Polynomial representations of rational Cherednik algebras of type A in positive characteristic $p \mid n$

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Abstract

We study the polynomial representation of the ratioal Cherednik algebra of type A_{n-1} with generic parameter in characteristic p for $p \mid n$. We give explicit formulas for generators for the maximal ideal, show that they cut out a complete intersection, and thus compute the Hilbert series of the irreducible quotient. Our methods are motivated by generalizing from characteristic 0 to characteristic p.

1 Introduction

In this paper we study lowest-weight representations of rational Cherednik algebras associated to the symmetric group Σ_n in characteristic p dividing n. This problem is an interesting test case for moving results from characteristic 0 to positive characteristic. Cherednik algebras, originally double affine Hecke algebras, were introduced by Cherednik in 1993; for an introduction to these algebras we refer to [4]. The representation theory of these algebras has been studied more extensively in characteristic 0 than in positive characteristic, including the case of Cherednik algebras of type A_{n-1} (associated to the symmetric group Σ_n). In general new techniques are required to study positive characteristc as opposed to characteristic 0. In this paper we directly connect the characteristc 0 case to the positive characteristic case; instead of taking complex residues, we consider the coefficients of formal power series in positive characteristic. While we only consider the case of the symmetric group Σ_n in this paper, this technique could also be used for the general case of complex reflection groups.

1.1 Definitions

Given the symmetric group Σ_n, we let $\mathcal{S}$ be the set of reflections in Σ_n. Let $\mathfrak{h}$ be the irreducible $(n-1)$-dimensional representation of Σ_n, realized as the submodule of k^n where the sum of the coordinates is 0. For each $s \in \mathcal{S}$ we assign a vector $\alpha_s \in \mathfrak{h}^*$ spanning the image of $1-s$, and choose $\alpha_s^\vee \in \mathfrak{h}$ so that $(1-s)x = \langle \alpha_s^\vee, x\rangle \alpha_s$ for all $x \in \mathfrak{h}^*$, where $\langle \cdot, \cdot \rangle$ indicates the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

Let $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ be the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$. We choose $\hbar, c \in k$. Then we define the *rational Cherednik algebra of type A* $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ as the quotient of $k[\Sigma_n] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \quad [y, y' = 0], \quad [y, x] = \hbar\langle y, x\rangle - \sum_{s \in \mathcal{S}} c\langle y, \alpha_s\rangle\langle \alpha_s^\vee, x\rangle s$$

for all $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$. We can give $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ a $\mathbb{Z}$-grading by setting $\deg x = 1$ for $x \in \mathfrak{h}^*$, $\deg y = -1$ for $y \in \mathfrak{h}$, and $\deg g = 0$ for $g \in k[G]$. We get the PBW-type decomposition $\mathcal{H}_{\hbar,c}(\mathfrak{h}) = \mathrm{Sym}(\mathfrak{h}) \otimes_k k[\Sigma_n] \otimes_k \mathrm{Sym}(\mathfrak{h}^*)$ ([4], section 3.2).

In general, for any $\alpha \neq 0$, $\mathcal{H}_{\hbar,c}(\mathfrak{h}) \simeq \mathcal{H}_{\alpha\hbar,\alpha c}(\mathfrak{h})$, so we restrict to $\hbar = 0$ or $\hbar = 1$.

1.2 Representations of rational Cherednik algebras

We are concerned with the *polynomial representation* $\mathrm{Sym}(\mathfrak{h}^*)$ of $\mathcal{H}_{\hbar,c}$, which is a polynomial ring. We refer to the polynomial representation as A; we can give this a $\mathbb{Z}$-grading in an obvious way.

As described in Section 2.5 of [1], $A = \mathrm{Sym}(\mathfrak{h}^*)$ has a unique maximal graded proper submodule J which can be realized as the kernel of the contravariant form $\beta_c : \mathrm{Sym}(\mathfrak{h}^*) \times \mathrm{Sym}(\mathfrak{h}) \to k$; β_c can be characterized by the property that for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}, f \in \mathrm{Sym}(\mathfrak{h}^*), g \in \mathrm{Sym}(\mathfrak{h})$:

$$\beta_c(xf, g) = \beta_c(f, xg), \quad \beta_c(f, yg) = \beta_c(yf, g), \quad \beta_c(1,1) = 1.$$

The quotient A/J is a finite-dimensional irreducible $\mathbb{Z}$-graded representation of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$. The *Hilbert series* $h_{A/J}(t)$ of A/J is $\sum_{j=0}^{\infty}(\dim_k L_j)t^k$ where L_j is the j-graded factor of A/J.

