CSC 24 month for joint PHD student.

1. Letter of Invation
2. Another new offer.

} Emphasis on a 2 year exchange abroad as a joint PhD student.

1 hour early

Eg. Toric Varieties.

X d-dim smooth proj Toric Var.

$N_{\mathbb{R}} \simeq \mathbb{R}^d$. $T = N \otimes_{\mathbb{Z}} k^*$.

$D \in \mathrm{Div}_{T_N}(X)$, $M_{\mathbb{R}} = N^*_{\mathbb{R}}$. χ^u on X.

$P_D \subset M_{\mathbb{R}}$. $:\{u \in M \mid D + \mathrm{div}(\chi^u) \geq 0\}$

$P_D \cap M \Rightarrow H^0(X, \mathcal{O}_X(D))$

$P_{D+\mathrm{div}(\chi^w)} = P_D - w \quad w \in M$.

$P_{mD} = mP_D$. $m \in \mathbb{Z}_{\geq 0}$.

$i \leq d$.

Y. prime Div_{T_N} $D_1, \dots, D_d$ of X. $Y_i = D_1 \cap \dots \cap D_i$

$v_i \sim D_i$, $\{v_i\} \Rightarrow N$ basis

primitive generator of ray. generate maximal cone σ. in fan of X.

$\mathbb{Z}^d \simeq N$. dual iso $\phi: M \longrightarrow \mathbb{Z}^d$,

$u \mapsto \phi(u) = (\langle u, v_i \rangle)_{1 \leq i \leq d}$.

$\phi_{\mathbb{R}}: M_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^d$.

Smooth Toric Vari

$0 \longrightarrow M \xrightarrow{\iota} \mathbb{Z}^r \xrightarrow{q} \mathrm{Pic}(X) \longrightarrow 0$.

$\qquad\qquad\qquad\qquad || $

$\qquad\qquad\qquad N^1(X)$. no torsion.

$$\parallel$$
$N^1(X)$. no torsion.

$$D \longmapsto [D].$$

$$u \longmapsto \operatorname{div}(\chi^u)$$
$$\parallel$$
$$\sum_{i=1}^{s} \langle u, v_i \rangle D_i.$$

$$\psi : \mathbb{Z}^d \times \operatorname{Pic}(X) \xrightarrow{\cong} \mathbb{Z}^s.$$

$$\psi^{-1}(D) = (p(D), q(D)).$$ $p : \mathbb{Z}^s \longrightarrow \mathbb{Z}^d$ first d componen

Prop X smooth. proj T var.
$Y_\cdot$

(i). $\forall\ L \in \operatorname{Big}\operatorname{Lbd}(X)$, $D \in \operatorname{Div}_{T_N}(X)$. s.t.

$$L \cong \mathcal{O}_X(D), \quad D|_{u_\sigma} = 0.$$

$$\Rightarrow \quad \Delta(L) = \phi_R(P_D).$$

(vi) σ maximal cone $\to u_\sigma$.

(ii). The global Okounkov body $\Delta(X)$. is inverse image
§ 4.2. under

$$\psi_{\mathbb{R}} : \mathbb{R}^d \times N^1(X)_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^s$$

of the non-negative orthant $\mathbb{R}_+^s \subseteq \mathbb{R}^s$.

pf:
(i). X smooth. $\sum D_i$ D_1 D_2 D_3

$$s \in H^0(X, L). \longrightarrow \sum a_i D_i$$

$$\Rightarrow \quad \nu_{Y_\cdot}(s) = (a_1, \dots, a_d)$$

$$u \in P_D. \longrightarrow \chi^u \in H^0(X, \mathcal{O}_X(D)).$$
zero locus
$$D + \sum_{i=1}^{r} \langle u, v_i \rangle D_i$$

$D|_{u_\sigma} = 0$. $\quad \nu_{Y_\cdot}(\chi^u) = \phi(u)$. $\to$ injective.

$h^0(L)$ lattice points in $P_D \cap M$.

$h^0(L)$ lattice points in $P_D \cap M$.

$\Rightarrow \mathrm{Im}\left((H^0(X,L) - \{0\}) \xrightarrow{\nu_{Y_\bullet}} \mathbb{Z}^d\right) = \phi(P_D \cap M)$.

$\forall \underset{\in \mathbb{Z}}{m} \geq 0$, s.t. mP_D.

