Lecture 17 More on Matrix Subspaces

Prop 2.41 The following are equivalent for an $n \times n$ matrix A:

(i) $\ker(A) = \{0\}$

(ii) $\operatorname{rank}(A) = n$ (recall $\operatorname{rank}(A) = \dim(\operatorname{rng}(A))$)

(iii) The system $Ax = b$ has no free variables

(iv) The system $Ax = b$ has a unique solution for all $b \in \operatorname{rng}(A)$

Pf We will prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

Suppose that $\ker(A) = \{0\}$. Then $\dim(\ker(A)) = 0$, so by the Fundamental Theorem of Linear Algebra, $\operatorname{rank}(A) = n$.

Suppose that $\operatorname{rank}(A) = n$. By definition, there is a pivot in every column, so the system has no free variables.

Suppose that $Ax = b$ has no free variables. By chapter 1, the solution to $Ax = b$ is unique when it exists.

Suppose that $Ax = b$ has a unique solution for all $b \in \operatorname{rng}(A)$. In particular, this is true for $b = 0$. Because $A0 = 0$, $\ker(A) = \{0\}$.

Prop 2.42 Suppose that A is an $n \times n$ matrix. The following are equivalent:

(i) A is nonsingular

(ii) A is invertible

(iii) $\operatorname{rank}(A) = n$

(iv) $\ker(A) = \{0\}$

(v) $\operatorname{rng}(A) = \mathbb{R}^n$

(vi) $\det(A) \neq 0$

(vii) Writing A in column vector form, $A = (v_1 \dots v_n)$, the vectors $v_1 \dots v_n$ form a basis of $\mathbb{R}^n$.

The Superposition Principle

The principle of superposition is the observation that if

$$A x^{(1)} = b^{(1)} \quad \text{and} \quad A x^{(2)} = b^{(2)}$$

then the system

$$A x = c_1 b^{(1)} + c_2 b^{(2)}$$

has the solution

$$x^* = c_1 x^{(1)} + c_2 x^{(2)}.$$

More generally, if

$$A x^{(i)} = b^{(i)} \quad \text{for } i = 1, 2, \ldots, k$$

then

$$A x = \sum_{i=1}^{k} c_i b^{(i)}$$

has the solution

$$x^* = \sum_{i=1}^{k} c_i x^{(i)} .$$

One important application of this is in solving the system

$$Ax = b$$

for many b. If A is $m \times n$, and we need to solve the system $Ax = b$ for the <u>same</u> A but more than m vectors b, it is more efficient to solve

$$A x = e^{(i)} \quad \text{for } i = 1 \ldots m \qquad \left(e^{(i)} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th position} \right).$$

Then we write each $b^{(j)}$ as

$$b^{(j)} = \sum_{i=1}^{m} b_i e^{(i)}$$

and we construct the solution

$$x^{*(j)} = \sum_{i=1}^{m} b_i x^{(i)}$$

as the solution to each equation.

To recap, we use the following techniques for efficient solving:

$Ax=b$ once: Use Gaussian Elimination

$Ax=b$ more than once but less than m times: use a permuted LU factorization

$Ax=b$ more than m times: solve $Ax=e^{(j)}$ for ~~all~~ the m basis vectors $e^{(j)}$ and reconstruct the rest of our solutions.