1.3 Main results

For a formal power series or Laurent series $r(z)$, we write the coefficient of z^ℓ in r as $[z^\ell]r(z)$.

We call c *generic* if we do not specify its value. We will be concerned with the case $\hbar = 1$, c generic, and the characteristic p of k dividing n. We realize $\mathfrak{h}$ as the subspace of k^n with the sum of the coordinates 0, which is an irreducible representation of Σ_n of dimension $n-1$. We state our result for this case below:

Theorem 3.6. *The irreducible representation A/J of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ is a complete intersection with Hilbert series $\left(\frac{1-t^p}{1-t}\right)^{n-1}$. The maximal ideal J is generated by $[z^p]F_a(z)$ for the formal power series*

$$F_a(z) = \frac{1}{1-x_a z}\prod_{j=0}^{n-1}\left(\sum_{k=0}^{p-1}\binom{c}{k}(-1+\prod_{j=0}^{n-1}(1-x_j z))^k\right)$$

for $a = 0, \ldots, n-2$.

We assume $p > 2$ since the case $p = 2$ is fully characterized in [5].

1.4 Connection to characteristic 0 results

In this paper we consider the case where the characteristic of k divides n. This is related to the case in characteristic 0 where c takes the specific value p/n, as described in [2].

Proposition 1.1 ([2], Proposition 3.1)**.** *Let the characteristic of k be 0 and set $c = p/n$. The ideal $J \subseteq A$ where A is the polynomial representation of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ is generated by $\mathrm{Res}_\infty\left[\frac{dz}{z-x_a}\prod_{i=0}^{n-1}(z-x_i)^c\right]$ for $a = 0, \ldots, n-2$.*

To get a similar result in positive characteristic, we must consider formal power series in z with coefficients from A. The formal power series would be $\frac{1}{z^{p+1}}\frac{1}{1-x_a z}\prod_{j=0}^{n-1}(1-x_j z)^c$, with the corresponding generator as the coefficient of $1/z$. We simplify and truncate the formal power series so we can define it in positive characteristic.

1.5 Relation to previous work

2 The rational Cherednik algebra

2.1 Dunkl operators

To understand the action of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ on A, we can use the PBW decompositions. The action of $\mathrm{Sym}(\mathfrak{h}^*)$ on A is by left multiplication; $k[\Sigma_n]$ acts by the diagonal action, and $\mathrm{Sym}(\mathfrak{h})$ acts via *Dunkl operators*. For $y \in \mathfrak{h}$, the Dunkl operator D_y acts on A by:

$$D_y(f) = \hbar \partial_y f - \sum_{s \in \mathcal{S}} c \frac{(y, \alpha_s)}{\alpha_s}(1-s).f.$$

We choose bases for $\mathfrak{h}, \mathfrak{h}^*$ in the following way: let V be the vector space spanned by $y_0, \ldots, y_{n-1}$ and let $\mathfrak{h}$ be the subspace spanned by $y_i - y_j$ for $i \neq j$; Σ_n acts by permuting indices. Then if $x_0, \ldots, x_{n-1}$ is the dual basis for V^*, we see that $\mathfrak{h}^*$ is the span of $x_0, \ldots, x_{n-1}$ under the relation $x_0 + \cdots + x_{n-1} = 0$; alternatively we can consider $\mathfrak{h}^*$ as the span of $x_0, \ldots, x_{n-2}$ with x_{n-1} defined as $-x_0 - \cdots - x_{n-2}$. If D_i is the Dunkl operator corresponding to y_i, the Dunkl operators for the elements of $\mathfrak{h}$ are spanned by $D_i - D_j$ with $i \neq j$.

For a transposition $s_{ij} \in \Sigma_n$ with $i < j$, we let the corresponding vector $\alpha_{s_{ij}} \in \mathfrak{h}^*$ be $x_i - x_j$.

3 Proof of the main result

We let $g(z) = \prod_{j=0}^{n-1}(1 - x_j z)$. Let $F_a(z)$ for $a = 0, \ldots, n-1$ be the formal power series in z defined by $F_a(z) = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$ where $\binom{c}{k} = \frac{c(c-1)\ldots(c-k+1)}{k!}$.

