1 $p = 3, 3 \mid n$, $n-1$ case full argument

1.1 Generators

Let $x_1, \ldots, x_{n-1}$ be a basis for the $n-1$ dimensional representation of S_n derived from the standard representation with basis $e_0, \ldots, e_{n-1}$ by the relation $x_i = e_i - e_0$ and the action of the symmetric group defined accordingly. Let s_{ij} be the transposition of i and j, for $0 \leq i < j \leq n-1$.

For $1 \leq i < j \leq n-1$, the relevant eigenvectors are $x_i - x_j$. For s_{0i} for $1 \leq i \leq n-1$, the relevant eigenvector is x_i.

Let f_a for $a = 1, \ldots, n-1$ be $f_a = -x_a^3 + c\left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_a\right)$. We will show that all the f_a are killed by Dunkl operators. Because of their symmetry, we need only consider f_1. Since f_1 is symmetric in the x_i excluding x_1, the only Dunkl operators we need consider are D_1, D_2.

We note that f_1 is preserved by s_{ij} for all $i, j > 1$. We consider s_{1a} for $2 \leq a$. We see that $s_{1a} f_1 = -x_a^3 + c\left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_a\right) = f_a$. Then:

$$\begin{aligned} f_1 - f_a &= -(x_1 - x_a)^3 + c\left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) - x_a x_i (x_a - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j (x_1 - x_a)\right) \\ &= -(x_1 - x_a)^3 + c\left(\sum_{i=1}^{n-1} x_i (x_1 + x_a - x_i)(x_1 - x_a)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j (x_1 - x_a)\right) \end{aligned}$$

Then $\frac{f_1 - s_{1a} f_1}{x_1 - x_a} = -(x_1 - x_a)^2 + c\left(\sum_{i=1}^{n-1} x_i (x_1 + x_a - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right)$.

We consider s_{0a} for $2 \leq a$. We see that s_{0a} sends x_a to $-x_a$ and x_i for $i \neq a$ to $x_i - x_a$. We see that:

$$\begin{aligned} s_{0a} f_1 &= s_{0a}\left(-x_1^3 + c\left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_1\right)\right) \\ &= s_{0a}\left(-x_1^3 + c\left(\sum_{i \neq 1, a} x_1 x_i (x_1 - x_i)\right) + c x_1 x_a (x_1 - x_a) + c^2\left(\sum_{i \neq a}\sum_{j \neq a} x_i x_j x_1\right) + 2c^2\left(\sum_{i \neq a} x_i x_a x_1\right) + c^2 x_a^2 x_1\right) \\ &= -(x_1 - x_a)^3 + c\left(\sum_{i \neq 1, a} (x_1 - x_a)(x_i - x_a)(x_1 - x_i)\right) + c(x_1 - x_a)(-x_a)(x_1) + \\ &\quad c^2\left(\sum_{i \neq a}\sum_{j \neq a} (x_i - x_a)(x_j - x_a)(x_1 - x_a)\right) + 2c^2\left(\sum_{i \neq a} (x_i - x_a)(-x_a)(x_1 - x_a)\right) + c^2(-x_a)^2(x_1 - x_a) \end{aligned}$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i\neq 1,a} x_1(x_i - x_a)(x_1 - x_i)\right) - c\left(\sum_{i\neq 1,a} x_a(x_a - x_i)(x_i - x_1)\right) + c(x_1 - x_a)(-x_a)(x_1) +$$

$$c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i - x_a)(x_j - x_a)(x_1 - x_a)\right) + 2c^2\left(\sum_{i\neq a}(x_i - x_a)(-x_a)(x_1 - x_a)\right) + c^2(-x_a)^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i\neq 1,a} x_1 x_i(x_1 - x_i)\right) - c\left(\sum_{i\neq 1,a} x_a x_i(x_a - x_i)\right) + c\left(\sum_{i\neq 1,a} -x_1^2 x_a + x_a^2 x_1\right) + c(x_1 - x_a)(-x_a)(x_1) +$$

$$c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i - x_a)(x_j - x_a)(x_1 - x_a)\right) + 2c^2\left(\sum_{i\neq a}(x_i - x_a)(-x_a)(x_1 - x_a)\right) + c^2(-x_a)^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i\neq 1,a} x_1 x_i(x_1 - x_i)\right) - c\left(\sum_{i\neq 1,a} x_a x_i(x_a - x_i)\right) - c x_1 x_a(x_1 - x_a) +$$

$$c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i - x_a)(x_j - x_a)(x_1 - x_a)\right) + 2c^2\left(\sum_{i\neq a}(x_i - x_a)(-x_a)(x_1 - x_a)\right) + c^2(-x_a)^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i\neq 1,a} x_1 x_i(x_1 - x_i)\right) - c\left(\sum_{i\neq 1,a} x_a x_i(x_a - x_i)\right) + c x_1 x_a(x_1 - x_a) - c x_1 x_a(x_a - x_1) +$$

