White Paper

Group 1:
Project: Risk model: Generalized Autoregressive Conditional Heteroscedastic (GARCH)-GJR Model
Applied Quantitative Risk Management – ECON 6295
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Investment Case

Our client wants to forecast the portfolio risks involving shorting 100 stocks of Spy (S & P 500) and long hedging 100 stocks of Apple. Our client has a cash reserve of $10,000 and our respective individual stock prices are $186.79 for Apple and $281.12 for Spy (S & P 500) .

To analyze the portfolio risks for these long and short hedging strategies, we also analyzed the historic, adjusted returns of each trading day over a period of one year. This distribution of adjusted returns provides context, and an estimated baseline of the distribution of each security's volatility for the given year. However, each year's volatility changes due to circumstances, so we wanted to simulate an expected daily range of volatility over the same period of time. Simulating risks generalizes the expected volatility any given year, which allows for a robust estimate of risk.

We used the GARCH-GJR risk model to simulate the distribution of daily volatility over a one year window. In brief, GARCH modeling involves calculating the current time period volatility based on the previous time period's volatility. Unlike GARCH (1,1), we used GARCH-GJR because we wanted more conservative estimates on daily returns, and more liberal estimates on daily risks. When GARCH-GJR calculates the current period's volatility it has higher weights on the previous day's risks, than on previous day's returns. This essentially preserves the multi-day effects of risks than expected multi-day returns in a time period, which emphasizes a risk-averse model when hedging the client's portfolio. We then simulated a 10-day forecasted volatility based on our simulated distributions.

Once we obtained simulated distributions of volatility, value at risk analysis was performed using the hedging specifications and portfolio holdings of the client to estimate loss (in dollars) for this hedging operation.

Exploratory Data Analysis

Our dataset comes from Yahoo finance's API for Apple and Spy (S & P 500 index). We used one year of historical adjusted closing price from the year 2022-01-01 to 2023-11-01. We then calculated the daily historical returns for each security using the adjusted closing price. Overall, Spy is appraised at a higher price than Apple, which is not a surprise since it is a benchmark security of the S & P 500 index.

Figure 1: Box-plot of Apple and SPY

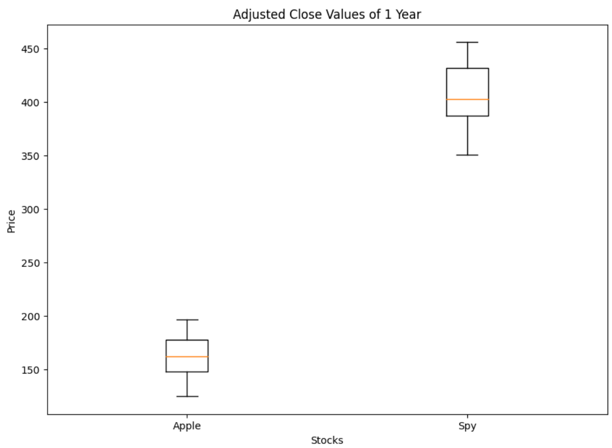
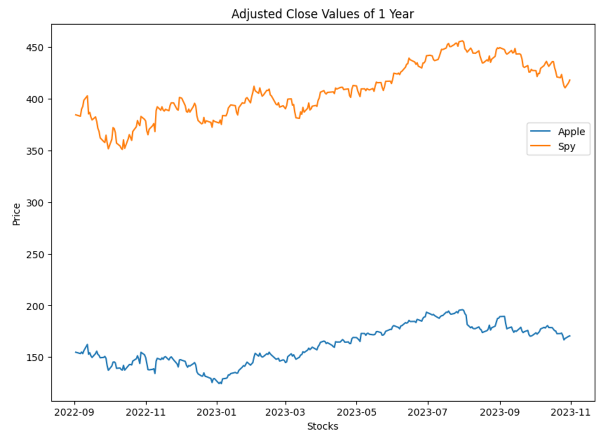


Figure 2: Historical adjusted price for Apple and SPY*



* Historical adjusted close prices for Apple and Spy from September, 2022 to November, 2023.

