Lecture 10 §2.1 & §2.2

More Vector space examples

Useful examples of vector spaces occur in functional analysis, a.k.a. the quantitative study of functions.

ex
- For $n = 0, 1, \ldots$ define $\mathcal{P}^{(n)} = \{p(x) : p \text{ is a polynomial of degree at most } n\}$.

  We recall that the degree of a polynomial is the largest exponent, e.g., the degree of $3x^2 + 2x - 1$ is 2.

  It is a simple exercise in the rules of high school algebra to see that $\mathcal{P}^{(n)}$ is a vector space.

- Let $\mathcal{P} = \{p(x) : p \text{ is a polynomial}\} = \bigcup_{i=0}^{\infty} \mathcal{P}^{(n)}$.

  "set theoretic union"

  Then $\mathcal{P}$ is also a vector space.

- For $n = 0, 1, 2, \ldots$ and $d = 1, 2, 3, \ldots$ we define

$$\mathcal{P}^{(n,d)} = \left\{p(x_1, \ldots, x_d) : \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \mathbb{R}^d,\ p \text{ is a polynomial of degree at most } n\right\}.$$

  That is, $\mathcal{P}^{(n)}$ is the vector space of single variable polynomials with degree $\leq n$, and $\mathcal{P}^{(n,d)}$ is the vector space of d-variable polynomials with degree $\leq d$.

  Again, the verification of $\mathcal{P}^{(n,d)}$ being a vector space is an exercise in elementary algebra.

- Let $I \subset \mathbb{R}$ be an interval; that is, $I = (a,b), (a,b], [a,b),$ or $[a,b]$ for some $a,b \in \mathbb{R}$. Let $C^0(I)$ be the space of continuous functions on I. Then $C^0(I)$ is a vector space. Indeed, the zero function is continuous, and plays the role of the identity. Moreover, the sum of two continuous functions is still continuous. The rest of the verification is left as an exercise.

§2.2 Subspaces

Defn Let V be a vector space. A subspace of V is a subset $W \subset V$ which is a vector space under the same addition and scalar multiplication as V.

Our first observation codifies a piece of intuition we used in the examples of section 2.1.

Prop 2.9 A nonempty set $W \subset V$ is a subspace if and only if the following hold:

(a) (closure of $+$) for all $v, w \in W$, $v + w \in W$

(b) (closure of $\cdot$) for all $v \in W$ and $c \in \mathbb{R}$, $cv \in W$.

Before giving the proof, we give some elementary facts

Facts If V is a vector space, then

(1) $cv = 0$ if and only if $c = 0$ or $v = 0$

(2) $-1 \cdot v = -v$.

Proof of facts (1) We must show two statements are equivalent:

(S1) $cv = 0$
(S2) $c = 0$ or $v = 0$.

Aside The most direct way to do this is to show that $S1 \Rightarrow S2$ (read "S1 implies S2") and $S2 \Rightarrow S1$ (read "S2 implies S1").

$S1 \Rightarrow S2$ || Assume that $cv = 0$. hypothesis Either $c=0$ or $c \neq 0$. Because we are hoping to prove that $c=0$ or $v=0$, we notice that if $c=0$ we are done. Hence, we assume $c \neq 0$. Thus, using axioms (a)-(g), we establish.

$$v \underset{(g)}{=} 1 \cdot v \underset{(\text{arithmetic}, c\neq 0)}{=} \left(\tfrac{1}{c} c\right) \cdot v \underset{(f)}{=} \tfrac{1}{c}(cv) \underset{(\text{assumption})}{=} \tfrac{1}{c} \cdot 0.$$

We now claim that $\frac{1}{c} \cdot 0 = 0$. Indeed, if λ is any constant,

(*1) $$\lambda 0 \underset{(c)}{=} \lambda(0+0) \underset{(e)}{=} \lambda 0 + \lambda 0 \underset{(d)}{\Rightarrow} \lambda 0 + -(\lambda 0) = \lambda 0 + \lambda 0 + (-(\lambda 0))$$

$$\underset{(d)\,\&\,(c)}{\Rightarrow} 0 = \lambda 0.$$ Taking $\lambda = \frac{1}{c}$ proves that

$$v = \tfrac{1}{c} \cdot 0 = 0.$$ $\boxed{\text{Hence, either } c=0 \text{ or } v=0.}$ conclusion

$S2 \Rightarrow S1$ || $\boxed{\text{Assume that } c=0 \text{ or } v=0}$ hypothesis.

case 1 $\boxed{c=0}$ $\boxed{\text{Assume } c=0}$ Mini hypothesis. Then

$$cv \underset{(\text{assumption})}{=} 0 \cdot v \underset{(\text{arithmetic})}{=} (0+0)\cdot v \underset{(e)}{=} 0v + 0v \underset{\text{assumption}}{=} cv + cv.$$ Thus, by (d) & (e),

$$\underset{(d)}{\,} cv + cv + -(cv) = cv + -(cv) \Rightarrow$$

$$cv + 0 \underset{(d)}{=} 0 \underset{(e)}{\Rightarrow} \boxed{cv = 0.}$$ Mini conclusion.

Case 2 $\boxed{v=0}$ $\boxed{\text{Assume } v=0}$. mini hypothesis. Then $cv=0$ by (*1) earlier in the proof with $\lambda = c$.

(2) We must now show that $-1\cdot v = -v$. We compute

$$-v + (-1)\cdot(-1)v \underset{(f)}{=} -v + 1\cdot v \underset{(g)}{=} -v + v \underset{(d)}{=} 0 .$$

Thus,

$$-v + -1\cdot -1\cdot v + -1\cdot v = 0 + -1\cdot v \qquad \text{(from previous equation)}$$

$$\Rightarrow \quad -v + (1-1)v = -1\cdot v \qquad \text{(from arithmetic \& (d))}$$

$$\Rightarrow \quad -v + 0\cdot v = -1\cdot v \qquad \text{(arithmetic)}$$

$$\Rightarrow \quad -v + 0 = -1\cdot v \qquad \text{(fact (1))}$$

$$\Rightarrow \quad -v = -1\cdot v . \qquad \text{(c)}$$

Proof of Prop 2.9 Suppose that $W \subset V$ is a subspace. Then (a) & (b) hold trivially.

Suppose that (a) & (b) hold for $W \subset V$. Then (i) & (ii) in definition 2.1 hold. Moreover, (a),(b),(e),(f) & (g) are true for all $v, w \in W$ because $W \subseteq V$. Moreover, by (b), $0 \in W$ because if w is any element in W, $0 = 0\cdot w$ by fact 1, so by (b), $0 \in W$. Similarly, $-w \in W$ because $-w = -1\cdot w$.