

Computations

$$\bar{h} = 1$$

1 S_2 , char 2, τ trivial

1.1 Hilbert polynomial

$$t^4 + 2t^3 + 2t^2 + 2t + 1 = (t + 1)^2(t^2 + 1).$$

1.2 Generators

$$x_0^2 + x_1^2$$

$$x_0^4.$$

2 S_3 , char 3, τ trivial

2.1 Hilbert polynomial

$$t^{12} + 3t^{11} + 6t^{10} + 8t^9 + 9t^8 + 9t^7 + 9t^6 + 9t^5 + 9t^4 + 8t^3 + 6t^2 + 3t + 1 = (t^2 + t + 1)^3(t^6 + t^3 + 1).$$

2.2 Generators

$$x_0^3 + x_1^3 + x_2^3$$

$$\begin{aligned} & ((2c + 2)/c)x_0^3 \\ & + ((c + 1)/c)x_1^3 \\ & - x_0^2x_1 - x_1^2x_2 - x_0x_2^2 \\ & + x_0x_1^2 + x_0^2x_2 + x_1x_2^2 \end{aligned}$$

$$x_0^9.$$

3 S_4 , char 2, τ trivial

3.1 Hilbert polynomial

$$t^6 + 4t^5 + 7t^4 + 8t^3 + 7t^2 + 4t + 1 = (t + 1)^4(t^2 + 1).$$

3.2 Generators

$$x_0^2 + x_1^2 + x_2^2 + x_3^2$$

$$((c+1)/c)x_0^2 + x_0x_1 + x_1x_2 + ((c+1)/c)x_2^2 + x_0x_3 + x_2x_3$$

$$((c+1)/c)x_0^2 + ((c+1)/c)x_1^2 + x_0x_2 + x_1x_2 + x_0x_3 + x_1x_3$$

$$x_0^4.$$

4 S_5 , char 5, τ trivial

4.1 Hilbert polynomial

PARTIAL

$$1 + 5t + 15t^2 + 35t^3 + 70t^4 + 122t^5 + 190t^6 + 270t^7 + \dots$$

4.2 Generators

PARTIAL

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5$$

$$\begin{aligned} & ((4c+4)/c)x_0^5 \\ & + ((c+1)/c)x_3^5 \\ & + x_0^4x_1 + x_0^4x_2 + x_1^4x_3 + x_2^4x_3 + x_0^4x_4 + x_3x_4^4 \\ & + 2x_0^4x_3 \\ & + 3x_0x_3^4 \\ & - x_0x_1^4 - x_0x_2^4 - x_1x_3^4 - x_2x_3^4 - x_3x_4^4 - x_0x_4^4 \\ & + ((c+4)/(c+2))x_0^2x_1^3 + ((c+4)/(c+2))x_0^2x_2^3 + ((c+4)/(c+2))x_1^2x_3^3 + ((c+4)/(c+2))x_2^2x_3^3 + ((c+4)/(c+2))x_3^3x_4^2 + ((c+4)/(c+2))x_0^2x_4^3 \\ & + ((2c+3)/(c+2))x_0^2x_3^3 \\ & + ((3c+2)/(c+2))x_0^3x_2^2 \\ & + ((4c+1)/(c+2))x_0^3x_1^2 + ((4c+1)/(c+2))x_0^3x_2^2 + ((4c+1)/(c+2))x_1^3x_3^2 + ((4c+1)/(c+2))x_2^3x_3^2 + ((4c+1)/(c+2))x_3^3x_4^2 + ((4c+1)/(c+2))x_0^2x_4^3 \\ & + (c/(c+2))x_0x_1^3x_2 + (c/(c+2))x_0x_1x_2^3 + (c/(c+2))x_0x_1^3x_4 + (c/(c+2))x_0x_2^3x_4 + (c/(c+2))x_0x_1x_4^3 + (c/(c+2))x_0x_2x_4^3 \\ & + (2c/(c+2))x_0^3x_1x_3 + (2c/(c+2))x_0^3x_2x_3 + (2c/(c+2))x_1x_2x_3^3 + (2c/(c+2))x_0^3x_3x_4 + (2c/(c+2))x_1x_3^3x_4 + (2c/(c+2))x_2x_3^3x_4 \\ & + (3c/(c+2))x_0^3x_1x_2 + (3c/(c+2))x_0x_1x_3^3 + (3c/(c+2))x_0x_2x_3^3 + (3c/(c+2))x_0^3x_1x_4 + (3c/(c+2))x_0^3x_2x_4 + (3c/(c+2))x_0x_3^3x_4 \\ & + (4c/(c+2))x_1^3x_2x_3 + (4c/(c+2))x_1x_2^3x_3 + (4c/(c+2))x_1^3x_3x_4 + (4c/(c+2))x_2^3x_3x_4 + (4c/(c+2))x_1x_3^3x_4 + (4c/(c+2))x_2x_3^3x_4 \\ & + ((c^2+4c)/(c^2+1))x_1^2x_2^2x_3 + ((c^2+4c)/(c^2+1))x_0x_1^2x_3^2 + ((c^2+4c)/(c^2+1))x_0x_2^2x_3^2 + ((c^2+4c)/(c^2+1))x_1^2x_3x_4^2 + ((c^2+4c)/(c^2+1))x_2^2x_3x_4^2 + ((c^2+4c)/(c^2+1))x_0x_3^2x_4^2 \\ & + ((2c^2+3c)/(c^2+1))x_1^2x_2x_3^2 + ((2c^2+3c)/(c^2+1))x_1x_2^2x_3^2 + ((2c^2+3c)/(c^2+1))x_1^2x_3^2x_4 + ((2c^2+3c)/(c^2+1))x_2^2x_3^2x_4 + ((2c^2+3c)/(c^2+1))x_1x_2^2x_4^2 + ((2c^2+3c)/(c^2+1))x_2x_2^2x_4^2 \\ & + ((3c^2+2c)/(c^2+1))x_0^2x_1^2x_2 + ((3c^2+2c)/(c^2+1))x_0^2x_1x_2^2 + ((3c^2+2c)/(c^2+1))x_0^2x_1^2x_4 + ((3c^2+2c)/(c^2+1))x_0^2x_2^2x_4 + ((3c^2+2c)/(c^2+1))x_1^2x_2^2x_4 + ((3c^2+2c)/(c^2+1))x_2^2x_2^2x_4 \end{aligned}$$

