HW1, Electromagnetism II, Spring 2017

Deheng Song

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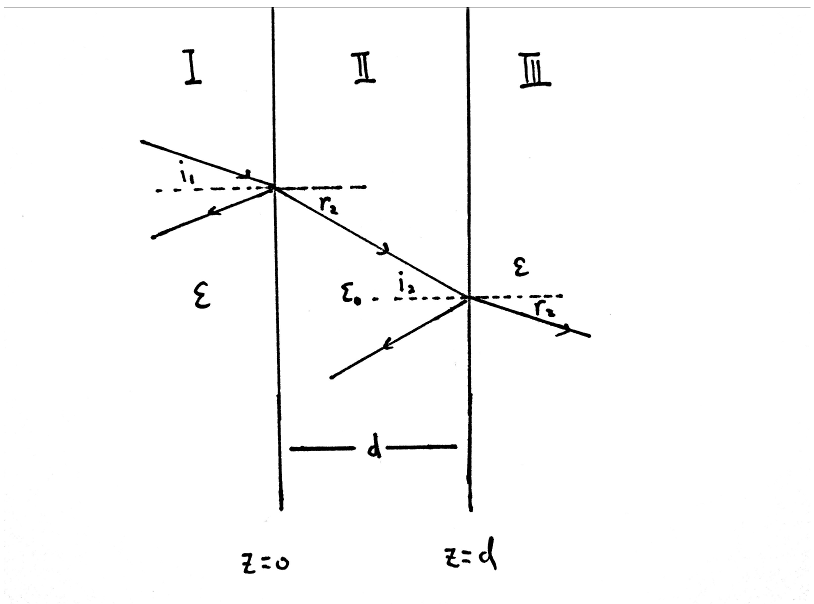
Jackson 7.3 (a)



The path and symbols are shown as above.The waves in three regions are:

Region I:

$$\vec{E}_1 = \vec{E}_1 e^{i\vec{k}_1\cdot\vec{x}-i\omega t} \qquad \vec{B}_1 = \sqrt{\mu_0\varepsilon}\frac{\vec{k}_1\times\vec{E}_1}{k_1}$$

$$\vec{E}'_1 = \vec{E}'_1 e^{i\vec{k}'_1\cdot\vec{x}-i\omega t} \qquad \vec{B}'_1 = \sqrt{\mu_0\varepsilon}\frac{\vec{k}'_1\times\vec{E}'_1}{k'_1}$$

Region II:

$$\vec{E}_2 = \vec{E}_2 e^{i\vec{k}_2\cdot\vec{x}-i\omega t} \qquad \vec{B}_2 = \sqrt{\mu_0\varepsilon_0}\frac{\vec{k}_2\times\vec{E}_2}{k_2}$$

$$\vec{E}'_2 = \vec{E}'_2 e^{i\vec{k}'_2\cdot\vec{x}-i\omega t} \qquad \vec{B}'_2 = \sqrt{\mu_0\varepsilon_0}\frac{\vec{k}'_2\times\vec{E}'_2}{k'_2}$$

Region III:

$$\vec{E}_3 = \vec{E}_3 e^{i\vec{k}_3\cdot\vec{x}-i\omega t} \qquad \vec{B}_3 = \sqrt{\mu_0\varepsilon}\frac{\vec{k}_3\times\vec{E}_3}{k_3}$$

where $k_1 = k_1' = k_3 = \omega\sqrt{\mu_0\varepsilon}$ and $k_2 = k_2' = \omega\sqrt{\mu_0\varepsilon_0}$. The phases match on the plane at $z = 0$ and $z = d$,

$$k_1 \sin i_1 = k_1' \sin i_1 = k_2 \sin r_1$$
$$k_2 \sin i_2 = k_2' \sin i_2 = k_2 \sin r_2$$
$$r_1 = i_2$$

Next we consider the boundary conditions at $z = 0$ and $z = d$.

