HW2, Electromagnetism II, Spring 2017

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Jackson 8.2 (a)

For TEM wave, we have

$$\vec{E} = \vec{E}(\rho, \phi) e^{i(kz-\omega t)}$$
$$\vec{B} = \vec{B}(\rho, \phi) e^{i(kz-\omega t)}$$

with $E_z = 0$ and $B_z = 0$.

By apply the Gauss's law,

$$2\pi\rho l E = \frac{\lambda l}{\varepsilon}$$
$$\vec{E} = \frac{\lambda}{2\pi\rho\varepsilon}\hat{\rho}$$

where λ is the linear charge density alone the line. While

$$\vec{B} = \sqrt{\mu\varepsilon}\hat{z} \times \vec{E} = \sqrt{\mu\varepsilon}\frac{\lambda}{2\pi\varepsilon\rho}\hat{\phi}$$

The peak value of the azimuthal magnetic field at the surface of the inner conductor is given by

$$H_0 = \frac{B(a)}{\mu} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{\lambda}{2\pi\varepsilon a}$$

$$E = \sqrt{\frac{\mu}{\varepsilon}} H_0 \frac{a}{\rho} \qquad B = \mu H_0 \frac{a}{\rho}$$

The time-averaged flux of energy is described by

$$\vec{S} = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}} |H_0|^2 \frac{a^2}{\rho^2}\hat{z}$$

While the total power flow P,

$$P = \int_A \vec{S} \cdot \hat{z} da = \int_0^{2\pi} \int_a^b \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}} H_0^2 \frac{a^2}{\rho} d\rho d\phi$$

$$\boxed{P = \sqrt{\frac{\mu}{\varepsilon}}\pi a^2 H_0^2 \ln(b/a)}$$

Jackson 8.2 (b)

The time-averaged power absorbed per unit area is given by (8.12),

$$\frac{dP_{loss}}{da} = \frac{dP}{2\pi\rho dz} = \frac{\mu_c\omega\delta}{4}|H_{||}|^2$$

For two boundaries $\rho = a$ and $\rho = b$,

$$\frac{dP_{loss}}{dz} = \frac{\pi\mu_c\omega\delta}{2}|H_0|^2 a(1+\frac{a}{b})$$

Assume the transmitted power has the form,

$$P(z) = P_0 e^{-2\gamma z}$$

$$\gamma = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{2P}\frac{dP_{loss}}{dz} = \frac{1}{2[\sqrt{\frac{\mu}{\varepsilon}}\pi a^2 H_0^2 \ln(b/a)]}\frac{\pi\mu_c\omega\delta}{2}|H_0|^2 a(1+\frac{a}{b})$$

$$\boxed{\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\frac{(1/a+1/b)}{\ln(b/a)}}$$

where we used $\delta = \sqrt{2\mu\omega/\sigma}$.

Jackson 8.2 (c)

By definition, the characteristic impedance Z_0 is given by,

$$Z_0 = \frac{\int_a^b E d\rho}{I_a} = \frac{\sqrt{\mu/\varepsilon}H_0 a \ln(b/a)}{I_a}$$

While by applying the Ampere's law inside the dielectric,

$$I_a = 2\pi a H_0$$

Substitute to the equation above, we find

$$\boxed{Z_0 = \frac{\sqrt{\mu/\varepsilon}}{2\pi}\ln(b/a)}$$

Jackson 8.2 (d)

The power loss on the line,

$$U = \frac{1}{2} I^2 R = \frac{dP_{loss}}{dz}$$
$$R = \frac{2}{I_a^2} \frac{dP_{loss}}{dz}$$

Insert the results we found above,

$$\boxed{R = \frac{1}{2\pi\sigma\delta}(\frac{1}{a} + \frac{1}{b})}$$

Refer to (5.157),

$$L = \frac{1}{I^2} \int \frac{\vec{B} \cdot \vec{B}}{\mu} d^3x$$

We already found the azimuthal magnetic field on the boundaries,

$$H_a = H_0 \qquad H_b = H_0 \frac{a}{b}$$

The field penetrate into the conductor is,

$$H_c = H_{||} e^{-\rho/\delta} e^{i\rho/\delta}$$

$$\begin{aligned} L &= \frac{2\pi}{I^2} |H_0|^2 [\mu_c \int_0^a e^{-2(a-\rho)/\delta} \rho d\rho + \mu \int_a^b \frac{1}{\rho} d\rho + \mu_c \frac{a^2}{b^2} \int_0^\infty e^{-2(\rho - b)/\delta} \rho d\rho] \\ &= \frac{2\pi}{I_a^2} |H_0|^2 [\mu a^2 \ln(b/a) + \mu_c a^2 (\frac{1}{a} + \frac{1}{b}) \frac{\delta}{2} + \frac{1}{4} \mu_c e^{-2a/\delta} \delta^2] \\ &\approx \frac{\mu}{2\pi} \ln(\frac{b}{a}) + \frac{\mu_c \delta}{4\pi} (\frac{1}{a} + \frac{1}{b}) \end{aligned}$$

The last term vanishes since $a >> \delta$.

Jackson 8.5 (a)

For TE waves, $\psi = H_z$ satisfies

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2)\psi = 0$$

with the boundary conditions $\frac{\partial \psi}{\partial n} = 0$ at $x = a$, $y = 0$ and $x = y$.

