

# Computations

$$\bar{h} = 1$$

## 1 $S_2$ , char 2, $\tau$ trivial

### 1.1 Hilbert polynomial

$$t^4 + 2t^3 + 2t^2 + 2t + 1 = (t + 1)^2(t^2 + 1).$$

### 1.2 Generators

$$x_0^2 + x_1^2$$

$$x_0^4.$$

## 2 $S_3$ , char 3, $\tau$ trivial

### 2.1 Hilbert polynomial

$$t^{12} + 3t^{11} + 6t^{10} + 8t^9 + 9t^8 + 9t^7 + 9t^6 + 9t^5 + 9t^4 + 8t^3 + 6t^2 + 3t + 1 = (t^2 + t + 1)^3(t^6 + t^3 + 1).$$

### 2.2 Generators

$$x_0^3 + x_1^3 + x_2^3$$

$$\begin{aligned} & ((2c + 2)/c)x_0^3 \\ & + ((c + 1)/c)x_1^3 \\ & - x_0^2x_1 - x_1^2x_2 - x_0x_2^2 \\ & + x_0x_1^2 + x_0^2x_2 + x_1x_2^2 \end{aligned}$$

$$x_0^9.$$

## 3 $S_4$ , char 2, $\tau$ trivial

### 3.1 Hilbert polynomial

$$t^6 + 4t^5 + 7t^4 + 8t^3 + 7t^2 + 4t + 1 = (t + 1)^4(t^2 + 1).$$

### 3.2 Generators

$$x_0^2 + x_1^2 + x_2^2 + x_3^2$$

$$((c+1)/c)x_0^2 + x_0x_1 + x_1x_2 + ((c+1)/c)x_2^2 + x_0x_3 + x_2x_3$$

$$((c+1)/c)x_0^2 + ((c+1)/c)x_1^2 + x_0x_2 + x_1x_2 + x_0x_3 + x_1x_3$$

$$x_0^4.$$

## 4 $S_5$ , char 5, $\tau$ trivial

### 4.1 Hilbert polynomial

PARTIAL

$$1 + 5t + 15t^2 + 35t^3 + 70t^4 + 122t^5 + 190t^6 + 270t^7 + \dots$$

### 4.2 Generators

PARTIAL

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5$$

$$\begin{aligned} & ((4c+4)/c)x_0^5 \\ & + ((c+1)/c)x_3^5 \\ & + x_0^4x_1 + x_0^4x_2 + x_1^4x_3 + x_2^4x_3 + x_0^4x_4 + x_3x_4^4 \\ & + 2x_0^4x_3 \\ & + 3x_0x_3^4 \\ & - x_0x_1^4 - x_0x_2^4 - x_1x_3^4 - x_2x_3^4 - x_3x_4^4 - x_0x_4^4 \\ & + ((c+4)/(c+2))x_0^2x_1^3 + ((c+4)/(c+2))x_0^2x_2^3 + ((c+4)/(c+2))x_1^2x_3^3 + ((c+4)/(c+2))x_2^2x_3^3 + ((c+4)/(c+2))x_3^3x_4^2 + ((c+4)/(c+2))x_0^2x_4^3 \\ & + ((2c+3)/(c+2))x_0^2x_3^3 \\ & + ((3c+2)/(c+2))x_0^3x_2^2 \\ & + ((4c+1)/(c+2))x_0^3x_1^2 + ((4c+1)/(c+2))x_0^3x_2^2 + ((4c+1)/(c+2))x_1^3x_3^2 + ((4c+1)/(c+2))x_2^3x_3^2 + ((4c+1)/(c+2))x_3^3x_4^2 + ((4c+1)/(c+2))x_0^2x_4^3 \\ & + (c/(c+2))x_0x_1^3x_2 + (c/(c+2))x_0x_1x_2^3 + (c/(c+2))x_0x_1^3x_4 + (c/(c+2))x_0x_2^3x_4 + (c/(c+2))x_0x_1x_4^3 + (c/(c+2))x_0x_2x_4^3 \\ & + (2c/(c+2))x_0^3x_1x_3 + (2c/(c+2))x_0^3x_2x_3 + (2c/(c+2))x_1x_2x_3^3 + (2c/(c+2))x_0^3x_3x_4 + (2c/(c+2))x_1x_3^3x_4 + (2c/(c+2))x_2x_3^3x_4 \\ & + (3c/(c+2))x_0^3x_1x_2 + (3c/(c+2))x_0x_1x_3^3 + (3c/(c+2))x_0x_2x_3^3 + (3c/(c+2))x_0^3x_1x_4 + (3c/(c+2))x_0^3x_2x_4 + (3c/(c+2))x_0x_3^3x_4 \\ & + (4c/(c+2))x_1^3x_2x_3 + (4c/(c+2))x_1x_2^3x_3 + (4c/(c+2))x_1^3x_3x_4 + (4c/(c+2))x_2^3x_3x_4 + (4c/(c+2))x_1x_3^3x_4 + (4c/(c+2))x_2x_3^3x_4 \\ & + ((c^2+4c)/(c^2+1))x_1^2x_2^2x_3 + ((c^2+4c)/(c^2+1))x_0x_1^2x_3^2 + ((c^2+4c)/(c^2+1))x_0x_2^2x_3^2 + ((c^2+4c)/(c^2+1))x_1^2x_3x_4^2 + ((c^2+4c)/(c^2+1))x_2^2x_3x_4^2 + ((c^2+4c)/(c^2+1))x_0x_3^2x_4^2 \\ & + ((2c^2+3c)/(c^2+1))x_1^2x_2x_3^2 + ((2c^2+3c)/(c^2+1))x_1x_2^2x_3^2 + ((2c^2+3c)/(c^2+1))x_1^2x_3^2x_4 + ((2c^2+3c)/(c^2+1))x_2^2x_3^2x_4 + ((2c^2+3c)/(c^2+1))x_1x_2^2x_4^2 + ((2c^2+3c)/(c^2+1))x_2x_2^2x_4^2 \\ & + ((3c^2+2c)/(c^2+1))x_0^2x_1^2x_2 + ((3c^2+2c)/(c^2+1))x_0^2x_1x_2^2 + ((3c^2+2c)/(c^2+1))x_0^2x_1^2x_4 + ((3c^2+2c)/(c^2+1))x_0^2x_2^2x_4 + ((3c^2+2c)/(c^2+1))x_1^2x_2^2x_4 + ((3c^2+2c)/(c^2+1))x_2^2x_2^2x_4 \end{aligned}$$

