Polynomial representations of rational Cherednik algebras of type A in positive characteristic $p \mid n$

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Abstract

We study the polynomial representation of the rational Cherednik algebra of type A_{n-1} with generic parameter in characteristic p for $p \mid n$. We give explicit formulas for generators for the maximal ideal, show that they cut out a complete intersection, and thus compute the Hilbert series of the irreducible quotient. Our methods are motivated by generalizing from characteristic 0 to characteristic p.

1 Introduction

In this paper we study lowest-weight representations of rational Cherednik algebras associated to the symmetric group Σ_n in characteristic p dividing n. This problem is an interesting test case for moving results from characteristic 0 to positive characteristic. Cherednik algebras, originally double affine Hecke algebras, were introduced by Cherednik in 1993; for an introduction to these algebras we refer to [4]. The representation theory of these algebras has been studied more extensively in characteristic 0 than in positive characteristic, including the case of Cherednik algebras of type A_{n-1} (associated to the symmetric group Σ_n). In general new techniques are required to study positive characteristic as opposed to characteristic 0. In this paper we directly connect the characteristic 0 case for the polynomial representation to the positive characteristic case; the irreducible representation of the Cherednik algebra in this case is the quotient of the polynomial representation by an ideal generated by complex residues. In the positive characteristic case, we instead let the generators be the coefficients of formal power series. In particular our argument that this quotient is a complete intersection is directly analogous to the corresponding argument in [2], though with additional 'error terms' that emerge because we must truncate the power series so that we can define it in positive characteristic. We are able to truncate the power series because we realize the $(n-1)$-dimensional representation $\mathfrak{h}$ of Σ_n as a quotient by the sum of the basis elements rather than as the reflection representation.

While we only consider the case of the symmetric group Σ_n in this paper, this technique could also be used for the general case of complex reflection groups.

1.1 Definitions

Given the symmetric group Σ_n, we let $\mathcal{S}$ be the set of reflections in Σ_n. Let $\mathfrak{h}$ be the irreducible $(n-1)$-dimensional representation of Σ_n, realized as the submodule of k^n where the sum of the coordinates is 0. For each $s \in \mathcal{S}$ we assign a vector $\alpha_s \in \mathfrak{h}^*$ spanning the image of $1-s$, and choose $\alpha_s^\vee \in \mathfrak{h}$ so that $(1-s)x = \langle \alpha_s^\vee, x\rangle \alpha_s$ for all $x \in \mathfrak{h}^*$, where $\langle \cdot, \cdot \rangle$ indicates the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

Let $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ be the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$. We choose $\hbar, c \in k$. Then we define the *rational Cherednik algebra of type A* $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ as the quotient of $k[\Sigma_n] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \quad [y, y' = 0], \quad [y, x] = \hbar\langle y, x\rangle - \sum_{s\in\mathcal{S}} c\langle y, \alpha_s\rangle\langle\alpha_s^\vee, x\rangle s$$

for all $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$. We can give $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ a $\mathbb{Z}$-grading by setting $\deg x = 1$ for $x \in \mathfrak{h}^*$, $\deg y = -1$ for $y \in \mathfrak{h}$, and $\deg g = 0$ for $g \in k[G]$. We get the PBW-type decomposition $\mathcal{H}_{\hbar,c}(\mathfrak{h}) = \mathrm{Sym}(\mathfrak{h}) \otimes_k k[\Sigma_n] \otimes_k \mathrm{Sym}(\mathfrak{h}^*)$ ([4], section 3.2).

In general, for any $\alpha \neq 0$, $\mathcal{H}_{\hbar,c}(\mathfrak{h}) \simeq \mathcal{H}_{\alpha\hbar,\alpha c}(\mathfrak{h})$, so we restrict to $\hbar = 0$ or $\hbar = 1$.

1.2 Representations of rational Cherednik algebras

We are concerned with the *polynomial representation* $\mathrm{Sym}(\mathfrak{h}^*)$ of $\mathcal{H}_{\hbar,c}$, which is a polynomial ring. We refer to the polynomial representation as A; we can give this a $\mathbb{Z}$-grading in an obvious way.

As described in Section 2.5 of [1], $A = \mathrm{Sym}(\mathfrak{h}^*)$ has a unique maximal graded proper submodule J which can be realized as the kernel of the contravariant form $\beta_c : \mathrm{Sym}(\mathfrak{h}^*) \times \mathrm{Sym}(\mathfrak{h}) \to k$; β_c can be characterized by the property that for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}, f \in \mathrm{Sym}(\mathfrak{h}^*), g \in \mathrm{Sym}(\mathfrak{h})$:

$$\beta_c(xf, g) = \beta_c(f, xg), \quad \beta_c(f, yg) = \beta_c(yf, g), \quad \beta_c(1, 1) = 1.$$

The quotient A/J is a finite-dimensional irreducible $\mathbb{Z}$-graded representation of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$. The *Hilbert series* $h_{A/J}(t)$ of A/J is $\sum_{j=0}^{\infty}(\dim_k L_j)t^k$ where L_j is the j-graded factor of A/J.

