

1 $p = 3, 3 \mid n, n - 1$ case full argument

1.1 Generators

Let x_1, \dots, x_{n-1} be a basis for the $n - 1$ dimensional representation of S_n derived from the standard representation with basis e_0, \dots, e_{n-1} by the relation $x_i = e_i - e_0$ and the action of the symmetric group defined accordingly. Let s_{ij} be the transposition of i and j , for $0 \leq i < j \leq n - 1$.

For $1 \leq i < j \leq n - 1$, the relevant eigenvectors are $x_i - x_j$. For s_{0i} for $1 \leq i \leq n - 1$, the relevant eigenvector is x_i .

Let f_a for $a = 1, \dots, n - 1$ be $f_a = -x_a^3 + c \left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j x_a \right)$. We will show that all the f_a are killed by Dunkl operators. Because of their symmetry, we need only consider f_1 . Since f_1 is symmetric in the x_i excluding x_1 , the only Dunkl operators we need consider are D_1, D_2 .

We note that f_1 is preserved by s_{ij} for all $i, j > 1$. We consider s_{1a} for $2 \leq a$. We see that $s_{1a} f_1 = -x_a^3 + c \left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j x_a \right) = f_a$. Then:

$$\begin{aligned} f_1 - f_a &= -(x_1 - x_a)^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) - x_a x_i (x_a - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j (x_1 - x_a) \right) \\ &= -(x_1 - x_a)^3 + c \left(\sum_{i=1}^{n-1} x_i (x_1 + x_a - x_i) (x_1 - x_a) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j (x_1 - x_a) \right) \end{aligned}$$

$$\text{Then } \frac{f_1 - s_{1a} f_1}{x_1 - x_a} = -(x_1 - x_a)^2 + c \left(\sum_{i=1}^{n-1} x_i (x_1 + x_a - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right).$$

We consider s_{0a} for $2 \leq a$. We see that s_{0a} sends x_a to $-x_a$ and x_i for $i \neq a$ to $x_i - x_a$. We see that:

$$\begin{aligned} s_{0a} f_1 &= s_{0a} \left(-x_1^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j x_1 \right) \right) \\ &= s_{0a} \left(-x_1^3 + c \left(\sum_{i \neq 1, a} x_1 x_i (x_1 - x_i) \right) + c x_1 x_a (x_1 - x_a) + c^2 \left(\sum_{i \neq a} \sum_{j \neq a} x_i x_j x_1 \right) + 2c^2 \left(\sum_{i \neq a} x_i x_a x_1 \right) + c^2 x_a^2 x_1 \right) \\ &= -(x_1 - x_a)^3 + c \left(\sum_{i \neq 1, a} (x_1 - x_a) (x_i - x_a) (x_1 - x_i) \right) + c(x_1 - x_a)(-x_a)(x_1) + \\ &\quad c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i - x_a) (x_j - x_a) (x_1 - x_a) \right) + 2c^2 \left(\sum_{i \neq a} (x_i - x_a) (-x_a) (x_1 - x_a) \right) + c^2 (-x_a)^2 (x_1 - x_a) \end{aligned}$$