Because of the definition of the contravariant form β_c, showing that an element f of A is in the kernel of β_c is equivalent to showing that the Dunkl operators corresponding to the basis elements of $\mathfrak{h}$ annihilate f.

Proposition 3.1. *Let $f_a = [z^p]F_a(z)$. Then f_a for $a = 0, \ldots, n-1$ are annihilated by the Dunkl operators.*

Proof. Since $x_{n-1} = -x_0 - \cdots - x_{n-2}$, we see that

$$\frac{\partial g}{\partial x_i} = -\frac{zg}{1 - x_i z} + \frac{zg}{1 - x_{n-1} z}$$

for all $0 \leq i < n-1$. Let

$$F(z) = \left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right).$$

Note that $F(z)$ is symmetric and $F_a(z) = \frac{F(z)}{1-x_a z}$ for all a. Then we see that for all $0 \leq i < n-1$ we have

$$
\begin{aligned}
\frac{\partial F}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right) \\
&= \left(\sum_{k=1}^{p-1}k\binom{c}{k}(g-1)^{k-1}\right)\frac{\partial g}{\partial x_i} \\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=1}^{p-1}k\binom{c}{k}(g-1)^{k-1}\right)g \\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=0}^{p-2}(k+1)\binom{c}{k+1}(g-1)^{k}\right)g \\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)(g-1+1) \\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)+\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k+1}\right)\right) \\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)+\left(\sum_{k=1}^{p-1}c\binom{c-1}{k-1}(g-1)^{k}\right)\right) \\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(-c\binom{c-1}{p-1}(g-1)^{p-1}+\sum_{k=0}^{p-1}c\binom{c}{k}(g-1)^{k}\right) \\
&= \left(-\frac{zc}{1-x_iz}+\frac{zc}{1-x_{n-1}z}\right)\left(-\binom{c-1}{p-1}(g-1)^{p-1}+F(z)\right).
\end{aligned}
$$

We define

$$G_i(z) = \left(-\frac{zc}{1-x_iz}+\frac{zc}{1-x_{n-1}z}\right)\left(-\binom{c-1}{p-1}(g-1)^{p-1}\right).$$

Then we see that

$$\frac{\partial F}{\partial x_i} = G_i(z) + \left(-\frac{zc}{1-x_iz}+\frac{zc}{1-x_{n-1}z}\right)F(z).$$

We can now calculate $\frac{\partial F_0}{\partial x_1}, \frac{\partial F_0}{\partial x_0}$.

$$
\begin{aligned}
\frac{\partial F_0}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(\frac{1}{1-x_0z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\right) \\
&= \frac{1}{1-x_0z}\frac{\partial F}{\partial x_1} \\
&= \frac{1}{1-x_0z}\left(-\frac{zc}{1-x_1z}+\frac{zc}{1-x_{n-1}z}\right)F(z)+\frac{G_1(z)}{1-x_0z} \\
&= \left(-\frac{zc}{1-x_1z}+\frac{zc}{1-x_{n-1}z}\right)F_0(z)+\frac{G_1(z)}{1-x_0z}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F_0}{\partial x_0} &= \frac{\partial}{\partial x_1}\left(\frac{1}{1-x_0z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\right)\\
&= \frac{z}{(1-x_0z)^2}F(z) + \frac{1}{1-x_0z}\frac{\partial F}{\partial x_0}\\
&= \frac{z}{1-x_0z}F_0(z) + \frac{1}{1-x_0z}\left(-\frac{zc}{1-x_0z} + \frac{zc}{1-x_{n-1}z}\right)F(z) + \frac{G_0(z)}{1-x_0z}\\
&= \frac{z}{1-x_0z}F_0(z) + \left(-\frac{zc}{1-x_0z} + \frac{zc}{1-x_{n-1}z}\right)F_0(z) + \frac{G_0(z)}{1-x_0z}\\
&= \left(\frac{z(1-c)}{1-x_0z} + \frac{zc}{1-x_{n-1}z}\right)F_0(z) + \frac{G_0(z)}{1-x_0z}.
\end{aligned}
$$