1.3 Main results

For a formal power series or Laurent series $r(z)$, we write the coefficient of z^ℓ in r as $[z^\ell]r(z)$.

We call c *generic* if we do not specify its value. We will be concerned with the case $\hbar = 1$, c generic, and the characteristic p of k dividing n. We realize $\mathfrak{h}$ as the subspace of k^n with the sum of the coordinates 0, which is an irreducible representation of Σ_n of dimension $n-1$. We state our result for this case below:

Theorem 3.6. *The irreducible representation A/J of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ is a complete intersection with Hilbert series $\left(\frac{1-t^p}{1-t}\right)^{n-1}$. The maximal ideal J is generated by $[z^p]F_a(z)$ for the formal power series*

$$F_a(z) = \frac{1}{1 - x_a z} \prod_{j=0}^{n-1} \left(\sum_{k=0}^{p-1} \binom{c}{k} (-1 + \prod_{j=0}^{n-1}(1 - x_j z))^k \right)$$

for $a = 0, \ldots, n-2$.

1.4 Connection to characteristic 0 results

In this paper we consider the case where the characteristic of k divides n. This is related to the case in characteristic 0 where c takes the specific value p/n, as described in [2].

Proposition 1.1 ([2], Proposition 3.1)**.** *Let the characteristic of k be 0 and set $c = p/n$. The ideal $J \subseteq A$ where A is the polynomial representation of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ is generated by $\mathrm{Res}_\infty \left[\frac{dz}{z - x_a} \prod_{i=0}^{n-1}(z - x_i)^c\right]$ for $a = 0, \ldots, n-2$.*

To get a similar result in positive characteristic, we must consider formal power series in z with coefficients from A. The formal power series would be $\frac{1}{z^{p+1}} \frac{1}{1-x_a z} \prod_{j=0}^{n-1}(1-x_j z)^c$, with the corresponding generator as the coefficient of $1/z$. We simplify and truncate the formal power series so we can define it in positive characteristic.

1.5 Relation to previous work

2 The rational Cherednik algebra

2.1 Dunkl operators

To understand the action of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ on A, we can use the PBW decompositions. The action of $\mathrm{Sym}(\mathfrak{h}^*)$ on A is by left multiplication; $k[\Sigma_n]$ acts by the diagonal action, and $\mathrm{Sym}(\mathfrak{h})$ acts via *Dunkl operators*. For $y \in \mathfrak{h}$, the Dunkl operator D_y acts on A by:

$$D_y(f) = \hbar \partial_y f - \sum_{s \in \mathcal{S}} c \frac{(y, \alpha_s)}{\alpha_s}(1-s).f.$$

We choose bases for $\mathfrak{h}, \mathfrak{h}^*$ in the following way: let V be the vector space spanned by $y_0, \ldots, y_{n-1}$ and let $\mathfrak{h}$ be the subspace spanned by $y_i - y_j$ for $i \neq j$; Σ_n acts by permuting indices. Then if $x_0, \ldots, x_{n-1}$ is the dual basis for V^*, we see that $\mathfrak{h}^*$ is the span of $x_0, \ldots, x_{n-1}$ under the relation $x_0 + \cdots + x_{n-1} = 0$; alternatively we can consider $\mathfrak{h}^*$ as the span of $x_0, \ldots, x_{n-2}$ with x_{n-1} defined as $-x_0 - \cdots - x_{n-2}$. If D_i is the Dunkl operator corresponding to y_i, the Dunkl operators for the elements of $\mathfrak{h}$ are spanned by $D_i - D_j$ with $i \neq j$.

For a transposition $s_{ij} \in \Sigma_n$ with $i < j$, we let the corresponding vector $\alpha_{s_{ij}} \in \mathfrak{h}^*$ be $x_i - x_j$.

3 Proof of the main result

We let $g(z) = \prod_{j=0}^{n-1}(1-x_j z)$. Let $F_a(z)$ for $a = 0, \ldots, n-1$ be the formal power series in z defined by $F_a(z) = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$ where $\binom{c}{k} = \frac{c(c-1)\ldots(c-k+1)}{k!}$.

Because of the definition of the contravariant form β_c, showing that an element f of A is in the kernel of β_c is equivalent to showing that the Dunkl operators corresponding to the basis elements of $\mathfrak{h}$ annihilate f.

