Lecture 30 § 5.2 Constructing Orthogonal Bases: The Grahm-Schmidt Process

In these notes, we discuss an algorithm to convert a basis $w_1 \dots w_n$ of an inner product space $(V, \langle , \rangle)$ into an orthonormal basis of V.

ex Consider the basis of $\mathbb{R}^2$ with the dot product,

$$v_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

In order to construct an orthonormal basis, we will first normalize v_1. Set

$$w_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ so that } \|w_1\| = 1.$$

Next, observe what happens when we set

$$\tilde{w}_2 = v_2 - \langle v_2, w_1 \rangle w_1$$

We compute $\langle \tilde{w}_2, w_1 \rangle$ to be

$$\langle \tilde{w}_2, w_1 \rangle \underset{\substack{\uparrow \\ \text{Definition}}}{=} \langle v_2 - \langle v_2, w_1 \rangle w_1, w_1 \rangle \underset{\substack{\uparrow \\ \text{linearity in the first entry}}}{=} \langle v_2, w_1 \rangle - \langle v_2, w_1 \rangle \langle w_1, w_1 \rangle$$

Because $1 = \|w_1\|$, $1 = \|w_1\|^2 = \langle w_1, w_1 \rangle$, so the above equation becomes

$$\langle \tilde{w}_2, w_1 \rangle = \langle v_2, w_1 \rangle - \langle v_2, w_1 \rangle = 0.$$

Thus $\tilde{w}_2$ is orthogonal to w_1. In our example,

$$\tilde{w}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We then set

$$w_2 = \frac{\tilde{w}_2}{\|\tilde{w}_2\|} \text{ so that } \|w_2\| = 1.$$ (In our example, $\|\tilde{w}_2\| = 1$ by coincidence).

This process generalizes to an arbitrary number of basis vectors. Consider $v_1 \dots v_n$ a basis of an inner product space V. We follow the following process to convert $v_1 \dots v_n$ into an orthonormal basis $w_1 \dots w_n$.

Step 1 Set $w_1 = \frac{v_1}{\|v_1\|}$ so that $\|w_1\| = 1$.

Step 2 Set $\tilde{w}_2 = v_2 - \langle v_2, w_1\rangle w_1$ and $w_2 = \frac{\tilde{w}_2}{\|\tilde{w}_2\|}$.

This ensures $\langle w_1, w_2\rangle = 0$ & $\|w_2\| = 1$.

Step 3 Set $\tilde{w}_3 = v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2$. This ensures that $\langle w_3, w_1\rangle = \langle w_3, w_2\rangle = 0$ & $\|w_3\| = 1$.

$\vdots$

Step k Set $\tilde{w}_k = v_k - \sum_{j=1}^{k-1} \langle v_k, w_j\rangle w_j$ and $w_k = \frac{\tilde{w}_k}{\|w_k\|}$. This ensures that

$$\langle w_k, w_j\rangle = 0 \text{ for } j = 1 \dots k-1 \text{ and } \|w_k\| = 1.$$

$\vdots$

Step n Set $\tilde{w}_n = v_n - \sum_{j=1}^{n-1} \langle v_n, w_j\rangle w_j$ and $w_n = \frac{\tilde{w}_n}{\|\tilde{w}_n\|}$.

This ensures that

$$\langle w_n, w_j\rangle = 0 \text{ for } j = 1 \dots n-1 \text{ and } \|w_n\| = 1.$$

~~Pf (that $\langle w_k, w_j\rangle = 0$ for $j = 1 \dots k-1$). We compute based on the above formulae~~

$$\cancel{\langle w_k, w_i\rangle = \frac{1}{\|\tilde{w}_k\|}\langle v_k - \sum_{j=1}^{k-1}\langle v_k, w_j\rangle w_j, w_i\rangle = \frac{1}{\|\tilde{w}_k\|}\left(\langle v_k, w_i\rangle - \sum_{j=1}^{k-1}\langle v_k, w_j\rangle\langle v_n\right.}$$

Rk Notice that if $\tilde{\omega}_k = 0$ for any k, our algorithm will break! In fact, if $v_1 \dots v_n$ is a basis, linear independence ensures this will not happen. However, if $v_1 \dots v_n$ is linearly dependent, then the algorithm will break, as $\tilde{\omega}_k = 0$ for some k.

