# Midterm 2

Math 4242 010, Au 2014

11/24/2014

Please write your name in the top left corner of the exam.

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Answer all questions completely and write neatly.

You may have an 8.5" ×11" sheets of notes. If you need clarification on a question, you may ask.

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| Total: | 40 | |

1. Let $K = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$.

(a) (5 points) Is $K$ positive definite?

$K \xrightarrow{r_2 - r_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 3 \end{pmatrix}$. We see that $K$ is not regular since using ERO1 yields a 0 on the diagonal. Thus $\boxed{\text{K is not positive definite.}}$

(b) (5 points) Does $p(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 4$ have a unique minimizer? Why or why not?

$\boxed{\text{No}}$. $p(x)$ will have a unique minimizer if and only if $K$ is positive definite.

2. Let $\langle , \rangle$ be the $L^2([0,1])$ inner product on $C^0([0,1])$. Let $f(x) = x^2$ and $g(x) = 2x - 1$.

(a) (3 points) Compute $\langle f, g \rangle$.

$$\langle f, g \rangle = \int_0^1 x^2(2x-1)\,dx = \int_0^1 2x^3 - x^2\,dx = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$$

(b) (2 points) Compute $||f||$.

$$\|f\| = \sqrt{\int_0^1 (x^2)^2\,dx} = \sqrt{\frac{1}{5}} = \boxed{\frac{1}{\sqrt{5}}}$$

(c) (2 points) Compute $||g||$.

$$\|g\| = \sqrt{\int_0^1 (2x-1)^2\,dx} = \sqrt{\int_0^1 4x^2 - 4x + 1\,dx} = \sqrt{\frac{4}{3} - 2 + 1} = \sqrt{\frac{1}{3}} = \boxed{\frac{1}{\sqrt{3}}}.$$

(d) (3 points) Explain what the Cauchy-Schwarz inequality says for the functions $f$ and $g$ above.

The Cauchy-Schwarz inequality states that $|\langle u, v \rangle| \leq \|u\| \|v\|$ for any inner product space. So in this case

$$\boxed{\frac{1}{6} = \left| \int_0^1 f(x) g(x)\,dx \right| = \langle f, g \rangle \leq \underbrace{\|f\|}_{\sqrt{\int_0^1 f(x)^2}} \underbrace{\|g\|}_{\sqrt{\int_0^1 g(x)^2\,dx}} = \frac{1}{\sqrt{5}} \frac{1}{\sqrt{3}}}$$

or, perhaps just

$$\boxed{\frac{1}{6} = \left| \int_0^1 x^2(2x-1)\,dx \right| \leq \sqrt{\int_0^1 (x^2)^2\,dx} \sqrt{\int_0^1 (2x-1)^2\,dx} = \frac{1}{\sqrt{5}\sqrt{3}}}$$

3. (10 points) Let $V = \{\mathbf{x} \in \mathbb{R}^3 : x_1 - 2x_2 + x_3 = 0\}$. Define an inner product on $\mathbb{R}^3$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T C \mathbf{y} \quad \text{where} \quad C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}.$$

(You take as a given that $C$ is positive definite.) Let $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. Find the nearest point $\mathbf{v}^* \in V$ to $\mathbf{b}$, and compute the distance from $\mathbf{b}$ to $\mathbf{v}^*$.

Choose basis of $V$ by setting free variables:

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \quad \text{Set } A = (v_1 \; v_2) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Set
$$K = A^T C A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & -2 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -4 & 6 \end{pmatrix}$$

$$f = A^T C b = \begin{pmatrix} 2 & 2 & -2 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}.$$

Then for $v = x_1 v_1 + x_2 v_2$,

$$\|v - b\|^2 = x^T K x - 2x^T f + \|b\|^2,$$

and the minimizer $x^*$ is given by solving $Kx = f$. So we solve the augmented system

$$(K \,\vdots\, f) = \left(\begin{array}{cc|c} 6 & -4 & 6 \\ -4 & 6 & -6 \end{array}\right) \overset{r_2 + \frac{2}{3} r_1}{\sim} \left(\begin{array}{cc|c} 6 & -4 & 6 \\ 0 & \frac{10}{3} & -2 \end{array}\right). \rightsquigarrow$$

$6 - \frac{8}{3} = \frac{10}{3}$

$\frac{10}{3} x_2^* = -2 \quad x_2^* = -\frac{3}{5}, \quad 6x_1^* = 4x_2^* + 6 \rightsquigarrow x_1^* = \frac{2}{3} x_2^* + 1 = -\frac{2}{5} + 1 = \frac{3}{5}.$

Thus, $\boxed{v^* = \frac{3}{5} v_1 - \frac{3}{5} v_2 = \frac{3}{5}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \frac{3}{5}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}}$

The distance ~~is~~ can be computed as $d^* = \|v^* - b\| = \sqrt{\|b\|^2 - f^T x^*}$

We compute

$$\|b\|^2 = b^T C b = (3\ 0\ 0)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = (3\ 0\ -3)\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 9$$

$$f^T x^* = (6\ -6)\begin{pmatrix} \frac{3}{5} \\ -\frac{3}{5} \end{pmatrix} = \frac{36}{5}$$

$$\|b\|^2 - f^T x^* = 9 - \frac{36}{5} = \frac{45}{5} - \frac{36}{5} = \frac{9}{5},$$

So $\boxed{d^* = \frac{3}{\sqrt{5}}}$ is the distance from $v^*$ to $b$.

4. (10 points) Find the least squares line of best fit to the data points

| $t_i$ | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| $y_i$ | -2 | 0 | 1 | 3 |

.

We set $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$, $x = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 3 \end{pmatrix}$, and

find the least squares solution to $Ax = \vec{y}$.

So ~~x is~~ the least squares solution is given by

$$A^T A x = A^T \vec{y}.$$

$A^T A = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$, $A^T y = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$. Solving, we find

$$\left(\begin{array}{cc|c} 4 & 2 & 2 \\ 2 & 6 & 9 \end{array}\right) \xrightarrow{r_1/2} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 2 & 6 & 9 \end{array}\right) \xrightarrow{r_2 - r_1} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 5 & 8 \end{array}\right) \rightsquigarrow$$

$5\alpha_1 = 8 \quad \alpha_1 = \frac{8}{5}, \quad 2\alpha_0 = -\alpha_1 + 1 = -\frac{8}{5} + 1 = -\frac{3}{5}, \quad \alpha_0 = -\frac{3}{10}.$

Thus the line of best fit is given by

$$\boxed{y(t) = -\frac{3}{10} + \frac{8}{5} t}$$