GTM52 Solution

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Contents

Introduction

This article mainly compiles the exercises and omitted proofs from Chapter 2 onwards of GTM52. The problems in the book are labeled with "Q," and the exercises at the end of the book are labeled with "Exercise."

It is necessary to make the following statement regarding some personal notations I use.

Math Symbol	**Meaning**
$X.\tau$	Return all open sets of topological space X.
X.**connect**	Return all connected sets of topological space X.
$X.\mathscr{U}(A)$	Return all open sets containing A.
$X.\mathscr{U}(p)$	Return neighbourhood systems of point p.
$s\|_U$	It is the image under the restriction, which omit $\rho_{-,-}$.
$\mathbf{Mor}_{-}(*_1, *_2)/\mathbf{Mor}$	Return all morphisms of type "-" from $*_1$ to $*_2$.
Mor.**Iso**/**Iso**	Return all isomorphisms.
Mor.**Inj**/**Inj**	Return all injective morphisms.
Mor.**Sur**/**Sur**	Return all surjective morphisms.
Shf(X,-)	Return all sheaves over X with - algebraic structure.
Grp	Return all (abelian) groups.
Rng	Return all (commutative with identity) ring.
Rng.Local	Return all local ring.
Rng.Grad	Return all graded ring.
Top	Return all topological space.
$\vdots$	$\vdots$

If there is not ambigous, I will not declare a new symbol any more.

2 Scheme

2.1 Sheaves

Q 2.1.1. Give a presheaf that satisfy the first property of sheaf but not second.

Solve:

We can let $X = \{a, b\}, U = \{a\}, X.\tau = \{\emptyset, X, U\}$.

We can have a presheaf $\mathscr{F}$, with sections

$$\mathscr{F}(X) = \mathbb{Z} \xrightarrow{\rho_{XU}=2id} \mathscr{F}(U) = \mathbb{Z}.$$

The remaining restriction maps are all zero maps.

Buf if we let $3 \in \mathscr{F}(U)$, we cannot find a element that $x \in \mathscr{F}(X), \rho_{XU}(x) = 3$. □

Q 2.1.2. Let X be a topological space, and A an abelian group, $\mathscr{A}$ a constant sheaf on X determined by A, as the definition of "constant sheaf" in the book, it satisfies the conditions of a sheaf. Besides, $\forall U \in X.\tau \cap X.$**connected**, $\mathscr{A}(U) \cong A$. If U is an open set whose connected components are open (which is always true on a locally connected topological space), then $\mathscr{A}(U)$ is a direct product of copies of A, one for each connected component of U.

Proof:

Firstly, it should be a presheaf. $U \in X.\tau$, we have an abelian group $\mathscr{A}(U) = A$. Trivially, $\mathscr{A}(\emptyset) = 0$. Take $\forall s \in \mathscr{A}(U)$, has the following properties:

$$\forall B \subseteq A, s^{-1}(B) \in U.\tau.$$

If we select $a_i \in A$ and U is connected, we have a open cover of U that

$$U = \bigcup_{a \in A} s^{-1}(a),$$

there must be $s^{-1}(a) \neq \emptyset, s^{-1}(a') \neq \emptyset$ satisfying $s^{-1}(a) \cap s^{-1}(a') \neq \emptyset$. According to the definition of map

$$\forall x \in s^{-1}(a) \cap s^{-1}(a'), \exists! a'' \in A, s(x) = a'' = a = a'.$$

That's mean there is only one element that corresponds to U, namely,

$$\mathscr{A}(U) \cong A.$$

Since each element inside is a map, restriction is simply limiting the map itself. Natrually, we have

$$\forall U, V \in X.\tau, V \subseteq U, \rho_{U,V} : \mathscr{A}(U) \to \mathscr{A}(V)$$

Through the above discussion, we know that the restriction is an identity, with the meaning of groups. Therefore, all presheaf properties are satisfied. To determine whether it satisfies the sheaf, two additional conditions need to be verified, and these are evident for the restriction of maps.

For the case of U satisfying

$$U = \bigcup_{\lambda \in \Lambda} U_\lambda,$$

$$\forall \lambda_1, \lambda_2 \in \Lambda, \lambda_1 \neq \lambda_2, U_{\lambda_1} \cap U_{\lambda_2} = \emptyset.$$

We can see that on each connected component, it is isomorphic to A. Then we can get $\{s_\lambda\}$ which $\forall \lambda \in \Lambda, s_\lambda \in \mathscr{A}(U_\lambda)$, and

$$s_{\lambda_1}|_{U_{\lambda_1} \cap U_{\lambda_2}} = s_{\lambda_2}|_{U_{\lambda_1} \cap U_{\lambda_2}} = 0.$$

Based on the properties of sheaves, we can obtain

$$s \in \mathscr{A}(U), s|_{U_\lambda} = s_\lambda, s \in \prod_{\lambda \in \Lambda} A$$

□

Q 2.1.3. If $U \in X.\tau$ has multiple connected components, that is $U = \cup_\lambda U_\lambda$, and $\mathscr{F}$ is a sheaf over X. Then

$$\mathscr{F}(U) = \prod_{\lambda \in \Lambda} \mathscr{F}(U_\lambda).$$

Proof:

We can use restriction in different component that

$$\rho_{UU_\lambda} : \mathscr{F}(U) \to \mathscr{F}(U_\lambda).$$

Then

$$\varphi = \prod_{\lambda \in \Lambda} \rho_{UU_\lambda} : \mathscr{F}(U) \to \prod_{\lambda \in \Lambda} \mathscr{F}(U_\lambda).$$

Combining the restriction maps and axioms of sheaves, we can demonstrate that they are isomorphic (Injectivity is proved by first axiom, the surjectivity is proved by second axiom.). $\square$

Q 2.1.4. Denote X is a topological space, $\mathscr{F}, \mathscr{G}$ are sheaves,there is a morphism of sheaves

$$\varphi : \mathscr{F} \to \mathscr{G}.$$

then $ker\varphi$ is a sheaf.

Proof:

We need to check that $\forall U \in X.\tau, \forall \{U_\lambda\}_{\lambda\in\Lambda} \in U.\mathbf{cover}, \forall s_\lambda \in ker(\varphi(U_\lambda))$ with

$$s_{\lambda_1}|_{U_{\lambda_1} \cap U_{\lambda_2}} = s_{\lambda_2}|_{U_{\lambda_1} \cap U_{\lambda_2}}.$$

$\exists s \in \mathscr{F}(U), s|_{U_\lambda} = s_\lambda$. We need to verify $\varphi(U)(s) = 0$. By commutative diagram,

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\mathscr{F}(U_\lambda) \arrow[r] & \mathscr{G}(U_\lambda)
\end{tikzcd}$$

and sheaf property of $\mathscr{G}$, we obtain $\varphi(U)(s) = 0$, so $s \in ker(\varphi(U))$, $ker\varphi$ is sheaf. $\square$

Q 2.1.5. Let $X \in \mathbf{Top}, \mathscr{F} \in \mathbf{Shf}, \forall U \in X.\tau, \forall V \in U.\tau$, we have

$$\mathscr{F}|_U(V) = \mathscr{F}(V).$$

And $\forall p \in U$, we have

$$(\mathscr{F}|_U)_p = \mathscr{F}_p.$$

Obviously, if $\forall U \in X.\tau, \forall V \in U.\tau$

$$(\mathscr{F}|_U)|_V = \mathscr{F}|_V.$$

Similarly, let $f : X \to Y$, $\mathscr{F}$ is a sheaf over Y.

$$(f^{-1}\mathscr{F})_p = \mathscr{F}_{f(p)}$$

Proof:

We let $i : U \hookrightarrow X$,

$$\begin{aligned}
\mathscr{F}|_U(V) &= i^{-1}\mathscr{F}(V) \\
&= \varinjlim_{W \in U.\mathscr{U}(i(V))} \mathscr{F}(W) \\
&= \varinjlim_{W \in U.\mathscr{U}(V)} \mathscr{F}(W) \\
&= \mathscr{F}(V).
\end{aligned}$$

This is because we can restrict sections on open sets to be restricted to V. According to the equivalence of direct limits, all elements on the components can be viewed as elements of the sections on V.

$$
\begin{aligned}
(\mathscr{F}|_U)_p &= \varinjlim_{V \in U.\mathscr{U}(p)} \mathscr{F}|_U(V) \\
&= \varinjlim_{V \in U.\mathscr{U}(p)} \left(\varinjlim_{W \in U.\mathscr{U}(i(V))} \mathscr{F}(W) \right) \\
&= \varinjlim_{V \in U.\mathscr{U}(p)} \mathscr{F}(V) \\
&= \varinjlim_{V \in X.\mathscr{U}(p)} \mathscr{F}(V) \\
&= \mathscr{F}_p
\end{aligned}
$$

Under the equivalence of direct limits, we often only need to consider those groups with the largest indices. Adding groups with smaller indices does not affect the final result.

Repeating the above process will prove the next statements.

(NOTE: We should notice that if $p \notin U$, we cannot make sure that $(\mathscr{F}|_U)_p = 0$ or not, such as the case $\forall W \in X.\mathscr{U}(p) \Rightarrow U \in X.\mathscr{U}(U)$.) □

Q 2.1.6. Let $f : X \to Y$ continuous map, $\mathscr{F}$ is a sheaf over X. To determine the form of $\forall q \in Y$,

$$
(f_*\mathscr{F})_q.
$$

Solve:

We check it by definition.

$$
\begin{aligned}
(f_*\mathscr{F})_q &= \varinjlim_{V \in Y.\mathscr{U}(q)} (f_*\mathscr{F})(V) \\
&= \varinjlim_{V \in Y.\mathscr{U}(q)} \mathscr{F}(f^{-1}(V))
\end{aligned}
$$

Consider the connected components of $U \in X.\mathscr{U}(f^{-1}(q))$. We can always select a subset $V \subseteq U, f^{-1}(q) \subseteq V$ that V may have more components. If this process is stable, $(f_*\mathscr{F})_q$ can be written as decomposion of product. Otherwise, we can always select $x \in (f_*\mathscr{F})_q$ which has another coordinate. □

Q 2.1.7. On X, we have three sheaves, then prove that composition of morphism is also morphism of sheaves.

$$
\begin{tikzcd}
 & \mathscr{H} \\
\mathscr{F} \arrow[ur, "g \circ f"] \arrow[r, "f"'] & \mathscr{G} \arrow[u, "g"']
\end{tikzcd}
$$

Proof:

$$
\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[r] \arrow[d] & \mathscr{H}(U) \arrow[d] \\
\mathscr{F}(V) \arrow[r] & \mathscr{G}(V) \arrow[r] & \mathscr{H}(V)
\end{tikzcd}
$$

According to the definition, it is commutative from $\mathscr{F}(U)$ to $\mathscr{G}(V)$ and from $\mathscr{G}(U)$ to $\mathscr{H}(V)$. In sum, it is commutative from $\mathscr{F}(U)$ to $\mathscr{H}(V)$. That means $g \circ f \in \mathbf{Mor}_{Shf}(\mathscr{F}, \mathscr{H})$. □

Q 2.1.8. Let $X \in \mathbf{Top}$, and two sheaves $\mathscr{F}, \mathscr{G}$ and $\varphi \in \mathbf{Mor}_{Shf}(\mathscr{F}, \mathscr{G})$. If $\varphi \in \mathbf{Iso}$, then $\forall U \in X.\tau, \varphi(U) \in \mathbf{Iso}$.

Proof:

$\varphi \in \mathbf{Iso} \Rightarrow \varphi, \varphi^{-1} \in \mathbf{Inj} \Rightarrow \forall U \in X.\tau, \varphi(U), \varphi^{-1}(U) \in \mathbf{Inj}$,

$$
\varphi(U) \in \mathbf{Iso}.
$$

□

Q 2.1.9. Two sheaf axioms are equivalent to below separately:

- $\forall U \in X.\tau, s|_U = 0 \Rightarrow s = 0 \in \mathscr{F}(X)$
- $\forall U' \in X.\tau, \exists s' \in \mathscr{F}(U'), \forall U'' \in X.\tau, \exists s \in \mathscr{F}(U'')$,

$$s'|_{U'\cap U''} = s''|_{U'\cap U''}.$$

$\Rightarrow \exists! s \in \mathscr{F}(X)$.

Proof:

(1) ($\Leftarrow$) By definition, if $\forall U \in X.\tau, s|_U = 0$, we can let a family of $\{U\}$ to become a cover of X, then $s = 0 \in \mathscr{F}(X)$.

($\Rightarrow$) If we have a cover $\{U_i\} \in X.\mathbf{cover}, s|_{U_i} = 0$, then $\forall U \in X.\tau, \{U \cap U_i\} \in U.\mathbf{cover}$,

$$s|_{U\cap U_i} = (s|_{U_i})|_{U\cap U_i} = 0,$$

so $s|_U = 0 \Rightarrow s = 0 \in \mathscr{F}(X)$. □

(2) ($\Leftarrow$) By definition, we can let U, V to be a cover of X.

($\Rightarrow$) If we have $\{U_i\} \in X.\mathbf{cover}, s_i \in \mathscr{F}(U_i), s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}, \forall U' \in X$, we can let $U' = X$, then we obtain element $s' \in U'$, and $s'' \in U''$ as well. Furthermore,

$$\forall i, (s'|_{U'})_{U'\cap U_i} = s_i|_{U'\cap U_i} = (s''|_{U''})|_{U''\cap U_i}.$$

□

Q 2.1.10. We let $x \in X$ a point in a topological space, and $\mathscr{F} \in \mathbf{Shf}(X)$. Denote the restriction by inclusion $\{x\} \to X$ is $\mathscr{F}|_{\{x\}}$. By definiton, we have directly

$$\mathscr{F}|_{\{x\}} \cong \mathscr{F}_x.$$

Exercise 2.1.1. Proof:

Note that **the constant presheaf associatted to** A is different from **constant sheaf**, because here it directly returns an abelian group but not a continuous map group. We denote this presheaf as $\mathscr{F}$, then to prove that

$$\mathscr{F}^+ \cong \mathscr{A}$$

(ps: Intuitively, when there is more than one connected component in U, the property of being a sheaf is not satisfied.)

There is a simple way is to verify that after sheafification, each open set section is mapped continuously to A. $\forall U \in X.\tau, \forall s \in \mathscr{F}^+(U)$, if we equip A with the discrete topology, we only need to prove that s is a continuous mapping. In other word, we need to show: $\forall a \in A, s^{-1}(a) \in U.\tau$. According to the definition of sheafification, $\forall P \in U, \exists V \in U.\tau, P \in V, \exists t \in \mathscr{F}^+(V) = A, s.t. \forall Q \in V, t_Q \equiv t = s(Q)$. So $\exists S \subseteq U, s(S) = \{a\}$, then we obtain a family of open sets $s^{-1}(a) = \{V_Q\}_{Q\in S}$, and $\bigcup_{Q\in S} V_Q \in U.\tau$ is what we want. □

Exercise 2.1.2. Proof:

(a)The problem

$$ker(\varphi_P) = (ker\varphi)_P$$

is equivalent to proving $ker(\varphi_P) \subseteq (ker\varphi)_P$ and $(ker\varphi)_P \subseteq ker(\varphi_P)$. $(im\varphi)_P = im(\varphi_P)$ is also similar.

