# Lecture 9 § 2.1 Vector Spaces

In Chapter 2, we begin developing the ideas and theories of vector spaces which underlie advanced Linear Algebra.

In this section, we develop the abstract notion of a vector space. The advantage of an abstract, or sometimes called axiomatic, approach in mathematics serves to distil the important qualities of an object, while making it objectively veriable if two definitions are the same.

**Defn 2.1** A <u>real vector space</u> (or just <u>vector space</u>) is a set $V$ equipped with two operations,

(addition) for any $v, w \in V$, $v+w$ defines another vector in $V$, and

(scalar multiplication) for any $c \in \mathbb{R}$, $v \in V$, $c \cdot v$ defines a vector in $V$,

such that the following axioms hold for all $u, v, w \in V$, $c, d \in \mathbb{R}$:

(a) $v+w = w+v$ (commutativity of +)

(b) $(u+v)+w = u+(v+w)$ (associativity of +)

(c) there is an element in $V$, called zero and denoted $0$, such that $0+v = v$.
(additive identity)

(d) there is an element $-v$ such that $v + -v = 0$.
(additive inverse)

(e) $(c+d)v = cv + dv$, $c(v+w) = cv + cw$ (distributivity)

(f) $c(dv) = (cd)v$ (associativity of $\cdot$)

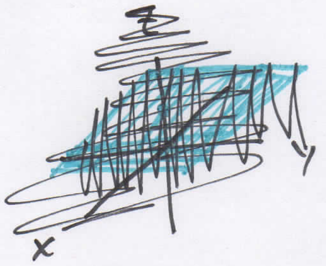
(g) $1v = v$ (multiplicative identity)

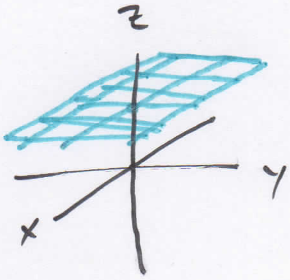
**ex** $\mathbb{R}^m = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} : x_i \in \mathbb{R} \right\}$. Here, $+$ & $\cdot$ are the normal addition & scalar multiplication: $\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_m + y_m \end{pmatrix}$ *, $c\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} cx_1 \\ \vdots \\ cx_m \end{pmatrix}$ *

Note that when $m=1$, definition 2.1 is simply the axioms of real valued arithmetic (or, rather, most of the axioms). Extending these rules from $m=1$ to general $m$ is done "component by component." *

**non-example** $V = \mathbb{R}^m$, but with addition defined in some kooky way like $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 + y_2 \\ x_2 + y_1 \end{pmatrix}$. Which axioms are okay, and which fail? We can verify 1 by 1 that (a) & (b) fail, but (c)–(g) are true. However, the failure of even one axiom means that $V$ is not a vector space under our kooky addition.

**ex** Consider $V = \left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}$. Is $V$ a vector space? Graphically, $V$ looks like




Indeed, $V$ is a plane, but this is not quite the same thing as a vector space. We check that operations (i) & (ii) fail to exist! addition, as defined in $\mathbb{R}^3$ fails the property that $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + \begin{pmatrix} z \\ w \\ 1 \end{pmatrix}$ is still in $V$.

ex $V = \left\{ \begin{pmatrix} x \\ y \\ x+y \end{pmatrix} : x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$. In this case, it is easily verified that (i) & (ii) hold for + & $\cdot$ on $\mathbb{R}^3$:

$$c\begin{pmatrix} x \\ y \\ x+y \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ c(x+y) \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cx+cy \end{pmatrix} \in V, \quad \begin{pmatrix} x \\ y \\ x+y \end{pmatrix} + \begin{pmatrix} u \\ v \\ u+v \end{pmatrix} = \begin{pmatrix} x+u \\ y+v \\ x+y+u+v \end{pmatrix}$$

$$= \begin{pmatrix} x+u \\ y+v \\ (x+u)+(y+v) \end{pmatrix} \in V.$$

Moreover, (a) - (g) hold because they hold on $\mathbb{R}^3$.