1 Introduction and Definitions

In this paper we study lowest-weight representations of rational Cherednik algebras associated to the symmetric group Σ_n in characteristic p dividing n.

Given a vector space $\mathfrak{h}$, an element $s \in \mathrm{GL}(\mathfrak{h})$ is an *reflection* if it has finite order and $\mathrm{rank}(1-s) = 1$. A finite subgroup of $\mathrm{GL}(\mathfrak{h})$ that is generated by reflections is a *reflection group*. In particular the symmetric group Σ_n in n variables is a reflection group (the reflections are the transpositions).

Given a reflection group $G \subset \mathrm{GL}(\mathfrak{h})$ and a vector space $\mathfrak{h}$ over a field k, we let $\mathcal{S}$ be the set of reflections in G. For each $s \in \mathcal{S}$ we assign a vector $\alpha_s \in \mathfrak{h}^*$ spanning the image of $1-s$, and choose $\alpha_s^\vee \in \mathfrak{h}$ so that $(1-s)x = \langle \alpha_s^\vee, x\rangle \alpha_s$ for all $x \in \mathfrak{h}^*$, where $\langle \cdot, \cdot \rangle$ indicates the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. We choose $\hbar \in k$ a number and $c : \mathcal{S} \to k$ a function so that $c(s) = c(s')$ whenever s, s' are conjugate. Let $\bar{c}$ be the function defined by $\bar{c}(s) = c(s^{-1})$.

Let $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ be the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$. Then we define the *rational Cherednik algebra* $\mathrm{H}_{\hbar,c}(G, \mathfrak{h})$ as the quotient of $k[G] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the following relations for all $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$:

$$[x, x'] = 0, \quad [y, y' = 0], \quad [y, x] = \hbar\langle y, x\rangle - \sum_{s\in\mathcal{S}} c(s)\langle y, \alpha_s\rangle\langle \alpha_s^\vee, x\rangle s.$$

We can give $\mathrm{H}_{\hbar,c}(G, \mathfrak{h})$ a $\mathbb{Z}$-grading by setting $\deg x = 1$ for $x \in \mathfrak{h}^*$, $\deg y = -1$ for $y \in \mathfrak{h}$, and $\deg g = 0$ for $g \in k[G]$. We get the PBW-type decomposition $\mathrm{H}_{\hbar,c}(G, \mathfrak{h}) = \mathrm{Sym}(\mathfrak{h}) \otimes_k k[G] \otimes_k \mathrm{Sym}(\mathfrak{h}^*)$ ([4], section 3.2).

In general, for any $\alpha \neq 0$, $\mathrm{H}_{\hbar,c}(G, \mathfrak{h}) \simeq \mathrm{H}_{\alpha\hbar,\alpha c}(G, \mathfrak{h})$. Then we can assume $\hbar = 0$ or $\hbar = 1$.

Let τ be a representation of G. The *Verma module* $M_{\hbar,c}(G, \mathfrak{h}, \tau)$ is defined as $\mathrm{H}_{\hbar,c}(G, \mathfrak{h}) \otimes_{k[G]\ltimes \mathrm{Sym}(\mathfrak{h})} \tau$. Using the PBW decomposition of the Cherednik algebra, we see that $M_{\hbar,c}(G, \mathfrak{h}, \tau) = \mathrm{Sym}(\mathfrak{h}^*) \otimes_k \tau$ as a k-vector space; we can give this a $\mathbb{Z}$-grading in an obvious way.

As described in section 2.5 of [1], $M_{\hbar,c}(G, \mathfrak{h}, \tau)$ has a unique maximal graded proper submodule $J_{\hbar,c}(G, \mathfrak{h}, \tau)$ which can be realized as the kernel of the contravariant form $\beta_c : M_{\hbar,c}(G, \mathfrak{h}, \tau) \times M_{\hbar,\bar{c}}(G, \mathfrak{h}^*, \tau^*) \to k$; β_c can be characterized by the property that for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}, f \in M_{\hbar,c}(G, \mathfrak{h}, \tau), g \in M_{\hbar,\bar{c}}(G, \mathfrak{h}^*, \tau^*), v \in \tau, w \in \tau^*$:

$$\beta_c(xf, g) = \beta_c(f, xg), \quad \beta_c(f, yg) = \beta_c(yf, g), \quad \beta_c(v, w) = \langle v, w\rangle.$$

The quotient $L_{\hbar,c}(G, \mathfrak{h}, \tau) = M_{\hbar,c}(G, \mathfrak{h}, \tau)/J_{\hbar,c}(G, \mathfrak{h}, \tau)$ is a finite-dimensional irreducible $\mathbb{Z}$-graded representation of $\mathrm{H}_{\hbar,c}(G, \mathfrak{h})$.