$$\begin{aligned}
& 1))x_0^2x_2^2x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2x_4^2 \\
& + ((4c^2 + c)/(c^2 + 1))x_0x_2^2x_2^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1^2x_3 + ((4c^2 + c)/(c^2 + 1))x_0^2x_2^2x_3 + ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_2^2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_3x_4^2 \\
& + (c^2/(c^2 + 1))x_0^2x_1x_2x_4 \\
& + (2c^2/(c^2 + 1))x_0x_1x_2x_3^2 + (2c^2/(c^2 + 1))x_1^2x_2x_3x_4 + (2c^2/(c^2 + 1))x_1x_2^2x_3x_4 + (2c^2/(c^2 + 1))x_0x_1x_3^2x_4 + (2c^2/(c^2 + 1))x_0x_2x_3^2x_4 + (2c^2/(c^2 + 1))x_1x_2x_3x_4^2 \\
& + (3c^2/(c^2 + 1))x_0^2x_1x_2x_3 + (3c^2/(c^2 + 1))x_0x_1^2x_2x_4 + (3c^2/(c^2 + 1))x_0x_1x_2^2x_4 + (3c^2/(c^2 + 1))x_0^2x_1x_3x_4 + (3c^2/(c^2 + 1))x_0^2x_2x_3x_4 + (3c^2/(c^2 + 1))x_0x_1x_2x_4^2 \\
& + (4c^2/(c^2 + 1))x_1x_2x_3^2x_4
\end{aligned}$$

$$\begin{aligned}
& ((4c+4)/c)x_0^5 + x_0^4x_1 + ((4c+1)/(c+2))x_0^3x_1^2 + ((c+4)/(c+2))x_0^2x_1^3 - x_0x_1^4 + 2x_0^4x_2 + (2c/(c+2))x_0^3x_1x_2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1^2x_2 + x_1^4x_2 + ((3c+2)/(c+2))x_0^3x_2^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_1^2x_2^2 + ((4c+1)/(c+2))x_1^3x_2^2 + ((2c+3)/(c+2))x_0^2x_3^2 + (3c/(c+2))x_0x_1x_3^2 + ((c+4)/(c+2))x_1^2x_3^2 + 3x_0x_2^4 - x_1x_2^4 + ((c+1)/c)x_0^5 + x_0^4x_3 + (3c/(c+2))x_0^3x_1x_3 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1^2x_3 + (c/(c+2))x_0x_1^3x_3 + (2c/(c+2))x_0^3x_2x_3 + (3c^2/(c^2 + 1))x_0^2x_1x_2x_3 + (4c/(c+2))x_1^3x_2x_3 + (2c^2/(c^2 + 1))x_0x_1x_2^2x_3 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2^2x_3 + (3c/(c+2))x_0x_3^2x_3 + (2c/(c+2))x_1x_3^2x_3 - x_1^4x_3 + ((4c+1)/(c+2))x_0^3x_3^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1x_3^2 + ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_3^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_2x_3^2 + ((c^2 + 4c)/(c^2 + 1))x_1^2x_2x_3^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_2^2x_3^2 + ((2c^2 + 3c)/(c^2 + 1))x_1x_2^2x_3^2 + ((c+4)/(c+2))x_2^3x_3^2 + ((c+4)/(c+2))x_0^2x_3^3 + (c/(c+2))x_0x_1x_3^3 + (4c/(c+2))x_1x_2x_3^3 + ((4c+1)/(c+2))x_2^2x_3^3 - x_0x_1^4 + x_2x_4^4 + x_1^4x_4 + (3c/(c+2))x_0^3x_1x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_1^2x_4 + (c/(c+2))x_0x_1^3x_4 + (2c/(c+2))x_0^3x_2x_4 + (3c^2/(c^2 + 1))x_0^2x_1x_2x_4 + (4c/(c+2))x_1^3x_2x_4 + (2c^2/(c^2 + 1))x_0x_1x_2^2x_4 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2^2x_4 + (3c/(c+2))x_0x_3^2x_4 + (2c/(c+2))x_1x_3^2x_4 - x_1^4x_4 + (3c/(c+2))x_0^3x_3x_4 + (c^2/(c^2 + 1))x_0^2x_1x_3x_4 + (3c^2/(c^2 + 1))x_0x_1^2x_3x_4 + (3c^2/(c^2 + 1))x_0^2x_2x_3x_4 + (2c^2/(c^2 + 1))x_1^2x_2x_3x_4 + (2c^2/(c^2 + 1))x_0x_2^2x_3x_4 + (4c^2/(c^2 + 1))x_1x_2^2x_3x_4 + (2c/(c+2))x_2^3x_3x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3^2x_4 + (3c^2/(c^2 + 1))x_0x_1x_2^2x_4 + (2c^2/(c^2 + 1))x_1x_2x_3^2x_4 + ((2c^2 + 3c)/(c^2 + 1))x_2^2x_3^2x_4 + (c/(c+2))x_0x_3^3x_4 + (4c/(c+2))x_2x_3^3x_4 + ((4c+1)/(c+2))x_0^3x_1^2x_4 + ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_1^2x_2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_2^2x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_1x_2^2x_4^2 + ((c+4)/(c+2))x_2^3x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3x_4^2 + (3c^2/(c^2 + 1))x_0x_1x_3x_4^2 + (2c^2/(c^2 + 1))x_1x_2x_3x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_2^2x_3x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_3^2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_2x_3^2x_4^2 + ((c+4)/(c+2))x_0^2x_4^3 + (c/(c+2))x_0x_1x_4^3 + (4c/(c+2))x_1x_2x_4^3 + ((4c+1)/(c+2))x_2^2x_4^3 + (c/(c+2))x_0x_3x_4^3 + (4c/(c+2))x_2x_3x_4^3 - x_0x_4^4 + x_2x_4^4
\end{aligned}$$

