1 $p = 3, 3 \mid n$, $n-1$ case full argument

1.1 Generators

Let $x_1, \ldots, x_{n-1}$ be a basis for the $n-1$ dimensional representation of S_n derived from the standard representation with basis $e_0, \ldots, e_{n-1}$ by the relation $x_i = e_i - e_0$ and the action of the symmetric group defined accordingly. Let s_{ij} be the transposition of i and j, for $0 \leq i < j \leq n-1$.

For $1 \leq i < j \leq n-1$, the relevant eigenvectors are $x_i - x_j$. For s_{0i} for $1 \leq i \leq n-1$, the relevant eigenvector is x_i.

Let f_a for $a = 1, \ldots, n-1$ be $f_a = -x_a^3 + c\left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_a\right)$. We will show that all the f_a are killed by Dunkl operators. Because of their symmetry, we need only consider f_1. Since f_1 is symmetric in the x_i excluding x_1, the only Dunkl operators we need consider are D_1, D_2.

We note that f_1 is preserved by s_{ij} for all $i, j > 1$. We consider s_{1a} for $2 \leq a$. We see that $s_{1a} f_1 = -x_a^3 + c\left(\sum_{i=1}^{n-1} x_a x_i (x_a - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j x_a\right) = f_a$. Then:

$$
\begin{aligned}
f_1 - f_a &= -(x_1 - x_a)^3 + c\left(\sum_{i=1}^{n-1} x_1 x_i (x_1 - x_i) - x_a x_i (x_a - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j (x_1 - x_a)\right) \\
&= -(x_1 - x_a)^3 + c\left(\sum_{i=1}^{n-1} x_i (x_1 + x_a - x_i)(x_1 - x_a)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j (x_1 - x_a)\right)
\end{aligned}
$$

Then $\frac{f_1 - s_{1a} f_1}{x_1 - x_a} = -(x_1 - x_a)^2 + c\left(\sum_{i=1}^{n-1} x_i (x_1 + x_a - x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right)$.

We consider s_{0a} for $2 \leq a$. We see that s_{0a} sends x_a to $-x_a$ and x_i for $i \neq a$ to $x_a - x_1$. We see that:

$$
\begin{aligned}
s_{0a}f_1 &= s_{0a}\left(-x_1^3 + c\left(\sum_{i=1}^{n-1} x_1x_i(x_1-x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_ix_jx_1\right)\right)\\
&= s_{0a}\left(-x_1^3 + c\left(\sum_{i\neq a} x_1x_i(x_1-x_i)\right) + c^2\left(\sum_{i\neq a}\sum_{j\neq a} x_ix_jx_1\right) + cx_1x_a(x_1-x_i) + c^2x_a^2x_1 + 2c^2\sum_{i\neq a} x_ix_ax_1\right)\\
&= -(x_1-x_a)^3 + c\left(\sum_{i\neq a}(x_1-x_a)(x_i-x_a)(x_1-x_a-x_i+x_a)\right) +\\
&c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i-x_a)(x_j-x_a)(x_1-x_a)\right) + c(x_1-x_a)(-x_a)(x_1-x_a-x_i+x_a) + c^2(-x_a)^2(x_1-x_a)+\\
&2c^2\sum_{i\neq a}(x_i-x_a)(-x_a)(x_1-x_a)\\
&= -x_1^3 + x_a^3 + c\left(\sum_{i\neq a}(x_1-x_a)(x_i-x_a)(x_1-x_i)\right) +\\
&c^2\left(\sum_{i\neq a}\sum_{j\neq a}(x_i-x_a)(x_j-x_a)(x_1-x_a)\right) + c(x_1-x_a)(-x_a)(x_1-x_i) + c^2x_a^2(x_1-x_a)+\\
&2c^2\sum_{i\neq a}(x_i-x_a)(-x_a)(x_1-x_a)
\end{aligned}
$$

