This article is mainly extracted from David Cox's Toric Varieties Chap6.1 with some detailed added.

Def. Let D be a Cartier divisor on a complete normal variety X. $W = \Gamma(X, \mathcal{O}_X(D))$ is finite-dimension.

a) This divisor D and the line bundle $\mathcal{O}_X(D)$ are **very ample** when D has no base-points and $\varphi_D = \varphi_{\mathcal{O}_X(D),W}: X \to \mathbb{P}(W^\wedge)$ is a closed embedding.

b) D and $\mathcal{O}_X(D)$ are **ample** when kD is very ample for some integer $k > 0$.

Basepoint Free Divisor. Consider the toric variety X_Σ of a complete fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$ and let $D = \sum_\rho a_\rho D_\rho$ be a torus-invariant Cartier divisor on X_Σ. We have the global sections

$$\Gamma\left(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)\right) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m$$

where $P_D \subset M_\mathbb{R}$ is the polytope defined by

$$P_D = \{m \in M_\mathbb{R} : \langle m, u_\rho \rangle \geq -a_\rho, \forall \rho \in \Sigma(1)\}$$

We first study when $D = \sum_\rho a_\rho D_\rho$ is basepoint free. For every $\sigma \in \Sigma$, there is $m_\sigma \in M$ with

$$\langle m_\sigma, u_\rho \rangle = -a_\rho, \rho \in \sigma(1)$$

Furthermore, D is uniquely determined by the Cartier data $\{m_\sigma\}_{\sigma \in \Sigma(n)}$ since Σ is complete.

Prop. The following are equivalent:

a) D has no basepoints, i.e., $\mathcal{O}_{X_\Sigma}(D)$ is generated by global sections.

b) $m_\sigma \in P_D, \forall \sigma \in \Sigma(n)$.

Pf:

First suppose that D is generated by global sections and take $\sigma \in \Sigma(n)$. The T_N-orbit corresponding to σ is a fixed point p of the T_N-action, and by the Orbit-Cone Correspondence,

$$\{p\} = \bigcap_{\rho \in \sigma(1)} D_\rho$$

There is a global section s such that p is not in the support of the divisor of zeros $div_0(s)$ of s. Since $\Gamma\left(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)\right)$ is spanned by χ^m for $m \in P_D \cap M$, we can assume that s is given by χ^m for some $m \in P_D \cap M$. The divisor of zeros of s is

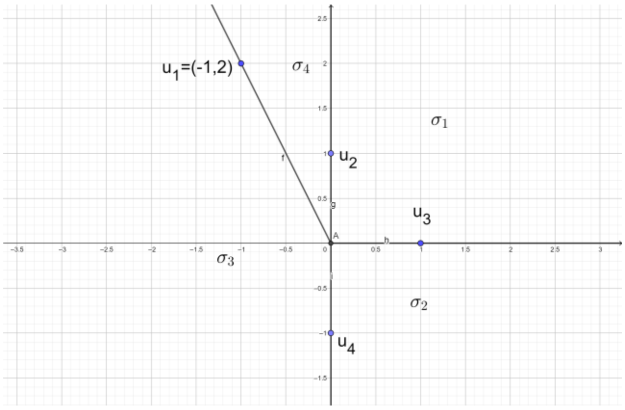
$$div_0(s) = D + div(\chi^m) = \sum_\rho \left(a_\rho + \langle m, u_\rho \rangle\right) D_\rho$$

The point p is not in the support of $div_0(s)$ yet lies in D_ρ for every $\rho \in \sigma(1)$. This forces $a_\rho + \langle m, u_\rho \rangle = 0, \rho \in \sigma(1)$. Since σ is n-dimension, we conclude that.

