§1.4 Lecture 4

- Follow up from last time.

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & -1 \\ 0 & 1 & -2 \end{pmatrix} \underset{}{\overset{r_2 - r_1}{\sim}} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \\ 0 & 1 & -2 \end{pmatrix} \underset{r_3 - r_2}{\sim} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{pmatrix}$$

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

It is not true that $\cancel{E_2 E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}$ $\quad L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

But it is true $(E_2 E_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. As long as we move left to right, this will happen. (Can you figure out why?)

Pivoting & permutations

Many linear systems $Ax = b$ will have solutions even though A has a zero pivot

ex $\quad A = \begin{pmatrix} 0 & 2 \\ 3 & -4 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 0 & 2 & 1 \\ 3 & -4 & 2 \end{array}\right) \overset{(1\,2)}{\sim} \left(\begin{array}{cc|c} 3 & -4 & 2 \\ 0 & 2 & 1 \end{array}\right) \qquad \boxed{x_2 = \tfrac{1}{2} \quad x_1 = 4/3}$$

What to do? Swap rows!

ex

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ -2 & -4 & 3 & -1 \\ -1 & 1 & 1 & 2 \end{array}\right) \underset{r_3 + r_1}{\overset{r_2 + 2r_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 3 & 7 \\ 0 & 3 & 1 & 6 \end{array}\right) \overset{(2\,3)}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 3 & 7 \end{array}\right) \qquad \boxed{\begin{aligned} x_3 &= 7/3 \\ x_2 &= 11/9 \\ x_1 &= 14/9 \end{aligned}}$$

ERO 2 swap two rows.

~~By applying ERO 1 repeatedly, we can arrive at an A=LU factorization. Is there an equivalent for using ERO 1 & ERO 2?~~

Defn A square matrix is called <u>nonsingular</u> if it can be reduced to upper triangular form with nonzero elements along the diagonal by ERO1 & ERO 2. Otherwise it is called <u>singular</u>

By applying ERO 1 repeatedly, we can find the $A = LU$ factorization

What if we use ERO1 & ERO 2?

Permuted LU factorization

Defn A <u>permutation matrix</u> is one obtained by reordering the rows of the identity matrix.

ex $n=3$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ all fixed $\quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $(1\,2)$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $(2\,3)$ $\quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $(1\,3)$

$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $(1\,3)$ then $(2\,3)$ $\quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $(1\,2)$ then $(2\,3)$

Multiplying A on the left by a permutation matrix P is the same as reording the rows of A

ex

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

<u>Big idea</u> we will make a permutation matrix P s.t. PA is regular, and thus PA has an LU factorization.

<u>Step 1</u> Do GE w/ ERO 1 & 2.

<u>Step 2</u> Use the record of operations to make P & L.

ex

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ -2 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \underset{\sim}{\begin{matrix} r_2 - 2r_1 \\ r_3 + 2r_1 \\ r_4 - r_1 \end{matrix}} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 6 & -2 & 1 \\ 0 & -2 & 1 & 1 \end{pmatrix} \underset{\sim}{(2\ 3)} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 6 & -2 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

$$\underset{\sim}{r_4 + \frac{r_2}{3}} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 6 & -2 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 1/3 & 4/3 \end{pmatrix} \underset{\sim}{r_4 - \frac{1}{12} r_3} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 6 & -2 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 13/12 \end{pmatrix} = U$$

Making P & L

$$P: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = P$$

PA = LU:

$$PA = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -2 & 2 & 0 & 1 \\ 2 & 4 & 2 & 3 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$L: I \rightsquigarrow \begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ -2 & 0 & 1 & \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 2 & 0 & 1 & \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 2 & 0 & 1 & \\ 1 & -\frac{1}{3} & [illegible] & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 2 & 0 & 1 & \\ 1 & \frac{1}{3} & \frac{1}{12} & 1 \end{pmatrix}$$

$$LU = \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 2 & 0 & 1 & \\ 1 & -1/3 & 1/12 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 6 & -2 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 13/12 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -2 & 2 & 0 & 1 \\ 2 & 4 & 2 & 3 \\ 1 & 0 & 0 & \end{pmatrix}$$

$$-1 - 2 \cdot -\frac{1}{3} + \frac{4}{12} = -1 + \frac{2}{3} + \frac{1}{3} \qquad -\frac{1}{3} + \frac{1}{4} + \frac{13}{12}$$

$$-\frac{4}{12} + \frac{3}{12} + \frac{13}{12} = 1$$

Solving $Ax = b$ with the permuted LU factorization

Solving $Ax = b$

$PAx = Pb = \hat{b}$ ← permute entries

$LUx = \hat{b}$ solve this as normal.

ex

$PA = LU$ from before. $b = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$

$Pb = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}$. $LUx = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}$

$Lc = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}$. $\left(\begin{array}{cccc|c} 1 & & & & 1 \\ -2 & 1 & & & 2 \\ 2 & 0 & 1 & & -1 \\ 1 & -1/3 & 1/12 & 1 & -2 \end{array}\right)$

$\boxed{c_1 = 1}$

$\boxed{c_2 = 4}$

$2 + c_3 = -1$ $\quad \boxed{c_3 = -3}$

$1 - \overset{16/12}{\frac{4}{3}} - \frac{3}{12} + c_4 = -2$

$-\frac{7}{12} + c_4 = -2$

$\boxed{c_4 = -\frac{17}{12}}$

$Ux = c$ $\quad \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ & 6 & -2 & 1 & 4 \\ & & 4 & 3 & -3 \\ & & & 13/12 & -17/12 \end{array}\right)$

$\boxed{x_4 = -\frac{17}{13}}$

$4x_3 - \frac{51}{13} = -\frac{39}{13}$

$4x_3 = \frac{12}{13}$ $\quad \boxed{x_3 = \frac{3}{13}}$

$6x_2 - \frac{6}{13} - \frac{17}{13} = 4$

$6x_2 - \frac{23}{13} = \frac{52}{13}$

$6x_2 = \frac{75}{13}$ $\quad \boxed{x_2 = \frac{25}{39}}$

$x_1 + \frac{50}{39} - \frac{3}{13} = 1$

$x_1 + \frac{41}{39} = 1$

$\boxed{x_1 = \frac{2}{39}}$