$$\begin{aligned}
& 1))x_0^2x_2^2x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2x_4^2 \\
& + ((4c^2 + c)/(c^2 + 1))x_0x_2^2x_2^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1^2x_3 + ((4c^2 + c)/(c^2 + 1))x_0^2x_2^2x_3 + ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_2^2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_3x_4^2 \\
& + (c^2/(c^2 + 1))x_0^2x_1x_2x_4 \\
& + (2c^2/(c^2 + 1))x_0x_1x_2x_3^2 + (2c^2/(c^2 + 1))x_1^2x_2x_3x_4 + (2c^2/(c^2 + 1))x_1x_2^2x_3x_4 + (2c^2/(c^2 + 1))x_0x_1x_3^2x_4 + (2c^2/(c^2 + 1))x_0x_2x_3^2x_4 + (2c^2/(c^2 + 1))x_1x_2x_3x_4^2 \\
& + (3c^2/(c^2 + 1))x_0^2x_1x_2x_3 + (3c^2/(c^2 + 1))x_0x_1^2x_2x_4 + (3c^2/(c^2 + 1))x_0x_1x_2^2x_4 + (3c^2/(c^2 + 1))x_0^2x_1x_3x_4 + (3c^2/(c^2 + 1))x_0^2x_2x_3x_4 + (3c^2/(c^2 + 1))x_0x_1x_2x_4^2 \\
& + (4c^2/(c^2 + 1))x_1x_2x_3^2x_4
\end{aligned}$$

$$\begin{aligned}
& ((4c+4)/c)x_0^5 + x_0^4x_1 + ((4c+1)/(c+2))x_0^3x_1^2 + ((c+4)/(c+2))x_0^2x_1^3 - x_0x_1^4 + 2x_0^4x_2 + (2c/(c+2))x_0^3x_1x_2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1^2x_2 + x_1^4x_2 + ((3c+2)/(c+2))x_0^3x_2^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_1^2x_2^2 + ((4c+1)/(c+2))x_1^3x_2^2 + ((2c+3)/(c+2))x_0^2x_3^2 + (3c/(c+2))x_0x_1x_3^2 + ((c+4)/(c+2))x_1^2x_3^2 + 3x_0x_2^4 - x_1x_2^4 + ((c+1)/c)x_0^5 + x_0^4x_3 + (3c/(c+2))x_0^3x_1x_3 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1^2x_3 + (c/(c+2))x_0x_1^3x_3 + (2c/(c+2))x_0^3x_2x_3 + (3c^2/(c^2 + 1))x_0^2x_1x_2x_3 + (4c/(c+2))x_1^3x_2x_3 + (2c^2/(c^2 + 1))x_0x_1x_2^2x_3 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2^2x_3 + (3c/(c+2))x_0x_3^2x_3 + (2c/(c+2))x_1x_3^2x_3 - x_1^4x_3 + ((4c+1)/(c+2))x_0^3x_3^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1x_3^2 + ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_3^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_2x_3^2 + ((c^2 + 4c)/(c^2 + 1))x_1^2x_2x_3^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_2^2x_3^2 + ((2c^2 + 3c)/(c^2 + 1))x_1x_2^2x_3^2 + ((c+4)/(c+2))x_2^3x_3^2 + ((c+4)/(c+2))x_0^2x_3^3 + (c/(c+2))x_0x_1x_3^3 + (4c/(c+2))x_1x_2x_3^3 + ((4c+1)/(c+2))x_2^2x_3^3 - x_0x_1^4 + x_2x_4^4 + x_1^4x_4 + (3c/(c+2))x_0^3x_1x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1^2x_4 + (c/(c+2))x_0x_1^3x_4 + (2c/(c+2))x_0^3x_2x_4 + (3c^2/(c^2 + 1))x_0^2x_1x_2x_4 + (4c/(c+2))x_1^3x_2x_4 + (2c^2/(c^2 + 1))x_0x_1x_2^2x_4 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2^2x_4 + (3c/(c+2))x_0x_3^2x_4 + (2c/(c+2))x_1x_3^2x_4 - x_1^4x_4 + (3c/(c+2))x_0^3x_3x_4 + (c^2/(c^2 + 1))x_0^2x_1x_3x_4 + (3c^2/(c^2 + 1))x_0x_1^2x_3x_4 + (3c^2/(c^2 + 1))x_0^2x_2x_3x_4 + (2c^2/(c^2 + 1))x_1^2x_2x_3x_4 + (2c^2/(c^2 + 1))x_0x_2^2x_3x_4 + (4c^2/(c^2 + 1))x_1x_2^2x_3x_4 + (2c/(c+2))x_2^3x_3x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3^2x_4 + (3c^2/(c^2 + 1))x_0x_1x_2^2x_4 + (2c^2/(c^2 + 1))x_1x_2x_3^2x_4 + ((2c^2 + 3c)/(c^2 + 1))x_2^2x_3^2x_4 + (c/(c+2))x_0x_3^3x_4 + (4c/(c+2))x_2x_3^3x_4 + ((4c+1)/(c+2))x_0^3x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_1^2x_2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_2^2x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_1x_2^2x_4^2 + ((c+4)/(c+2))x_2^3x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3x_4^2 + (3c^2/(c^2 + 1))x_0x_1x_3x_4^2 + (2c^2/(c^2 + 1))x_1x_2x_3x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_2^2x_3x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_2^3x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_2x_3^2x_4^2 + ((c+4)/(c+2))x_0^2x_4^3 + (c/(c+2))x_0x_1x_4^3 + (4c/(c+2))x_1x_2x_4^3 + ((4c+1)/(c+2))x_2^2x_4^3 + (c/(c+2))x_0x_3x_4^3 + (4c/(c+2))x_2x_3x_4^3 - x_0x_4^4 + x_2x_4^4
\end{aligned}$$

