Lecture 15 Bases & Dimension

A set of vectors $v_1 \dots v_n$ spans V if all $v \in V$ can be written as $v = c_1 v_1 + \dots + c_n v_n$. The set of vectors is linearly independent if $c_1 v_1 + \dots + c_n v_n = 0$ implies $c_1 = \dots = c_n = 0$. The first condition, intuitively, says $v_1 \dots v_n$ cover all of V with their linear combinations. The second says they are not redundant.

Defn A basis of a vector space V is a collection $v_1, \dots, v_n$ ~~such that~~ which
(a) spans V, and
(b) is linearly independent.

ex
- The standard basis of $\mathbb{R}^n$ is $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$
- The space $P^{(2)} = \{a_2 x^2 + a_1 x + a_0 : a_i \in \mathbb{R}\}$ has a basis $\{1, x, x^2\}$.

First, we prove ~~an~~ a "fundamental fact" about bases.

Theorem 2.29 All bases have the same size. That is, if $v_1 \dots v_k$ is one basis for a vector space V and $w_1 \dots w_l$ is another, then $k = l$.

To prove this Theorem, we will use the following lemma.

Lemma 2.30 Suppose that $v_1 \dots v_k$ spans a vector space V. Then any set of more than k elements is linearly dependent.

Pf The idea of this proof is to turn an abstract ~~linear~~ vector space question into a concrete question about matrix algebra.

Suppose $V = \text{span}\{v_1, \dots v_k\}$ and $w_1, \dots, w_l \in V$ for some $l > k$.

Because the v_i span V, we may write each

$$w_j = a_{1j}v_1 + \dots + a_{kj}v_k \text{ for some } a_{ij} \in \mathbb{R}.$$

Consider the matrix $A = (a_{ij})$. We notice that

(2)
$$c_1w_1 + \dots + c_\ell w_\ell = \sum_{j=1}^{\ell} c_j w_j = \sum_{j=1}^{\ell} c_j(a_{1j}v_1 + \dots + a_{kj}v_k)$$

$$= \cancel{a}\ \boxed{\sum_{j=1}^{\ell} a_{1j}c_jv_1} + \sum_{j=1}^{\ell} a_{2j}c_jv_2 + \dots + \sum_{j=1}^{\ell} a_{kj}v_k.$$

Also, ~~A~~ letting $c = \begin{pmatrix} c_1 \\ \vdots \\ c_\ell \end{pmatrix}$, we see that

$$Ac = \begin{pmatrix} \boxed{\sum_{j=1}^{\ell} a_{1j}c_j} \\ \vdots \\ \sum_{j=1}^{\ell} a_{kj}c_j \end{pmatrix}.$$

That is, the ith entry of Ac is the coefficient of v_i in equation (2). We note now that A is $k \times \ell$ where $\ell > k$. Thus, there is a nontrivial solution c to $Ac = 0$. But by (2), this implies that ~~the~~ such a vector c gives

$$c_1w_1 + \dots + c_\ell w_\ell = (Ac)_1v_1 + \dots + (Ac)_kv_k = 0.$$

Hence, $w_1 \dots w_\ell$ are linearly dependendent. //

<u>Proof</u> (of Thm 2.29) Let $v_1 \dots v_k \in V$ & $w_1 \dots w_\ell \in V$ both be bases. Then by Lemma 2.30, because $v_1 \dots v_k$ spans V & $w_1 \dots w_\ell$ is linearly independent, $\ell \le k$. By symmetry, $k \le \ell$ as well. So $\ell = k$. //

<u>Thm 2.31</u> Suppose that V is an n-dimensional vectorspace. Then

(a) Every set of more than n elements is linearly dependent.

(b) No set of ~~less than~~ fewer than n elements spans V.

(c) A set of n elements is linearly independent if and only if it spans V if and only if it is a basis.

<u>Pf</u> Let V be an n-dimensional vectorspace. Let $v_1, \dots, v_n \in V$ be a basis.

(a) Part (a) follows from Lemma 2.30 because $v_1 \dots v_n$ span V.

(b) Suppose $w_1 \dots w_m$ span V, $m < n$. Then by Lemma 2.30, $v_1, \dots, v_n$ is linearly dependent, which is a contradiction because it is a basis.

(c) Let $w_1 \dots w_n \in V$. Suppose that $w_1 \dots w_n$ span V but are linearly dependent. Then there is some i such that w_i may be removed without changing the span;

$$\text{span}\{w_1 \dots w_n\} = \text{span}\{w_1 \dots w_{i-1}, w_{i+1} \dots, w_n\}.$$

(see the remark on page 45). But because $\text{span}\{w_1 \dots w_n\} = V$, this contradicts part (b).

Similarly, suppose $w_1 \dots w_n$ ~~span~~ are linearly independent but don't span V. Then there exists $w_{n+1} \in V$ such that $w_{n+1} \notin \text{Span}\{w_1 \dots w_n\}$. It follows that $w_1 \dots w_{n+1}$ is

reads, "is not in"

also linearly independent. ~~so~~ However, this ~~~~ contradicts Lemma 2.30 because $v_1 \dots v_n$ span V.

Thus, $w_1 \dots w_n$ spans V if and only if it is a linearly ~~independent~~ independent. (c) follows.

Writing a vector in terms of a basis

Lemma 2.34 The elements $v_1, \dots, v_n \in V$ form a basis if and only if every $v \in V$ can be written uniquely as a linear combination of $v_1, \dots, v_n$.