The only remaining case to consider is s_{01}. We see that s_{01} sends x_1 to $-x_1$ and x_a for $a \geq 2$ to $x_a - x_1$. Then $s_{01}f_1$ goes to

$$
\begin{aligned}
&-(-x_1)^3 + c\left(\sum_{i=2}^{n-1}(-x_1)(x_i-x_1)(-x_1-(x_i-x_1))\right) + c^2\left(\sum_{i=2}^{n-1}\sum_{j=2}^{n-1}(x_i-x_1)(x_j-x_1)(-x_1)\right) +\\
&2c^2\left(\sum_{i=2}^{n-1}(x_i-x_1)x_1^2\right) + c^2(-x_1)^3\\
&= x_1^3 + c\left(\sum_{i=2}^{n-1} x_1x_i(x_i-x_1)\right) + c^2\left(\sum_{i=2}^{n-1}\sum_{j=2}^{n-1} -x_ix_jx_1 + x_ix_1^2 + x_jx_1^2 - x_1^3\right) +\\
&2c^2\left(\sum_{i=2}^{n-1}(x_i-x_1)x_1^2\right) + 2c^2x_1^3\\
&= x_1^3 + c\left(\sum_{i=2}^{n-1} x_1x_i(x_i-x_1)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} -x_ix_jx_1\right)\\
&= -f_1
\end{aligned}
$$

The above used the fact that $n-2 \equiv 1 \mod 3$. Therefore $\frac{f_1 - s_{01}f_1}{x_1} = 2f_1/x_1 = -f_1/x_1 = x_1^2 + c\left(\sum_{i=1}^{n-1} x_i(x_i-x_1)\right) -$

$c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}x_ix_j\right)$.

We can use these to calculate the effects of the Dunkl operators.

1.2 Dunkl operators

1.2.1 D_1f_1

We first consider $D_1f_1 = \partial_1 f_1 - c\sum_{a>1}\frac{f_1 - s_{1a}f_1}{x_1-x_a} - c\frac{f_1-s_{01}f_1}{x_1}$. Let $G_1 = \sum_{a>1}\frac{f_1-s_{1a}f_1}{x_1-x_a}, G_2 = \frac{f_1-s_{01}f_1}{x_1}$. We wish to show that $\partial_1 f_1 = cG_1 + cG_2$, since then $D_1f_1 = 0$. We calculate $\partial_1 f_1$:

$$\begin{aligned}
\partial_1 f_1 &= \partial_1\left(-x_1^3 + c\left(\sum_{i=1}^{n-1}x_1x_i(x_1-x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}x_ix_jx_1\right)\right)\\
&= c\left(\sum_{i=2}^{n-1}x_1x_i + x_i(x_1-x_i)\right) + c^2\left(\sum_{i=2}^{n-1}\sum_{j=2}^{n-1}x_ix_j\right) + 2c^2\left(\sum_{i=2}^{n-1}2x_ix_1\right)\\
&= c\left(\sum_{i=2}^{n-1}x_1x_i + x_ix_1 - x_i^2\right) + c^2\left(\sum_{i=2}^{n-1}\sum_{j=2}^{n-1}x_ix_j\right) + c^2\left(\sum_{i=2}^{n-1}x_ix_1\right)\\
&= c\left(\sum_{i=2}^{n-1}-x_1x_i - x_i^2\right) + c^2\left(\sum_{i=2}^{n-1}\sum_{j=1}^{n-1}x_ix_j\right)
\end{aligned}$$

We also see that $G_1 = \sum_{a>1}\frac{f_1-s_{1a}f_1}{x_1-x_a} = \sum_{a=2}^{n-1}-(x_1-x_a)^2 + c\left(\sum_{i=1}^{n-1}x_i(x_1+x_a-x_i)\right) + c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}x_ix_j\right)$ and $G_2 = x_1^2 + c\left(\sum_{i=1}^{n-1}x_i(x_i-x_1)\right) - c^2\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}x_ix_j\right)$.

Since $n-2 \equiv 1 \mod 3$, we see that the c^2 terms cancel in G_1+G_2. We also note that $x_i(x_i-x_1) = x_i(x_1+x_1-x_i) - x_i^2$ for all i, so $G_1+G_2 = x_1^2 - c\left(\sum_{i=1}^{n-1}x_i^2\right) + \sum_{a=1}^{n-1}-(x_1-x_a)^2 + c\left(\sum_{i=1}^{n-1}x_i(x_1+x_a-x_i)\right)$.