$$c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i - x_a)(x_j - x_a)(x_1 - x_a)\right) + 2c^2\left(\sum_{i\neq a}(x_i - x_a)(-x_a)(x_1 - x_a)\right) + c^2(-x_a)^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i=1}^{n-1} x_1 x_i(x_1 - x_i)\right) - c\left(\sum_{i=1}^{n-1} x_a x_i(x_a - x_i)\right) +$$

$$c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i - x_a)(x_j - x_a)(x_1 - x_a)\right) + 2c^2\left(\sum_{i\neq a}(x_i - x_a)(-x_a)(x_1 - x_a)\right) + c^2(-x_a)^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i=1}^{n-1} x_1 x_i(x_1 - x_i)\right) - c\left(\sum_{i=1}^{n-1} x_a x_i(x_a - x_i)\right) +$$

$$c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i x_j - x_a x_j - x_a x_i + x_a^2)(x_1 - x_a)\right) + 2c^2\left(\sum_{i\neq a}(x_a^2 - x_a x_i)(x_1 - x_a)\right) + c^2 x_a^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i=1}^{n-1} x_1 x_i(x_1 - x_i)\right) - c\left(\sum_{i=1}^{n-1} x_a x_i(x_a - x_i)\right) +$$

$$c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i x_j + x_a^2)(x_1 - x_a)\right) + c^2\left(\sum_{i\neq a} x_a x_i(x_1 - x_a)\right) + c^2\left(\sum_{i\neq a}(x_a^2 - x_a x_i)(x_1 - x_a)\right) + c^2 x_a^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i=1}^{n-1} x_1 x_i(x_1 - x_i)\right) - c\left(\sum_{i=1}^{n-1} x_a x_i(x_a - x_i)\right) +$$

$$c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i x_j + x_a^2)(x_1 - x_a)\right) + c^2\left(\sum_{i\neq a} x_a^2(x_1 - x_a)\right) + c^2 x_a^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i=1}^{n-1} x_1x_i(x_1 - x_i)\right) - c\left(\sum_{i=1}^{n-1} x_ax_i(x_a - x_i)\right) + c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_ix_j)(x_1 - x_a)\right) +$$

$$2c^2\left(\sum_{i\neq a} x_a^2(x_1 - x_a)\right) + c^2x_a^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i=1}^{n-1} x_1x_i(x_1 - x_i)\right) - c\left(\sum_{i=1}^{n-1} x_ax_i(x_a - x_i)\right) + c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_ix_j)(x_1 - x_a)\right) +$$

$$2c^2x_a^2(x_1 - x_a) + c^2x_a^2(x_1 - x_a)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i=1}^{n-1} x_1x_i(x_1 - x_i)\right) - c\left(\sum_{i=1}^{n-1} x_ax_i(x_a - x_i)\right) + c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_ix_j)(x_1 - x_a)\right)$$

$$= -x_1^3 + x_a^3 + c\left(\sum_{i=1}^{n-1} x_1x_i(x_1 - x_i)\right) - c\left(\sum_{i=1}^{n-1} x_ax_i(x_a - x_i)\right) + c^2\left(\sum_{i\neq a}\sum_{j\neq a} x_ix_jx_1 - x_ix_jx_a\right)$$

$$= f_1 - f_a$$

Then $\frac{f_1 - s_{0a}f_1}{x_a} = f_a/x_a = -x_a^2 - c\left(\sum_{i=1}^{n-1} x_i(x_i - x_a)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_ix_j\right)$.

The only remaining case to consider is s_{01}. We see that s_{01} sends x_1 to $-x_1$ and x_a for $a \geq 2$ to $x_a - x_1$. Then $s_{01}f_1$ goes to

$$-(-x_1)^3 + c\left(\sum_{i=2}^{n-1}(-x_1)(x_i - x_1)(-x_1 - (x_i - x_1))\right) + c^2\left(\sum_{i=2}^{n-1}\sum_{j=2}^{n-1}(x_i - x_1)(x_j - x_1)(-x_1)\right) +$$

$$2c^2\left(\sum_{i=2}^{n-1}(x_i - x_1)x_1^2\right) + c^2(-x_1)^3$$

$$= x_1^3 + c\left(\sum_{i=2}^{n-1} x_1x_i(x_i - x_1)\right) + c^2\left(\sum_{i=2}^{n-1}\sum_{j=2}^{n-1} -x_ix_jx_1 + x_ix_1^2 + x_jx_1^2 - x_1^3\right) +$$

$$2c^2\left(\sum_{i=2}^{n-1}(x_i - x_1)x_1^2\right) + 2c^2x_1^3$$

$$= x_1^3 + c\left(\sum_{i=2}^{n-1} x_1x_i(x_i - x_1)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} -x_ix_jx_1\right)$$

$$= -f_1$$

The above used the fact that $n-2 \equiv 1 \mod 3$. Therefore $\frac{f_1 - s_{01}f_1}{x_1} = 2f_1/x_1 = -f_1/x_1 = x_1^2 + c\left(\sum_{i=1}^{n-1} x_i(x_i - x_1)\right) - c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_ix_j\right)$.