$$\begin{aligned}
& ((4c+4)/c)x_0^5 + 2x_0^4x_1 + ((3c+2)/(c+2))x_0^3x_1^2 + ((2c+3)/(c+2))x_0^2x_1^3 + 3x_0x_1^4 + ((c+1)/c)x_0^5 + x_0^4x_2 + (2c/(c+2))x_0^3x_1x_2 + (3c/(c+2))x_0x_1^2x_2 - x_1^4x_2 + ((4c+1)/(c+2))x_0^3x_2^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1x_2^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_1^2x_2^2 + ((c+4)/(c+2))x_1^3x_2^2 + ((c+4)/(c+2))x_0^2x_3^2 + ((4c+1)/(c+2))x_1^2x_3^2 - x_0x_2^4 + x_1x_2^4 + x_0^4x_3 + (2c/(c+2))x_0^3x_1x_3 + (3c/(c+2))x_0x_1^2x_3 - x_1^4x_3 + (3c/(c+2))x_0^3x_2x_3 + (3c^2/(c^2 + 1))x_0^2x_1x_2x_3 + (2c^2/(c^2 + 1))x_0x_1^2x_2x_3 + (2c/(c+2))x_1^3x_2x_3 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2^2x_3 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2^2x_3 + (c/(c+2))x_0x_2^2x_3 + (4c/(c+2))x_1x_2^2x_3 + ((4c+1)/(c+2))x_0^3x_3^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1x_3^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_1^2x_3^2 + ((c+4)/(c+2))x_1^3x_3^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2x_3^2 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2x_3^2 + ((4c^2 + c)/(c^2 + 1))x_0x_2^2x_3^2 + ((c^2 + 4c)/(c^2 + 1))x_1x_2^2x_3^2 + ((c+4)/(c+2))x_0^2x_3^3 + ((4c+1)/(c+2))x_1^2x_3^3 + (c/(c+2))x_0x_2x_3^3 + (4c/(c+2))x_1x_2x_3^3 - x_0x_3^4 + x_1x_3^4 + x_0^4x_4 + (2c/(c+2))x_0^3x_1x_4 + (3c/(c+2))x_0x_1^2x_4 - x_1^4x_4 + (3c/(c+2))x_0^3x_2x_4 + (3c^2/(c^2 + 1))x_0^2x_1x_2x_4 + (2c^2/(c^2 + 1))x_0x_1^2x_2x_4 + (2c/(c+2))x_1^3x_2x_4 + (2c/(c+2))x_1^3x_2x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2^2x_4 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2^2x_4 + (c/(c+2))x_0x_3^2x_4 + (4c/(c+2))x_1x_3^2x_4 + (3c/(c+2))x_0^3x_3x_4 + (3c^2/(c^2 + 1))x_0^2x_1x_3x_4 + (2c^2/(c^2 + 1))x_0x_1^2x_3x_4 + (2c/(c+2))x_1^3x_3x_4 + (c^2/(c^2 + 1))x_0^2x_2x_3x_4 + (4c^2/(c^2 + 1))x_1^2x_2x_3x_4 + (3c^2/(c^2 + 1))x_0x_2^2x_3x_4 + (2c^2/(c^2 + 1))x_1x_2^2x_3x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3^2x_4 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_3^2x_4 + (3c^2/(c^2 + 1))x_0x_2x_3^2x_4 + (2c^2/(c^2 + 1))x_1x_2x_3^2x_4 + (c/(c+2))x_0x_3^3x_4 + (4c/(c+2))x_2x_3^3x_4 + ((4c+1)/(c+2))x_0^3x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_1^2x_4^2 + ((c+4)/(c+2))x_1^3x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_2^2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_1x_2^2x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_3x_4^2 + (3c^2/(c^2 + 1))x_0x_2x_3x_4^2 + (2c^2/(c^2 + 1))x_1x_2x_3x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_3^2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_1x_3^2x_4^2 + ((c+4)/(c+2))x_0^2x_4^3 + ((4c+1)/(c+2))x_1^2x_4^3 + (c/(c+2))x_0x_2x_4^3 + (4c/(c+2))x_1x_2x_4^3 + (c/(c+2))x_0x_3x_4^3 + (4c/(c+2))x_1x_3x_4^3 - x_0x_4^4 + x_1x_4^4
\end{aligned}$$

conjecture:  $x_0^{25}$

## 5 $S_6$ , char 3, $\tau$ trivial

### 5.1 Hilbert polynomial

PARTIAL

$$1 + 6t + 21t^2 + 51t^3 + 96t^4 + 147t^5 + 192t^6 + 222t^7 + \dots$$

### 5.2 Generators

PARTIAL

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3$$

$$\begin{aligned} & ((2c+2)/c)x_0^3 \\ & + ((c+1)/c)x_4^3 \\ & + x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + x_1^2x_4 + x_2^2x_4 + x_3^2x_4 + x_0x_4^2 + x_0^2x_5 + x_4x_5^2 \\ & - x_0x_1^2 - x_0x_2^2 - x_0x_3^2 - x_0^2x_4 - x_1x_4^2 - x_2x_4^2 - x_3x_4^2 - x_4^2x_5 - x_0x_5^2 \\ & + (c/(c+2))x_0x_1x_2 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 \\ & + (2c/(c+2))x_1x_2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 + (2c/(c+2))x_1x_4x_5 + (2c/(c+2))x_2x_4x_5 + \\ & (2c/(c+2))x_3x_4x_5 \end{aligned}$$

$$\begin{aligned} & ((2c+2)/c)x_0^3 \\ & + ((c+1)/c)x_3^3 \\ & + x_0^2x_1 + x_0^2x_2 + x_1^2x_3 + x_2^2x_3 + x_0x_3^2 + x_0^2x_4 + x_3x_4^2 + x_0^2x_5 + x_3x_5^2 \\ & - x_0x_1^2 - x_0x_2^2 - x_0^2x_3 - x_1x_3^2 - x_2x_3^2 - x_3^2x_4 - x_0x_4^2 - x_3^2x_5 - x_0x_5^2 \\ & + (c/(c+2))x_0x_1x_2 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_4x_5 \\ & + (2c/(c+2))x_1x_2x_3 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 + (2c/(c+2))x_1x_3x_5 + (2c/(c+2))x_2x_3x_5 + \\ & (2c/(c+2))x_3x_4x_5 \end{aligned}$$

$$\begin{aligned} & ((2c+2)/c)x_0^3 \\ & + ((c+1)/c)x_2^3 \\ & + x_0^2x_1 + x_1^2x_2 + x_0x_2^2 + x_0^2x_3 + x_2x_3^2 + x_0^2x_4 + x_2x_4^2 + x_0^2x_5 + x_2x_5^2 \\ & - x_0x_1^2 - x_0^2x_2 - x_1x_2^2 - x_2^2x_3 - x_0x_3^2 - x_2^2x_4 - x_0x_4^2 - x_2^2x_5 - x_0x_5^2 \\ & + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_3x_4 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5 \\ & + (2c/(c+2))x_1x_2x_3 + (2c/(c+2))x_1x_2x_4 + (2c/(c+2))x_2x_3x_4 + (2c/(c+2))x_1x_2x_5 + (2c/(c+2))x_2x_3x_5 + \\ & (2c/(c+2))x_2x_4x_5 \end{aligned}$$

$$\begin{aligned} & ((2c+2)/c)x_0^3 \\ & + ((c+1)/c)x_1^3 \\ & - x_0^2x_1 - x_1^2x_2 - x_0x_2^2 - x_0^2x_3 - x_0x_3^2 - x_1^2x_4 - x_0x_4^2 - x_1^2x_5 - x_0x_5^2 \\ & + x_0x_1^2 + x_0^2x_2 + x_1x_2^2 + x_0^2x_3 + x_1x_3^2 + x_0^2x_4 + x_1x_4^2 + x_0^2x_5 + x_1x_5^2 \\ & + (c/(c+2))x_0x_2x_3 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_3x_4 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5 \\ & + (2c/(c+2))x_1x_2x_3 + (2c/(c+2))x_1x_2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_1x_2x_5 + (2c/(c+2))x_1x_3x_5 + \\ & (2c/(c+2))x_1x_4x_5 \end{aligned}$$

conjecture:  $x_0^9$ .

