Computations

$\bar{h} = 1$

1 S_2, char 2, τ trivial

1.1 Hilbert polynomial

$t^4 + 2t^3 + 2t^2 + 2t + 1 = (t+1)^2(t^2+1)$.

1.2 Generators

$x_0^2 + x_1^2$

x_0^4.

2 S_3, char 3, τ trivial

2.1 Hilbert polynomial

$t^{12} + 3t^{11} + 6t^{10} + 8t^9 + 9t^8 + 9t^7 + 9t^6 + 9t^5 + 9t^4 + 8t^3 + 6t^2 + 3t + 1 = (t^2+t+1)^3(t^6+t^3+1)$.

2.2 Generators

$x_0^3 + x_1^3 + x_2^3$

$((2c+2)/c)x_0^3$
$+ ((c+1)/c)x_1^3$
$- x_0^2x_1 - x_1^2x_2 - x_0x_2^2$
$+ x_0x_1^2 + x_0^2x_2 + x_1x_2^2$

x_0^9.

3 S_4, char 2, τ trivial

3.1 Hilbert polynomial

$t^6 + 4t^5 + 7t^4 + 8t^3 + 7t^2 + 4t + 1 = (t+1)^4(t^2+1)$.

3.2 Generators

$x_0^2 + x_1^2 + x_2^2 + x_3^2$

$((c+1)/c)x_0^2 + x_0x_1 + x_1x_2 + ((c+1)/c)x_2^2 + x_0x_3 + x_2x_3$

$((c+1)/c)x_0^2 + ((c+1)/c)x_1^2 + x_0x_2 + x_1x_2 + x_0x_3 + x_1x_3$

x_0^4.

4 S_5, char 5, τ trivial

4.1 Hilbert polynomial

PARTIAL

$1 + 5t + 15t^2 + 35t^3 + 70t^4 + 122t^5 + 190t^6 + 270t^7 + \dots$

4.2 Generators

PARTIAL

$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5$

$((4c+4)/c)x_0^5$
$+ ((c+1)/c)x_3^5$
$+ x_0^4x_1 + x_0^4x_2 + x_1^4x_3 + x_2^4x_3 + x_0^4x_4 + x_3x_4^4$
$+ 2x_0^4x_3$
$+ 3x_0x_3^4$
$- x_0x_1^4 - x_0x_2^4 - x_1x_3^4 - x_2x_3^4 - x_3^4x_4 - x_0x_4^4$
$+ ((c+4)/(c+2))x_0^2x_1^3 + ((c+4)/(c+2))x_0^2x_2^3 + ((c+4)/(c+2))x_1^2x_3^3 + ((c+4)/(c+2))x_2^2x_3^3 + ((c+4)/(c+2))x_3^3x_4^2 + ((c+4)/(c+2))x_0^2x_4^3$
$+ ((2c+3)/(c+2))x_0^2x_3^3$
$+ ((3c+2)/(c+2))x_0^3x_3^2$
$+ ((4c+1)/(c+2))x_0^3x_1^2 + ((4c+1)/(c+2))x_0^3x_2^2 + ((4c+1)/(c+2))x_1^3x_3^2 + ((4c+1)/(c+2))x_2^3x_3^2 + ((4c+1)/(c+2))x_0^3x_4^2 + ((4c+1)/(c+2))x_3^2x_4^3$
$+ (c/(c+2))x_0x_1^3x_2 + (c/(c+2))x_0x_1x_2^3 + (c/(c+2))x_0x_1^3x_4 + (c/(c+2))x_0x_2^3x_4 + (c/(c+2))x_0x_1x_4^3 + (c/(c+2))x_0x_2x_4^3$
$+ (2c/(c+2))x_0^3x_1x_3 + (2c/(c+2))x_0^3x_2x_3 + (2c/(c+2))x_1x_2x_3^3 + (2c/(c+2))x_0^3x_3x_4 + (2c/(c+2))x_1x_3^3x_4 + (2c/(c+2))x_2x_3^3x_4$
$+ (3c/(c+2))x_0^3x_1x_2 + (3c/(c+2))x_0x_1x_3^3 + (3c/(c+2))x_0x_2x_3^3 + (3c/(c+2))x_0^3x_1x_4 + (3c/(c+2))x_0^3x_2x_4 + (3c/(c+2))x_0x_3^3x_4$
$+ (4c/(c+2))x_1^3x_2x_3 + (4c/(c+2))x_1x_2^3x_3 + (4c/(c+2))x_1^3x_3x_4 + (4c/(c+2))x_2^3x_3x_4 + (4c/(c+2))x_1x_3x_4^3 + (4c/(c+2))x_2x_3x_4^3$
$+ ((c^2+4c)/(c^2+1))x_1^2x_2^2x_3 + ((c^2+4c)/(c^2+1))x_0x_1^2x_3^2 + ((c^2+4c)/(c^2+1))x_0x_2^2x_3^2 + ((c^2+4c)/(c^2+1))x_1^2x_3x_4^2 + ((c^2+4c)/(c^2+1))x_2^2x_3x_4^2 + ((c^2+4c)/(c^2+1))x_0x_3^2x_4^2$
$+ ((2c^2+3c)/(c^2+1))x_1^2x_2x_3^2 + ((2c^2+3c)/(c^2+1))x_1x_2^2x_3^2 + ((2c^2+3c)/(c^2+1))x_1^2x_3^2x_4 + ((2c^2+3c)/(c^2+1))x_2^2x_3^2x_4 + ((2c^2+3c)/(c^2+1))x_1x_3^2x_4^2 + ((2c^2+3c)/(c^2+1))x_2x_3^2x_4^2$
$+ ((3c^2+2c)/(c^2+1))x_0^2x_1^2x_2 + ((3c^2+2c)/(c^2+1))x_0^2x_1x_2^2 + ((3c^2+2c)/(c^2+1))x_0^2x_1^2x_4 + ((3c^2+2c)/(c^2+$

$1))x_0^2x_2^2x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2x_4^2$
$+ ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_2^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1^2x_3 + ((4c^2 + c)/(c^2 + 1))x_0^2x_2^2x_3 + ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_2^2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_3x_4^2$
$+ (c^2/(c^2 + 1))x_0^2x_1x_2x_4$
$+ (2c^2/(c^2 + 1))x_0x_1x_2x_3^2 + (2c^2/(c^2 + 1))x_1^2x_2x_3x_4 + (2c^2/(c^2 + 1))x_1x_2^2x_3x_4 + (2c^2/(c^2 + 1))x_0x_1x_3^2x_4 + (2c^2/(c^2 + 1))x_0x_2x_3^2x_4 + (2c^2/(c^2 + 1))x_1x_2x_3x_4^2$
$+ (3c^2/(c^2 + 1))x_0^2x_1x_2x_3 + (3c^2/(c^2 + 1))x_0x_1^2x_2x_4 + (3c^2/(c^2 + 1))x_0x_1x_2^2x_4 + (3c^2/(c^2 + 1))x_0^2x_1x_3x_4 + (3c^2/(c^2 + 1))x_0^2x_2x_3x_4 + (3c^2/(c^2 + 1))x_0x_1x_2x_4^2$
$+ (4c^2/(c^2 + 1))x_1x_2x_3^2x_4$