Through the neighborhood system of P, we can obtain a positive set

$$\{ker\varphi(U)\}_{P\in U}.$$

And we have

$$(ker\varphi)_P = \varinjlim_{U\in X.\mathscr{P}} ker\varphi(U).$$

For any $x \in ker(\varphi_P)$, there exist a finite number of coordinates that are non-zero, that means $\exists \{U_i\}_{i\in I}, \exists t_{U_i} \in \mathscr{F}(U_i)$,

$$\varphi_P(x) = \varinjlim_{U\in X.\mathscr{P}} \varphi(U)(t_U) = 0,$$

which $\exists W \in X.\mathscr{U}(P), \forall i \in I, W \subseteq U_i$,

$$\sum_{i\in I} \rho'_{U_iW}(\varphi(U_i)(t_{U_i})) = 0 \Rightarrow \exists t' \in \mathscr{F}(W) \cap ker\varphi(W), t' = \sum_{i\in I} \rho_{U_iW}(t_{U_i}).$$

For any $x \in (ker\varphi)_P$,there exists a finite number of coordinates that non-zero, that $\exists \{U_i\}_{i\in I}, \exists t_{U_i} \in ker\varphi(U_i)$,

$$x = \varinjlim_{U\in X.\mathscr{P}} t_{U_i}$$

$(ker\varphi)_P \subseteq ker(\varphi_P)$ is obvious, because each coordinate is in the kernel and remains in ker under restriction. On the other hand, $\forall x \in ker(\varphi_P), \exists W \in X.\mathscr{U}(P), \exists t \in \mathscr{F}(W)$,

$$x \sim [t, W] \in (\ker\varphi)_P.$$

So the equation holds.

For the case of im, the discussion is similar to that of ker. Although im is the sheafification of the presheaf obtained from the morphism, according to the construction, there is no change in the local properties compared to before sheafification.(NOTICE:$\mathscr{F}, \mathscr{G}$ do not necessarily have to be sheaves, but the subsequent problems rely on the properties of sheaves.) □

(b)Using the conclusion from the previous question, the $(\Rightarrow)$ follows naturally for both cases. Conversely, using Proposition 1.1 in the book and the existence of the natural map

$$i : ker\varphi \to \mathscr{F}, j : im\varphi \to \mathscr{G},$$

the proposition holds. □

(c)Using the conclusion from the previous question and definition of exact sequence,

$$\begin{aligned} exact &\Leftrightarrow im\varphi_{i-1} = ker\varphi_i \\ &\Leftrightarrow \forall P \in X, im((\varphi_{i-1})_P) = (im\varphi_{i-1})_P = (ker\varphi_i)_P = ker((\varphi_i)_P) \end{aligned}$$

□

Exercise 2.1.3. Proof:

(a)

$(\Rightarrow)$ Taking into account the construction of the sheafification, if φ is surjective, then $\forall U \in X.\tau, \forall s \in \mathscr{G}(U)$, we can find an element in $im\varphi^+(U)$, that is $\forall P \in U, \exists V_P \in U.\mathscr{U}(P), \exists s_{V_P} \in im(\varphi(V)), \forall Q \in V, \exists W \in V.\mathscr{U}(Q)$,

$$s|_W = s_{V_P}|_W.$$

Obviously, $\exists t_{W_P} \in \mathscr{F}(W_P)$ satisfies $\varphi(V)(t_{W_P}) = s_{V_P}|_{W_P}$ and $\{W_P\}_{P\in U} \in U.$**cover**,

$$\varphi(V_P)(t_{W_P}) = s|_{W_P}$$

$(\Leftarrow)$ $\forall P \in X$, we have $U \in X.\tau, U_i \subseteq U, t_i \in \mathscr{F}(U_i)$, s.t.

$$\varphi(U_i)(t_i) = s|_{U_i}.$$

Then $[t_i, U_i] \in \mathscr{F}_P$ is the one of the preimage of $[s, U_i] \in \mathscr{G}_P$. □

(b) omitted □

Exercise 2.1.4. Proof

(a)

The text discusses injective morphisms between sheaves, while this question is equivalent to asking whether if the morphism between presheaves is injective, then does it induce an injective morphism between sheaves. Since they are sheaves, we can verify it according to the definition of injective morphisms between sheaves. So $\forall U \in X.\tau$, we only need to verify that

$$ker(\varphi^+(U)) = 0$$

If not, $\exists x \in \mathscr{F}^+(U), x \neq 0$, that means $\exists P \in U, \exists V \in U.\mathscr{U}(P), \exists x_V \in \mathscr{F}(V) - \{0\}, x_V = x|_V$,

$$\varphi(V)(x_V) = 0$$

which contradicts the assumption of injectivity. □

(b)

According to the definition, we need to check two points:

- $\forall U \in X.\tau, (im\varphi)(U)$ is the subgrouop of $\mathscr{G}(U)$
- Restriction of $im\varphi$ is induced by $\mathscr{G}$

For the second point is obvious by the propertiy of morphism of sheaves. For the first point, the main concern is whether the elements obtained through sheafification can still be represented as sections over some open sets (as element in a group), that is $\forall U \in X.\tau, \forall x \in (im\varphi)(U) \Rightarrow x \in \mathscr{G}(U)$?

$\forall P \in U, \exists V \in U.\mathscr{U}(P), \exists x_V \in im(\varphi(V)) \subseteq \mathscr{G}(V), \forall Q \in V, \exists W_Q \in V.\mathscr{U}(Q)$,

$$x|_{W_Q} = x_V|_{W_Q}.$$

Under the action of restriction maps of sheaf, we have $\forall V_1, V_2 \in U.\mathscr{U}(Q), W_Q \subseteq V_1 \cap V_2$,

$$x|_{W_Q} = x_{V_1}|_{W_Q} = x_{V_2}|_{W_Q}$$

By the properties of sheaves, we can obtain that $x \in \mathscr{G}(\mathscr{U})$ with $x|_V = x_V$. (Without sheaf, then this proposition **hardly holds**!) $\square$

Exercise 2.1.5. Proof:

For morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ or $\varphi \in \mathbf{Mor}_{Shf}(\mathscr{F}, \mathscr{G})$,

$$\begin{aligned}\varphi \in \mathbf{Iso} &\Leftrightarrow \forall P \in X, \varphi_P \in \mathbf{Iso}\\ &\Leftrightarrow \varphi_P \in \mathbf{Inj} \cap \mathbf{Sur}\\ &\Leftrightarrow (ker\varphi)_P = ker(\varphi_P) = 0, (im\varphi)_P = im(\varphi_P) = \mathscr{G}_P\\ &\Leftrightarrow \varphi \in \mathbf{Inj} \cap \mathbf{Sur}\end{aligned}$$

$\square$

Exercise 2.1.6. Proof:

(a)

we denote the natural map

$$i : \mathscr{F} \to \mathscr{F}/\mathscr{F}',$$

with $\forall U \in X.\tau$,

$$\begin{aligned} i(U) : \mathscr{F}(U) &\to \mathscr{F}(U)/\mathscr{F}'(U)\\ a &\mapsto \bar{a}\end{aligned}$$

After sheafification, $\forall \bar{y} \in (\mathscr{F}/\mathscr{F}')(U)$, we can find $\{U_\lambda\}_{\lambda\in\Lambda} \in U.\mathbf{cover}, \exists \bar{y}_\lambda \in \mathscr{F}(U)/\mathscr{F}'(U)$, that

$$\bar{y}|_{U_\lambda} = \bar{y}_\lambda.$$

And obviously $i(U_\lambda) \in \mathbf{Mor}_{Rng}(\mathscr{F}(U_\lambda), \mathscr{F}(U_\lambda)/\mathscr{F}'(U_\lambda)).\mathbf{Sur}$. So $\exists y_\lambda \in \mathscr{F}(U_\lambda)$, and this follows the definition of surjective of sheavf morphisms.

(ps: $U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$ is presheaf, $U \mapsto (\mathscr{F}/\mathscr{F}')(U)$ is sheaf associated before presheaf. In detail, these two cannot be equated.)

Besides, sheafification doesn't change zero element (because zero element exists uniquely). So the kernel of i is a sheaf that $U \mapsto ker(i(U))$. Next, we need to check that after sheafification, image of $\mathscr{F}'$ is only zero. That's obviously following the property of sheaf (Only the elements in the section of $\mathscr{F}'$ will contribute to the zero element.). $\square$

(b)

We know that im is a subsheaf, and we only need to show that $\mathscr{F}'$ is isomorphic to im. The surjectivity naturally holds, so we only need to demonstrate injectivity. Due to the exactness, injectivity holds. Therefore, $\mathscr{F}'$ is isomorphic to the sublayer. We may denote this sublayer as $\mathscr{F}'$ for convenience.

We denote $\varphi \in \mathbf{Mor}_{Shf}(\mathscr{F}, \mathscr{F}'').\mathbf{Sur}$ with $ker\varphi = \mathscr{F}'$. This is equivalent to stating that

$$im\varphi/ker\varphi \cong \mathscr{F}''.$$

It can be proved by check $\forall P \in X$,

$$im(\varphi_P)/ker(\varphi_P) \cong (im\varphi)_P/(ker\varphi)_P \cong (im\varphi/ker\varphi)_P \cong \mathscr{F}''_P.$$

This is true because of the exactness. $\square$

Exercise 2.1.7. Proof:

(a)

First, both the image and the quotient are sheaves associated with the presheaf. And it is obviously that

$$\forall U \in X.\tau, (\mathscr{F}(U)/ker\varphi(U) \to im(\varphi(U))) \in \mathbf{Inj}$$

So $(\mathscr{F}/ker\varphi \to im\varphi) \in \mathbf{Inj}$. The surjectivity property is obvious.. In conclusion,

$$im\varphi \cong \mathscr{F}/ker\varphi$$

□

(b)

Same as above. □

Exercise 2.1.8. Proof:

We denote

$$0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$$

Then, we need to proof:

- $\varphi(U) \in \mathbf{Inj}$
- $im(\varphi(U)) = ker(\psi(U))$

The first one is obvious by definition. For the second, we have

$$im(\varphi(U)) \subseteq (im\varphi)(U) = (ker\psi)(U) = (ker\psi(U)).$$

So we only need to prove the converse case. $\forall x \in (ker\psi)(U), \exists \{U_\lambda\}_{\lambda\in\Lambda} \in U.\mathbf{cover}, \exists t_\lambda \in \mathscr{F}'(U_\lambda)$,

$$\varphi(U_\lambda)(t_\lambda) = x|_{U_\lambda}, \psi(U_\lambda)(\varphi(U_\lambda)(t_\lambda)) = \psi(U_\lambda)(x|_{U_\lambda}) = 0.$$

Then we have a commutative diagram

$$\begin{tikzcd}
\mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}''(U) \arrow[d] \\
\mathscr{F}'(U_\lambda) \arrow[r] & \mathscr{F}(U_\lambda) \arrow[r] & \mathscr{F}''(U_\lambda)
\end{tikzcd}$$

In order to get an element in $im(\varphi(U))$, we should glue all the elements $\{t_\lambda\}_{\lambda\in\Lambda}$ from each U_λ to U. Thus we need to check that $\forall \lambda_1, \lambda_2 \in \Lambda, \lambda_1 \neq \lambda_2$,

$$t_{\lambda_1}|_{U_{\lambda_1}\cap U_{\lambda_2}} = t_{\lambda_2}|_{U_{\lambda_1}\cap U_{\lambda_2}}.$$

Due to φ is injective for every open set, then there exists a unique element satisfying that

$$\varphi^{-1}(U_{\lambda_1} \cap U_{\lambda_2})(s|_{U_{\lambda_1}\cap U_{\lambda_2}}) = t_{\lambda_1}|_{U_{\lambda_1}\cap U_{\lambda_2}} = t_{\lambda_2}|_{U_{\lambda_1}\cap U_{\lambda_2}}.$$

That means $\exists t \in \mathscr{F}'(U), \forall \lambda \in \Lambda, t|_{U_\lambda} = t_\lambda$, and $\psi(U)(\varphi(U)(t)) = 0$. So $im(\varphi(U)) = ker(\psi(U))$. □

(NOTE: $im(\varphi(U))$ is different $(im\varphi)(U)$, because im is the sheafification of $U \mapsto im(\varphi(U))$. Generally speaking, in the sense of bijection, $(im\varphi)(U)$ is larger than $im(\varphi(U))$.)

Exercise 2.1.9. Proof:

We will proof that $U \mapsto \mathscr{F} \oplus \mathscr{G}$ is a sheaf. Here, we will provide the proof from two points as defined in the book.

- $\forall U \in X.\tau, \forall \{U_\lambda\}_{\lambda\in\Lambda} \in U.\mathbf{cover}, \forall (f,g) \in \mathscr{F}(U) \oplus \mathscr{G}(U)$, if $\forall \lambda \in \Lambda, (f,g)|_{U|\lambda} = (0,0)$, that means $\forall \lambda \in \Lambda f|_{U_\lambda} = 0, g|_{U_\lambda} = 0$, then $f = 0, g = 0$.
$$(f,g) = (0,0)$$

- $\forall U \in X.\tau, \forall \{U_\lambda\}_{\lambda\in\Lambda} \in U.$**cover**. If we have $(f_\lambda, g_\lambda) \in \mathscr{F}(U_\lambda) \oplus \mathscr{G}(U_\lambda)$ and $\forall \lambda_1, \lambda_2 \in \Lambda, \lambda_1 \neq \lambda_2$, we have $(f_{\lambda_1}, g_{\lambda_1})|_{U_{\lambda_1 \cap \lambda_2}} = (f_{\lambda_2}, g_{\lambda_2})|_{U_{\lambda_1 \cap \lambda_2}}$, that means $f_{\lambda_1}|_{U_{\lambda_1\cap\lambda_2}} = f_{\lambda_2}|_{U_{\lambda_1\cap\lambda_2}}, g_{\lambda_1}|_{U_{\lambda_1\cap\lambda_2}} = g_{\lambda_2}|_{U_{\lambda_1\cap\lambda_2}}$, so we have $f \in \mathscr{F}(U), g \in \mathscr{G}(U), f|_{U_\lambda} = f_\lambda, g|_{U_\lambda} = g_\lambda$. That means we obtain

$$(f, g) \in \mathscr{F}(U) \oplus \mathscr{G}(U), (f, g)|_{U_\lambda} = (f_\lambda, g_\lambda)$$

For category part, we need to prove two points as below:

- $\exists i \in \mathbf{Mor}(\mathscr{F}, \mathscr{F} \oplus \mathscr{G}).\mathbf{Inj}, j \in \mathbf{Mor}(\mathscr{G}, \mathscr{F} \oplus \mathscr{G}).\mathbf{Inj}$
- $\forall \mathscr{F}' \in \mathbf{Sch}(X, \mathbf{Grp}), \forall \alpha \in \mathbf{Mor}(\mathscr{F}, \mathscr{F}'), \forall \beta \in \mathbf{Mor}(\mathscr{G}, \mathscr{F}'), \exists! \gamma \in \mathbf{Mor}(\mathscr{F} \oplus \mathscr{F}, \mathscr{F}`)$, that

$$\gamma \circ i = \alpha,$$

$$\gamma \circ j = \beta,$$

For the first one, we can give the difinition of i, and j is similar.

$$\begin{aligned} i(U) : \mathscr{F}(U) &\to \mathscr{F}(U) \oplus \mathscr{G}(U) \\ a &\mapsto (a, 0) \end{aligned}$$

with diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \oplus \mathscr{G}(U) \arrow[d] \\ \mathscr{F}(V) \arrow[r] & \mathscr{F}(V) \oplus \mathscr{G}(V) \end{tikzcd}$$

with any $V \subseteq U$. For j we have similar result

$$\begin{aligned} j(U) : \mathscr{G}(U) &\to \mathscr{F}(U) \oplus \mathscr{G}(U) \\ b &\mapsto (0, b) \end{aligned}$$

with diagram

$$\begin{tikzcd} \mathscr{G}(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \oplus \mathscr{G}(U) \arrow[d] \\ \mathscr{G}(V) \arrow[r] & \mathscr{F}(V) \oplus \mathscr{G}(V) \end{tikzcd}$$

For the second one, it can be verified by checking over every open sets. $\forall U \in X.\tau$, we let

$$\gamma(U)(a, b) = \gamma(U)(i(U)(a) + j(U)(b)) = \alpha(U)(a) + \beta(U)(b).$$

It exists uniquely by group theory. We have commutative diagram,

$$\begin{tikzcd} & & \mathscr{F}(U) \text{ or } \mathscr{G}(U) \arrow[dll] \arrow[dl] \arrow[ddd] \\ (\mathscr{F} \oplus \mathscr{G})(U) \arrow[r, "\exists!"'] \arrow[d] & \mathscr{F}'(U) \arrow[d] & \\ (\mathscr{F} \oplus \mathscr{G})(V) \arrow[r, "\exists!"] & \mathscr{F}'(V) & \\ & & \mathscr{F}(V) \text{ or } \mathscr{G}(V) \arrow[ull] \arrow[ul] \end{tikzcd}$$

So $\Gamma \in \mathbf{Mor}_{Shf}$. $\square$

Exercise 2.1.10. Proof:

Denote that $\varphi_i : \mathscr{F}_i \to \mathscr{G}$, canonical map $\psi_i : \mathscr{F}_i \to \varinjlim_{i\in I} \mathscr{F}_i$ and transition map $f_{ij} : \mathscr{F}_i \to \mathscr{F}_j, j \geq i, \forall i, j \in I$ where I is direct set. For $\varinjlim_{i\in I} \mathscr{F}_i$, it can be viewed as the direct sum of $\mathscr{F}_i$, followed by taking the quotient by the equivalence relation induced by f_{ij}.