For this converse, take $\sigma \in \Sigma(n)$. Since $m_\sigma \in P_D$, the character χ^{m_σ} gives a global section s whose divisor of zeros is $div_0(s) = D + div(\chi^{m_\sigma})$. The support of $div_0(s)$ misses U_σ, so that s is nonvanishing on U_σ. Then we done since the U_σ cover X_Σ.■

Eg: the fan for the Hirzebruch surface $\mathcal{H}_2$

$$D = D_4, D' = D_2 + D_4$$



For divisor D, in σ_1, we calculate $m_{\sigma_1} = [x \quad y]$ (variable x, y are temporary),

$$\langle m_{\sigma_1}, u_2 \rangle = 0, \langle m_{\sigma_1}, u_3 \rangle = 0$$

$$\Rightarrow \left([x \quad y] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \right) \wedge \left([x \quad y] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \right)$$

$$\Rightarrow m_{\sigma_1} = [0 \quad 0]$$

In σ_2, we calculate $m_{\sigma_2} = [x \quad y]$,

$$\langle m_{\sigma_2}, u_3 \rangle = 0, \langle m_{\sigma_2}, u_4 \rangle = -1$$

$$\Rightarrow \left([x \quad y] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \right) \wedge \left([x \quad y] \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \right)$$

$$\Rightarrow m_{\sigma_2} = [0 \quad 1]$$

In σ_3, we calculate $m_{\sigma_3} = [x \quad y]$,

$$\langle m_{\sigma_3}, u_4 \rangle = -1, \langle m_{\sigma_3}, u_1 \rangle = 0$$

$$\Rightarrow \left([x \quad y] \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \right) \wedge \left([x \quad y] \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0 \right)$$

$$\Rightarrow m_{\sigma_3} = [2 \quad 1]$$

In σ_4, we calculate $m_{\sigma_4} = [x \quad y]$,

$$\langle m_{\sigma_4}, u_1 \rangle = 0, \langle m_{\sigma_4}, u_2 \rangle = 0$$

$$\Rightarrow \left([x \quad y] \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0 \right) \wedge \left([x \quad y] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \right)$$

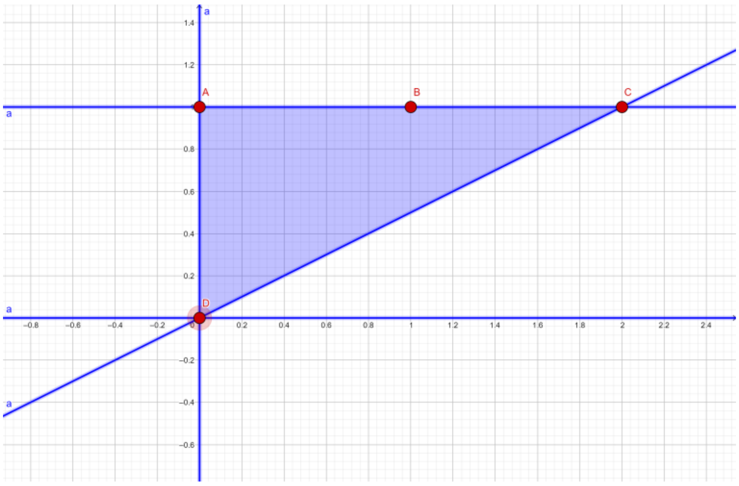
$$\Rightarrow m_{\sigma_4} = [0 \quad 0]$$

So we have

$$\psi_D = \begin{cases} \langle [0 \quad 0], n \rangle, n \in \sigma_1 \\ \langle [0 \quad 1], n \rangle, n \in \sigma_2 \\ \langle [2 \quad 1], n \rangle, n \in \sigma_3 \\ \langle [0 \quad 0], n \rangle, n \in \sigma_4 \end{cases}$$

Then

$$\begin{aligned} P_D &= \{ m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho, \rho \in \Sigma(1) \} \\ &= \{ m \in M_{\mathbb{R}} : \langle m, u_1 \rangle \geq 0, \langle m, u_2 \rangle \geq 0, \langle m, u_3 \rangle \geq 0, \langle m, u_4 \rangle \geq -1 \} \\ &= \{ [0 \quad 0], [0 \quad 1], [1 \quad 1], [2 \quad 1] \} \end{aligned}$$



For divisor D', in σ_1, we calculate $m_{\sigma_1} = [x \quad y]$,

$$\langle m_{\sigma_1}, u_2 \rangle = -1, \langle m_{\sigma_1}, u_3 \rangle = 0$$
$$\Rightarrow \left([x \quad y] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \right) \wedge \left([x \quad y] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \right)$$
$$\Rightarrow m_{\sigma_1} = [0 \quad -1]$$