$((4c+4)/c)x_0^5 + x_0^4x_1 + ((4c+1)/(c+2))x_0^3x_1^2 + ((c+4)/(c+2))x_0^2x_1^3 - x_0x_1^4 + 2x_0^4x_2 + (2c/(c+2))x_0^3x_1x_2 + ((4c^2 + c)/(c^2+1))x_0^2x_1^2x_2 + x_1^4x_2 + ((3c+2)/(c+2))x_0^3x_2^2 + ((c^2+4c)/(c^2+1))x_0x_1^2x_2^2 + ((4c+1)/(c+2))x_1^3x_2^2 + ((2c+3)/(c+2))x_0^2x_2^3 + (3c/(c+2))x_0x_1x_2^3 + ((c+4)/(c+2))x_1^2x_2^3 + 3x_0x_2^4 - x_1x_2^4 + ((c+1)/c)x_2^5 + x_0^4x_3 + (3c/(c+2))x_0^3x_1x_3 + ((3c^2+2c)/(c^2+1))x_0^2x_1^2x_3 + (c/(c+2))x_0x_1^3x_3 + (2c/(c+2))x_0^3x_2x_3 + (3c^2/(c^2+1))x_0^2x_1x_2x_3 + (4c/(c+2))x_1^3x_2x_3 + (2c^2/(c^2+1))x_0x_1x_2^2x_3 + ((2c^2+3c)/(c^2+1))x_1^2x_2^2x_3 + (3c/(c+2))x_0x_2^3x_3 + (2c/(c+2))x_1x_2^3x_3 - x_2^4x_3 + ((4c+1)/(c+2))x_0^3x_3^2 + ((3c^2+2c)/(c^2+1))x_0^2x_1x_3^2 + ((4c^2+c)/(c^2+1))x_0x_1^2x_3^2 + ((4c^2 + c)/(c^2+1))x_0^2x_2x_3^2 + ((c^2+4c)/(c^2+1))x_1^2x_2x_3^2 + ((c^2+4c)/(c^2+1))x_0x_2^2x_3^2 + ((2c^2+3c)/(c^2+1))x_1x_2^2x_3^2 + ((c+4)/(c+2))x_2^3x_3^2 + ((c+4)/(c+2))x_0^2x_3^3 + (c/(c+2))x_0x_1x_3^3 + (4c/(c+2))x_1x_2x_3^3 + ((4c+1)/(c+2))x_2^2x_3^3 - x_0x_3^4 + x_2x_3^4 + x_0^4x_4 + (3c/(c+2))x_0^3x_1x_4 + ((3c^2+2c)/(c^2+1))x_0^2x_1^2x_4 + (c/(c+2))x_0x_1^3x_4 + (2c/(c+2))x_0^3x_2x_4 + (3c^2/(c^2+1))x_0^2x_1x_2x_4 + (4c/(c+2))x_1^3x_2x_4 + (2c^2/(c^2+1))x_0x_1x_2^2x_4 + ((2c^2+3c)/(c^2+1))x_1^2x_2^2x_4 + (3c/(c+2))x_0x_2^3x_4 + (2c/(c+2))x_1x_2^3x_4 - x_2^4x_4 + (3c/(c+2))x_0^3x_3x_4 + (c^2/(c^2+1))x_0^2x_1x_3x_4 + (3c^2/(c^2+1))x_0x_1^2x_3x_4 + (3c^2/(c^2+1))x_0^2x_2x_3x_4 + (2c^2/(c^2+1))x_1^2x_2x_3x_4 + (2c^2/(c^2+1))x_0x_2^2x_3x_4 + (4c^2/(c^2+1))x_1x_2^2x_3x_4 + (2c/(c+2))x_2^3x_3x_4 + ((3c^2+2c)/(c^2+1))x_0^2x_3^2x_4 + (3c^2/(c^2+1))x_0x_1x_3^2x_4 + (2c^2/(c^2+1))x_1x_2x_3^2x_4 + ((2c^2+3c)/(c^2+1))x_2^2x_3^2x_4 + (c/(c+2))x_0x_3^3x_4 + (4c/(c+2))x_2x_3^3x_4 + ((4c+1)/(c+2))x_0^3x_4^2 + ((3c^2+2c)/(c^2+1))x_0^2x_1x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_1^2x_2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_2^2x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_1x_2^2x_4^2 + ((c + 4)/(c + 2))x_2^3x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3x_4^2 + (3c^2/(c^2 + 1))x_0x_1x_3x_4^2 + (2c^2/(c^2 + 1))x_1x_2x_3x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_2^2x_3x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_3^2x_4^2 + ((c^2 + 4c)/(c^2+1))x_2x_3^2x_4^2 + ((c+4)/(c+2))x_0^2x_4^3 + (c/(c+2))x_0x_1x_4^3 + (4c/(c+2))x_1x_2x_4^3 + ((4c+1)/(c+2))x_2^2x_4^3 + (c/(c + 2))x_0x_3x_4^3 + (4c/(c + 2))x_2x_3x_4^3 - x_0x_4^4 + x_2x_4^4$

$((4c+4)/c)x_0^5 + 2x_0^4x_1 + ((3c+2)/(c+2))x_0^3x_1^2 + ((2c+3)/(c+2))x_0^2x_1^3 + 3x_0x_1^4 + ((c+1)/c)x_1^5 + x_0^4x_2 + (2c/(c+2))x_0^3x_1x_2 + (3c/(c+2))x_0x_1^3x_2 - x_1^4x_2 + ((4c+1)/(c+2))x_0^3x_2^2 + ((4c^2+c)/(c^2+1))x_0^2x_1x_2^2 + ((c^2+4c)/(c^2+1))x_0x_1^2x_2^2 + ((c+4)/(c+2))x_1^3x_2^2 + ((c+4)/(c+2))x_0^2x_2^3 + ((4c+1)/(c+2))x_1^2x_2^3 - x_0x_2^4 + x_1x_2^4 + x_0^4x_3 + (2c/(c+2))x_0^3x_1x_3 + (3c/(c+2))x_0x_1^3x_3 - x_1^4x_3 + (3c/(c+2))x_0^3x_2x_3 + (3c^2/(c^2+1))x_0^2x_1x_2x_3 + (2c^2/(c^2+1))x_0x_1^2x_2x_3 + (2c/(c+2))x_1^3x_2x_3 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2^2x_3 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2^2x_3 + (c/(c+2))x_0x_2^3x_3 + (4c/(c+2))x_1x_2^3x_3 + ((4c + 1)/(c + 2))x_0^3x_3^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1x_3^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_1^2x_3^2 + ((c + 4)/(c+2))x_1^3x_3^2 + ((3c^2+2c)/(c^2+1))x_0^2x_2x_3^2 + ((2c^2+3c)/(c^2+1))x_1^2x_2x_3^2 + ((4c^2+c)/(c^2+1))x_0x_2^2x_3^2 + ((c^2+4c)/(c^2+1))x_1x_2^2x_3^2 + ((c+4)/(c+2))x_0^2x_3^3 + ((4c+1)/(c+2))x_1^2x_3^3 + (c/(c+2))x_0x_2x_3^3 + (4c/(c+2))x_1x_2x_3^3 - x_0x_3^4 + x_1x_3^4 + x_0^4x_4 + (2c/(c+2))x_0^3x_1x_4 + (3c/(c+2))x_0x_1^3x_4 - x_1^4x_4 + (3c/(c+2))x_0^3x_2x_4 + (3c^2/(c^2+1))x_0^2x_1x_2x_4 + (2c^2/(c^2+1))x_0x_1^2x_2x_4 + (2c/(c+2))x_1^3x_2x_4 + ((3c^2+2c)/(c^2+1))x_0^2x_2^2x_4 + ((2c^2+3c)/(c^2+1))x_1^2x_2^2x_4 + (c/(c+2))x_0x_2^3x_4 + (4c/(c+2))x_1x_2^3x_4 + (3c/(c+2))x_0^3x_3x_4 + (3c^2/(c^2+1))x_0^2x_1x_3x_4 + (2c^2/(c^2+1))x_0x_1^2x_3x_4 + (2c/(c+2))x_1^3x_3x_4 + (c^2/(c^2 + 1))x_0^2x_2x_3x_4 + (4c^2/(c^2 + 1))x_1^2x_2x_3x_4 + (3c^2/(c^2 + 1))x_0x_2^2x_3x_4 + (2c^2/(c^2 + 1))x_1x_2^2x_3x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3^2x_4 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_3^2x_4 + (3c^2/(c^2 + 1))x_0x_2x_3^2x_4 + (2c^2/(c^2 + 1))x_1x_2x_3^2x_4 + (c/(c+2))x_0x_3^3x_4 + (4c/(c+2))x_1x_3^3x_4 + ((4c+1)/(c+2))x_0^3x_4^2 + ((4c^2+c)/(c^2+1))x_0^2x_1x_4^2 + ((c^2+4c)/(c^2+1))x_0x_1^2x_4^2 + ((c+4)/(c+2))x_1^3x_4^2 + ((3c^2+2c)/(c^2+1))x_0^2x_2x_4^2 + ((2c^2+3c)/(c^2+1))x_1^2x_2x_4^2 + ((4c^2+c)/(c^2+1))x_0x_2^2x_4^2 + ((c^2+4c)/(c^2+1))x_1x_2^2x_4^2 + ((3c^2+2c)/(c^2+1))x_0^2x_3x_4^2 + ((2c^2+3c)/(c^2+1))x_1^2x_3x_4^2 + (3c^2/(c^2+1))x_0x_2x_3x_4^2 + (2c^2/(c^2+1))x_1x_2x_3x_4^2 + ((4c^2+c)/(c^2+1))x_0x_3^2x_4^2 + ((c^2+4c)/(c^2+1))x_1x_3^2x_4^2 + ((c+4)/(c+2))x_0^2x_4^3 + ((4c+1)/(c+2))x_1^2x_4^3 + (c/(c+2))x_0x_2x_4^3 + (4c/(c+2))x_1x_2x_4^3 + (c/(c+2))x_0x_3x_4^3 + (4c/(c+2))x_1x_3x_4^3 - x_0x_4^4 + x_1x_4^4$

conjecture: x_0^{25}

5 S_6, char 3, τ trivial

5.1 Hilbert polynomial

PARTIAL

$1 + 6t + 21t^2 + 51t^3 + 96t^4 + 147t^5 + 192t^6 + 222t^7 + \dots$

5.2 Generators

PARTIAL

$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3$

$((2c+2)/c)x_0^3$
$+ ((c+1)/c)x_4^3$
$+ x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + x_1^2x_4 + x_2^2x_4 + x_3^2x_4 + x_0x_4^2 + x_0^2x_5 + x_4x_5^2$
$- x_0x_1^2 - x_0x_2^2 - x_0x_3^2 - x_0^2x_4 - x_1x_4^2 - x_2x_4^2 - x_3x_4^2 - x_4^2x_5 - x_0x_5^2$
$+ (c/(c+2))x_0x_1x_2 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5$
$+ (2c/(c+2))x_1x_2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 + (2c/(c+2))x_1x_4x_5 + (2c/(c+2))x_2x_4x_5 + (2c/(c+2))x_3x_4x_5$

$((2c+2)/c)x_0^3$
$+ ((c+1)/c)x_3^3$
$+ x_0^2x_1 + x_0^2x_2 + x_1^2x_3 + x_2^2x_3 + x_0x_3^2 + x_0^2x_4 + x_3x_4^2 + x_0^2x_5 + x_3x_5^2$
$- x_0x_1^2 - x_0x_2^2 - x_0^2x_3 - x_1x_3^2 - x_2x_3^2 - x_3^2x_4 - x_0x_4^2 - x_3^2x_5 - x_0x_5^2$
$+ (c/(c+2))x_0x_1x_2 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_4x_5$
$+ (2c/(c+2))x_1x_2x_3 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 + (2c/(c+2))x_1x_3x_5 + (2c/(c+2))x_2x_3x_5 + (2c/(c+2))x_3x_4x_5$