## 6 $S_6$ , char 2, $\tau$ trivial

### 6.1 Hilbert polynomial

$$t^8 + 6t^7 + 16t^6 + 26t^5 + 30t^4 + 26t^3 + 16t^2 + 6t + 1 = (t+1)^6(t^2+1)$$

### 6.2 Generators

$$\begin{aligned} & x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \\ & ((c+1)/c)x_0^2 + x_0x_1 + x_0x_2 + x_0x_3 + x_1x_4 + x_2x_4 + x_3x_4 + ((c+1)/c)x_4^2 + x_0x_5 + x_4x_5 \\ & ((c+1)/c)x_0^2 + x_0x_1 + x_0x_2 + x_1x_3 + x_2x_3 + ((c+1)/c)x_3^2 + x_0x_4 + x_3x_4 + x_0x_5 + x_3x_5 \\ & ((c+1)/c)x_0^2 + x_0x_1 + x_1x_2 + ((c+1)/c)x_2^2 + x_0x_3 + x_2x_3 + x_0x_4 + x_2x_4 + x_0x_5 + x_2x_5 \\ & ((c+1)/c)x_0^2 + ((c+1)/c)x_1^2 + x_0x_2 + x_1x_2 + x_0x_3 + x_1x_3 + x_0x_4 + x_1x_4 + x_0x_5 + x_1x_5 \\ & x_0^4. \end{aligned}$$

## 7 $S_9$ , char 3, $\tau$ trivial

### 7.1 Hilbert polynomial

PARTIAL

$$\dots + 157t^3 + 45t^2 + 9t + 1$$

### 7.2 Generators

PARTIAL

$$\begin{aligned} & x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 + x_8^3 \\ & ((2c+2)/c)x_0^3 \\ & + ((c+1)/c)x_7^3 \\ & + x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + x_0^2x_4 + x_0^2x_5 + x_0^2x_6 + x_1^2x_7 + x_2^2x_7 + x_3^2x_7 + x_4^2x_7 + x_5^2x_7 + x_6^2x_7 + x_0x_7^2 + x_0^2x_8 + x_7x_8^2 \\ & - x_0x_1^2 - x_0x_2^2 - x_0x_3^2 - x_0x_4^2 - x_0x_5^2 - x_0x_6^2 - x_0^2x_7 - x_1x_7^2 - x_2x_7^2 - x_3x_7^2 - x_4x_7^2 - x_5x_7^2 - x_6x_7^2 - x_7^2x_8 - x_0x_8^2 \\ & + (c/(c+2))x_0x_1x_2 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_3x_4 \\ & + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5 + (c/(c+2))x_0x_1x_6 + (c/(c+2))x_0x_2x_6 \\ & + (c/(c+2))x_0x_3x_6 + (c/(c+2))x_0x_4x_6 + (c/(c+2))x_0x_5x_6 + (c/(c+2))x_0x_1x_8 + (c/(c+2))x_0x_2x_8 + (c/(c+2))x_0x_3x_8 \\ & + (c/(c+2))x_0x_4x_8 + (c/(c+2))x_0x_5x_8 + (c/(c+2))x_0x_6x_8 \end{aligned}$$