In σ_2, we calculate $m_{\sigma_2} = [x \quad y]$,

$$\langle m_{\sigma_2}, u_3 \rangle = 0, \langle m_{\sigma_2}, u_4 \rangle = -1$$
$$\Rightarrow \left([x \quad y] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \right) \wedge \left([x \quad y] \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \right)$$
$$\Rightarrow m_{\sigma_2} = [0 \quad 1]$$

In σ_3, we calculate $m_{\sigma_3} = [x \quad y]$,

$$\langle m_{\sigma_3}, u_4 \rangle = -1, \langle m_{\sigma_3}, u_1 \rangle = 0$$
$$\Rightarrow \left([x \quad y] \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \right) \wedge \left([x \quad y] \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0 \right)$$
$$\Rightarrow m_{\sigma_3} = [2 \quad 1]$$

In σ_4, we calculate $m_{\sigma_4} = [x \quad y]$,

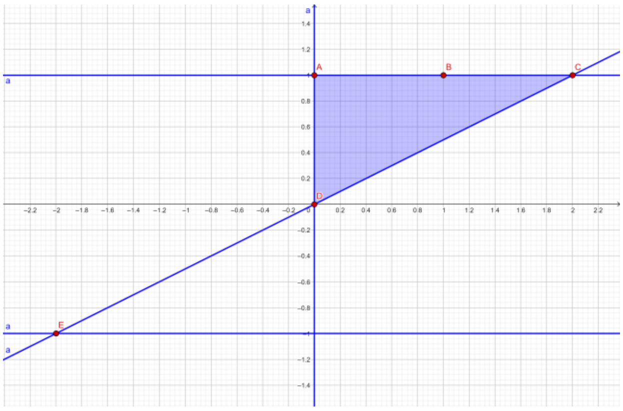
$$\langle m_{\sigma_4}, u_1 \rangle = 0, \langle m_{\sigma_4}, u_2 \rangle = -1$$
$$\Rightarrow \left([x \quad y] \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0 \right) \wedge \left([x \quad y] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \right)$$
$$\Rightarrow m_{\sigma_4} = [-2 \quad -1]$$

So we have

$$\psi_D = \begin{cases} \langle [0 \quad -1], n \rangle, n \in \sigma_1 \\ \langle [0 \quad 1], n \rangle, n \in \sigma_2 \\ \langle [2 \quad 1], n \rangle, n \in \sigma_3 \\ \langle [-2 \quad -1], n \rangle, n \in \sigma_4 \end{cases}$$

Then

$$\begin{aligned} P_D &= \{ m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho, \rho \in \Sigma(1) \} \\ &= \{ m \in M_{\mathbb{R}} : \langle m, u_1 \rangle \geq 0, \langle m, u_2 \rangle \geq -1, \langle m, u_3 \rangle \geq 0, \langle m, u_4 \rangle \geq -1 \} \\ &= \{ [0 \quad 0], [0 \quad 1], [1 \quad 1], [2 \quad 1] \} \end{aligned}$$



Very Ample Polytopes. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ a full dimensional lattice polytope with facet presentation

$$P = \{m \in M_{\mathbb{R}}: \langle m, u_F \rangle \geq -a_F, F \in facets\}$$

This gives the complete normal fan Σ_P and the toric variety X_P. Write

$$P \cap M = \{m_1, \dots, m_s\}$$

A vertex $m_i \in P$ corresponds to a maximal cone

$$\sigma_i = Cone(P \cap M - m_i)^{\vee} \in \Sigma_P(n)$$

$D_P = \sum_F a_F D_F$ is Cartier since $\langle m_i, u_F \rangle = -a_F$ when $m_i \in F$. P is **very ample** if for every vertex $m_i \in P$, the semigroup $\mathbb{N}(P \cap M - m_I)$ is saturated in M.

Prop. Let X_P and D_P be as above. Then:

a) D_P is ample and basepoint free.

b) If $n \geq 2$, then kD_P is very ample for every $k \geq n - 1$.

c) D_P is very ample if and only if P is a very ample polytope.

Support Functions and Convexity. Let $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor on a complete toric variety X_Σ. Support function $\varphi_D: N_{\mathbb{R}} \to \mathbb{R}$ is determined by the following properties:

- φ_D is linear on each cone $\sigma \in \Sigma$.
- $\varphi_D(u_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$.

The explicit formula for $\varphi_D|_\sigma$ is given by $\varphi_D(u) = \langle m_\sigma, u \rangle, \forall u \in \sigma$.