To understand the action of $\mathrm{H}_{\hbar,c}(G, \mathfrak{h})$ on $M_{\hbar,c}(G, \mathfrak{h}, \tau)$, we can use the PBW decompositions. The action of $\mathrm{Sym}(\mathfrak{h}^*)$ on $M_{\hbar,c}(G, \mathfrak{h}, \tau) = \mathrm{Sym}(h^*) \otimes_k \tau$ is by left multiplication; $k[G]$ acts by the diagonal action, and $\mathrm{Sym}(\mathfrak{h})$ acts via *Dunkl operators*. For $y \in \mathfrak{h}$, the Dunkl operator D_y acts on $M_{\hbar,c}(G, \mathfrak{h}, \tau)$ by:

$$D_y(f \otimes v) = \hbar\partial_y f \otimes v - \sum_{s\in\mathcal{S}} c(s)\frac{(y, \alpha_s)}{\alpha_s}(1-s).f \otimes s.v.$$

Throughout the paper we let $G = \Sigma_n$ and τ the trivial representation. There is only one conjugacy class of reflections in Σ_n, so c is an element of k for our purposes. We call c *generic* if we do not specify its value. We will be concerned with the case $\hbar = 1$ and c generic in this paper. (Note that in particular this case means that $\bar{c} = c$, since there is only one conjugacy class of reflections.) The characteristic of the field k is $p > 0$. We let V be the vector space spanned by $y_0, \ldots, y_{n-1}$ and let $\mathfrak{h}$ be the subspace spanned by $y_i - y_j$ for $i \neq j$; Σ_n acts by permuting indices. Then if $x_0, \ldots, x_{n-1}$ is the dual basis for V^*, we see that $\mathfrak{h}^*$ is the span of $x_0, \ldots, x_{n-1}$ under the relation $x_0 + \cdots + x_{n-1} = 0$; alternatively we can consider $\mathfrak{h}^*$ as the span of $x_0, \ldots, x_{n-2}$ with x_{n-1} defined as $-x_0 - \cdots - x_{n-2}$. For a transposition $s_{ij} \in \Sigma_n$ with $i < j$, we let the corresponding vector $\alpha_{s_{ij}} \in \mathfrak{h}^*$ be $x_i - x_j$.

In this case, since τ is trivial, $M_{\hbar,c}(G,\mathfrak{h},\tau)$ is a polynomial ring $k[x_0,\ldots,x_{n-2}]$; we call this polynomial ring A. Similarly we refer to $J_{\hbar,c}(G,\mathfrak{h},\tau)$ as J, which is an ideal in A, and A/J is the irreducible representation of the Cherednik algebra we desire to find. Because of the definition of the contravariant form β_c, showing that an element f of A is in the kernel of β_c is equivalent to showing that the Dunkl operators corresponding to the basis elements of $\mathfrak{h}$ annihilate f. If D_i is the Dunkl operator corresponding to y_i, the Dunkl operators for the elements of $\mathfrak{h}$ are spanned by $D_i - D_j$ with $i \neq j$.

We assume $p > 2$ since the case $p = 2$ is fully characterized in [5].

2 Representations of rational Cherednik algebras of Σ_n in positive characteristic $p \mid n$

In this paper we consider the case where the characteristic p of k divides n. This is related to the case in characteristic 0 where c takes the specific value p/n, as described in [2]. In that case the generators of the ideal J were the residues of some function in z. To get a similar result in positive characteristic, we must consider formal power series in z with coefficients from A.

We let $g = \prod_{j=0}^{n-1}(1-x_j z)$. Let F_a for $a = 0,\ldots,n-1$ be the formal power series in z defined by $F_a = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$ where $\binom{c}{k} = \frac{c(c-1)\ldots(c-k+1)}{k!}$.

Proposition 2.1. *Let f_a be the coefficient of z^p in the power series F_a. Then f_a for $a = 0,\ldots,n-1$ are annihilated by the Dunkl operators.*

Proof. Taking the Dunkl operator of an element of A consists of taking derivatives in the x_i, dividing by polynomials in the x_i, and letting the symmetric group act on the x_i, in addition to linear operations. We see that this means we can apply the Dunkl operators to F_a and check that the coefficient of z^p in the result is 0 to show that the Dunkl operators annihilate the f_a.

We note that each F_a is symmetric in the x_i not including x_a, and that for any transposition $s_{ab} \in \Sigma_n$, $s_{ab}F_a = F_b$. Therefore we need only consider the action of the Dunkl operators on F_0. We also note that $\mathfrak{h}$ is spanned by $y_i - y_0$ for $0 < i \leq n-1$; then using the fact that F_0 is symmetric in the x_i with $i \neq 0$, we need only show that $(D_1 - D_0)(F_0)$ has z^p coefficient 0 to show that all of the f_a are annihilated by the Dunkl operators.