$$\begin{aligned}
& + (2c/(c+2))x_1x_2x_7 + (2c/(c+2))x_1x_3x_7 + (2c/(c+2))x_2x_3x_7 + (2c/(c+2))x_1x_4x_7 + (2c/(c+2))x_2x_4x_7 + \\
& (2c/(c+2))x_3x_4x_7 + (2c/(c+2))x_1x_5x_7 + (2c/(c+2))x_2x_5x_7 + (2c/(c+2))x_3x_5x_7 + (2c/(c+2))x_4x_5x_7 + \\
& (2c/(c+2))x_1x_6x_7 + (2c/(c+2))x_2x_6x_7 + (2c/(c+2))x_3x_6x_7 + (2c/(c+2))x_4x_6x_7 + (2c/(c+2))x_5x_6x_7 + \\
& (2c/(c+2))x_1x_7x_8 + (2c/(c+2))x_2x_7x_8 + (2c/(c+2))x_3x_7x_8 + (2c/(c+2))x_4x_7x_8 + (2c/(c+2))x_5x_7x_8 + \\
& (2c/(c+2))x_6x_7x_8
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 + x_0^2x_2 + (c/(c+2))x_0x_1x_2 - x_0x_2^2 + x_0^2x_3 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 - \\
& x_0x_3^2 + x_0^2x_4 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_3x_4 - x_0x_4^2 + x_0^2x_5 + (c/(c+2))x_0x_1x_5 + \\
& (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5 - x_0x_5^2 - x_0^2x_6 + x_1^2x_6 + (2c/(c+2))x_1x_2x_6 + x_2^2x_6 + \\
& (2c/(c+2))x_1x_3x_6 + (2c/(c+2))x_2x_3x_6 + x_3^2x_6 + (2c/(c+2))x_1x_4x_6 + (2c/(c+2))x_2x_4x_6 + (2c/(c+2))x_3x_4x_6 + \\
& x_4^2x_6 + (2c/(c+2))x_1x_5x_6 + (2c/(c+2))x_2x_5x_6 + (2c/(c+2))x_3x_5x_6 + (2c/(c+2))x_4x_5x_6 + x_5^2x_6 + x_0x_6^2 - x_1x_6^2 - \\
& x_2x_6^2 - x_3x_6^2 - x_4x_6^2 - x_5x_6^2 + ((c+1)/c)x_6^3 + x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (c/(c+2))x_0x_2x_7 + (c/(c+2))x_0x_3x_7 + \\
& (c/(c+2))x_0x_4x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_1x_6x_7 + (2c/(c+2))x_2x_6x_7 + (2c/(c+2))x_3x_6x_7 + \\
& (2c/(c+2))x_4x_6x_7 + (2c/(c+2))x_5x_6x_7 - x_6^2x_7 - x_0x_7^2 + x_6x_7^2 + x_0^2x_8 + (c/(c+2))x_0x_1x_8 + (c/(c+2))x_0x_2x_8 + \\
& (c/(c+2))x_0x_3x_8 + (c/(c+2))x_0x_4x_8 + (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_1x_6x_8 + (2c/(c+2))x_2x_6x_8 + (2c/(c+2))x_3x_6x_8 + \\
& (2c/(c+2))x_4x_6x_8 + (2c/(c+2))x_5x_6x_8 - x_6^2x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_6x_7x_8 - x_0x_8^2 + x_6x_8^2
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 + x_0^2x_2 + (c/(c+2))x_0x_1x_2 - x_0x_2^2 + x_0^2x_3 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 - \\
& x_0x_3^2 + x_0^2x_4 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_3x_4 - x_0x_4^2 - x_0^2x_5 + x_1^2x_5 + (2c/(c+2))x_1x_2x_5 + \\
& x_2^2x_5 + (2c/(c+2))x_1x_3x_5 + (2c/(c+2))x_2x_3x_5 + x_3^2x_5 + (2c/(c+2))x_1x_4x_5 + (2c/(c+2))x_2x_4x_5 + \\
& (2c/(c+2))x_3x_4x_5 + x_4^2x_5 + x_0x_5^2 - x_1x_5^2 - x_2x_5^2 - x_3x_5^2 - x_4x_5^2 + ((c+1)/c)x_5^3 + x_0^2x_6 + (c/(c+2))x_0x_1x_6 + (c/(c+2))x_0x_2x_6 + \\
& (c/(c+2))x_0x_3x_6 + (c/(c+2))x_0x_4x_6 + (2c/(c+2))x_1x_5x_6 + (2c/(c+2))x_2x_5x_6 + (2c/(c+2))x_3x_5x_6 + \\
& (2c/(c+2))x_4x_5x_6 - x_5^2x_6 - x_0x_6^2 + x_5x_6^2 + x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (c/(c+2))x_0x_2x_7 + (c/(c+2))x_0x_3x_7 + \\
& (c/(c+2))x_0x_4x_7 + (2c/(c+2))x_1x_5x_7 + (2c/(c+2))x_2x_5x_7 + (2c/(c+2))x_3x_5x_7 + (2c/(c+2))x_4x_5x_7 - x_5^2x_7 + \\
& (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_5x_6x_7 - x_0x_7^2 + x_5x_7^2 + x_0^2x_8 + (c/(c+2))x_0x_1x_8 + (c/(c+2))x_0x_2x_8 + (c/(c+2))x_0x_3x_8 + \\
& (c/(c+2))x_0x_4x_8 + (2c/(c+2))x_1x_5x_8 + (2c/(c+2))x_2x_5x_8 + (2c/(c+2))x_3x_5x_8 + (2c/(c+2))x_4x_5x_8 - x_5^2x_8 + (c/(c+2))x_0x_6x_8 + \\
& (2c/(c+2))x_5x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_5x_7x_8 - x_0x_8^2 + x_5x_8^2
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 + x_0^2x_2 + (c/(c+2))x_0x_1x_2 - x_0x_2^2 + x_0^2x_3 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 - \\
& x_0x_3^2 - x_0^2x_4 + x_1^2x_4 + (2c/(c+2))x_1x_2x_4 + x_2^2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 + x_3^2x_4 + x_0x_4^2 - \\
& x_1x_4^2 - x_2x_4^2 - x_3x_4^2 + ((c+1)/c)x_4^3 + x_0^2x_5 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 + (2c/(c+2))x_1x_4x_5 + \\
& (2c/(c+2))x_2x_4x_5 + (2c/(c+2))x_3x_4x_5 - x_4^2x_5 - x_0x_5^2 + x_4x_5^2 + x_0^2x_6 + (c/(c+2))x_0x_1x_6 + (c/(c+2))x_0x_2x_6 + (c/(c+2))x_0x_3x_6 + \\
& (2c/(c+2))x_1x_4x_6 + (2c/(c+2))x_2x_4x_6 + (2c/(c+2))x_3x_4x_6 - x_4^2x_6 + (c/(c+2))x_0x_5x_6 + (2c/(c+2))x_4x_5x_6 - x_0x_6^2 + x_4x_6^2 + \\
& x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (c/(c+2))x_0x_2x_7 + (c/(c+2))x_0x_3x_7 + (2c/(c+2))x_1x_4x_7 + (2c/(c+2))x_2x_4x_7 + (2c/(c+2))x_3x_4x_7 - \\
& x_4^2x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_4x_5x_7 + (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_4x_6x_7 - x_0x_7^2 + x_4x_7^2 + x_0^2x_8 + (c/(c+2))x_0x_1x_8 + \\
& (c/(c+2))x_0x_2x_8 + (c/(c+2))x_0x_3x_8 + (2c/(c+2))x_1x_4x_8 + (2c/(c+2))x_2x_4x_8 + (2c/(c+2))x_3x_4x_8 - x_4^2x_8 + (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_4x_5x_8 + \\
& (c/(c+2))x_0x_6x_8 + (2c/(c+2))x_4x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_4x_7x_8 - x_0x_8^2 + x_4x_8^2
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 + x_0^2x_2 + (c/(c+2))x_0x_1x_2 - x_0x_2^2 - x_0^2x_3 + x_1^2x_3 + (2c/(c+2))x_1x_2x_3 + x_2^2x_3 + \\
& x_0x_3^2 - x_1x_3^2 - x_2x_3^2 + ((c+1)/c)x_3^3 + x_0^2x_4 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 - \\
& x_3^2x_4 - x_0x_4^2 + x_3x_4^2 + x_0^2x_5 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (2c/(c+2))x_1x_3x_5 + (2c/(c+2))x_2x_3x_5 - x_3^2x_5 + \\
& (c/(c+2))x_0x_4x_5 + (2c/(c+2))x_3x_4x_5 - x_0x_5^2 + x_3x_5^2 + x_0^2x_6 + (c/(c+2))x_0x_1x_6 + (c/(c+2))x_0x_2x_6 + (2c/(c+2))x_1x_3x_6 + \\
& (2c/(c+2))x_2x_3x_6 - x_3^2x_6 + (c/(c+2))x_0x_4x_6 + (2c/(c+2))x_3x_4x_6 + (c/(c+2))x_0x_5x_6 + (2c/(c+2))x_3x_5x_6 - x_0x_6^2 + x_3x_6^2 + \\
& x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (c/(c+2))x_0x_2x_7 + (2c/(c+2))x_1x_3x_7 + (2c/(c+2))x_2x_3x_7 - x_3^2x_7 + (c/(c+2))x_0x_4x_7 + \\
& (2c/(c+2))x_3x_4x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_3x_5x_7 + (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_3x_6x_7 - x_0x_7^2 + x_3x_7^2 + x_0^2x_8 + \\
& (c/(c+2))x_0x_1x_8 + (c/(c+2))x_0x_2x_8 + (2c/(c+2))x_1x_3x_8 + (2c/(c+2))x_2x_3x_8 - x_3^2x_8 + (c/(c+2))x_0x_4x_8 + (2c/(c+2))x_3x_4x_8 + \\
& (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_3x_5x_8 + (c/(c+2))x_0x_6x_8 + (2c/(c+2))x_3x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_3x_7x_8 - x_0x_8^2 + x_3x_8^2
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 - x_0^2x_2 + x_1^2x_2 + x_0x_2^2 - x_1x_2^2 + ((c+1)/c)x_2^3 + x_0^2x_3 + (c/(c+2))x_0x_1x_3 + (2c/(c+2))x_1x_2x_3 - \\
& x_2^2x_3 - x_0x_3^2 + x_2x_3^2 + x_0^2x_4 + (c/(c+2))x_0x_1x_4 + (2c/(c+2))x_1x_2x_4 - x_2^2x_4 + (c/(c+2))x_0x_3x_4 + (2c/(c+2))x_2x_3x_4 - \\
& x_0x_4^2 + x_2x_4^2 + x_0^2x_5 + (c/(c+2))x_0x_1x_5 + (2c/(c+2))x_1x_2x_5 - x_2^2x_5 + (c/(c+2))x_0x_3x_5 +
\end{aligned}$$