$$\begin{aligned}
& ((4c+4)/c)x_0^5 + 2x_0^4x_1 + ((3c+2)/(c+2))x_0^3x_1^2 + ((2c+3)/(c+2))x_0^2x_1^3 + 3x_0x_1^4 + ((c+1)/c)x_0^5 + x_0^4x_2 + (2c/(c+2))x_0^3x_1x_2 + (3c/(c+2))x_0x_1^2x_2 - x_1^4x_2 + ((4c+1)/(c+2))x_0^3x_2^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1x_2^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_1^2x_2^2 + ((c+4)/(c+2))x_1^3x_2^2 + ((c+4)/(c+2))x_0^2x_3^2 + ((4c+1)/(c+2))x_1^2x_3^2 - x_0x_2^4 + x_1x_2^4 + x_0^4x_3 + (2c/(c+2))x_0^3x_1x_3 + (3c/(c+2))x_0x_1^2x_3 - x_1^4x_3 + (3c/(c+2))x_0^3x_2x_3 + (3c^2/(c^2 + 1))x_0^2x_1x_2x_3 + (2c^2/(c^2 + 1))x_0x_1^2x_2x_3 + (2c/(c+2))x_1^3x_2x_3 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2^2x_3 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2^2x_3 + (c/(c+2))x_0x_2^2x_3 + (4c/(c+2))x_1x_2^2x_3 + ((4c+1)/(c+2))x_0^3x_3^2 + ((4c^2 + c)/(c^2 + 1))x_0^2x_1x_3^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_1^2x_3^2 + ((c+4)/(c+2))x_1^3x_3^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2x_3^2 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2x_3^2 + ((4c^2 + c)/(c^2 + 1))x_0x_2^2x_3^2 + ((c^2 + 4c)/(c^2 + 1))x_1x_2^2x_3^2 + ((c+4)/(c+2))x_0^2x_3^3 + ((4c+1)/(c+2))x_1^2x_3^3 + (c/(c+2))x_0x_2x_3^3 + (4c/(c+2))x_1x_2x_3^3 - x_0x_3^4 + x_1x_3^4 + x_0^4x_4 + (2c/(c+2))x_0^3x_1x_4 + (3c/(c+2))x_0x_1^2x_4 - x_1^4x_4 + (3c/(c+2))x_0^3x_2x_4 + (3c^2/(c^2 + 1))x_0^2x_1x_2x_4 + (2c^2/(c^2 + 1))x_0x_1^2x_2x_4 + (2c/(c+2))x_1^3x_2x_4 + (2c/(c+2))x_1^3x_2x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2^2x_4 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2^2x_4 + (c/(c+2))x_0x_3^2x_4 + (4c/(c+2))x_1x_3^2x_4 + (3c/(c+2))x_0^3x_3x_4 + (3c^2/(c^2 + 1))x_0^2x_1x_3x_4 + (2c^2/(c^2 + 1))x_0x_1^2x_3x_4 + (2c/(c+2))x_1^3x_3x_4 + (c^2/(c^2 + 1))x_0^2x_2x_3x_4 + (4c^2/(c^2 + 1))x_1^2x_2x_3x_4 + (3c^2/(c^2 + 1))x_0x_2^2x_3x_4 + (2c^2/(c^2 + 1))x_1x_2^2x_3x_4 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3^2x_4 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_3^2x_4 + (3c^2/(c^2 + 1))x_0x_2x_3^2x_4 + (2c^2/(c^2 + 1))x_1x_2x_3^2x_4 + (c/(c+2))x_0x_3^3x_4 + (4c/(c+2))x_2x_3^3x_4 + ((4c+1)/(c+2))x_0^3x_1^2x_4 + ((4c^2 + c)/(c^2 + 1))x_0x_1^2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_0x_2^2x_4^2 + ((c+4)/(c+2))x_1^3x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_2x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_2x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_2^2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_1x_2^2x_4^2 + ((3c^2 + 2c)/(c^2 + 1))x_0^2x_3x_4^2 + ((2c^2 + 3c)/(c^2 + 1))x_1^2x_3x_4^2 + (3c^2/(c^2 + 1))x_0x_2x_3x_4^2 + (2c^2/(c^2 + 1))x_1x_2x_3x_4^2 + ((4c^2 + c)/(c^2 + 1))x_0x_3^2x_4^2 + ((c^2 + 4c)/(c^2 + 1))x_1x_3^2x_4^2 + ((c+4)/(c+2))x_0^2x_4^3 + ((4c+1)/(c+2))x_1^2x_4^3 + (c/(c+2))x_0x_2x_4^3 + (4c/(c+2))x_1x_2x_4^3 + (c/(c+2))x_0x_3x_4^3 + (4c/(c+2))x_1x_3x_4^3 - x_0x_4^4 + x_1x_4^4
\end{aligned}$$

conjecture: x_0^{25}

5 S_6 , char 3, τ trivial

5.1 Hilbert polynomial

PARTIAL

$$1 + 6t + 21t^2 + 51t^3 + 96t^4 + 147t^5 + 192t^6 + 222t^7 + \dots$$

5.2 Generators

PARTIAL

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3$$

$$\begin{aligned} & ((2c+2)/c)x_0^3 \\ & + ((c+1)/c)x_4^3 \\ & + x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + x_1^2x_4 + x_2^2x_4 + x_3^2x_4 + x_0x_4^2 + x_0^2x_5 + x_4x_5^2 \\ & - x_0x_1^2 - x_0x_2^2 - x_0x_3^2 - x_0^2x_4 - x_1x_4^2 - x_2x_4^2 - x_3x_4^2 - x_4^2x_5 - x_0x_5^2 \\ & + (c/(c+2))x_0x_1x_2 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 \\ & + (2c/(c+2))x_1x_2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 + (2c/(c+2))x_1x_4x_5 + (2c/(c+2))x_2x_4x_5 + \\ & (2c/(c+2))x_3x_4x_5 \end{aligned}$$

$$\begin{aligned} & ((2c+2)/c)x_0^3 \\ & + ((c+1)/c)x_3^3 \\ & + x_0^2x_1 + x_0^2x_2 + x_1^2x_3 + x_2^2x_3 + x_0x_3^2 + x_0^2x_4 + x_3x_4^2 + x_0^2x_5 + x_3x_5^2 \\ & - x_0x_1^2 - x_0x_2^2 - x_0^2x_3 - x_1x_3^2 - x_2x_3^2 - x_3^2x_4 - x_0x_4^2 - x_3^2x_5 - x_0x_5^2 \\ & + (c/(c+2))x_0x_1x_2 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_4x_5 \\ & + (2c/(c+2))x_1x_2x_3 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 + (2c/(c+2))x_1x_3x_5 + (2c/(c+2))x_2x_3x_5 + \\ & (2c/(c+2))x_3x_4x_5 \end{aligned}$$

$$\begin{aligned} & ((2c+2)/c)x_0^3 \\ & + ((c+1)/c)x_2^3 \\ & + x_0^2x_1 + x_1^2x_2 + x_0x_2^2 + x_0^2x_3 + x_2x_3^2 + x_0^2x_4 + x_2x_4^2 + x_0^2x_5 + x_2x_5^2 \\ & - x_0x_1^2 - x_0^2x_2 - x_1x_2^2 - x_2^2x_3 - x_0x_3^2 - x_2^2x_4 - x_0x_4^2 - x_2^2x_5 - x_0x_5^2 \\ & + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_3x_4 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5 \\ & + (2c/(c+2))x_1x_2x_3 + (2c/(c+2))x_1x_2x_4 + (2c/(c+2))x_2x_3x_4 + (2c/(c+2))x_1x_2x_5 + (2c/(c+2))x_2x_3x_5 + \\ & (2c/(c+2))x_2x_4x_5 \end{aligned}$$

$$\begin{aligned} & ((2c+2)/c)x_0^3 \\ & + ((c+1)/c)x_1^3 \\ & - x_0^2x_1 - x_1^2x_2 - x_0x_2^2 - x_0^2x_3 - x_0x_3^2 - x_1^2x_4 - x_0x_4^2 - x_1^2x_5 - x_0x_5^2 \\ & + x_0x_1^2 + x_0^2x_2 + x_1x_2^2 + x_0^2x_3 + x_1x_3^2 + x_0^2x_4 + x_1x_4^2 + x_0^2x_5 + x_1x_5^2 \\ & + (c/(c+2))x_0x_2x_3 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_3x_4 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5 \\ & + (2c/(c+2))x_1x_2x_3 + (2c/(c+2))x_1x_2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_1x_2x_5 + (2c/(c+2))x_1x_3x_5 + \\ & (2c/(c+2))x_1x_4x_5 \end{aligned}$$

conjecture: x_0^9 .