We can use these to calculate the effects of the Dunkl operators.

1.2 Dunkl operators

1.2.1 $D_1 f_1$

We first consider $D_1 f_1 = \partial_1 f_1 - c\sum_{a>1} \frac{f_1 - s_{1a} f_1}{x_1 - x_a} - c\frac{f_1 - s_{01} f_1}{x_1}$. Let $G_1 = \sum_{a>1} \frac{f_1 - s_{1a} f_1}{x_1 - x_a}, G_2 = \frac{f_1 - s_{01} f_1}{x_1}$. We wish to show that $\partial_1 f_1 = cG_1 + cG_2$, since then $D_1 f_1 = 0$. We calculate $\partial_1 f_1$:

$$\begin{aligned}
\partial_1 f_1 &= \partial_1 \left(-x_1^3 + c\left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i)\right) + c^2 \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_1\right)\right) \\
&= c\left(\sum_{i=2}^{n-1} x_1 x_i + x_i(x_1 - x_i)\right) + c^2 \left(\sum_{i=2}^{n-1}\sum_{j=2}^{n-1} x_i x_j\right) + 2c^2 \left(\sum_{i=2}^{n-1} 2x_i x_1\right) \\
&= c\left(\sum_{i=2}^{n-1} x_1 x_i + x_i x_1 - x_i^2\right) + c^2 \left(\sum_{i=2}^{n-1}\sum_{j=2}^{n-1} x_i x_j\right) + c^2 \left(\sum_{i=2}^{n-1} x_i x_1\right) \\
&= c\left(\sum_{i=2}^{n-1} -x_1 x_i - x_i^2\right) + c^2 \left(\sum_{i=2}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right)
\end{aligned}$$

We also see that $G_1 = \sum_{a>1} \frac{f_1 - s_{1a} f_1}{x_1 - x_a} = \sum_{a=2}^{n-1} -(x_1 - x_a)^2 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_a - x_i)\right) + c^2 \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right)$ and $G_2 = x_1^2 + c\left(\sum_{i=1}^{n-1} x_i(x_i - x_1)\right) - c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right)$.

Since $n - 2 \equiv 1 \mod 3$, we see that the c^2 terms cancel in $G_1 + G_2$. We also note that $x_i(x_i - x_1) = x_i(x_1 + x_1 - x_i) - x_i^2$ for all i, so $G_1 + G_2 = x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{a=1}^{n-1} -(x_1 - x_a)^2 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_a - x_i)\right)$.

$$
\begin{aligned}
G_1 + G_2 &= x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{a=1}^{n-1} -(x_1 - x_a)^2 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_a - x_i)\right) \\
&= x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{a=1}^{n-1} -x_1^2 - x_a^2 + 2x_a x_1 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_a - x_i)\right) \\
&= x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{j=1}^{n-1} -x_1^2 - x_j^2 + 2x_j x_1 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_j - x_i)\right) \\
&= x_1^2 + -(n-1)x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{j=1}^{n-1} -x_j^2 - x_j x_1 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_j - x_i)\right) \\
&= 2x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{j=1}^{n-1} \left(-x_j^2 - x_j x_1\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_1 + x_i x_j - x_i^2\right) \\
&= \sum_{j=2}^{n-1} \left(-x_j^2 - x_j x_1\right) - c\left(\sum_{i=1}^{n-1} x_i^2\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_1 + x_i x_j - x_i^2\right) \\
&= \sum_{j=2}^{n-1} \left(-x_j^2 - x_j x_1\right) - c\left(\sum_{i=1}^{n-1} x_i^2\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right) - c\left(\sum_{i=1}^{n-1} x_i x_1 - x_i^2\right) \\
&= \sum_{j=2}^{n-1} \left(-x_j^2 - x_j x_1\right) - c\left(\sum_{i=1}^{n-1} x_i^2\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=2}^{n-1} x_i x_j\right) + c\left(\sum_{i=1}^{n-1} x_i^2\right) \\
&= \sum_{j=2}^{n-1} \left(-x_j^2 - x_j x_1\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=2}^{n-1} x_i x_j\right)
\end{aligned}
$$

Then a simple change of indices tells us that $cG_1 + cG_2 = \partial_1 f_1$ as desired.

1.2.2 $D_2 f_1$

We see that $D_2 f_1 = \partial_2 f_1 - c\frac{f_1 - s_{12} f_1}{x_2 - x_1} - c\frac{f_1 - s_{02} f_1}{x_2}$.