We also note that adding powers z^k with $k > p$ will not change the value of the z^p coefficient in $(D_1 - D_0)(F_0)$. In particular, we note that since $x_0 + \cdots + x_{n-1} = 0$ divides the coefficient of z in g, we have $z^2 \mid g-1$. Then since $p > 2$, we note that $z^{p+1} \mid z^{2p-2} \mid (g-1)^{p-1}$. Therefore we can add multiples of $(g-1)^{p-1}$ when taking the Dunkl operator's action on F_0, since even when multipled by another power series it cannot contribute anything to the coefficient of z^p. We also note that we can add n times any multiple of F_0 since $n \equiv 0 \bmod p$.

Using the allowed manipulations and the fact that $x_{n-1} = -x_0 + \cdots - x_{n-2}$, we see that up to the z^p coefficient,

$$\frac{\partial F_0}{\partial x_1} = \left(\frac{zc}{1-x_{n-1}z} - \frac{zc}{1-x_1 z}\right)F_0, \quad \frac{\partial F_0}{\partial x_0} = \left(\frac{zc}{1-x_{n-1}z} + \frac{z(1-c)}{1-x_1 z}\right)F_0.$$

We note that when $0 < i, j$ we have $\frac{1-s_{ij}}{x_i - x_j}(F_0) = 0$. We also see that for $0 < i \leq n-1$ we have $\frac{1-s_{ij}}{x_i - x_j}(F_0) = \frac{z}{1-x_i z}F_0$.

We also consider $\frac{\partial F_0}{\partial z}$; up to the addition of some multiple of $(g-1)^{p-1}$, this is equal to $\frac{x_0}{1-x_0 z}F_0 - \sum_{j\geq 0} \frac{-x_j c}{1-x_j z}F_0$.

Then we see that:

$$
\begin{aligned}
(D_1 - D_0)(F_0) &= \frac{\partial F_0}{\partial x_1} - \frac{\partial F_0}{\partial x_0} - c\frac{1-s_{01}}{x_1 - x_0}(F_0) + c\sum_{j>0}\frac{1-s_{0j}}{x_0-x_j}(F_0) \\
&= \left(\frac{zc}{1-x_{n-1}z} - \frac{zc}{1-x_1 z}\right)F_0 - \left(\frac{zc}{1-x_{n-1}z} + \frac{z(1-c)}{1-x_1 z}\right)F_0 + \frac{zc}{1-x_1 z}F_0 + \sum_{j>0}\frac{zc}{1-x_j z}F_0 \\
&= \frac{z(c-1)}{1-x_0 z}F_0 + \sum_{j>0}\frac{zc}{1-x_j z}F_0 \\
&= -\frac{z}{1-x_0 z}f_0 + \sum_{j}\frac{zc}{1-x_j z}f_0 \\
&= -\frac{z}{1-x_0 z}f_0 + \left(\sum_{j}\frac{zc}{1-x_j z}f_0\right) - nzcf_0 \\
&= -zf_0 + \frac{z-x_0 z^2}{1-x_0 z}f_0 - \frac{z}{1-x_0 z}f_0 + \left(\sum_{j}\frac{-zc + x_j c z^2}{1-x_j z}f_0 + \frac{zc}{1-x_j z}f_0\right) \\
&= -zf_0 + \frac{-x_0 z^2}{1-x_0 z}f_0 + \left(\sum_{j}\frac{x_j c z^2}{1-x_j z}f_0\right) \\
&= -zf_0 - z^2\frac{\partial f_0}{\partial z}.
\end{aligned}
$$

Then if b is the coefficient of z^{p-1} in f_0, we see that the coefficient of z^p in $-zf_0$ is $-b$, and the coefficient of z^p in $-z^2\frac{\partial f_0}{\partial z}$ is $-(p-1)b = b$. Therefore the coefficient of z^p in $(D_1 - D_0)(F_0)$ is $-b+b=0$; this means that $(D_1 - D_0)(f_0) = 0$.

Then as we discussed above, the symmetry of the f_a means that all the f_a are annihilated by the Dunkl operators, as desired.

$\square$

Proposition 2.2. *The f_a for $a = 0, \dots, n-2$ are linearly independent homogeneous polynomials of degree p.*

Proof. We recall that $F_a = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right) = \left(\sum_{k=0}^{\infty} x_a^k z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$. It is then clear that f_a is a homogeneous polynomial in the x_i of degree p, since the coefficient of z^k for any k in F_a is a homogeneous polynomial in the x_i of degree k for all k (this follows from the fact that this is true in both multiplicands in F_a).

Since c is generic, we can write the f_a as polynomials in c with coefficients from the polynomial ring A; we can therefore consider the 'constant term' of f_a as a polynomial in c. Recall that f_a is the coefficient of z^p in $F_a = \left(\sum_{k=0}^{\infty} x_a^k z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$. Then as polynomials it is clear that $c \mid \binom{c}{k}$ for all $k>0$; therefore when trying to find the constant term of the coefficient of z^p, we can ignore the terms with $k>0$ in the second multiplicand. The term for $k=0$ is just 1; it is then clear that the constant term (the coefficient of c^0) of f_a is x_a^p.

