Lecture 6 § 1.8 General Linear Systems

So far, we've focused on systems with n variables & n equations. These are the most intuitive systems, especially when the coefficient matrix A is nonsingular. Indeed if A is nonsingular then $Ax=b$ has exactly one solution for every choice of b.

Today we'll look at general systems with m equations in n variables. The coefficient matrix will be $m \times n$.

ex

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 5 \\ 6x_1 - x_2 + x_3 &= 1 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 6 & -1 & 1 & 1 \end{array}\right) \xrightarrow{r_2 - 2r_1} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 5 \\ 0 & -5 & 3 & -9 \end{array}\right)$$

Notice that in this simple example, any choice of x_3 will lead to a solution in x_1 & x_2. In fact, we can solve for x_1 & x_2 in terms of x_3.

2^{nd} row: $-5x_2 + 3x_3 = -9 \Rightarrow \boxed{x_2 = \dfrac{9+3x_3}{5}}$

1^{st} row: $3x_1 + 2\left(\dfrac{9+3x_3}{5}\right) - x_3 = 5$

$\Rightarrow 3x_1 + \dfrac{18}{5} + \dfrac{x_3}{5} = \dfrac{25}{5} \Rightarrow \boxed{x_1 = \dfrac{7-x_3}{15}}$

In this case, we say x_3 is a free variable

"Defns" A matrix is said to be in row echelon form (REF) if it "looks like"

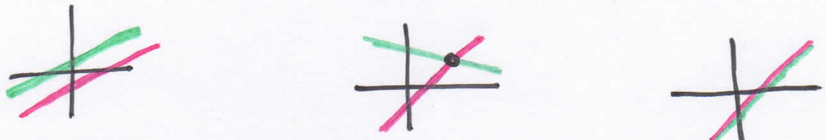
free variables, x_3, x_5, x_6, x_8

$$\text{pivots} \quad \begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

basic variables x_1, x_2, x_4, x_7

A pivot is the first nonzero entry in each row. We say that a column with a pivot corresponds to a basic variable, and the rest are free variables.

Thm A linear system $Ax = b$ will have either 0, 1, or infinitely many solutions



0 solutions   1 solution ∞'ly many solutions

We will now see how to generalize our previous techniques to general linear systems.

Gaussian Elimination

Our elementary row operations 1, 2 & 3 still work as before (recall what they represent in terms of linear equations).

Our answers will be of the form

$$\boxed{\text{basic variables} = \text{formula in terms of free variables.}}$$

However, we will need to also check consistency of our linear system.

ex
$$\left(\begin{array}{ccccc|c} 1 & 3 & 2 & -1 & 0 & 0 \\ 0 & 0 & -3 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$
Consider the linear system represented by this augmented matrix. The final row represents the equation $0 = 1$.

Thus, this linear system has no solution.

Fact A linear system in row echelon form is inconsistent if and only if it has a row of the form $(0\ 0 \ldots 0 \mid a)$ where $a \neq 0$.

modified ex
$$\left(\begin{array}{ccccc|c} 1 & 3 & 2 & -1 & 0 & 0 \\ 0 & 0 & -3 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$
This linear system is consistent. It has free variables x_2 & x_4. We perform back substitution with the free variables.

$4x_5 = 3 \Rightarrow \boxed{x_5 = 3/4.}$

$-3x_3 + 6x_4 + 9/4 = 3 \Rightarrow -3x_3 = 3/4 - 6x_4 \Rightarrow \boxed{x_3 = -\frac{1}{4} + 2x_4}$

$x_1 + 3x_2 + 2(-\frac{1}{4} + 2x_4) - x_4 = 0 \Rightarrow x_1 + 3x_2 - \frac{1}{2} + 3x_4 = 0$

$\Rightarrow \boxed{x_1 = -3x_2 - 3x_4 + \frac{1}{2}}$

Generalized LU factorization

We will write $A = LU$ where $L = m\times m$ lowertriangular & U is $m\times n$ in row echelon form.

ex

$$\begin{pmatrix} 3 & 1 & 4 & 5 \\ -3 & 2 & 2 & 0 \\ 1 & 0 & 1 & -2 \end{pmatrix} = A. \quad \begin{matrix} r_2 + r_1 \\ r_3 - \frac{r_1}{3} \end{matrix} \begin{pmatrix} 3 & 1 & 4 & 5 \\ 0 & 3 & 6 & 5 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{11}{3} \end{pmatrix} \; r_3 + \frac{r_2}{9} \begin{pmatrix} 3 & 1 & 4 & 5 \\ 0 & 3 & 6 & 5 \\ 0 & 0 & \frac{1}{3} & -\frac{28}{9} \end{pmatrix} = U.$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{9} & 1 \end{pmatrix}$$

Solving with an LU factorization also works similarly.

From $Ax = b$ and $A = LU$, we have

$L(Ux) = b$. We split this into 2 equations

$Lc = b$

$Ux = c$.

Because L is lower triangular, we solve the first equation with forward substitution. Take $b = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$

$c_1 = 4$

$-c_1 + c_2 = 0 \Rightarrow c_2 = 4$

$\frac{1}{3} c_1 - \frac{1}{9} c_2 + c_3 = -1 \Rightarrow \frac{4}{3} - \frac{4}{9} + c_3 = -1 \Rightarrow c_3 = -\frac{17}{9}$.

We then perform back substitution, keeping the free variables unassigned, to solve $Ux = c$.

$\frac{1}{3} x_3 - \frac{28}{9} x_4 = -\frac{17}{9} \Rightarrow \boxed{x_3 = -\frac{17}{3} + \frac{28}{3} x_4}$

$3x_2 + 6\left(-\frac{17}{3} + \frac{28}{3} x_4\right) + 5x_4 = 4 \Rightarrow$

$x_2 + -\frac{34}{3} + \frac{56}{3} x_4 + \frac{15}{3} x_4 = \frac{12}{3} \Rightarrow$

$\boxed{x_2 = \frac{46}{3} + \frac{71}{3} x_4}$

$3x_1 + \frac{46}{3} + \frac{71}{3} x_4 + 4\left(-\frac{17}{3} + \frac{28}{3} x_4\right) + 5x_4 = 4 \Rightarrow$

$3x_1 + -\frac{22}{3} + \frac{183}{3} x_4 + 5x_4 = 4 \Rightarrow$

$3x_1 = \frac{40}{3} - 66x_4 \quad \boxed{x_1 = \frac{40}{9} - 22x_4}$