We let $g = \prod_{j=0}^{n-1}(1 - x_j z)$. Let F_a for $a = 0, \ldots, n-1$ be the formal power series in z defined by $F_a = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$ where $\binom{c}{k} = \frac{c(c-1)\ldots(c-k+1)}{k!}$. We note that the coefficients of this power series lie in A; we will get the generators of the ideal J from these coefficients.

Proposition 0.1. *Let f_a be the coefficient of z^p in the power series F_a. Then f_a for $a = 0, \ldots, n-1$ are annihilated by the Dunkl operators.*

Proof. Taking the Dunkl operator of an element of A consists of taking derivatives in the x_i, dividing by polynomials in the x_i, and letting the symmetric group act on the x_i, in addition to linear operations. We see that this means we can apply the Dunkl operators to F_a and check that the coefficient of z^p in the result is 0 to show that the Dunkl operators annihilate the f_a.

We note that each F_a is symmetric in the x_i not including x_a, and that for any transposition $s_{ab} \in \Sigma_n$, $s_{ab}F_a = F_b$. Therefore we need only consider the action of the Dunkl operators on F_0. We also note that $\mathfrak{h}$ is spanned by $y_i - y_0$ for $0 < i \leq n-1$; if the Dunkl operator corresponding to y_j is D_j, then using the fact that F_0 is symmetric in the x_i with $i \neq 0$, we need only show that $(D_1 - D_0)(F_0)$ has z^p coefficient 0 to show that all of the f_a are annihilated by the Dunkl operators.

We also note that adding powers z^k with $k > p$ will not change the value of the z^p coefficient in $(D_1 - D_0)(F_0)$. In particular, we note that since $x_0 + \cdots + x_{n-1} = 0$ divides the coefficient of z in g, we have $z^2 \mid g - 1$. Then since $p > 2$, we note that $z^{p+1} \mid z^{2p-2} \mid (g-1)^{p-1}$. Therefore we can add multiples of $(g-1)^{p-1}$ when taking the Dunkl operator's action on F_0, since even when multipled by another power series it cannot contribute anything to the coefficient of z^p. We also note that we can add n times any multiple of F_0 since $n \equiv 0 \bmod p$.

Using the allowed manipulations and the fact that $x_{n-1} = -x_0 + \cdots - x_{n-2}$, we see that $\frac{\partial F_0}{\partial x_1} = \left(\frac{zc}{1-x_{n-1}z} - \frac{zc}{1-x_1 z}\right)F_0$ and $\frac{\partial F_0}{\partial x_0} = \left(\frac{zc}{1-x_{n-1}z} + \frac{z(1-c)}{1-x_1 z}\right)F_0$ up to the z^p coefficient, which is all that we need.

We note that when $0 < i, j$ we have $\frac{1-s_{ij}}{x_i - x_j}(F_0) = 0$. We also see that for $0 < i \leq n-1$ we have $\frac{1-s_{ij}}{x_i - x_j}(F_0) = \frac{z}{1-x_i z}F_0$.

We also consider $\frac{\partial F_0}{\partial z}$; up to the addition of some multiple of $(g-1)^{p-1}$, this is equal to $\frac{x_0}{1-x_0 z}F_0 - \sum_{j \geq 0}\frac{-x_j c}{1-x_j z}F_0$.

Then we see that:

$$
\begin{aligned}
(D_1 - D_0)(F_0) &= \frac{\partial F_0}{\partial x_1} - \frac{\partial F_0}{\partial x_0} - c\frac{1-s_{01}}{x_1 - x_0}(F_0) + c\sum_{j>0}\frac{1-s_{0j}}{x_0 - x_j}(F_0) \\
&= \left(\frac{zc}{1-x_{n-1}z} - \frac{zc}{1-x_1 z}\right)F_0 - \left(\frac{zc}{1-x_{n-1}z} + \frac{z(1-c)}{1-x_1 z}\right)F_0 + \frac{zc}{1-x_1 z}F_0 + \sum_{j>0}\frac{zc}{1-x_j z}F_0 \\
&= \frac{z(c-1)}{1-x_0 z}F_0 + \sum_{j>0}\frac{zc}{1-x_j z}F_0 \\
&= -\frac{z}{1-x_0 z}f_0 + \sum_{j}\frac{zc}{1-x_j z}f_0 \\
&= -\frac{z}{1-x_0 z}f_0 + \left(\sum_{j}\frac{zc}{1-x_j z}f_0\right) - nzcf_0 \\
&= -zf_0 + \frac{z - x_0 z^2}{1-x_0 z}f_0 - \frac{z}{1-x_0 z}f_0 + \left(\sum_{j}\frac{-zc + x_j c z^2}{1-x_j z}f_0 + \frac{zc}{1-x_j z}f_0\right) \\
&= -zf_0 + \frac{-x_0 z^2}{1-x_0 z}f_0 + \left(\sum_{j}\frac{x_j c z^2}{1-x_j z}f_0\right) \\
&= -zf_0 - z^2\frac{\partial f_0}{\partial z}
\end{aligned}
$$

Then if b is the coefficient of z^{p-1} in f_0, we see that the coefficient of z^p in $-zf_0$ is $-b$, and the coefficient of z^p in $-z^2\frac{\partial f_0}{\partial z}$ is $-(p-1)b = b$. Therefore the coefficient of z^p in $(D_1 - D_0)(F_0)$ is $-b + b = 0$; this means that $(D_1 - D_0)(f_0) = 0$.