$$
\begin{aligned}
G_1 + G_2 &= x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{a=1}^{n-1} -(x_1 - x_a)^2 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_a - x_i)\right) \\
&= x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{a=1}^{n-1} -x_1^2 - x_a^2 + 2x_a x_1 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_a - x_i)\right) \\
&= x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{j=1}^{n-1} -x_1^2 - x_j^2 + 2x_j x_1 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_j - x_i)\right) \\
&= x_1^2 + -(n-1)x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{j=1}^{n-1} -x_j^2 - x_j x_1 + c\left(\sum_{i=1}^{n-1} x_i(x_1 + x_j - x_i)\right) \\
&= 2x_1^2 - c\left(\sum_{i=1}^{n-1} x_i^2\right) + \sum_{j=1}^{n-1} \left(-x_j^2 - x_j x_1\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_1 + x_i x_j - x_i^2\right) \\
&= \sum_{j=2}^{n-1} \left(-x_j^2 - x_j x_1\right) - c\left(\sum_{i=1}^{n-1} x_i^2\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_1 + x_i x_j - x_i^2\right) \\
&= \sum_{j=2}^{n-1} \left(-x_j^2 - x_j x_1\right) - c\left(\sum_{i=1}^{n-1} x_i^2\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i x_j\right) - c\left(\sum_{i=1}^{n-1} x_i x_1 - x_i^2\right) \\
&= \sum_{j=2}^{n-1} \left(-x_j^2 - x_j x_1\right) - c\left(\sum_{i=1}^{n-1} x_i^2\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=2}^{n-1} x_i x_j\right) + c\left(\sum_{i=1}^{n-1} x_i^2\right) \\
&= \sum_{j=2}^{n-1} \left(-x_j^2 - x_j x_1\right) + c\left(\sum_{i=1}^{n-1}\sum_{j=2}^{n-1} x_i x_j\right)
\end{aligned}
$$

Then a simple change of indices tells us that $cG_1 + cG_2 = \partial_1 f_1$ as desired.

1.2.2 $D_2 f_1$

We see that $D_2 f_1 = \partial_2 f_1 - c\frac{f_1 - s_{12} f_1}{x_2 - x_1} + \dots$.

1.3 Complete intersection

Then we see that the ideal I generated by $f_1, \dots, f_{n-1}$ has $I \subseteq J$ where J is the kernel of the β form, since the generators $f_1, \dots, f_{n-1}$ are killed by the Dunkl operators.

We note that the f_i are linearly independent, since x_i^3 has nonzero coefficient in f_j only when $i = j$. We will show that A/I, where $A = k[x_1, \dots, x_{n-1}]$, is a complete intersection by showing that $f_1(x) = \dots = f_{n-1}(x) = 0$ implies $x = 0$.

A simple algebraic manipulation lets us see that $f_a = -x_a^3 + cx_a^2\left(\sum_{i=1}^{n-1} x_i\right) + (c^2+2c)x_a\left(\sum_{i=1}^{n-1} x_i\right)^2 + x_a\left(\sum_{1 \le i < j \le n-1} 2cx_i x_j\right)$. We also note that $\sum_{i=1}^{n-1} f_i = (c^2+2)\sum_{i=1}^{n-1} x_i^3 = (c^2+2)\left(\sum_{i=1}^{n-1} x_i\right)^3$ Therefore if $f_1(x) = \cdots = f_{n-1}(x) = 0$, we see that $\sum_{i=1}^{n-1} x_i = 0$. Modulo this new relation, we see that $f_a = x_a \left(-x_a^2 + \sum_{1\le i<j\le n-1} 2cx_ix_j\right)$. Therefore if $f_a = 0$ for all a, we see that for each a, either $x_a = 0$ or x_a is a square root of $C = \sum_{1\le i<j\le n-1} 2cx_ix_j$. If $C = 0$ or C has no square roots, then all the x_a are 0 and we are done; so we assume C has two square roots which are additive inverses of each other. Then each pair x_ix_j either multiply to $\pm C$ or 0, so $C = 2cmC$ for some integer m. Then $(1 + cm)C = 0$; if $C \neq 0$, then $1 + cm = 0$; however, m is an integer, so this is impossible. Therefore $C = 0$, so all the x_a are 0.

Therefore $f_1(x) = \cdots = f_{n-1}(x) = 0$ implies $x = 0$, so A/I is a complete intersection; it then must have Hilbet polynomial $h_{A/I}(t) = (t^2 + t + 1)^{n-1}$.

By Proposition 3.4 in http://arxiv.org/abs/1107.0504, we see that the Hilbert polynomial of A/J is $(t^2+t+1)^{n-1}h(t^3)$ for some polynomial h with nonnegative integer coefficients; since $I \subseteq J$, we see that $h_{A/I}(t) \geq h_{A/J}(t)$ coefficientwise; however by this restriction of the form of $h_{A/J}(t)$, we see that the only possible choice for h is $h(t) = 1$. Therefore $h_{A/I}(t) = h_{A/J}(t)$, so $I = J$ and these $n-1$ generators generate the whole ideal.