I-II:

$$[\varepsilon(\vec{E}_1 + \vec{E}_1') - \varepsilon_0\vec{E}_2 - \varepsilon_0\vec{E}_2'] \cdot \vec{n} = 0$$
$$[\vec{k}_2 \times \vec{E}_1 + \vec{k}_1' \times \vec{E}_1' - \vec{k}_2 \times \vec{E}_2 - \vec{k}_2' \times \vec{E}_2'] \cdot \vec{n} = 0$$
$$[\vec{E}_1 + \vec{E}_1' - \vec{E}_2 - \vec{E}_2'] \times \vec{n} = 0$$
$$[\vec{k}_1 \times \vec{E}_1 + \vec{k}_1' \times \vec{E}_1' - \vec{k}_2 \times \vec{E}_2 - \vec{k}_2' \times \vec{E}_2'] \times \vec{n} = 0$$

II-III:

$$[\varepsilon_0(\vec{E}_2 + \vec{E}_2') - \varepsilon\vec{E}_3] \cdot \vec{n} = 0$$
$$[\vec{k}_2 \times \vec{E}_2 + \vec{k}_2' \times \vec{E}_2' - \vec{k}_3 \times \vec{E}_3] \cdot \vec{n} = 0$$
$$[\vec{E}_2 + \vec{E}_2' - \vec{E}_3] \times \vec{n} = 0$$
$$[\vec{k}_2 \times \vec{E}_2 + \vec{k}_2' \times \vec{E}_2' - \vec{k}_3 \times \vec{E}_3] \times \vec{n} = 0$$

Let's discuss two different cases.

Case I

When the electric field perpendicular to the plane of incidence, the boundary conditions turn to,

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -\sqrt{\varepsilon}\cos i_1 & -\sqrt{\varepsilon_0}\cos r_1 & \sqrt{\varepsilon_0}\cos r_1 & 0 \\ 0 & e^{i\phi} & e^{-i\phi} & -1 \\ 0 & \sqrt{\varepsilon_0}\cos i_2 e^{i\phi} & -\sqrt{\varepsilon_0}\cos i_2 e^{-i\phi} & -\sqrt{\varepsilon}\cos r_2 \end{bmatrix} \begin{bmatrix} E_1' \\ E_2 \\ E_2' \\ E_3 \end{bmatrix} = \begin{bmatrix} -E_1 \\ \sqrt{\varepsilon}E_1\cos i_1 \\ 0 \\ 0 \end{bmatrix}$$

where the phase ϕ is given by $\phi = kd\cos r_1 = \frac{\omega d\sqrt{1-n^2\sin^2 i_1}}{c}$. Our aim is to find,

$$\frac{E_1'}{E_1} = \frac{i(1-\alpha^2)\sin\phi}{2\alpha\cos\phi - i(1+\alpha^2)\sin\phi}$$
$$\frac{E_3}{E_1} = \frac{2\alpha}{2\alpha\cos\phi - i(1+\alpha^2)\sin\phi}$$

where $\alpha = \frac{n\cos i_1}{\sqrt{1-n^2\sin^{i_1}}}$. To calculate the transmission and reflection coefficients:

$$T = \frac{|E_3|^2}{|E_1|^2} = \frac{4\alpha^2}{4\alpha^2 + (1-\alpha^2)^2\sin^2\phi}$$
$$R = \frac{|E_1'|^2}{|E_1|^2} = \frac{(1-\alpha^2)^2\sin^2\phi}{4\alpha^2 + (1-\alpha^2)^2\sin^2\phi}$$

Case II

Similarly, we can write the boundary condition in matrix whenthe electric field is parallel to the plane of incidence.

$$\begin{bmatrix} -\cos i_1 & -\cos r_1 & \cos r_1 & 0 \\ \sqrt{\varepsilon} & -\sqrt{\varepsilon_0} & -\sqrt{\varepsilon_0} & 0 \\ 0 & \cos i_2 e^{i\phi} & -\cos i_2 e^{-i\phi} & -\cos r_2 \\ 0 & \sqrt{\varepsilon_0} e^{i\phi} & \sqrt{\varepsilon_0} e^{-i\phi} & \sqrt{\varepsilon} \end{bmatrix} \begin{bmatrix} E_1' \\ E_2 \\ E_2' \\ E_3 \end{bmatrix} = \begin{bmatrix} -\cos i_1 E_1 \\ -\sqrt{\varepsilon} E_1 \\ 0 \\ 0 \end{bmatrix}$$

In this case,

$$T = \frac{|E_3|^2}{|E_1|^2} = \frac{4\beta^2}{4\beta^2 + (1-\beta^2)^2 \sin^2 \phi}$$

$$R = \frac{|E_1'|^2}{|E_1|^2} = \frac{(1-\beta^2)^2 \sin^2 \phi}{4\beta^2 + (1-\beta^2)^2 \sin^2 \phi}$$

where $\beta = \frac{\cos i_1}{n\sqrt{1-n^2 \sin^2 i_1}}$.