$\underline{\text{Conclude } \frac{1}{m} \phi(mP_D \cap M)) = \phi_{\mathbb{R}}(P_D)}$.

$\phi_{\mathbb{R}}(D) = \Delta(D)$. shows i_2 in $\mathrm{Pic}(X)_{\mathbb{Q}}$.

(ii). $S \subseteq M \times \mathrm{Pic}(X)$.

consisting (u, L) : $\exists$ D unique T_N-Div $\left\{ \begin{array}{l} \mathcal{O}(D) \cong L. \\ D|_{U_\sigma} = 0. \end{array} \right\}$ Cond. (i) in

$\Rightarrow u \in P_D$.

To show: $\phi(S) = N^s$

$\phi : M \times \mathrm{Pic}(X) \longrightarrow \mathbb{Z}^s$,

$\psi \circ (\phi, id)$.

$\Phi^{-1}(E) = (u, [E])$. $u \in M$. $E|_{U_\sigma} = \mathrm{div}(\chi^u)|_{U_\sigma}$.

$\Rightarrow (u, [E]) \in S \Leftrightarrow u \in P_{E - \mathrm{div}(\chi^u)} = u + P_E$.

$\Leftrightarrow 0 \in P_E$. E is effective.

$\Rightarrow \phi^{-1}(E) \in S \Leftrightarrow E \in N^s$.

$\mathbb{R}^d \times N^1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}^r$ N^s.

big $\subset$ Eff

$D_1, \dots, D_n \sim N^1(X)$. $\Delta_{Y_\bullet}\left(\frac{\vec{a} \cdot D}{t}\right)$

$\Sigma a_i D_i$.

2.2. Global Linear Series.

Lemma. X proj. variety $\dim = d$. $Y_\bullet$ admissible flag

$D \in$ big $\mathrm{Div}(X)$ ⇝ graded semigroup.

$$\Gamma = \Gamma_{Y_\bullet}(D) \subseteq \mathbb{N}^{d+1}$$

satisfies (i) ~ (iii)

pf:

(i). $\Gamma_0 = 0$, is clear.

(ii). $\exists\, b \gg 0$. s.t.

$$\nu_i(s) \leq mb, \quad 1 \leq i \leq d, \quad \forall\, s \underset{\neq 0}{\in} H^0(X, \mathcal{O}_X(mD))$$

$B \subseteq \mathbb{N}^{d+1}$ generated by $(a_1, \dots, a_d, 1) \in \mathbb{N}^{d+1}$

with $0 \leq a_i \leq b$.

(iii). $D = A - B$ add very ample to A, B.

Let A, B very ample

$$s_0 \in H^0(X, \mathcal{O}_X(A)), \quad t_i \in H^0(X, \mathcal{O}_X(B))$$

$$\nu(s_0) = \nu(t_0) = 0, \quad \nu(t_i) = \underline{e_i} \qquad (1 \leq i \leq d)$$

$$= (0, \dots, \overset{i\text{th}}{1}, \dots, 0).$$

$t_i \neq 0$ on Y_{i-1}. $\quad t_i|_{Y_{i-1}}(Y_i) \equiv 0$.

neighborhood of Y_d.

D is big, $m_0 = m_0(D)$ s.t. $\exists$ Kodaira lemma (in Laz)

$$\underline{mD - B \equiv_{\mathrm{lin}} F_m}, \quad m \geq m_0$$

big cone int. point D

$$mD \equiv_{\mathrm{lin}} B + F_m.$$

$\nu(s)$ (F_m section)

point. D $\quad mD \equiv_{lin} B + F_m$.

Kodaira. $D \neq \emptyset$, $\quad f_m \in \mathbb{Z}^d \xleftarrow{\nu(\cdot)}$ (F_m section)

$(f_m, m), (f_m+e_1, m), \ldots, (f_m+e_d, m) \in T$.

$$(m+1)D \underset{lin}{\equiv} B + F_m + \underset{\substack{\| \\ A-B}}{D} \equiv A + F_m.$$

$(f_m, m+1) \in T$. $\longrightarrow$ standard basis of $\mathbb{Z}^{d+1}$

Euclidean alg. $(e_1, \ldots\ldots e_{d+1})$

$\square$

Thm: D bigDiv(X) $\rightarrow$ proj. var. dim $= d$.