We calculate $\partial_2 f_1$:

$$\partial_2 f_1 = \partial_2 \left(-x_1^3 + c\left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i)\right) + c^2 \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_1\right)\right)$$
$$= \partial_2 \left(-x_1^3 + c\left(\sum_{i=3}^{n-1} x_1 x_i (x_1 - x_i)\right) + c x_1 x_2 (x_1 - x_2) + c^2 \left(\sum_{i\neq 2}\sum_{j\neq 2} x_i x_j x_1\right) + 2c^2 \left(\sum_{i\neq 2} x_i x_1 x_2\right) + c^2 x_2^2 x_1\right)$$
$$= c x_1^2 - 2c x_1 x_2 + 2c^2 \left(\sum_{i\neq 2} x_i x_1\right) + 2c^2 x_2 x_1$$
$$= c x_1^2 + c x_1 x_2 + 2c^2 \left(\sum_{i=1}^{n-1} x_i x_1\right)$$

Let $G_1 = \frac{f_1 - s_{12} f_1}{x_2 - x_1}, G_2 = \frac{f_1 - s_{02} f_1}{x_2}$. If $c(G_1 + G_2) = \partial_2 f_1$, then $D_2 f_1 = 0$.

$$G_1 + G_2 = \frac{f_1 - s_{12} f_1}{x_2 - x_1} + \frac{f_1 - s_{02} f_1}{x_2}$$
$$= (x_1 - x_2)^2 - c\left(\sum_{i=1}^{n-1} x_i (x_1 + x_2 - x_i)\right) - c^2 \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right) - x_2^2 - c\left(\sum_{i=1}^{n-1} x_i (x_i - x_2)\right) + c^2 \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right)$$
$$= x_1^2 + x_1 x_2 + x_2^2 - c\left(\sum_{i=1}^{n-1} x_i (x_1 + x_2 - x_i)\right) - x_2^2 - c\left(\sum_{i=1}^{n-1} x_i (x_i - x_2)\right)$$
$$= x_1^2 + x_1 x_2 - c\left(\sum_{i=1}^{n-1} x_i (x_1 + x_2 - x_i)\right) - c\left(\sum_{i=1}^{n-1} x_i (x_i - x_2)\right)$$
$$= x_1^2 + x_1 x_2 - c\left(\sum_{i=1}^{n-1} x_i (x_1 + x_2 - x_i + x_i - x_2)\right)$$
$$= x_1^2 + x_1 x_2 - c\left(\sum_{i=1}^{n-1} x_i x_1\right)$$

Then we see easily that $\partial_2 f_1 = cG_1 + cG_2$, so $D_2 f_1 = 0$ as desired.

1.3 Complete intersection

Then we see that the ideal I generated by $f_1, \ldots, f_{n-1}$ has $I \subseteq J$ where J is the kernel of the β form, since the generators $f_1, \ldots, f_{n-1}$ are killed by the Dunkl operators.

We note that the f_i are linearly independent, since x_i^3 has nonzero coefficient in f_j only when $i = j$. We will show that A/I, where $A = k[x_1, \ldots, x_{n-1}]$, is a complete intersection by showing that $f_1(x) = \cdots = f_{n-1}(x) = 0$ implies $x = 0$.

Recall $f_a = -x_a^3 + c\left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i)\right) + c^2 \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_a\right)$.

We also note that $\left(\sum_{i=1}^{n-1} x_i\right)^2 = \left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right)$, and $\left(\sum_{i=1}^{n-1} x_i\right)^2 + \sum_{1\leq i<j\leq n-1} x_i x_j = \sum_{i=1}^{n-1} x_i^2$ modulo 3.

Therefore:

$$\begin{aligned}
f_a &= -x_a^3 + c\left(\sum_{i=1}^{n-1} x_a x_i(x_a - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_a\right) \\
&= -x_a^3 + c\left(\sum_{i=1}^{n-1} x_a^2 x_i - x_a x_i^2\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_a\right) \\
&= -x_a^3 + cx_a^2\left(\sum_{i=1}^{n-1} x_i\right) - cx_a\left(\sum_{i=1}^{n-1} x_i^2\right) + c^2 x_a\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right) \\
&= -x_a^3 + cx_a^2\left(\sum_{i=1}^{n-1} x_i\right) + 2cx_a\left(\sum_{i=1}^{n-1} x_i\right)^2 + 2cx_a\left(\sum_{1\leq i<j\leq n-1} x_i x_j\right) + c^2 x_a\left(\sum_{i=1}^{n-1} x_i\right)^2 \\
&= -x_a^3 + cx_a^2\left(\sum_{i=1}^{n-1} x_i\right) + (c^2+2c)x_a\left(\sum_{i=1}^{n-1} x_i\right)^2 + x_a\left(\sum_{1\leq i<j\leq n-1} 2cx_i x_j\right)
\end{aligned}$$

We see that $f_a = -x_a^3 + cx_a^2\left(\sum_{i=1}^{n-1} x_i\right) + (c^2+2c)x_a\left(\sum_{i=1}^{n-1} x_i\right)^2 + x_a\left(\sum_{1\leq i<j\leq n-1} 2cx_i x_j\right)$.