Figure 1 illustrates the box-plot of the Apple and SPY stock price. It shows the stocks maximum, minimum, average, and percentile prices. Apple has the highest price around $200, where SPY has around $450, more than twice. Interestingly, none of the stock has the outliers in the price during the one year period.

Figure 2 shows the movement of the stock price for a year. It can be seen that SPY prices start from approximately $380, and Apple starts from around $155.

Normal Volatility

Table 1: Normal Volatility for Apple and SPY

S.no.	Volatility Duration	Apple	SPY
1	Daily	1.75%	1.10%
2	Monthly	8.01%	5.06%
3	Yearly	27.76%	17.53%

Volatility for the both stocks were calculated based on the trading days by: day, months, and year. Usually, there are 21 trading days in a month, and the trading floor opens for 252 days in a year. Based on the above information, the daily volatility for Apple and SPY was 1.75% and 1.10% respectively. Likewise, the monthly volatility and yearly volatility are 8.01%, 27.76% for Apple and 5.06% and 17.53% for SPY respectively.

The volatility rate for Apple is quite high compared to the SPY because the SPY has more or less 500 stocks and it takes the average of it, if one stock goes down another goes up. In other words, negative movement of one or more stocks canceled by the positive movement of one or other stocks and vice versa. Apple is a single index and is more volatile in the case of good and bad news. As a result, significantly changes the stock prices and variance.

Model Description

In this analysis, we chose the asymmetric Generalized Autoregressive Conditional Heteroscedastic (GARCH) model, Glosten, Jagannathan and Runkle (GARCH GJR). The main differences between GARCH (1,1) and GARCH GJR, is that the latter reflects the asymmetric impact of past volatility shocks on current volatility. This allows the model to account for the fact that negative and positive shocks are weighted differently on volatility. In short, the parameter (gamma) weights daily losses more than daily returns, which makes the model more conservative by inflating volatility. For reference, GJR-GARCH is calculated when p = q = 1, as follows:

$$\sigma^2_t = \omega + (\alpha + \gamma I_{t-1})\varepsilon^2_{t-1} + \beta\sigma^2_{t-1}$$

Whereas,

σ^2_t = Conditional Variance

ω = Omega , α = Alpha , γ = Gamma , β = Beta

ε^2_{t-1} = Error terms in previous period or time points

In our model, we chose to model daily volatility, so we can expect our model to simulate daily volatility higher than the historic daily volatility. Using GARCH GJR offers more generalizable daily volatility for any given year for our securities. This means that our results are robust for any given year, compared to relying on the intricacies of historical daily volatility.

Based on the equation, daily volatility is calculated from the prior day's volatility plus any returns and loss on the return. The Garch GJR allows seasonal patterns in the volatility, such as positive and negative unanticipated returns. In the above model, we can gauge the positivity and negativity of conditional variance. The positivity of conditional variance is assured by when, Omega > 0 , Beta >=0, Alpha >=0 and Alpha + Gamma >= 0. The stationary of variance can be witnessed when the sum of the alpha,beta, and 0.5(gamma) is less than 1.

Moreover, the "I" is an indicator function expressed by:

$$I_{t-1} = \begin{cases} 0 \text{ if } R_{t-1} >= 0 \text{, (good news)} \\ 1 \text{ if } R_{t-1} < 0 \text{. (bad news)} \end{cases}$$

When Y (gamma) = 0 , it reduces the standard Garch model, which treats good news and bad news the same way. In other words, the impact of good and bad news on conditional variance remains the same. Likewise, when Y is not equal to Zero, the news has an asymmetric impact on conditional variance. The bad news and good news have different impacts on conditional variance. Similarly, when Y(gamma) >0, bad news has a larger impact on conditional variance than the good news.