5.3 $n - 1$ case

6 S_6 , char 2, τ trivial

6.1 Hilbert polynomial

$$t^8 + 6t^7 + 16t^6 + 26t^5 + 30t^4 + 26t^3 + 16t^2 + 6t + 1 = (t + 1)^6(t^2 + 1)$$

6.2 Generators

$$\begin{aligned} & x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \\ & ((c + 1)/c)x_0^2 + x_0x_1 + x_0x_2 + x_0x_3 + x_1x_4 + x_2x_4 + x_3x_4 + ((c + 1)/c)x_4^2 + x_0x_5 + x_4x_5 \\ & ((c + 1)/c)x_0^2 + x_0x_1 + x_0x_2 + x_1x_3 + x_2x_3 + ((c + 1)/c)x_3^2 + x_0x_4 + x_3x_4 + x_0x_5 + x_3x_5 \\ & ((c + 1)/c)x_0^2 + x_0x_1 + x_1x_2 + ((c + 1)/c)x_2^2 + x_0x_3 + x_2x_3 + x_0x_4 + x_2x_4 + x_0x_5 + x_2x_5 \\ & ((c + 1)/c)x_0^2 + ((c + 1)/c)x_1^2 + x_0x_2 + x_1x_2 + x_0x_3 + x_1x_3 + x_0x_4 + x_1x_4 + x_0x_5 + x_1x_5 \\ & x_0^4. \end{aligned}$$

7 S_9 , char 3, τ trivial

7.1 Hilbert polynomial

PARTIAL

$$\dots + 157t^3 + 45t^2 + 9t + 1$$

7.2 Generators

PARTIAL

$$\begin{aligned} & x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 + x_8^3 \\ & ((2c + 2)/c)x_0^3 \\ & + ((c + 1)/c)x_7^3 \\ & + x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + x_0^2x_4 + x_0^2x_5 + x_0^2x_6 + x_1^2x_7 + x_2^2x_7 + x_3^2x_7 + x_4^2x_7 + x_5^2x_7 + x_6^2x_7 + x_0x_7^2 + x_0^2x_8 + x_7x_8^2 \\ & - x_0x_1^2 - x_0x_2^2 - x_0x_3^2 - x_0x_4^2 - x_0x_5^2 - x_0x_6^2 - x_1x_7^2 - x_2x_7^2 - x_3x_7^2 - x_4x_7^2 - x_5x_7^2 - x_6x_7^2 - x_7x_8^2 - x_0x_8^2 \\ & + (c/(c + 2))x_0x_1x_2 + (c/(c + 2))x_0x_1x_3 + (c/(c + 2))x_0x_2x_3 + (c/(c + 2))x_0x_1x_4 + (c/(c + 2))x_0x_2x_4 + (c/(c + 2))x_0x_3x_4 + (c/(c + 2))x_0x_1x_5 + (c/(c + 2))x_0x_2x_5 + (c/(c + 2))x_0x_3x_5 + (c/(c + 2))x_0x_4x_5 + (c/(c + 2))x_0x_5^2 + (c/(c + 2))x_1x_5^2 + (c/(c + 2))x_2x_5^2 + (c/(c + 2))x_3x_5^2 + (c/(c + 2))x_4x_5^2 + (c/(c + 2))x_5x_8^2 + (c/(c + 2))x_7x_8^2 + (c/(c + 2))x_8^3 \end{aligned}$$