Jackson 7.12 (a)

In absence of external magnetic field, the equations are given as below:

$$\vec{J} = \sigma(\omega)\vec{E}$$
$$\frac{\partial \rho}{\partial t} + \nabla \times \vec{J} = 0$$
$$\nabla \times \vec{E} = \frac{\rho}{\varepsilon_0}$$

Substitute the first and the third equations to the second one, we find

$$\frac{\partial \rho}{\partial t} + \frac{\sigma(\omega)}{\varepsilon_0}\rho = 0$$

The time-Fourier-transformed charge density is

$$\rho(\vec{x}, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho(\vec{x}, t)e^{-i\omega t}dt$$

Inserting to the continuity equation, we can rewrite it as,

$$\boxed{[-i\omega + \frac{\sigma(\omega)}{\varepsilon_0}]\rho(\vec{x}, \omega) = 0]}$$

Jackson 7.12 (b)

From the continuity equation above, we may write

$$\sigma(\omega) - i\omega\varepsilon_0 = 0$$

while $\sigma(\omega) = \sigma_0/(1 - i\omega\tau)$ and $\sigma_0 = \varepsilon_0\omega_p^2\tau$. The equation above can be rewrite as,

$$\tau\varepsilon_0\omega^2 - i\varepsilon_0\omega - \varepsilon + 0\omega_p^2\tau$$

which can be solved in the form of

$$\omega = \frac{i\varepsilon_0 \pm \sqrt{-\varepsilon_0 + 4\varepsilon_0\omega_p^2\tau^2}}{2\tau\varepsilon_0}$$

In the approximation $\omega_p\tau \gg 1$,

$$\sqrt{-\varepsilon_0 + 4\varepsilon_0\omega_p^2\tau^2} \approx 2\varepsilon_0\omega_p\tau$$

The solutions for ω turn to

$$\boxed{\omega = \frac{1}{2\tau} \pm \omega_p}$$

This can be interpret as oscillations at the plasma frequency while decay with a factor $1/2\tau$.

Jackson 7.16 (a)

Write the electric field as,

$$\vec{E} = \vec{E}_0 e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

Then calculate,

$$\nabla \times \vec{E} = i\vec{k} \times \vec{E}, \qquad \frac{\partial}{\partial t}\vec{E} = -i\omega\vec{E}$$

From the Maxwell equations,

$$\nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0$$

$$\nabla \times \vec{H} - \frac{\partial}{\partial t}\vec{D} = 0$$

By taking the curl of the first equation and substitute the second one, we find

$$\nabla \times (\nabla \times \vec{E}) + \mu_0 \frac{\partial^2}{\partial t^2}\vec{D} = 0$$

Using the results above,

$$\boxed{\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0 \omega^2 \vec{D} = 0}$$

Jackson 7.16 (b)

With $\vec{k} = k\vec{n}$ and $\vec{D} = \varepsilon\vec{E}$($\varepsilon$ is a 3 by 3 matrix), we may write the result in part (a) as

$$k^2[\vec{n}(\vec{n}\cdot\vec{E}) - \vec{E}(\vec{n}\cdot\vec{n})] + \mu_0\omega^2\varepsilon\vec{E} = 0$$

When choosing the principle axes as the coordinate axes, the components of displacement can be written as $D_i = \varepsilon_i E_i$. Form the equation above in matrix,

$$\begin{bmatrix} n_1^2-1 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2^2-1 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3^2-1 \end{bmatrix}\vec{E} + v^2\begin{bmatrix} 1/v_1^2 & & \\ & 1/v_2^2 & \\ & & 1/v_3^2 \end{bmatrix}\vec{E} = 0$$

where we used $v = \omega/k$ and $v_i = 1/\sqrt{\mu_0\varepsilon_0}$. The nontrivial solution exist only when the determinant is equal to zero.