Generally speaking, it is similar to the previous question (as the direct sum appears in both constructions), but the definition of the positive limit sheaf involves a sheafification process. Therefore, before sheafification, $\forall U \in X.\tau$,we have

$$\forall i \in I, \varphi_i^-(U) : \mathscr{F}_i(U) \xrightarrow{\psi_i^-(U)} \varinjlim_{i\in I}(\mathscr{F}_i(U)) \xrightarrow{\exists! \gamma^-(U)} \mathscr{G}(U).$$

from the universal property of group direct limit. When sheafifying $\varinjlim_{i\in I} \mathscr{F}_i$, the uniqueness of γ is determined, and this is the universal property of sheafification. Next, here is a commutative diagram naturally

$$\begin{tikzcd}
\mathscr{F}_i \arrow[r, "\psi_i^-"] \arrow[rr, bend left=80, "\varphi_i"] \arrow[dr, "\psi_i"'] & (\varinjlim_{i\in I} \mathscr{F}_i)^- \arrow[r, "\gamma^-"] \arrow[d, "\theta_i"] & \mathscr{G} \\
& \varinjlim_{i\in I} \mathscr{F}_i \arrow[ur, "\exists!\gamma"'] &
\end{tikzcd}$$

□

Exercise 2.1.11. Proof:

We only need to prove the properties of sheaves in the order presented in the book.

- $\forall U \in X.\tau, \forall \{U_\lambda\}_{\lambda\in\Lambda} \in U.$**cover**$, \forall s \in \varinjlim_{i\in I} \mathscr{F}_i(U)$,
$$\forall \lambda \in \Lambda, s|_{U_\lambda} = 0 \Rightarrow s = 0.$$
- $\forall U \in X.tau, \forall \{U_\lambda\}_{\lambda\in\Lambda} \in U.$**cover**,
$$\forall \lambda, \lambda_1, \lambda_2 \in \Lambda, \lambda_1 \neq \lambda_2, \exists s_\lambda \in \varinjlim_{i\in I} \mathscr{F}_i(U_\lambda), s_{\lambda_1}|_{U_{\lambda_1}\cap U_{\lambda_2}} = s_{\lambda_2}|_{U_{\lambda_1}\cap U_{\lambda_2}}$$
$$\Rightarrow \exists s \in \varinjlim_{i\in I} \mathscr{F}_i(U), \forall \lambda \in \Lambda, s|_{U_\lambda} = s_\lambda$$

Because of noetherian topological, we can assume the cover is finite $\{U_k\}_{k=1}^N$. For the first assertion, we can denote $s = \sum_{j=1}^M \bar{s}_j$, where $\bar{s}_j = \psi_j(U)(s_j), s_j \in \mathscr{F}_j(U)$. $\forall k \in \{1, 2, ..., N\}$, if $s|_{U_k} = 0$,

$$\Rightarrow \exists j = j_k, s_{j_k} \in \mathscr{F}_{j_k}(U_k), \bar{s}_{j_k} = (\sum_{j=1}^M \bar{s}_j)|_{U_k} = \sum_{j=1}^M (\bar{s}_j|_{U_k}) = 0$$

Because of finiteness, we can selec $m \in I, m \geq j_k$ as $\forall k$ go over $\{1, 2, ..., N\}$. In order to distinguish different open set, we denote $m = m_k$ that

$$\bar{s}_{j_k} = \bar{s}_{m_k},$$

where $s_{m_k} \in \mathscr{F}_{m_k}(U_k)$. Due to $\{\mathscr{F}_i\}$ is the set of sheaves. So there must be an element $s_m \in \mathscr{F}_m(U), s_m = 0$. By the equivalence of direct limits, we can obtain

$$s = \bar{s}_m = 0.$$

For the second one, similarly, due to finiteness, we can reduce the problem to an index $m = m_k \in I$ which is big enough and we only need to consider the sheaf properties of $s_{m_k} \in \mathscr{F}_{m_k}(U_k), s_k = \bar{s}_{m_k}$. The process is roughly the same as before. □

Exercise 2.1.12. Proof:

Following the same approach as the previous question, assume $s|_{U_\lambda} = 0 \in \varprojlim_{i\in I}(\mathscr{F}_i(U_\lambda))$, then every coordinate is 0. That's means every coordinate of $s \in \varprojlim_{i\in I}(\mathscr{F}_i(U))$ is 0 (By verifying each index's s_i). And the second point could be proved as above.

(NOTE: In inverse limite case, each coordinate of any element is effectively concatenated by the transition maps of the inverse system. However, in the case of direct limit, each element can only be determined by an element in a larger indexed group, and when restricting to different open sets, this index may change.)

Exercise 2.1.13. Proof:

We only need to analyze $Spe(\mathscr{F})$. Consider continuous sections of $Spe(\mathscr{F})$ over U, they have form like $s \in \mathscr{F}(U), \bar{s}$ is continuous. That's mean, $\forall V' \in Spe(\mathscr{F}).\tau \Rightarrow \bar{s}^{-1}(V') \in U.\tau$. And we know that

$$V' = \{s'_q : s' \in \mathscr{F}(U), q \in V\} = \{s'_q : s' \in \mathscr{F}(U), q \in V, s'_q = s_q\}.$$

So, $V \in U.\tau$. That means, for any continuous section $\bar{s}, \forall p \in U, \exists V \in U.\mathscr{U}(P), \forall Q \in V, \bar{s}(Q) = s'_Q$. That follows the definition of sheafification.

Based on the previous discussion and the universal property of sheafification, the proposition can be proved. $\square$

Exercise 2.1.14. Proof:

$\forall p \notin Supp(s), s_p = 0, \Rightarrow \exists V \in U.\mathscr{U}(P), \rho_{UV}(s) = 0 \in \mathscr{F}(V)$.

$$\Rightarrow Supp(s)^c \in U.\tau^c \Leftrightarrow Supp(s) \in U.\tau.$$

We consider $Supp(\mathscr{F})^c$. $\forall p \in Supp(\mathscr{F})^c, \forall U \in X.\mathscr{U}(p), \forall s \in \mathscr{F}(U), p \in Supp(s)^c$,

$$\Rightarrow p \in \bigcap_{U\in X.\mathscr{U}(p)} \bigcap_{s\in\mathscr{F}(U)} Supp(s)^c.$$

$Supp(s)^c \in X.\tau$ and it is difficult to ensure that there are only finitely many terms. Therefore, it may not necessarily be an open set or a closed set. $\square$

Exercise 2.1.15. Proof:

First, let's determine the form of each section. $\forall \varphi \in Hom(\mathscr{F}|_U, \mathscr{G}|_U)$, it has $\forall V \in U.\tau, \forall s \in \mathscr{F}|_U(V) = \mathscr{F}(V), \exists! t \in \mathscr{G}|_U(V) = \mathscr{G}(V)$,

$$\varphi(V)(s) = t.$$

Therefore, we can provide the definition of addition.

$$\forall \varphi, \psi \in Hom(\mathscr{F}|_U, \mathscr{G}|_U), \forall V \in U.\tau, \forall s \in \mathscr{F}(V),$$

$$(\varphi + \psi)(V)(s) = (\varphi(V) + \psi(V))(s) = \varphi(V)(s) + \psi(V)(s),$$

$$\Rightarrow \varphi + \psi \in Hom(\mathscr{F}|_U, \mathscr{G}|_U).$$

And zero element is that $\forall s \in \mathscr{F}(V), 0(V)(s) = 0$.

For the sake of convenience, we will refer to the open sets in the open cover of U as V.

- If $\varphi|_V = 0$
  $$\Rightarrow \forall W \in V.\tau, \forall s \in \mathscr{F}(W), \varphi|_V(W)(s) = 0.$$
  This directly satisfies the definition of the zero element.
- If $\exists \varphi_V \in Hom(\mathscr{F}|_V, \mathscr{G}|_V), \forall V' \neq V, \varphi_V|_{V\cap V'} = \varphi_{V'}|_{V\cap V'}$. We need to construct such a correspondence through mapping definitions. $\forall W \in U.\tau, \forall s \in \mathscr{F}(W)$, we can consider its restriction $s|_V$ (Here, we assume that V is contained in W as cover.). Through φ_V, there exists a unique t_V that corresponds to it. $\mathscr{G}$ is a sheaf, so $\exists! t_V \in \mathscr{G}(V)$ and we can define a morphism $\varphi(V)(s|_V) = t_V$. According to the assumption, there is no ambiguity in the image of the intersection, so
  $$\exists \varphi \in Hom(\mathscr{F}|_U, \mathscr{G}|_U), \forall V, \varphi|_V = \varphi_V.$$

It is a sheaf. $\square$

Exercise 2.1.16. Proof:

(a) Every irreducible topological space is connected. So $\forall U \in X.\tau, \mathscr{A}(U) \cong A$. $\forall V \in U.\tau, \forall s \in \mathscr{A}(V)$,

$$\exists! a \in A, \forall p \in V_|, s(p) = a.$$

We can construct a continuous map that $s', \forall p \in U, s'(p) = a, s' \in \mathscr{A}(U)$ and it simultaneously satisfies the condition that the restriction equals s. So it is flasque. □

(b) We denote that

$$0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0.$$

According to the previous work, we only to prove that $\forall U \in X.\tau, \psi(U) \in \mathbf{Mor}.\mathbf{Sur}$. $\psi \in \mathbf{Mor}.Since\mathbf{Sur}$, we have

$$\forall t \in \mathscr{F}''(U), \exists \{U_\lambda\}_{\lambda \in \Lambda} \in U.\mathbf{cover}, \exists s_\lambda \in \mathscr{F}(U_\lambda), \psi(U_\lambda)(s_\lambda) = t|_{U_\lambda}$$

In order to find $s \in \mathscr{F}(U), \psi(U)(s) = t$, we first make sure that

$$s_\lambda|_{U_{\lambda\mu}} = s_\mu|_{U_{\lambda\mu}}, U_{\lambda\mu} = U_\lambda \cap U_\mu.$$

Otherwise,

$$0 \neq s_\lambda|_{U_{\lambda\mu}} - s_\mu|_{U_{\lambda\mu}} \in ker\psi(U_{\lambda_\mu}).$$

Due to exactness, we have

$$\exists r_{\lambda\mu} \in \mathscr{F}'(U_{\lambda\mu}), \varphi(U_{\lambda\mu})(r_{\lambda\mu}) = s_\lambda|_{U_{\lambda\mu}} - s_\mu|_{U_{\lambda\mu}}.$$

Due to flasque, we have $\exists r_\lambda \in \mathscr{F}'(U_\lambda)$. We can let

$$s'_\lambda = s_\lambda - \varphi(U_\lambda)(r_\lambda).$$

Then $\forall \mu, \lambda, s'_\lambda|_{U_{\lambda\mu}} - s_\mu|_{U_{\lambda\mu}} = 0$.

Next, we need to find a (U_λ, s'_λ) as U_λ big enough. We can give a partial order set $\mathscr{T} = \{(U_\lambda, s_\lambda) : s \in \mathscr{F}(U)\}$. According to Zorn lemma, $\exists (U^*, s^*)$ as maximal element. Replace (U_λ, s_λ) with $\exists (U^*, s^*)$. We can get a bigger (U', t') by patching $s'_\lambda|_{U^* \cap U_\mu}, U_\lambda \cap U^* \neq \emptyset$. Finally, we can get $s \in \mathscr{F}(U)$ by repeating above process that is what we want. □

(c) According to above discussion, $\forall U \in X.\tau, \psi(U) \in \mathbf{Mor}.\mathbf{Sur}$, then $\forall V \in U.\tau, \forall t \in \mathscr{F}''(V), \exists s \in \mathscr{F}(V)$,

$$\psi(V)(s) = t.$$

Because $\mathscr{F}$ is flasque, then $\exists s' \in \mathscr{F}(U), s'|_V = s$. So, $\exists! t' \in \mathscr{F}(U), \psi(U)(s') = t', t'|_V = t$. □

(d) We denote $\forall U \in X.\tau, \forall V \in U.\tau, \rho : \mathscr{F}(U) \to \mathscr{F}(V) \in \mathbf{Sur}$. If $\forall U' \in Y.\tau, \forall V' \in U'.\tau, f^{-1}(V') \in f^{-1}(U').\tau$, so

$$f_*\mathscr{F}(U') \to f_*\mathscr{F}(V') \in \mathbf{Sur}.$$

□

(e) We need to show that $\forall U \in X.\tau, \forall V \in U.\tau, \mathscr{G}(U) \to \mathscr{G}(V) \in \mathbf{Sur}$. So

$$\forall s \in \mathscr{G}(V), s : V \to \bigcup_{p \in V} \mathscr{F}_p.$$

We can consturct a map that

$$s'(p) = \begin{cases} s(p) & \text{if } p \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $s' : U \to \bigcup_{p \in U} \mathscr{F}_p, s \in \mathscr{G}(U)$. □

Exercise 2.1.17. Proof:

We need to verify $\varinjlim_{U \in X.\mathscr{U}(Q)} i_P(A)(U)$ in 2 cases. When $Q \in \overline{\{P\}}, \forall U \in X.\tau$, we can assert that $Q \in U \Rightarrow P \in U$. Otherwise, $P \in U^c, \overline{\{P\}} \neq \overline{\{P\}} \cap U^c \in X.\tau^c$. Thus

$$Q \in \overline{\{P\}}, \varinjlim_{U \in X.\mathscr{U}(Q)} i_P(A)(U) = \varinjlim_{U \in X.\mathscr{U}(P)} i_P(A)(U) = A,$$

$$Q \notin \overline{\{P\}}, \varinjlim_{U \in X.\mathscr{U}(Q)} i_P(A)(U) = \varinjlim_{U \in \overline{\{P\}}.\tau} i_P(A)(U) = 0.$$

Next,

$$\forall U \in X.\mathscr{U}(P), i_*(\mathscr{A})(U) = \mathscr{A}(i^{-1}(U)) = \mathscr{A}(U \cap \overline{\{P\}}) = A,$$
$$\forall U \in \overline{\{P\}}^c.\tau, i_*(\mathscr{A})(U) = \mathscr{A}(i^{-1}(U)) = \mathscr{A}(\emptyset) = 0.$$

This is consistent with the definition of $i_P(A)$. □

Exercise 2.1.18. Proof:

We determine the form of $f^{-1}f_*\mathscr{F}$ on each $U \in X.\tau$.

$$\begin{aligned} f^{-1}f_*\mathscr{F}(U) &= \varinjlim_{V \in Y.\mathscr{U}(f(U))} (f_*\mathscr{F})(V) \\ &= \varinjlim_{V \in Y.\mathscr{U}(f(U))} \mathscr{F}(f^{-1}(V)) \end{aligned}$$

So we have $\forall x \in (f^{-1}f_*\mathscr{F})(U), \exists U' \in X.\mathscr{U}(U), x' \in \mathscr{F}(U')$,

$$x \sim [x'].$$

This is because we cannot make sure $\exists W \in Y.\tau, f^{-1}(W) = U$. Naturally, we can give a map that

$$\begin{array}{ccc} (f^{-1}f_*\mathscr{F})(U) & \to & \mathscr{F}(U) \\ x & \mapsto & x'|_U \end{array}$$

Then the commutative diagram is hold by composition of restriction,

$$\begin{array}{ccc} (f^{-1}f_*\mathscr{F})(U) & \longrightarrow & \mathscr{F}(U) \\ \downarrow & & \downarrow \\ (f^{-1}f_*\mathscr{F})(V) & \longrightarrow & \mathscr{F}(V) \end{array}$$

So the morphism of sheaves exists naturally (Actually it is larger than the original).