$((2c+2)/c)x_0^3$
$+ ((c+1)/c)x_2^3$
$+ x_0^2x_1 + x_1^2x_2 + x_0x_2^2 + x_0^2x_3 + x_2x_3^2 + x_0^2x_4 + x_2x_4^2 + x_0^2x_5 + x_2x_5^2$
$- x_0x_1^2 - x_0^2x_2 - x_1x_2^2 - x_2^2x_3 - x_0x_3^2 - x_2^2x_4 - x_0x_4^2 - x_2^2x_5 - x_0x_5^2$
$+ (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_3x_4 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5$
$+ (2c/(c+2))x_1x_2x_3 + (2c/(c+2))x_1x_2x_4 + (2c/(c+2))x_2x_3x_4 + (2c/(c+2))x_1x_2x_5 + (2c/(c+2))x_2x_3x_5 + (2c/(c+2))x_2x_4x_5$

$((2c+2)/c)x_0^3$
$+ ((c+1)/c)x_1^3$
$- x_0^2x_1 - x_1^2x_2 - x_0x_2^2 - x_1^2x_3 - x_0x_3^2 - x_1^2x_4 - x_0x_4^2 - x_1^2x_5 - x_0x_5^2$
$+ x_0x_1^2 + x_0^2x_2 + x_1x_2^2 + x_0^2x_3 + x_1x_3^2 + x_0^2x_4 + x_1x_4^2 + x_0^2x_5 + x_1x_5^2$
$+ (c/(c+2))x_0x_2x_3 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_3x_4 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5$
$+ (2c/(c+2))x_1x_2x_3 + (2c/(c+2))x_1x_2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_1x_2x_5 + (2c/(c+2))x_1x_3x_5 + (2c/(c+2))x_1x_4x_5$

conjecture: x_0^9.

5.3 $n-1$ case

6 S_6, char 2, τ trivial

6.1 Hilbert polynomial

$t^8+6t^7+16t^6+26t^5+30t^4+26t^3+16t^2+6t+1=(t+1)^6(t^2+1)$

6.2 Generators

$x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2$

$((c+1)/c)x_0^2+x_0x_1+x_0x_2+x_0x_3+x_1x_4+x_2x_4+x_3x_4+((c+1)/c)x_4^2+x_0x_5+x_4x_5$

$((c+1)/c)x_0^2+x_0x_1+x_0x_2+x_1x_3+x_2x_3+((c+1)/c)x_3^2+x_0x_4+x_3x_4+x_0x_5+x_3x_5$

$((c+1)/c)x_0^2+x_0x_1+x_1x_2+((c+1)/c)x_2^2+x_0x_3+x_2x_3+x_0x_4+x_2x_4+x_0x_5+x_2x_5$

$((c+1)/c)x_0^2+((c+1)/c)x_1^2+x_0x_2+x_1x_2+x_0x_3+x_1x_3+x_0x_4+x_1x_4+x_0x_5+x_1x_5$

x_0^4.

7 S_9, char 3, τ trivial

7.1 Hilbert polynomial

PARTIAL

$\cdots+157t^3+45t^2+9t+1$

7.2 Generators

PARTIAL

$x_0^3+x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3+x_7^3+x_8^3$

$((2c+2)/c)x_0^3$
$+((c+1)/c)x_7^3$
$+x_0^2x_1+x_0^2x_2+x_0^2x_3+x_0^2x_4+x_0^2x_5+x_0^2x_6+x_1^2x_7+x_2^2x_7+x_3^2x_7+x_4^2x_7+x_5^2x_7+x_6^2x_7+x_0x_7^2+x_0^2x_8+x_7x_8^2$
$-x_0x_1^2-x_0x_2^2-x_0x_3^2-x_0x_4^2-x_0x_5^2-x_0x_6^2-x_0^2x_7-x_1x_7^2-x_2x_7^2-x_3x_7^2-x_4x_7^2-x_5x_7^2-x_6x_7^2-x_7^2x_8-x_0x_8^2$
$+(c/(c+2))x_0x_1x_2+(c/(c+2))x_0x_1x_3+(c/(c+2))x_0x_2x_3+(c/(c+2))x_0x_1x_4+(c/(c+2))x_0x_2x_4+(c/(c+$

$$2))x_0x_3x_4+(c/(c+2))x_0x_1x_5+(c/(c+2))x_0x_2x_5+(c/(c+2))x_0x_3x_5+(c/(c+2))x_0x_4x_5+(c/(c+2))x_0x_1x_6+(c/(c+2))x_0x_2x_6+(c/(c+2))x_0x_3x_6+(c/(c+2))x_0x_4x_6+(c/(c+2))x_0x_5x_6+(c/(c+2))x_0x_1x_8+(c/(c+2))x_0x_2x_8+(c/(c+2))x_0x_3x_8+(c/(c+2))x_0x_4x_8+(c/(c+2))x_0x_5x_8+(c/(c+2))x_0x_6x_8$$

$$+(2c/(c+2))x_1x_2x_7+(2c/(c+2))x_1x_3x_7+(2c/(c+2))x_2x_3x_7+(2c/(c+2))x_1x_4x_7+(2c/(c+2))x_2x_4x_7+(2c/(c+2))x_3x_4x_7+(2c/(c+2))x_1x_5x_7+(2c/(c+2))x_2x_5x_7+(2c/(c+2))x_3x_5x_7+(2c/(c+2))x_4x_5x_7+(2c/(c+2))x_1x_6x_7+(2c/(c+2))x_2x_6x_7+(2c/(c+2))x_3x_6x_7+(2c/(c+2))x_4x_6x_7+(2c/(c+2))x_5x_6x_7+(2c/(c+2))x_1x_7x_8+(2c/(c+2))x_2x_7x_8+(2c/(c+2))x_3x_7x_8+(2c/(c+2))x_4x_7x_8+(2c/(c+2))x_5x_7x_8+(2c/(c+2))x_6x_7x_8$$

$$((2c+2)/c)x_0^3+x_0^2x_1-x_0x_1^2+x_0^2x_2+(c/(c+2))x_0x_1x_2-x_0x_2^2+x_0^2x_3+(c/(c+2))x_0x_1x_3+(c/(c+2))x_0x_2x_3-x_0x_3^2+x_0^2x_4+(c/(c+2))x_0x_1x_4+(c/(c+2))x_0x_2x_4+(c/(c+2))x_0x_3x_4-x_0x_4^2+x_0^2x_5+(c/(c+2))x_0x_1x_5+(c/(c+2))x_0x_2x_5+(c/(c+2))x_0x_3x_5+(c/(c+2))x_0x_4x_5-x_0x_5^2-x_0^2x_6+x_1^2x_6+(2c/(c+2))x_1x_2x_6+x_2^2x_6+(2c/(c+2))x_1x_3x_6+(2c/(c+2))x_2x_3x_6+x_3^2x_6+(2c/(c+2))x_1x_4x_6+(2c/(c+2))x_2x_4x_6+(2c/(c+2))x_3x_4x_6+x_4^2x_6+(2c/(c+2))x_1x_5x_6+(2c/(c+2))x_2x_5x_6+(2c/(c+2))x_3x_5x_6+(2c/(c+2))x_4x_5x_6+x_5^2x_6+x_0x_6^2-x_1x_6^2-x_2x_6^2-x_3x_6^2-x_4x_6^2-x_5x_6^2+((c+1)/c)x_6^3+x_0^2x_7+(c/(c+2))x_0x_1x_7+(c/(c+2))x_0x_2x_7+(c/(c+2))x_0x_3x_7+(c/(c+2))x_0x_4x_7+(c/(c+2))x_0x_5x_7+(2c/(c+2))x_1x_6x_7+(2c/(c+2))x_2x_6x_7+(2c/(c+2))x_3x_6x_7+(2c/(c+2))x_4x_6x_7+(2c/(c+2))x_5x_6x_7-x_6^2x_7-x_0x_7^2+x_6x_7^2+x_0^2x_8+(c/(c+2))x_0x_1x_8+(c/(c+2))x_0x_2x_8+(c/(c+2))x_0x_3x_8+(c/(c+2))x_0x_4x_8+(c/(c+2))x_0x_5x_8+(2c/(c+2))x_1x_6x_8+(2c/(c+2))x_2x_6x_8+(2c/(c+2))x_3x_6x_8+(2c/(c+2))x_4x_6x_8+(2c/(c+2))x_5x_6x_8-x_6^2x_8+(c/(c+2))x_0x_7x_8+(2c/(c+2))x_6x_7x_8-x_0x_8^2+x_6x_8^2$$