$$\begin{aligned}
&\begin{vmatrix} n_1^2-1+\frac{v^2}{v_1^2} & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2^2-1+\frac{v^2}{v_2^2} & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3^2-1+\frac{v^2}{v_3^2} \end{vmatrix} \\
=&\frac{v^6}{v_1^2v_2^2v_3^2} - (\frac{n_1^2+n_2^2}{v_1^2v_2^2} + \frac{n_1^2+n_3^2}{v_1^2v_3^2} + \frac{n_2^2+n_3^2}{v_2^2+v_3^2})v^4 + (\frac{n_1^2}{v_1^2} + \frac{n_2^2}{v_2^2} + \frac{n_3^2}{v_3^2})v^2 \\
=&\frac{v^2}{v_1^2v_2^2v_3^2}[n_1^2(v^2-v_2^2)(v^2-v_3^2) + n_2^2(v^2-v_1^2)(v^2-v_3^2) + n_3^2(v^2-v_1^2)(v^2-v_2^2)] \\
=&\frac{v^2}{v_1^2v_2^2v_3^2}\frac{1}{(v^2-v_1^2)(v^2-v_2^2)(v^2-v_3^2)}[\frac{n_1^2}{v^2-v_1^2} + \frac{n_2^2}{v^2-v_2^2} + \frac{n_3^2}{v^2-v_3^2}] = 0
\end{aligned}$$

Except for trivial solution $v = 0$, we have two distinct solutions for v, given by the quadratic equation. For each solution, the phase velocity satisfies the Fresnel equation,

$$\boxed{\sum_{i=1}^{3}\frac{n_i^2}{v^2-v_i^2} = 0}$$

Jackson 7.16 (c)

From the equation in part a, we have

$$\vec{n} \times (\vec{n} \times \vec{E}_a) = -\mu_0 v_a^2 \vec{D}_a$$

Hence the dot product of the two displacement can be written as,

$$\begin{aligned}
\vec{D}_a \cdot \vec{D}_b &= -\frac{1}{\mu_0 v_a^2}(\vec{n} \times (\vec{n} \times \vec{E}_a)) \cdot \vec{D}_b \\
&= -\frac{1}{\mu_0 v_a^2}(\vec{n}(\vec{n} \cdot \vec{E}_a) - \vec{E}_a) \cdot \vec{D}_b \\
&= \frac{1}{\mu_0 v_a^2}\vec{E}_a \cdot \vec{D}_b
\end{aligned}$$

Meanwhile we can also write the product as

$$\vec{D}_a \cdot \vec{D}_b = \frac{1}{\mu_0 v_b^2}\vec{D}_a \cdot E_b$$

Thus,

$$\mu_0(v_a^2 - v_b^2)\vec{D}_a \cdot \vec{D}_b = \vec{E}_a \cdot \vec{D}_b - \vec{D}_a \cdot \vec{E}_b$$

where $\vec{E}_a \cdot \vec{D}_b - \vec{D}_a \cdot \vec{E}_b = E_i^a \varepsilon_i E_i^b - \varepsilon_i E_i^a E_i^b = 0$. Since v_a and v_b are distinct solution for the wave, $v_a^2 \neq v_b^2$. Hence,

$$\boxed{\vec{D}_a \cdot \vec{D}_b = 0}$$

Jackson 7.28

Since the wave is circularly polarized, we can write the electric filed as

$$\vec{E}(x,y,z,t) \simeq [E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + F(x,y)\vec{e}_3]e^{ikz-i\omega t}$$

The field need to satisfy $\nabla \cdot \vec{E} = 0$,

$$\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y} + ikF(x,y) = 0$$

$$F(x,y) = \frac{i}{k}(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y})$$

Thus the electric are given approximately by

$$\vec{E}(x,y,z,t) \simeq [E_0(x,y)(\vec{e}_1 \pm i\vec{e}_2) + \frac{i}{k}(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y})\vec{e}_3]e^{ikz-i\omega t}$$

For the magnetic field, we have

$$\nabla \times \vec{E} - i\omega\vec{B} = 0$$

Since the amplitude changes slowly, the second order derivative for E_0 can be neglect.

$$\begin{aligned}
\vec{B} &\simeq \frac{1}{i\omega}\nabla \times E_0(\vec{e}_1 \pm i\vec{e}_2)e^{ikz-i\omega t} \\
&= \frac{k}{i\omega}[\pm iE_0\vec{e}_1 + E_0\vec{e}_2 + (\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y})]e^{ikz-i\omega t} \\
&= \mp\frac{k}{\omega}\vec{E}
\end{aligned}$$

where $k/\omega = \sqrt{\mu\varepsilon}$