Def. Let $S \subseteq N_{\mathbb{R}}$ be convex. A function $\varphi: S \to \mathbb{R}$ is convex if

$$\varphi(tu + (1-t)v) \geq t\varphi(u) + (1-t)\varphi(v)$$

for all $u, v \in S, t \in [0,1]$.

Lemma. For the support function φ_D, the following are equivalent:

a) φ_D is convex

b) $\varphi_D(u) \leq \langle m_\sigma, u \rangle, u \in N_{\mathbb{R}}, \sigma \in \Sigma(n)$

c) $\varphi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u \rangle, u \in N_{\mathbb{R}}$

d) For every wall $\tau = \sigma \cap \sigma'$, there is $u_0 \in \sigma' \setminus \sigma$ with $\varphi_D(u_0) \leq \langle m_\sigma, u_0 \rangle$.

Lemma. Let Σ be a fan and $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor on X_Σ. Then

$$P_D = \{m \in M_{\mathbb{R}}: \varphi_D(u) \leq \langle m, u \rangle, \forall u \in |\Sigma|\}$$

Thm. Assume Σ is complete and let φ_D be the support function of a Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on X_Σ. Then the following are equivalent:

a) D is a basepoint free.

b) $m_\sigma \in P_D, \forall \sigma \in \Sigma(n)$.

c) $P_D = Conv(m_\sigma : \sigma \in \Sigma(n))$.

d) $\{m_\sigma : \sigma \in \Sigma(n)\}$ is the set of vertices of P_D

e) $\varphi_D(u) = \min_{m \in P_D} \langle m, u \rangle, \forall u \in N_\mathbb{R}$

f) $\varphi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u \rangle, \forall u \in N_\mathbb{R}$.

g) $\varphi_D : N_\mathbb{R} \to \mathbb{R}$ is convex.

Pf:

The equivalences $(a) \Leftrightarrow (b)$ and $(f) \Leftrightarrow (g)$ were proved. Furthermore,

$$\varphi_D \text{ is convex} \Leftrightarrow \varphi_D(u) \le \langle m_\sigma, u \rangle, \forall \sigma \in \Sigma(n), u \in N_\mathbb{R}$$
$$\Leftrightarrow m_\sigma \in P_D, \forall \sigma \in \Sigma(n)$$

This proves $(g) \Leftrightarrow (b)$, so that $(a), (b), (f), (g)$ are equivalent.

$(b) \Rightarrow (e)$. $m_\sigma \in P_D$ and $\varphi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u \rangle$. Combining these, we obtain

$$\varphi_D(u) \le \min_{m \in P_D} \langle m, u \rangle \le \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u \rangle = \varphi_D(u)$$

The implication $(e) \Rightarrow (g)$ follows since the minimum of a compact set of linear functions is convex. So $(a) \Leftrightarrow (b) \Leftrightarrow (e) \Leftrightarrow (f) \Leftrightarrow (g)$.

Consider (d). The implications $(d) \Rightarrow (c) \Rightarrow (b)$ are clear. For $(b) \Rightarrow (d)$, take $\sigma \in \Sigma(n)$. Let u be in the interior of σ and set $a = \varphi_D(u)$. $H_{u,a} = \{m \in M_\mathbb{R} : \langle m, u \rangle = a\}$ is a supporting hyperplane of P_D and