Proposition 3.1. *Let $f_a = [z^p]F_a(z)$. Then f_a for $a = 0, \ldots, n-1$ are annihilated by the Dunkl operators.*

Proof. Since $x_{n-1} = -x_0 - \cdots - x_{n-2}$, we see that

$$\frac{\partial g}{\partial x_i} = -\frac{zg}{1-x_i z} + \frac{zg}{1-x_{n-1}z}$$

for all $0 \leq i < n-1$. Let

$$F(z) = \left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right).$$

Note that $F(z)$ is symmetric and $F_a(z) = \frac{F(z)}{1-x_a z}$ for all a. Then we see that for all $0 \leq i < n-1$ we have

$$
\begin{aligned}
\frac{\partial F}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\\
&= \left(\sum_{k=1}^{p-1}k\binom{c}{k}(g-1)^{k-1}\right)\frac{\partial g}{\partial x_i}\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=1}^{p-1}k\binom{c}{k}(g-1)^{k-1}\right)g\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=0}^{p-2}(k+1)\binom{c}{k+1}(g-1)^{k}\right)g\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)(g-1+1)\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)+\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k+1}\right)\right)\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(\left(\sum_{k=0}^{p-2}c\binom{c-1}{k}(g-1)^{k}\right)+\left(\sum_{k=1}^{p-1}c\binom{c-1}{k-1}(g-1)^{k}\right)\right)\\
&= \left(-\frac{z}{1-x_iz}+\frac{z}{1-x_{n-1}z}\right)\left(-c\binom{c-1}{p-1}(g-1)^{p-1}+\sum_{k=0}^{p-1}c\binom{c}{k}(g-1)^{k}\right)\\
&= \left(-\frac{zc}{1-x_iz}+\frac{zc}{1-x_{n-1}z}\right)\left(-\binom{c-1}{p-1}(g-1)^{p-1}+F(z)\right).
\end{aligned}
$$

We define

$$G_i(z)=\left(-\frac{zc}{1-x_iz}+\frac{zc}{1-x_{n-1}z}\right)\left(-\binom{c-1}{p-1}(g-1)^{p-1}\right).$$

Then we see that

$$\frac{\partial F}{\partial x_i}=G_i(z)+\left(-\frac{zc}{1-x_iz}+\frac{zc}{1-x_{n-1}z}\right)F(z).$$

We can now calculate $\frac{\partial F_0}{\partial x_1}, \frac{\partial F_0}{\partial x_0}$.

$$
\begin{aligned}
\frac{\partial F_0}{\partial x_1} &= \frac{\partial}{\partial x_1}\left(\frac{1}{1-x_0z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\right)\\
&= \frac{1}{1-x_0z}\frac{\partial F}{\partial x_1}\\
&= \frac{1}{1-x_0z}\left(-\frac{zc}{1-x_1z}+\frac{zc}{1-x_{n-1}z}\right)F(z)+\frac{G_1(z)}{1-x_0z}\\
&= \left(-\frac{zc}{1-x_1z}+\frac{zc}{1-x_{n-1}z}\right)F_0(z)+\frac{G_1(z)}{1-x_0z}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F_0}{\partial x_0} &= \frac{\partial}{\partial x_1}\left(\frac{1}{1-x_0 z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)\right) \\
&= \frac{z}{(1-x_0 z)^2}F(z) + \frac{1}{1-x_0 z}\frac{\partial F}{\partial x_0} \\
&= \frac{z}{1-x_0 z}F_0(z) + \frac{1}{1-x_0 z}\left(-\frac{zc}{1-x_0 z} + \frac{zc}{1-x_{n-1} z}\right)F(z) + \frac{G_0(z)}{1-x_0 z} \\
&= \frac{z}{1-x_0 z}F_0(z) + \left(-\frac{zc}{1-x_0 z} + \frac{zc}{1-x_{n-1} z}\right)F_0(z) + \frac{G_0(z)}{1-x_0 z} \\
&= \left(\frac{z(1-c)}{1-x_0 z} + \frac{zc}{1-x_{n-1} z}\right)F_0(z) + \frac{G_0(z)}{1-x_0 z}.
\end{aligned}
$$