Next we need to determine the sheaf $f_*f^{-1}\mathscr{G}$ on each $U \in X.\tau$,

$$\begin{aligned} (f_*f^{-1}\mathscr{G})(U) &= (f^{-1}\mathscr{G})(f^{-1}(U)) \\ &= \varinjlim_{V \in Y.\mathscr{U}(f \circ f^{-1}(U))} \mathscr{G}(V) \\ &= \varinjlim_{V \in Y.\mathscr{U}((U))} \mathscr{G}(V) \\ &= \mathscr{G}(U). \end{aligned}$$

So we can let identity map for this.

First, we check $Hom_X(f^{-1}\mathscr{G}, \mathscr{F}) \subseteq Hom_Y(\mathscr{G}, f_*\mathscr{F})$. $\forall \varphi \in Hom_X(f^{-1}\mathscr{G}, \mathscr{F})$,

$$\varphi'(U) : \mathscr{G}(U) \to f_*f^{-1}\mathscr{G}(U) = (f^{-1}\mathscr{G})(f^{-1}(U)) \xrightarrow{\varphi(f^{-1}(U))} \mathscr{F}(f^{-1}(U)) = f_*\mathscr{F}(U).$$

Conversely, $\forall \psi \in Hom_Y(\mathscr{G}, f_*\mathscr{F})$,

$$\psi'(U) : f^{-1}\mathscr{G}(U) \to \varinjlim_{V \in X.\mathscr{U}(f(U))} \mathscr{G}(V) \xrightarrow{\varinjlim_V \psi(V)} \varinjlim_{V \in X.\mathscr{U}(f(U))} f_*\mathscr{F}(V) = f^{-1}f_*\mathscr{F}(U) \to \mathscr{F}(U).$$

For bijectivity, after the above discussion, we can obtain the following commutative diagram

$$\varphi''(U) : f^{-1}\mathscr{G}(U) \to \mathscr{F}(U)$$

$$\begin{tikzcd} f^{-1}\mathscr{G}(U) \arrow[dd] \arrow[dr, dashed] & & & \varinjlim_{V \in X.\mathscr{U}(U)} f_*\mathscr{F}(V) \arrow[dd] \\ & \varinjlim_{V \in X.\mathscr{U}(U)} (f^{-1}\mathscr{G})(f^{-1}(V)) \arrow[r, "\varphi'(U)"] & \varinjlim_{V \in X.\mathscr{U}(U)} \mathscr{F}(f^{-1}(V)) \arrow[ur] \arrow[dr, dashed] & \\ \varinjlim_{V \in X.\mathscr{U}(U)} \mathscr{G}(V) \arrow[ur] & & & \mathscr{F}(U) \end{tikzcd}$$

By chasing every element over U, we can demonstrate that $\varphi = \varphi''$. □

Exercise 2.1.19. Proof:

(a)

$$(i_*\mathscr{F})_p = \varinjlim_{V\in X.\mathscr{U}(p)} (i_*\mathscr{F})(V) = \varinjlim_{V\in X.\mathscr{U}(p)} (\mathscr{F})(i^{-1}(V)) \begin{cases} \varinjlim_{V\in Z.\mathscr{U}(p)}(\mathscr{F})(V) = \mathscr{F}_p, & \text{if } p \in U, \\ \varinjlim_{\emptyset}(\mathscr{F})(\emptyset) = 0, & \text{if } p \notin U. \end{cases}$$

□

(b) If $p \in U$, there is always an open set that $W \in X.\mathscr{U}(p), W \subseteq U$,

$$(j_!\mathscr{F})_p = \varinjlim_{V\in X.\mathscr{U}(p)} (j_!\mathscr{F})(V) = \varinjlim_{V\in W.\mathscr{U}(p)} (j_!\mathscr{F})(V) = \varinjlim_{V\in W.\mathscr{U}(p)} \mathscr{F}(V) = \mathscr{F}_p.$$

Otherwise, there is always an open set that $W \in X.\mathscr{U}(p), W \nsubseteq U$

$$(j_!\mathscr{F})_p = \varinjlim_{V\in X.\mathscr{U}(p)} (j_!\mathscr{F})(V) = \varinjlim_{V\in W.\mathscr{U}(p)} (j_!\mathscr{F})(V) = \varinjlim_{V\in W.\mathscr{U}(p)} 0 = 0.$$

□

(c) $\forall W \in X.\tau$, if $W \subset U$, we have

$$\Rightarrow 0 \to \mathscr{F}(W) \to \mathscr{F}(W) \to 0 \to 0.$$

Otherwise,

$$\Rightarrow 0 \to 0 \to \mathscr{F}(W) \to \mathscr{F}(W) \to 0.$$

Then we have defined the morphism of sheaves. Exactness can be verified by stalk of each point $p \in X$,

$$0 \to \mathscr{F}_p \to \mathscr{F}_p \to 0 \to 0, p \in U,$$

$$0 \to 0 \to \mathscr{F}_p \to \mathscr{F}_p \to 0, p \notin U,$$

So we have exact sequence

$$0 \to (j_!(\mathscr{F}|_U)) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$$

(NOTE:) □

Exercise 2.1.20. Proof:

We can give a brief definition

$$\Gamma_Z(X,\mathscr{F}) = \{s \in \Gamma(X,\mathscr{F}) : Supp(s) \subseteq Z\} \subseteq \Gamma(X,\mathscr{F}).$$

Actually, it is a subgroup. Because

- $Supp(0) = \emptyset \subseteq Z$,
- $\forall s_1, s_2 \in \Gamma_Z(X,\mathscr{F}), \forall p \notin Z, (s_1)_p = 0 = (s_2)_p \Rightarrow (s_1 - s_2)_p = 0 \Rightarrow Supp(s_1 + s_2) \subseteq Z.$

(a) Check the axioms of sheaf,

- $\forall U \in X.\tau, \forall \{U_\lambda\}_{\lambda\in\Lambda} \in U.\mathbf{cover}, \forall s \in \Gamma_{U\cap Z}(U, \mathscr{F}|_U), \forall \lambda \in \Lambda, s|_{U_\lambda} = 0,$

$$s = 0 \in \Gamma(U,\mathscr{F}|_U) \Rightarrow s \in \Gamma_{U\cap Z}(U,\mathscr{F}|_U).$$

- $\forall U \in X.\tau, \forall \{U_\lambda\}_{\lambda\in\Lambda} \in U.\mathbf{cover}, \forall \lambda, \exists s_\lambda \in \Gamma_{U_\lambda\cap Z}(U_\lambda, \mathscr{F}|_{U_\lambda}), \forall \lambda, \mu \in \Lambda, s_\lambda|_{U_{\lambda\mu}} = s_\mu|_{U_{\lambda\mu}}, U_{\lambda\mu} = U_\lambda \cap U_\mu,$

$$\exists s \in \Gamma(U,\mathscr{F}|_U), \forall \lambda \in \Lambda, s|_{U_\lambda} = s_\lambda.$$

$\forall p \notin U \cap Z, \forall \lambda \in \Lambda,$

$$s_p = s|_{U_\lambda p} = (s_\lambda)_p = 0,$$
$$\Rightarrow s \in \Gamma_{U\cap Z}(U,\mathscr{F}|_U).$$

The above existence of conclusion is mainly obtained by F being a sheaf. $\square$

(b) First, we should give morphisms of sheaves. If $W \subseteq U$,

$$0 \to 0 \to \mathscr{F}(U) \xrightarrow{\rho_{UW}} \mathscr{F}|_U(j^{-1}(W)) = \mathscr{F}((W)).$$

Otherwise,

$$0 \to \Gamma_{W\cap Z}(W, \mathscr{F}|_W) \to \Gamma(W, \mathscr{F}) \xrightarrow{\rho_{W,W\cap U}} \Gamma(W \cap U, \mathscr{F}|_U).$$

Exactness can be verified by stalk of each point, $\forall p \in X$, if $p \in Z$,

$$ker(\mathscr{H}_Z^0(\mathscr{F})_p \to \mathscr{F}_p) = 0,$$

if $p \notin Z$,

$$ker(\mathscr{H}_Z^0(\mathscr{F})_p \to \mathscr{F}_p) = ker(0 \to \mathscr{F}_p) = 0,$$

So injectivity is proved.

For middle exactness,$\forall p \notin Z$, the answer is obvious by check stalk.

If $\mathscr{F}$ is flasque, then $\forall V \subseteq U \in X.\tau, \rho_{UV} \in \mathbf{Sur}$, then $\varinjlim \rho \in \mathbf{Sur}$ as well at stalk of every point. $\square$

Exercise 2.1.21. Proof:

(a) According to the illustration,

$$\mathscr{I}_Y(U) = \left\{ \frac{f}{g} : f, g \in k[x_1, ..., x_n], f(Y) = \{0\}, 0 \notin g(U) \right\}.$$

In order to prove $\mathscr{I}_Y$ is a sheaf, actually, we only need to prove that if we get $x \in \mathscr{O}_X(U)$ from $x_i \in \mathscr{I}_Y(U_i)$ locally, then $x \in \mathscr{I}_Y(U)$. Suppose $\exists p \in Y, x(p) \neq 0$,

$$\exists U_i, p \in U_i, 0 \neq x(p)|_{U_i} = x_i(p) = \frac{f_i}{g_i}(p) = 0,$$

which is contradiction. $\square$

(b) $\forall U \in X.\tau$, if $U \cap Y = \emptyset$,

$$\varphi(U) : \mathscr{O}_X(U)/\mathscr{I}_Y(U) = 0 \cong 0 = \mathscr{O}_Y(\emptyset) = i_*\mathscr{O}_Y(U).$$

If $U \cap Y \neq \emptyset$,

$$\varphi(U) : \mathscr{O}_X(U)/\mathscr{I}_Y(U) \to i_*\mathscr{O}_Y(U),$$

$\varphi(U)\left(\frac{f}{g}\right) = \frac{f}{g}$. We can check it at each point by definition. $\square$

(c) $\Gamma(X, \mathscr{O}_X) = k, \Gamma(X, \mathscr{F}) = k \oplus k$. It cannot be surjective.

(d) Obviously, $\forall U \in X.\tau$, $\frac{f}{g} \in \mathscr{O}_X(U) \Rightarrow \frac{f}{g} \in \mathscr{K}(U)$.

We can get an exact sequence,

$$0 \to \mathscr{O}_X \to \mathscr{K} \to \mathscr{K}/\mathscr{O}_X \to 0.$$

Select each stalk of p, we will get

$$(\mathscr{K}/\mathscr{O}_X)_p \cong \mathscr{K}_p/\mathscr{O}_{p,X} = K/\mathscr{O}_p = I_p.$$

Patch these results of p, we will get $\sum_{p\in X} i_p(I_p)$. $\square$

(e) Omitted. $\square$

Exercise 2.1.22. Proof:

We should say that if $U_i \cap U_j = \emptyset, \varphi_{ij} = 0$. We can prove this proposition by construction. First we can define a presheaf,

$$U \mapsto \mathscr{F}(U) = \{a = (a^i)_{i\in I} \in \prod_{i\in I} \mathscr{F}_i(U_i \cap U) : a^i \sim a^j, \forall i, j \in I\},$$

which $\sim$ means $\forall a \in \mathscr{F}(U), \forall i, j \in I, \varphi_{ij}(U_i \cap U_j \cap U)(a^i) = a^j$. For reflexivity and symmetry, they hold due to the isomorphism of φ_{ij}. As for transitivity, it is established based on the condition that $\forall i, j, k \in I, \varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ holds. Then we need to verify it is a sheaf.

- $\forall U \in X.\tau, \forall a \in \mathscr{F}(U), \forall V \in U.\tau, a|_V = 0,$

$$\Rightarrow \forall i \in I, a^i|_V = 0 \Rightarrow a^i = 0 \Rightarrow a = 0.$$

- $\forall U \in X.\tau, \forall V_1, V_2 \in U.\tau, \exists a_1 \in \mathscr{F}(V_1), a_2 \in \mathscr{F}(V_2), a_1|_{V_1\cap V_2} = a_2|_{V_1\cap V_2}$,

$$\Rightarrow \forall i \in I, a_1^i|_{V_1\cap V_2} = a_2^i|_{V_1\cap V_2} \Rightarrow \exists a^i \in \mathscr{F}(U), a^i|_{V_1} = a_1^i.$$

We need to prove that such $a = (a^i) \in \mathscr{F}(U)$. According to the commutative diagram,

$$\begin{tikzcd}
\mathscr{F}_i(U\cap U_i\cap U_j) \arrow[r] \arrow[d] & \mathscr{F}_j(U\cap U_i\cap U_j) \arrow[d] \\
\mathscr{F}_i(V\cap U_i\cap U_j) \arrow[r] & \mathscr{F}_j(V\cap U_i\cap U_j)
\end{tikzcd}$$

we can say that $\forall a^i \sim a^j$, then $a \in \mathscr{F}(U)$. That is a sheaf. $\forall U \in U_i.\tau$, we define

$$\psi_i(U) = pr_i : \mathscr{F}(U) \xrightarrow{\cong} \mathscr{F}_i(U),$$

naturally we obtain

$$\psi_j = \varphi_{ij} \circ \psi_i.$$

$\square$

2.2 Schemes

Q 2.2.1. Give an example that is ring space but not a locally ring space.
Proof:
We just get a ring sheaf that $\exists p \in X, \mathscr{O}_{p,X} \notin \mathbf{Rng}.\mathbf{Local}.T$
Let $X = \{a, b\}, U = \{a\}, X.\tau = \{\emptyset, X, U\}$ and ring sheaf

$$\mathscr{O}(X) = k[x,y]_{(\mathfrak{p}_1)} \oplus k[x,y]_{(\mathfrak{p}_2)} = \mathscr{O}(U), \mathfrak{p}_1 \neq \mathfrak{p}_2,$$

$$\mathscr{O}(X) \xrightarrow{id} \mathscr{O}(U).$$

Obviously,

$$\mathscr{O}_{a,X} = \mathscr{O}_{b,X} = k[x,y]_{(\mathfrak{p}_1)} \oplus k[x,y]_{(\mathfrak{p}_2)} \notin \mathbf{Rng}.\mathbf{Local}.$$

$\square$

Exercise 2.2.1. Proof:
According to commutative algebra theory,

$$D(f) \cong Spec A_f,$$

because $\mathfrak{p} \leftrightarrow \mathfrak{p}A_f$ is bijective when $(f) \nsubseteq \mathfrak{p}$. $\forall g \in A - (f)$,

$$\mathscr{O}_X|_{D(f)}(D(g)) = \mathscr{O}_X(D(fg)) = A_{fg} = (A_f)_g.$$

So $(D(f), \mathscr{O}_X|_{D(f)}) \cong (Spec(A_f), \mathscr{O}_{Spec(A_f)})$. $\square$

Exercise 2.2.2. Proof:
Because X is a scheme, then $\forall p \in U \subseteq X, \exists A, Spec(A) \in X.\mathscr{U}(p) \Rightarrow Spec(A) \cap U \in Spec(A).\tau$

$$\Rightarrow \exists f \in A, Spec(A) \cap U \cong D(f) \cong Spec(A_f).$$

Because $(\mathscr{O}_X|_U)|_{U\cap D(f)} \cong \mathscr{O}_X|_{U\cap D(f)}$, then $(Spec(A_f), \mathscr{O}_{Spec(A_f)})$ is a affine cover of $p \in U$. $\square$

Exercise 2.2.3. Proof:
(a) $\forall p \in X, \exists A, Spec(A) \in X.\mathscr{U}(p)$, so $p \sim \mathfrak{p}$.

$$\mathscr{O}_{X,p} \cong A_{\mathfrak{p}}.$$

Because of $rad(A) = 0 \Leftrightarrow rad(A_{\mathfrak{p}}) = 0$, proposition is proved. $\square$

(b) $\forall p \in X, \exists U = Spec(A) \in X.\mathscr{U}(p)$. According to the commutative algebra theory, we can obtain a homeomorphism $U = Spec(A) \cong Spec(A/rad(A))$. Next, we need to prove that

$$\mathscr{O}_{Spec(A/rad(A))} \cong (\mathscr{O}_X)_{red}|_U.$$

$\forall f \in A, D(f) \in U.\tau$, we have

$$\mathscr{O}_{Spec(A/rad(A))}(D(\overline{f})) = (A/rad(A))_{\overline{f}} \cong A_f/rad(A)_f = \mathscr{O}_X)_{red}|_U(D(f)).$$

which $\overline{f}$ is the image of $A \to A/rad(A)$. Both of the restrictions follow by $\mathscr{O}_X$. So $(X,(O)_X)$ is a scheme.