Construct the general solution as,

$$\psi_{mn} = H_0(\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a} + \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{a})$$

Check the boundary conditions:

- At $x = a$:

$$\frac{\partial \psi}{\partial n} = \frac{\partial \psi}{\partial x} = H_0(-\frac{m\pi}{a}\sin m\pi\cos\frac{n\pi y}{a} - \frac{n\pi}{a}\sin n\pi\cos\frac{m\pi y}{a}) = 0$$

- At y=0:

$$\frac{\partial \psi}{\partial n} = -\frac{\partial \psi}{\partial y} \sim \sin 0 + \sin 0 = 0$$

- At x=y:

$$\begin{aligned}\frac{\partial \psi}{\partial n} &= (-\partial_x + \partial_y)\psi \\ = H_0(\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a} &+ \frac{n\pi}{a}\sin\frac{n\pi x}{a}\cos\frac{m\pi x}{a} \\ -\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a} &- \frac{n\pi}{a}\sin\frac{n\pi x}{a}\cos\frac{m\pi x}{a}) = 0\end{aligned}$$

So the solution satisfies all the boundary conditions. The cut off frequency is given by,

$$\gamma_{mn} = \frac{\pi}{a}\sqrt{m^2 + n^2}$$

$$\omega_{mn} = \frac{\pi c}{a}\sqrt{m^2 + n^2}$$

For TM waves, similarly we can construct the solution be of the form:

$$\psi_{mn} = E_0(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} - \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a})$$

With the same cut off frequency:

$$\omega_{mn} = \frac{\pi c}{a}\sqrt{m^2 + n^2}$$

Jackson 8.7 (a)

From section 8.9, we can find the differential equation for $u_l(r)$,

$$\frac{\partial^2 u_l(r)}{\partial r^2} + [\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}]u_l(r) = 0$$

with the boundary condition $\frac{\partial u_l(r)}{\partial r} = 0$, for $r = a$ and $r = b$. The solutions of $u_l(r)$ are r times the spherical Bessel functions:

$$u_l(r) = r(A_l j_l(kr) + B_l n_l(kr))$$

where $k = \omega/c$. Substitution into the boundary condition yields:

$$\begin{aligned} A_l[j_l(ka) + kaj_l'(ka)] + B_l[n_l(ka) + kan_l'(ka)] = 0 \\ A_l[j_l(kb) + kbj_l'(kb)] + B_l[n_l(kb) + kbn_l'(kb)] = 0 \end{aligned}$$

Non-trivial solutions for A_l and B_l require:

$$\begin{vmatrix} j_l(ka) + kaj_l'(ka) & n_l(ka) + kan_l'(ka) \\ j_l(kb) + kbj_l'(kb) & n_l(kb) + kbn_l'(kb) \end{vmatrix} = 0$$

So we can write the transcendental equation for the characteristic frequencies as:

$$\boxed{[j_l(ka) + kaj_l'(ka)][n_l(kb) + kbn_l'(kb)] - [n_l(ka) + kan_l'(ka)][j_l(kb) + kbj_l'(kb)] = 0}$$

Jackson 8.7 (b)

When $l = 1$,

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

Insert to the transcendental equation, we find:

$$\begin{aligned} [(ka)(kb) + (1-(ka)^2)(1-(kb)^2)]sin(kb-ka) \\ +[ka(1-(kb)^2) - kb(1-(ka)^2)]\cos(kb-ka) = 0 \end{aligned}$$

With $h = b - a$,

$$\begin{aligned} \tan kh &= \frac{(ka)(kb) + (1-(ka)^2)(1-(kb)^2)}{ka(1-(kb)^2) - kb(1-(ka)^2} \\ &= kh\frac{k^2 + 1/ab}{k^2 + ab(k^2 - 1/a^2)(k^2 - 1/b^2)} \end{aligned}$$

Jackson 8.7 (c)

When $l = 1$, the characteristic frequency found in section 8.9 is:

$$\omega = \sqrt{2}\frac{c}{a}$$

For $h/a << 1$, that is $a \to b$, the results from part b shows:

$$\frac{\tan kh}{kh} = \frac{k^2 + 1/a^2}{k^2 + a^2(k^2 - 1/a^2)^2} = 1$$

Rearrange the equation we find,

$$k^2 = \frac{\omega^2}{c^2} = \frac{2}{a^2}$$

$$\boxed{\omega = \sqrt{2}\frac{c}{a}}$$

Jackson 8.18 (a)

Two dimensional Green's theorem indicates:

$$\int_A [\phi\nabla^2\psi - \psi\nabla^2\phi]da = \int_C [\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}]dl$$

By using $\phi = \psi_\lambda$ and $\psi = \psi_\mu$, we find

$$(\gamma_\lambda^2 - \gamma_\mu^2)\int_A \psi_\lambda\psi_\mu da = \int_S [\psi_\lambda\partial_n\psi_\mu - \psi_\mu\partial_n\psi_\lambda]dl$$

Here we substituted the wave equation:

$$(\nabla^2 + \gamma^2)\psi = 0$$

Since the boundary conditions are given by $\psi = 0$ for TM waves and $\partial_n\psi = 0$ for TE waves, the right hand side of the equation always vanishes.

For $\lambda \neq \mu$, $\gamma_\lambda^2 - \gamma_\mu^2$ is not zero. We shall have the orthogonal condition:

$$\int_A \psi_\lambda\psi_\mu da = 0$$

For TM modes, $\psi_{\lambda,\mu}$ corresponds to $E_{z\lambda,\mu}$. For TE modes, $\psi_{\lambda,\mu} = H_{z\lambda,\mu}$.