§ 3.4 Positive Definite Matrices   Lecture 22

Let $\langle , \rangle$ be an inner product on $\mathbb{R}^n$. We now show how to represent $\langle , \rangle$ as a matrix.

Let $x, y \in \mathbb{R}^n$, & expand each in the standard basis

$$x = \sum_{i=1}^{n} x_i e_i \qquad y = \sum_{j=1}^{n} y_j e_j .$$

Applying bilinearity, we get

$$\langle x, y \rangle = \left\langle \sum_{i=1}^{n} x_i e_i , \sum_{j=1}^{n} y_j e_j \right\rangle = \sum_{i=1}^{n} x_i \left\langle e_i , \sum_{j=1}^{n} y_j e_j \right\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \langle e_i, e_j \rangle = \sum_{i,j=1}^{n} x_i y_j \langle e_i, e_j \rangle .$$

Set $k_{ij} = \langle e_i, e_j \rangle$, & define the $n \times n$ matrix $K = (k_{ij})$. By the previous equation & the definition of matrix multiplication, we see

$$\langle x, y \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i y_j = \sum_{i=1}^{n} (y^T K)_i x_i = y^T K x .$$

For example, if $\langle , \rangle$ is the standard dot product, then

$$\langle e_i, e_j \rangle = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

and so

$$K = I_n, \quad x \cdot y = y^T I x = y^T x$$

is the familiar representation of the dot product. If $\langle , \rangle$ is the weighted dot product $\langle x, y \rangle = \sum_{i=1}^{n} w_i x_i y_i$, then

$$K = \begin{pmatrix} w_1 & & 0 \\ & w_2 & \\ 0 & & \ddots \; w_n \end{pmatrix} \qquad \langle x, y \rangle = y^T K x .$$

For the inner product of problem 3.12 c, $\langle x, y \rangle = (x_1 + x_2)(y_1 + y_2)$, we have

$$K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

We observe that the matrix K representing an inner product $\langle , \rangle$ is symmetric, because

$$k_{ij} = \langle e_i, e_j \rangle = \langle e_j, e_i \rangle = k_{ji}.$$

Finally, we notice that

$$x^T K x = \langle x, x \rangle > 0 \quad \text{for all } x \neq 0$$

because $\langle , \rangle$ is positive.

Defn 1 An $n \times n$ matrix K is said to be positive definite if it is symmetric and

$$x^T K x > 0 \quad \text{for all } x \neq 0.$$

We will use the notation $K > 0$ to say K is positive definite.

Thm 3.21 A function $\langle , \rangle$, accepting two vectors in $\mathbb{R}^n$ as inputs, is an inner product if and only if

$$\langle x, y \rangle = x^T K y$$

for a positive definite matrix K.

Pf In the preceding paragraphs, we have shown that if $\langle , \rangle$ is an inner product, then $\langle x, y \rangle = x^T K y$ for a positive definite matrix K given by $K = (\langle e_i, e_j \rangle)$.

Suppose that $\langle x, y \rangle = x^T K y$ for a positive definite matrix K. ~~Therefore~~ We check the inner product conditions.

bilinearity The property of bilinearity follows from linearity of matrix multiplication.

Symmetry. Let $x, y \in \mathbb{R}^n$. Then

$$\langle x, y \rangle = x^T K y = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i y_j = \sum_{j=1}^{n} \sum_{i=1}^{n} k_{ij} y_j x_i.$$

By the assumption that K is symmetric (see defn 1), we have $k_{ij} = k_{ji}$ ~~and~~ whence

$$\sum_{j=1}^{n} \sum_{i=1}^{n} k_{ij} y_j x_i = \sum_{j=1}^{n} \sum_{i=1}^{n} k_{ji} y_j x_i = y^T K x = \langle y, x \rangle.$$

So $\langle , \rangle$ is symmetric.

positivity Positivity follows from the fact that K is positive definite, as

$$\langle x, x \rangle = x^T K x > 0 \quad \text{for } x \neq 0.$$

ex Let $K = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$. Then

$$x^T K x = 4x_1^2 - 4x_1x_2 + 3x_2^2 = (2x_1 - x_2)^2 + 2x_2^2 > 0$$

for $x \neq 0$. Hence, K is positive definite. Moreover,

$$x^T K y = (x_1 \ x_2) \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 4x_1y_1 - 2x_1y_2 - 2y_1x_2 + 3y_2x_2$$

defines an inner product.

non-ex Let $K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Then

$$x^T K x = x_1^2 + 4x_1x_2 + x_2^2.$$

Let $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Then

$$x^T K x = 1 - 4 + 1 = -2 < 0$$

So K is not positive definite.

Thm 3.37 A symmetric matrix K is positive definite if and only if it is regular and has all positive pivots.