If $\sum_{a=0}^{n-2} \lambda_a f_a = 0$ for some λ_a rational functions in c, we can multiply through by a least common denominator and assume the λ_a are polynomials in c. We assume that not all of the λ_a are 0. Then we can let e be the smallest nonnegative integer such that there exists an index b with the coefficient of c^e in λ_b nonzero. We can divide all of the λ_a by c^e, so that λ_b for some b must have nonzero constant term.

The constant term of the sum is $\sum_{a=0}^{n-2} \mu_a x_a^p$ where μ_a is the constant term of λ_a. Since the x_a^p for $a = 0, \dots, n-2$ are clearly linearly independent, we see that μ_a must be 0 for $a = 0, \dots, n-2$. Then in particular the constant term μ_b of λ_b is 0, a contradiction. This means that our assumption that not all of the λ_a were 0 is false, so $\lambda_a = 0$ for all a.

Then since $\sum_{a=0}^{n-2} \lambda_a f_a = 0$ means all the λ_a are 0, we see that f_a for $a = 0, \dots, n-2$ are linearly independent as desired.

□

Proposition 2.3. *Let $I \subseteq A$ be the ideal generated by f_a for $a = 0, \dots, n-2$. A/I is a complete intersection.*

Proof. We write x for the vector $\langle x_0, \dots, x_{n-2} \rangle$, where the x_i are taken from the rational function field in c over k. Then we can consider f_a as a function on these vectors x for all a. For any rational function $u(c)$, we let $u(c)x = \langle u(c)x_0, \dots, u(c)x_{n-2} \rangle$.

To show that A/I is a complete intersection, we will show that if $f_a(x) = 0$ for $a = 0, \dots, n-2$, then $x = 0$, which is an equivalent condition.

We showed that for all a, f_a is a homogeneous polynomial in the x_i of degree p. Then for any rational function $u(c)$, we see that $f_a(u(c)x) = u(c)^p f_a(x)$. In particular, if $f_a(x) = 0$, then for any rational function $u(c)$ we have $f_a(u(c)x) = 0$ as well. Therefore if $f_a(x) = 0$ for all $a = 0, \dots, n-2$, then by choosing a particular polynomial $v(c)$ such that $v(c)x_i$ is a polynomial for all $i = 0, \dots, n-2$ (a least common denominator), we see that since $f_a(v(c)x) = 0$ that we can just assume the x_i are polynomials in c. We assume that not all of the x_a are 0. Then we can find the smallest nonnegative integer e such that there exists an b with the coefficient of c^e in x_b nonzero. Then since $f_a(\frac{1}{c^e}x) = 0$ and x_a/c^e is still a polynomial for any a, we see that we can assume that the constant term of x_b for some b is nonzero.

For any a, we can then consider $f_a(x)$ to be a polynomial in c. Since this is zero, we can in particular consider the constant term, which must be 0. The constant term of the coefficient of z^p is the constant term of x_a^p; this must be the constant term of x_a raised to the p power. If this is zero, then the constant term of x_a must be 0.

Then if $f_a(x) = 0$ for $a = 0, \dots, n-2$, we see that the constant terms in all the x_a are 0. In particular, x_b has zero constant term, a contradiction. This means that our assumption that not all of the x_a were 0 is false, so $x_a = 0$ for all a.

Therefore $f_0(x) = \cdots = f_{n-2}(x) = 0$ implies $x = 0$, so A/I is a complete intersection. □

Theorem 2.4. *The f_a generate the ideal J; A/J has Hilbert series $\left(\frac{1-t^p}{1-t}\right)^{n-1}$.*

Proof. By Propositions 2.2, 2.3, A/I is a complete intersection with $n-1$ generators of degree p. It then must have Hilbert series $h_{A/I}(t) = \left(\frac{1-t^p}{1-t}\right)^{n-1}$. By Proposition 2.1, the generators of I are annihilated by the Dunkl operators, so $I \subseteq J$.

By Proposition 3.4 in [1], we see that the Hilbert series of A/J is $\left(\frac{1-t^p}{1-t}\right)^{n-1} h(t^p)$ for some polynomial h with nonnegative integer coefficients; since $I \subseteq J$, we see that $h_{A/I}(t) \geq h_{A/J}(t)$ coefficientwise; however by this restriction of the form of $h_{A/J}(t)$, we see that the only possible choice for h is $h(t) = 1$. Therefore $h_{A/I}(t) = h_{A/J}(t)$, so $I = J$ and these $n-1$ generators generate the whole ideal J.

$\square$

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