We note that $F_0(z)$ is invariant under s_{ij} where $0 < i, j$. Therefore for transpositions we need only consider transpositions of the form s_{0i} for $0 < i \leq n-1$. We see that

$$
\begin{aligned}
\frac{1-s_{0i}}{x_0-x_i}(F_0) &= \frac{1}{x_0-x_i}\left(\frac{F(z)}{1-x_0z} - \frac{F(z)}{1-x_iz}\right)\\
&= \frac{1}{x_0-x_i}\left(\frac{1}{1-x_0z} - \frac{1}{1-x_iz}\right)F(z)\\
&= \frac{1}{x_0-x_i}\left(\frac{(1-x_iz)-(1-x_0z)}{(1-x_0z)(1-x_iz)}\right)F(z)\\
&= \frac{x_0z-x_iz}{(1-x_0z)(1-x_iz)(x_0-x_i)}F(z)\\
&= \frac{z}{(1-x_iz)(1-x_0z)}F(z)\\
&= \frac{z}{1-x_iz}F_0(z).
\end{aligned}
$$

We recall that we need only consider the action of $D_1 - D_0$ on $F_0(z)$. We consider D_0F_0, D_1F_0 separately first. We see that

$$
D_0 = \left(\frac{\partial}{\partial x_0} - c\sum_{j>0}\frac{1-s_{0j}}{x_0-x_j}\right), D_1 = \left(\frac{\partial}{\partial x_1} - c\frac{1-s_{01}}{x_1-x_0}\right)
$$

since F_0 is invariant under s_{ij} where $0 < i, j$.

$$
\begin{aligned}
D_0F_0 &= \left(\frac{\partial}{\partial x_0} - c\sum_{j>0}\frac{1-s_{0j}}{x_0-x_j}\right)(F_0)\\
&= \frac{\partial F_0}{\partial x_0} - c\sum_{j>0}\frac{1-s_{0j}}{x_0-x_j}(F_0)\\
&= \frac{G_0(z)}{1-x_0z} + \left(\frac{z(1-c)}{1-x_0z} + \frac{zc}{1-x_{n-1}z}\right)F_0(z) - \sum_{j>0}\frac{zc}{1-x_jz}F_0(z).
\end{aligned}
$$

$$
\begin{aligned}
D_1 F_0 &= \left(\frac{\partial}{\partial x_1} - c\frac{1-s_{01}}{x_1 - x_0}\right)(F_0) \\
&= \frac{\partial F_0}{\partial x_1} + c\frac{1-s_{01}}{x_0 - x_1}(F_0) \\
&= \frac{G_1(z)}{1-x_0 z} + \left(-\frac{zc}{1-x_1 z} + \frac{zc}{1-x_{n-1}z}\right)F_0(z) + \frac{zc}{1-x_1 z}F_0(z) \\
&= \frac{G_1(z)}{1-x_0 z} + \frac{zc}{1-x_{n-1}z}F_0(z).
\end{aligned}
$$

It is then easy to see that

$$
(D_1 - D_0)(F_0) = \frac{G_1 - G_0}{1-x_0 z} + \frac{z(c-1)}{1-x_0 z}F_0 + \sum_{j>0}\frac{zc}{1-x_j z}F_0.
$$

In order to show that the pth coefficient in this formal power series is 0, we must consider $\frac{\partial F_0}{\partial z}$.

We see easily that

$$
\frac{\partial g}{\partial z} = g(z)\sum_j \frac{-x_j}{1-x_j z}.
$$

We now consider $\frac{\partial F}{\partial z}$.

$$
\begin{aligned}
\frac{\partial F}{\partial z} &= \frac{\partial}{\partial z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right) \\
&= \left(\sum_{k=1}^{p-1} k\binom{c}{k}(g-1)^{k-1}\right)\frac{\partial g}{\partial z} \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\sum_{k=1}^{p-1} k\binom{c}{k}(g-1)^{k-1}\right)g(z) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\sum_{k=0}^{p-2}(k+1)\binom{c}{k+1}(g-1)^{k}\right)g(z) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)(g-1+1) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right) + \left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k+1}\right)\right) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right) + \left(\sum_{k=1}^{p-1}c\binom{c-1}{k-1}(g-1)^{k}\right)\right) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(-c\binom{c-1}{p-1}(g-1)^{p-1} + \sum_{k=0}^{p-1}c\binom{c}{k}(g-1)^{k}\right) \\
&= \left(\sum_j \frac{-cx_j}{1-x_j z}\right)F(z) + \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(-c\binom{c-1}{p-1}(g-1)^{p-1}\right).
\end{aligned}
$$

