Lecture 19 Inner Products §3.1

Let V be a vector space.

Defn An <u>inner product</u> on V is a map, indicated by the brackets $\langle \cdot, \cdot \rangle$, such that for any $v, w \in V$, $\langle v, w \rangle \in \mathbb{R}$ satisfying the following properties:

(i) Bilinearity

$$\left.\begin{aligned} \langle cu + dv, w \rangle &= c\langle u, w \rangle + d\langle v, w \rangle \\ \langle u, cv + dw \rangle &= c\langle u, v \rangle + d\langle u, w \rangle \end{aligned}\right\} \text{ for any } u, v, w \in V.\ c, d \in \mathbb{R}$$

(*) see footnote

(ii) Symmetry

$$\langle v, w \rangle = \langle w, v \rangle \quad \text{for all } v, w \in V$$

(iii) Positivity

$$\langle v, v \rangle > 0 \quad \text{for all } v \neq 0 \quad \text{and} \quad \langle 0, 0 \rangle = 0.$$

ex1 The most basic inner product is the dot product on $\mathbb{R}^n$:

$$\langle u, v \rangle := u \cdot v = \sum_{i=1}^{n} u_i v_i$$

The axioms (i)–(iii) are easily seen to be satisfied:

(i) $(cu + dv) \cdot w = \sum_{i=1}^{n} (cu_i + dv_i) w_i = \sum_{i=1}^{n} cu_i w_i + dv_i w_i = c\sum_{i=1}^{n} u_i w_i + d\sum_{i=1}^{n} v_i w_i$

$$= c\, u \cdot w + d\, v \cdot w$$

(ii) $u \cdot v = \sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} v_i u_i = v \cdot u$

(iii) $u \cdot u = \sum_{i=1}^{n} u_i^2$. If $u = 0$, $u \cdot u = 0$. If $u \neq 0$, then

$$\sum_{i=1}^{n} u_i^2 > 0.$$

Rk A vector space with an inner product $(V, \langle , \rangle)$ is called an <u>inner product space</u>.

(*) Note that the second condition, $\langle u, cv + dw \rangle = c\langle u, v \rangle + d\langle u, w \rangle$, can be proven from the first by using (ii).

ex2 Let $c_1 \dots c_n$ be positive numbers. Define ~~the~~ a weighted inner product ~~by~~ on $\mathbb{R}^n$ by

$$\langle u, v \rangle = \sum_{i=1}^{n} c_i u_i v_i .$$

The proof that $\langle , \rangle$ is an inner product is identical to ~~~~ the previous proof.

ex3 Let $V = C^0([a,b]) := \{ f:[a,b] \to \mathbb{R} : f \text{ is continuous} \}$, where $a, b \in \mathbb{R}$.

As discussed ~~~~ in Lecture 10, $C^0([a,b])$. Is a vector space under the usual addition & scalar multiplication. ~~~~ Define an inner product on V by

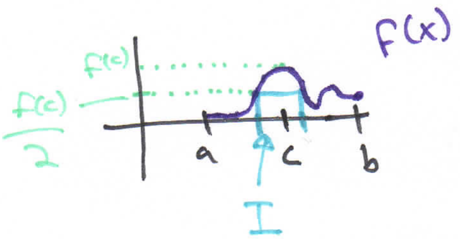
$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx = \int_a^b fg .$$

Because f and g are continuous on a closed interval $[a,b]$, this guarantees that fg is continuous on $[a,b]$ and so the Riemann integral $\int_a^b fg$ exists & is finite. We call $\mathcal{L}^2([a,b])$ the inner product space of $\left(C^0([a,b]), \langle , \rangle \right)$.

(* pronounced "L two of the interval $[a,b]$", or just "L two".)

To verify that $\langle , \rangle$ is an inner product, (i) & (ii) follow similarly to ex 1 (where "sums" are replaced by "integrals").

positivity ~~~~ First, we note that $\langle 0, 0 \rangle = \int_a^b 0^2 = 0$. Next, we consider $f:[a,b] \to \mathbb{R}$ which is nonzero. Let $c \in [a,b]$, $f(c) \neq 0$. Then ~~then~~ because f is continuous, there is some interval $I \subseteq [a,b]$, $c \in I$, where $|f(c)| > \frac{|f(c)|}{2}$, as diagrammed below. Let $I = [a', b']$.



Because $f(x)^2 \geq 0$, we get

$$\langle f, f\rangle = \int_a^b f(x)^2\,dx \geq \int_{a'}^{b'} f(x)^2\,dx \geq \int_{a'}^{b'} \left|\frac{f(c)}{2}\right|^2 = \frac{f(c)^2}{4}\cdot(b'-a') > 0.$$

An important application of this to the theory of oridinary and partial differential equations (ODE & PDE) is that on the integral $[a,b] = [0,\pi]$

$$\langle \sin x, \cos x\rangle = \int_0^{\pi} \sin x \cos x = \frac{1}{2}\sin^2 x \Big|_{x=0}^{\pi} = 0.$$

More generally

$$\langle \sin nx, \cos mx\rangle = 0 \quad \text{for all } m,n = 1,2,\dots$$

$$\langle \sin nx, \sin mx\rangle = \langle \cos nx, \cos mx\rangle = 0 \quad \text{for all } m \neq n.$$

This allows a "fourier series" solution of many problems.

ex Let $w : [a,b] \to \mathbb{R}$ be a positive continuous function. Define ~~the~~ a weighted L two space $L^2([a,b], w\,dx)$ to be the inner product space with

$$\langle f, g\rangle := \int_a^b f(x)g(x)w(x)\,dx.$$

Defn The norm associated to an inner product space $(V, \langle,\rangle)$ is

$$\|v\| := \sqrt{\langle v, v\rangle}$$

The L^2 norm of a function is related to ~~known as~~ the "standard deviation" in probability theory. If $w(x)\,dx$ is a probability distribution on $[a,b]$, and $X : [a,b] \to \mathbb{R}$ is a random variable, then the standard deviation of X is

$$\|X - E[X]\|$$

where $E[X] = \int_a^b X(x)w(x)\,dx$ is the "expectation" of X.