We note that $F_0(z)$ is invariant under s_{ij} where $0 < i, j$. Therefore for transpositions we need only consider transpositions of the form s_{0i} for $0 < i \le n-1$. We see that

$$
\begin{aligned}
\frac{1-s_{0i}}{x_0 - x_i}(F_0) &= \frac{1}{x_0 - x_i}\left(\frac{F(z)}{1-x_0 z} - \frac{F(z)}{1-x_i z}\right) \\
&= \frac{1}{x_0 - x_i}\left(\frac{1}{1-x_0 z} - \frac{1}{1-x_i z}\right)F(z) \\
&= \frac{1}{x_0 - x_i}\left(\frac{(1-x_i z) - (1-x_0 z)}{(1-x_0 z)(1-x_i z)}\right)F(z) \\
&= \frac{x_0 z - x_i z}{(1-x_0 z)(1-x_i z)(x_0 - x_i)}F(z) \\
&= \frac{z}{(1-x_i z)(1-x_0 z)}F(z) \\
&= \frac{z}{1-x_i z}F_0(z).
\end{aligned}
$$

We recall that we need only consider the action of $D_1 - D_0$ on $F_0(z)$. We consider $D_0 F_0, D_1 F_0$ separately first. We see that

$$
D_0 = \left(\frac{\partial}{\partial x_0} - c\sum_{j>0}\frac{1-s_{0j}}{x_0 - x_j}\right), D_1 = \left(\frac{\partial}{\partial x_1} - c\frac{1-s_{01}}{x_1 - x_0}\right)
$$

since F_0 is invariant under s_{ij} where $0 < i, j$.

$$
\begin{aligned}
D_0 F_0 &= \left(\frac{\partial}{\partial x_0} - c\sum_{j>0}\frac{1-s_{0j}}{x_0 - x_j}\right)(F_0) \\
&= \frac{\partial F_0}{\partial x_0} - c\sum_{j>0}\frac{1-s_{0j}}{x_0 - x_j}(F_0) \\
&= \frac{G_0(z)}{1-x_0 z} + \left(\frac{z(1-c)}{1-x_0 z} + \frac{zc}{1-x_{n-1} z}\right)F_0(z) - \sum_{j>0}\frac{zc}{1-x_j z}F_0(z).
\end{aligned}
$$

$$
\begin{aligned}
D_1 F_0 &= \left(\frac{\partial}{\partial x_1} - c\frac{1-s_{01}}{x_1-x_0}\right)(F_0) \\
&= \frac{\partial F_0}{\partial x_1} + c\frac{1-s_{01}}{x_0-x_1}(F_0) \\
&= \frac{G_1(z)}{1-x_0 z} + \left(-\frac{zc}{1-x_1 z} + \frac{zc}{1-x_{n-1}z}\right) F_0(z) + \frac{zc}{1-x_1 z} F_0(z) \\
&= \frac{G_1(z)}{1-x_0 z} + \frac{zc}{1-x_{n-1}z} F_0(z).
\end{aligned}
$$

It is then easy to see that

$$
(D_1 - D_0)(F_0) = \frac{G_1 - G_0}{1-x_0 z} + \frac{z(c-1)}{1-x_0 z} F_0 + \sum_{j>0} \frac{zc}{1-x_j z} F_0.
$$

In order to show that the pth coefficient in this formal power series is 0, we must consider $\frac{\partial F_0}{\partial z}$.

We see easily that

$$
\frac{\partial g}{\partial z} = g(z) \sum_j \frac{-x_j}{1-x_j z}.
$$

We now consider $\frac{\partial F}{\partial z}$.

$$
\begin{aligned}
\frac{\partial F}{\partial z} &= \frac{\partial}{\partial z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right) \\
&= \left(\sum_{k=1}^{p-1} k\binom{c}{k}(g-1)^{k-1}\right)\frac{\partial g}{\partial z} \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\sum_{k=1}^{p-1} k\binom{c}{k}(g-1)^{k-1}\right) g(z) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\sum_{k=0}^{p-2} (k+1)\binom{c}{k+1}(g-1)^{k}\right) g(z) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\sum_{k=0}^{p-2} c\binom{c-1}{k}(g-1)^{k}\right) (g-1+1) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\left(\sum_{k=0}^{p-2} c\binom{c-1}{k}(g-1)^{k}\right) + \left(\sum_{k=0}^{p-2} c\binom{c-1}{k}(g-1)^{k+1}\right)\right) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(\left(\sum_{k=0}^{p-2} c\binom{c-1}{k}(g-1)^{k}\right) + \left(\sum_{k=1}^{p-1} c\binom{c-1}{k-1}(g-1)^{k}\right)\right) \\
&= \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(-c\binom{c-1}{p-1}(g-1)^{p-1} + \sum_{k=0}^{p-1} c\binom{c}{k}(g-1)^{k}\right) \\
&= \left(\sum_j \frac{-cx_j}{1-x_j z}\right) F(z) + \left(\sum_j \frac{-x_j}{1-x_j z}\right)\left(-c\binom{c-1}{p-1}(g-1)^{p-1}\right).
\end{aligned}
$$

We define

$$V(z) = \left(\sum_j \frac{-x_j}{1 - x_j z} \right) \left(-c \binom{c-1}{p-1} (g-1)^{p-1} \right).$$

