HW2 Classical Electromagnetism, Fall 2016

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Jackson 1.14

Apply Green's theorem with $\phi = G(\vec{x}, \vec{y})$ and $\psi = G(\vec{x}', \vec{y})$,

$$\int_V -4\pi(G(\vec{x}, \vec{y})\delta(\vec{y} - \vec{x}') + G(\vec{x}', y')\delta(\vec{y} - \vec{x}))d^3y = \oint_S [G(\vec{x}, \vec{y})\frac{\partial G(\vec{x}', \vec{y})}{\partial n} - G(\vec{x}', \vec{y})\frac{\partial G(\vec{x}, \vec{y})}{\partial n}]da$$

Thus,

$$\boxed{G(\vec{x}, \vec{x}') - G(\vec{x}', \vec{x}) = -\frac{1}{4\pi}\oint_S [G(\vec{x}, \vec{y})\frac{\partial G(\vec{x}', \vec{y})}{\partial n} - G(\vec{x}', \vec{y})\frac{\partial G(\vec{x}, \vec{y})}{\partial n}]da}$$

(a).For the Dirichlet bounddary conditions:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')Gd^3x' - \frac{1}{4\pi}\oint_S \Phi\frac{\partial G}{\partial n'}da'$$

For the Dirichlet boundary condition, the Green function $G_D = 0$ on the surface. We have

$$\begin{aligned}
G(\vec{x}, \vec{x}') - G(\vec{x}', \vec{x}) &= -\frac{1}{4\pi}\oint_S [G(\vec{x}, \vec{y})\frac{\partial G(\vec{x}', \vec{y})}{\partial n} - G(\vec{x}', \vec{y})\frac{\partial G(\vec{x}, \vec{y})}{\partial n}]da \\
&= -\frac{1}{4\pi}\oint_S [0\frac{\partial G(\vec{x}', \vec{y})}{\partial n} - 0\frac{\partial G(\vec{x}, \vec{y})}{\partial n}]da \\
&= 0
\end{aligned}$$

The Green function G_D is symmetric.

(b).Eq(1.45) tells us that, for Neumann boundary condition,

$$\frac{\partial G_N}{\partial n'} = -\frac{4\pi}{S}$$

We find

$$G(\vec{x}, \vec{x}') - G(\vec{x}', \vec{x}) = \frac{1}{S}\oint_S [G_N(\vec{x}, \vec{y}) - G_N(\vec{x}', \vec{y})]da_y$$

And

$$G(\vec{x}, \vec{x}') - \frac{1}{S}\oint_S G_N(\vec{x}, \vec{y})da_y = G(\vec{x}', \vec{x}) - \frac{1}{S}\oint_S G_N(\vec{x}', \vec{y})da_y$$

Thus $G_N(\vec{x}, \vec{x}') - F(x)$ is symmetric.

(c).Add $F(x)$ to the solution under Neumann Boundary Condition:

$$\begin{aligned}\Phi'(\vec{x}) &=< \phi >_S + \frac{4\pi\varepsilon_0}{\int}_V \rho(\vec{x}')(G_N + F)d^3x' + \frac{1}{4\pi}\oint_S (G_N + F)\frac{\partial \Phi}{\partial n'}da' \\ &= \Phi(\vec{x}) + \frac{F(\vec{x})}{4\pi}\left(\frac{\int_V \rho(\vec{x}')d^3x'}{\varepsilon_0} + \oint_S \frac{\partial \Phi}{\partial n'}da'\right)\end{aligned}$$

With Gauss's law and $\partial\Phi/\partial n' = -\vec{E}\cdot\vec{n}'$, we find

$$\frac{\int_V \rho(\vec{x}')d^3x'}{\varepsilon_0} + \oint_S \frac{\partial \Phi}{\partial n'}da' = \oint_S \vec{E}\cdot\vec{n}'da' - \oint_S \vec{E}\cdot\vec{n}'da' = 0$$

Thus,

$$\boxed{\Phi'(\vec{x}) = \Phi(\vec{x})}$$

The addition of $F(\vec{x})$ does not affect the potential $\Phi(\vec{x})$

Jackson 2.2

(a).Let the center of the shpere lies on the orgin. Consider an image charge q' outside the sphere at the position $\vec{y}'$, the potential due to the charges q and q' is:

$$\Phi(\vec{x}) = \frac{q/4\pi\varepsilon_0}{|\vec{x}-\vec{y}|} + \frac{q'/4\pi\varepsilon_0}{|\vec{x}-\vec{y}'|}$$