$$\begin{aligned}
&= -x_1^3 + x_a^3 + c \left(\sum_{i \neq 1, a} x_1(x_i - x_a)(x_1 - x_i) \right) - c \left(\sum_{i \neq 1, a} x_a(x_a - x_i)(x_i - x_1) \right) + c(x_1 - x_a)(-x_a)(x_1) + \\
&c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i - x_a)(x_j - x_a)(x_1 - x_a) \right) + 2c^2 \left(\sum_{i \neq a} (x_i - x_a)(-x_a)(x_1 - x_a) \right) + c^2(-x_a)^2(x_1 - x_a) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i \neq 1, a} x_1 x_i(x_1 - x_i) \right) - c \left(\sum_{i \neq 1, a} x_a x_i(x_a - x_i) \right) + c \left(\sum_{i \neq 1, a} -x_1^2 x_a + x_a^2 x_1 \right) + c(x_1 - x_a)(-x_a)(x_1) + \\
&c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i - x_a)(x_j - x_a)(x_1 - x_a) \right) + 2c^2 \left(\sum_{i \neq a} (x_i - x_a)(-x_a)(x_1 - x_a) \right) + c^2(-x_a)^2(x_1 - x_a) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i \neq 1, a} x_1 x_i(x_1 - x_i) \right) - c \left(\sum_{i \neq 1, a} x_a x_i(x_a - x_i) \right) - c x_1 x_a(x_1 - x_a) + \\
&c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i - x_a)(x_j - x_a)(x_1 - x_a) \right) + 2c^2 \left(\sum_{i \neq a} (x_i - x_a)(-x_a)(x_1 - x_a) \right) + c^2(-x_a)^2(x_1 - x_a) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i \neq 1, a} x_1 x_i(x_1 - x_i) \right) - c \left(\sum_{i \neq 1, a} x_a x_i(x_a - x_i) \right) + c x_1 x_a(x_1 - x_a) - c x_1 x_a(x_a - x_1) + \\
&c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i - x_a)(x_j - x_a)(x_1 - x_a) \right) + 2c^2 \left(\sum_{i \neq a} (x_i - x_a)(-x_a)(x_1 - x_a) \right) + c^2(-x_a)^2(x_1 - x_a) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i(x_1 - x_i) \right) - c \left(\sum_{i=1}^{n-1} x_a x_i(x_a - x_i) \right) + \\
&c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i - x_a)(x_j - x_a)(x_1 - x_a) \right) + 2c^2 \left(\sum_{i \neq a} (x_i - x_a)(-x_a)(x_1 - x_a) \right) + c^2(-x_a)^2(x_1 - x_a) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i(x_1 - x_i) \right) - c \left(\sum_{i=1}^{n-1} x_a x_i(x_a - x_i) \right) + \\
&c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i x_j - x_a x_j - x_a x_i + x_a^2)(x_1 - x_a) \right) + 2c^2 \left(\sum_{i \neq a} (x_a^2 - x_a x_i)(x_1 - x_a) \right) + c^2 x_a^2(x_1 - x_a) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i(x_1 - x_i) \right) - c \left(\sum_{i=1}^{n-1} x_a x_i(x_a - x_i) \right) + \\
&c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i x_j + x_a^2)(x_1 - x_a) \right) + c^2 \left(\sum_{i \neq a} x_a x_i(x_1 - x_a) \right) + c^2 \left(\sum_{i \neq a} (x_a^2 - x_a x_i)(x_1 - x_a) \right) + c^2 x_a^2(x_1 - x_a) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i(x_1 - x_i) \right) - c \left(\sum_{i=1}^{n-1} x_a x_i(x_a - x_i) \right) + \\
&c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i x_j + x_a^2)(x_1 - x_a) \right) + c^2 \left(\sum_{i \neq a} x_a^2(x_1 - x_a) \right) + c^2 x_a^2(x_1 - x_a)
\end{aligned}$$

$$\begin{aligned}
&= -x_1^3 + x_a^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) \right) - c \left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i) \right) + c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i x_j) (x_1 - x_a) \right) + \\
&2c^2 \left(\sum_{i \neq a} x_a^2 (x_1 - x_a) \right) + c^2 x_a^2 (x_1 - x_a) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) \right) - c \left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i) \right) + c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i x_j) (x_1 - x_a) \right) + \\
&2c^2 x_a^2 (x_1 - x_a) + c^2 x_a^2 (x_1 - x_a) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) \right) - c \left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i) \right) + c^2 \left(\sum_{i \neq a} \sum_{j \neq a} (x_i x_j) (x_1 - x_a) \right) \\
&= -x_1^3 + x_a^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) \right) - c \left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i) \right) + c^2 \left(\sum_{i \neq a} \sum_{j \neq a} x_i x_j x_1 - x_i x_j x_a \right) \\
&= f_1 - f_a
\end{aligned}$$

Then $\frac{f_1 - s_{0a} f_1}{x_a} = f_a / x_a = -x_a^2 - c \left(\sum_{i=1}^{n-1} x_i (x_i - x_a) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right)$.