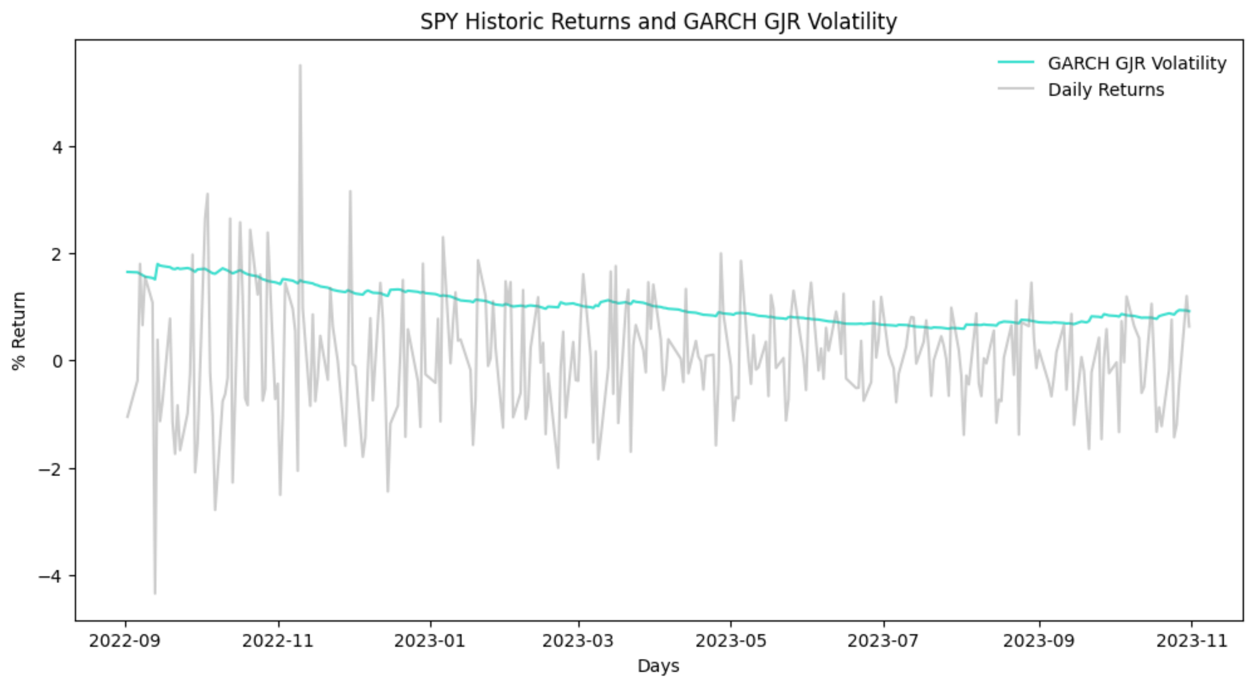
Moreover, bad news has an impact of $\boldsymbol{\alpha} + \gamma$ on conditional variance while good news has impact of α on conditional variance.

Model interpretation:

Table 2: Gamma (Y) value along P-Value

No.	**Index**	**Gamma(Y) Value**	**P value**
1	Apple	0.039	0.24
2	SPY (S & P 500 index)	0.053	0.16

The gamma value for Apple is 0.039>0 , which means the bad news has a larger impact on the conditional variance of Apple stock than the good news. While P value is more than 0.05, show it is not statistically significant. Likewise, the Gamma value for the SPY is higher than zero and not statistically significant. As a result, it can be indicated past volatility does not have any association or effect on future volatility. However, along the Value at Risk (VaR) it captures the asymmetric effects.