Let $\vec{n}$ be the unit vector in the direction $\vec{x}$ and $\vec{n}'$ in the direction $\vec{y}'$.
The sphere is conducting and grounded, thus the potential vanishes at $|\vec{x}| = a$. Make x out of the first term and y' the second, the potential at a is:

$$\Phi(x=a) = \frac{q/4\pi\varepsilon_0}{a|\vec{n} - \frac{y}{a}\vec{n}'|} + \frac{q'/4\pi\varepsilon_0}{y'|\vec{n}' - \frac{a}{y'}\vec{n}|} = 0$$

We find

$$q' = -\frac{a}{y}q, \qquad y' = \frac{a^2}{y}$$

The result is same as when the charge is outside the sphere.

$$\boxed{\Phi(\vec{x}) = \frac{q/4\pi\varepsilon_0}{|\vec{x}-\vec{y}|} - \frac{aq/4\pi y\varepsilon_0}{|\vec{x} - (\frac{a}{y})^2\vec{y}|}}$$

(b). For the case that charge q outside the sphere, the normal direction point inward the sphere, and we get $\vec{E}\cdot\vec{n} = \sigma/\varepsilon_0$. When q inside the sphere, the normal direction outward and we have $-\vec{E}\cdot\vec{n} = \sigma/\varepsilon_0$.

The induced charge density on the surface of sphere:

$$\boxed{\sigma = \varepsilon_0 \frac{\partial\Phi}{\partial n}|_{x=a} = \frac{q}{4\pi a^2}\frac{a}{y}\frac{1-a^2/y^2}{(1+\frac{a^2}{y^2} - 2\frac{a}{y}\cos\gamma)^{3/2}}}$$

Also γ is the angle between $\vec{x}$ and $\vec{y}$.

(c).We can calculate the force on charge q due to the induced charges by calculate the force between q and the image charge q' instead. Since q and q' have different sign, the image charge "attract" q outward the sphere. So the direction of the force is the same as the normal direction $\vec{n}$. According to Coulomb's law,

$$\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2}(\frac{a}{y})^3(1-\frac{a^2}{y^2})^{-2}\vec{n}}$$

Jackson 2.3

(a).Set three image charges λ_1 at $(-x_0, y_0)$, λ_2 at $(-x_0, -y_0)$ and λ_3 at $(x_0, -y_0)$. Remove the conducting boundary, the potential in the x-y plane

$$\begin{aligned}\Phi(x,y) =& \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{R^2}{(x-x_0)^2+(y-y_0)^2} + \frac{\lambda_1}{4\pi\varepsilon_0}\ln\frac{R^2}{(x+x_0)^2+(y-y_0)^2} \\ &+ \frac{\lambda_2}{4\pi\varepsilon_0}\ln\frac{R^2}{(x+x_0)^2+(y+y_0)^2} + \frac{\lambda_3}{4\pi\varepsilon_0}\ln\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\end{aligned}$$

The potential vanishes at the intersecting planes:

$$\begin{aligned}\Phi(0,y) &= \frac{1}{4\pi\varepsilon_0}\ln(\frac{R^2}{x_0^2+(y-y_0)^2})^{\lambda+\lambda_1} + \frac{1}{4\pi\varepsilon_0}\ln(\frac{R^2}{x_0^2+(y+y_0)^2})^{\lambda_2+\lambda_3} = 0 \\ \Phi(x,0) &= \frac{1}{4\pi\varepsilon_0}\ln(\frac{R^2}{(x-x_0)^2+y_0^2})^{\lambda+\lambda_3} + \frac{1}{4\pi\varepsilon_0}\ln(\frac{R^2}{(x+x_0)^2+y_0^2})^{\lambda_1+\lambda_2} = 0\end{aligned}$$

To keep $\Phi = 0$, we have

$$\begin{aligned}\lambda + \lambda_1 = 0 \qquad \lambda_2 + \lambda_3 = 0 \\ \lambda + \lambda_3 = 0 \qquad \lambda_2 + \lambda_1 = 0\end{aligned}$$