We now consider the sum of the f_a:

$$
\begin{aligned}
\sum_{a=1}^{n-1} f_a &= \sum_{a=1}^{n-1}\left(-x_a^3 + cx_a^2\left(\sum_{i=1}^{n-1} x_i\right) + (c^2+2c)x_a\left(\sum_{i=1}^{n-1} x_i\right)^2 + x_a\left(\sum_{1\le i<j\le n-1} 2cx_ix_j\right)\right)\\
&= \left(\sum_{a=1}^{n-1} -x_a^3\right) + c\left(\sum_{a=1}^{n-1} x_a^2\right)\left(\sum_{i=1}^{n-1} x_i\right) + (c^2+2c)\left(\sum_{a=1}^{n-1} x_a\right)\left(\sum_{i=1}^{n-1} x_i\right)^2 + \left(\sum_{a=1}^{n-1} x_a\right)\left(\sum_{1\le i<j\le n-1} 2cx_ix_j\right)\\
&= 2\left(\sum_{i=1}^{n-1} x_i^3\right) + c\left(\sum_{i=1}^{n-1} x_i^2\right)\left(\sum_{i=1}^{n-1} x_i\right) + (c^2+2c)\left(\sum_{i=1}^{n-1} x_i\right)\left(\sum_{i=1}^{n-1} x_i\right)^2 + \left(\sum_{i=1}^{n-1} x_i\right)\left(\sum_{1\le i<j\le n-1} 2cx_ix_j\right)\\
&= 2\left(\sum_{i=1}^{n-1} x_i^3\right) + c\left(\left(\sum_{i=1}^{n-1} x_i\right)^2 + \left(\sum_{1\le i<j\le n-1} x_ix_j\right)\right)\left(\sum_{i=1}^{n-1} x_i\right) + (c^2+2c)\left(\sum_{i=1}^{n-1} x_i\right)^3 +\\
&\quad 2c\left(\sum_{i=1}^{n-1} x_i\right)\left(\sum_{1\le i<j\le n-1} x_ix_j\right)\\
&= 2\left(\sum_{i=1}^{n-1} x_i^3\right) + c\left(\sum_{i=1}^{n-1} x_i\right)^3 + c\left(\sum_{1\le i<j\le n-1} x_ix_j\right)\left(\sum_{i=1}^{n-1} x_i\right) + (c^2+2c)\left(\sum_{i=1}^{n-1} x_i\right)^3 +\\
&\quad 2c\left(\sum_{i=1}^{n-1} x_i\right)\left(\sum_{1\le i<j\le n-1} x_ix_j\right)\\
&= (c^2+2)\left(\sum_{i=1}^{n-1} x_i\right)^3
\end{aligned}
$$

We see that $\sum_{a=1}^{n-1} f_a = (c^2+2)\left(\sum_{i=1}^{n-1} x_i\right)^3$.

Therefore if $f_1(x) = \cdots = f_{n-1}(x) = 0$, we see that $\sum_{i=1}^{n-1} x_i = 0$. Modulo this new relation, we see that $f_a = x_a\left(-x_a^2 + \sum_{1\le i<j\le n-1} 2cx_ix_j\right)$. Therefore if $f_a = 0$ for all a, we see that for each a, either $x_a = 0$ or x_a is a square root of $C = \sum_{1\le i<j\le n-1} 2cx_ix_j$. If $C = 0$ or C has no square roots, then all the x_a are 0 and we are done; so we assume C has two square roots which are additive inverses of each other. Then each pair x_ix_j either multiply to $\pm C$ or 0, so $C = 2cmC$ for some integer m. Then $(1+cm)C = 0$; if $C \neq 0$, then $1 + cm = 0$; however, m is an integer, so this is impossible. Therefore $C = 0$, so all the x_a are 0.

Therefore $f_1(x) = \cdots = f_{n-1}(x) = 0$ implies $x = 0$, so A/I is a complete intersection; it then must have Hilbet polynomial $h_{A/I}(t) = (t^2+t+1)^{n-1}$.

By Proposition 3.4 in http://arxiv.org/abs/1107.0504, we see that the Hilbert polynomial of A/J is $(t^2+t+1)^{n-1}h(t^3)$ for some polynomial h with nonnegative integer coefficients; since $I \subseteq J$, we see that $h_{A/I}(t) \geq h_{A/J}(t)$ coefficientwise; however by this restriction of the form of $h_{A/J}(t)$, we see that the only possible choice for h is $h(t) = 1$. Therefore $h_{A/I}(t) = h_{A/J}(t)$, so $I = J$ and these $n-1$ generators generate the whole ideal.

