Lecture 31 The complex numbers $\mathbb{C}$ and the Complex vector space $\mathbb{C}^n$ §3.6

So far we've focused on real (meaning real numbers) vector spaces. However, some areas will be better studied with complex numbers. We will not focus on a full development of complex linear algebra; just enough to discuss signal processing and the Fourier transform.

Idea We define the number i to be a solution to $x^2 = -1$. We the extend this to all numbers of the form $x+iy$ where $x, y \in \mathbb{R}$, so that

$$(1) \quad \begin{aligned} (x+iy)+(u+iv) &= (x+u)+i(y+v) \\ (x+iy)(u+iv) &= xu + ixv + iyu + i^2 yv \\ &= (xu - yv) + i(xv + yu). \end{aligned}$$

Defn The complex number system $\mathbb{C}$ is the set of all numbers $x+iy$ for $x, y \in \mathbb{R}$ with addition and multiplication as defined in equation (1).

Why mathematicians love $\mathbb{C}$ (not required to understand deeply, but good to know in general)

Fact The complex numbers are algebraically closed. That is, if

$$p(z) = \alpha_0 + \alpha_1 z + \dots + \alpha_n z^n$$

is a polynomial with coefficients $\alpha_0 \dots \alpha_n \in \mathbb{C}$, then p has n complex valued solutions to $p(z) = 0$.

This fails for $\mathbb{R}$, where $x^2+1=0$ has no solutions.

Q: What is a pirate's favorite number system?
A: $\mathbb{R}$!
Response: Oh, thats true, we do like $\mathbb{R}$. But we really like the mighty $\mathbb{C}$!

ex $3+\sqrt{2}\,i + -9+3\sqrt{2}\,i = -6+4\sqrt{2}\,i$

$$(3+\sqrt{2}\,i)(-9+3\sqrt{2}\,i) = (-27-6)+i(9\sqrt{2}-9\sqrt{2}) = -33$$

The complex Exponential

We now define the exponential function $e^{i\theta}$ for $\theta \in \mathbb{R}$.

For real numbers, we have the power series representation,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ where } 0! = 1, \text{ and for } n>0,\ n! = 1\cdot 2\cdot \ldots \cdot n, \text{ and } x \in \mathbb{R}$$

If we plug in $x = i\theta$ into the above power series, we get

$$\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{\theta^n}{n!}.$$

Now, when n is even, $i^n = \pm 1$. When n is odd, $i^n = \pm i$. That is,

$$i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i,\ i^8 = 1 \ldots$$

Thus, we can split the above sum into

$$\sum_{n=0}^{\infty} i^n \frac{\theta^n}{n!} = \sum_{\substack{n \text{ even} \\ n \geq 0}}^{\infty} i^n \frac{\theta^n}{n!} + \sum_{\substack{n \text{ odd} \\ n \geq 0}}^{\infty} i^n \frac{\theta^n}{n!}$$

$$= \sum_{m=0}^{\infty} i^{2m} \frac{\theta^{2m}}{(2m)!} + \sum_{m=0}^{\infty} i^{2m+1} \frac{\theta^{2m}}{(2m+1)!}$$

$2m = n$ $\qquad$ $2m+1 = n$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2m}}{(2m+1)!}$$

$i^2 = -1$ $\qquad$ $i^2 = -1$, pull i out in front

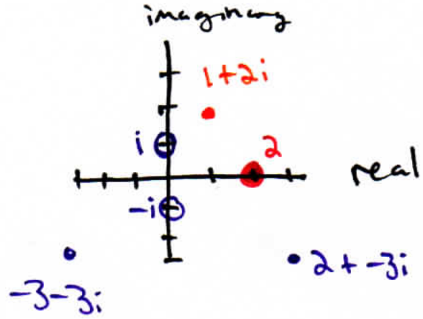
$$= \cos\theta + i\sin\theta.$$

So we end up with the remarkable realization through power series that the proper definition of $e^{i\theta}$ is

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Visualizing $\mathbb{C}$ and using Polar Coordinates.

To draw $\mathbb{C}$, we draw a "real axis" and an "imaginary axis" and plot points according to their real and imaginary parts



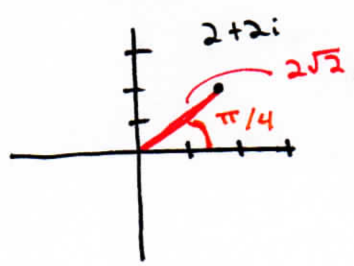
Every number can also be represented in the form

$re^{i\theta}$.