$$((2c+2)/c)x_0^3+x_0^2x_1-x_0x_1^2+x_0^2x_2+(c/(c+2))x_0x_1x_2-x_0x_2^2+x_0^2x_3+(c/(c+2))x_0x_1x_3+(c/(c+2))x_0x_2x_3-x_0x_3^2+x_0^2x_4+(c/(c+2))x_0x_1x_4+(c/(c+2))x_0x_2x_4+(c/(c+2))x_0x_3x_4-x_0x_4^2-x_0^2x_5+x_1^2x_5+(2c/(c+2))x_1x_2x_5+x_2^2x_5+(2c/(c+2))x_1x_3x_5+(2c/(c+2))x_2x_3x_5+x_3^2x_5+(2c/(c+2))x_1x_4x_5+(2c/(c+2))x_2x_4x_5+(2c/(c+2))x_3x_4x_5+x_4^2x_5+x_0x_5^2-x_1x_5^2-x_2x_5^2-x_3x_5^2-x_4x_5^2+((c+1)/c)x_5^3+x_0^2x_6+(c/(c+2))x_0x_1x_6+(c/(c+2))x_0x_2x_6+(c/(c+2))x_0x_3x_6+(c/(c+2))x_0x_4x_6+(2c/(c+2))x_1x_5x_6+(2c/(c+2))x_2x_5x_6+(2c/(c+2))x_3x_5x_6+(2c/(c+2))x_4x_5x_6-x_5^2x_6-x_0x_6^2+x_5x_6^2+x_0^2x_7+(c/(c+2))x_0x_1x_7+(c/(c+2))x_0x_2x_7+(c/(c+2))x_0x_3x_7+(c/(c+2))x_0x_4x_7+(2c/(c+2))x_1x_5x_7+(2c/(c+2))x_2x_5x_7+(2c/(c+2))x_3x_5x_7+(2c/(c+2))x_4x_5x_7-x_5^2x_7+(c/(c+2))x_0x_6x_7+(2c/(c+2))x_5x_6x_7-x_0x_7^2+x_5x_7^2+x_0^2x_8+(c/(c+2))x_0x_1x_8+(c/(c+2))x_0x_2x_8+(c/(c+2))x_0x_3x_8+(c/(c+2))x_0x_4x_8+(2c/(c+2))x_1x_5x_8+(2c/(c+2))x_2x_5x_8+(2c/(c+2))x_3x_5x_8+(2c/(c+2))x_4x_5x_8-x_5^2x_8+(c/(c+2))x_0x_6x_8+(2c/(c+2))x_5x_6x_8+(c/(c+2))x_0x_7x_8+(2c/(c+2))x_5x_7x_8-x_0x_8^2+x_5x_8^2$$

$$((2c+2)/c)x_0^3+x_0^2x_1-x_0x_1^2+x_0^2x_2+(c/(c+2))x_0x_1x_2-x_0x_2^2+x_0^2x_3+(c/(c+2))x_0x_1x_3+(c/(c+2))x_0x_2x_3-x_0x_3^2-x_0^2x_4+x_1^2x_4+(2c/(c+2))x_1x_2x_4+x_2^2x_4+(2c/(c+2))x_1x_3x_4+(2c/(c+2))x_2x_3x_4+x_3^2x_4+x_0x_4^2-x_1x_4^2-x_2x_4^2-x_3x_4^2+((c+1)/c)x_4^3+x_0^2x_5+(c/(c+2))x_0x_1x_5+(c/(c+2))x_0x_2x_5+(c/(c+2))x_0x_3x_5+(2c/(c+2))x_1x_4x_5+(2c/(c+2))x_2x_4x_5+(2c/(c+2))x_3x_4x_5-x_4^2x_5-x_0x_5^2+x_4x_5^2+x_0^2x_6+(c/(c+2))x_0x_1x_6+(c/(c+2))x_0x_2x_6+(c/(c+2))x_0x_3x_6+(2c/(c+2))x_1x_4x_6+(2c/(c+2))x_2x_4x_6+(2c/(c+2))x_3x_4x_6-x_4^2x_6+(c/(c+2))x_0x_5x_6+(2c/(c+2))x_4x_5x_6-x_0x_6^2+x_4x_6^2+x_0^2x_7+(c/(c+2))x_0x_1x_7+(c/(c+2))x_0x_2x_7+(c/(c+2))x_0x_3x_7+(2c/(c+2))x_1x_4x_7+(2c/(c+2))x_2x_4x_7+(2c/(c+2))x_3x_4x_7-x_4^2x_7+(c/(c+2))x_0x_5x_7+(2c/(c+2))x_4x_5x_7+(c/(c+2))x_0x_6x_7+(2c/(c+2))x_4x_6x_7-x_0x_7^2+x_4x_7^2+x_0^2x_8+(c/(c+2))x_0x_1x_8+(c/(c+2))x_0x_2x_8+(c/(c+2))x_0x_3x_8+(2c/(c+2))x_1x_4x_8+(2c/(c+2))x_2x_4x_8+(2c/(c+2))x_3x_4x_8-x_4^2x_8+(c/(c+2))x_0x_5x_8+(2c/(c+2))x_4x_5x_8+(c/(c+2))x_0x_6x_8+(2c/(c+2))x_4x_6x_8+(c/(c+2))x_0x_7x_8+(2c/(c+2))x_4x_7x_8-x_0x_8^2+x_4x_8^2$$

$$((2c+2)/c)x_0^3+x_0^2x_1-x_0x_1^2+x_0^2x_2+(c/(c+2))x_0x_1x_2-x_0x_2^2-x_0^2x_3+x_1^2x_3+(2c/(c+2))x_1x_2x_3+x_2^2x_3+x_0x_3^2-x_1x_3^2-x_2x_3^2+((c+1)/c)x_3^3+x_0^2x_4+(c/(c+2))x_0x_1x_4+(c/(c+2))x_0x_2x_4+(2c/(c+2))x_1x_3x_4+(2c/(c+2))x_2x_3x_4-x_3^2x_4-x_0x_4^2+x_3x_4^2+x_0^2x_5+(c/(c+2))x_0x_1x_5+(c/(c+2))x_0x_2x_5+(2c/(c+2))x_1x_3x_5+(2c/(c+2))x_2x_3x_5-x_3^2x_5+(c/(c+2))x_0x_4x_5+(2c/(c+2))x_3x_4x_5-x_0x_5^2+x_3x_5^2+x_0^2x_6+(c/(c+2))x_0x_1x_6+(c/(c+2))x_0x_2x_6+(2c/(c+2))x_1x_3x_6+(2c/(c+2))x_2x_3x_6-x_3^2x_6+(c/(c+2))x_0x_4x_6+(2c/(c+2))x_3x_4x_6+(c/(c+2))x_0x_5x_6+(2c/(c+2))x_3x_5x_6-x_0x_6^2+x_3x_6^2+x_0^2x_7+(c/(c+2))x_0x_1x_7+(c/(c+2))x_0x_2x_7+(2c/(c+2))x_1x_3x_7+(2c/(c+2))x_2x_3x_7-x_3^2x_7+(c/(c+2))x_0x_4x_7+(2c/(c+2))x_3x_4x_7+(c/(c+2))x_0x_5x_7+(2c/(c+2))x_3x_5x_7+(c/(c+2))x_0x_6x_7+(2c/(c+2))x_3x_6x_7-x_0x_7^2+x_3x_7^2+x_0^2x_8+(c/(c+2))x_0x_1x_8+(c/(c+2))x_0x_2x_8+(2c/(c+2))x_1x_3x_8+(2c/(c+2))x_2x_3x_8-x_3^2x_8+(c/(c+2))x_0x_4x_8+(2c/(c+2))x_3x_4x_8+(c/(c+2))x_0x_5x_8+(2c/(c+2))x_3x_5x_8+(c/(c+2))x_0x_6x_8+(2c/(c+2))x_3x_6x_8+(c/(c+2))x_0x_7x_8+(2c/(c+2))x_3x_7x_8-x_0x_8^2+x_3x_8^2$$

$((2c+2)/c)x_0^3+x_0^2x_1-x_0x_1^2-x_0^2x_2+x_1^2x_2+x_0x_2^2-x_1x_2^2+((c+1)/c)x_2^3+x_0^2x_3+(c/(c+2))x_0x_1x_3+(2c/(c+2))x_1x_2x_3-x_2^2x_3-x_0x_3^2+x_2x_3^2+x_0^2x_4+(c/(c+2))x_0x_1x_4+(2c/(c+2))x_1x_2x_4-x_2^2x_4+(c/(c+2))x_0x_3x_4+(2c/(c+2))x_2x_3x_4-x_0x_4^2+x_2x_4^2+x_0^2x_5+(c/(c+2))x_0x_1x_5+(2c/(c+2))x_1x_2x_5-x_2^2x_5+(c/(c+2))x_0x_3x_5+(2c/(c+2))x_2x_3x_5+(c/(c+2))x_0x_4x_5+(2c/(c+2))x_2x_4x_5-x_0x_5^2+x_2x_5^2+x_0^2x_6+(c/(c+2))x_0x_1x_6+(2c/(c+2))x_1x_2x_6-x_2^2x_6+(c/(c+2))x_0x_3x_6+(2c/(c+2))x_2x_3x_6+(c/(c+2))x_0x_4x_6+(2c/(c+2))x_2x_4x_6+(c/(c+2))x_0x_5x_6+(2c/(c+2))x_2x_5x_6-x_0x_6^2+x_2x_6^2+x_0^2x_7+(c/(c+2))x_0x_1x_7+(2c/(c+2))x_1x_2x_7-x_2^2x_7+(c/(c+2))x_0x_3x_7+(2c/(c+2))x_2x_3x_7+(c/(c+2))x_0x_4x_7+(2c/(c+2))x_2x_4x_7+(c/(c+2))x_0x_5x_7+(2c/(c+2))x_2x_5x_7+(c/(c+2))x_0x_6x_7+(2c/(c+2))x_2x_6x_7-x_0x_7^2+x_2x_7^2+x_0^2x_8+(c/(c+2))x_0x_1x_8+(2c/(c+2))x_1x_2x_8-x_2^2x_8+(c/(c+2))x_0x_3x_8+(2c/(c+2))x_2x_3x_8+(c/(c+2))x_0x_4x_8+(2c/(c+2))x_2x_4x_8+(c/(c+2))x_0x_5x_8+(2c/(c+2))x_2x_5x_8+(c/(c+2))x_0x_6x_8+(2c/(c+2))x_2x_6x_8+(c/(c+2))x_0x_7x_8+(2c/(c+2))x_2x_7x_8-x_0x_8^2+x_2x_8^2$