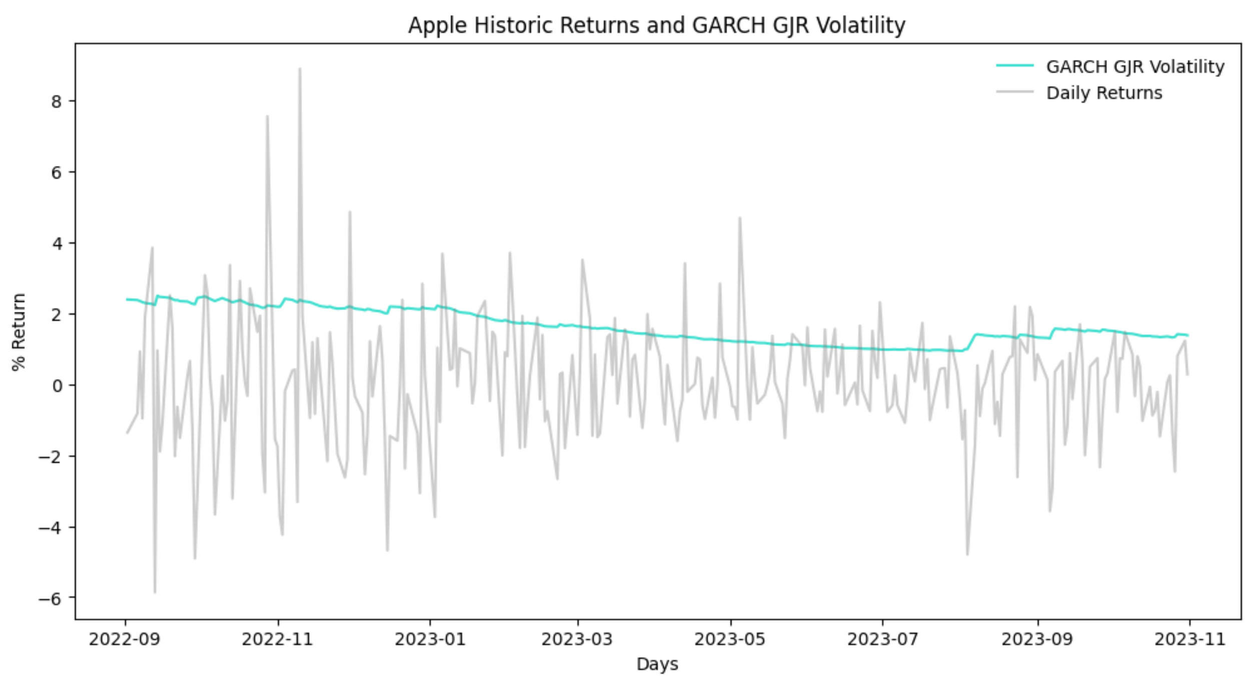


Figure 3: GARCH GRJ (in blue) results on daily volatility (in grey) for Apple and Spy from September, 2022 to November, 2023.

We then forecast 10 trading day volatilities, where the 10th day volatility is used in calculating the portfolio risk in our shorting of Spy and longing of Apple. We compared this portfolio value to the daily simulated volatility from our GARCH GJR models to construct loss distributions. We calculated the probability of our portfolio loosing in Value-at-Risk (VaR) to examine and evaluate the market risk of holdings. Due to the lack of daily profit and loss data of holdings, we take one year of sample to backward testing. And further, used to forecast the 10 days value at risk. Aftermath, we conclude the GJR- GARCH works well in Value at Risk (VaR) forecasting.

The generalized autoregressive conditional heteroskedasticity (GARCH) process is an econometric term developed in 1982 by Robert F. Engle, who later won the Nobel Memorial Prize for Economics in 2003. GARCH describes an approach to estimate volatility in financial markets. Later Glosten, Jagannathan and Runkle advanced it and named it GARCH-GJR. The GJR-GARCH model extends the basic GARCH(1,1) by weighting the previous time period's volatility more than the previous time period's returns for each calculation of the current time period. In the GARCH, (1,1) in parentheses notation refers to the first number as how many ARCH or autoregressive lags appear in the equation, while the second one means how many moving average lags are specified in number of GARCH terms. In some cases, more than one lag is needed to have good variance forecasts.