Thus,

$$\lambda_1 = \lambda_3 = -\lambda \qquad \lambda_2 = \lambda$$

So the potential in the first quadrant

$$\boxed{\begin{aligned}\Phi(x,y) =& \frac{\lambda}{4\pi\varepsilon}(\ln\frac{R^2}{(x-x_0)^2+(y-y_0)^2} + \ln\frac{R^2}{(x+x_0)^2+(y+y_0)^2}) \\ &-\frac{\lambda}{4\pi\varepsilon_0}(\ln\frac{R^2}{(x+x_0)^2+(y-y_0)^2} + \ln\frac{R^2}{(x-x_0)^2+(y+y_0)^2})\end{aligned}}$$

It's easy to find $\Phi = 0$ when $x = 0$ or $y = 0$. The tangential electric field on the boundary

$$E_x = -\frac{\partial\Phi}{\partial x}|_{x=0} = \frac{\lambda}{4\pi\varepsilon_0}[(-2x_0+2x_0)/[x_0^2+(y-y_0)^2] + (2x_0-2x_0)/[x_0^2+(y+y_0)^2]] = 0$$

Similarly the tangential electric field vanishes when $y = 0$.

(c).The charge density on the plane $y = 0, x \geq 0$

$$\sigma = -\varepsilon_0\frac{\partial\Phi}{\partial y}|_{y=0} = \frac{\lambda}{4\pi}[\frac{-4y_0}{(x-x_0)^2+y^2} + \frac{4y_0}{(x+x_0)+y_0^2}]$$

Integral the charge density along the +x-axis,

$$\begin{aligned}Q_x &= -\frac{\lambda y_0}{\pi}\int_0^\infty[\frac{1}{(x-x_0)^2+y_0^2} - frac1(x+x_0)^2+y_0^2]dx \\ &= -\frac{\lambda y_0}{\pi}[-\frac{\tan^{-1}(\frac{x_0-x}{y_0})}{y_0} + \frac{\tan^{-1}(\frac{x_0+x}{y_0})}{y_0}]|_0^\infty \\ &= -\frac{2\lambda}{\pi}\tan^{-1}(x_0/y_0)\end{aligned}$$

The total charge on the plane $y = 0$ $x \geq 0$,

$$\boxed{Q_x = \frac{2\lambda}{\pi} \tan^{-1} \frac{x_0}{y_0}}$$

(d).

Jackson 2.7

(a).The Green function:

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$$

With Dirichlet boundary conditions on the plan $z' = 0$, we have

$$G(\vec{x},\vec{x}')|_{z=0} = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + F(\vec{x},\vec{x}') = 0$$

$$F(\vec{x},\vec{x}')| = -\frac{1}{\sqrt{(x-x')^2+(y-y')^2+z^2}}$$

$$\boxed{G(\vec{x},\vec{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+z^2}}}$$

(b).The potential for Dirichlet boundary conditions is:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G}{\partial n'}da'$$

The first term vanishes in free space for $\rho = 0$. In cylindrical coordinate, the second integral on plane $z' = 0$ can be written as

$$\begin{aligned}
\Phi(\vec{x}) &= -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\infty}\Phi\frac{\partial G}{\partial n'}\rho' d\rho' d\varphi' \\
&= \frac{V}{4\pi}\int_0^{2\pi}\int_0^{a}\frac{\partial}{\partial z'}\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z-z')^2}} - \frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2}}\rho' d\rho' d\varphi' \\
&= \frac{V}{4\pi}\int_0^{2\pi}\int_0^{a}\frac{(z-z')}{(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z-z')^2)^{3/2}} - \frac{z}{(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2)^{3/2}}\rho' d\rho' d\varphi'
\end{aligned}$$

For $z' = 0$,

$$\boxed{\Phi(\vec{x}) == \frac{Vz}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{\rho'}{(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2)^{3/2}}d\rho' d\varphi'}$$

(c).When $\rho = 0$,

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{Vz}{2\pi}\int_0^{2\pi}d\varphi'\int_0^{a}\frac{\rho'}{(\rho'^2+z^2)^{3/2}}d\rho' \\
&= Vz\int_0^{a}\frac{1}{2}(\rho'^2+z^2)^{-3/2}d\rho'^2 \\
&= Vz(\rho'^2+z^2)^{-1/2}|_0^a \\
&= V(1-\frac{z}{\sqrt{a^2+z^2}})
\end{aligned}$$

The potential along the axis of the circle

$$\boxed{\Phi = V(1-\frac{z}{\sqrt{a^2+z^2}})}$$

(d).