Pf Let $v_1 \dots v_n \in V$. Suppose that $v_1 \dots v_n$ form a basis. Let $v \in V$, & suppose that

$$v = \sum_{i=1}^{n} c_i v_i \quad \text{and} \quad v = \sum_{i=1}^{n} d_i v_i .$$

By subtraction,

$$0 = v - v = \sum_{i=1}^{n} c_i v_i - \sum_{i=1}^{n} d_i v_i = \sum_{i=1}^{n} (c_i - d_i) v_i .$$

Because the v_i are a basis, they are linearly independent. Hence, $c_i - d_i = 0$ for all i, so $c_i = d_i$ for all i. Moreover, $\operatorname{span}\{v_1 \dots v_n\} = V$. Hence, every $v \in V$ has a unique representation as $v = \sum_{i=1}^{n} c_i v_i$.

Suppose, conversely, that every $v \in V$ has a unique representation as $v = \sum_{i=1}^{n} c_i v_i$. Then $\operatorname{span}\{v_1 \dots v_n\} = V$. Note also that of the representation of 0, $0 = \sum_{i=1}^{n} 0 v_i$, so by uniqueness the v_i are linearly independent.

ex Consider the basis of $\mathbb{R}^4$ given by

$$\underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} -2 \\ -3 \\ -7 \\ -1 \end{pmatrix}}, \underset{v_4}{\begin{pmatrix} 1 \\ 5 \\ 2 \\ 3 \end{pmatrix}}.$$

Find the representation of $v = \sum_{i=1}^{4} c_i v_i$ for $v = \begin{pmatrix} 6 \\ 7 \\ 8 \\ 2 \end{pmatrix}$

By the correspondance $(v_1 \dots v_n)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c_1 v_1 + \dots + c_n v_n$, we see that we are solving

$$\left(v_1\, v_2\, v_3\, v_4\right)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = v.$$

To this end, we apply Gaussian Elimination to the corresponding augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 6 \\ 1 & 1 & -3 & 5 & 7 \\ -1 & 4 & -7 & 9 & 8 \\ 0 & 1 & -1 & 3 & 9 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 6 \\ 0 & -2 & -1 & 4 & 1 \\ 0 & 7 & -9 & 10 & 15 \\ 0 & 1 & -1 & 3 & 9 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 6 \\ 0 & 1 & -1 & 3 & 9 \\ 0 & 7 & -9 & 10 & 15 \\ 0 & -2 & -1 & 4 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 6 \\ 0 & 1 & -1 & 3 & 9 \\ 0 & 0 & -2 & -11 & -48 \\ 0 & 0 & -3 & 10 & 19 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 6 \\ 0 & 1 & -1 & 3 & 9 \\ 0 & 0 & 1 & -21 & -67 \\ 0 & 0 & -3 & 10 & 19 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 6 \\ 0 & 1 & -1 & 3 & 9 \\ 0 & 0 & 1 & -21 & -67 \\ 0 & 0 & 0 & -53 & -182 \end{array}\right) \Rightarrow c_4 \approx 3.434$$

$$c_3 = -67 + 21c_4 \approx 5.113$$

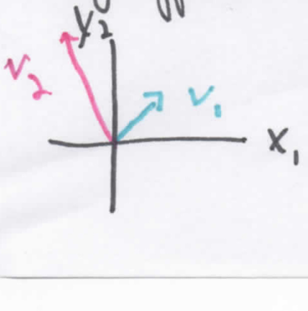
$$c_2 = c_3 - 3c_4 + 9 \approx 3.811$$

$$c_1 = 6 - 3c_2 + 2c_3 - c_4 \approx 1.358$$

The process of expanding an element in terms of its basis is like having a coordinate reference relative to a given basis. To visualize this, consider the following example.

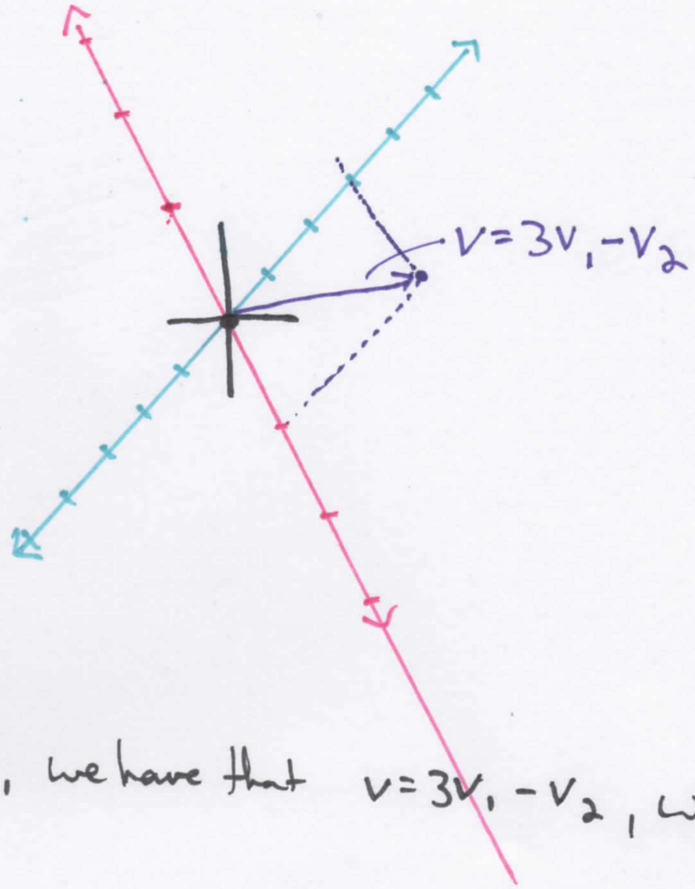
ex Consider the basis of $\mathbb{R}^2$ given by $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$.

Plotted, they appear as



Zooming out, we add coordinate axes parallel to v_1 & v_2,

$\bullet = 0$



So if we take $v = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, we have that $v = 3v_1 - v_2$, which can be visualized as above.