We can let $\varphi : X_{red} \to X$, which $\varphi^\# : \mathscr{O}_X \to \varphi_*(\mathscr{O}_X)_{red} = (\mathscr{O}_X)_{red}$. Homeomorphism property has been proved. Morphism of sheaves can be give as following natural map:

$$\forall U \in X.\tau, \mathscr{O}_X(U) \to \mathscr{O}_X(U)/rad(\mathscr{O}_X(U)).$$

$\square$

(c) For underlying topological space,

$$\begin{tikzcd}
\text{X} \arrow[rr, dashed, "\exists! g"] \arrow[dr] & & Y_{red} \arrow[dl] \\
& \text{Y} &
\end{tikzcd}$$

$\forall p \in X, \exists! q \in Y, f(p) = q$. Because Y is homeomorphism to Y_{red} on the underlying topological spaces, $\exists! q' \in Y_{red}, p \mapsto q'$. So there is a unique continuous map g that

$$g(p) = q' \in Y_{red}.$$

For morphism of sheaves,

$$\begin{tikzcd}
\mathscr{O}_Y \arrow[r] \arrow[dr] & f_*\mathscr{O}_X \\
& (\mathscr{O}_{red}) \arrow[u, dashed, "\exists! g^\#"']
\end{tikzcd}$$

the existence can be determined by properties of ring. $\square$

Exercise 2.2.4. Proof:

($\Rightarrow$) $\forall f : X \to Spec(A)$, we can select global section,

$$f^\#(Spec(A)) : \mathscr{O}_{Spec(A)}(Spec(A)) \to f_*\mathscr{O}_X(Spec(A)),$$

$$f^\#(Spec(A)) : A \to \mathscr{O}_X(X) = \Gamma(X, \mathscr{O}_X).$$

This is determined uniquely.

($\Leftarrow$) If $X = Spec(B)$, then the result is obvious by commutative algebra theory.

For general case, we can choice a family of open affine cover $\{U_\lambda = Spec(B_\lambda)\}_{\lambda\in\Lambda}$. According to the axioms of presheaf of restriction, we have a commutative diagram for $i, j \in \Lambda, W = Spec(B_{ij}) \in (U_i \cap U_j).\tau$,

$$\begin{tikzcd}
& & \Gamma(U_i, \mathscr{O}_X|_{U_i}) = B_i \arrow[dll, "\rho_{U_i,W}"'] \\
\Gamma(W, \mathscr{O}_X|_W) = B_{ij} & A \arrow[l, "\exists!"'] \arrow[r, "\forall\varphi"] \arrow[ur, "\exists!\varphi_i"] \arrow[dr, "\exists!\varphi_j"'] & \Gamma(X, \mathscr{O}_X) \arrow[u, "\rho_{X,U_i}"'] \arrow[d, "\rho_{X,U_j}"] \\
& & \Gamma(U_j, \mathscr{O}_X|_{U_j}) = B_j \arrow[ull, "\rho_{U_i,W}"]
\end{tikzcd}$$

Then we have diagram of (affine) schemes,

$$
\begin{tikzcd}
& & U_i \arrow[dl, "f_i"'] \arrow[d, "\mu_j"] \\
W \arrow[urr] \arrow[r] \arrow[drr] & Spec(A) & X \arrow[l, dashed, "\exists! f"'] \\
& & U_j \arrow[ul, "f_j"] \arrow[u, "\mu_j"']
\end{tikzcd}
$$

In terms of topology, we can assume that

$$X = \left(\coprod_{i \in \Lambda} U_i \right) / \sim_1,$$

which $\forall p_i \in U_i, \forall, p_j \in U_j$,

$$p_i \sim_1 p_j \Leftrightarrow \mu_i(p_i) = \mu_j(p_j).$$

So, we can define $f : X \to Spec(A)$ as following,

$$f = \coprod_{i \in \Lambda} f_i : X = \left(\coprod_{i \in \Lambda} U_i \right) / \sim_1 \to \left(\coprod_{i \in \Lambda} Spec(A) \right) / \sim_2 \cong Spec(A).$$

For $\sim_2$, we can select $\forall \mathfrak{p}_i \subset B_i, \forall \mathfrak{p}_j \subset B_j$ to represent point in U_i, U_j respectively. If they satisfy the equivalence of $\sim_1$, then $\exists \mathfrak{p} \subset B_{ij}, \rho^{-1}_{W,U_i}(\mathfrak{p}) = \mathfrak{p}_i, \rho^{-1}_{W,U_j}(\mathfrak{p}) = \mathfrak{p}_j$. According to the first commutative diagram, we have

$$\varphi_i^{-1}(\mathfrak{p}_i) = \varphi_i^{-1} \circ \rho^{-1}_{W,U_i}(\mathfrak{p}) = \varphi^{-1} \circ \rho^{-1}_{X,W}(\mathfrak{p}) = \varphi_j^{-1} \circ \rho^{-1}_{W,U_j}(\mathfrak{p}) = \varphi_j^{-1}(\mathfrak{p}_j),$$

which is also give the definition of $\sim_2$. Meanwhile, the homeomorphism of $\left(\coprod_{i \in \Lambda} Spec(A)\right) / \sim_2 \cong Spec(A)$ is also simultaneously established.

Moreover, $\forall U \in Spec(A).\tau, f_i^{-1}(U) \in U_i.\tau \Rightarrow f^{-1}(U) = \cup_{i \in \Lambda} f_i^{-1}(U) \in X.\tau$. So f is a well defined continuous map.

Then I try to show that $f^{\#} : \mathscr{O}_{Spec(A)} \to f_* \mathscr{O}_X$ is well defined morphism of ring sheaves. Actually, we can take $\mathscr{O}_X$ as glueing sheaves of $\{\mathscr{O}_X|_{U_i}\}$. For each $\mathscr{O}_X|_{U_i}$, we have $f_i^{\#} : \mathscr{O}_{Spec(A)} \to (f_i)_* \mathscr{O}_X|_{U_i}$ is well defined. Then $f^{\#} = \prod_{i \in \Lambda} (f_i)^{\#}$ is also well defined according to the construction of glueing sheaves and universal property of product. □

Exercise 2.2.5. Proof:

In a unital commutative ring, $\mathbb{Z}$ is an initial object, so the same holds for $\Gamma(X, \mathscr{O}_X)$ as well. Based on the conclusion from the previous question, there exists a unique morphism $X \to Spec(\mathbb{Z})$. □

Exercise 2.2.6. Proof:

Spectrum of zero ring is empty, so in the category of topological spaces, it is the initial object. Therefore for any scheme $(X, \mathscr{O}_X)$, we have

$$\emptyset \to X, \mathscr{O}_X \to \mathscr{O}_{Spec(\{0\})}.$$

Then spectrum of zero ring is initial object.

Exercise 2.2.7. Proof:

($\Rightarrow$) For morphism $\forall f : Spec(K) = \{(0)\} \to X$, $\exists! x \in X$,

$$f : (0) \mapsto x.$$

Then, $\forall U \in X.\mathscr{U}(x)$,

$$f^{\#}(U) : \mathscr{O}_X(U) \to \mathscr{O}_{Spec(K)}(f^{-1}(U)) = \mathscr{O}_{Spec(K)}((0)) = K,$$

$$f_x^{\#}(U) : \mathscr{O}_{X,x} \to K.$$

Because X is a scheme, it is also a locally ringed space. So $\mathscr{O}_{X,x} \in \mathbf{localRng}, K \in \mathbf{localRng}$. So $(f_x^{\#})^{-1}((0)) = \mathfrak{m}_x$ which is a local morphism. Thus

$$k(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x \cong im(f_x^{\#}))/ker(f_x^{\#})) \cong K.$$

($\Leftarrow$) We let $x \in X, k(x) = K$, that means we have a local morphism

$$f : \mathscr{O}_{X,x} \to K.$$

which $f^{-1}((0)) = \mathfrak{m}_x$. We can define an inclusion of topological spaces

$$i : Spec(K) = \{(0)\} \hookrightarrow X, (0) \mapsto x.$$

For ring sheaf, we can give a morphism

$$i^{\#}(U) : \mathscr{O}_X(U) \to K = \mathscr{O}_{Spec(K)}(i^{-1}(U)) = i_* \mathscr{O}_{Spec(K)}(U)$$

$$\begin{tikzcd}
\mathscr{O}_X(U) \arrow[r, "i^{\#}(U)"] \arrow[d, "\rho_{UV}"'] & \text{K} \arrow[d, "id"] \\
\mathscr{O}_X(V) \arrow[r, "i^{\#}(V)"] & \text{K}
\end{tikzcd}$$

We get an inclusion of schemes:

$$i : Spec(K) \to X.$$

$\square$

Exercise 2.2.8. Proof:

($\Rightarrow$) If we have a morphism $Spec(k[\epsilon]/(\epsilon^2)) \to X$, that means $\exists! x \in X, (\epsilon) \mapsto x$ on topological space. Then we have local morphism of local rings,

$$\varphi : \mathscr{O}_{X,x} \to k[\epsilon]/(\epsilon^2),$$

which $\varphi^{-1}((\epsilon)) = \mathfrak{m}_x$. So we have

$$k(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x \cong (k[\epsilon]/(\epsilon^2))/(\epsilon) \cong k.$$

For k-morphism φ, $\forall \alpha \in \mathfrak{m}_x, \varphi(\alpha) \in (\epsilon)$. If $\alpha \in \mathfrak{m}_x^2, \varphi(\alpha) = 0$. That means

$$\varphi : \mathfrak{m}_x/\mathfrak{m}_x^2 \to (\epsilon).$$

Because φ is a local morphism, $\exists \alpha \in \mathfrak{m}_x/\mathfrak{m}_x^2, \exists t \in k, \varphi(\alpha) = t\epsilon$. So $\varphi(\alpha/t) = \epsilon.$, we can denote α. $\exists \alpha^* \in \mathfrak{m}_x/\mathfrak{m}_x^2$,

$$\varphi(\beta) = \alpha^*(\beta) \cdot \epsilon.$$

($\Leftarrow$) Following the process from the previous question, this step is similarly proven. $\square$

Exercise 2.2.9. Proof:

Suppose there are two different points $\zeta_1, \zeta_2 \in Z$, that $\overline{\{\zeta_1\}} = \overline{\{\zeta_2\}} = Z$.

If these two points are in different affine open sets,

$$\zeta_1 \in U_1, \zeta_1 \notin U_2,$$

$$\zeta_2 \in U_2, \zeta_2 \notin U_1.$$

However, $\zeta_2 \in U_1^c \cap Z \in X.\tau^c \Rightarrow \overline{\{\zeta_1\}} \subseteq U_1^c \cap Z \neq Z$. This contradicts irreducibility of Z.

That means $\forall U = SpecA \in X.\tau, U \cap Z \neq \emptyset, \zeta_1, \zeta_2 \in U$. And $Z \cap U \in U.\tau^c$, so $\exists \mathfrak{a} \subset A, U \cap Z \cong SpecA/\mathfrak{a}$. Assume the point ζ is a generic point defined by prime ideal $\mathfrak{p}$ and Z affine, it should be

$$\begin{aligned} V((0)) = Z = \overline{\{\mathfrak{p}\}} &= \bigcap_{W \in X.\mathscr{U}^c(p)} W \\ &= \bigcap_{\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{p}}} V(\mathfrak{a}) \\ &= V\left(\sum_{\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{p}}} \mathfrak{a}\right) \\ \Rightarrow \sqrt{(0)} &= \sqrt{\sum_{\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{p}}} \mathfrak{a}} \end{aligned}$$

Actually, $\sqrt{\mathfrak{p}} = \sum_{\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{p}}} \mathfrak{a}$. This is because $\forall x \in \sum_{\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{p}}} \mathfrak{a}, \exists \{a_i\}_{fin}, a_i \in \mathfrak{a}_i, a_i^{r_i} \in \mathfrak{p}, x = \sum a_i$, so $\exists r, x^r \in \mathfrak{p}$. Conversely, if $x \in \sqrt{\mathfrak{p}}, x \notin \sum_{\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{p}}} \mathfrak{a} \Rightarrow x(1) = (1)$, which is contradict that $\mathfrak{p}$ is an ideal.

Then, we have

$$\sqrt{(0)} = \sqrt{\sqrt{\mathfrak{p}}} = \sqrt{\mathfrak{p}}.$$

Because Z is irreducible, then $\sqrt{(0)}$ is a prime ideal, so as $\sqrt{\mathfrak{p}}$.

$$\sqrt{(0)} \subseteq \mathfrak{p} \subseteq \sqrt{\mathfrak{p}} = \sqrt{(0)}.$$

That means $\mathfrak{p} = \sqrt{(0)}$ is defined uniquely. $\square$

Exercise 2.2.10. Solve:

$Spec(\mathbb{R}[x]) = \{f(x) : f \in \mathbb{R}[x].\mathbf{Irr}\} deg(f) = 1, 2$.

$Spec(\mathbb{R}) = \{(0)\}$.

$Spec(\mathbb{C}) = \{(0)\}$. $\square$

Exercise 2.2.11. Proof:

This problem is equivalent to calculating the number of irreducible polynomials in $\mathbb{F}_p[x]$.

$$\#\mathbb{F}_p[x].\mathbf{Irr} = \sum_{k=1}^{p-1} \frac{(p^k - p)(p-1)}{k} + p(p-1)$$

$\square$

Exercise 2.2.12. Proof:

We will prove it by construction. First, we now glue the topological spaces by together.

$$X = \coprod_{\lambda \in \Lambda} X_i / \sim.$$

We should construct relation "$\sim$" as an equivalence relation as following,

$$\sim := \{(a_i, a_j) : \forall i, j \in \Lambda, U_{ij} \neq \emptyset, a_j = \varphi_{ij}(a_i)\}.$$

According to the definition of equivalence relation:

- (Reflexivity) We let $U_{ii} = X_i, \varphi_{ii} = id$, this is obvious.
- (Symmetry) $a_j = \varphi_{ij}(a_i) \Leftrightarrow a_j = \varphi_{ji}^{-1}(a_i) \Leftrightarrow \varphi_{ji}(a_j) = a_i$.
- (Transitivity) $a_i \sim a_j, a_j \sim a_k \Rightarrow a_k = \varphi_{jk}(a_j) = \varphi_{jk} \circ \varphi_{ij}(a_i) = \varphi_{ik}(a_i)$.

So we obtain a topological space X with cover $\{X_\lambda\}_{\lambda \in \Lambda}$. Meanwhile, we can say that $\forall i, j \in \Lambda, X_i \cap X_j \cong U_{ij} \cong U_{ji}$. When we view each covering as scheme, we can obtain a structure sheaf family $\{\mathscr{O}_{X_\lambda}\}_{\lambda \in \Lambda}$ over each X_λ. Similarily, we have a family of isomorphism of sheaves

$$\varphi_{ji}^{\#} : \mathscr{O}_{X_i}|_{X_i \cap X_j} \to \mathscr{O}_{X_j}|_{X_i \cap X_j},$$

such that

- $\varphi_{ii}^{\#} = id$,
- $\varphi_{ki}^{\#} = \varphi_{kj}^{\#} \circ \varphi_{ji}^{\#}, X_i \cap X_j \cap X_k$.