$((2c+2)/c)x_0^3-x_0^2x_1+x_0x_1^2+((c+1)/c)x_1^3+x_0^2x_2-x_1^2x_2-x_0x_2^2+x_1x_2^2+x_0^2x_3-x_1^2x_3+(c/(c+2))x_0x_2x_3+(2c/(c+2))x_1x_2x_3-x_0x_3^2+x_1x_3^2+x_0^2x_4-x_1^2x_4+(c/(c+2))x_0x_2x_4+(2c/(c+2))x_1x_2x_4+(c/(c+2))x_0x_3x_4+(2c/(c+2))x_1x_3x_4-x_0x_4^2+x_1x_4^2+x_0^2x_5-x_1^2x_5+(c/(c+2))x_0x_2x_5+(2c/(c+2))x_1x_2x_5+(c/(c+2))x_0x_3x_5+(2c/(c+2))x_1x_3x_5+(c/(c+2))x_0x_4x_5+(2c/(c+2))x_1x_4x_5-x_0x_5^2+x_1x_5^2+x_0^2x_6-x_1^2x_6+(c/(c+2))x_0x_2x_6+(2c/(c+2))x_1x_2x_6+(c/(c+2))x_0x_3x_6+(2c/(c+2))x_1x_3x_6+(c/(c+2))x_0x_4x_6+(2c/(c+2))x_1x_4x_6+(c/(c+2))x_0x_5x_6+(2c/(c+2))x_1x_5x_6-x_0x_6^2+x_1x_6^2+x_0^2x_7-x_1^2x_7+(c/(c+2))x_0x_2x_7+(2c/(c+2))x_1x_2x_7+(c/(c+2))x_0x_3x_7+(2c/(c+2))x_1x_3x_7+(c/(c+2))x_0x_4x_7+(2c/(c+2))x_1x_4x_7+(c/(c+2))x_0x_5x_7+(2c/(c+2))x_1x_5x_7+(c/(c+2))x_0x_6x_7+(2c/(c+2))x_1x_6x_7-x_0x_7^2+x_1x_7^2+x_0^2x_8-x_1^2x_8+(c/(c+2))x_0x_2x_8+(2c/(c+2))x_1x_2x_8+(c/(c+2))x_0x_3x_8+(2c/(c+2))x_1x_3x_8+(c/(c+2))x_0x_4x_8+(2c/(c+2))x_1x_4x_8+(c/(c+2))x_0x_5x_8+(2c/(c+2))x_1x_5x_8+(c/(c+2))x_0x_6x_8+(2c/(c+2))x_1x_6x_8+(c/(c+2))x_0x_7x_8+(2c/(c+2))x_1x_7x_8-x_0x_8^2+x_1x_8^2$

conjecture: x_0^9

8 Conjecture for $p = 3, 3 \mid n$

Variables are $x_0, \ldots, x_{n-1}$.

Generators are:

x_0^9 in degree 9, and $\sum x_i^3$ in degree 3. There are $n-2$ remaining generators in degree 3, each with the following form:

$\frac{c+1}{c}(x_1^3 - x_0^3) + (x_1 - x_0)(x_0x_1) + (x_1 - x_0)\left(\sum_{i\geq 2} x_i^2 - x_i(x_1 + x_0)\right) + \frac{2c}{c+2}(x_1 - x_0)\left(\sum_{i,j\geq 2; i<j} x_ix_j\right).$

(The other generators are created from this one by switching x_1 with x_k for some $k \geq 2$.)

We note that since $3 \mid n$ that $\sum_{i<j}(x_i - x_j)^2 = \sum_{i<j} x_ix_j + \sum_{i<j} x_i^2 + x_j^2 = -\sum_i x_i^2 + \sum_{i<j} x_ix_j$.

We also note that $\sum_i (x_i^2 - x_ix_1 - x_ix_0) = x_0x_1 + \sum_{i\geq 2}(x_i^2 - x_ix_1 - x_ix_0)$.

We also note that $nx_1^2 = nx_0^2 = nx_0x_1 = 0$.

$$\frac{c+1}{c}(x_1^3 - x_0^3) + (x_1 - x_0)(x_0x_1) + (x_1 - x_0)\left(\sum_{i\geq 2} x_i^2 - x_i(x_1 + x_0)\right) + \frac{2c}{c+2}(x_1 - x_0)\left(\sum_{i,j\geq 2; i<j} x_ix_j\right) =$$

$$= \frac{x_1 - x_0}{c(c+2)}\left((c+1)(c+2)(x_0^2 + x_0x_1 + x_1^2) + c(c+2)(x_0x_1) + c(c+2)\left(\sum_{i\geq 2} x_i^2 - x_i(x_1 + x_0)\right) + 2c^2\left(\sum_{i,j\geq 2; i<j} x_ix_j\right)\right)$$

$$= \frac{x_1 - x_0}{c(c+2)}\left((c^2 - 1)(x_0^2 + x_0x_1 + x_1^2) + (c^2 - c)(x_0x_1) + (c^2 - c)\left(\sum_{i\geq 2} x_i^2 - x_i(x_1 + x_0)\right) - c^2\left(\sum_{i,j\geq 2; i<j} x_ix_j\right)\right)$$

$$= \frac{x_1 - x_0}{c(c+2)}((c^2 - 1)(x_0^2 + x_0x_1 + x_1^2) + (c^2 - c)(x_0x_1) + c^2\left(\sum_{i\geq 2} x_i^2\right) - c^2\left(\sum_{i\geq 2} x_ix_1 + x_ix_0\right)$$

$$- c\left(\sum_{i\geq 2} x_i^2 - x_ix_1 - x_ix_0\right) - c^2\left(\sum_{i,j\geq 2; i<j} x_ix_j\right))$$

$$= \frac{x_1 - x_0}{c(c+2)}((c^2 - 1)(x_0^2 + x_0x_1 + x_1^2) + (c^2 - c)(x_0x_1) + c^2\left(\sum_{i\geq 2} x_i^2\right) + c^2x_0x_1 - c\left(\sum_{i\geq 2} x_i^2 - x_ix_1 - x_ix_0\right)$$

$$- c^2\left(\sum_{i<j} x_ix_j\right))$$

$$= \frac{x_1 - x_0}{c(c+2)}\left(c^2x_0^2 + c^2x_1^2 - x_0^2 - x_1^2 - x_0x_1 - cx_0x_1 + c^2\left(\sum_{i\geq 2} x_i^2\right) - c\left(\sum_{i\geq 2} x_i^2 - x_ix_1 - x_ix_0\right) - c^2\left(\sum_{i<j} x_ix_j\right)\right)$$

$$= \frac{x_1 - x_0}{c(c+2)}\left(-x_0^2 - x_1^2 - x_0x_1 + c^2\left(\sum_{i} x_i^2\right) - c\left(\sum_{i} x_i^2 - x_ix_1 - x_ix_0\right) - c^2\left(\sum_{i<j} x_ix_j\right)\right)$$

$$= \frac{x_1 - x_0}{c(c+2)}\left(-(x_0 - x_1)^2 + c^2\left(\sum_{i} x_i^2\right) - c\left(\sum_{i} x_i^2 - x_ix_1 - x_ix_0\right) - c^2\left(\sum_{i<j} x_ix_j\right)\right)$$

$$= \frac{x_1 - x_0}{c(c+2)}\left(-(x_0 - x_1)^2 - c\left(\sum_{i} x_i^2 - x_ix_1 - x_ix_0 + x_0x_1\right) - c^2\left(\sum_{i<j} (x_i - x_j)^2\right)\right)$$

$$= \frac{x_1 - x_0}{c(c+2)}\left(-(x_0 - x_1)^2 - c\left(\sum_{i} (x_i - x_1)(x_i - x_0)\right) - c^2\left(\sum_{i<j} (x_i - x_j)^2\right)\right)$$

$$= \frac{1}{c(c+2)}\left(x_0^3 - x_1^3 - c\left(\sum_{i} (x_1 - x_0)(x_i - x_1)(x_i - x_0)\right) - c^2\left(\sum_{i<j} (x_1 - x_0)(x_i - x_j)^2\right)\right)$$

(Checked with Sage)

Let this generator equal g. We see that for $\{i,j\} \cap \{0,1\} = \emptyset$ that $s_{ij}g = g$. Therefore $\frac{g - s_{ij}g}{x_i - x_j} = 0$.

We see easily that $s_{0k}g$ for $k \neq 1$ is $\frac{x_1 - x_k}{c(c+2)}\left(-(x_k - x_1)^2 - c\sum_i (x_i - x_1)(x_i - x_k) - c^2\sum_{i<j}(x_i - x_j)^2\right)$. From there further simple algebra tells us that $(x_1 - x_0)(x_i - x_0)(x_i - x_1) - (x_1 - x_k)(x_i - x_1)(x_i - x_k) =$

$(x_1-x_i)(x_0-x_1+x_k-x_i)(x_k-x_0)$ for all i. From this we see easily that

$\frac{g-s_{0k}g}{x_k-x_0}=\frac{1}{c(c+2)}\left(-(x_k-x_0)^2-c\sum_i(x_1-x_i)(x_0-x_1+x_k-x_i)-c^2\sum_{i<j}(x_i-x_j)^2\right)$.

Similar algebra shows us that for $k\neq 0,1$:

$\frac{g-s_{1k}g}{x_k-x_1}=\frac{1}{c(c+2)}\left((x_k-x_1)^2+c\sum_i(x_0-x_i)(x_1-x_0+x_k-x_i)+c^2\sum_{i<j}(x_i-x_j)^2\right)$.