2 $p \mid n$, $n-1$ case full argument

We assume $p > 2$ since $p = 2$ has been fully characterized.

2.1 Generators

Let $x_0, \ldots, x_{n-2}$ be a basis for the $n-1$ dimensional representation of S_n derived from the standard representation with basis $e_0, \ldots, e_{n-1}$ by taking the quotient by $e_0 + \cdots + e_{n-1}$; assume x_i is the representative for e_i in the quotient. We see that we can in fact say that the representation is spanned by $x_0, \ldots, x_{n-1}$ with the relation $x_0 + \cdots + x_{n-1} = 0$, or $x_{n-1} = -x_0 - \cdots - x_{n-2}$. Let s_{ij} be the transposition of i and j for $0 \le i < j \le n-1$. For $0 \le i < j \le n-1$ the relevant eigenvectors are $x_i - x_j$.

We let $g = \prod_{j=0}^{n-1}(1 - x_j z)$, where z is another variable.

Let f_a for $a = 0, \ldots, n-1$ be the formal power series in z defined by $f_a = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$ where $\binom{c}{k} = \frac{c(c-1)\ldots(c-k+1)}{k!}$. We will show that the coefficient of z^p in f_a is killed by the Dunkl operators for all a. Since the Dunkl operators consist of taking derivatives in the x_i, dividing by polynomials in the x_i, linear operations, and the action of the symmetric group on the x_i, we see that we can just apply the Dunkl operators to the f_a and check that the coefficient of z^p in the resulting formal power series is 0.

Let $y_0, \ldots, y_{n-1}$ be a basis for the dual of the standard representation of S_n. Then the dual of the $(n-1)$-dimensional quotient we are considering is spanned by $y_i - y_0$ for $0 < i \le n-1$. Therefore if D_i is the Dunkl operator corresponding to y_i acting on the standard representation of S_n, we see that the Dunkl operators on our quotient representation are $D_i - D_0$ for $0 < i \le n-1$

Because of the symmetry in the f_a, we need only consider the action of the Dunkl operators on f_0. Then again by symmetry we need only consider the action of $D_1 - D_0$ on f_0.

We also note that we can add powers of z greater than z^p at any stage of taking the Dunkl operator, since those will not affect the final result.

2.2 Dunkl operators

We note $z^2 \mid g - 1$ since $x_0 + \cdots + x_{n-1} = 0$ divides the coefficient of z in g. Therefore when $p > 2$ we see that $z^{p+1} \mid z^{2p-2} \mid (g-1)^{p-1}$. Therefore we can add multiples of $(g-1)^{p-1}$ at any stage, since even when multipled by other power series it cannot contribute anything to the coefficient of z^p.

Since $x_{n-1} = -x_0 - \cdots - x_{n-2}$, we see that $\frac{\partial g}{\partial x_i} = -\frac{zg}{1-x_i z} + \frac{zg}{1-x_{n-1}z}$ for all $0 \le i < n-1$. Let $F = f_0(1-x_1 z) = \left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$. Note that F is symmetric. Then we see that for all $0 \le i < n-1$:

$$
\begin{aligned}
\frac{\partial F}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\\
&= \left(\sum_{k=1}^{p-1}k\binom{c}{k}(g-1)^{k-1}\right)\frac{\partial g}{\partial x_i}\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=1}^{p-1}k\binom{c}{k}(g-1)^{k-1}\right)g\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=0}^{p-2}(k+1)\binom{c}{k+1}(g-1)^{k}\right)g\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)(g-1+1)\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)+\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k+1}\right)\right)\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)+\left(\sum_{k=1}^{p-1}c\binom{c-1}{k-1}(g-1)^{k}\right)\right)
\end{aligned}
$$

(We see that for $k=1,\dots,p-1$ we have $\binom{c-1}{k}+\binom{c-1}{k-1}=\binom{c}{k}$, that $\binom{c-1}{0}=\binom{c}{0}$ as a polynomial, and that we can add a multiple of $(g-1)^{p-1}$).

$$
\begin{aligned}
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=0}^{p-1}c\binom{c}{k}(g-1)^{k}\right)\\
&= \left(-\frac{zc}{1-x_iz}+\frac{zc}{1-x_{n-1}z}\right)F
\end{aligned}
$$