Therefore, based on the previous conclusion, $\exists! \mathscr{O}_X, \forall \lambda \in \Lambda, \mathscr{O}_X|_{X_\lambda} \cong \mathscr{O}_{X_\lambda}$. $\forall p \in X$, the existence of its affine cover can be traced back to the cover of X_i, so X is a scheme. With the customization of the obtained structural sheaf and the property of induced scheme, the proposition is proven.

If X_i is disjoint, then

$$X = \coprod_{\lambda \in \Lambda} X_i / \sim = \coprod_{\lambda \in \Lambda} X_i.$$

So as scheme. $\square$

Exercise 2.2.13. Proof:

(a) ($\Rightarrow$) Consider that

- A noetherian topological space is quasi-compact.
- Any subset of a noetherian topological space is noetherian space in its induced topology

So it is obvious.

($\Leftarrow$) We need to check that $\{U_i\}_{i \in I} \subset X.\tau, \forall i \in I, U_i \subsetneq U_{i+1}, \exists n, \forall j \geq n, U_j = U_{j+1}$. We denote $U = \cup_{i \in I} U_i$. So $\{U_i\}_{i \in I} \in U.$**cover**. Because U is quasi-compact, there is a finite open sets that cover U. In other word,

$$\exists n, U_n = U \Rightarrow \forall j \geq n, U = U_n \subseteq U_j \subseteq U.$$

$\square$

(b)

$$\begin{aligned}
\forall E \subseteq A, \cup_{f_i \in E} D(f) = Spec(A) &\Rightarrow \cap_{f_i \in E} V((f_i)) = \emptyset = V((1)) \\
&\Rightarrow V(\sum_{f_i \in E} (f_i)) = V((1)) \\
&\Rightarrow \sqrt{\sum_{f_i \in E} (f_i)} = (1) \\
&\Rightarrow \exists I, \# I < +\infty, \forall i \in I, g_i \in (f_i), \sum_{i \in I} g_i = 1 \\
&\Rightarrow \cup_{i \in I} D(g_i) = Spec(A).
\end{aligned}$$

$Spec(\mathbb{C}[x_1, x_2, ...])$ is not noetherian, because

$$V((x_1)) \supset V((x_1, x_2)) \supset ...$$

doesn't stop. $\square$

(c)

$$\begin{aligned}
&\forall \{Y_i\}_{i \in I} \subset Spec(A).\tau^c, \forall i \in I, Y_i \supset Y_{i+1} \\
&\Rightarrow \forall i \in I, \exists J_i \in A.\mathbf{Ideal}, V(J_i) = Y_i, J_i \subset J_{i+1} \\
&\Rightarrow \exists n, \forall j \geq n, J_j = J_{j+1} && (A \in \mathbf{Rng}.\mathbf{Noetherian}) \\
&\Rightarrow \forall j \geq n, Y_j = Y_{j+1}.
\end{aligned}$$

$\square$

(d) $Spec(\mathbb{C}[x_1, x_2, ...]/(x_1^2, x_2^2, ...)) = \{(0)\}$, but it is not noetherian ring. $\square$

Exercise 2.2.14. Proof:

(a)

$$\begin{aligned}
\forall f \in S_+, \exists n \in \mathbb{Z}^+, f^n = 0 &\Leftrightarrow \forall \mathfrak{p} \in S.\mathbf{prime}, \mathfrak{p} \supseteq S_+ \\
&\Leftrightarrow Proj(S) = \emptyset.
\end{aligned}$$

$\square$

(b) We need to prove that $U^c = \{\mathfrak{p} \in Proj(T) : \mathfrak{p} \supseteq \varphi(S_+)\} \in Proj(T).\tau^c$.

$$\begin{aligned}\forall \mathfrak{p} \supseteq \varphi(S_+) &\Rightarrow \mathfrak{p} = \mathfrak{p} \cdot T \supseteq \varphi(S_+) \cdot T \in T.\mathbf{ideal}\\ &\Rightarrow U^c = V(\varphi(S_+) \cdot T) \in Proj(T).\tau^c\\ &\Rightarrow U \in Proj(T).\tau.\end{aligned}$$

$\forall \mathfrak{p} \in U, \varphi^{-1}(\mathfrak{p}) \not\supseteq S_+ \Rightarrow \exists! \mathfrak{q} = \varphi^{-1}(\mathfrak{p}) \in Proj(S)$. And it is a continuous map. Because $\forall V \in Proj(S).\tau^c, \exists \mathfrak{a} \in S.\mathbf{ideal}, V(\mathfrak{a}) = V$. Then $V' = \{\mathfrak{q} \supseteq \varphi(\mathfrak{a}) \cdot T : \mathfrak{q} \not\supseteq \varphi(S_+)\} \in U.\tau^c$. Obviously, V' is the preimage of V. □

(c) The main problem is $\forall \mathfrak{q} \in Proj(S), \exists \mathfrak{p}, \mathfrak{p}' \in Proj(T)$, when $d \geq d_0, \mathfrak{p} \cap T_d = \mathfrak{p}' \cap T_d \cong \mathfrak{q} \cap S_d$. But when $d < d_0$, maybe $\mathfrak{p} \cap T_d \neq \mathfrak{p}' \cap T_d$. If it is true, $\exists s \in \mathfrak{p} \cap T_d, d < d_0, s^{d_0} \in \mathfrak{p} \cap T_{d*d_0} = \mathfrak{p}' \cap T_{d*d_0}$. Due to $\mathfrak{p}'$ is prime ideal, so $s \in \mathfrak{p}'$. That means φ determines a isomorphism $Proj(S) \cong Proj(T)$. □

(d) For scheme $(t(V), \alpha_*(\mathscr{O}_V))$, which t is a functor mentioned, $\alpha : V \to t(V)$ is a continuous map and $\mathscr{O}_V$ is regular function sheaf over V. In terms of topology,

$$\begin{aligned}\forall t(W) \in t(V).\tau^c &\Leftrightarrow t(W) = \{W' \in W.\tau^c.\mathbf{irr}\}\\ &\Leftrightarrow \exists \mathfrak{a} \in S.\mathbf{ideal}, \exists \mathfrak{p} \in S.\mathbf{prime}, \mathfrak{p} \not\supseteq S_+, \mathfrak{p} \supseteq \mathfrak{a}, \mathfrak{p} \sim W'\\ &\Leftrightarrow t(W) \sim V(\mathfrak{a}) \in Proj(S).\tau^c.\end{aligned}$$

So it is homeomorphism of topological spaces.

In terms of ring sheaves,

$$\begin{aligned}\forall U \in t(V).\tau, \forall x \in \alpha_*\mathscr{O}_V(U) = \mathscr{O}_V(\alpha^{-1}(U)), &\exists s, t \in S, deg(s) = deg(t), t(\alpha^{-1}(U)) \neq 0, x = s/t\\ &\Leftrightarrow x = s/t \in \mathscr{O}_{Proj(S)}(U)\end{aligned}$$

□

Exercise 2.2.15. Proof:

(a) We can choose an affine cover $\exists A = k[x_1, ..., x_n]/I, Spec(A) \in t(V).\mathscr{U}(P)$. We denote that $\mathfrak{m} \in A.\mathbf{prime}, \mathfrak{m} \sim P$. Because P is a closed point, so

$$\begin{aligned}\overline{\{P\}} = \{P\} &\Leftrightarrow \forall \mathfrak{p} \supseteq \mathfrak{m}, \mathfrak{p} = \mathfrak{m}\\ &\Leftrightarrow \mathfrak{m} \in A.\mathbf{maxideal}.\end{aligned}$$

Thus, $k(P) = A/\mathfrak{m} = k$. □

(b) First, we have a commutative diagram of schemes,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & Spec(k) & \end{tikzcd}$$

Then, we obtain a commutative diagram of sheaves at P,

$$\begin{tikzcd} \mathscr{O}_{Y,f(P)} \arrow[rr, "f_P^\#"] & & \mathscr{O}_{X,P} \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Choose the residue field, we have

$$\begin{tikzcd} \mathscr{O}_{Y,f(P)}/\mathfrak{m}_{f(P)} \arrow[rr, "\overline{f_P^\#}"] & & \mathscr{O}_{X,P}/\mathfrak{m}_P = k \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

If $k_Y(f(P)) \neq k$, then $\exists x \notin k, k(x) \subseteq k_Y(f(P))$. For k homomorphism $\overline{f_P^{\#}}$, we have $\overline{f_P^{\#}}(x) = \overline{f_P^{\#}}\left(\overline{f_P^{\#}}(x)\right)$. That means

$$x - \overline{f_P^{\#}}(x) \in ker\left(\overline{f_P^{\#}}\right).$$

However, this contradicts condition $ker\left(\overline{f_P^{\#}}\right) = \{0\}$. □

(c) Omitted. □

Exercise 2.2.16. Proof:

(a) We denote $(\overline{f}) = f|_U \in B$.

$$\begin{aligned}\forall \mathfrak{p} \in U \cap X_f, \mathfrak{p} \subset B, \overline{f}_{\mathfrak{p}} \notin \mathfrak{p} \subseteq A_{\mathfrak{p}} &\Leftrightarrow \overline{f} \notin \mathfrak{p} \subseteq A \\ &\Leftrightarrow \mathfrak{p} \in D(\overline{f}).\end{aligned}$$

$$\{Spec(B_i)\} \in X.\mathbf{cover}, Spec(B_i) \cap X_f \in Spec(B_i).\tau \Rightarrow X_f = \cup_i (Spec(B_i) \cap X_f) \in X.\tau.$$

□

(b) Because X is quasi-compact, then $\exists I \subset \mathbb{Z}, \#I < +\infty$ is a index set,

$$X = \cup_{i \in I} U_i, U_i = Spec(A_i).$$

$$\begin{aligned} a|_{X_f} = 0 &\Rightarrow \forall i \in I, (a|_{X_f})|_{U_i} = a|_{X_f \cap U_i} = 0 \\ &\Rightarrow \exists n_i, a|_{X_f \cap U_i} \cdot (f_{X_f \cap U_i})^{n_i} = 0 \in A_i && \text{(axioms of local ring)} \\ &\Rightarrow n = \max_{i \in I}\{n_i\}, (a \cdot f^n)|_{X_f \cap U_i} = a|_{X_f \cap U_i} \cdot (f_{X_f \cap U_i})^{n_i} = 0 \\ &\Rightarrow f^n \cdot a = 0 && \text{(first axiom of sheaf)} \end{aligned}$$

(c) We denote the affine open cover is $\{Spec(B_i)\}_{i \in I}$ that

$$X = \cup_{i \in I} Spec(B_i).$$

We let $f_i = f|_{X_f \cap Spec(B_i)}$. Then $D(f_i) \cong Spec(B_i) \cap X_f$. $\forall i \in I, \exists n_i \in \mathbb{Z}, \exists b_i \in B_i$,

$$b_i|_{D(f_i)} = (f|_{D(f_i)})^{n_i} \cdot b|_{D(f_i)} = ((f^{n_i})|_{X_f} \cdot b)|_{D(f_i)}.$$

Because $\#I < +\infty$, then we can let $n = max_{i \in I}\{n_i\}$,

$$b_i|_{D(f_i)} = ((f^n)|_{X_f} \cdot b)|_{D(f_i)}.$$

$\forall i, j \in I$, we need to let $b_i \in B_i, b_j \in B_j, b_i|_{Spec(B_i) \cap Spec(B_j)} = s_j|_{Spec(B_i) \cap Spec(B_j)}$. We know that $\exists J \subset \mathbb{Z}, \#J_{ij} < +\infty$,

$$Spec(B_i) \cap Spec(B_j) = \cup_{k \in J_{ij}} D(g_k),$$

which g_k was in both image of inclusion of B_i, B_j. $\forall k \in J_{ij}, b_i|_{D(g_k)} \in Spec\left((B_i)_{g_k}\right), b_j|_{D(g_k)} \in Spec\left((B_j)_{g_k}\right)$. We let $D(g_k f_i) = D(g_k) \cap D(f_i), b_i|_{D(g_k f_i)} = b_j|_{D(g_k f_j)}$. According to the axiom of localization, $\exists m_k > 0, \exists b_i' \in D(g_k) \subseteq Spec(B_i), \exists b_j' \in D(g_k) \subseteq Spec(B_j)$,

$$b_i'|_{D(g_k f_i)} = (f_i^{m_k} b_i)|_{D(g_k f_i)} = (f_j^{m_k} b_j)|_{D(g_k f_j)} = b_j'|_{D(g_k f_j)}.$$

Because $\#J < +\infty, m_{ij} = max_{k \in J_{ij}} m_k$,

$$\forall k \in J_{ij}, b_i'|_{D(g_k f_i)} = (f_i^{m_{ij}} b_i)|_{D(g_k f_i)}.$$

Then, we can let $b_i \in B_i, b_j \in B_j$,

$$b_i|_{Spec(B_i) \cap Spec(B_j)} = b_j|_{Spec(B_i) \cap Spec(B_j)}.$$

Due to finiteness of I, J_{ij}, then $m = max_{i,j\in I}\{m_{ij}\}, n$ is finite. So, according to the axiom of sheaf, $\exists s \in \Gamma(X, \mathscr{O}_X), s|_{X_f} = f^{m+n}b$. □

(NOTE: Here we denote that $\Gamma(X_f, \mathscr{O}_{X_f}) = \mathscr{O}_X(X_f))$

(d) Naturally, we have a commutative diagram as following,

$$\begin{tikzcd} A \arrow[rr, "\rho"] \arrow[dr, "j"'] & & \Gamma(X_f, \mathscr{O}_{X_f}) \\ & A_f \arrow[ur, "\varphi"'] & \end{tikzcd}$$

which ρ is restriction, j is a natural map of localization.

First, we prove that $\varphi \in \mathbf{Inj}$. That means, $\forall a, b \in A, \forall n_1, n_2 \in \mathbb{Z}^+$,

$$\begin{aligned} &\varphi\left(\frac{a}{f^{n_1}} - \frac{b}{f^{n_2}}\right) = 0 \\ \Rightarrow &(a|_{X_f}) \cdot (f|_{X_f})^{-n_1} - (b|_{X_f}) \cdot (f|_{X_f})^{-n_2} = 0 \\ \Rightarrow &\exists m \in \mathbb{Z}^+ a \cdot f^{m-n_1} - b \cdot f^{m-n_2} = 0 && \text{((b))} \\ \Rightarrow &\frac{a}{f^{n_1}} = \frac{b}{f^{n_2}} && \text{(axiom of localization)} \end{aligned}$$

So, it is injective.

According to the result of (c),$\forall b \in \Gamma(X_f, \mathscr{O}_{X_f}), \exists n \in \mathbb{Z}^+, \exists b' \in A, b'|_{X_f} = b \cdot (f|_{X_f})^n$. So we can obtain an element $\frac{j(b')}{f^n} \in A_f$. Repeat same method, we can determine that

$$\varphi\left(\frac{j(b')}{f^n}\right) = b.$$

In sum, φ is an isomorphism. □

Exercise 2.2.17. Proof:

(a) In topology, we aim to prove its homeomorphism. Obviously, f is continuous map. We denote $V_i = f^{-1}(U_i)$. Because $f|_{V_i}$ is isomorphism, so f is also homeomorphism in topological space. $\forall y \in Y, \exists U_i \in Y.\mathscr{U}(y)$,

$$\exists! x \in X, \{x\} = f^{-1}(y) \subset V_i.$$

So we can define the inverse f^{-1} as a map of $Y \to X$. $\forall V \in X.\tau, (f^{-1}|_{V_i})^{-1}(V_i \cap V) = f(V_i \cap V) \in Y.\tau$,

$$(f^{-1})^{-1}(V) = \cup_i (f^{-1}|_{V_i})^{-1}(V \cap V_i) \in Y.\tau.$$

So f is an homeomorphism of $X \to Y$.