Fact The Grahm schmidt process will always yield an orthonormal basis $\omega_1 \dots \omega_n$ if given a basis $v_1 \dots v_n$.

Pf (Sketch) Suppose $v_1 \dots v_n$ is a basis of an inner product space $(V, \langle , \rangle)$.

Claim 1 $\tilde{\omega}_k \neq 0$ for any k.

Pf by induction $v_1 \neq 0$ because $v_1 \dots v_n$ are LI. Thus, $\tilde{\omega}_1 = v_1 \neq 0$. So we may define $\omega_1 = \frac{\tilde{\omega}_1}{\|\tilde{\omega}_1\|}$.

Suppose $\tilde{\omega}_1 \dots \tilde{\omega}_{k-1}$ are not 0. We set $\omega_i = \frac{\tilde{\omega}_i}{\|\tilde{\omega}_i\|}$ for each $i = 1, \dots, k-1$. Defining

$$\tilde{\omega}_k = v_k - \sum_{j=1}^{k-1} \langle v_k, \omega_j \rangle \omega_j ,$$

we note that $\tilde{\omega}_k$ is a nonzero linear combination of $v_k, \omega_1, \dots, \omega_{k-1}$. But each ω_i is a linear combination of $v_1 \dots v_i$, so $\tilde{\omega}_k$ is a linear combination of $v_k, v_1 \dots v_{k-1}$. Linear independence implies $\tilde{\omega}_k \neq 0$, because the only v_k term has a nonzero coefficient.

Claim 2 $\langle \omega_k, \omega_j \rangle = 0$ for $j = 1 \dots k-1$.

Pf We compute

$$\langle \omega_k, \omega_j \rangle = \frac{1}{\|\tilde{\omega}_k\|} \langle \tilde{\omega}_k, \omega_j \rangle = \frac{1}{\|\tilde{\omega}_k\|} \langle v_k - \sum_{i=1}^{k-1} \langle v_k, \omega_i \rangle \omega_i, \omega_j \rangle$$

$$= \frac{1}{\|\tilde{\omega}_k\|} \left(\langle v_k, \omega_j \rangle - \sum_{i=1}^{k-1} \langle v_k, \omega_i \rangle \langle \omega_i, \omega_j \rangle \right)$$

linearity

Assuming $\langle \omega_i, \omega_j \rangle = 0$ for $i \neq j$, $i, j \leq k-1$ implies the above says

$$\langle \omega_k, \omega_j \rangle = \frac{1}{\|\tilde{\omega}_k\|} \left(\langle v_k, \omega_j \rangle - \langle v_k, \omega_j \rangle \right) = 0 .$$

ex ~~[illegible]~~ Consider $\mathbb{R}^3$ with the dot product, and let

$$v_1 = \begin{pmatrix}1\\0\\1\end{pmatrix},\ v_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix},\ v_3 = \begin{pmatrix}0\\2\\1\end{pmatrix}.$$

Step 1 $\|v_1\| = \sqrt{2}$, so $w_1 = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$.

Step 2 $\widetilde{w}_2 = v_2 - \langle v_2, w_1\rangle w_1$. $\langle v_2, w_1\rangle = \frac{2}{\sqrt{2}} = \sqrt{2}$, so

$$\widetilde{w}_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix} - \sqrt{2}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix}. \quad \|\widetilde{w}_2\| = 1,$$

$$w_2 = \frac{\widetilde{w}_2}{\|\widetilde{w}_2\|} = \begin{pmatrix}0\\-1\\0\end{pmatrix}.$$

Step 3 $\widetilde{w}_3 = v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2$.