We also see that:

$$
\begin{aligned}
\frac{\partial f_0}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(\frac{1}{1-x_0z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\right)\\
&= \frac{1}{1-x_0z}\frac{\partial F}{\partial x_1}\\
&= \frac{1}{1-x_0z}\left(-\frac{zc}{1-x_1z}+\frac{zc}{1-x_{n-1}z}\right)F\\
&= \left(-\frac{zc}{1-x_1z}+\frac{zc}{1-x_{n-1}z}\right)f_0
\end{aligned}
$$

and that:

$$
\begin{aligned}
\frac{\partial f_0}{\partial x_0} &= \frac{\partial}{\partial x_1}\left(\frac{1}{1-x_0z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\right)\\
&= \frac{z}{(1-x_0z)^2}F + \frac{1}{1-x_0z}\frac{\partial F}{\partial x_0}\\
&= \frac{z}{1-x_0z}f_0 + \frac{1}{1-x_0z}\left(-\frac{zc}{1-x_0z}+\frac{zc}{1-x_{n-1}z}\right)F\\
&= \frac{z}{1-x_0z}f_0 + \left(-\frac{zc}{1-x_0z}+\frac{zc}{1-x_{n-1}z}\right)f_0\\
&= \left(\frac{z(1-c)}{1-x_0z}+\frac{zc}{1-x_{n-1}z}\right)f_0
\end{aligned}
$$

We note that f_0 is invariant under s_{ij} where $0 < i, j$. Therefore for transpositions we need only consider transpositions of the form s_{0i} for $0 < i \leq n-1$.

$$
\begin{aligned}
\frac{1-s_{0i}}{x_0-x_i}(f_0) &= \frac{1}{x_0-x_i}\left(\frac{F}{1-x_0z}-\frac{F}{1-x_iz}\right)\\
&= \frac{1}{x_0-x_i}\left(\frac{1}{1-x_0z}-\frac{1}{1-x_iz}\right)F\\
&= \frac{1}{x_0-x_i}\left(\frac{(1-x_iz)-(1-x_0z)}{(1-x_0z)(1-x_iz)}\right)F\\
&= \frac{x_0z-x_iz}{(1-x_0z)(1-x_iz)(x_0-x_i)}F\\
&= \frac{z}{(1-x_iz)(1-x_0z)}F\\
&= \frac{z}{1-x_iz}f_0
\end{aligned}
$$

We recall that we need only consider the action of $D_1 - D_0$ on f_0. We consider $D_0 f_0, D_1 f_0$ separately first. We see that $D_0 = \left(\frac{\partial}{\partial x_0} - c\sum_{j>0}\frac{1-s_{0j}}{x_0-x_j}\right), D_1 = \left(\frac{\partial}{\partial x_1} - c\frac{1-s_{01}}{x_1-x_0}\right)$ since f_0 is invariant under s_{ij} where $0 < i, j$.

$$
\begin{aligned}
D_0 f_0 &= \left(\frac{\partial}{\partial x_0} - c\sum_{j>0}\frac{1-s_{0j}}{x_0-x_j}\right)(f_0)\\
&= \frac{\partial f_0}{\partial x_0} - c\sum_{j>0}\frac{1-s_{0j}}{x_0-x_j}(f_0)\\
&= \left(\frac{z(1-c)}{1-x_0z}+\frac{zc}{1-x_{n-1}z}\right)f_0 - \sum_{j>0}\frac{zc}{1-x_jz}f_0
\end{aligned}
$$

$$
\begin{aligned}
D_1 f_0 &= \left(\frac{\partial}{\partial x_1} - c\frac{1-s_{01}}{x_1-x_0}\right)(f_0)\\
&= \frac{\partial f_0}{\partial x_1} + c\frac{1-s_{01}}{x_0-x_1}(f_0)\\
&= \left(-\frac{zc}{1-x_1 z} + \frac{zc}{1-x_{n-1}z}\right) f_0 + \frac{zc}{1-x_1 z} f_0\\
&= \frac{zc}{1-x_{n-1}z} f_0
\end{aligned}
$$

It is then easy to see that $(D_1 - D_0)(f_0) = \frac{z(c-1)}{1-x_0 z} f_0 + \sum_{j>0} \frac{zc}{1-x_j z} f_0$.

In order to show that the pth coefficient in this formal power series is 0, we must consider $\frac{\partial f_0}{\partial z}$.

We see easily that $\frac{\partial g}{\partial z} = g \sum_j \frac{-x_j}{1-x_j z}$. We now consider $\frac{\partial F}{\partial z}$:

$$
\begin{aligned}
\frac{\partial F}{\partial z} &= \frac{\partial}{\partial z}\left(\sum_{k=0}^{p-1} \binom{c}{k} (g-1)^k\right)\\
&= \left(\sum_{k=1}^{p-1} k\binom{c}{k}(g-1)^{k-1}\right)\frac{\partial g}{\partial z}\\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\sum_{k=1}^{p-1} k\binom{c}{k}(g-1)^{k-1}\right) g
\end{aligned}
$$