We define

$$V(z) = \left(\sum_j \frac{-x_j}{1 - x_j z}\right)\left(-c\binom{c-1}{p-1}(g-1)^{p-1}\right).$$

Then we see that

$$\frac{\partial F}{\partial z} = V(z) + \left(\sum_j \frac{-cx_j}{1 - x_j z}\right) F(z).$$

From this it follows that

$$\begin{aligned}
\frac{\partial F_0}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{F(z)}{1 - x_0 z}\right) \\
&= \frac{1}{1 - x_0 z}\frac{\partial F}{\partial z} + \frac{x_0}{1 - x_0 z} F(z) \\
&= \frac{V(z)}{1 - x_0 z} + \frac{1}{1 - x_0 z}\left(\sum_j \frac{-cx_j}{1 - x_j z}\right) F(z) + \frac{x_0}{(1 - x_0 z)^2} F(z) \\
&= \frac{V(z)}{1 - x_0 z} + \left(\sum_j \frac{-cx_j}{1 - x_j z}\right) F_0(z) + \frac{x_0}{1 - x_0 z} F_0(z).
\end{aligned}$$

We now again consider $(D_1 - D_0)(F_0)$. Recall that $n \equiv 0 \bmod p$, so in particular we can add n times any multiple of F_0 since that is 0 in characteristic p.

$$
\begin{aligned}
(D_1 - D_0)(F_0) &= \frac{G_1(z) - G_0(z)}{1 - x_0 z} + \frac{z(c-1)}{1 - x_0 z} F_0 + \sum_{j>0} \frac{zc}{1 - x_j z} F_0 \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - \frac{z}{1 - x_0 z} F_0 + \sum_{j} \frac{zc}{1 - x_j z} F_0 \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - \frac{z}{1 - x_0 z} F_0 + \left(\sum_{j} \frac{zc}{1 - x_j z} F_0 \right) - nzcF_0 \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 + zF_0 - \frac{z}{1 - x_0 z} F_0 + \left(\sum_{j} -zcF_0 + \frac{zc}{1 - x_j z} F_0 \right) \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 + \frac{z - x_0 z^2}{1 - x_0 z} F_0 - \frac{z}{1 - x_0 z} F_0 + \left(\sum_{j} \frac{-zc + x_j c z^2}{1 - x_j z} F_0 + \frac{zc}{1 - x_j z} F_0 \right) \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 + \frac{-x_0 z^2}{1 - x_0 z} F_0 + \left(\sum_{j} \frac{x_j c z^2}{1 - x_j z} F_0 \right) \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 - z^2 \left(\frac{x_0}{1 - x_0 z} F_0 + \left(\sum_{j} - \frac{x_j c}{1 - x_j z} F_0 \right) \right) \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 - z^2 \frac{\partial F_0}{\partial z} + z^2 \frac{V(z)}{1 - x_0 z} \\
&= \frac{G_1(z) - G_0(z) + z^2 V(z)}{1 - x_0 z} - zF_0 - z^2 \frac{\partial F_0}{\partial z}.
\end{aligned}
$$

We consider $V(z), G_1(z), G_0(z)$; since $z^{p+1} \mid z^{2p-2} \mid (g-1)^{p-1}$ and $(g-1)^{p-1}$ divides $V(z), G_1(z), G_0(z)$, we see that for $0 \leq \ell \leq p$ that

$$
[z^\ell]V(z) = [z^\ell]G_0(z) = [z^\ell]G_1(z) = 0.
$$

Then we also have

$$
[z^\ell](G_1(z) - G_0(z) + z^2 V(z)) = 0.
$$

Then $[z^p]\left(\frac{G_1 - G_0 + z^2 V}{1 - x_0 z}\right)$ is 0 since it is a linear combination of $[z^\ell](G_1(z) - G_0(z) + z^2 V(z))$ for $0 \leq \ell \leq p$, so

$$
[z^p]\left((D_1 - D_0)(F_0)\right) = [z^p]\left(-zF_0 - z^2 \frac{\partial F_0}{\partial z}\right).
$$