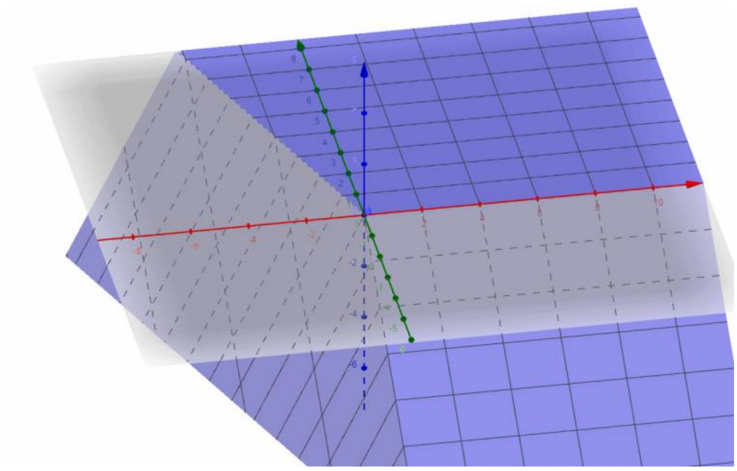
$$H_{u,a} \cap P_D = \{m_\sigma\}$$

This implies that m_σ is a vertex of P_D. Conversely, let $H_{u,a}$ be a supporting hyperplane of a vertex $v \in P_D$. This means $\langle m, n \rangle \ge a$ for all $m \in P_D$, with equality iff $m = v$. Since (b) holds, we also have $(e), (f)$. By (e), $\varphi_D(u) = \min_{m \in P_D} \langle m, u \rangle = \langle m, v \rangle = a$. Combining this with (f), we obtain

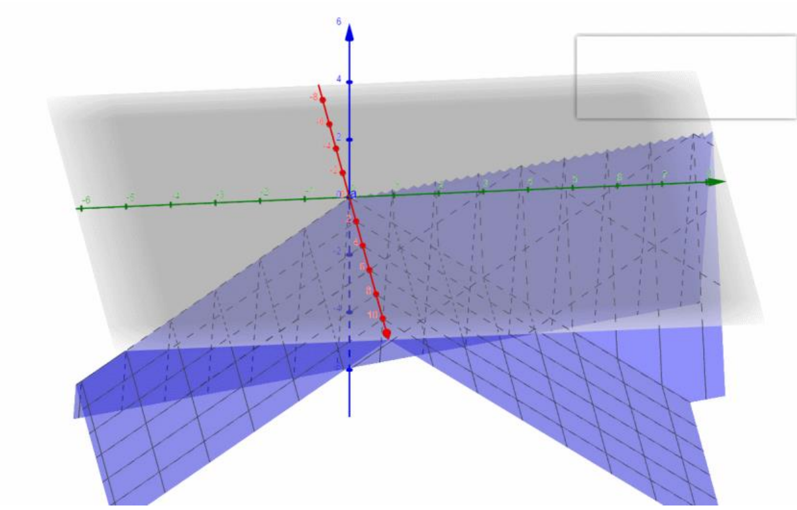
$$\varphi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u \rangle = a$$

Hence $\langle m_\sigma, u \rangle = a$ must occur for some $\sigma \in \Sigma(n)$, which forces $v = m_\sigma$.■

Consider the example above again, $\mathcal{H}_2$.



φ_D is convex, D is basepoint free



$\varphi_{D'}$ is not convex, D is not basepoint free

Ampleness and Strict Convexity. Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on X_Σ is ample. The Cartier data $\{m_\sigma\}_{\sigma\in\Sigma(n)}$ of D satisfies

$$\langle m_\sigma, u\rangle = \varphi_D(u), \forall u \in \sigma$$

Lemma. For the support function φ_D the following are equivalent:

a) φ_D is strictly convex.

b) $\varphi_D(u) < \langle m_\sigma, u\rangle, \forall u \notin \sigma, \sigma \in \Sigma(n)$.

c) For every wall $\tau = \sigma \cap \sigma'$, there is $u_0 \in \sigma' \setminus \sigma$ with $\varphi_D(u_0) < \langle m_\sigma, u_0\rangle$

d) φ_D is convex and $m_\sigma \neq m_{\sigma'}$, when $\sigma \neq \sigma'$ in $\Sigma(n)$ and $\sigma \cap \sigma'$ is wall.

e) φ_D is convex and $m_\sigma \neq m_{\sigma'}$, when $\sigma \neq \sigma'$ in $\Sigma(n)$.

f) $\langle m_\sigma, u_\rho\rangle > -a_\rho, \forall \rho \in \Sigma(1) \setminus \sigma(1)$ and $\sigma \in \Sigma(n)$

g) $\varphi_D(u+v) > \varphi_D(u) + \varphi_D(v), \forall u, v \in N_\mathbb{R}$ not in the same cone of Σ.

Thm. Assume that φ_D is the support function of a Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on a complete toric variety X_Σ. Then

$$D \textit{ is ample} \Leftrightarrow \varphi_D \textit{ is strictly convex}$$

Furthermore, if $n \geq 2$ and D is ample, then kD is very ample for all $k \geq n-1$..