The final case is $s_{01}g$. We see easily that all the terms inside the largest parentheses are left untouched by s_{01}. Therefore $s_{01}g=-g$, so $\frac{g-s_{01}g}{x_1-x_0}=\frac{g--g}{x_1-x_0}=\frac{2g}{x_1-x_0}=-\frac{g}{x_1-x_0}$; this is just

$\frac{1}{c(c+2)}\left((x_0-x_1)^2+c\left(\sum_i(x_i-x_1)(x_i-x_0)\right)+c^2\left(\sum_{i<j}(x_i-x_j)^2\right)\right)$.

We can use these to calculate the values for the Dunkl operators. We need only check D_0g, D_1g, D_2g, because the rest are essentially equivalent to D_2g.

8.1 D_0g

We start with D_0g. We see that $D_0g=\partial_0 g-c\sum_{i\geq 1}\frac{g-s_{0i}g}{x_0-x_i}$.

We consider the partial derivative.

$$\begin{aligned}
\partial_0 g &= \partial_0\left(\frac{1}{c(c+2)}\left(x_0^3-x_1^3-c\left(\sum_i(x_1-x_0)(x_i-x_1)(x_i-x_0)\right)-c^2\left(\sum_{i<j}(x_1-x_0)(x_i-x_j)^2\right)\right)\right)\\
&=\frac{1}{c(c+2)}\left(-c\left(\sum_{i\geq 2}(x_1-x_i)(x_0+x_1+x_i)\right)-c^2\left(\sum_{i<j}\partial_0((x_1-x_0)(x_i-x_j)^2)\right)\right)\\
&=\frac{1}{c(c+2)}\left(-c\left(\sum_{i\geq 2}(x_1-x_i)(x_0+x_1+x_i)\right)-c^2\left(\sum_{i\geq 1}\partial_0((x_1-x_0)(x_i-x_0)^2)\right)-c^2\left(\sum_{0<i<j}\partial_0((x_1-x_0)(x_i-x_j)^2)\right)\right)\\
&=\frac{1}{c(c+2)}\left(-c\left(\sum_{i\geq 2}(x_1-x_i)(x_0+x_1+x_i)\right)-c^2\left(\sum_{i\geq 1}\partial_0((x_1-x_0)(x_i-x_0)^2)\right)+c^2\left(\sum_{0<i<j}(x_i-x_j)^2\right)\right)\\
&=\frac{1}{c(c+2)}\left(-c\left(\sum_{i\geq 2}(x_1-x_i)(x_0+x_1+x_i)\right)+c^2\left(\sum_{i\geq 2}(x_1-x_i)(x_0-x_i)\right)+c^2\left(\sum_{0<i<j}(x_i-x_j)^2\right)\right)\\
&=\frac{c}{c(c+2)}\left(-\left(\sum_{i\geq 2}(x_1-x_i)(x_0+x_1+x_i)\right)+c\left(\sum_{i\geq 2}(x_1-x_i)(x_0-x_i)\right)+c\left(\sum_{0<i<j}(x_i-x_j)^2\right)\right)
\end{aligned}$$

We note that in $\sum_{0<i<j}(x_i-x_j)^2$ that in each term at least one of the x_i, x_j has index ≥ 2. Therefore $\sum_{0<i<j}(x_i-x_j)^2=\sum_{0<i<j}-2(x_i-x_j)^2=-\sum_{0<i,j;i\neq j}(x_i-x_j)^2=-\left(\sum_{i\geq 2}\sum_{j\geq 1}(x_i-x_j)^2\right)-\left(\sum_{i\geq 2}(x_i-x_1)^2\right)$ because of the double-counting. Then since $n-1=-1$, we see that this is equal to $-\left(\sum_{i\geq 2}\sum_{j\geq 1}(x_i-x_j)^2-(x_i-x_1)^2\right)=\sum_{i\geq 2}\sum_{j\geq 1}(x_i-x_1)^2-(x_i-x_j)^2=\sum_{i\geq 2}\sum_{j\geq 1}(x_1-x_j)(x_1+$

$x_j + x_i)$.

Therefore we have (there is a switch of indices):

$$\begin{aligned}
\partial_0 g &= \frac{c}{c(c+2)}\left(-\left(\sum_{i\geq 2}(x_1-x_i)(x_0+x_1+x_i)\right)+c\left(\sum_{i\geq 2}(x_1-x_i)(x_0-x_i)\right)+c\left(\sum_{i\geq 2}\sum_{j\geq 1}(x_1-x_j)(x_1+x_j+x_i)\right)\right)\\
&= \frac{c}{c(c+2)}\left(-\left(\sum_{i\geq 2}(x_1-x_i)(x_0+x_1+x_i)\right)+c\left(\sum_{i\geq 2}(x_1-x_i)(x_0-x_i)\right)+c\left(\sum_{i\geq 2}\sum_{j\geq 2}(x_1-x_i)(x_1+x_j+x_i)\right)\right)\\
&= \frac{c}{c(c+2)}\left(\sum_{i\geq 2}\left(-\left((x_1-x_i)(x_0+x_1+x_i)\right)+c\left((x_1-x_i)(x_0-x_i)\right)+c\left(\sum_{j\geq 2}(x_1-x_i)(x_1+x_j+x_i)\right)\right)\right)\\
&= \frac{c}{c(c+2)}\left(\sum_{i\geq 2}(x_i-x_1)\left((x_0+x_1+x_i)-c\left(x_0-x_i\right)-c\left(\sum_{j\geq 2}(x_1+x_j+x_i)\right)\right)\right)\\
&= \frac{c}{c(c+2)}\left(\sum_{i\geq 2}(x_i-x_1)\left((x_0+x_1+x_i)-cx_0+cx_i-cx_1-cx_i-c\left(\sum_{j\geq 2}x_j\right)\right)\right)\\
&= \frac{c}{c(c+2)}\left(\sum_{i\geq 2}(x_i-x_1)\left((x_0+x_1+x_i)-c\left(\sum_{j}x_j\right)\right)\right)
\end{aligned}$$

Let $G_1 = \frac{g-s_{01}g}{x_0-x_1} = -\frac{1}{c(c+2)}\left((x_0-x_1)^2+c\left(\sum_i(x_i-x_1)(x_i-x_0)\right)+c^2\left(\sum_{i<j}(x_i-x_j)^2\right)\right)$. Let $G_2 = \sum_{i\geq 2}\frac{g-s_{0i}g}{x_0-x_i}$.

$$\begin{aligned}
G_2 &= \sum_{i\geq 2}\frac{g-s_{0i}g}{x_0-x_i}\\
&= \sum_{k\geq 2}\left(\frac{1}{c(c+2)}\left((x_k-x_0)^2+c\sum_i(x_1-x_i)(x_0-x_1+x_k-x_i)+c^2\sum_{i<j}(x_i-x_j)^2\right)\right)\\
&= \frac{1}{c(c+2)}\left(\sum_{k\geq 2}\left((x_k-x_0)^2+c\sum_i(x_1-x_i)(x_0-x_1+x_k-x_i)\right)+(n-2)c^2\sum_{i<j}(x_i-x_j)^2\right)\\
&= \frac{1}{c(c+2)}\left(\sum_{k\geq 2}\left((x_k-x_0)^2+c\sum_i(x_1-x_i)(x_0-x_1+x_k-x_i)\right)+c^2\sum_{i<j}(x_i-x_j)^2\right)\\
&= \frac{1}{c(c+2)}\left(c\sum_{k\geq 2}\sum_i(x_1-x_i)(x_0-x_1+x_k-x_i)+c^2\sum_{i<j}(x_i-x_j)^2+\left(\sum_{k\geq 2}(x_k-x_0)^2\right)\right)
\end{aligned}$$

We note that $n-2-1=0$ since $3 \mid n$. Therefore:

$$G_2+G_1=\frac{1}{c(c+2)}\left(\sum_{k\geq 2}\left((x_k-x_0)^2+c\sum_i(x_1-x_i)(x_0-x_1+x_k-x_i)\right)+(n-2)c^2\sum_{i<j}(x_i-x_j)^2\right)$$
$$-\frac{1}{c(c+2)}\left((x_0-x_1)^2+c\left(\sum_i(x_i-x_1)(x_i-x_0)\right)+c^2\left(\sum_{i<j}(x_i-x_j)^2\right)\right)$$
$$=\frac{1}{c(c+2)}(\sum_{k\geq 2}\left((x_k-x_0)^2+c\sum_i(x_1-x_i)(x_0-x_1+x_k-x_i)\right)$$
$$-(x_0-x_1)^2-c\left(\sum_i(x_i-x_1)(x_i-x_0)\right))$$

We consider $\sum_{k\geq 2}\sum_i(x_1-x_i)(x_0-x_1+x_k-x_i)-(\sum_i(x_i-x_1)(x_i-x_0))$. We note that $n-2=1$, so this is equal to $\sum_{k\geq 2}\sum_i(x_1-x_i)(x_0-x_1+x_k-x_i)-\left(\sum_{k\geq 2}\sum_i(x_i-x_1)(x_i-x_0)\right)$; then this is equal to $\sum_{k\geq 2}\sum_i((x_1-x_i)(x_0-x_1+x_k-x_i)-(x_i-x_1)(x_i-x_0))$. This simplifies to $\sum_{k\geq 2}\sum_i(x_1-x_i)(x_k-x_1)$.

Therefore we see that (again using $n-2=1$):

$$G_1+G_2=\frac{1}{c(c+2)}\left(\sum_{k\geq 2}\left((x_k-x_0)^2-(x_1-x_0)^2+c\sum_i(x_1-x_i)(x_k-x_1)\right)\right)$$
$$=\frac{1}{c(c+2)}\left(\sum_{k\geq 2}\left((x_k-x_0+x_1-x_0)(x_k-x_1)-c\sum_i x_i(x_k-x_1)\right)\right)$$
$$=\frac{1}{c(c+2)}\left(\sum_{k\geq 2}(x_k-x_1)\left(x_k+x_1+x_0-c\sum_i x_i\right)\right)$$

Then we can relabel indices and see that $c(G_1+G_2)=\partial_0 g$, so the Dunkl operator is 0 as desired.

