§1.3 Lecture 3

The $A = LU$ factorization

Recall that a "regular" matrix is one whose pivots are nonzero during Gaussian Elimination (GE).

ex Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Solve for x in the equation

$Ax = b$.

(1) $$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & -1 \end{array}\right) \underset{r_3 - 7r_1}{\overset{r_2 - 4r_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & -6 & -18 & -8 \end{array}\right) \underset{r_3 - 2r_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & -6 & 0 \end{array}\right)$$

Back substitution

$-6x_3 = 0$	$-3x_2 - 6 \cdot 0 = -4$	$x_1 + 2 \cdot 4/3 + 3 \cdot 0 = 1$
$\boxed{x_3 = 0}$	$3x_2 = 4$	$x_1 + 8/3 = 3/3$
"~~[illegible]~~ third row"	$\boxed{x_2 = 4/3}$	$\boxed{x_1 = -5/3}$
	"second row"	"first row"

Each application of ERO1 (in blue above) corresponds to multiplying on the left by the corresponding elementary matrix

~~eg~~ $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$.

So the process of GE (1) can be written as

$$E_3 E_2 E_1 (A \mid b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & -6 & 0 \end{array}\right).$$

Fact Each elementary row operation is invertible;

$$(E_3 E_2 E_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}.$$

In general, this holds true for GE moving from left to right

~~Thus~~ Set $L = (E_3 E_2 E_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}$.

(L is for lower triangular).

Let $U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -6 \end{pmatrix}$ (U is for upper triangular).

In our example, we worked out that

$$E_3 E_2 E_1 A = U.$$

Multiplying on the left by L, we get

$$L(E_3 E_2 E_1) A = LU$$

$$(E_3 E_2 E_1)^{-1} (E_3 E_2 E_1) A = LU$$

$$\boxed{A = LU}$$

This is the "LU" factorization of A. We can use an LU factorization to solve $Ax = b$. How?

$$Ax = b \iff LUx = b \iff Lc = b \text{ \& } Ux = c.$$

"is equivalent to"

<u>ex</u> (continued) Using our same A & b, $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & a \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -6 \end{pmatrix}$

$A = LU$.

<u>Step 1</u> Solve $Lc = b$ using "<u>forward substitution</u>"

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 \\ 7 & 2 & 1 & -1 \end{array}\right)$$

$c_1 = 1$ "first row"

$4 \cdot 1 + c_2 = 0 \quad c_2 = -4$ "second row"

$7 \cdot 1 + 2 \cdot -4 + c_3 = -1 \quad c_3 = 0$ "third row"

<u>Step 2</u> Now that we know c, we solve $Ux = c$ using "back substitution"

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & -6 & 0 \end{array}\right)$$

$$\boxed{\begin{aligned} x_3 &= 0 \\ x_2 &= 4/3 \\ x_1 &= -5/3 \end{aligned}}$$

When is using Gaussian Elimination on $(A|b)$ good?

To solve $Ax=b$ once.

When is using GE to get an LU factorization good?

To solve $Ax=b$ for same A, lots of b.

Roughly speaking, for an $n\times n$ matrix, Gaussian Elimination takes

$O(n^3)$ steps.

However, forward & back substitution each require

$O(n^2)$ steps.

The computational savings are staggering for matrices w/

$n \approx 10{,}000;\ 1{,}000{,}000;$ etc....

ex $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & -1 \\ 0 & 1 & -2 \end{pmatrix} \overset{r_2 - r_1}{\sim} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \\ 0 & 1 & -2 \end{pmatrix} \underset{r_3 - r_2}{\sim} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{pmatrix}$ ✓

Because we worked left to right (see ✻)

$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, so $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{pmatrix}$

ex & Solve $Ax = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\underline{Lc=b}$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$ $\boxed{c_1 = 0}$

$c_1 + c_2 = 1$ $\boxed{c_2 = 1}$

$c_2 + c_3 = 1$ $\boxed{c_3 = 0}$

$1 + c_3 = 1$

$\underline{Ux=c}$ $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right)$ $2x_3 = 0$ $\boxed{x_3 = 0}$

$x_2 - 4\cdot 0 = 1$ $\boxed{x_2 = 1}$

$x_1 + 1\cdot 1 + 3\cdot 0 = 0$ $\boxed{x_1 = -1}$

<u>Check</u> $A\,x = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & -1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = b$ ✓