Then we see that

$$\frac{\partial F}{\partial z} = V(z) + \left(\sum_j \frac{-c x_j}{1 - x_j z} \right) F(z).$$

From this it follows that

$$\begin{aligned}
\frac{\partial F_0}{\partial z} &= \frac{\partial}{\partial z} \left(\frac{F(z)}{1 - x_0 z} \right) \\
&= \frac{1}{1 - x_0 z} \frac{\partial F}{\partial z} + \frac{x_0}{1 - x_0 z} F(z) \\
&= \frac{V(z)}{1 - x_0 z} + \frac{1}{1 - x_0 z} \left(\sum_j \frac{-c x_j}{1 - x_j z} \right) F(z) + \frac{x_0}{(1 - x_0 z)^2} F(z) \\
&= \frac{V(z)}{1 - x_0 z} + \left(\sum_j \frac{-c x_j}{1 - x_j z} \right) F_0(z) + \frac{x_0}{1 - x_0 z} F_0(z).
\end{aligned}$$

We now again consider $(D_1 - D_0)(F_0)$. Recall that $n \equiv 0 \bmod p$, so in particular we can add n times any multiple of F_0 since that is 0 in characteristic p.

$$
\begin{aligned}
(D_1 - D_0)(F_0) &= \frac{G_1(z) - G_0(z)}{1 - x_0 z} + \frac{z(c-1)}{1 - x_0 z} F_0 + \sum_{j>0} \frac{zc}{1 - x_j z} F_0 \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - \frac{z}{1 - x_0 z} F_0 + \sum_{j} \frac{zc}{1 - x_j z} F_0 \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - \frac{z}{1 - x_0 z} F_0 + \left(\sum_{j} \frac{zc}{1 - x_j z} F_0 \right) - nzcF_0 \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 + zF_0 - \frac{z}{1 - x_0 z} F_0 + \left(\sum_{j} -zcF_0 + \frac{zc}{1 - x_j z} F_0 \right) \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 + \frac{z - x_0 z^2}{1 - x_0 z} F_0 - \frac{z}{1 - x_0 z} F_0 + \left(\sum_{j} \frac{-zc + x_j c z^2}{1 - x_j z} F_0 + \frac{zc}{1 - x_j z} F_0 \right) \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 + \frac{-x_0 z^2}{1 - x_0 z} F_0 + \left(\sum_{j} \frac{x_j c z^2}{1 - x_j z} F_0 \right) \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 - z^2 \left(\frac{x_0}{1 - x_0 z} F_0 + \left(\sum_{j} - \frac{x_j c}{1 - x_j z} F_0 \right) \right) \\
&= \frac{G_1(z) - G_0(z)}{1 - x_0 z} - zF_0 - z^2 \frac{\partial F_0}{\partial z} + z^2 \frac{V(z)}{1 - x_0 z} \\
&= \frac{G_1(z) - G_0(z) + z^2 V(z)}{1 - x_0 z} - zF_0 - z^2 \frac{\partial F_0}{\partial z}.
\end{aligned}
$$

We consider $z^2 V(z)$ and $G_i(z)$ for $i = 0, 1$. We note that $(g-1)^{p-1}$ divides $V(z)$ and $z(g-1)^{p-1}$ divides $G_i(z)$. Since the sum $\sum_{a=0}^{n-1} x_a$ is 0, we see that $z^2 \mid g(z) - 1$; therefore $z^{2p-2} \mid (g-1)^{p-1}$. Therefore $z^{p+1} \mid z^{2p-1} \mid G_i(z)$ and $z^{p+1} \mid z^{2p} \mid z^2 V(z)$. (We note that $z^p \mid z^{2p-2} \mid V(z)$; this will be relevant later.)

Then we see that for $0 \le \ell \le p$ we have

$$
[z^\ell](z^2 V(z)) = [z^\ell] G_0(z) = [z^\ell] G_1(z) = 0.
$$

Then we also have

$$
[z^\ell](G_1(z) - G_0(z) + z^2 V(z)) = 0.
$$

Then $[z^p] \left(\frac{G_1 - G_0 + z^2 V}{1 - x_0 z} \right)$ is 0 since it is a linear combination of $[z^\ell](G_1(z) - G_0(z) + z^2 V(z))$ for $0 \le \ell \le p$, so

$$
[z^p] \left((D_1 - D_0)(F_0) \right) = [z^p] \left(-zF_0 - z^2 \frac{\partial F_0}{\partial z} \right).
$$