The only remaining case to consider is s_{01} . We see that s_{01} sends x_1 to $-x_1$ and x_a for $a \geq 2$ to $x_a - x_1$. Then $s_{01} f_1$ goes to

$$\begin{aligned}
&-(-x_1)^3 + c \left(\sum_{i=2}^{n-1} (-x_1) (x_i - x_1) (-x_1 - (x_i - x_1)) \right) + c^2 \left(\sum_{i=2}^{n-1} \sum_{j=2}^{n-1} (x_i - x_1) (x_j - x_1) (-x_1) \right) + \\
&2c^2 \left(\sum_{i=2}^{n-1} (x_i - x_1) x_1^2 \right) + c^2 (-x_1)^3 \\
&= x_1^3 + c \left(\sum_{i=2}^{n-1} x_1 x_i (x_i - x_1) \right) + c^2 \left(\sum_{i=2}^{n-1} \sum_{j=2}^{n-1} -x_i x_j x_1 + x_i x_1^2 + x_j x_1^2 - x_1^3 \right) + \\
&2c^2 \left(\sum_{i=2}^{n-1} (x_i - x_1) x_1^2 \right) + 2c^2 x_1^3 \\
&= x_1^3 + c \left(\sum_{i=2}^{n-1} x_1 x_i (x_i - x_1) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} -x_i x_j x_1 \right) \\
&= -f_1
\end{aligned}$$

The above used the fact that $n-2 \equiv 1 \pmod{3}$. Therefore $\frac{f_1 - s_{01} f_1}{x_1} = 2f_1 / x_1 = -f_1 / x_1 = x_1^2 + c \left(\sum_{i=1}^{n-1} x_i (x_i - x_1) \right) - c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right)$.

We can use these to calculate the effects of the Dunkl operators.

1.2 Dunkl operators

1.2.1 $D_1 f_1$

We first consider $D_1 f_1 = \partial_1 f_1 - c \sum_{a>1} \frac{f_1 - s_{1a} f_1}{x_1 - x_a} - c \frac{f_1 - s_{01} f_1}{x_1}$. Let $G_1 = \sum_{a>1} \frac{f_1 - s_{1a} f_1}{x_1 - x_a}$, $G_2 = \frac{f_1 - s_{01} f_1}{x_1}$. We wish to show that $\partial_1 f_1 = cG_1 + cG_2$, since then $D_1 f_1 = 0$. We calculate $\partial_1 f_1$:

$$\begin{aligned}
\partial_1 f_1 &= \partial_1 \left(-x_1^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j x_1 \right) \right) \\
&= c \left(\sum_{i=2}^{n-1} x_1 x_i + x_i (x_1 - x_i) \right) + c^2 \left(\sum_{i=2}^{n-1} \sum_{j=2}^{n-1} x_i x_j \right) + 2c^2 \left(\sum_{i=2}^{n-1} 2x_i x_1 \right) \\
&= c \left(\sum_{i=2}^{n-1} x_1 x_i + x_i x_1 - x_i^2 \right) + c^2 \left(\sum_{i=2}^{n-1} \sum_{j=2}^{n-1} x_i x_j \right) + c^2 \left(\sum_{i=2}^{n-1} x_i x_1 \right) \\
&= c \left(\sum_{i=2}^{n-1} -x_1 x_i - x_i^2 \right) + c^2 \left(\sum_{i=2}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right)
\end{aligned}$$

We also see that $G_1 = \sum_{a>1} \frac{f_1 - s_{1a} f_1}{x_1 - x_a} = \sum_{a=2}^{n-1} -(x_1 - x_a)^2 + c \left(\sum_{i=1}^{n-1} x_i (x_1 + x_a - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right)$ and $G_2 = x_1^2 + c \left(\sum_{i=1}^{n-1} x_i (x_i - x_1) \right) - c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right)$.

Since $n - 2 \equiv 1 \pmod{3}$, we see that the c^2 terms cancel in $G_1 + G_2$. We also note that $x_i(x_i - x_1) = x_i(x_1 + x_1 - x_i) - x_i^2$ for all i , so $G_1 + G_2 = x_1^2 - c \left(\sum_{i=1}^{n-1} x_i^2 \right) + \sum_{a=1}^{n-1} -(x_1 - x_a)^2 + c \left(\sum_{i=1}^{n-1} x_i (x_1 + x_a - x_i) \right)$.