For example, $z = 2+2i = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

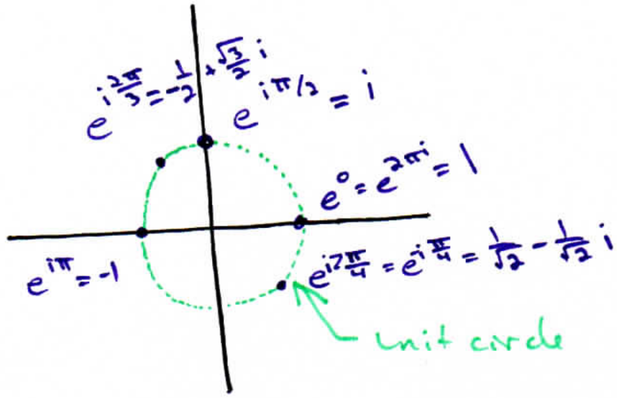
$$= 2\sqrt{2}\, e^{i\frac{\pi}{4}}.$$

In polar coordinates, this means that z is distance $2\sqrt{2}$ from the origin and at an angle of $\frac{\pi}{4}$



This holds in general: if $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, we have that z has a coordinate of $r\cos\theta$ along the real axis and $r\sin\theta$ on the imaginary axis. Thus, if $z = re^{i\theta}$ then (r,θ) are the polar coordinates of z!

ex



To convert between the Euclidean form $x+iy$ and the polar coordinate form $re^{i\theta}$, we simply reuse the polar coordinate formulas

Starting from $x+iy$

$$r = \sqrt{x^2+y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad (+\,\pi, \text{ depending on the exact point})$$

(except when $x=0$; in this case, $\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ depending on whether y is positive or negative)

Starting from $re^{i\theta}$

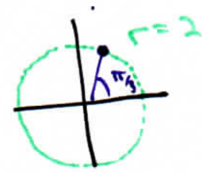
$$x = r\cos\theta$$

$$y = r\sin\theta$$

For example,

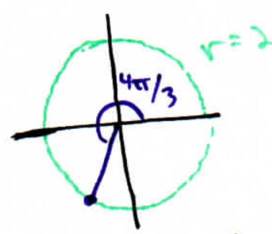
$z = 1+\sqrt{3}i$, $r = \sqrt{1+3} = \sqrt{4} = 2$,

$\theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$.



However, if $z = -1-\sqrt{3}i$, then $r=2$ but

$\theta = \arctan(\sqrt{3}) + \pi = \frac{4\pi}{3}$



The ambiguity comes from the fact that $\frac{y}{x} = \frac{-y}{-x}$, and relatedly, that arctan always returns (by definition) a number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. If $x>0$, an angle between $-\frac{\pi}{2}$ & $\frac{\pi}{2}$ is accurate. If $x<0$, we add π to move into the other quadrants.

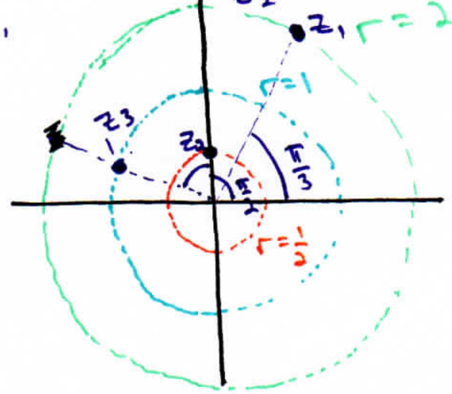
Multiplication with Complex exponentials

Multiplication with $e^{i\theta}$ works as it does with real exponentials:

$$\boxed{re^{i\theta}se^{i\varphi}=rs\,e^{i(\theta+\varphi)}}$$

For example,

$$\underbrace{2e^{i\pi/3}}_{z_1}\cdot\underbrace{\tfrac{1}{2}e^{i\pi/2}}_{z_2}=\underbrace{e^{i5\frac{\pi}{6}}}_{z_3}$$



Division

To divide by a complex number in Euclidean form, it is easiest to use conjugates.

Defn The complex conjugate of $z=x+iy$ is

$$\overline{z}=x-iy.$$

RK $z\cdot\overline{z}=x^2+y^2+i(xy-xy)=x^2+y^2$.

So, to divide by complex numbers, we use the following trick:

$$\frac{3+4i}{1-2i}=\frac{3+4i}{1-2i}\cdot\frac{1+2i}{1+2i}=\frac{(3+4i)(1+2i)}{1+4}=\frac{3-8+i(5+4)}{5}$$

$$=\frac{-5+i(9)}{5}=-1+\frac{9}{5}i.$$

So, in general, to compute $\frac{z}{w}$, we compute this as $\frac{z}{w}=\frac{z}{w}\frac{\overline{w}}{\overline{w}}$, and then use the fact that $w\overline{w}$ is a real number.

The n^{th} roots of 1

The n^{th} roots of 1 are the n complex numbers which are solutions to the equation $z^n = 1$.

To solve this algebraically, we set $z = re^{i\theta}$, and find

$$z^n = r^n e^{in\theta} = 1.$$

Now, $1 = 1 \cdot e^{i \cdot 0} = 1 \cdot e^{2\pi k}$ for $k = 0, 1, 2, 3, \ldots$

So we get that $r = 1$ and $n\theta = 2\pi k$. This tells us $\theta = \frac{2\pi k}{n}$.