In ring sheaf, according to the previous result,

$$\mathscr{O}_Y \cong f_*\mathscr{O}_X \Leftrightarrow \forall p \in X, \mathscr{O}_{Y,f(p)} \cong \mathscr{O}_{X,p}.$$

And $\forall p \in X, \exists V_i \in X.\mathscr{U}(p)$,

$$\mathscr{O}_{Y,f(p)} = \mathscr{O}_Y|_{U_i,f(p)} \cong \mathscr{O}_X|_{V_i,p} = \mathscr{O}_{X,p}$$

In sum, f is an isomorphism of schemes. □

(NOTE: Without assertion of f being a morphism of schemes, this is glueing lemma.)

(b)

($\Rightarrow$) It is obvious by commutative algebra theory.

($\Leftarrow$) $\{X_{f_i}\}_{i=1}^r$ is a open affine cover of X with $X_{f_i} = Spec(B_i) = U_i$. $\forall i, j \in I = \{1, ..., r\}$, we need to determine if $X_{f_i} \cap X_{f_j}$ is quasi-compact or not. Fixing $i \in I, U_i \cap U_j = D(f_j|_{U_i}) = Spec((B_i)_{f_j|_{U_i}}) \subseteq U_i$. So it is quasi-compact, then $\forall i \in I, U_i = Spec(A_{f_i}), B_i = A_{f_i}$.

From identity map of $A \to \Gamma(X, \mathscr{O}_X)$, we can acquire a morphism of schemes uniquely that $f : X \to Spec(A)$, with isomorphism $\forall i \in I, X_{f_i} \cong Spec(A_{f_i})$. □

(Next two exercises are from commutative algebra exercises, so I'll set these issues aside for future handling.)

Exercise 2.2.18. Proof:
(a) If there is a prime ideal $\mathfrak{p} \subset A$ that

$$\begin{aligned}\mathfrak{p} \in D(f) &\Leftrightarrow f \notin \mathfrak{p} \\ &\Leftrightarrow f^n \notin \mathfrak{p} && \text{(axiom of prime ideal)} \\ &\Leftrightarrow 0 \notin \mathfrak{p}. && \text{(nilpotent of } f\text{)}\end{aligned}$$

That is contradiction. □
(b)
(c)

Exercise 2.2.19. Proof:

2.3 First Properties of Schemes

Q 2.3.1. We let denote a morphism of affine schemes,

$$f : Spec(B) \to Spec(A).$$

Then we have that $\forall g \in A, f^{-1}(D(g)) = Spec(B_{f\#(Spec(A))(g)})$.
Proof: f is a continuous on topological space, so we only need to verify the form of $f^{-1}(D(f))$. We can denote that f induces a homomorphism

$$\varphi : A \to B.$$

$$\begin{aligned}\forall \mathfrak{p} \in f^{-1}(D(g)) &\Leftrightarrow f(\mathfrak{p}) \in D(g) \\ &\Leftrightarrow \varphi^{-1}(\mathfrak{p}) \in D(g) \\ &\Leftrightarrow g \notin \varphi^{-1}(\mathfrak{p}) \\ &\Leftrightarrow \varphi(g) \notin \mathfrak{p} \\ &\Leftrightarrow \mathfrak{p} \in D(\varphi(g)).\end{aligned}$$

So $f^{-1}(D(g)) = D(\varphi(g)) = Spec(B_{f\#(Spec(A))(g)})$. □

Q 2.3.2. We let $X = Spec(A)$, and we have finite element $\{f_i\}_i \subset A$ which principle open set can cover X, then $\exists \{a_i\}_i \subset A, \sum_i a_i \cdot f_i = 1$.
Proof:

$$\begin{aligned}X = \cup_i D(f_i) &\Leftrightarrow \emptyset = \cap_i V((f_i)) \\ &\Leftrightarrow V((1)) = V(\sum_i (f_i)) \\ &\Leftrightarrow \sqrt{(1)} = \sqrt{\sum_i (f_i)} \\ &\Leftrightarrow (1) = \sum_i (f_i) \\ &\Leftrightarrow \exists \{a_i\} \subset A, \sum_i a_i \cdot f_i = 1\end{aligned}$$

□

Q 2.3.3. Let $f : X \to Y$ be a morphism of schemes, $y \in Y$. We have fibre of y

$$X_y = X \times_Y Spec(k(y)).$$

Consider the open affine set $Spec(A) \in Y.\mathscr{U}(y), Spec(B) \in f^{-1}(Spec(A)).\tau$. Then If $Spec(B) \cap f^{-1}(y) = \emptyset$, then $Spec(B) \times_{Spec(A)} Spec(k(y)) = \emptyset$.
Proof:

We denote $\varphi : A \to B$ induced by f, and $y \sim \mathfrak{p} \subset A$. Because $Spec(B) \cap f^{-1}(y) = \emptyset$,

$$f^{-1}(y) = f^{-1}(V_A(\mathfrak{p})) = V_B(\varphi(\mathfrak{p})) = \emptyset = V_B((1)).$$

That means $\forall \alpha \in \mathfrak{p}, \varphi^{-1}(\alpha) \in B$. $\forall a \in B, \forall b \in A_{\mathfrak{p}}/\mathfrak{p}, 0 = \overline{\alpha} \in A_{\mathfrak{p}}/\mathfrak{p}$,

$$\begin{aligned} a \otimes_A b &= \big(\varphi(\alpha) \cdot \varphi^{-1}(\alpha) \cdot a\big) \otimes_A b \\ &= \big(\varphi^{-1}(\alpha) \cdot a\big) \otimes_A (\overline{\alpha} \cdot b) = 0. \end{aligned}$$

So

$$B \otimes_A (A_{\mathfrak{p}}/\mathfrak{p}) = \{0\} \Leftrightarrow Spec(B) \times_{Spec(A)} Spec(k(y)) = \emptyset.$$

$\square$

(NOTE: If $Spec(A) \notin Y.\mathscr{U}(y)$, such fibre product doesn't exist.)

Q 2.3.4. We let $f : X \to Y$ be a morphism of schemes, $y \in Y, \forall Spec(A), Spec(A') \in Y.\mathscr{U}(y)$. Then

$$S = X \times_{Spec(A)} Spec(k(y)) \cong X \times_{Spec(A')} Spec(k(y)) = S'.$$

Proof:

For $\{Spec(B_i)\}_{i \in I} \in X.$**cover**, by the preimage of first projection of $S \to X, S' \to X$, we can get open affine cover $\{Spec(B_i) \times_{Spec(A)} Spec(k(y))\} \in S.$**cover**, $\{Spec(B_i) \times_{Spec(A')} Spec(k(y))\} \in S'$. Besides, we have $A_{\mathfrak{p}} \cong \mathscr{O}_{Y,y} \cong A'_{\mathfrak{p}}$. Next, we should show that

$$Spec(B) \times_{Spec(A)} Spec(k(y)) \cong Spec(B) \times_{Spec(A')} Spec(k(y)).$$

If $f^{-1}(y) \notin Spec(B)$, the result is obvious. Otherwise,

Exercise 2.3.1. Proof:

($\Leftarrow$) It is obvious by assigning such affine open set to a cover of Y.

($\Rightarrow$) We let $Y = \cup_{i \in I} Spec(B_i), f^{-1}(Spec(B_i)) = \cup_{j \in J_i} Spec(A_{ij}), A_{ij} \in$ **Alg**(B_i).**FinGen**. We denote the induced ring homomorphisms that $\varphi_{ij} : B_i \to A_{ij}$.

$\forall U = Spec(B) \in Y.\tau$, we have affine open cover of $U = \cup_{i \in I} \cup_{k \in K_i} Spec((B_i)_{g_{ik}}), g_{ik} \in B_i$, which $U \cap Spec(B_i) = \cup_{k \in K_i} Spec((B_i)_{g_{ik}})$. Because U is quasi-compact, so we can assume that K_i are all finite set.

Then

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\cup_{i \in I}(U \cap Spec(B_i))) \\ &= \cup_{i \in I} f^{-1}(\cup_{k \in K_i} Spec((B_i)_{g_{ik}})) \\ &= \cup_{i \in I} \cup_{k \in K_i} f^{-1}(Spec((B_i)_{g_{ik}})) \\ &= \cup_{i \in I} \cup_{k \in K_i} \big(\cup_{j \in J_i} Spec((A_{ij})_{\varphi_{ij}(g_{ik})})\big) \\ &= \cup_{i \in I} \cup_{k \in K_i} \cup_{j \in J_i} Spec((A_{ij})_{\varphi_{ij}(g_{ik})}). \end{aligned}$$

So each $Spec((A_{ij})_{\varphi_{ij}(g_{ik})})$ can form a cover of $f^{-1}(U)$. Next, we need to show that $\forall i \in I, \forall j \in J_i, \forall k \in K_i, (A_{ij})_{\varphi_{ij}(g_{ik})} \in$ **Alg**(B).**FinGen**.

According to the commutative diagram, we have

$$\begin{tikzcd} Spec((A_{ij})_{\varphi_{ij}(g_{ik})}) \arrow[r, "f"] \arrow[dr] & Spec((B_i)_{g_{ik}}) \arrow[d] \\ & Spec(B) \end{tikzcd}$$

which $Spec((B_i)_{g_{ik}}) \subseteq Spec(B) \cap Spec(B_i)$. Then we can assume $g_k \in B$. So

$$B \hookrightarrow B_{g_{ik}} \cong (B_i)_{g_{ik}} \to (A_{ij})_{\varphi_{ij}(g_{ik})}.$$

We have $B_{g_{ik}} \cong B[x]/(xg_{ik} - 1)$ and $A_{ij} \cong B_i[x_1, ..., x_n], \{x_i\}_{i=1}^n \subset A_{ij}$, so

$$(A_{ij})_{\varphi_{ij}(g_{ik})} \cong B_i \left[x_1, ..., x_n, \frac{1}{\varphi_{ij}(g_{ik})}\right] \cong (B_i)_{g_{ik}}[x_1, ..., x_n] \cong B_{g_{ik}}[x_1, ..., x_n] \cong B \left[x_1, ..., x_n, \frac{1}{\varphi_{ij}(g_{ik})}\right].$$

$\square$

Exercise 2.3.2. Proof:

($\Leftarrow$) It is obvious according to previous disscussion.

($\Rightarrow$) First, we have $\{V_i = Spec(B_i)\}_{i\in I}, Y = \cup_{i\in I} V_i, \forall\{U_{ij} = Spec(A_{ij})\}_{j\in J_i}, f^{-1}(V_i) = \cup_{j\in J_i} U_{ij}$, which $\forall i \in I, \exists J_i' \subseteq J, \#J_i' < +\infty$, such that

$$f^{-1}(V_i) = \cup_{j\in J_i'} U_{ij}.$$

So $\forall V = Spec(B) \in Y.\tau$, we have a cover of it that $\{V_i \cap V\}_{i\in I} \in V.\mathbf{cover}$. $\forall i \in I, \exists\{g_{ik}\}_{k\in K_i} \subset B_i, V_i \cap V = \cup_{k\in K_i} Spec((B_i)_{g_{ik}})$. So $\{Spec((B_i)_{g_{ik}})\}_{i\in I, k\in K_i} \in V.\mathbf{cover}$. Because V is quasi-compact, we can assume I, K are both finite.

In a certain affine open set $U_{ij} \in X.\tau, \exists Spec((B_i)_{g_{ik}}) \in Y.\tau, Spec((B_i)_{g_{ik}}) \subseteq f(U_{ij}) \subseteq V_i$. So

$$f^{-1}(Spec((B_i)_{g_{ik}})) = Spec((A_{ij})_{f\#(Spec((B_i)_{g_{ik}}))(g_{ik})}),$$

which is affine and quasi-compact as well.

In sum,

$$\begin{aligned} f^{-1}(V) &= \cup_{i\in I} \left(f^{-1}(V \cap V_i)\right) \\ &= \cup_{i\in I} \cup_{k\in K_i} \left(f^{-1}(Spec((B_i)_{g_{ik}}))\right) \\ &= \cup_{i\in I} \cup_{k\in K_i} \cup_{j\in J_i} \left(Spec((A_{ij})_{f\#(Spec((B_i)_{g_{ik}}))(g_{ik})})\right). \end{aligned}$$

As we have verified, I, K_i, J_i are all finite, then $f^{-1}(V)$ is the union of a finite number of quasi-compact spaces, thus quasi-compact. $\square$

Exercise 2.3.3. Proof:

(a) Obviously, "locally finite type" and "finite type" are distinguished only by "quasi-compact" (preimage of $f : X \to Y$) according to the definitions of them. $\square$

(b)

$$\begin{aligned} \forall(f : X \to Y) \in \mathbf{FiniteType} &\Leftrightarrow (f \in \mathbf{LocallyFiniteType}) \wedge (f \in \mathbf{Quasi\text{-}Compact}) \\ &\Rightarrow \forall V = Spec(B) \in Y.\tau, f^{-1}(V) = \cup_{i\in I} Spec(A_i), A_i \in B.\mathbf{FinGen}, \#I < +\infty \end{aligned}$$

$\square$

(c) For a certain $V = Spec(B) \subseteq Y, \forall U = Spec(A) \subseteq f^{-1}(V)$, we have finite cover

$$f^{-1}(V) = \cup_{i\in I} U_i, U_i = Spec(A_i), A_i \in B.\mathbf{FinGen}, \#I < +\infty,$$

and

$$U = \cup_{i\in I} \cup_{j\in J_i} Spec((A_i)_{g_{ij}}), g_{ij} \in A_i, \forall i \in I, \#J_i < +\infty.$$

We note that $W_{ij} = Spec((A_i)_{g_{ij}})$.

Because $\forall i, j, W_{ij} \hookrightarrow U$, then we have homomorphisms induced by restrictions and morphism f,

$$\varphi_{ij} : A \to (A_i)_{g_{ij}},$$

which $\exists g_{ij}' \in A, A_{g_{ij}'} \cong (A_i)_{g_{ij}}$, so we assume that g_{ij} is also an element of A. Thus we have $A_{g_{ij}} \in B.\mathbf{FinGen}$, that means

$$A_{g_{ij}} \cong B\left[x_i^1, ..., x_i^{n_i}, \frac{1}{g_{ij}}\right], \{x_i^k\}_{i=1}^{n_i} \subset A_i.$$

If $\forall x \in A$, its image in $A_{g_{ij}}$ must have form like

$$F_{ij}\left(x_i^1, ..., x_i^{n_i}, \frac{1}{g_{ij}}\right) \in B\left[x_i^1, ..., x_i^{n_i}, \frac{1}{g_{ij}}\right].$$

And we let $F_{ij}' = x \cdot g_{ij}^{n_{ij}} = g_{ij}^{n_{ij}} F_{ij} \in B\left[x_i^1, ..., x_i^{n_i}\right]$. Because I, J_i are all finite, we can let $N = max_{i\in I, j\in J_i}\{n_{ij}\}, F_{ij}' = x \cdot g_{ij}^N$. Because $D(g_{ij})$ can form a cover of U, $\exists\{a_{ij}\}_{i,j} \subset A, \sum_{ij} a_{ij} \cdot g_{ij}^N = 1$.

Finally, we have

$$x = \sum_{i,j} a_{ij} \cdot F_{ij} \cdot g_{ij}^N,$$

which $\{x_{ij}^k\}$ is a finite set. $\square$

Exercise 2.3.4. Proof:

($\Leftarrow$) It is obvious that apply it for any cover of Y.

