HW3, Electromagnetism II, Spring 2017

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Jackson 9.1 (a)

For the rotating charges, the charge density

$$\rho(\vec{x},t)=\rho(r,\theta,\phi-\omega t)$$

Using this charge density to calculate the time-dependent multipole moments directly,

$$\begin{aligned}
q_{lm}(t) &= \int r^l Y^*_{lm}(\theta',\phi')\rho(r',\theta',\phi'-\omega t)d^3x' \\
&= \int r^l Y^*_{lm}(\theta',\phi'-\omega t)\rho(r',\theta',\phi')d^3x' \\
&= [\int r^l Y^*_{lm}(\theta',\phi')\rho(r',\theta',\phi')d^3x']e^{-im\omega t} \\
&= q_{lm}e^{-im\omega t}
\end{aligned}$$

where q_{lm} is the electrostatics multipole moments.With the identity $Y_{l,-m}=(-1)^m Y*_{lm}$, we can write $q_{l,-m}=(-1)^m q_{lm}$.For $m>0$,find the real time denpendence,

$$q_{lm}(t)Y_{lm}+q_{l,-m}(t)Y_{l,-m}=q_{lm}(t)Y_{lm}+(-1)^{2m}q_{lm}(t)Y^*_{lm}=Re[2q_{lm}Y_{lm}]\cos m\omega t$$

while when $m=0$ the time dependence vanishes.

Jackson 9.1 (b)

For the time dependent charge density $\rho(\vec{x}, t)$ with period $T = 2\pi/omega$, we can write out the Fourier series directly,

$$\rho(x,t) = \rho_0(x) + \sum_{n=1}^{\infty} Re[2\rho_n(x)e^{-in\omega t}]$$

where

$$\rho_n(x) = \frac{1}{T}\int_0^T \rho(x,t)e^{in\omega t}dt$$

If we use this density to calculate the multipole moments by the equation,

$$q_{lm}(t) = \int r^l Y_{lm}^* \rho(\vec{x}', t) d^3x'$$

It's easy to check that

- the firt term in the series corresponding to the $m = 0$ terms in part a, for they all have no time dependence.
- the higher order terms corespond to each others when $m = n$, both with a factor 2.

So the results in part a and part b are connected together.

Jackson 9.1 (c)

Write the charge density as,

$$\rho(\vec{x},t) = \frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)$$

For part a, calculate the multipole moments directly,

$$\begin{aligned}
q_{lm} &= \int r^l Y^*_{lm}(\theta,\phi)\frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi) \\
&= qR^l Y^*_{lm}(\frac{\pi}{2},0) \\
&= qR^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)
\end{aligned}$$

while for part b, calculate the $\rho_n(\vec{x})$ first,

$$\begin{aligned}
\rho_n(\vec{x}) &= \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}}\frac{q}{R^2}\delta(r-R)\delta(\cos\theta)\delta(\phi-\omega_0 t)e^{in\omega_0 t}dt \\
&= \frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)e^{in\phi}
\end{aligned}$$

and the multipole moments are,

$$\begin{aligned}
q_{lm} &= \int r^l Y^*_{lm}\frac{q}{R^2}\delta(r-R)\frac{q}{2\pi R^2}\delta(r-R)\delta(\cos\theta)e^{in\phi} \\
&= \frac{qR^l}{2\pi}\int_0^2 \pi Y^*_{lm}(\frac{\pi}{2},\phi)e^{in\phi}d\phi \\
&= \frac{qR^l}{2\pi}\int_0^2 \pi Y^*_{lm}(\frac{\pi}{2},0)e^{i(n-m)\phi}d\phi \\
&= qR^l\delta_{mn}Y^*_{lm}(\frac{\pi}{2},0)
\end{aligned}$$

which gives the same results from part a when $m-n$. For $l=0$ and $l=1$, we find

$$q_{00} = \frac{1}{4\pi}q, \quad 2q_{11} = -\frac{3}{2\pi}qR$$

Jackson 9.3

For electrostatics case, we already know the solution for scalar potential from chapter 3. Write out the potential outside the sphere to the first term,

$$\Phi = V\frac{3}{2}\frac{a^2}{r^2}P_1(\cos\theta) = \frac{3VR^2}{2r^2}\cos\theta$$