8.2 $D_1 g$

We see that $D_1 g=\partial_1 g-c\sum_{i\neq 1}\frac{g-s_{1i}g}{x_1-x_i}$.

We consider the partial derivative.

$$
\begin{aligned}
\partial_1 g &= \partial_1 \left(\frac{1}{c(c+2)} \left(x_0^3 - x_1^3 - c \left(\sum_i (x_1 - x_0)(x_i - x_1)(x_i - x_0) \right) - c^2 \left(\sum_{i<j} (x_1 - x_0)(x_i - x_j)^2 \right) \right) \right) \\
&= \frac{1}{c(c+2)} \left(c \left(\sum_{i \geq 2} (x_0 - x_i)(x_0 + x_1 + x_i) \right) - c^2 \left(\sum_{i<j} \partial_1((x_1 - x_0)(x_i - x_j)^2) \right) \right) \\
&= \frac{1}{c(c+2)} \left(c \left(\sum_{i \geq 2} (x_0 - x_i)(x_0 + x_1 + x_i) \right) - c^2 \left(\sum_{i \neq 1} \partial_1((x_1 - x_0)(x_i - x_1)^2) \right) - c^2 \left(\sum_{1 \neq i < j \neq 1} \partial_1((x_1 - x_0)(x_i - x_j)^2) \right) \right) \\
&= \frac{1}{c(c+2)} \left(c \left(\sum_{i \geq 2} (x_0 - x_i)(x_0 + x_1 + x_i) \right) - c^2 \left(\sum_{i \geq 2} (x_i - x_1)(x_i - x_0) \right) - c^2 \left(\sum_{1 \neq i < j \neq 1} \partial_1((x_1 - x_0)(x_i - x_j)^2) \right) \right) \\
&= \frac{1}{c(c+2)} \left(c \left(\sum_{i \geq 2} (x_0 - x_i)(x_0 + x_1 + x_i) \right) - c^2 \left(\sum_{i \geq 2} (x_i - x_1)(x_i - x_0) \right) - c^2 \left(\sum_{1 \neq i < j \neq 1} (x_i - x_j)^2 \right) \right) \\
&= \frac{c}{c(c+2)} \left(\left(\sum_{i \geq 2} (x_0 - x_i)(x_0 + x_1 + x_i) \right) - c \left(\sum_{i \geq 2} (x_i - x_1)(x_i - x_0) \right) - c \left(\sum_{1 \neq i < j \neq 1} (x_i - x_j)^2 \right) \right)
\end{aligned}
$$

We note that in $\sum_{1 \neq i < j \neq 1} (x_i - x_j)^2$ that in each term at least one of the x_i, x_j has index ≥ 2. Therefore $\sum_{1 \neq i < j \neq 1} (x_i - x_j)^2 = \sum_{1 \neq i < j \neq 1} -2(x_i - x_j)^2 = -\sum_{1 \neq i,j; i \neq j} (x_i - x_j)^2 = -\left(\sum_{i \geq 2} \sum_{j \neq 1} (x_i - x_j)^2 \right) - \left(\sum_{i \geq 2} (x_i - x_0)^2 \right)$ because of the double-counting. Then since $n - 1 = -1$, we see that this is equal to $-\left(\sum_{i \geq 2} \sum_{j \neq 1} (x_i - x_j)^2 - (x_i - x_0)^2 \right) = \sum_{i \geq 2} \sum_{j \neq 1} (x_i - x_0)^2 - (x_i - x_j)^2 = \sum_{i \geq 2} \sum_{j \neq 1} (x_0 - x_j)(x_0 + x_j + x_i) = \sum_{i \geq 2} \sum_{j \geq 2} (x_0 - x_j)(x_0 + x_j + x_i)$.

Therefore we have (there is a switch of indices):

$$
\begin{aligned}
\partial_1 g &= \frac{c}{c(c+2)} \left(\left(\sum_{i \geq 2} (x_0 - x_i)(x_0 + x_1 + x_i) \right) - c \left(\sum_{i \geq 2} (x_i - x_1)(x_i - x_0) \right) - c \left(\sum_{i \geq 2} \sum_{j \geq 2} (x_0 - x_i)(x_0 + x_j + x_i) \right) \right) \\
&= \frac{c}{c(c+2)} \left(\sum_{i \geq 2} ((x_0 - x_i)(x_0 + x_1 + x_i)) - c\,((x_i - x_1)(x_i - x_0)) - c \left(\sum_{j \geq 2} (x_0 - x_i)(x_0 + x_j + x_i) \right) \right) \\
&= \frac{c}{c(c+2)} \left(\sum_{i \geq 2} (x_0 - x_i) \left((x_0 + x_1 + x_i) - c(x_1 - x_i) - c \left(\sum_{j \geq 2} (x_0 + x_j + x_i) \right) \right) \right) \\
&= \frac{c}{c(c+2)} \left(\sum_{i \geq 2} (x_0 - x_i) \left((x_0 + x_1 + x_i) - cx_1 + cx_i - cx_0 - cx_i - c \left(\sum_{j \geq 2} x_j \right) \right) \right) \\
&= \frac{c}{c(c+2)} \left(\sum_{i \geq 2} (x_0 - x_i) \left((x_0 + x_1 + x_i) - c \left(\sum_j x_j \right) \right) \right)
\end{aligned}
$$

Let $G_1 = \frac{g - s_{01}g}{x_1 - x_0} = \frac{1}{c(c+2)} \left((x_0 - x_1)^2 + c \left(\sum_i (x_i - x_1)(x_i - x_0) \right) + c^2 \left(\sum_{i<j} (x_i - x_j)^2 \right) \right)$. Let $G_2 = \sum_{i \geq 2} \frac{g - s_{1i}g}{x_1 - x_i}$.

$$
\begin{aligned}
G_2 &= \sum_{k \geq 2} \frac{g - s_{1k}g}{x_1 - x_k} \\
&= -\sum_{k \geq 2} \left(\frac{1}{c(c+2)} \left((x_k - x_1)^2 + c \sum_i (x_0 - x_i)(x_1 - x_0 + x_k - x_i) + c^2 \sum_{i<j} (x_i - x_j)^2 \right) \right) \\
&= -\frac{1}{c(c+2)} \left(\sum_{k \geq 2} \left((x_k - x_1)^2 + c \sum_i (x_0 - x_i)(x_1 - x_0 + x_k - x_i) \right) + (n-2)c^2 \sum_{i<j} (x_i - x_j)^2 \right) \\
&= -\frac{1}{c(c+2)} \left(\sum_{k \geq 2} \left((x_k - x_1)^2 + c \sum_i (x_0 - x_i)(x_1 - x_0 + x_k - x_i) \right) + c^2 \sum_{i<j} (x_i - x_j)^2 \right)
\end{aligned}
$$

We note that $n - 2 - 1 = 0$ since $3 \mid n$. Therefore:

$$
\begin{aligned}
G_2 + G_1 = &-\frac{1}{c(c+2)} \left(\sum_{k \geq 2} \left((x_k - x_1)^2 + c \sum_i (x_0 - x_i)(x_1 - x_0 + x_k - x_i) \right) + c^2 \sum_{i<j} (x_i - x_j)^2 \right) \\
&+ \frac{1}{c(c+2)} \left((x_0 - x_1)^2 + c \left(\sum_i (x_i - x_1)(x_i - x_0) \right) + c^2 \left(\sum_{i<j} (x_i - x_j)^2 \right) \right) \\
&= \frac{1}{c(c+2)} (- \left(\sum_{k \geq 2} \left((x_k - x_1)^2 + c \sum_i (x_0 - x_i)(x_1 - x_0 + x_k - x_i) \right) \right) \\
&+ \left((x_0 - x_1)^2 + c \left(\sum_i (x_i - x_1)(x_i - x_0) \right) \right))
\end{aligned}
$$

We consider $-\sum_{k \geq 2} \sum_i (x_0 - x_i)(x_1 - x_0 + x_k - x_i) + (\sum_i (x_i - x_1)(x_i - x_0))$. We note that $n - 2 = 1$, so this is equal to $-\sum_{k \geq 2} \sum_i (x_0 - x_i)(x_1 - x_0 + x_k - x_i) + \left(\sum_{k \geq 2} \sum_i (x_1 - x_i)(x_0 - x_i) \right)$; then this is equal to $\sum_{k \geq 2} \sum_i ((x_0 - x_i)(x_0 - x_k))$. Therefore we see that (again using $n - 2 = 1$):

$$
\begin{aligned}
G_1 + G_2 &= -\frac{1}{c(c+2)}\left(\sum_{k\geq 2}\left((x_k - x_1)^2 - (x_0 - x_1)^2 - c\sum_i (x_0 - x_i)(x_0 - x_k)\right)\right) \\
&= -\frac{1}{c(c+2)}\left(\sum_{k\geq 2}\left((x_k - x_1 + x_0 - x_1)(x_k - x_0) - c\sum_i x_i(x_k - x_0)\right)\right) \\
&= -\frac{1}{c(c+2)}\left(\sum_{k\geq 2}(x_k - x_0)\left(x_k + x_1 + x_0 - c\sum_i x_i\right)\right) \\
&= \frac{1}{c(c+2)}\left(\sum_{k\geq 2}(x_0 - x_k)\left(x_k + x_1 + x_0 - c\sum_i x_i\right)\right)
\end{aligned}
$$

Then we can relabel indices and see that $c(G_1 + G_2) = \partial_1 g$, so the Dunkl operator is 0 as desired.