$$\begin{aligned}
G_1 + G_2 &= x_1^2 - c \left(\sum_{i=1}^{n-1} x_i^2 \right) + \sum_{a=1}^{n-1} -(x_1 - x_a)^2 + c \left(\sum_{i=1}^{n-1} x_i(x_1 + x_a - x_i) \right) \\
&= x_1^2 - c \left(\sum_{i=1}^{n-1} x_i^2 \right) + \sum_{a=1}^{n-1} -x_1^2 - x_a^2 + 2x_a x_1 + c \left(\sum_{i=1}^{n-1} x_i(x_1 + x_a - x_i) \right) \\
&= x_1^2 - c \left(\sum_{i=1}^{n-1} x_i^2 \right) + \sum_{j=1}^{n-1} -x_1^2 - x_j^2 + 2x_j x_1 + c \left(\sum_{i=1}^{n-1} x_i(x_1 + x_j - x_i) \right) \\
&= x_1^2 + -(n-1)x_1^2 - c \left(\sum_{i=1}^{n-1} x_i^2 \right) + \sum_{j=1}^{n-1} -x_j^2 - x_j x_1 + c \left(\sum_{i=1}^{n-1} x_i(x_1 + x_j - x_i) \right) \\
&= 2x_1^2 - c \left(\sum_{i=1}^{n-1} x_i^2 \right) + \sum_{j=1}^{n-1} (-x_j^2 - x_j x_1) + c \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_1 + x_i x_j - x_i^2 \right) \\
&= \sum_{j=2}^{n-1} (-x_j^2 - x_j x_1) - c \left(\sum_{i=1}^{n-1} x_i^2 \right) + c \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_1 + x_i x_j - x_i^2 \right) \\
&= \sum_{j=2}^{n-1} (-x_j^2 - x_j x_1) - c \left(\sum_{i=1}^{n-1} x_i^2 \right) + c \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right) - c \left(\sum_{i=1}^{n-1} x_i x_1 - x_i^2 \right) \\
&= \sum_{j=2}^{n-1} (-x_j^2 - x_j x_1) - c \left(\sum_{i=1}^{n-1} x_i^2 \right) + c \left(\sum_{i=1}^{n-1} \sum_{j=2}^{n-1} x_i x_j \right) + c \left(\sum_{i=1}^{n-1} x_i^2 \right) \\
&= \sum_{j=2}^{n-1} (-x_j^2 - x_j x_1) + c \left(\sum_{i=1}^{n-1} \sum_{j=2}^{n-1} x_i x_j \right)
\end{aligned}$$

Then a simple change of indices tells us that $cG_1 + cG_2 = \partial_1 f_1$ as desired.

1.2.2 $D_2 f_1$

We see that $D_2 f_1 = \partial_2 f_1 - c \frac{f_1 - s_{12} f_1}{x_2 - x_1} - c \frac{f_1 - s_{02} f_1}{x_2}$.

We calculate $\partial_2 f_1$:

$$\begin{aligned}
\partial_2 f_1 &= \partial_2 \left(-x_1^3 + c \left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j x_1 \right) \right) \\
&= \partial_2 \left(-x_1^3 + c \left(\sum_{i=3}^{n-1} x_1 x_i (x_1 - x_i) \right) + c x_1 x_2 (x_1 - x_2) + c^2 \left(\sum_{i \neq 2} \sum_{j \neq 2} x_i x_j x_1 \right) + 2c^2 \left(\sum_{i \neq 2} x_i x_1 x_2 \right) + c^2 x_2^2 x_1 \right) \\
&= c x_1^2 - 2c x_1 x_2 + 2c^2 \left(\sum_{i \neq 2} x_i x_1 \right) + 2c^2 x_2 x_1 \\
&= c x_1^2 + c x_1 x_2 + 2c^2 \left(\sum_{i=1}^{n-1} x_i x_1 \right)
\end{aligned}$$

Let $G_1 = \frac{f_1 - s_{12} f_1}{x_2 - x_1}$, $G_2 = \frac{f_1 - s_{02} f_1}{x_2}$. If $c(G_1 + G_2) = \partial_2 f_1$, then $D_2 f_1 = 0$.