Let $b = [z^{p-1}](F_0)$. We see that $[z^p](-zF_0) = -b$. Then $[z^{p-2}]\left(\frac{\partial F_0}{\partial z}\right) = (p-1)b = -b$, so $[z^p]\left(-z^2 \frac{\partial F_0}{\partial z}\right) = b$. Therefore $[z^p]\left(-zF_0 - z^2 \frac{\partial F_0}{\partial z}\right) = -b + b = 0$, so

$$
[z^p]((D_1 - D_0)(F_0)) = (D_1 - D_0)([z^p]F_0) = (D_1 - D_0)f_0 = 0.
$$

By the symmetry of the F_a, this is the only Dunkl operator we need consider; it is clear that all the f_a are killed by the Dunkl operators. $\square$

Proposition 3.2. *The f_a for $a = 0, \ldots, n-2$ are linearly independent homogeneous polynomials of degree p.*

Proof. We recall that $F_a(z) = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right) = \left(\sum_{k=0}^{\infty} x_a^k z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$. It is then clear that f_a is a homogeneous polynomial in the x_i of degree p, since $[z^k]F_a(z)$ is a homogeneous polynomial in the x_i of degree k for all k (this follows from the fact that this is true in both multiplicands in F_a).

Assume $\sum_{a=0}^{n-2}\lambda_a f_a = 0$ for some $\lambda_a \in k$. Then in particular this must be true when we set $x_a = 1$ and $x_i = 0$ for $i \neq a, i < n-1$. (We let $x_{n-1} = -1$ so that $\sum x_i = 0$.) Note that if this is the case that $g(z) = (1-z)(1+z) = 1-z^2$, so

$$F_a(z) = \left(\sum_{k=0}^{\infty} z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(-z^2)^k\right)$$

and

$$F_i(z) = \left(\sum_{k=0}^{\infty} 0^k z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(-z^2)^k\right) = \sum_{k=0}^{p-1}\binom{c}{k}(-z^2)^k$$

for $n-1 > i \neq a$.

Therefore we have $[z^p]F_a(z) = f_a = \sum_{k=0}^{(p-1)/2}(-1)^k\binom{c}{k} = \binom{c-1}{(p-1)/2}$ and $[z^p]F_i(z) = f_i = 0$ for $n-1 > i \neq a$ (since $p > 2$). Since c is generic, we can assume $\binom{c-1}{(p-1)/2} \neq 0$. Then since $\sum_{a=0}^{n-2}\lambda_a f_a = 0$, we see that $\lambda_a\binom{c-1}{(p-1)/2} = 0$; then we must have $\lambda_a = 0$.

By setting $x_a = 1, x_{n-1} = -1$ and $x_i = 0$ for $i \neq a, i \neq n-1$ for all $a = 0, \ldots, n-1$ in succession, we find that $\lambda_a = 0$ for all a. Therefore the f_a for $a = 0, \ldots, n-2$ are linearly independent as desired.

□

Proposition 3.3. *Let $y_0, \ldots, y_{r-1}$ be distinct elements of k. Then the polynomials $b_i(z) = \prod_{j\neq i}(1-y_j z)$ are linearly independent.*

Proof. Assume that for some $\lambda_i \in k$ we have $\sum_i \lambda_i b_i(z) = 0$. Then $\left(\prod_j(1-y_j z)\right)\left(\sum_i \frac{\lambda_i}{1-y_i z}\right)$ must be 0 in the ring of formal power series. Therefore since the ring of power series is an integral domain, $\sum_i \frac{\lambda_i}{1-y_i z} = 0$. Then in particular for $\ell = 0, \ldots, r-1$ we have $\sum_i \lambda_i y_i^\ell = 0$.

This implies that the product of the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ y_0 & y_1 & \cdots & y_{r-1} \\ \cdots & \cdots & \cdots & \cdots \\ y_0^{r-1} & y_1^{r-1} & \cdots & y_{r-1}^{r-1} \end{pmatrix}$$

with the column vector $(\lambda_0, \ldots, \lambda_{r-1})$ is 0; however since all the y_i are distinct, the Vandermonde matrix has nonzero determinant $\prod_{0\leq i<j\leq r-1}(y_i - y_j)$, implying that $\lambda_i = 0$ for all i.

Therefore the polynomials $b_i(z)$ are linearly independent as desired.

□

Proposition 3.4. *Let $I \subseteq A$ be the ideal generated by f_a for $a = 0, \ldots, n-2$. A/I is a complete intersection.*

Proof. We will show that if $x_0, \ldots, x_{n-1} \in k$ satisfy $f_i(x_0, \ldots, x_{n-1}) = 0$ for all i, then $x_0 = \cdots = x_{n-1} = 0$.