$$(2c/(c+2))x_2x_3x_5 + (c/(c+2))x_0x_4x_5 + (2c/(c+2))x_2x_4x_5 - x_0x_5^2 + x_2x_5^2 + x_0^2x_6 + (c/(c+2))x_0x_1x_6 + (2c/(c+2))x_1x_2x_6 - x_2^2x_6 + (c/(c+2))x_0x_3x_6 + (2c/(c+2))x_2x_3x_6 + (c/(c+2))x_0x_4x_6 + (2c/(c+2))x_2x_4x_6 + (c/(c+2))x_0x_5x_6 + (2c/(c+2))x_2x_5x_6 - x_0x_6^2 + x_2x_6^2 + x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (2c/(c+2))x_1x_2x_7 - x_2^2x_7 + (c/(c+2))x_0x_3x_7 + (2c/(c+2))x_2x_3x_7 + (c/(c+2))x_0x_4x_7 + (2c/(c+2))x_2x_4x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_2x_5x_7 + (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_2x_6x_7 - x_0x_7^2 + x_2x_7^2 + x_0^2x_8 + (c/(c+2))x_0x_1x_8 + (2c/(c+2))x_1x_2x_8 - x_2^2x_8 + (c/(c+2))x_0x_3x_8 + (2c/(c+2))x_2x_3x_8 + (c/(c+2))x_0x_4x_8 + (2c/(c+2))x_2x_4x_8 + (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_2x_5x_8 + (c/(c+2))x_0x_6x_8 + (2c/(c+2))x_2x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_2x_7x_8 - x_0x_8^2 + x_2x_8^2$$

$$((2c+2)/c)x_0^3 - x_0^2x_1 + x_0x_1^2 + ((c+1)/c)x_1^3 + x_0^2x_2 - x_1^2x_2 - x_0x_2^2 + x_1x_2^2 + x_0^2x_3 - x_1^2x_3 + (c/(c+2))x_0x_2x_3 + (2c/(c+2))x_1x_2x_3 - x_0x_3^2 + x_1x_3^2 + x_0^2x_4 - x_1^2x_4 + (c/(c+2))x_0x_2x_4 + (2c/(c+2))x_1x_2x_4 + (c/(c+2))x_0x_3x_4 + (2c/(c+2))x_1x_3x_4 - x_0x_4^2 + x_1x_4^2 + x_0^2x_5 - x_1^2x_5 + (c/(c+2))x_0x_2x_5 + (2c/(c+2))x_1x_2x_5 + (c/(c+2))x_0x_3x_5 + (2c/(c+2))x_1x_3x_5 + (c/(c+2))x_0x_4x_5 + (2c/(c+2))x_1x_4x_5 - x_0x_5^2 + x_1x_5^2 + x_0^2x_6 - x_1^2x_6 + (c/(c+2))x_0x_2x_6 + (2c/(c+2))x_1x_2x_6 + (c/(c+2))x_0x_3x_6 + (2c/(c+2))x_1x_3x_6 + (c/(c+2))x_0x_4x_6 + (2c/(c+2))x_1x_4x_6 + (c/(c+2))x_0x_5x_6 + (2c/(c+2))x_1x_5x_6 - x_0x_6^2 + x_1x_6^2 + x_0^2x_7 - x_1^2x_7 + (c/(c+2))x_0x_2x_7 + (2c/(c+2))x_1x_2x_7 + (c/(c+2))x_0x_3x_7 + (2c/(c+2))x_1x_3x_7 + (c/(c+2))x_0x_4x_7 + (2c/(c+2))x_1x_4x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_1x_5x_7 + (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_1x_6x_7 - x_0x_7^2 + x_1x_7^2 + x_0^2x_8 - x_1^2x_8 + (c/(c+2))x_0x_2x_8 + (2c/(c+2))x_1x_2x_8 + (c/(c+2))x_0x_3x_8 + (2c/(c+2))x_1x_3x_8 + (c/(c+2))x_0x_4x_8 + (2c/(c+2))x_1x_4x_8 + (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_1x_5x_8 + (c/(c+2))x_0x_6x_8 + (2c/(c+2))x_1x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_1x_7x_8 - x_0x_8^2 + x_1x_8^2$$

conjecture:  $x_0^9$

## 8 Conjecture for $p = 3, 3 \mid n$

Variables are  $x_0, \dots, x_{n-1}$ .

Generators are:

$x_0^9$  in degree 9, and  $\sum x_i^3$  in degree 3. There are  $n - 2$  remaining generators in degree 3, each with the following form:

$$\frac{c+1}{c}(x_1^3 - x_0^3) + (x_1 - x_0)(x_0x_1) + (x_1 - x_0) \left( \sum_{i \geq 2} x_i^2 - x_i(x_1 + x_0) \right) + \frac{2c}{c+2}(x_1 - x_0) \left( \sum_{i,j \geq 2; i < j} x_ix_j \right).$$

(The other generators are created from this one by switching  $x_1$  with  $x_k$  for some  $k \geq 2$ .)

We note that since  $3 \mid n$  that  $\sum_{i < j} (x_i - x_j)^2 = \sum_{i < j} x_ix_j + \sum_{i < j} x_i^2 + x_j^2 = -\sum_i x_i^2 + \sum_{i < j} x_ix_j$ .

We also note that  $\sum_i (x_i^2 - x_ix_1 - x_ix_0) = x_0x_1 + \sum_{i \geq 2} (x_i^2 - x_ix_1 - x_ix_0)$ .

We also note that  $nx_1^2 = nx_0^2 = nx_0x_1 = 0$ .

$$\begin{aligned}
& \frac{c+1}{c}(x_1^3 - x_0^3) + (x_1 - x_0)(x_0x_1) + (x_1 - x_0) \left( \sum_{i \geq 2} x_i^2 - x_i(x_1 + x_0) \right) + \frac{2c}{c+2}(x_1 - x_0) \left( \sum_{i,j \geq 2; i < j} x_ix_j \right) = \\
& = \frac{x_1 - x_0}{c(c+2)} \left( (c+1)(c+2)(x_0^2 + x_0x_1 + x_1^2) + c(c+2)(x_0x_1) + c(c+2) \left( \sum_{i \geq 2} x_i^2 - x_i(x_1 + x_0) \right) + 2c^2 \left( \sum_{i,j \geq 2; i < j} x_ix_j \right) \right) \\
& = \frac{x_1 - x_0}{c(c+2)} \left( (c^2 - 1)(x_0^2 + x_0x_1 + x_1^2) + (c^2 - c)(x_0x_1) + (c^2 - c) \left( \sum_{i \geq 2} x_i^2 - x_i(x_1 + x_0) \right) - c^2 \left( \sum_{i,j \geq 2; i < j} x_ix_j \right) \right) \\
& = \frac{x_1 - x_0}{c(c+2)} \left( (c^2 - 1)(x_0^2 + x_0x_1 + x_1^2) + (c^2 - c)(x_0x_1) + c^2 \left( \sum_{i \geq 2} x_i^2 \right) - c^2 \left( \sum_{i \geq 2} x_ix_1 + x_ix_0 \right) \right. \\
& \quad \left. - c \left( \sum_{i \geq 2} x_i^2 - x_ix_1 - x_ix_0 \right) - c^2 \left( \sum_{i,j \geq 2; i < j} x_ix_j \right) \right) \\
& = \frac{x_1 - x_0}{c(c+2)} \left( (c^2 - 1)(x_0^2 + x_0x_1 + x_1^2) + (c^2 - c)(x_0x_1) + c^2 \left( \sum_{i \geq 2} x_i^2 \right) + c^2 x_0x_1 - c \left( \sum_{i \geq 2} x_i^2 - x_ix_1 - x_ix_0 \right) \right. \\
& \quad \left. - c^2 \left( \sum_{i < j} x_ix_j \right) \right) \\
& = \frac{x_1 - x_0}{c(c+2)} \left( c^2 x_0^2 + c^2 x_1^2 - x_0^2 - x_1^2 - x_0x_1 - cx_0x_1 + c^2 \left( \sum_{i \geq 2} x_i^2 \right) - c \left( \sum_{i \geq 2} x_i^2 - x_ix_1 - x_ix_0 \right) - c^2 \left( \sum_{i < j} x_ix_j \right) \right) \\
& = \frac{x_1 - x_0}{c(c+2)} \left( -x_0^2 - x_1^2 - x_0x_1 + c^2 \left( \sum_i x_i^2 \right) - c \left( \sum_i x_i^2 - x_ix_1 - x_ix_0 \right) - c^2 \left( \sum_{i < j} x_ix_j \right) \right) \\
& = \frac{x_1 - x_0}{c(c+2)} \left( -(x_0 - x_1)^2 + c^2 \left( \sum_i x_i^2 \right) - c \left( \sum_i x_i^2 - x_ix_1 - x_ix_0 \right) - c^2 \left( \sum_{i < j} x_ix_j \right) \right) \\
& = \frac{x_1 - x_0}{c(c+2)} \left( -(x_0 - x_1)^2 - c \left( \sum_i x_i^2 - x_ix_1 - x_ix_0 + x_0x_1 \right) - c^2 \left( \sum_{i < j} (x_i - x_j)^2 \right) \right) \\
& = \frac{x_1 - x_0}{c(c+2)} \left( -(x_0 - x_1)^2 - c \left( \sum_i (x_i - x_1)(x_i - x_0) \right) - c^2 \left( \sum_{i < j} (x_i - x_j)^2 \right) \right)
\end{aligned}$$