$$\begin{aligned}
G_1 + G_2 &= \frac{f_1 - s_{12} f_1}{x_2 - x_1} + \frac{f_1 - s_{02} f_1}{x_2} \\
&= (x_1 - x_2)^2 - c \left(\sum_{i=1}^{n-1} x_i (x_1 + x_2 - x_i) \right) - c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right) - x_2^2 - c \left(\sum_{i=1}^{n-1} x_i (x_i - x_2) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right) \\
&= x_1^2 + x_1 x_2 + x_2^2 - c \left(\sum_{i=1}^{n-1} x_i (x_1 + x_2 - x_i) \right) - x_2^2 - c \left(\sum_{i=1}^{n-1} x_i (x_i - x_2) \right) \\
&= x_1^2 + x_1 x_2 - c \left(\sum_{i=1}^{n-1} x_i (x_1 + x_2 - x_i) \right) - c \left(\sum_{i=1}^{n-1} x_i (x_i - x_2) \right) \\
&= x_1^2 + x_1 x_2 - c \left(\sum_{i=1}^{n-1} x_i (x_1 + x_2 - x_i + x_i - x_2) \right) \\
&= x_1^2 + x_1 x_2 - c \left(\sum_{i=1}^{n-1} x_i x_1 \right)
\end{aligned}$$

Then we see easily that $\partial_2 f_1 = cG_1 + cG_2$, so $D_2 f_1 = 0$ as desired.

1.3 Complete intersection

Then we see that the ideal I generated by f_1, \dots, f_{n-1} has $I \subseteq J$ where J is the kernel of the β form, since the generators f_1, \dots, f_{n-1} are killed by the Dunkl operators.

We note that the f_i are linearly independent, since x_i^3 has nonzero coefficient in f_j only when $i = j$. We will show that A/I , where $A = k[x_1, \dots, x_{n-1}]$, is a complete intersection by showing that $f_1(x) = \dots = f_{n-1}(x) = 0$ implies $x = 0$.

Recall $f_a = -x_a^3 + c \left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j x_a \right)$.

We also note that $\left(\sum_{i=1}^{n-1} x_i\right)^2 = \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j\right)$, and $\left(\sum_{i=1}^{n-1} x_i\right)^2 + \sum_{1 \leq i < j \leq n-1} x_i x_j = \sum_{i=1}^{n-1} x_i^2$ modulo 3.

Therefore:

$$\begin{aligned}
f_a &= -x_a^3 + c \left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i) \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j x_a \right) \\
&= -x_a^3 + c \left(\sum_{i=1}^{n-1} x_a^2 x_i - x_a x_i^2 \right) + c^2 \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j x_a \right) \\
&= -x_a^3 + c x_a^2 \left(\sum_{i=1}^{n-1} x_i \right) - c x_a \left(\sum_{i=1}^{n-1} x_i^2 \right) + c^2 x_a \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i x_j \right) \\
&= -x_a^3 + c x_a^2 \left(\sum_{i=1}^{n-1} x_i \right) + 2c x_a \left(\sum_{i=1}^{n-1} x_i \right)^2 + 2c x_a \left(\sum_{1 \leq i < j \leq n-1} x_i x_j \right) + c^2 x_a \left(\sum_{i=1}^{n-1} x_i \right)^2 \\
&= -x_a^3 + c x_a^2 \left(\sum_{i=1}^{n-1} x_i \right) + (c^2 + 2c) x_a \left(\sum_{i=1}^{n-1} x_i \right)^2 + x_a \left(\sum_{1 \leq i < j \leq n-1} 2c x_i x_j \right)
\end{aligned}$$

We see that $f_a = -x_a^3 + c x_a^2 \left(\sum_{i=1}^{n-1} x_i \right) + (c^2 + 2c) x_a \left(\sum_{i=1}^{n-1} x_i \right)^2 + x_a \left(\sum_{1 \leq i < j \leq n-1} 2c x_i x_j \right)$.