$$\mathrm{Vol}_{\mathbb{R}^d}(\Delta(D)) = \frac{1}{d!}\mathrm{Vol}_X(D)$$

$$:= \lim_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}$$

Newton Okounkov body, $Y_\bullet$

*: take limsup (Laz)

pf: $\Gamma = \Gamma(D) = \Gamma_{Y_\bullet}(D)$.

$$\mathrm{Vol}_{\mathbb{R}^d}(\Delta(D)) = \lim_{m\to\infty} \frac{\#\Gamma(D)_m}{m^d}$$

$$\#\Gamma(D)_m = h^0(X, \mathcal{O}_X(mD))$$

$\square$

2.3. $D \xrightarrow{\text{bigDiv}}$ Graded Linear Series

X irr var dim $= d$. $W_\bullet$ graded linear series $\sim D$.

$Y_\bullet$

$\Gamma_{Y_\bullet}(W_\bullet) \subseteq \mathbb{N}^{d+1}$ $\quad$ (i) $\sim$ (iii)

Def (Cond(A)).

$W_\bullet$ w.r.t $Y_\bullet$: if $\exists\, b > 0$. s.t. $\forall\, \underset{\neq 0}{s} \in W_m$.

$$\nu_i(s) \leq mb, \quad 1 \leq i \leq d.$$

Rmk: Cond(A) True if X is proj

Def (Cond(B)).

$W_\bullet$: if $W_m \neq 0$, $\forall m >> 0$ and m.

$$\phi_m : X \dashrightarrow \mathbb{P} = \mathbb{P}(W_m)$$

defined by $|W_m|$. is birational onto its image.

$W_k \neq 0$. $k >> 0$. ϕ_m. birat.

Lemma: $W_\bullet$ Cond(B). $\exists$ $Y_\bullet$ on X w.r.t.

$\Gamma_{Y_\bullet}(W_\bullet) \subseteq \mathbb{N}^{d+1}$ generates $\mathbb{Z}^{d+1}$ as a group.

(iii).

pf: $|W_\ell| \to \phi = \phi_\ell : X \dashrightarrow \mathbb{P} = \mathbb{P}(W_\ell)$

$y \in X$ smooth pt. & ϕ_ℓ is defined, locally iso to im(ϕ)

$y \notin$ base locus of $|W_q|$,

for fixed large q, $(q, \ell) = 1$.

$Y_\bullet$: $X = Y_1 \supseteq \cdots \supseteq Y_{d-1} \supseteq Y_d = \{y\}$.

(hypersurface in $\mathbb{P}$)

$p >> 0$, $t_0, t_1, \cdots, t_d \in W_{p\ell}$. s.t.

$\nu_{Y_\bullet}(t_0) = 0$, $\nu_{Y_\bullet}(t_i) = e_i$ $(1 \le i \le d)$

$\exists s_0 \in W_q$. s.t. $\nu_{Y_\bullet}(s_0) = 0$.

$(0, p\ell), \cdots, (e_d, p\ell), (0, q) \in \mathbb{N}^{d+1}$

$\Downarrow$

$\mathbb{Z}^{d+1}$ as group. $\square$

Def (Cond(C)).

X proj. $W_\bullet \sim D$ (big), $W_\bullet$ satisfies cond(C) if

(i) $\forall m >> 0$ $\exists$ $F_m \in$ eff Div(X). s.t.

(i). $\forall m \gg 0$. $\exists F_m \in$ eff $\mathrm{Div}(X)$. s.t.

$$\underbrace{A_m}_{\text{ample}} =_{\det} mD - F_m.$$

(ii). $\forall p \gg 0$.

$$H^0(X, \mathcal{O}_X(pA_m)) = H^0(X, \mathcal{O}_X(pmD - pF_m))$$
$$\subseteq W_{pm} \subseteq H^0(X, \mathcal{O}_X(pmD))$$

Rmk: (Alternate) Cond (C).

(i) (ii) $\cdot m_0$ of m. $W_k \neq 0$, $\forall k \gg 0$.