Value at Risk (VaR)

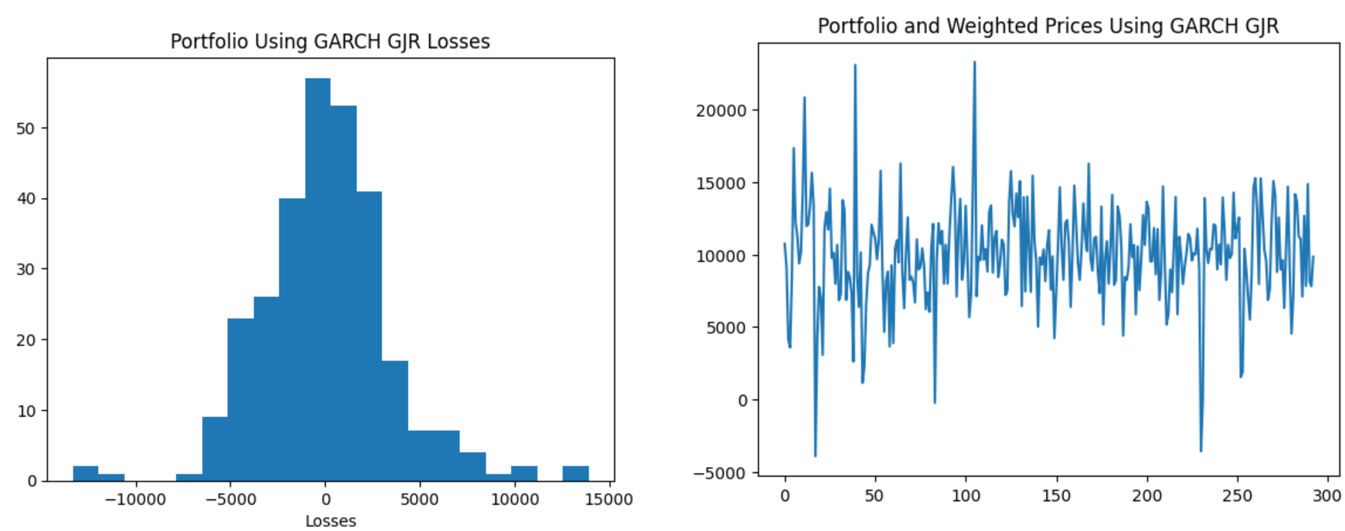


Figure 4: GARCH GRJ loss distribution (left) and GARCH GJR weighted prices (right) from the portfolio.

Value at risk is the percentile of the loss distribution or vice versa. Specific confidence levels of the loss distribution correspond to dollar amounts of loss or gain and their probabilities. It was found that the expected loss at the 5% risk at the next 10 days is $5793.16 or greater.

Backtesting, Assumptions and Limitations:

Index	MAE	MSE	MAPE
Apple	0.935	0.878	0.409
SPY	0.385	0.152	0.782

Backtesting is used to measure the performance of VaR that has been estimated. There are different methods for the backtesting, however, we used 10 days forecasted variance and 10 days implied variance. Using actual and forecasted variance, the mean absolute percentage error (MAPE) has been calculated along the mean absolute error (MAE) and mean square error (MSE). In the finance realm, the MAPE is more representative as it is in percentage form.

As provided in Table 3, the MAPE criteria is most indicative of a successful model if the weight is above .30, so for both Apple and Spy it seems our MAPE value meets this criteria so the model is sound enough for use. Additionally, compared to historical daily volatility mu, both models reflect a similar mu with the expected difference that GJR simulates 'hot' or slightly overshoots the model when penalizing risks more than returns.

Though VaR can offer volatility probabilities, it fails to account how volatility clusters during the year, and from internal and external shocks. For example, VaR may not accurately predict the losses during periods of heightened market volatility. Essentially, it is during this period that the distributions of volatility are changing in some way, and VaR's current loss distribution may not reflect those changes. As a result, extreme and or frequent losses break the expected predicted value from the VaR loss distributions. However, VaR with GARCH capture the conditional volatility into the account and make the prediction more accurate than before.

References

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estimation:

Estimates all the parameters(μ,ω,α,γ,β)

We estimate all the parameters(μ,ω,α,γ,β) simultaneously, by maximizing the log likelihood.

Return equation using constant mean models:

$R_t = u + E_t$

R_t= return in assets at time t

u = mean of asset return

E_t= shock term where, E_t = σtzt

is conditional volatility and Z_t ~ N(0,1)

The volatility is more likely to be high at time t if it was also high at time t−1. Another way of seeing this is noting that a shock at time t−1 also impacts the variance at time t.