Matrix Inverses §1.5 Lecture 5

Defn Let A be an $n \times n$ matrix. A matrix X is called the inverse of A if

$$XA = I \quad \& \quad AX = I.$$

We denote the inverse matrix by A^{-1}.

ex Elementary matrices type 1

$$\begin{pmatrix} 1 & & \\ 0 & 1 & \\ -2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}$$

Permutation matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Fact If P is a permutation matrix, $P^{-1} = P^T$ ← transpose
switch $(i,j) \to (j,i)$
"reflected"

ex $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$, $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = X$

$$AX = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$ax+bz = 1 \qquad ay+bw = 0$

$cx+dz = 0 \qquad cy+dw = 1$

$$\left(\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ c & 0 & d & 0 & 0 \\ 0 & a & 0 & b & 0 \\ 0 & c & 0 & d & 1 \end{array}\right) \text{ can solve by GE}$$

$$x = \frac{d}{ad-bc} \qquad y = -\frac{b}{ad-bc} \qquad z = -\frac{c}{ad-bc} \qquad w = \frac{a}{ad-bc}$$

$$X = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \quad AX = I$$

$$XA = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & db-bd \\ -ac+ac & -cb+ad \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Lemma 1.19 If A has an inverse, then it is unique.

PF $\boxed{\text{Suppose } X \text{ \& } Y \text{ are inverses for } A}$. Then hypothesis

$$AX = I = XA$$
$$AY = I = YA.$$

From $AX = I$,

we multiply by Y on the left $\quad Y(AX) = YI$

associativity $\quad (YA)X = YI$

hypothesis $\quad IX = YI$

prop of Identity $\quad \boxed{X = Y}$ ← conclusion.

Lemma 1.20 If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$. hypothesis

PF $\boxed{\text{Suppose } A \text{ is invertible}}$. Then A^{-1} exists and

$$AA^{-1} = I \quad \text{\&} \quad A^{-1}A = I.$$

But this is the definition of A being the inverse of A^{-1}. conclusion

Lemma 1.21 If A & B are $n \times n$ & invertible, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

PF $\boxed{\text{Suppose } A \text{ \& } B \text{ are } n\times n \text{ \& invertible.}}$ hypothesis

Then AB ~~matrix~~ is defined & $n \times n$.

$$(B^{-1}A^{-1})(AB) \overset{\text{associativity}}{=} (B^{-1}(A^{-1}A))B \overset{\text{property of inverses}}{=} (B^{-1}I)B$$

$$\underset{\text{property of } I}{=} B^{-1}B \underset{\text{property of inverses.}}{=} I$$

Gauss - Jordan Elimination

$$AX = I. \quad X = (x_1, x_2 \ldots x_n)$$

column vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

= columns of I_n.

$$A x_1 = e_1 \quad A x_2 = e_2 \ldots$$

Do gaussian elimination to solve each one, but the work would all be the same! Do them all at the same time.

$$(A \mid I)$$

Elementary Row operation 3 Multiply a row by a nonzero number.

(corresponds to multiplying the corresponding equation)

Using ERO 1, 2, & 3, we get $(A \mid I) \sim (I \mid X)$ where

$$AX = I.$$

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 8 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$r_3 - \tfrac{2}{3} r_1 \left(\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 8 & 0 & 1 & 0 \\ 0 & 1/3 & -1/3 & -\frac{2}{3} & 0 & 1 \end{array}\right) \quad r_3 + \tfrac{1}{3} r_2 \left(\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 8 & 0 & 1 & 0 \\ 0 & 0 & 7/3 & -\frac{2}{3} & \frac{1}{3} & 1 \end{array}\right)$$

$$r_3 \cdot \tfrac{3}{7} \left(\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 8 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2/7 & 1/7 & 3/7 \end{array}\right) \quad \begin{array}{l} r_2 - 8 r_3 \\ r_1 - 2 r_3 \end{array} \left(\begin{array}{ccc|ccc} 3 & 1 & 0 & 11/7 & -2/7 & -6/7 \\ 0 & -1 & 0 & 16/7 & -1/7 & -24/7 \\ 0 & 0 & 1 & -2/7 & 1/7 & 3/7 \end{array}\right)$$

$$r_2 \cdot -1 \left(\begin{array}{ccc|ccc} 3 & 1 & 0 & 11/7 & -2/7 & -6/7 \\ 0 & 1 & 0 & -16/7 & +1/7 & 24/7 \\ 0 & 0 & 1 & -2/7 & 1/7 & 3/7 \end{array}\right) \quad r_1 - r_2 \left(\begin{array}{ccc|ccc} 3 & 0 & 0 & -5/7 & -1/7 & 18/7 \\ 0 & 1 & 0 & 16/7 & -1/7 & -24/7 \\ 0 & 0 & 1 & -2/7 & 1/7 & 3/7 \end{array}\right)$$

$$r_1 / 3 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/21 & -1/21 & 18/21 \\ 0 & 1 & 0 & 16/7 & -1/7 & -24/7 \\ 0 & 0 & 1 & -2/7 & 1/7 & 3/7 \end{array}\right)$$

Thm If A is nonsingular, then it is invertible

Pf $\boxed{\text{Suppose } A \text{ is nonsingular}}$. hypothesis

By defn, there are elementary matrices $E_1, \dots E_n$ s.t.

$$E_n \cdot \dots \cdot E_1 A = U$$

where U is upper triangular w/ nonzero entries on the diagonal.

It follows there are e.m. $E_{n+1} \dots E_M$ s.t.

$$E_M \cdot \dots \cdot E_{n+1} \cdot E_n \cdot \dots \cdot E_1 A = I.$$

Let $X = E_M \cdot \dots \cdot E_1$. Then X^{-1} exists and

$$X^{-1} = E_1^{-1} \cdot \dots \cdot E_M^{-1}.$$

From $XA = I$, we get $A = X^{-1}$. Thus, by Lemma 1.20,

A is invertible and $A^{-1} = (X^{-1})^{-1} = X$. //

Remark Once we have A^{-1}, to solve $Ax = b$

$$A^{-1} A x = A^{-1} b$$
$$x = A^{-1} b.$$

But this is numerically unstable!