8.3 $D_2 g$

We see that $D_2 g = \partial_2 g - c\frac{g - s_{02}g}{x_2 - x_0} - c\frac{g - s_{12}g}{x_2 - x_1}$ because the rest cancel. Let $G = \frac{g - s_{02}g}{x_2 - x_0} + \frac{g - s_{12}g}{x_2 - x_1}$. From the above we see that:

$$
\begin{aligned}
G &= \frac{g - s_{02}g}{x_2 - x_0} + \frac{g - s_{12}g}{x_2 - x_1} \\
&= \frac{1}{c(c+2)}\left((x_2 - x_1)^2 - (x_2 - x_0)^2 + c\sum_i (x_0 - x_i)(x_1 - x_0 + x_2 - x_i) - (x_1 - x_i)(x_0 - x_1 + x_2 - x_i)\right) \\
&= \frac{1}{c(c+2)}\left((x_2 - x_1 + x_2 - x_0)(x_0 - x_1) + c\sum_i (x_1 - x_0)(x_0 + x_1 - x_2 - x_i)\right) \\
&= \frac{1}{c(c+2)}\left((x_2 - x_1 + x_2 - x_0)(x_0 - x_1) + c\sum_i x_i(x_0 - x_1)\right)
\end{aligned}
$$

We consider the partial derivative.

$$
\begin{aligned}
\partial_2 g &= \frac{x_1 - x_0}{c(c+2)} \left(-c(x_2 - x_1 + x_2 - x_0) - c^2 \sum_{i \neq 2} 2(x_2 - x_i) \right) \\
&= \frac{x_1 - x_0}{c(c+2)} \left(-c(x_2 - x_1 + x_2 - x_0) + c^2 \sum_{i \neq 2} x_2 - x_i \right) \\
&= \frac{x_1 - x_0}{c(c+2)} \left(-c(x_2 - x_1 + x_2 - x_0) + c^2 \sum_{i \neq 2} -x_i + c^2(n-1)x_2 \right) \\
&= \frac{x_1 - x_0}{c(c+2)} \left(-c(x_2 - x_1 + x_2 - x_0) - c^2 \sum_{i} x_i \right)
\end{aligned}
$$

We see that $\partial_2 g = cG$, so $D_2 g = 0$ as desired.

9 Conjecture for $p = 5, 5 \mid n$

Variables are $x_0, \ldots, x_{n-1}$.

Generators are:

x_0^{25} in degree 25, and $\sum x_i^5$ in degree 5. There are $n-2$ remaining generators in degree 3, each with the following form:

$\frac{c+1}{c}(x_1^5 - x_0^5) + 2x_0^4 x_1 + 3x_0 x_1^4 + \left(\sum_{i \geq 2} x_0^4 x_i + x_1 x_i^4 - x_0 x_i^4 - x_1^4 x_i \right) + \frac{2c+3}{c+2} x_0^2 x_1^3 + \frac{3c+2}{c+2} x_0^3 x_1^2 +$
$\frac{c+4}{c+2} \left(\sum_{i \geq 2} x_0^2 x_i^3 + x_1^3 x_i^2 - x_0^3 x_i^2 - x_1^2 x_i^3 \right) + \frac{c}{c+2} \left(\sum_{i,j \geq 2; i<j} x_0 x_i^3 x_j + x_0 x_i x_j^3 - x_1 x_i^3 x_j - x_1 x_i x_j^3 \right) +$
$\frac{2c}{c+2} \left(\sum_{i \geq 2} x_0^3 x_1 x_i - x_0 x_1^3 x_i \right) + \frac{2c}{c+2} \left(\sum_{i,j \geq 2; i<j} x_i x_j (x_1^3 - x_0^3) \right) +$
$\frac{c^2+4c}{c^2+1} \left(\sum_{i \geq 2} x_0 x_1^2 x_i^2 - x_0^2 x_1 x_i^2 \right) + \frac{c^2+4c}{c^2+1} \left(\sum_{i,j \geq 2; i<j} x_1 x_i^2 x_j^2 - x_0 x_i^2 x_j^2 \right) +$
$\frac{2c^3+3c}{c^2+1} \left(\sum_{i,j \geq 2; i<j} x_1^2 x_i^2 x_j + x_1^2 x_i x_j^2 - x_0^2 x_i x_j^2 - x_0^2 x_i^2 x_j \right) + \frac{c^2}{c^2+1} \left(\sum_{i,j,k \geq 2; i<j<k} x_0^2 x_i x_j x_k - x_1^2 x_i x_j x_k \right) +$
$\frac{2c^2}{c^2+1} \left(\sum_{i,j \geq 2; i<j} x_0 x_1^2 x_i x_j - x_0^2 x_1 x_i x_j \right) +$
$\frac{2c^2}{c^2+1} \left(\sum_{i,j,k \geq 2; i<j<k} x_1 x_i^2 x_j x_k + x_1 x_i x_j^2 x_k + x_1 x_i x_j x_k^2 - x_0 x_i^2 x_j x_k - x_0 x_i x_j^2 x_k - x_0 x_i x_j x_k^2 \right)$

It is defintely possible to factor out $x_1 - x_0$ from this.

(The other generators are created from this one by switching x_1 with x_k for some $k \geq 2$.)

9.1 x_0^9

This is the final generator. We note that all partial derivatives are 0, so we disregard them. For Dunkl operators other than D_0, the only relevant term is $\frac{x_0^9 - x_k^9}{x_k - x_0} = (x_k - x_0)^8$. We will show this is in the ideal. We note that then $D_0(x_0^9) = -c \sum_{k \geq 1} \frac{x_0^9 - x_k^9}{x_0 - x_k}$ will also be in the ideal.

We therefore need only consider the case D_1 since all others will be symmetric and follow.

$\frac{x_0^9-x_1^9}{x_1-x_0} = -\sum_{i=0}^{8} x_0^i x_1^{8-i}$.

We note that $D_0(x_0^9) = -c\sum_{k\geq 1} \frac{x_0^9-x_k^9}{x_0-x_k} = -c\sum_{i=0}^{8} x_0^i(\sum_j x_j^{8-i})$.

Let $m_{i,j} = \sum_{k=0}^{8} x_i^k x_j^{8-k}$ and $g_{i,j}$ be the generators described above. Ex: $g_{0,1}$ is the generator we were working with above.

We wish to write $m_{0,1} = \sum_{i=1}^{5} p_i g_{0,i}$ where p_i is a homogeneous degree 5 polynomial for all i.

We must consider the symmetries. We note that $g_{i,j} = -g_{j,i}, g_{i,j} - g_{k,j} = g_{i,k}$ and $g_{i,j} - g_{i,k} = g_{k,j}$ for all i, j, k. We also note that $g_{i,i} = 0$ for all i, and $\sum_{j=0}^{5} g_{i,j} = \frac{c+1}{c}\sum_{j=0}^{5} x_j^3$ for all i.

Let s_{ij} be the transposition switching x_i and x_j. By letting these act on $m_{0,1}$ we see that we must have for all $i, j \geq 2$ that $s_{ij}p_k = p_k$ for $k \neq i, k \neq j$ and $s_{ij}p_i = p_j, s_{ij}p_j = p_i$.

We also note that $s_{01}m_{0,1} = m_{0,1}$ and $s_{01}g_{0,1} = -g_{0,1}$, while $s_{0,1}g_{0,k}$ for $k \geq 2$ is $g_{1,k} = g_{0,k} - g_{0,1}$.

Then $m_{0,1} = -(s_{0,1} * p_1) * g_{0,1} + \sum_{i=2}^{5}(s_{0,1}p_i) * (g_{0,i} - g_{0,1})$.

From this we see that $-\sum_{i=1}^{5} s_{0,1}p_i = p_1$ and $s_{0,1}p_i = p_i$ for $i \geq 2$. This means that $p_1 + s_{0,1}p_1 + \sum_{i=2}^{5} p_i = 0$.

10 Conjecture for general $p \mid n$

Variables are $x_0, \ldots, x_{n-1}$.

Generators are:

$x_0^{p^2}$ in degree p^2, and $\sum x_i^p$ in degree p. There are $n-2$ remaining generators in degree p. It is clear that each such generator contains a term of the form $\frac{c+1}{c}(x_k^p - x_0^p)$. If we assume we are in the generator with $k = 1$, then the generator also contains a term $\left(\sum_{i\geq 2} x_0^{p-1}x_i + x_1 x_i^{p-1} - x_0 x_i^{p-1} - x_1^{p-1}x_i\right)$

11 $n-1$ case, $n-2$ case

In the case where we use the $n-1$ representation, the Hilbert polynomial appears to be $\left(\frac{1-t^p}{1-t}\right)^{n-1}$.

There are $n-1$ generators in degree 3. If we choose as basis $e_i = x_i - x_0$ for $i = 1, \ldots, n-1$ we will find that these generators are the same as the ones in the n-dimensional case, which we can express in terms of the e_i.

In the case $3 \mid n$ where we use the $n-2$ representation, the Hilbert polynomial appears to be $\left(\frac{1-t^p}{1-t}\right)^{n-2}$.

There are $n-3$ generators in degree 3. If we choose as basis $e_i = x_i - x_0$ for $i = 1, \ldots, n-2$ with $e_{n-1} = -\sum_{i=1}^{n-2} e_i$, we will find that these generators are the same as the ones in the n-dimensional case and the $n-1$ dimensional case, which we can express in terms of the e_i (the last generator is the negative of the sum of the others and therefore can be disregarded).