We now consider the sum of the f_a :

$$\begin{aligned}
\sum_{a=1}^{n-1} f_a &= \sum_{a=1}^{n-1} \left(-x_a^3 + cx_a^2 \left(\sum_{i=1}^{n-1} x_i \right) + (c^2 + 2c)x_a \left(\sum_{i=1}^{n-1} x_i \right)^2 + x_a \left(\sum_{1 \leq i < j \leq n-1} 2cx_i x_j \right) \right) \\
&= \left(\sum_{a=1}^{n-1} -x_a^3 \right) + c \left(\sum_{a=1}^{n-1} x_a^2 \right) \left(\sum_{i=1}^{n-1} x_i \right) + (c^2 + 2c) \left(\sum_{a=1}^{n-1} x_a \right) \left(\sum_{i=1}^{n-1} x_i \right)^2 + \left(\sum_{a=1}^{n-1} x_a \right) \left(\sum_{1 \leq i < j \leq n-1} 2cx_i x_j \right) \\
&= 2 \left(\sum_{i=1}^{n-1} x_i^3 \right) + c \left(\sum_{i=1}^{n-1} x_i^2 \right) \left(\sum_{i=1}^{n-1} x_i \right) + (c^2 + 2c) \left(\sum_{i=1}^{n-1} x_i \right) \left(\sum_{i=1}^{n-1} x_i \right)^2 + \left(\sum_{i=1}^{n-1} x_i \right) \left(\sum_{1 \leq i < j \leq n-1} 2cx_i x_j \right) \\
&= 2 \left(\sum_{i=1}^{n-1} x_i^3 \right) + c \left(\left(\sum_{i=1}^{n-1} x_i \right)^2 + \left(\sum_{1 \leq i < j \leq n-1} x_i x_j \right) \right) \left(\sum_{i=1}^{n-1} x_i \right) + (c^2 + 2c) \left(\sum_{i=1}^{n-1} x_i \right)^3 + \\
&\quad 2c \left(\sum_{i=1}^{n-1} x_i \right) \left(\sum_{1 \leq i < j \leq n-1} x_i x_j \right) \\
&= 2 \left(\sum_{i=1}^{n-1} x_i^3 \right) + c \left(\sum_{i=1}^{n-1} x_i \right)^3 + c \left(\sum_{1 \leq i < j \leq n-1} x_i x_j \right) \left(\sum_{i=1}^{n-1} x_i \right) + (c^2 + 2c) \left(\sum_{i=1}^{n-1} x_i \right)^3 + \\
&\quad 2c \left(\sum_{i=1}^{n-1} x_i \right) \left(\sum_{1 \leq i < j \leq n-1} x_i x_j \right) \\
&= (c^2 + 2) \left(\sum_{i=1}^{n-1} x_i \right)^3
\end{aligned}$$

We see that $\sum_{a=1}^{n-1} f_a = (c^2 + 2) \left(\sum_{i=1}^{n-1} x_i \right)^3$.

Therefore if $f_1(x) = \cdots = f_{n-1}(x) = 0$, we see that $\sum_{i=1}^{n-1} x_i = 0$. Modulo this new relation, we see that $f_a = x_a \left(-x_a^2 + \sum_{1 \leq i < j \leq n-1} 2cx_i x_j \right)$. Therefore if $f_a = 0$ for all a , we see that for each a , either $x_a = 0$ or x_a is a square root of $C = \sum_{1 \leq i < j \leq n-1} 2cx_i x_j$. If $C = 0$ or C has no square roots, then all the x_a are 0 and we are done; so we assume C has two square roots which are additive inverses of each other. Then each pair $x_i x_j$ either multiply to $\pm C$ or 0, so $C = 2cmC$ for some integer m . Then $(1 + cm)C = 0$; if $C \neq 0$, then $1 + cm = 0$; however, m is an integer, so this is impossible. Therefore $C = 0$, so all the x_a are 0.

Therefore $f_1(x) = \cdots = f_{n-1}(x) = 0$ implies $x = 0$, so A/I is a complete intersection; it then must have Hilbert polynomial $h_{A/I}(t) = (t^2 + t + 1)^{n-1}$.

By Proposition 3.4 in <http://arxiv.org/abs/1107.0504>, we see that the Hilbert polynomial of A/J is $(t^2 + t + 1)^{n-1} h(t^3)$ for some polynomial h with nonnegative integer coefficients; since $I \subseteq J$, we see that $h_{A/I}(t) \geq h_{A/J}(t)$ coefficientwise; however by this restriction of the form of $h_{A/J}(t)$, we see that the only possible choice for h is $h(t) = 1$. Therefore $h_{A/I}(t) = h_{A/J}(t)$, so $I = J$ and these $n - 1$ generators generate the whole ideal.