Then as we discussed above, the symmetry of the f_a means that all the f_a are annihilated by the Dunkl operators, as desired.

$\square$

Proposition 0.2. *The f_a for $a = 0, \ldots, n-2$ are linearly independent, with $f_{n-1} = -\sum_{a=0}^{n-2} f_a$.*

Proof. We recall that $F_a = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right) = \left(\sum_{k=0}^{\infty} x_a^k z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$. It is then clear that f_a is a homogeneous polynomial in the x_i of degree p, since the coefficient of z^k for any k in F_a is a homogeneous polynomial in the x_i of degree k for all k (this follows from the fact that this is true in both multiplicands in F_a).

We also note that:

$$
\begin{aligned}
\sum_{a=0}^{n-1} F_a &= \left(\sum_a \frac{1}{1-x_a z}\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\\
&= \left(\sum_a \frac{1}{1-x_a z}\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right) - n\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\\
&= \left(\sum_a \frac{x_a z-1}{1-x_a z}+\frac{1}{1-x_a z}\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\\
&= \left(\frac{-x_a z}{1-x_a z}\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\\
&= \frac{z}{c}\frac{\partial g}{\partial z}\left(\sum_{k=0}^{p-1} c\binom{c}{k}(g-1)^k\right)\\
&= z\frac{\partial}{\partial z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)
\end{aligned}
$$

This equality only holds up to the z^p coefficient, since we implicitly add a multiple of $(g-1)^{p-1}$ in the last step.

Then the coefficient of z^p in this sum is the coefficient of z^{p-1} in $\frac{\partial}{\partial z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$, which is p times the coefficient of z^p in $\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k$, which must be 0 since we are in characteristic p. The coefficient of z^p in this sum is also $\sum_{a=0}^{n-1} f_a$, so we have $\sum_{a=0}^{n-1} f_a = 0$, and $f_{n-1} = -\sum_{a=0}^{n-2} f_a$.

We note that we can write the f_a as polynomials in c with coefficients from the polynomial ring $\mathbb{F}_p[x_i]$; we can therefore consider the 'constant term' of f_a as a polynomial in c. Recall that f_a is the coefficient of z^p in $F_a = \left(\sum_{k=0}^{\infty} x_a^k z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$. Then as polynomials it is clear that $c \mid \binom{c}{k}$ for all $k>0$; therefore when trying to find the constant term of the coefficient of z^p, we can ignore the terms with $k>0$ in the second multiplicand. The term for $k=0$ is just 1; it is then clear that the constant term (the coefficient of c^0) of f_a is x_a^p.

If $\sum_{a=0}^{n-2}\lambda_a f_a = 0$ for some λ_a rational functions in c, we can multiply through by a least common denominator and assume the λ_a are polynomials in c. Then the constant term of this sum is $\sum_{a=0}^{n-2} C(\lambda_a)x_a^p$ where $C(\lambda_a)$ represents the constant term of λ_a. Since the x_a^p for $a = 0,\ldots,n-2$ are clearly linearly independent, we see that $C(\lambda_a)$ must be 0 for $a=0,\ldots,n-2$. Then we can factor out c from all of the λ_a, since if $\sum_{a=0}^{n-2}\lambda_a f_a = 0$, we have $\sum_{a=0}^{n-2}\frac{1}{c}\lambda_a f_a = 0$ as well; we have that $\frac{1}{c}\lambda_a$ is a polynomial since the constant terms in all the λ_a are 0. Then we see that by the same logic, the constant terms of λ_a/c for all a are 0, so the coefficient of c in all of the λ_a is 0. This means that $c^2 \mid \lambda_a$ for all a, so we can perform the same calculation again to get that the coefficient of c^2 is 0 and so forth. If $e = \max_a \deg\lambda_a$, we need only apply this logic e times to show that the coefficients of $e = 1, c, \ldots, c^e$ in all of the λ_a are 0, meaning all of the λ_a are 0.

Then since $\sum_{a=0}^{n-2}\lambda_a f_a = 0$ means all the λ_a are 0, we see that f_a for $a=0,\ldots,n-2$ are linearly independent as desired.

□

let I the ideal generated by f_a

Proposition 0.3. *A/I is a complete intersection.*

Theorem 0.4. *The f_a generate the ideal J; A/J has Hilbert series $\left(\frac{1-t^p}{1-t}\right)^{n-1}$.*