While a electric dipole along the z-axis has a potential,

$$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{pz\cos\theta}{r^3}$$

Campare the two results, we find

$$\vec{p} = 6\pi\varepsilon_0 VR^2\hat{z}$$

In the radiation zone the fields take the forms,

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r} = \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}6\pi\varepsilon_0 VR^2(\hat{n}\times\hat{z}) = -\frac{3ck^2\varepsilon_0 VR^2}{2}\frac{e^{ikr}}{r}\sin\theta\hat{\phi}$$

$$\vec{E} = Z_0\vec{H}\times\hat{n} = -\frac{3k^2VR^2}{2}\frac{e^{ikr}}{r}\sin\theta\hat{\phi}\times\hat{n} = -\frac{3k^2VR^2}{2}\frac{e^{ikr}}{r}\sin\theta\sin\theta\hat{\theta}$$

The angular distribution for radiated power,

$$\frac{dP}{d\Omega} = \frac{c^2Z_0}{32\pi^2}k^4|\vec{p}|^2\sin^2\theta = \frac{c^2Z_0}{32\pi^2}k^4 36\pi^2\varepsilon_0^2V^2R^4\sin^2\theta = \frac{9k^4V^2R^4}{8Z_0}\sin^2\theta$$

and integrate over the angular to find the total radiated power

$$P = \int d\Omega\frac{9k^4V^2R^4}{8Z_0}\sin^2\theta = \frac{9k^4V^2R^4}{8Z_0}\int_0^{2\pi}d\phi\int_{-1}^{1}(1-\cos^2\theta)d\cos\theta = \frac{3\pi k^4V^2R^4}{Z_0}$$

Jackson 9.5 (a)

The solution for the scalar potential for the Lorentz gauge is,

$$\Phi(\vec{x},t) = \int d^3x' \int t' \frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta(t+\frac{|\vec{x}-\vec{x}'|}{c}-t)$$

With the sinusoidal time dependence on the charge density, we found

$$\Phi(\vec{x},t) = \Phi(\vec{x})e^{-i\omega t}$$

where

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3x'$$

For long-wavelenth limit,

$$|\vec{x}-\vec{x}'| = r - \hat{n}\cdot\vec{x}'$$

We can write the scalar potential in power series,

$$\lim_{kr\to\infty}\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}(1+\frac{\hat{n}\cdot\vec{x}'}{r})\sum_n \frac{(-ik)^n}{n!}\int \rho(\vec{x}')(\hat{n}\cdot\vec{x}')^n d^3$$

Consider only the dipole terms,

$$\Phi(\vec{x}) = \frac{e^{ikr}}{4\pi\varepsilon_0 r^2}\hat{n}\cdot\vec{p}(1-ikr)$$

where

$$\vec{p} = \int \vec{x}'\rho(\vec{x}')d^3x'$$

Also can show that,

$$\vec{A}(\vec{x}) = -i\frac{\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\vec{p}$$

Jackson 9.5 (b)

The magnetic field is given by

$$\begin{aligned}
\vec{H} &= \frac{1}{\mu_0}\nabla \times \vec{A} = -\frac{i\omega}{4\pi}\nabla \times \frac{e^{ikr}}{r}\vec{p} \\
&= -\frac{i\omega}{4\pi}[\hat{n}\frac{\partial}{\partial r}\frac{e^{ikr}}{r} \times \vec{p} + \frac{e^{ikr}}{r}\nabla \times \vec{p}] \\
&= -\frac{ick}{4\pi}[\frac{e^{ikr}}{r}(ik - \frac{1}{r^2})\hat{n} \times \vec{p} \\
&= \frac{ck^2}{4\pi}(\hat{n} \times \vec{p})\frac{e^{ikr}}{r}(1 - \frac{1}{ikr})
\end{aligned}$$