(i) (ii) hold for $m = m_0$, $E_k \sim S_k \in W_k$ ∴

$E_k \equiv_{\text{lin}} kD$. multiplication by $s_k^{\otimes p}$

l, $W_l \subseteq W_{l+kp}$

$m = m_0 + k$, by taking

$F_{m_0+k} = F_{m_0} + E_k$.

$$A_m = (m_0+k)D - F_{m_0+k} \equiv_{\text{lin}} A_{m_0} \text{ ample}$$
$$H^0(X, \mathcal{O}_X(pA_{m_0})) \subseteq W_{pm_0} \subseteq W_{pm_0+pk}.$$

Lemma. $W_\bullet \sim$ Cond (C). $\forall$ $Y_\bullet$ on X.

$\Gamma_{Y_\bullet}(W_\bullet)$ generate $\mathbb{Z}^{d+1}$.

pf: m, $p \gg 0$. $\Gamma = \Gamma_{Y_\bullet}(W_\bullet)$.

(pf_m, pm), $(pf_m + e_1, pm)$, $\cdots$, $(pf_m + e_d, pm) \in \mathbb{N}^{d+1}$

$f_m \sim F_m$, e_i standard basis.

$(qf_l, ql) \in \Gamma$, $f_l \in \mathbb{N}^d$. ☐

Thm. $W_\bullet$ satisfies (A), (B), (C), $Y_\bullet$

Thm. $W_\bullet$ satisfies (A), (B), (C), $Y_\bullet$

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta(W_\bullet)) = \frac{1}{d!} \cdot \underline{\mathrm{vol}(W_\bullet)}$$

where $\mathrm{vol}(W_\bullet) =_{\det} \lim_{m\to\infty} \frac{\dim W_m}{m^d/d!}$ v2

2.4. Restricted Linear Series.

V proj var D big on V.

$\underline{\mathbb{B}(D)} \subseteq V. \to \bigcap_m \mathrm{Bs}(|mD|)$

eg: $[D]_{num}$ $\overset{\times}{}$depend.

Augmented base locus $\mathbb{B}_+(D) \subseteq V$ to be

$$\mathbb{B}_+(D) = \mathbb{B}(D-A)$$

for any small ample $\mathbb{Q}$-div A. indep A sufficient small.

$\mathbb{B}_+(D) \sim [D]_{num}$.

ξ^{d} $\mathbb{B}_+(\xi)$.

$X \subseteq V$. be an irr subvar dim d.

$W_m = H^0(\underline{V|X}, \mathcal{O}_V(mD)) := \mathrm{Im}\,(H^0(V, \mathcal{O}_V(mD)) \xrightarrow{res} H^0(X, \mathcal{O}_X(mD)))$

high dim rest divisor

restricted complete series

Restricted volume of D from V to X.

$\mathrm{vol}_{V|X}(D) = \mathrm{vol}(W_\bullet)$

Lemma. $X \nsubseteq \underline{\mathbb{B}_+(D)}$. $W_\bullet$ satisfies $\underline{\text{Cond (C)}}$.

pf: A very ample div on V,

$A+D$ is also very ample. rational

$A+D$ is also very ample.

$X \not\subset \mathbb{B}_+(D) \Rightarrow X \not\subset \mathbb{B}(D-\varepsilon A)$ small rational $\varepsilon>0$.

$\exists\, m_0 \in \mathbb{N}$ s.t. $X \not\subset Bs(|m_0 D - A|)$.

$\Rightarrow E_{m_0} \in |m_0 D - A|$ meet X properly.

$F_{m_0} = E_{m_0}|_X$, $A_{m_0} = (m_0 D - F_{m_0})|_X$.

$\Rightarrow A_{m_0} \equiv_{lin} A|_X$ ample div on X.

$H^0(V, \mathcal{O}_V(pA)) \longrightarrow H^0(X, \mathcal{O}_X(pA)) \longrightarrow H^1(V,$ Gcm 5.2.

surjective when $p \gg 0$. (Serre vanishing)

Show $H^0(V|X, \mathcal{O}_V(mD)) \neq 0$. $m \gg 0$.

$\mathcal{O}_V(m_0 D) \neq 0$. since A is very ample.

$(m_0+1)D \equiv_{lin} \underbrace{(m_0 D - A)}_{\text{very}} + \underbrace{(A+D)}_{\text{very ample}}$

$A_{m_0} = (m_0 D - F_{m_0})|_X$

$\wr\wr\; A|_X$.

4.1 chapter 4

April 2.

5:00pm.

e.g. Surface.