Let $b = [z^{p-1}](F_0)$. We see that $[z^p](-zF_0) = -b$. Then $[z^{p-2}] \left(\frac{\partial F_0}{\partial z} \right) = (p-1)b = -b$, so $[z^p] \left(-z^2 \frac{\partial F_0}{\partial z} \right) = b$. Therefore $[z^p] \left(-zF_0 - z^2 \frac{\partial F_0}{\partial z} \right) = -b + b = 0$, so

$$
[z^p]((D_1 - D_0)(F_0)) = (D_1 - D_0)([z^p] F_0) = (D_1 - D_0) f_0 = 0.
$$

By the symmetry of the F_a, this is the only Dunkl operator we need consider; it is clear that all the f_a are killed by the Dunkl operators. $\square$

Proposition 3.2. *The f_a for $a = 0, \dots, n-2$ are linearly independent homogeneous polynomials of degree p.*

Proof. We recall that $F_a(z) = \frac{1}{1-x_a z}\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right) = \left(\sum_{k=0}^{\infty} x_a^k z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(g-1)^k\right)$. It is then clear that f_a is a homogeneous polynomial in the x_i of degree p, since $[z^k]F_a(z)$ is a homogeneous polynomial in the x_i of degree k for all k (this follows from the fact that this is true in both multiplicands in F_a).

Assume $\sum_{a=0}^{n-2}\lambda_a f_a = 0$ for some $\lambda_a \in k$. Then in particular this must be true when we set $x_a = 1$ and $x_i = 0$ for $i \neq a, i < n-1$. (We let $x_{n-1} = -1$ so that $\sum x_i = 0$.) Note that if this is the case that $g(z) = (1-z)(1+z) = 1 - z^2$, so

$$F_a(z) = \left(\sum_{k=0}^{\infty} z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(-z^2)^k\right)$$

and

$$F_i(z) = \left(\sum_{k=0}^{\infty} 0^k z^k\right)\left(\sum_{k=0}^{p-1}\binom{c}{k}(-z^2)^k\right) = \sum_{k=0}^{p-1}\binom{c}{k}(-z^2)^k$$

for $n-1 > i \neq a$.

Therefore we have $[z^p]F_a(z) = f_a = \sum_{k=0}^{(p-1)/2}(-1)^k\binom{c}{k} = \binom{c-1}{(p-1)/2}$ and $[z^p]F_i(z) = f_i = 0$ for $n-1 > i \neq a$ (since $p > 2$). Since c is generic, we can assume $\binom{c-1}{(p-1)/2} \neq 0$. Then since $\sum_{a=0}^{n-2}\lambda_a f_a = 0$, we see that $\lambda_a\binom{c-1}{(p-1)/2} = 0$; then we must have $\lambda_a = 0$.

By setting $x_a = 1, x_{n-1} = -1$ and $x_i = 0$ for $i \neq a, i \neq n-1$ for all $a = 0, \dots, n-1$ in succession, we find that $\lambda_a = 0$ for all a. Therefore the f_a for $a = 0, \dots, n-2$ are linearly independent as desired.

□

Proposition 3.3. *Let $y_0, \dots, y_{r-1}$ be distinct elements of k. Then the polynomials $b_i(z) = \prod_{j \neq i}(1 - y_j z)$ are linearly independent.*

Proof. Assume that for some $\lambda_i \in k$ we have $\sum_i \lambda_i b_i(z) = 0$. Then $\left(\prod_j (1 - y_j z)\right)\left(\sum_i \frac{\lambda_i}{1 - y_i z}\right)$ must be 0 in the ring of formal power series. Therefore since the ring of power series is an integral domain, $\sum_i \frac{\lambda_i}{1-y_i z} = 0$. Then in particular for $\ell = 0, \dots, r-1$ we have $\sum_i \lambda_i y_i^\ell = 0$.

This implies that the product of the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ y_0 & y_1 & \dots & y_{r-1} \\ \dots & \dots & \dots & \dots \\ y_0^{r-1} & y_1^{r-1} & \dots & y_{r-1}^{r-1} \end{pmatrix}$$

with the column vector $(\lambda_0, \dots, \lambda_{r-1})$ is 0; however since all the y_i are distinct, the Vandermonde matrix has nonzero determinant $\prod_{0 \le i < j \le r-1}(y_i - y_j)$, implying that $\lambda_i = 0$ for all i.

Therefore the polynomials $b_i(z)$ are linearly independent as desired.

□

Proposition 3.4. *Let $I \subseteq A$ be the ideal generated by f_a for $a = 0, \dots, n-2$. A/I is a complete intersection.*

Proof. We will show that if $x_0, \dots, x_{n-1} \in k$ satisfy $f_i(x_0, \dots, x_{n-1}) = 0$ for all i, then $x_0 = \cdots = x_{n-1} = 0$.