For the electric field, can show that

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E} = -\nabla\hat{n} \cdot \frac{e^{ikr}}{4\pi\varepsilon_0 r^2}(1 - ikr)\vec{p}$$

Use the identities,

$$\begin{aligned}
\nabla(\hat{n} \cdot (f(r)\vec{p}) &= f(r)\vec{p} + \hat{n}(\nabla \cdot (f(r)\vec{p}) + \hat{n} \times \nabla \times (f(r)\vec{p}) \\
\nabla \cdot (f(r)\vec{p}) &= \vec{p} \cdot \nabla f(r) + f(r)\nabla \cdot \vec{p} \\
\nabla \times (f(r)\vec{p}) &= \nabla f(r) \times \vec{p} + f(r)\nabla \times \vec{p}
\end{aligned}$$

Can show,

$$\vec{E}(\vec{x}) = f(r)\vec{p} + \hat{n}(\vec{p} \cdot \hat{n})\frac{\partial f}{\partial r} + \hat{n} \times \hat{n} \times \vec{p}\frac{\partial f}{\partial r}$$

Jackson 9.11

We start from the charge density:

$$\begin{aligned}\rho &= q[2\delta(x)\delta(y)\delta(z) - \delta(x)\delta(y)\delta(z - a\cos\omega t) - \delta(x)\delta(y)\delta(z + a\cos\omega t)] \\ &= \frac{q}{2\pi r^2 \sin\theta}[2\delta(r)\delta(\theta) - \delta(r - |a\cos\omega t|)\delta(\theta) - \delta(r + |a\cos\omega t|)\delta(\theta - \pi)]\end{aligned}$$

Obviously there is no magnetization in the system. Considering the electric multipole,

- The monopole is zero since the total charge vanishes.
- The dipole vanishes if we consider the symmetry on the z-axis:
$$p = qa\cos\omega t - qa\cos\omega t = 0.$$

So the lowers nonvanishing multipole moments start from a quadrupole. By azimuthal symmetry only $m = 0$ terms exist,

$$\begin{aligned}&Q_{20} = \int r^2 Y_{20}\rho d^3x \quad (9.170) \\ &= \int_0^\infty \int_0^\pi dr d\theta q r^2 \sqrt{\frac{5}{4\pi}}(\frac{3}{2}\cos^2\theta - \frac{1}{2})[2\delta(r)\delta(\theta) - \delta(r - |a\cos\omega t|)\delta(\theta) - \delta(r + |a\cos\omega t|)\delta(\theta - \pi)] \\ &= -2qa^2\cos^2(\omega t)\sqrt{\frac{5}{4\pi}} \\ &= -\sqrt{\frac{5}{\pi}} qa^2\cos^2(\omega t)\end{aligned}$$

which is the lowest multipole moment.

For a pure multipole, the angular distribution is given by (9.151),

$$\frac{dP(2,0)}{d\Omega} = \frac{Z_0}{2k^2}|a(2,0)|^2|X_{20}|^2$$

where

$$\begin{aligned}a_E(2,0) &= \frac{ck^4}{15i}\sqrt{\frac{3}{2}}Q_{20} \quad (9.169) \\ &= -\frac{ck^4}{15i}\sqrt{\frac{15}{2\pi}} qa^2\cos^2(\omega t)\end{aligned}$$

and

$$|X_{20}|^2 = \frac{15}{8\pi}\sin^2\theta\cos^2\theta$$

Finally we can find the time-averaged angular distribution,

$$\frac{dP(20)}{d\Omega} = \frac{Z_0 c^2 k^6 q^2 a^4}{128\pi^2}\sin^2\theta\cos^2\theta$$

While,

$$\int d\Omega \sin^2\theta\cos^2\theta = \int_0^{2\pi} d\phi \int_{-1}^{1} \cos^2\theta - \cos^4\theta d\cos\theta = \frac{8\pi}{15}$$

The total power radiated is

$$P = \frac{Z_0 c^2 k^6 q^2 a^4}{240\pi}$$