$$\begin{aligned}
& 2))x_0x_3x_4 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5 + (c/(c+2))x_0x_1x_6 + \\
& (c/(c+2))x_0x_2x_6 + (c/(c+2))x_0x_3x_6 + (c/(c+2))x_0x_4x_6 + (c/(c+2))x_0x_5x_6 + (c/(c+2))x_0x_1x_8 + (c/(c+2))x_0x_2x_8 + \\
& (c/(c+2))x_0x_3x_8 + (c/(c+2))x_0x_4x_8 + (c/(c+2))x_0x_5x_8 + (c/(c+2))x_0x_6x_8 \\
& + (2c/(c+2))x_1x_2x_7 + (2c/(c+2))x_1x_3x_7 + (2c/(c+2))x_2x_3x_7 + (2c/(c+2))x_1x_4x_7 + (2c/(c+2))x_2x_4x_7 + \\
& (2c/(c+2))x_3x_4x_7 + (2c/(c+2))x_1x_5x_7 + (2c/(c+2))x_2x_5x_7 + (2c/(c+2))x_3x_5x_7 + (2c/(c+2))x_4x_5x_7 + \\
& (2c/(c+2))x_1x_6x_7 + (2c/(c+2))x_2x_6x_7 + (2c/(c+2))x_3x_6x_7 + (2c/(c+2))x_4x_6x_7 + (2c/(c+2))x_5x_6x_7 + \\
& (2c/(c+2))x_1x_7x_8 + (2c/(c+2))x_2x_7x_8 + (2c/(c+2))x_3x_7x_8 + (2c/(c+2))x_4x_7x_8 + (2c/(c+2))x_5x_7x_8 + \\
& (2c/(c+2))x_6x_7x_8
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 + x_0^2x_2 + (c/(c+2))x_0x_1x_2 - x_0x_2^2 + x_0^2x_3 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 - \\
& x_0x_3^2 + x_0^2x_4 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_3x_4 - x_0x_4^2 + x_0^2x_5 + (c/(c+2))x_0x_1x_5 + \\
& (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 + (c/(c+2))x_0x_4x_5 - x_0x_5^2 - x_0^2x_6 + x_1^2x_6 + (2c/(c+2))x_1x_2x_6 + x_2^2x_6 + \\
& (2c/(c+2))x_1x_3x_6 + (2c/(c+2))x_2x_3x_6 + x_3^2x_6 + (2c/(c+2))x_1x_4x_6 + (2c/(c+2))x_2x_4x_6 + (2c/(c+2))x_3x_4x_6 + \\
& x_4^2x_6 + (2c/(c+2))x_1x_5x_6 + (2c/(c+2))x_2x_5x_6 + (2c/(c+2))x_3x_5x_6 + (2c/(c+2))x_4x_5x_6 + x_5^2x_6 + x_0x_6^2 - x_1x_6^2 - \\
& x_2x_6^2 - x_3x_6^2 - x_4x_6^2 - x_5x_6^2 + ((c+1)/c)x_0^3 + x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (c/(c+2))x_0x_2x_7 + (c/(c+2))x_0x_3x_7 + \\
& (c/(c+2))x_0x_4x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_1x_6x_7 + (2c/(c+2))x_2x_6x_7 + (2c/(c+2))x_3x_6x_7 + \\
& (2c/(c+2))x_4x_6x_7 + (2c/(c+2))x_5x_6x_7 - x_6^2x_7 - x_0x_7^2 + x_6x_7^2 + x_0^2x_8 + (c/(c+2))x_0x_1x_8 + (c/(c+2))x_0x_2x_8 + \\
& (c/(c+2))x_0x_3x_8 + (c/(c+2))x_0x_4x_8 + (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_1x_6x_8 + (2c/(c+2))x_2x_6x_8 + (2c/(c+2))x_3x_6x_8 + \\
& (2c/(c+2))x_4x_6x_8 + (2c/(c+2))x_5x_6x_8 - x_6^2x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_6x_7x_8 - x_0x_8^2 + x_6x_8^2
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 + x_0^2x_2 + (c/(c+2))x_0x_1x_2 - x_0x_2^2 + x_0^2x_3 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 - \\
& x_0x_3^2 + x_0^2x_4 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (c/(c+2))x_0x_3x_4 - x_0x_4^2 - x_0^2x_5 + x_1^2x_5 + (2c/(c+2))x_1x_2x_5 + \\
& x_2^2x_5 + (2c/(c+2))x_1x_3x_5 + (2c/(c+2))x_2x_3x_5 + x_3^2x_5 + (2c/(c+2))x_1x_4x_5 + (2c/(c+2))x_2x_4x_5 + \\
& (2c/(c+2))x_3x_4x_5 + x_4^2x_5 + x_0x_5^2 - x_1x_5^2 - x_2x_5^2 - x_3x_5^2 - x_4x_5^2 + ((c+1)/c)x_0^3 + x_0^2x_6 + (c/(c+2))x_0x_1x_6 + (c/(c+2))x_0x_2x_6 + \\
& (c/(c+2))x_0x_3x_6 + (c/(c+2))x_0x_4x_6 + (2c/(c+2))x_1x_5x_6 + (2c/(c+2))x_2x_5x_6 + (2c/(c+2))x_3x_5x_6 + \\
& (2c/(c+2))x_4x_5x_6 - x_5^2x_6 - x_0x_6^2 + x_5x_6^2 + x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (c/(c+2))x_0x_2x_7 + (c/(c+2))x_0x_3x_7 + \\
& (c/(c+2))x_0x_4x_7 + (2c/(c+2))x_1x_5x_7 + (2c/(c+2))x_2x_5x_7 + (2c/(c+2))x_3x_5x_7 + (2c/(c+2))x_4x_5x_7 - x_5^2x_7 + \\
& (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_5x_6x_7 - x_0x_7^2 + x_5x_7^2 + x_0^2x_8 + (c/(c+2))x_0x_1x_8 + (c/(c+2))x_0x_2x_8 + (c/(c+2))x_0x_3x_8 + \\
& (c/(c+2))x_0x_4x_8 + (2c/(c+2))x_1x_5x_8 + (2c/(c+2))x_2x_5x_8 + (2c/(c+2))x_3x_5x_8 + (2c/(c+2))x_4x_5x_8 - x_5^2x_8 + (c/(c+2))x_0x_6x_8 + \\
& (2c/(c+2))x_5x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_5x_7x_8 - x_0x_8^2 + x_5x_8^2
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 + x_0^2x_2 + (c/(c+2))x_0x_1x_2 - x_0x_2^2 + x_0^2x_3 + (c/(c+2))x_0x_1x_3 + (c/(c+2))x_0x_2x_3 - \\
& x_0x_3^2 - x_0^2x_4 + x_1^2x_4 + (2c/(c+2))x_1x_2x_4 + x_2^2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 + x_3^2x_4 + x_0x_4^2 - \\
& x_1x_4^2 - x_2x_4^2 - x_3x_4^2 + ((c+1)/c)x_0^3 + x_0^2x_5 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (c/(c+2))x_0x_3x_5 + (2c/(c+2))x_1x_4x_5 + \\
& (2c/(c+2))x_2x_4x_5 + (2c/(c+2))x_3x_4x_5 - x_2^2x_5 - x_0x_5^2 + x_4x_5^2 + x_0^2x_6 + (c/(c+2))x_0x_1x_6 + (c/(c+2))x_0x_2x_6 + (c/(c+2))x_0x_3x_6 + \\
& (2c/(c+2))x_1x_4x_6 + (2c/(c+2))x_2x_4x_6 + (2c/(c+2))x_3x_4x_6 - x_2^2x_6 + (c/(c+2))x_0x_5x_6 + (2c/(c+2))x_4x_5x_6 - x_0x_6^2 + x_4x_6^2 + \\
& x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (c/(c+2))x_0x_2x_7 + (c/(c+2))x_0x_3x_7 + (2c/(c+2))x_1x_4x_7 + (2c/(c+2))x_2x_4x_7 + (2c/(c+2))x_3x_4x_7 - \\
& x_4^2x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_4x_5x_7 + (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_4x_6x_7 - x_0x_7^2 + x_4x_7^2 + x_0^2x_8 + (c/(c+2))x_0x_1x_8 + \\
& (c/(c+2))x_0x_2x_8 + (c/(c+2))x_0x_3x_8 + (2c/(c+2))x_1x_4x_8 + (2c/(c+2))x_2x_4x_8 + (2c/(c+2))x_3x_4x_8 - x_4^2x_8 + (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_4x_5x_8 + \\
& (c/(c+2))x_0x_6x_8 + (2c/(c+2))x_4x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_4x_7x_8 - x_0x_8^2 + x_4x_8^2
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 + x_0^2x_2 + (c/(c+2))x_0x_1x_2 - x_0x_2^2 - x_0^2x_3 + x_1^2x_3 + (2c/(c+2))x_1x_2x_3 + x_2^2x_3 + \\
& x_0x_3^2 - x_1x_3^2 - x_2x_3^2 + ((c+1)/c)x_0^3 + x_0^2x_4 + (c/(c+2))x_0x_1x_4 + (c/(c+2))x_0x_2x_4 + (2c/(c+2))x_1x_3x_4 + (2c/(c+2))x_2x_3x_4 - \\
& x_3^2x_4 - x_0x_4^2 + x_3x_4^2 + x_0^2x_5 + (c/(c+2))x_0x_1x_5 + (c/(c+2))x_0x_2x_5 + (2c/(c+2))x_1x_3x_5 + (2c/(c+2))x_2x_3x_5 - x_3^2x_5 + \\
& (c/(c+2))x_0x_4x_5 + (2c/(c+2))x_3x_4x_5 - x_0x_5^2 + x_3x_5^2 + x_0^2x_6 + (c/(c+2))x_0x_1x_6 + (c/(c+2))x_0x_2x_6 + (2c/(c+2))x_1x_3x_6 + \\
& (2c/(c+2))x_2x_3x_6 - x_3^2x_6 + (c/(c+2))x_0x_4x_6 + (2c/(c+2))x_3x_4x_6 + (c/(c+2))x_0x_5x_6 + (2c/(c+2))x_3x_5x_6 - x_0x_6^2 + x_3x_6^2 + \\
& x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (c/(c+2))x_0x_2x_7 + (2c/(c+2))x_1x_3x_7 + (2c/(c+2))x_2x_3x_7 - x_3^2x_7 + (c/(c+2))x_0x_4x_7 + \\
& (2c/(c+2))x_3x_4x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_3x_5x_7 + (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_3x_6x_7 - x_0x_7^2 + x_3x_7^2 + x_0^2x_8 + \\
& (c/(c+2))x_0x_1x_8 + (c/(c+2))x_0x_2x_8 + (2c/(c+2))x_1x_3x_8 + (2c/(c+2))x_2x_3x_8 - x_3^2x_8 + (c/(c+2))x_0x_4x_8 + (2c/(c+2))x_3x_4x_8 + \\
& (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_3x_5x_8 + (c/(c+2))x_0x_6x_8 + (2c/(c+2))x_3x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_3x_7x_8 - x_0x_8^2 + x_3x_8^2
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 + x_0^2x_1 - x_0x_1^2 - x_0^2x_2 + x_1^2x_2 + x_0x_2^2 - x_1x_2^2 + ((c+1)/c)x_2^3 + x_0^2x_3 + (c/(c+2))x_0x_1x_3 + (2c/(c+2))x_1x_2x_3 - x_2^2x_3 - x_0x_3^2 + x_2x_3^2 + x_0^2x_4 + (c/(c+2))x_0x_1x_4 + (2c/(c+2))x_1x_2x_4 - x_2^2x_4 + (c/(c+2))x_0x_3x_4 + \\
& (2c/(c+2))x_2x_3x_4 - x_0x_4^2 + x_2x_4^2 + x_0^2x_5 + (c/(c+2))x_0x_1x_5 + (2c/(c+2))x_1x_2x_5 - x_2^2x_5 + (c/(c+2))x_0x_3x_5 + \\
& (2c/(c+2))x_2x_3x_5 + (c/(c+2))x_0x_4x_5 + (2c/(c+2))x_2x_4x_5 - x_0x_5^2 + x_2x_5^2 + x_0^2x_6 + (c/(c+2))x_0x_1x_6 + (2c/(c+2))x_1x_2x_6 - x_2^2x_6 + (c/(c+2))x_0x_3x_6 + (2c/(c+2))x_2x_3x_6 + (c/(c+2))x_0x_4x_6 + (2c/(c+2))x_2x_4x_6 + (c/(c+2))x_0x_5x_6 + (2c/(c+2))x_2x_5x_6 - x_0x_6^2 + x_2x_6^2 + x_0^2x_7 + (c/(c+2))x_0x_1x_7 + (2c/(c+2))x_1x_2x_7 - x_2^2x_7 + (c/(c+2))x_0x_3x_7 + (2c/(c+2))x_2x_3x_7 + (c/(c+2))x_0x_4x_7 + (2c/(c+2))x_2x_4x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_2x_5x_7 + \\
& (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_2x_6x_7 - x_0x_7^2 + x_2x_7^2 + x_0^2x_8 + (c/(c+2))x_0x_1x_8 + (2c/(c+2))x_1x_2x_8 - x_2^2x_8 + (c/(c+2))x_0x_3x_8 + (2c/(c+2))x_2x_3x_8 + (c/(c+2))x_0x_4x_8 + (2c/(c+2))x_2x_4x_8 + (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_2x_5x_8 + (c/(c+2))x_0x_6x_8 + (2c/(c+2))x_2x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_2x_7x_8 - x_0x_8^2 + x_2x_8^2
\end{aligned}$$