($\Rightarrow$) We let $\{U_i = Spec(B_i)\}_{i\in I} \in Y.\mathbf{cover}$, with $f^{-1}(U_i) = Spec(A_i), A_i \in \mathbf{Alg}(B_i)\cap\mathbf{Mod}(B_i).\mathbf{FinGen}$. Then $\forall U = Spec(B) \in Y.\tau$, so $\forall i \in I, \exists J_i, f_{ij} \in B_i, \cup_{j\in J_i} D(f_{ij}) = Spec(B)\cap Spec(B_i)$. Apart from these, we denote that

$$\varphi_i : B_i \to A_i.$$

Here we have a commutative diagram,

$$\begin{tikzcd}
Spec((A_i)_{\varphi_i(f_{ij})}) \arrow[r, "f"] \arrow[d] \arrow[dr, "f"] & Spec(B) \\
Spec((B_i)_{f_{ij}}) \arrow[r] & Spec(B)\cap Spec(B_i) \arrow[u]
\end{tikzcd}$$

$f_{ij} \in B\cap B_i, (B)_{f_{ij}} \cong (B_i)_{f_{ij}}$. $Spec(B)$ is quasi-compact, then we can assume that $\#I < \infty$ and $\forall i \in I, \#J_i < \infty$,

$$Spec(B) = \cup_{i\in I}\cup_{j\in J_i} Spec(B_{f_{ij}}), f^{-1}(Spec(B_{f_{ij}})) = Spec((A_i)_{\varphi_i(f_{ij})}).$$

Next, we need to show that $f^{-1}(Spec(B))$ is affine. For each f_{ij}, $\exists\{b_{ij}\}\subset B$,

$$\sum(b_{ij}f_{ij}) = 1 \in B.$$

That means, with ring homomorphism $\varphi := f^{\#}(Spec(B)) : B \to \mathscr{O}_X(f^{-1}(Spec(B)))$,

$$\varphi|_{(B)_{f_{ij}}} = \varphi_i|_{(B_i)_{f_{ij}}} : B_{f_{ij}} \to (A_i)_{\varphi_i(f_{ij})},$$

$$1 = \varphi(1) = \varphi(\sum(b_{ij}f_{ij})) = \sum(\varphi(b_{ij})\varphi(f_{ij})) \in \mathscr{O}_X(f^{-1}(Spec(B))),$$
$$\Leftrightarrow \cup(D(\varphi(f_{ij}))) = Spec(A), \cup(D(f_{ij})) = Spec(B).$$

so $f^{-1}(Spec(B))$ is affine which we can denote $f^{-1}(Spec(B)) = Spec(A)$. Then we need to show that $A \in \mathbf{Alg}(B)\cap\mathbf{Mod}(B).\mathbf{FinGen}$.

We assume the image of B can be taken as a subset of A, so $f_{ij} \in A$. $\forall D(f_{ij}), \exists\{e_{ij}^k\}_{k=1}^{N_{ij}} \subset A_{f_{ij}}, \forall x \in A_{f_{ij}}, \exists\{a_{ij}^k\}_{k=1}^{N_{ij}} \subset B_{f_{ij}}$,

$$x = \sum_k(a_{ij}^k e_{ij}^k).$$

Due to the axiom of localization, $\exists n_{ij} \in \mathbb{Z}^+$,

$$x\cdot f_{ij}^{n_{ij}} \in A.$$

So $\forall i,j,k$, we have $n_{ijk} \in \mathbb{Z}^+$ that $f_{ij}^{n_{ijk}}e_{ij}^k \in A$. We can let $n = max_{i,j,k}\{n_{ijk}\}, f_{ij}^n e_{ij}^k \in A$, which $\{f_{ij}^n e_{ij}^k\}$ is finite. We define the new B-base as $\bar{e}_{ij}^k = f_{ij}^n e_{ij}^k \in A$. $\forall x \in A, x|_{D(f_{ij})} = x_{ij} \in (A_i)_{f_{ij}}, \exists\{a_{ij}^k\}\subset B_{f_{ij}}$,

$$x_{ij} = \sum_k(a_{ij}^k e_{ij}^k) \in (A_i)_{f_{ij}},$$

and

$$\exists m_{ijk} \in \mathbb{Z}^+, f_{ij}^{m_{ijk}}a_{ij}^k \in A.$$

Similarily, we let $m = max_{i,j,k}\{m_{ijk}\}$, so $\bar{a}_{ij}^k = f_{ij}^m a_{ij}^k \in B$. Because $Spec(A) = \cup(D(f_{ij})) = \cup(D(f_{ij}^{m+n}))$, so $\exists\{b_{ij}\}\subset B$,

$$\sum(b_{ij}f_{ij}^{m+n}) = 1,$$

$$\Rightarrow x = x\left(\sum_{i,j}(b_{ij}f_{ij}^{m+n})\right) = \sum_{i,j}\left(b_{ij}f_{ij}^{m+n}\left(\sum_k(a_{ij}^k e_{ij}^k)\right)\right) = \sum_{i,j}\sum_k\left((b_{ij}f_{ij}^m a_{ij}^k)(f_{ij}^n e_{ij}^k)\right) = \sum_{i,j,k}(b_{ij}\bar{a}_{ij}^k\bar{e}_{ij}^k),$$

which means that x can be B-linear represented by $\{\bar{e}_{ij}^k\}$. $\square$

Exercise 2.3.5. Proof:

(a) $\forall y \in Y, \exists U = Spec(B) \in Y.\mathscr{U}(y), f^{-1}(U) = Spec(A), A \in \mathbf{Alg}(B) \cap \mathbf{Mod}(B).\mathbf{FinGen}$. We denote that $\varphi : A \to B$. Because $f^{-1}(y) \cong_{\mathbf{top}} Spec(A) \times_{Spec(B)} Spec(k(y))$, so we need only to show that there are finite prime ideals in $A \otimes_B k(y)$.

$\forall \mathfrak{p} \in A \otimes_B k(y), (A \otimes_B k(y))/\mathfrak{p} \in \mathbf{Rng}.\mathbf{Domain}$. Because it is also a finite $k(y)$-algebra, so it is a field. That means each prime ideal in $A \otimes_B k(y)$ is maximal ideal.

We suppose that $\{\mathfrak{m}_i\}_{i\in I}$ is a group of distinct prime ideals in $A\otimes_B k(y)$ and $k(y)$-bases $\{x_j\}_{j=1}^n$. We can choose each generator in $\mathfrak{m}_i, \forall i \in I$, that means $\forall i \in I, \exists \{a_i^j\}_{j=1}^n \subset k(y), \alpha_i = \sum_{j=1}^n a_i^j x_j \in \mathfrak{m}_i$. If $\#I$ is not finite set, then there are also infinite $k(y)$-independent elements $\{\alpha_i\}$. So it must be finite. $\square$

(b)

(c)

Exercise 2.3.6. Proof:

Because X is integral, so ξ is unique. And $\forall U \in X.\tau, \xi \in U$ according to the previous work. That means

$$\mathscr{O}_{X,\xi} = \varinjlim_{U\in X.\mathscr{U}(\xi)} \mathscr{O}_X(U) = \varinjlim_{U\in X.\tau} \mathscr{O}_X(U).$$

$\forall x \in \mathscr{O}_{X,\xi}, x \neq 0, \exists U = Spec(B) \in X.\tau, \exists b \in B - \{0\}, x = \bar{b} = \overline{(b|_{D(b)})}$. And $(b|_{D(b)}) \in Spec((B)_b).\mathbf{Unit}$. So $x^{-1} = \overline{(b|_{D(b)})^{-1}} \in \mathscr{O}_{X,\xi}$.

We denote that $U = Spec(A)$. So

$$\mathscr{O}_{X,\xi} = \varinjlim_{V\in X.\mathscr{U}(\xi)} \mathscr{O}_X(V) = \varinjlim_{V\in U.\mathscr{U}(\xi)} \mathscr{O}_X|_U(V) \cong A_{(0)}.$$

$\square$

Exercise 2.3.7. Proof:

Exercise 2.3.8.

Exercise 2.3.9.

Exercise 2.3.10. Proof:

(a) We can let $Spec(A) \in Y.\mathscr{U}(y), Spec(B) \in X.\tau, Spec(B) \cap f^{-1}(y) \neq \emptyset$ and $\mathfrak{p} \in A.\mathbf{Prime}, \mathfrak{p} \sim y$. According to the morphism f, we can obtain a homomorphism $\varphi : A \to B$. Obviously, the topology of $f^{-1}(y) \cap Spec(B) = Spec\left(B_{\varphi(\mathfrak{p})}/(\varphi(\mathfrak{p})B_{\varphi(\mathfrak{p})})\right)$ should be homeomorphism to $Spec(B) \times_{Spec(A)} Spec(A_\mathfrak{p}/\mathfrak{p})$. We have $A_\mathfrak{p} \otimes_A B \cong B_{\varphi(\mathfrak{p})}$ and $A/\mathfrak{p} \otimes_A B \cong B/(\varphi(\mathfrak{p})B)$. So

$$\begin{aligned}(A_\mathfrak{p}/(\mathfrak{p}A_\mathfrak{p})) \otimes_A B &\cong (A_\mathfrak{p} \otimes_A A/\mathfrak{p}) \otimes_A B \\ &\cong A_\mathfrak{p} \otimes_A (A/\mathfrak{p} \otimes_A B) \\ &\cong A_\mathfrak{p} \otimes_A B/(\varphi(\mathfrak{p})B) \\ &\cong B_{\varphi(\mathfrak{p})}/(\varphi(\mathfrak{p})B_{\varphi(\mathfrak{p})}).\end{aligned}$$

So we have isomorphism of affine scheme

$$Spec\left(B_{\varphi(\mathfrak{p})}/(\varphi(\mathfrak{p})B_{\varphi(\mathfrak{p})})\right) \cong Spec\left(A_\mathfrak{p}/(\mathfrak{p}A_\mathfrak{p})\right).$$

In sum, $f^{-1}(y) \cong X \times_Y Spec(k(y))$. $\square$

(b)

Exercise 2.3.11.

2.4 Separated and Proper Morphisms

2.5 Sheaves of Modules

Q 2.5.1. Denote $(X, \mathscr{O}_X)$ is a ring space, $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_X$-module sheaves over X, give an example that

$$U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}(U)} \mathscr{G}(U)$$

cannot be a sheaf.

Solve:

We consider thet topological space that $X.\tau = \{X, U_1, U_2, \emptyset\}, U_1 \cap U_2 = \emptyset$ and ring sheaf is $\mathscr{O}(X) = \mathscr{O}(U_1) = \mathscr{O}(U_2) = \mathbb{Z}$. We let

$$\mathscr{F}(X) = \mathbb{Z}^{\oplus 2}, \mathscr{F}(X) \xrightarrow{pr_1} \mathscr{F}(U_1), \mathscr{F}(X) \xrightarrow{pr_2} \mathscr{F}(U_2),$$

$$\mathscr{G}(X) = \mathbb{Z}^{\oplus 2}, \mathscr{G}(X) \xrightarrow{pr_1} \mathscr{G}(U_1), \mathscr{G}(X) \xrightarrow{pr_2} \mathscr{G}(U_2),$$

In X, $\mathscr{F}(X) \otimes \mathscr{G}(X) \cong \mathbb{Z}^{\oplus 4}$, but $\mathscr{F}(U_i) \otimes \mathscr{G}(U_i) \cong \mathbb{Z}^{\oplus 2}$. For example, if we select $a \oplus b \in \mathscr{F}(X), c \oplus d \in \mathscr{G}(X)$,

$$(a \otimes c) \oplus (a \otimes d) \oplus (b \otimes c) \oplus (b \otimes d) \in \mathscr{F}(X) \otimes \mathscr{G}(X).$$

We can let $a = 0, d = 0$, then it leaves $b \otimes c$, but

$$0 = a \otimes c \in \mathscr{F}(U_1) \otimes \mathscr{G}(U_1), 0 = b \otimes d \in \mathscr{F}(U_2) \otimes \mathscr{G}(U_2).$$

So $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}(U)} \mathscr{G}(U)$ is only a presheaf but not a sheaf. □

Q 2.5.2. An example that "tensor product of ideals isn't equal to product of ideals". $A = k[x, y], I = (x, y)$, prove that

$$I \otimes_A I \to I^2 \notin \mathbf{Inj}.$$

Solve: We denote that $\varphi : I \otimes_A I \to I^2 \notin, \forall f, g \in I$,

$$\varphi(f \otimes_A g) = fg.$$

$f \otimes g - g \otimes f \neq 0 \in I \otimes_A I$, but $\varphi(f \otimes g - g \otimes f) = \varphi(f \otimes g) - \varphi(g \otimes f) = fg - gf = 0 \in I^2$. □

Exercise 2.5.1.

Exercise 2.5.2.

Exercise 2.5.3.

Exercise 2.5.4.

Exercise 2.5.5.

2.6 Divisors

2.7 Projective Morhpisms

Q 2.7.1. Please verify the questions left by the construction of $\mathbf{Proj}\mathscr{S}$ in the book that

$$\pi_U^{-1}(U \cap V) \cong \pi_V^{-1}(U \cap V).$$

Proof:

$\forall U \in X.\tau, \mathscr{S}_1(U) = M$, where M is an finitely generated A-module by $\{m_i\}_{i=1}^N$. So

$$\mathscr{S}(U) = A[m_1, ..., m_N] \cong A \otimes_{\mathbb{Z}} \mathbb{Z}[m_1, ..., m_N] \in \mathbf{Rng}.\mathbf{Grad}.$$

Similarly, $V \in X.\tau, \mathscr{S}_1(V) = M'$, where M' is generated by $\{m'_j\}_{j=1}^{N'}$ as B-module. So

$$\mathscr{S}(V) = B[m'_1, ..., m'_{N'}] \cong B \otimes_{\mathbb{Z}} \mathbb{Z}[m'_1, ..., m'_{N'}] \in \mathbf{Rng}.\mathbf{Grad}.$$

Because they are both things on $U \cap V$, so there is alywas an isomorphism mapping

$$W = SpecA_f \cong SpecB_g \subseteq U \cap V.$$

So we denote it as $W = SpecC$, we need to prove for each $W \subseteq U \cap V$, there is always isomorphism

$$\pi_U^{-1}(W) \cong \pi_V^{-1}(W)$$

$$\Rightarrow C \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{m}_1, ..., \bar{m}_N] \cong C \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{m}'_1, ..., \bar{m}'_{N'}].$$

Because $M_f = \mathscr{S}_1|_U(W) = \mathscr{S}_1(W) = \mathscr{S}_1|_V(W) = M'_g$, $\exists \{c_{ij}\}_{i=j=1}^{i=N, j=N'} \in C$,

$$\begin{bmatrix} c_{11} & ... & c_{1N} \\ \vdots & \ddots & \vdots \\ c_{N'1} & ... & c_{N'N} \end{bmatrix} \begin{bmatrix} \bar{m}_1 \\ \vdots \\ \bar{m}_N \end{bmatrix} = \begin{bmatrix} \bar{m}'_1 \\ \vdots \\ \bar{m}'_{N'} \end{bmatrix},$$

and it can be inverse as C-module. As a $\mathbb{Z}$-module, it may not be linearly isomorphic, but it is after tensoring with C. □

2.8 Differentials

2.9 Formal Schemes

3 Cohomology

3.1 Derived Functors

3.2 Cohomology of Sheaves

Exercise 3.2.1. Proof:

(a) We can denote inclusion

$$j : X - \{P, Q\} = U \to X, i : \{P, Q\} = Z = X - U \to X.$$

Then we have exact sequence,

$$0 \to j_!(\mathbb{Z}|_U) \to \mathbb{Z} \to i_*(\mathbb{Z}|_Z) \to 0.$$

Combining the long exact sequence and the vanishing theorem, we obtain

$$... \to H^0(X, \mathbb{Z}) \to H^0(X, i_*(\mathbb{Z}|_Z)) \to H^1(X, j_!(\mathbb{Z}|_U)) \to H^1(X, \mathbb{Z}) \to H^1(X, i_*(\mathbb{Z}|_Z)) \to 0 \to ...$$

Because of $dimZ = 0$, $H^1(X, i_*(\mathbb{Z}|_Z)) = 0$. So we have

$$... \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H^1(X, \mathbb{Z}|_U) \to H^1(X, \mathbb{Z}) \to 0 \to ...$$

$(\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}) \notin \mathbf{Sur} \Rightarrow H^1(X, \mathbb{Z}|_U) \neq \emptyset$. $(H^1(X, \mathbb{Z}|_U) \to H^1(X, \mathbb{Z})) \in \mathbf{Sur} \Rightarrow H^1(X, \mathbb{Z}) \neq \emptyset$. $\square$

(b)

3.3 Cohomology of a Noetherian Affine Scheme

4 Curves

5 Surfaces