Lecture 16 The Fundamental Matrix Subspaces

Consider, again, equations of the form

(1) $\quad Ax = b$

Where A is an $m \times n$ matrix. We now seek to understand the solutions of (1).

Kernel & Range

Defn The range of a matrix A, $m \times n$, is defined as

$$\operatorname{rng}(A) = \{ b \in \mathbb{R}^m : Ax = b \text{ for some } x \in \mathbb{R}^n \} = \{ Ax : x \in \mathbb{R}^n \}.$$

The kernel of A is defined as

$$\ker(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}.$$

Lemma Let A be an $m \times n$ matrix. Then $\operatorname{rng}(A)$ is a subspace of $\mathbb{R}^m$ and $\ker(A)$ is a subspace of $\mathbb{R}^n$.

Pf By the subspace criteria, Prop 2.9, we need to demonstrate only closure under addition and scalar multiplication of $\operatorname{rng}(A)$ and $\ker(A)$.

We will prove this for $\operatorname{rng}(A)$, and leave the proof for $\ker(A)$ as an exercise.

Suppose $a, b \in \operatorname{rng}(A)$. Then $a = Ax$ and $b = Ay$ for some $x, y \in \mathbb{R}^n$. Then $a + b = Ax + Ay = A(x+y)$.

Thus, $a+b$ is also in $\operatorname{rng}(A)$, so $\operatorname{rng}(A)$ is closed under addition.

Moreover, let $c \in \mathbb{R}$. Then $c \cdot a = c \cdot Ax = A(c \cdot x)$, so $c \cdot a \in \operatorname{rng}(A)$. By the subspace criteria, $\operatorname{rng}(A)$ is a subspace. //

Thm 2.39 The linear system $Ax = b$ has a solution if and only if $b \in \operatorname{rng}(A)$. If $Ax = b$ has at least one solution x^*, then x is a solution to $Ax = b$ if and only if

$$x - x^* \in \ker(A).$$

That is, $\{ x : Ax = b \} = \{ x^* + z : z \in \ker(A) \}$.

ex Consider the system $Ax = b$ where

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

We perform Gaussian Elimination to the augmented system:

$$(A|b) = \left(\begin{array}{ccc|c} 2 & 3 & 4 & b_1 \\ 1 & 0 & -1 & b_2 \\ 0 & 1 & 1 & b_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_2 \\ 2 & 3 & 4 & b_1 \\ 0 & 1 & 1 & b_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_2 \\ 0 & 3 & 6 & b_1 - 2b_2 \\ 0 & 1 & 1 & b_3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_2 \\ 0 & 1 & 1 & b_3 \\ 0 & 3 & 6 & b_1 - 2b_2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_2 \\ 0 & 1 & 1 & b_3 \\ 0 & 0 & 3 & b_1 - 2b_2 - 3b_3 \end{array}\right).$$

Because $U = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$, the row echelon form of A, has 3 pivots, the right hand side b has a solution x for any b. So $\operatorname{rng}(A) = \mathbb{R}^3$. Moreover, because A has no free variables, $Ax = 0$ has only the trivial solution, so

$$\ker(A) = \{0\}.$$

ex* Consider $A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{pmatrix}$, $Ax = b$.

$$(A|b) = \left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & b_1 \\ 2 & -3 & -1 & -4 & b_2 \\ 3 & -5 & -1 & -1 & b_3 \end{array}\right) \underset{\boxed{r_3 - r_1 - r_2}}{\overset{\boxed{r_2 - 2r_1}}{\sim}} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & b_1 \\ 0 & 1 & -1 & -10 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right).$$

Then $Ax = b$ has a solution ~~on~~ if and only if the bottom row is compatible, i.e., if $b_3 - b_1 - b_2 = 0$. So

$$\operatorname{rng}(A) = \{ b \in \mathbb{R}^3 : b_3 - b_1 - b_2 = 0 \}.$$

On the other hand, to find the kernel of A, we set $b = 0$ and solve the system

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 1 & -1 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

* ex 2.38 continued.

By back substitution, $x_2 = x_3 + 10x_4$

$x_1 = 2x_2 - 3x_4 = 2x_3 + 17x_4$,

where x_3 & x_4 are free parameters. So

$$\ker(A) = \left\{ \begin{pmatrix} 2s + 17t \\ s + 10t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

For example, $b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \in \operatorname{rng}(A)$ because $b_3 - b_1 - b_2 = 0$.

Then our augmented matrix from before becomes

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 1 & -1 & -10 & -1 \end{array}\right)$$

where we omit the bottom row, $0 = 0$. By finding the solution with free variables set to 0,

we get

$$x_2 = -1$$
$$x_1 = 2x_2 + 1 = -1$$

So $x^* = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ is one solution. Thus, the general solution is

$$x = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2s + 17t \\ s + 10t \\ s \\ t \end{pmatrix}$$ by Thm 2.39.

The adjoint spaces

The adjoint system of a matrix A is the system

$$A^T y = f,$$

where we recall that $A^T = (a_{ji})$ where $A = (a_{ij})$.

Defn The corange of A is

$$\operatorname{corng}(A) = \operatorname{rng}(A^T).$$

The cokernel of A is

$$\operatorname{coker}(A) = \ker(A).$$

The cokernel & corange are key to many theoretical ~~applications~~ uses, but we will not focus on them. Here is an example of an application of cokernel.

Thm 5.5 Let A be an $m \times n$ matrix. Then $b \in \text{rng}(A)$ if and only if b is perpendicular to the cokernel of A; that is, $b \cdot v = 0$ for all $v \in \text{coker}(A)$ where $\cdot$ is the usual Euclidean dot product.

Rk The above theorem is not required class material, but rather an example for context.