$$\begin{aligned}
& ((2c+2)/c)x_0^3 - x_0^2x_1 + x_0x_1^2 + ((c+1)/c)x_1^3 + x_0^2x_2 - x_1^2x_2 - x_0x_2^2 + x_1x_2^2 + x_0^2x_3 - x_1^2x_3 + (c/(c+2))x_0x_2x_3 + \\
& (2c/(c+2))x_1x_2x_3 - x_0x_3^2 + x_1x_3^2 + x_0^2x_4 - x_1^2x_4 + (c/(c+2))x_0x_2x_4 + (2c/(c+2))x_1x_2x_4 + (c/(c+2))x_0x_3x_4 + \\
& (2c/(c+2))x_1x_3x_4 - x_0x_4^2 + x_1x_4^2 + x_0^2x_5 - x_1^2x_5 + (c/(c+2))x_0x_2x_5 + (2c/(c+2))x_1x_2x_5 + (c/(c+2))x_0x_3x_5 + \\
& (2c/(c+2))x_1x_3x_5 + (c/(c+2))x_0x_4x_5 + (2c/(c+2))x_1x_4x_5 - x_0x_5^2 + x_1x_5^2 + x_0^2x_6 - x_1^2x_6 + (c/(c+2))x_0x_2x_6 + \\
& (2c/(c+2))x_1x_2x_6 + (c/(c+2))x_0x_3x_6 + (2c/(c+2))x_1x_3x_6 + (c/(c+2))x_0x_4x_6 + (2c/(c+2))x_1x_4x_6 + (c/(c+2))x_0x_5x_6 + (2c/(c+2))x_1x_5x_6 - x_0x_6^2 + x_1x_6^2 + x_0^2x_7 - x_1^2x_7 + (c/(c+2))x_0x_2x_7 + (2c/(c+2))x_1x_2x_7 + (c/(c+2))x_0x_3x_7 + (2c/(c+2))x_1x_3x_7 + (c/(c+2))x_0x_4x_7 + (2c/(c+2))x_1x_4x_7 + (c/(c+2))x_0x_5x_7 + (2c/(c+2))x_1x_5x_7 + \\
& (c/(c+2))x_0x_6x_7 + (2c/(c+2))x_1x_6x_7 - x_0x_7^2 + x_1x_7^2 + x_0^2x_8 - x_1^2x_8 + (c/(c+2))x_0x_2x_8 + (2c/(c+2))x_1x_2x_8 + (c/(c+2))x_0x_3x_8 + (2c/(c+2))x_1x_3x_8 + (c/(c+2))x_0x_4x_8 + (2c/(c+2))x_1x_4x_8 + (c/(c+2))x_0x_5x_8 + (2c/(c+2))x_1x_5x_8 + (c/(c+2))x_0x_6x_8 + (2c/(c+2))x_1x_6x_8 + (c/(c+2))x_0x_7x_8 + (2c/(c+2))x_1x_7x_8 - x_0x_8^2 + x_1x_8^2
\end{aligned}$$