(Checked with Sage)

Let this generator equal  $g$ . We see that for  $\{i, j\} \cap \{0, 1\} = \emptyset$  that  $s_{ij}g = g$ . Therefore  $\frac{g - s_{ij}g}{x_i - x_j} = 0$ .

We see easily that  $s_{0k}g$  for  $k \neq 1$  is  $\frac{x_1 - x_k}{c(c+2)} \left( -(x_k - x_1)^2 - c \sum_i (x_i - x_1)(x_i - x_k) - c^2 \sum_{i < j} (x_i - x_j)^2 \right)$ . From there further simple algebra tells us that  $(x_1 - x_0)(x_i - x_0)(x_i - x_1) - (x_1 - x_k)(x_i - x_1)(x_i - x_k) = (x_1 - x_i)(x_0 - x_1 + x_k - x_i)(x_k - x_0)$  for all  $i$ . From this we see easily that

$$\frac{g - s_{0k}g}{x_k - x_0} = \frac{1}{c(c+2)} \left( -(x_k - x_0)^2 - c \sum_i (x_1 - x_i)(x_0 - x_1 + x_k - x_i) - c^2 \sum_{i < j} (x_i - x_j)^2 \right).$$



Similar algebra shows us that for  $k \neq 0, 1$ :

$$\frac{g-s_{1k}g}{x_k-x_1} = \frac{1}{c(c+2)} \left( (x_k-x_1)^2 + c \sum_i (x_0-x_i)(x_1-x_0+x_k-x_i) + c^2 \sum_{i<j} (x_i-x_j)^2 \right).$$

The final case is  $s_{01}g$ . We see easily that all the terms inside the largest parentheses are left untouched by  $s_{01}$ . Therefore  $s_{01}g = -g$ , so  $\frac{g-s_{01}g}{x_1-x_0} = \frac{g-(-g)}{x_1-x_0} = \frac{2g}{x_1-x_0} = -\frac{g}{x_1-x_0}$ ; this is just

$$\frac{1}{c(c+2)} \left( (x_0-x_1)^2 + c \left( \sum_i (x_i-x_1)(x_i-x_0) \right) + c^2 \left( \sum_{i<j} (x_i-x_j)^2 \right) \right).$$

We can use these to calculate the values for the Dunkl operators. We need only check  $D_0g, D_1g, D_2g$ , because the rest are essentially equivalent to  $D_2g$ .

We start with  $D_0g$ . We see that  $D_0g = \partial_0g - c \sum_{i \geq 1} \frac{g-s_{0i}g}{x_0-x_i}$ .

We consider the partial derivative.

$$\begin{aligned} \partial_0g &= \frac{1}{c(c+2)} \left( -c \sum_{i \geq 2} ((x_i-x_1)(x_0-x_i+x_0-x_1)) - c^2 \sum_{i \geq 1} ((x_1-x_0)(2(x_0-x_i)) - (x_0-x_i)^2) \right) \\ &= \frac{1}{c(c+2)} \left( -c \sum_{i \geq 2} ((x_i-x_1)(x_0-x_i+x_0-x_1)) + c^2 \sum_{i \geq 1} ((x_1-x_0)(x_0-x_i) + (x_0-x_i)^2) \right) \\ &= \frac{1}{c(c+2)} \left( -c \sum_{i \geq 2} ((x_i-x_1)(x_0-x_i+x_0-x_1)) + c^2 \sum_i ((x_1-x_i)(x_0-x_i)) \right) \end{aligned}$$

Let  $G_1 = \frac{g-s_{01}g}{x_0-x_1} = -\frac{1}{c(c+2)} \left( (x_0-x_1)^2 + c \left( \sum_i (x_i-x_1)(x_i-x_0) \right) + c^2 \left( \sum_{i<j} (x_i-x_j)^2 \right) \right)$ . Let  $G_2 = \sum_{i \geq 2} \frac{g-s_{0i}g}{x_0-x_i}$ .

$$\begin{aligned} G_2 &= \sum_{i \geq 2} \frac{g-s_{0i}g}{x_0-x_i} \\ &= \sum_{k \geq 2} \left( \frac{1}{c(c+2)} \left( (x_k-x_0)^2 + c \sum_i (x_1-x_i)(x_0-x_1+x_k-x_i) + c^2 \sum_{i<j} (x_i-x_j)^2 \right) \right) \\ &= \frac{1}{c(c+2)} \left( \sum_{k \geq 2} \left( (x_k-x_0)^2 + c \sum_i (x_1-x_i)(x_0-x_1+x_k-x_i) \right) + (n-2)c^2 \sum_{i<j} (x_i-x_j)^2 \right) \end{aligned}$$

We note that  $n-2-1=0$  since  $3 \mid n$ . Therefore:

$$\begin{aligned}
G_2 + G_1 &= \frac{1}{c(c+2)} \left( \sum_{k \geq 2} \left( (x_k - x_0)^2 + c \sum_i (x_1 - x_i)(x_0 - x_1 + x_k - x_i) \right) + (n-2)c^2 \sum_{i < j} (x_i - x_j)^2 \right) \\
&\quad - \frac{1}{c(c+2)} \left( (x_0 - x_1)^2 + c \left( \sum_i (x_i - x_1)(x_i - x_0) \right) + c^2 \left( \sum_{i < j} (x_i - x_j)^2 \right) \right) \\
&= \frac{1}{c(c+2)} \left( \sum_{k \geq 2} \left( (x_k - x_0)^2 + c \sum_i (x_1 - x_i)(x_0 - x_1 + x_k - x_i) \right) \right. \\
&\quad \left. - (x_0 - x_1)^2 - c \left( \sum_i (x_i - x_1)(x_i - x_0) \right) \right)
\end{aligned}$$

