Lecture 14 Linear Independence

Defn Let V be a vectorspace. $v_1 \dots v_k \in V$ are said to be <u>linearly dependent</u> if there exist $c_1 \dots c_k \in \mathbb{R}$, not all zero, such that $c_1 v_1 + \dots + c_k v_k = 0$.

The vectors $v_1 \dots v_k$ are said to be <u>linearly independent</u> if they are not linearly dependent.

So to spell the condition of linear independence out, we have that $v_1 \dots v_k$ are linearly independent if

$$c_1 v_1 + \dots + c_k v_k = 0$$

implies $c_1 = c_2 = \dots = c_k = 0$

<u>ex</u> The vectors $v_1 = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ -7 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} -8 \\ 2 \\ 28 \end{pmatrix}$ are linearly dependendent because

$$+2 v_1 + v_2 + \frac{1}{2} v_3 = 0.$$

<u>ex</u> If $v_i = 0$ for any i, then $v_1 \dots v_k$ is linearly dependent since

$$0 v_1 + \dots + 0 v_{i-1} + 1 \cdot v_i + 0 \cdot v_{i+1} + \dots + 0 v_k = 0.$$

<u>ex</u> Suppose $v, w \in V$. Then $\{v, w\}$ is linearly dependent only if v & w are parallel. Indeed, if $cv + dw = 0$ and $c \neq 0$, $v = -\frac{d}{c} w$. If $d \neq 0$, then $w = -\frac{c}{d} v$. In any case, we see the vectors are scalar multiples & thus parallel.

<u>Remark</u> A set of vectors $v_1 \ldots v_k$ being linearly dependent tells us there is a "redundancy". Indeed, the definition tells us that

$$c_1 v_1 + \ldots + c_k v_k = 0 \quad \text{for some } c_1 \ldots c_k \text{ not all zero.}$$

Suppose $c_i \neq 0$. Then we can solve

$$v_i = \frac{-1}{c_i}\left(c_1 v_1 + \ldots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \ldots + c_k v_k\right).$$

Thus,

$$\operatorname{Span}\{v_1, \ldots v_k\} = \operatorname{Span}\{v_1, \ldots v_{i-1}, v_{i+1}, \ldots, v_k\}.$$

To see why this is, we simply note that if

$$v = d_1 v_1 + \ldots + d_k v_k$$

is any linear combination, then

$$v = d_1 v_1 + \ldots + d_i\left(\frac{-1}{c_i}\left(c_1 v_1 + \ldots + c_{i-1} v_{i-1} + c_{i+1} v_i + \ldots + c_k v_k\right)\right) + \ldots + d_k v_k$$

$$= \left(d_1 - \frac{d_i c_1}{c_i}\right) v_1 + \ldots + \left(d_{i-1} - \frac{d_i c_{i-1}}{c_i}\right) v_{i-1} + \left(d_{i+1} - \frac{d_i c_{i+1}}{c_i}\right) v_{i+1} + \ldots + \left(d_k - \frac{d_i c_k}{c_i}\right) v_k,$$

So v is in the span of $v_1, \ldots v_{i-1}, v_{i+1}, \ldots, v_k$.

Thm 2.21 Let $v_1, \ldots v_k \in \mathbb{R}^n$ & $A = (v_1 \ldots v_k)$ be the corresponding $n \times k$ matrix

(a) The vectors $v_1, \ldots v_k \in \mathbb{R}^n$ are linearly dependent if and only if there is a nonzero solution to $Ac = 0$.

(b) The vectors are linearly independent if and only if there is only the trivial solution to $Ac = 0$.

(c) A vector $b \in V$ is in the span of $v_1 \ldots v_k$ if and only if $Ac = b$ has a solution.

Pf First, we note that if $c = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$, then

$$(1) \quad Ac = c_1 v_1 + \ldots + c_k v_k.$$

(a) By equation (1), we have that $Ac = 0$ has a nontrivial solution exactly when $c_1 v_1 + \ldots + c_k v_k = 0$ has a solution with the c_i not all 0.

(b) Part (a) and (b) are equivalent. Further explanation*

This is because part (a) is of the form "P if and only if Q" where P stipulates linear dependence of $v_1 \ldots v_k$ and Q stipulates a nontrivial solution to $Ac = 0$. Part (b) is then of the form "not P if and only if not Q." The logical equivalence follows.

* not required, say, on an exam.

(c) Part (c) also follows from equation (1).

Proposition 2.24 A set of vectors $v_1 \ldots v_k \in \mathbb{R}^n$ is linearly independent if and only if the matrix $A = (v_1 \ldots v_k)$ has rank k.

Pf Let U be the row echelon form of A. Then A has rank k if and only if U has k pivots if and only if $Ux = 0$ has no free variables if and only if $Ax = 0$ has only the trivial solution. By Theorem 2.21, $Ax = 0$ has only the trivial solution if and only if $v_1 \ldots v_k$ are linearly independent. The conclusion follows.