Note that above we showed that $\left(\sum_{k=1}^{p-1} k\binom{c}{k}(g-1)^{k-1}\right) g = \left(\sum_{k=0}^{p-1} c\binom{c}{k}(g-1)^k\right)$ up to the addition of multiples of z^{p+1}.

$$
\begin{aligned}
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\sum_{k=0}^{p-1} c\binom{c}{k}(g-1)^k\right)\\
&= \left(\sum_j \frac{-cx_j}{1-x_j z}\right) F
\end{aligned}
$$

From this it follows that:

$$
\begin{aligned}
\frac{\partial f_0}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{F}{1-x_0 z}\right) \\
&= \frac{1}{1-x_0 z}\frac{\partial F}{\partial z} + \frac{x_0}{1-x_0 z}F \\
&= \frac{1}{1-x_0 z}\left(\sum_j \frac{-cx_j}{1-x_j z}\right)F + \frac{x_0}{(1-x_0 z)^2}F \\
&= \left(\sum_j \frac{-cx_j}{1-x_j z}\right)f_0 + \frac{x_0}{1-x_0 z}f_0
\end{aligned}
$$

We now again consider $(D_1 - D_0)(f_0)$. Recall that $n \equiv 0 \bmod p$, so in particular we can add n times any multiple of f_1 since that is 0 in characteristic p.

$$
\begin{aligned}
(D_1 - D_0)(f_0) &= \frac{z(c-1)}{1-x_0 z}f_0 + \sum_{j>0}\frac{zc}{1-x_j z}f_0 \\
&= -\frac{z}{1-x_0 z}f_0 + \sum_j \frac{zc}{1-x_j z}f_0 \\
&= -\frac{z}{1-x_0 z}f_0 + + \left(\sum_j \frac{zc}{1-x_j z}f_0\right) - nzcf_0 \\
&= -zf_0 + zf_0 - \frac{z}{1-x_0 z}f_0 + \left(\sum_j -zcf_0 + \frac{zc}{1-x_j z}f_0\right) \\
&= -zf_0 + \frac{z-x_0 z^2}{1-x_0 z}f_0 - \frac{z}{1-x_0 z}f_0 + \left(\sum_j \frac{-zc + x_j c z^2}{1-x_j z}f_0 + \frac{zc}{1-x_j z}f_0\right) \\
&= -zf_0 + \frac{-x_0 z^2}{1-x_0 z}f_0 + \left(\sum_j \frac{x_j c z^2}{1-x_j z}f_0\right) \\
&= -zf_0 - z^2\left(\frac{x_0}{1-x_0 z}f_0 + \left(\sum_j -\frac{x_j c}{1-x_j z}f_0\right)\right) \\
&= -zf_0 - z^2\frac{\partial f_0}{\partial z}
\end{aligned}
$$

Let b be the coefficient of z^{p-1} in f_0. We see that the coefficient of z^p in $-zf_0$ is $-b$. Then the coefficient of z^{p-2} in $\frac{\partial f_0}{\partial z}$ is $(p-1)b = -b$, so the coefficient of z^p in $-z^2\frac{\partial f_0}{\partial z}$ is b. Therefore the coefficient of z^p in $-zf_0 - z^2\frac{\partial f_0}{\partial z}$ is $-b+b=0$, so the coefficient of z^p in $(D_1 - D_0)(f_0)$ is 0. Then if B is the coefficient of z^p in f_0, it is clear that $(D_1 - D_0)(B) = 0$ as desired. By the symmetry of the f_a, this is the only Dunkl operator we need consider; it is clear that the coefficients of z^p in all the f_a are killed by the Dunkl operators.

2.3 Complete intersection

Let B_a be the coefficient of z^p in f_a. We see that $f_a = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right) = \left(\sum_{k=0}^{\infty} x_a^k z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$. It is then clear that B_a is a homogeneous polynomial in the x_i of degree p, since the coefficient of z^k for any k is a homogeneous polynomial in the x_i of degree k for all k (this follows from the fact that this is true in both multiplicands).

We also note that:

$$
\begin{aligned}
\sum_{a=0}^{n-1} f_a &= \left(\sum_a \frac{1}{1-x_a z}\right) F \\
&= \left(\sum_a \frac{1}{1-x_a z}\right) F - nF \\
&= \left(\sum_a \frac{x_a z - 1}{1-x_a z} + \frac{1}{1-x_a z}\right) F \\
&= \left(\frac{-x_a z}{1-x_a z}\right) F \\
&= z\frac{\partial F}{\partial z}
\end{aligned}
$$

Then the coefficient of z^p in this sum is the coefficient of z^{p-1} in $\frac{\partial F}{\partial z}$, which is p times the coefficient of z^p in F, which must be 0 since we are in characteristic p. The coefficient of z^p in this sum is also $\sum_a B_a$, so we have $\sum_a B_a = 0$.