We suppose that the x_i take the distinct values $\{y_0, \ldots, y_{r-1}\}$ with multiplicity $m_i > 0$; then since $\sum_{i=0}^{n-1} x_i = 0$ we have $\sum_{i=0}^{r-1} m_i y_i = 0$.

Then $g(z) = (1 - y_i z)^{m_i}$. Recall

$$F(z) = \sum_{k=0}^{p-1} \binom{c}{k} (g-1)^k.$$

We assume that $y_i \neq 0$ for all i.

Define

$$a(z) = \frac{1}{\prod_{i=0}^{r-1}(1 - y_i z)} F(z).$$

Recall that we showed

$$F'(z) = V(z) + \left(\sum_j \frac{-cx_j}{1 - x_j z} F(z) \right) = V(z) + \left(\sum_j \frac{-cm_j y_j}{1 - y_j z} F(z) \right)$$

where $[z^\ell]V(z) = 0$ for $0 \leq \ell \leq p$. We then see that

$$\begin{aligned}
a'(z) &= \frac{d}{dz} \left(\frac{1}{\prod_{i=0}^{r-1}(1 - y_i z)} F(z) \right) \\
&= \frac{1}{\prod_{i=0}^{r-1}(1 - y_i z)} F'(z) + \sum_{j=0}^{r-1} \frac{y_j}{(1 - y_j z) \prod_{i=0}^{r-1}(1 - y_i z)} F(z) \\
&= \frac{1}{\prod_{i=0}^{r-1}(1 - y_i z)} \left(V(z) + \left(\sum_j \frac{-cm_j y_j}{1 - y_j z} F(z) \right) \right) + \sum_{j=0}^{r-1} \frac{y_j}{1 - y_j z} a(z) \\
&= \frac{V(z)}{\prod_{i=0}^{r-1}(1 - y_i z)} + \left(\sum_j \frac{-cm_j y_j}{1 - y_j z} a(z) \right) + \sum_{j=0}^{r-1} \frac{y_j}{1 - y_j z} a(z) \\
&= \frac{V(z)}{\prod_{i=0}^{r-1}(1 - y_i z)} - \left(\sum_j \frac{(m_j c - 1) y_j}{1 - y_j z} a(z) \right)
\end{aligned}$$

Since $[z^\ell]V(z) = 0$ for $0 \leq \ell \leq p$ we see that $\frac{V(z)}{\prod_{i=0}^{r-1}(1 - y_i z)} = z^p b(z)$ for some power series $b(z)$. Therefore

$$a'(z) = z^p b(z) - \sum_j \frac{(m_j c - 1) y_j}{1 - y_j z} a(z)$$

for some power series $b(z)$.

Define $b_i(z) = \prod_{j \neq i}(1 - y_j z)$ for $i = 0, \ldots, r-1$; then by Proposition 3.3 we see that the polynomials $b_i(z)$ are linearly independent.

Note that for any i, $a(z) b_i(z) = \frac{1}{1 - y_i z} F(z)$. Then in particular,

$$[z^p] a(z) b_i(z) = [z^p] \frac{1}{1 - y_i z} F(z) = 0.$$

Therefore for any $\lambda_i \in k$,

$$\sum_{i=0}^{r-1} \lambda_i [z^p] a(z) b_i(z) = [z^p] \left(a(z) \sum_{i=0}^{r-1} \lambda_i b_i(z) \right) = 0. \tag{3.5}$$

Recall that the $b_i(z)$ are linearly independent; they also have degree at most $r-1$. Therefore for $0 \leq k \leq r-1$ we can choose $\{\lambda_i^k\}$ such that

$$\sum_i \lambda_i^k b_i(z) = z^k.$$

Then if we use the λ_i^k in Equation 3.5, we see that $[z^p](a(z)z^k) = [z^{p-k}]a(z) = 0$ for $k = 0, \ldots, r-1$.