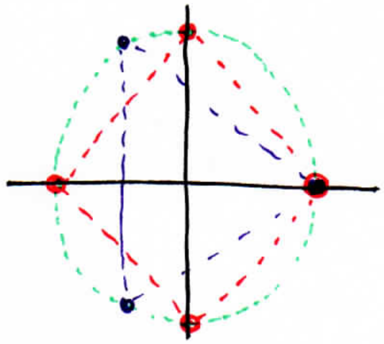
However, since $e^{i\theta} = e^{i(\theta + 2\pi)}$, we ~~may~~ only need to take
(think polar coordinates)

the first n solutions, $k = 0, 1, \ldots n-1$.

Conclusion:

$$\boxed{\text{There are } n \ n^{th} \text{ roots of } 1 \text{, given as } z = e^{\frac{2\pi k}{n} i} \text{ for } k = 0, 1, 2, \ldots, n-1.}$$

These roots are evenly spaced around the unit circle, forming a regular polygon



• = the third roots of 1
$e^0, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$

• = the fourth roots of 1
$e^0, e^{\frac{2\pi i}{4}}, e^{\frac{4\pi i}{4}}, e^{\frac{6\pi i}{4}}$

<u>Rk</u> While the polar representation $z = re^{i\theta}$ sheds light on multiplication and n^{th} roots, it is useless for addition. Addition needs to be done with the Euclidean representation $z = x + iy$.

The space $\mathbb{C}^n$

Defn The space $\mathbb{C}^n$ is the set consisting of all column vectors with complex entries:

$$\mathbb{C}^n = \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} : z_i \in \mathbb{C} \right\}.$$

Addition and scalar multiplication can be carried out entrywise.

ex Consider $\mathbb{C}^2$.

$$\begin{pmatrix} 1+2i \\ 3i \end{pmatrix} + \begin{pmatrix} -4-5i \\ 6-7i \end{pmatrix} = \begin{pmatrix} -3-3i \\ 6-4i \end{pmatrix}$$

$$(1+2i)\begin{pmatrix} 1+2i \\ 3i \end{pmatrix} = \begin{pmatrix} -3+4i \\ -6+3i \end{pmatrix}$$

$$2e^{\frac{4\pi}{3}i}\begin{pmatrix} e^0 \\ e^{\frac{2\pi}{3}i} \\ e^{\frac{4\pi}{3}i} \end{pmatrix} = \begin{pmatrix} 2e^{\frac{4\pi}{3}i} \\ 2e^{\frac{6\pi i}{3}} \\ 2e^{\frac{8\pi i}{3}} \end{pmatrix} = \begin{pmatrix} 2e^{\frac{4\pi}{3}i} \\ 2e^0 \\ 2e^{\frac{2\pi i}{3}} \end{pmatrix},$$

where we use the fact that $e^{2\pi i + \theta i} = e^{\theta i}$ (think polar coords)

Rk $\mathbb{C}^n$ is the standard "complex vectorspace". We will only delve into $\mathbb{C}^n$, not the general theory of complex vectorspaces.

The Complex Dot Product

Recall the complex conjugate of $z = x+iy$ is defined as $\overline{z} = x - iy$.

If $z \in \mathbb{C}^n$, $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$ is a complex vector, we define $\overline{z} = \begin{pmatrix} \overline{z_1} \\ \vdots \\ \overline{z_n} \end{pmatrix}$.

Defn The complex dot product of $z, w \in \mathbb{C}^n$ is define as

$$z \cdot w = z_1\overline{w_1} + z_2\overline{w_2} + \dots + z_n\overline{w_n}.$$

ex $\begin{pmatrix} 1+i \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2-i \\ 3+i \end{pmatrix} = (1+i)\overline{(2-i)} + (2)\overline{(3+i)} = (1+i)(2+i) + 2(3-i)$

$$= 1+3i + 6 - 2i = 7 - i.$$

Defn The ~~norm~~ absolute value of a complex number $x+iy$ is
$$|x+iy| = \sqrt{x^2+y^2}.$$

Observation $(x+iy)\overline{(x+iy)} = (x+iy)(x-iy) = x^2+y^2 = |x+iy|^2$.

Thus, for $z \in \mathbb{C}^n$,
$$z \cdot z = \sum_{i=1}^{n} z_i \overline{z_i} = \sum_{i=1}^{n} |z_i|^2 \geq 0. \text{ If } z \neq 0, \text{ then } z \cdot z > 0.$$

Thus, the complex dot product satisfies

(positivity) $\quad z \cdot z \geq 0$, and $z \cdot z > 0$ for $z \neq 0$

Moreover, the complex dot product satisfies

(Conjugate symmetry) $z \cdot w = \overline{w \cdot z}$

(sesqui linearity)

$(z+w) \cdot v = z \cdot v + w \cdot v \qquad$ for $z, w, v \in \mathbb{C}^n$

$(cz) \cdot (v) = c(z \cdot v) \qquad$ for $z, v \in \mathbb{C}^n$, $c \in \mathbb{C}$

$z \cdot (w+v) = z \cdot w + z \cdot v \qquad$ for $z, w, v \in \mathbb{C}^n$

$z \cdot (cv) = \overline{c}\,(z \cdot v) \qquad$ for $z, v \in \mathbb{C}^n$, $c \in \mathbb{C}$.