-1)cy_i}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_i^kz^k\right)\left(-(-1)^{p-1}\binom{c-1}{p-1}y_i^{p-1}z^{p-1}\right)+\frac{-y_i(c-1)}{1-y_iz}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)^{m_i-2}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_i^kz^k\right)\left(-(-1)^{p-1}\binom{c-2}{p-1}y_i^{p-1}z^{p-1}\right)$. Obviously z^{p-1} divides this as a power series. Then $z^{p-1} \mid \sum_i v_i(z)$. Note that since the m_i are integers and $\sum m_iy_i = 0$ that $\sum m_iy_i^p = 0$.

Also note that $(c-1)\binom{c-2}{p-1} = \frac{(c-1)(c-2)\ldots(c-2-(p-3))(c-2-(p-2))}{(p-1)!} = \frac{(c-1)(c-2)\ldots(c-2-(p-3))c)}{(p-1)!} = c\binom{c-1}{p-1}$.

$$\begin{aligned}
\sum_i [z^{p-1}]v_i(z) &= \sum_i -(m_i-1)cy_i(-1)^p\binom{c-1}{p-1}y_i^{p-1} - y_i(c-1)(-1)^p\binom{c-2}{p-1}y_i^{p-1} \\
&= \sum_i -cm_iy_i^p(-1)^p\binom{c-1}{p-1} + cy_i^p(-1)^p\binom{c-1}{p-1} - (c-1)y_i^p(-1)^p\binom{c-2}{p-1} \\
&= \sum_i cy_i^p(-1)^p\binom{c-1}{p-1} - (c-1)y_i^p(-1)^p\binom{c-2}{p-1} \\
&= \sum_i cy_i^p(-1)^p\binom{c-1}{p-1} - cy_i^p(-1)^p\binom{c-1}{p-1} \\
&= 0
\end{aligned}$$

Then

$$\begin{aligned}
a'(z) &= \frac{\partial}{\partial z}\left(\prod_{i=0}^{r-1} a_i(z)\right) \\
&= \sum_{i=0}^{r-1} a_i'(z)\frac{a(z)}{a_i(z)} \\
&= \sum_{i=0}^{r-1}\left(\frac{y_i(1-m_ic)}{1-y_iz}a_i(z) + v_i(z)\right)\frac{a(z)}{a_i(z)} \\
&= -\sum_{i=0}^{r-1}\frac{(m_ic-1)y_i}{1-y_iz}a(z) + \sum_{i=0}^{r-1} v_i(z)\frac{a(z)}{a_i(z)}
\end{aligned}$$

Recall that $a(z)/a_i(z)$ is a polynomial with constant term 1 for all i and $z^{p-1} \mid b_i(z)$ for all i. Therefore $[z^{p-1}]\sum_{i=0}^{r-1} b_i(z)\frac{a(z)}{a_i(z)} = [z^{p-1}]\sum_{i=0}^{r-1} b_i(z) = 0$. Then we can write $\sum_{i=0}^{r-1} b_i(z)\frac{a(z)}{a_i(z)}$ as $z^p b(z)$ for some power series b.

We write

$$\begin{aligned}
a(z)\prod_j(1-y_jz) &= \prod_j\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_j^kz^k\right)^{m_j-1}\left(\sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_j^kz^k\right)(1-y_jz) \\
&= \prod_j\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_j^kz^k\right)^{m_j-1}\left(\sum_{k=0}^{p-1}(-1)^{k+1}\binom{c-1}{k}y_j^{k+1}z^{k+1} + \sum_{k=0}^{p-1}(-1)^k\binom{c-1}{k}y_j^kz^k\right) \\
&= \prod_j\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_j^kz^k\right)^{m_j-1}\left((-1)^p\binom{c-1}{p-1}y_j^pz^p + \sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_j^kz^k\right) \\
&= g(z) + \prod_j\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_j^kz^k\right)^{m_j-1}\left((-1)^p\binom{c-1}{p-1}y_j^pz^p\right)
\end{aligned}$$

Then let $u(z) = \prod_j\left(\sum_{k=0}^{p-1}(-1)^k\binom{c}{k}y_j^kz^k\right)^{m_j-1}\left((-1)^p\binom{c-1}{p-1}y_j^pz^p\right)$. Note that $z^{pr} \mid u(z)$.