For any integer ℓ we define the Laurent polynomial

$$h_\ell(z) = z^{-\ell-r} \prod_i (1 - y_i z).$$

Then

$$\begin{aligned}
\frac{d}{dz}(h_\ell(z)a(z)) &= -(l+r)z^{-1}h_\ell(z)a(z) - \sum_i \frac{y_i}{1-y_iz} h_\ell(z)a(z) + h_\ell(z)a'(z) \\
&= -(\ell+r)z^{-1}h_\ell(z)a(z) - \sum_i \frac{y_i}{1-y_iz} h_\ell(z)a(z) - h_\ell(z)\sum_j \frac{(m_jc-1)y_i}{1-y_ijz} a(z) + z^p h_\ell(z)b(z) \\
&= -(\ell+r)z^{-1}h_\ell(z)a(z) - \sum_i \frac{y_im_ic}{1-y_iz} h_\ell(z)a(z) + z^p h_\ell(z)b(z) \\
&= -\left((\ell+r)z^{-1} + \sum_i \frac{y_im_ic}{1-y_iz}\right) h_\ell(z)a(z) + z^p h_\ell(z)b(z).
\end{aligned}$$

We note that

$$-\left((\ell+r)z^{-1} + \sum_i \frac{y_im_ic}{1-y_iz}\right) h_\ell(z)$$

is a Laurent polynomial with lowest degree term

$$-z^{-l-r-1}\left(\ell + r + \sum_i y_i m_i c\right) = -z^{-\ell-r-1}(\ell+r)$$

and highest degree term

$$-z^{-\ell-1}(-1)^r \prod_i y_i \left((\ell+r) - \sum_i m_i c\right).$$

The lowest degree term of the formal power series $z^p h_\ell(z)b(z)$ has degree at least $p-l-r$. We note that we must have $[z^{-1}]\left(\frac{d}{dz}(h_\ell(z)a(z))\right) = 0$. Therefore we have a linear dependence relation between $[z^\ell]a(z), [z^{\ell+1}]a(z), \ldots, [z^{\ell+r}]a(z)$ in which $[z^\ell]a(z)$ has nonzero coefficient $(-1)^r \prod_i y_i \left((\ell+r) - \sum_i m_i c\right)$ - we can assume this is nonzero since c is generic and none of the y_i are 0. Therefore we can write $[z^\ell]a(z)$ as a linear combination of $[z^{\ell+1}]a(z), \ldots, [z^{\ell+r}]a(z)$.

We see that since $[z^{p-r+1}]a(z) = \cdots = [z^p]a(z) = 0$, we can use induction starting with $\ell = p-r$ to show that $[z^\ell]a(z) = 0$ for $\ell = p-r, p-r-1, \ldots, 0$. Therefore $[z^0]a(z) = 0$; however, it is clear from the definition of $a(z)$ that this is not the case. Therefore we have a contradiction, so one of the y_i is 0.

We can assume without loss of generality that $y_{r-1} = 0$. Then if $r > 1$, we can go through the same argument again with $y_0, \ldots, y_{r-2}$ and find that one of the y_i with $i < r-1$ is 0, which gives a contradiction since the y_i are distinct. Therefore we must have $r = 1$ and $y_0 = 0$.

$\square$

Using these propositions, we are able to prove the main theorem.

Theorem 3.6. *The irreducible representation A/J of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ is a complete intersection with Hilbert series $\left(\frac{1-t^p}{1-t}\right)^{n-1}$. The maximal ideal J is generated by $[z^p]F_a(z)$ for the formal power series*

$$F_a(z) = \frac{1}{1-x_a z}\prod_{j=0}^{n-1}\left(\sum_{k=0}^{p-1}\binom{c}{k}(-1+\prod_{j=0}^{n-1}(1-x_j z))^k\right)$$

for $a = 0, \ldots, n-2$.

Proof. It suffices to show that the f_a generate the ideal J and that A/J has Hilbert series $\left(\frac{1-t^p}{1-t}\right)^{n-1}$.

By Propositions 3.2, 3.4, A/I is a complete intersection with $n-1$ generators of degree p. It then must have Hilbert series $h_{A/I}(t) = \left(\frac{1-t^p}{1-t}\right)^{n-1}$. By Proposition 3.1, the generators of I are annihilated by the Dunkl operators, so $I \subseteq J$.

By Proposition 3.4 in [1], we see that the Hilbert series of A/J is $\left(\frac{1-t^p}{1-t}\right)^{n-1} h(t^p)$ for some polynomial h with nonnegative integer coefficients; since $I \subseteq J$, we see that $h_{A/I}(t) \geq h_{A/J}(t)$ coefficientwise; however by this restriction of the form of $h_{A/J}(t)$, we see that the only possible choice for h is $h(t) = 1$. Therefore $h_{A/I}(t) = h_{A/J}(t)$, so $I = J$ and these $n-1$ generators generate the whole ideal J.

□

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