Lecture 21 § 3.3 Norms

<u>Defn</u> A <u>norm</u> on a vector space V is a function $\| \; \| : V \to \mathbb{R}$ such that for any $v, w \in V$, $c \in \mathbb{R}$

(a) Positivity: $\|v\| \geq 0$, with $\|v\| = 0$ only if $v = 0$.

(b) Homogeneity: $\|cv\| = |c| \, \|v\|$

(c) Triangle inequality: $\|v+w\| \leq \|v\| + \|w\|$.

<u>ex</u> In section 3.2, we defined the norm associated to an inner product $\langle , \rangle$ as

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

This is a norm in the sense above

<u>ex</u> The "Euclidean norm" of $\mathbb{R}^n$ is the most common,

$$\|v\| = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2} = \sqrt{v \cdot v}.$$

This is also known as the L^2 norm of $\mathbb{R}^n$

<u>ex</u> For any p, $1 \leq p < \infty$, we define the L^p norm of $\mathbb{R}^n$ by

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$$

So, for example,

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|$$

$$\|v\|_3 = \left(\sum_{i=1}^{n} |v_i|^3\right)^{1/3}.$$

<u>Note</u> Notice that positivity & homogeneity are automatic

(a) If $v \neq 0$, then $\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p} \geq \left(\max_{i=1\dots n} |v_i|^p\right)^{1/p} = \max_{i=1\dots n} |v_i| > 0.$

(b) $\|cv\|_p = \left(\sum_{i=1}^{n} |cv_i|^p\right)^{1/p} = \left(\sum_{i=1}^{n} |c|^p |v_i|^p\right)^{1/p} = |c| \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$

$= |c| \, \|v\|_p.$

The triangle inequality, however, is more difficult to prove, but is true.

ex The L^∞ norm on $\mathbb{R}^n$ is given by

$$\|v\|_\infty = \max_{i=1..n} |v_i|.$$

So $\left\| \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right\|_\infty = \max\left(|1|, |-2|, |0|\right) = 2$.

In this case, positivity, homogeneity & the triangle inequality are quickly established.

ex We define the normed function space $L^p([a,b])$ for $1 \le p < \infty$ by equipping the space $C^0([a,b])$ with the norm

$$\|f\|_p = \left(\int_a^b |f|^p\right)^{1/p}.$$

Similarly, $L^\infty([a,b])$ is defined as the space $C^0([a,b])$ with the norm

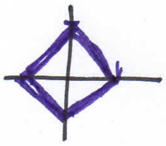
$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|.$$

This maximum exists and is finite because f is continuous on a closed interval.

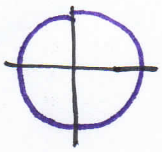
None of these norms except for L^2 on $\mathbb{R}^n$ & $L^2([a,b])$ come from an inner product

Geometric interpretation of L^p

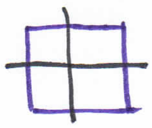
The "Unit" Sphere of radius 1 associated to a norm is defined on $\mathbb{R}^2$ to be $S_1 = \{x \in \mathbb{R}^n : |x| = 1\}$. The 1-spheres for $\|\cdot\|_1, \|\cdot\|_2, \|\ \|_\infty$ are



$\|\ \|_1$

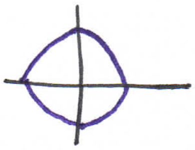


$\|\ \|_2$

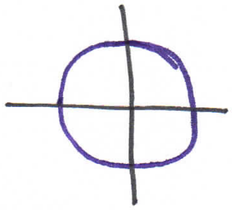


$\|\ \|_\infty$

For other values of p, the spheres are "in between" these cases:

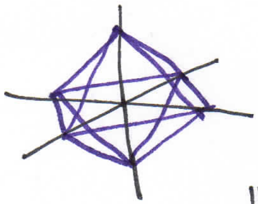


$\| \ \|_{3/2}$

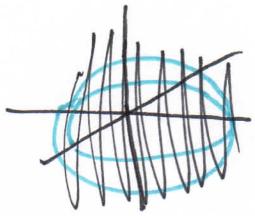


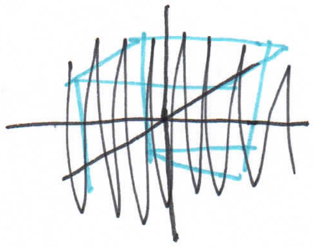
$\| \ \|_{7}$

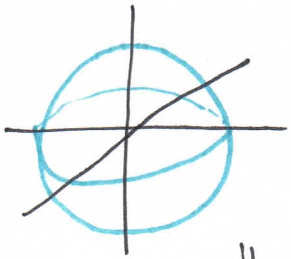
In $\mathbb{R}^3$, the 1-spheres look like



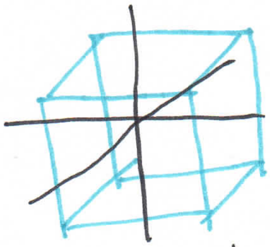
$\| \ \|_{1}$







$\| \ \|_{2}$



$\| \ \|_{\infty}$