Otherwise, we see that for all i, $a(z)\prod_{j\neq i}(1-y_jz) = \frac{1}{1-y_iz}g(z) + \frac{u(z)}{1-y_iz}$. Then since $u(z)$ is divisible by z^{pr}, $[z^p]\frac{u(z)}{1-y_iz} = 0$. Therefore $[z^p]a(z)\prod_{j\neq i}(1-y_jz) = [z^p]\frac{1}{1-y_iz}g(z) = [z^p]f_0(z) = 0$.

We write

We claim that $a(z)$ satisfies

- $a'(z) = -\sum_{i=0}^{r-1} \frac{(m_i c-1)y_i}{1-y_i z} a(z) + z^p b(z)$ for some polynomial $b(z)$;
- for any i, we have
$$[z^p]a(z)\prod_{j\neq i}(1-y_j z) = [z^p]\frac{1}{1-y_i z}g(z) = 0.$$

This implies that for any $\lambda_i \in k$, we have

$$[z^p]a(z)\sum_{i=0}^{r-1}\lambda_i\prod_{j\neq i}(1-y_j z) = 0. \tag{1}$$

Note that the r polynomials $b_i(z) = \prod_{j\neq i}(1-y_j z)$ have degree at most $r-1$ and are linearly independent by Lemma **??**. Therefore for $0 \le k \le r-1$, there exist $\{\lambda_i^k\}$ so that

$$\sum_i \lambda_i^k b_i(z) = z^k.$$

Substituting these choices of $\{\lambda_i^k\}$ into (**??**), we find that for $0 \le k \le r-1$, we have

$$[z^p]a(z)z^k = 0$$

and therefore that $[z^k]a(z) = 0$ for $p-r+1 \le k \le p$.

For each $l \in \mathbb{Z}$, consider now the Laurent polynomial

$$h_l(z) = z^{-l-r}\prod_i(1-y_i z)$$

and notice that

$$\begin{aligned}
\frac{d}{dz}h_l(z)a(z) &= -(l+r)z^{-1}h_l(z)a(z) - \sum_i \frac{y_i}{1-y_i z}h_l(z)a(z) + h_l(z)a'(z)\\
&= -(l+r)z^{-1}h_l(z)a(z) - \sum_i \frac{y_i}{1-y_i z}h_l(z)a(z) - h_l(z)\sum_i\frac{(m_i c-1)y_i}{1-y_i z}a(z) + z^p h_l(z)b(z)\\
&= -(l+r)z^{-1}h_l(z)a(z) - \sum_i \frac{y_i m_i c}{1-y_i z}h_l(z)a(z) + z^p h_l(z)b(z)\\
&= -\Big((l+r)z^{-1} + \sum_i \frac{y_i m_i c}{1-y_i z}\Big)h_l(z)a(z) + z^p h_l(z)b(z).
\end{aligned}$$

Notice that

$$-\Big((l+r)z^{-1} + \sum_i \frac{y_i m_i c}{1-y_i z}\Big)h_l(z)$$

is a Laurent polynomial with lowest degree term

$$-z^{-l-r-1}\Big(l+r+\sum_i y_i m_i c\Big) = -z^{-l-r-1}(l+r)$$

and highest degree term

$$-z^{-l-1}(-1)^r\prod_i y_i\Big((l+r)-\sum_i m_i c\Big).$$

Further, the lowest degree term of $z^p h_l(z)b(z)$ has degree at least $p-l-r$. Now, we see that $[z^{-1}]\frac{d}{dz}h_l(z)a(z) = 0$, so if $p-l-r > -1$ then $[z^l]a(z)$ is a linear combination of $[z^{l+r}]a(z), \ldots, [z^{l+1}]a(z)$. If all y_i are not equal to 0, then inducting down from $l = p-r$ to $l = 0$, we find

$$[z^0]a(z) = 0,$$

which is a contradiction. We conclude that one of the y_i has value 0.

Assume without loss of generality that $y_0 = 0$. Since $r > 1$, we may then run the entire argument again with $y_1, \ldots, y_{r-1}$ to conclude that one of $y_1, \ldots, y_{r-1}$ must be 0, which is a contradiction. $\square$