We consider  $\sum_{k \geq 2} \sum_i (x_1 - x_i)(x_0 - x_1 + x_k - x_i) - (\sum_i (x_i - x_1)(x_i - x_0))$ . We note that  $n - 2 = 1$ , so this is equal to  $\sum_{k \geq 2} \sum_i (x_1 - x_i)(x_0 - x_1 + x_k - x_i) - \left( \sum_{k \geq 2} \sum_i (x_i - x_1)(x_i - x_0) \right)$ ; then this is equal to  $\sum_{k \geq 2} \sum_i ((x_1 - x_i)(x_0 - x_1 + x_k - x_i) - (x_i - x_1)(x_i - x_0))$ . This simplifies to  $\sum_{k \geq 2} \sum_i (x_1 - x_i)(x_k - x_1)$ .

We see that  $D_2 g = \partial_2 g - c \frac{g - s_{02}g}{x_2 - x_0} - c \frac{g - s_{12}g}{x_2 - x_1}$  because the rest cancel. Let  $G = \frac{g - s_{02}g}{x_2 - x_0} + \frac{g - s_{12}g}{x_2 - x_1}$ . From the above we see that:

$$\begin{aligned}
G &= \frac{g - s_{02}g}{x_2 - x_0} + \frac{g - s_{12}g}{x_2 - x_1} \\
&= \frac{1}{c(c+2)} \left( (x_2 - x_1)^2 - (x_2 - x_0)^2 + c \sum_i (x_0 - x_i)(x_1 - x_0 + x_2 - x_i) - (x_1 - x_i)(x_0 - x_1 + x_2 - x_i) \right) \\
&= \frac{1}{c(c+2)} \left( (x_2 - x_1 + x_2 - x_0)(x_0 - x_1) + c \sum_i (x_1 - x_0)(x_0 + x_1 - x_2 - x_i) \right) \\
&= \frac{1}{c(c+2)} \left( (x_2 - x_1 + x_2 - x_0)(x_0 - x_1) + c \sum_i x_i(x_0 - x_1) \right)
\end{aligned}$$

We consider the partial derivative.

$$\begin{aligned}
\partial_2 g &= \frac{x_1 - x_0}{c(c+2)} \left( -c(x_2 - x_1 + x_2 - x_0) - c^2 \sum_{i \neq 2} 2(x_2 - x_i) \right) \\
&= \frac{x_1 - x_0}{c(c+2)} \left( -c(x_2 - x_1 + x_2 - x_0) + c^2 \sum_{i \neq 2} x_2 - x_i \right) \\
&= \frac{x_1 - x_0}{c(c+2)} \left( -c(x_2 - x_1 + x_2 - x_0) + c^2 \sum_{i \neq 2} -x_i + c^2(n-1)x_2 \right) \\
&= \frac{x_1 - x_0}{c(c+2)} \left( -c(x_2 - x_1 + x_2 - x_0) - c^2 \sum_i x_i \right)
\end{aligned}$$

We see that  $\partial_2 g = cG$ , so  $D_2 g = 0$  as desired.

## 9 Conjecture for $p = 5, 5 \mid n$

Variables are  $x_0, \dots, x_{n-1}$ .

Generators are:

$x_0^{25}$  in degree 25, and  $\sum x_i^5$  in degree 5. There are  $n-2$  remaining generators in degree 3, each with the following form:

$$\begin{aligned}
&\frac{c+1}{c}(x_1^5 - x_0^5) + 2x_0^4 x_1 + 3x_0 x_1^4 + \left( \sum_{i \geq 2} x_0^4 x_i + x_1 x_i^4 - x_0 x_i^4 - x_1^4 x_i \right) + \frac{2c+3}{c+2} x_0^2 x_1^3 + \frac{3c+2}{c+2} x_0^3 x_1^2 + \\
&\frac{c+4}{c+2} \left( \sum_{i \geq 2} x_0^2 x_i^3 + x_1^3 x_i^2 - x_0^3 x_i^2 - x_1^2 x_i^3 \right) + \frac{c}{c+2} \left( \sum_{i, j \geq 2; i < j} x_0 x_i^3 x_j + x_0 x_i x_j^3 - x_1 x_i^3 x_j - x_1 x_i x_j^3 \right) + \\
&\frac{2c}{c+2} \left( \sum_{i \geq 2} x_0^3 x_1 x_i - x_0 x_1^3 x_i \right) + \frac{2c}{c+2} \left( \sum_{i, j \geq 2; i < j} x_i x_j (x_1^3 - x_0^3) \right) + \\
&\frac{c^2+4c}{c^2+1} \left( \sum_{i \geq 2} x_0 x_1^2 x_i^2 - x_0^2 x_1 x_i^2 \right) + \frac{c^2+4c}{c^2+1} \left( \sum_{i, j \geq 2; i < j} x_1 x_i^2 x_j^2 - x_0 x_i^2 x_j^2 \right) + \\
&\frac{2c^3+3c}{c^2+1} \left( \sum_{i, j \geq 2; i < j} x_1^2 x_i^2 x_j + x_1^2 x_i x_j^2 - x_0^2 x_i x_j^2 - x_0^2 x_i^2 x_j \right) + \frac{c^2}{c^2+1} \left( \sum_{i, j, k \geq 2; i < j < k} x_0^2 x_i x_j x_k - x_1^2 x_i x_j x_k \right) + \\
&\frac{2c^2}{c^2+1} \left( \sum_{i, j \geq 2; i < j} x_0 x_1^2 x_i x_j - x_0^2 x_1 x_i x_j \right) + \\
&\frac{2c^2}{c^2+1} \left( \sum_{i, j, k \geq 2; i < j < k} x_1 x_i^2 x_j x_k + x_1 x_i x_j^2 x_k + x_1 x_i x_j x_k^2 - x_0 x_i^2 x_j x_k - x_0 x_i x_j^2 x_k - x_0 x_i x_j x_k^2 \right)
\end{aligned}$$

It is definitely possible to factor out  $x_1 - x_0$  from this.

(The other generators are created from this one by switching  $x_1$  with  $x_k$  for some  $k \geq 2$ .)

## 10 Conjecture for general $p \mid n$

Variables are  $x_0, \dots, x_{n-1}$ .

Generators are:

$x_0^{p^2}$  in degree  $p^2$ , and  $\sum x_i^p$  in degree  $p$ . There are  $n - 2$  remaining generators in degree  $p$ . It is clear that each such generator contains a term of the form  $\frac{c+1}{c}(x_k^p - x_0^p)$ . If we assume we are in the generator with  $k = 1$ , then the generator also contains a term  $\left(\sum_{i \geq 2} x_0^{p-1} x_i + x_1 x_i^{p-1} - x_0 x_i^{p-1} - x_1^{p-1} x_i\right)$