We suppose that the x_i take the distinct values $\{y_0, \dots, y_{r-1}\}$ with multiplicity $m_i > 0$; then since $\sum_{i=0}^{n-1} x_i = 0$ we have $\sum_{i=0}^{r-1} m_i y_i = 0$.

Then $g(z) = (1 - y_i z)^{m_i}$. Recall

$$F(z) = \sum_{k=0}^{p-1} \binom{c}{k} (g-1)^k.$$

We assume that $y_i \neq 0$ for all i.

Define

$$a(z) = \frac{1}{\prod_{i=0}^{r-1}(1 - y_i z)} F(z).$$

Recall that we showed

$$F'(z) = V(z) + \left(\sum_j \frac{-cx_j}{1 - x_j z} F(z) \right) = V(z) + \left(\sum_j \frac{-cm_j y_j}{1 - y_j z} F(z) \right)$$

where $[z^\ell]V(z) = 0$ for $0 \leq \ell \leq p$. We then see that

$$\begin{aligned}
a'(z) &= \frac{d}{dz} \left(\frac{1}{\prod_{i=0}^{r-1}(1 - y_i z)} F(z) \right) \\
&= \frac{1}{\prod_{i=0}^{r-1}(1 - y_i z)} F'(z) + \sum_{j=0}^{r-1} \frac{y_j}{(1 - y_j z) \prod_{i=0}^{r-1}(1 - y_i z)} F(z) \\
&= \frac{1}{\prod_{i=0}^{r-1}(1 - y_i z)} \left(V(z) + \left(\sum_j \frac{-cm_j y_j}{1 - y_j z} F(z) \right) \right) + \sum_{j=0}^{r-1} \frac{y_j}{1 - y_j z} a(z) \\
&= \frac{V(z)}{\prod_{i=0}^{r-1}(1 - y_i z)} + \left(\sum_j \frac{-cm_j y_j}{1 - y_j z} a(z) \right) + \sum_{j=0}^{r-1} \frac{y_j}{1 - y_j z} a(z) \\
&= \frac{V(z)}{\prod_{i=0}^{r-1}(1 - y_i z)} - \left(\sum_j \frac{(m_j c - 1) y_j}{1 - y_j z} a(z) \right)
\end{aligned}$$

Since $[z^\ell]V(z) = 0$ for $0 \leq \ell \leq p$ we see that $\frac{V(z)}{\prod_{i=0}^{r-1}(1 - y_i z)} = z^p b(z)$ for some power series $b(z)$. Therefore

$$a'(z) = z^p b(z) - \sum_j \frac{(m_j c - 1) y_j}{1 - y_j z} a(z)$$

for some power series $b(z)$.

Define $b_i(z) = \prod_{j \neq i}(1 - y_j z)$ for $i = 0, \dots, r-1$; then by Proposition 3.3 we see that the polynomials $b_i(z)$ are linearly independent.

Note that for any i, $a(z) b_i(z) = \frac{1}{1 - y_i z} F(z)$. Then in particular,

$$[z^p] a(z) b_i(z) = [z^p] \frac{1}{1 - y_i z} F(z) = 0.$$

Therefore for any $\lambda_i \in k$,

$$\sum_{i=0}^{r-1} \lambda_i [z^p] a(z) b_i(z) = [z^p] \left(a(z) \sum_{i=0}^{r-1} \lambda_i b_i(z) \right) = 0. \tag{3.5}$$

Recall that the $b_i(z)$ are linearly independent; they also have degree at most $r-1$. Therefore for $0 \le k \le r-1$ we can choose $\{\lambda_i^k\}$ such that

$$\sum_i \lambda_i^k b_i(z) = z^k.$$

Then if we use the λ_i^k in Equation 3.5, we see that $[z^p](a(z)z^k) = [z^{p-k}]a(z) = 0$ for $k = 0, \ldots, r-1$.

We will show that we must have $[z^0]a(z) = 0$, a contradiction. Note that if $r > p$ then we have already shown that; therefore we assume $r \le p$.