conjecture: x_0^9

8 S_4 , char 3, τ trivial

8.1 Polynomial

$$(t+1)^2(t^4+t^2+1)^2(t^2+t+1)^2(t^6+t^3+1)$$

8.2 Generators

$$x_0^3 + x_1^3 + x_2^3 + x_3^3$$

$$\begin{aligned}
& ((2c+2)/c)x_0^6 - x_0^5x_1 + x_0^4x_1^2 - x_0^2x_1^4 + x_0x_1^5 + ((c+1)/c)x_1^6 + x_0^5x_2 + x_0^3x_1^2x_2 - x_0^2x_1^3x_2 - x_1^5x_2 - x_0^4x_2^2 - x_0^3x_1x_2^2 + \\
& x_0x_1^3x_2^2 + x_1^4x_2^2 + ((c+1)/c)x_0^3x_2^3 + x_0^2x_1x_2^3 - x_0x_1^2x_2^3 + ((2c+2)/c)x_1^3x_2^3 + x_0^2x_2^4 - x_1^2x_2^4 - x_0x_2^5 + x_1x_2^5 + x_0^3x_1^2x_3 - \\
& x_0^2x_1^3x_3 - x_0^3x_2^2x_3 + x_1^3x_2^2x_3 + x_0^2x_2^3x_3 - x_1^2x_2^3x_3 - x_0^3x_1x_2^3 + x_0x_1^3x_2^3 + x_0^3x_2x_2^3 - x_1^3x_2x_2^3 - x_0x_2^3x_2^3 + x_1x_2^3x_2^3
\end{aligned}$$

$$x_0^6 + x_0^3x_1^3 + x_1^6 + x_0^3x_2^3 + x_1^3x_2^3 + x_2^6$$

$$x_0^9 + x_0^6x_1^3 + x_0^3x_1^6 + x_1^9$$

9 S_4 , char 3, τ trivial, $n-1$ representation

9.1 Polynomial

$$(t+1)^2(t^4+t^2+1)^2(t^2+t+1)(t^6+t^3+1)$$

$$x_0^3 x_1^3 + x_0^3 x_2^3 + x_1^3 x_2^3$$

10 S_7 , char 3, τ trivial

PARTIAL

PARTIAL

11 S_7 , char 3, τ trivial, $n - 1$ representation

PARTIAL

8

$$\begin{aligned}
& ((c^2+2)/c^2)x_0^3x_2^3 + ((2c+1)/c)x_0^2x_1x_2^3 + ((c+2)/c)x_0x_1^2x_2^3 + ((c+2)/c)x_0^3x_1^2x_3 + ((c+2)/c)x_0^2x_1^3x_3 + (2c/(c+2))x_0^2x_1^2x_2x_3 + x_0x_1^3x_2x_3 + ((2c+1)/c)x_0^3x_2^2x_3 + (c/(c+2))x_0^2x_1x_2^2x_3 + ((2c+1)/c)x_0^2x_2^3x_3 - x_0x_1x_2^3x_3 + ((2c+1)/c)x_0^3x_1x_2^3 + x_0^2x_1^2x_2^3 + ((2c+1)/c)x_0x_1^3x_2^3 + ((c+2)/c)x_0x_1^2x_2^3 + (2c/(c+2))x_0x_1^2x_2x_3 - x_0^2x_2^2x_3^2 + (c/(c+2))x_0x_1x_2^2x_3^2 + ((c+2)/c)x_0x_2^3x_3^2 + ((c+2)/c)x_0^3x_1^2x_4 + ((c+2)/c)x_0^2x_1^3x_4 + (2c/(c+2))x_0^2x_1^2x_2x_4 + x_0x_1^3x_2x_4 + ((2c+1)/c)x_0^3x_2^2x_4 + (c/(c+2))x_0^2x_1x_2^2x_4 + ((2c+1)/c)x_0^2x_2^3x_4 - x_0x_1x_2^3x_4 + x_0^3x_1x_3x_4 + (1/(c+2))x_0^2x_1^2x_3x_4 + x_0x_1^3x_3x_4 - x_0^3x_2x_3x_4 + (2c/(c+2))x_0x_1^2x_2x_3x_4 + (2/(c+2))x_0^2x_2^2x_3x_4 + (c/(c+2))x_0x_1x_2^2x_3x_4 - x_0x_2^3x_3x_4 + ((c+1)/(c+2))x_0^2x_1x_2^3x_4 + ((c+1)/(c+2))x_0x_1^2x_2^3x_4 + ((2c+2)/(c+2))x_0^2x_2x_2^3x_4 + ((2c+2)/(c+2))x_0x_2^2x_2^3x_4 + ((2c+1)/c)x_0^3x_1x_2^4 + x_0^2x_1^2x_2^4 + ((2c+1)/c)x_0x_1^3x_2^4 + ((c+2)/c)x_0^3x_2x_2^4 + (2c/(c+2))x_0x_1^2x_2x_2^4 - x_0^2x_2^2x_2^4 + (c/(c+2))x_0x_1x_2^2x_2^4 + ((c+2)/c)x_0x_2^3x_2^4 + ((c+1)/(c+2))x_0^2x_1x_2^3x_2^4 + ((c+1)/(c+2))x_0x_1^2x_2^3x_2^4 + ((2c+2)/(c+2))x_0^2x_2x_2^3x_2^4 - x_0x_1x_2^3x_2^4 + x_0x_2x_2^3x_2^4 + ((c+2)/c)x_0^3x_1^2x_5 + ((c+2)/c)x_0^2x_1^3x_5 + (2c/(c+2))x_0^2x_1^2x_2x_5 + x_0x_1^3x_2x_5 + ((2c+1)/c)x_0^3x_2^2x_5 + (c/(c+2))x_0^2x_1x_2^2x_5 + ((2c+1)/c)x_0^2x_3^2x_5 - x_0x_1x_2^3x_5 + x_0^3x_1x_3x_5 + (1/(c+2))x_0^2x_1^2x_3x_5 + x_0x_1^3x_3x_5 - x_0^3x_2x_3x_5 + (2c/(c+2))x_0x_1^2x_2x_3x_5 + (2/(c+2))x_0^2x_2^2x_3x_5 + (c/(c+2))x_0x_1x_2^2x_3x_5 - x_0x_2^3x_3x_5 + ((c+1)/(c+2))x_0^2x_1x_2^3x_5 + ((c+1)/(c+2))x_0x_1^2x_2^3x_5 + ((2c+2)/(c+2))x_0^2x_2x_2^3x_5 + ((2c+2)/(c+2))x_0x_2^2x_2^3x_5 + x_0^3x_1x_4x_5 + (1/(c+2))x_0^2x_1^2x_4x_5 + x_0x_1^3x_4x_5 - x_0^3x_2x_4x_5 + (2c/(c+2))x_0x_1^2x_2x_4x_5 + (2/(c+2))x_0^2x_2^2x_4x_5 + (c/(c+2))x_0x_1x_2^2x_4x_5 - x_0x_2^3x_4x_5 + (c/(c+2))x_0x_1x_2^3x_4x_5 + (2c/(c+2))x_0x_2x_2^3x_4x_5 + ((c+1)/(c+2))x_0^2x_1x_2^4x_5 + ((c+1)/(c+2))x_0x_1^2x_2^4x_5 + ((2c+2)/(c+2))x_0^2x_2x_2^4x_5 + ((2c+2)/(c+2))x_0x_2^2x_2^4x_5 + (c/(c+2))x_0x_1x_2x_2^4x_5 + (2c/(c+2))x_0x_2x_2x_2^4x_5 + ((2c+1)/c)x_0^3x_1x_2^5 + x_0^2x_1^2x_2^5 + (2c/(c+2))x_0x_1^2x_2x_2^5 - x_0^2x_2^2x_2^5 + (c/(c+2))x_0x_1x_2^2x_2^5 + ((c+2)/c)x_0x_2^3x_2^5 + ((c+1)/(c+2))x_0^2x_1x_2^3x_2^5 + ((c+1)/(c+2))x_0x_1^2x_2^3x_2^5 + ((2c+2)/(c+2))x_0^2x_2x_2^3x_2^5 - x_0x_1x_2^3x_2^5 + x_0x_2x_2^3x_2^5 + ((c+1)/(c+2))x_0^2x_1x_4x_2^5 + ((c+1)/(c+2))x_0x_1^2x_4x_2^5 + ((2c+2)/(c+2))x_0^2x_2x_4x_2^5 + ((2c+2)/(c+2))x_0x_2^2x_4x_2^5 + (c/(c+2))x_0x_1x_3x_4x_2^5 + (2c/(c+2))x_0x_2x_3x_4x_2^5 - x_0x_1x_4x_2^5 + x_0x_2x_4x_2^5
\end{aligned}$$