For any integer ℓ we define the Laurent polynomial

$$h_\ell(z) = z^{-\ell-r} \prod_i (1 - y_i z).$$

Then

$$\begin{aligned}
\frac{d}{dz}(h_\ell(z)a(z)) &= -(l+r)z^{-1}h_\ell(z)a(z) - \sum_i \frac{y_i}{1-y_i z} h_\ell(z)a(z) + h_\ell(z)a'(z) \\
&= -(\ell+r)z^{-1}h_\ell(z)a(z) - \sum_i \frac{y_i}{1-y_i z} h_\ell(z)a(z) - h_\ell(z)\sum_j \frac{(m_j c - 1)y_i}{1 - y_i j z} a(z) + z^p h_\ell(z)b(z) \\
&= -(\ell+r)z^{-1}h_\ell(z)a(z) - \sum_i \frac{y_i m_i c}{1-y_i z} h_\ell(z)a(z) + z^p h_\ell(z)b(z) \\
&= -\left((\ell+r)z^{-1} + \sum_i \frac{y_i m_i c}{1-y_i z} \right) h_\ell(z)a(z) + z^p h_\ell(z)b(z).
\end{aligned}$$

We note that

$$-\left((\ell+r)z^{-1} + \sum_i \frac{y_i m_i c}{1-y_i z} \right) h_\ell(z)$$

is a Laurent polynomial with lowest degree term

$$-z^{-l-r-1}\left(\ell + r + \sum_i y_i m_i c \right) = -z^{-\ell-r-1}(\ell+r)$$

and highest degree term

$$-z^{-\ell-1}(-1)^r \prod_i y_i \left((\ell+r) - \sum_i m_i c \right).$$

The lowest degree term of the formal power series $z^p h_\ell(z)b(z)$ has degree at least $p-\ell-r$. We note that we must have $[z^{-1}]\left(\frac{d}{dz}(h_\ell(z)a(z))\right) = 0$.

Then as long as $p - \ell - r \ge 0$, which is true when $\ell \le p - r$, we see that $[z^{-1}]\left(\frac{d}{dz}(h_\ell(z)a(z))\right) = 0$ is a linear combination of $[z^\ell]a(z), [z^{\ell+1}]a(z), \ldots, [z^{\ell+r}]a(z)$ in which $[z^\ell]a(z)$ has nonzero coefficient

$(-1)^r \prod_i y_i \left((\ell + r) - \sum_i m_i c\right)$ - we can assume this is nonzero since c is generic and none of the y_i are 0. Therefore we can write $[z^\ell]a(z)$ as a linear combination of $[z^{\ell+1}]a(z), \ldots, [z^{\ell+r}]a(z)$ when $\ell \leq p - r$.

We see that since $[z^{p-r+1}]a(z) = \cdots = [z^p]a(z) = 0$, we can use induction starting with $\ell = p - r$ to show that $[z^\ell]a(z) = 0$ for $\ell = p-r, p-r-1, \ldots, 0$. Therefore $[z^0]a(z) = 0$; however, it is clear from the definition of $a(z)$ that this is not the case. Therefore we have a contradiction, so one of the y_i is 0.

We can assume without loss of generality that $y_{r-1} = 0$. Then if $r > 1$, we can go through the same argument again with $y_0, \ldots, y_{r-2}$. We are able to do this because our calculation of $F'(z)$ and $V(z)$ so that $z^p \mid V(z)$ relies only on the fact that $\sum m_i y_i = 0$, which will still be true if we leave out $y_{r-1} = 0$. We will then find that one of the y_i with $i < r - 1$ is 0, which gives a contradiction since the y_i are distinct. Therefore we must have $r = 1$ and $y_0 = 0$.

□

Using these propositions, we are able to prove the main theorem.

Theorem 3.6. *The irreducible representation A/J of $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ is a complete intersection with Hilbert series $\left(\frac{1-t^p}{1-t}\right)^{n-1}$. The maximal ideal J is generated by $[z^p]F_a(z)$ for the formal power series*

$$F_a(z) = \frac{1}{1 - x_a z} \prod_{j=0}^{n-1} \left(\sum_{k=0}^{p-1} \binom{c}{k} (-1 + \prod_{j=0}^{n-1} (1 - x_j z))^k \right)$$

for $a = 0, \ldots, n-2$.

Proof. It suffices to show that the f_a generate the ideal J and that A/J has Hilbert series $\left(\frac{1-t^p}{1-t}\right)^{n-1}$.

By Propositions 3.2, 3.4, A/I is a complete intersection with $n-1$ generators of degree p. It then must have Hilbert series $h_{A/I}(t) = \left(\frac{1-t^p}{1-t}\right)^{n-1}$. By Proposition 3.1, the generators of I are annihilated by the Dunkl operators, so $I \subseteq J$.

By Proposition 3.4 in [1], we see that the Hilbert series of A/J is $\left(\frac{1-t^p}{1-t}\right)^{n-1} h(t^p)$ for some polynomial h with nonnegative integer coefficients; since $I \subseteq J$, we see that $h_{A/I}(t) \geq h_{A/J}(t)$ coefficient-wise; however by this restriction of the form of $h_{A/J}(t)$, we see that the only possible choice for h is $h(t) = 1$. Therefore $h_{A/I}(t) = h_{A/J}(t)$, so $I = J$ and these $n-1$ generators generate the whole ideal J.

□

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