12 Conjecture for $p = 3, 3 \mid n$

Variables are x_0, \dots, x_{n-1} .

Generators are:

x_0^9 in degree 9, and $\sum x_i^3$ in degree 3. There are $n - 2$ remaining generators in degree 3, each with the following form:

$$\frac{c+1}{c}(x_1^3 - x_0^3) + (x_1 - x_0)(x_0x_1) + (x_1 - x_0) \left(\sum_{i \geq 2} x_i^2 - x_i(x_1 + x_0) \right) + \frac{2c}{c+2}(x_1 - x_0) \left(\sum_{i,j \geq 2; i < j} x_ix_j \right).$$

13 Conjecture for general $p \mid n$

Variables are x_0, \dots, x_{n-1} .

Generators are:

$x_0^{p^2}$ in degree p^2 , and $\sum x_i^p$ in degree p . There are $n - 2$ remaining generators in degree p . It is clear that each such generator contains a term of the form $\frac{c+1}{c}(x_k^p - x_0^p)$. If we assume we are in the generator with $k = 1$, then the generator also contains a term $\left(\sum_{i \geq 2} x_0^{p-1}x_i + x_1x_i^{p-1} - x_0x_i^{p-1} - x_1^{p-1}x_i \right)$

14 Conjectures, $n = 1 \pmod 3, p = 3$

The degree 6 generators include ones of the form:

$$\begin{aligned}
& \frac{2c^2+1}{c^2} (x_0^3 x_1^3 - x_1^3 x_2^3) + \\
& \frac{c+2}{c} (x_0 x_1^3 x_2^2 + x_0^3 x_1^2 x_2 + x_0^2 x_1 x_2^3 - x_0^2 x_1^3 x_2 - x_0^3 x_1 x_2^2 - x_0 x_1^2 x_2^3) + \\
& \frac{c+2}{c} \left(\sum_{i>2} x_i (x_0^3 x_1^2 + x_0^2 x_1^3 - x_2^2 x_1^3 - x_2^3 x_1^2) + \sum_{i>2} x_i^2 (x_1^3 x_2 + x_1 x_2^3 - x_0^3 x_1 - x_0 x_1^3) \right) + \\
& \sum_{i>2} x_i (x_0^3 x_1 x_2 - x_0 x_1 x_2^3) + \sum_{2<i<j} x_i x_j (x_0^3 x_1 + x_0 x_1^3 - x_1^3 x_2 - x_1 x_2^3) + \sum_{i>2} x_i^2 (x_0^2 x_1^2 - x_1^2 x_2^2) + \sum_{2<i<j} x_i^2 x_j^2 (x_1 x_2 - x_0 x_1) + \\
& \frac{2c}{c+2} \left(\sum_{i>2} x_i (x_0^2 x_1^2 x_2 - x_0 x_1^2 x_2^2) + \sum_{i>2} x_i^2 (x_0^2 x_1 x_2 - x_0 x_1 x_2^2) + \sum_{2<i<j} x_i x_j (x_0^2 x_1 x_2 - x_0 x_1 x_2^2) \right) + \\
& (2c/(c+2)) \left(\sum_{2<i<j<k} x_1 (x_2 - x_0) (x_i x_j x_k^2 + x_i x_j^2 x_k + x_i^2 x_j x_k) \right) + \\
& \frac{1}{c+2} \left(\sum_{2<i<j} x_i x_j (x_0^2 x_1^2 - x_1^2 x_2^2) \right) + \frac{c+1}{c+2} \left(\sum_{2<i<j} (x_i x_j^2 + x_i^2 x_j) (x_0^2 x_1 + x_0 x_1^2 - x_1^2 x_2 - x_1 x_2^2) \right)
\end{aligned}$$

where x_0, x_1, x_2 can be swapped with any other three of the variables. TODO: figure out which set of these type of generators are linearly independent, simplify in some other fashion.

Another degree 6 generator is $\sum_{i<j} x_i^3 x_j^3$.

There is another type of degree 6 generator that I need to get a general form for.

Then there is a generator in degree 9 that I do not know the form of.