$\langle v_3, w_1\rangle = \frac{1}{\sqrt{2}}$, $\langle v_3, w_2\rangle = -2$. So

$$\widetilde{w}_3 = \begin{pmatrix}0\\2\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} - -2\cdot\begin{pmatrix}0\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - 2\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}. \quad \|\widetilde{w}_3\| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = \frac{1}{\sqrt{2}},$$

~~[illegible]~~ $w_3 = \frac{\widetilde{w}_3}{\|\widetilde{w}_3\|} = \sqrt{2}\,\frac{1}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$.

ex Consider the space $P^{(3)} = \{$polynomials of degree less than or equal to 3$\}$. Define an inner product on $P^{(3)}$ by the $L^2([0,1])$ inner product:

$\langle f, g\rangle = \int_0^1 fg\,dx$. Consider the monomial basis

$p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2$, $p_3(x) = x^3$. We find an orthonormal basis q_0, q_1, q_2, q_3.

Step 1 $\|p_0(x)\| = \sqrt{\int_0^1 1^2\,dx} = 1$. Take $q_0(x) = \frac{p_0(x)}{\|p_0(x)\|} = 1$.

Step 2 Set $\widetilde{q}_1(x) = p_1(x) - \langle p_1(x), q_0(x)\rangle q_0(x) = x - \left(\int_0^1 x\cdot 1\,dx\right) 1 = x - \frac{1}{2}$.

$\|\widetilde{q}_1(x)\| = \sqrt{\int_0^1 (x - \frac{1}{2})^2} = \sqrt{\int_0^1 x^2 - x + \frac{1}{4}} = \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} = \frac{1}{2\sqrt{3}}$

Let $q_1(x) = \dfrac{\tilde{q}_1(x)}{\|\tilde{q}_1\|} = 2\sqrt{3}\left(x - \frac{1}{2}\right)$

Step 3 Set $\tilde{q}_2(x) = p_2(x) - \langle p_2, q_0\rangle q_0(x) - \langle p_2, q_1\rangle q_1(x)$

We compute

$$\langle p_2, q_0\rangle = \int_0^1 x^2 \cdot 1 \, dx = \frac{1}{3}$$

$$\langle p_2, q_1\rangle = \int_0^1 x^2 \cdot 2\sqrt{3}\left(x - \frac{1}{2}\right) = 2\sqrt{3}\int_0^1 x^3 - \frac{1}{2}x^2$$

$$= 2\sqrt{3}\left(\frac{1}{4} - \frac{1}{6}\right) = 2\sqrt{3}\,\frac{1}{12} = \frac{\sqrt{3}}{6}$$

So $\tilde{q}_2(x) = x^2 - \frac{1}{3} - \frac{\sqrt{3}}{6}\cdot 2\sqrt{3}\left(x - \frac{1}{2}\right)$

$$= x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) = x^2 - x + \frac{1}{6}$$

Next we compute

$$\|\tilde{q}_2\| = \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2} = \sqrt{\frac{1}{180}} = \frac{1}{3\sqrt{20}}$$

So $q_2(x) = \dfrac{\tilde{q}_2(x)}{\|\tilde{q}_2\|} = 3\sqrt{20}\left(x^2 - x + \frac{1}{6}\right)$

Step 4 Set $\tilde{q}_3(x) = p_3(x) - \langle p_3, q_0\rangle q_0(x) - \langle p_3, q_1\rangle q_1(x) - \langle p_3, q_2\rangle q_2(x)$.

We compute

$$\langle p_3, q_0\rangle = \int_0^1 x^3 \cdot 1 \, dx = \frac{1}{4}$$

$$\langle p_3, q_1\rangle = \int_0^1 x^3\, 2\sqrt{3}\left(x - \frac{1}{2}\right) = \frac{3\sqrt{3}}{20}$$

$$\langle p_3, q_2\rangle = \int_0^1 x^3\, 3\sqrt{20}\left(x^2 - x + \frac{1}{6}\right) = \frac{1}{4\sqrt{5}}$$

So

$$\tilde{q}_3(x) = x^3 - \frac{1}{4} - \frac{3\sqrt{3}}{20}\, 2\sqrt{3}\left(x - \frac{1}{2}\right) - \frac{1}{4\sqrt{5}}\, 3\sqrt{20}\left(x^2 - x + \frac{1}{6}\right)$$

$$= x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$$

We compute

$\|\tilde{q}_3(x)\| = \sqrt{\frac{1}{2800}} = \frac{1}{20\sqrt{7}}$, So $q_3(x) = 20\sqrt{7}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right)$

Thus, the Gram Schmidt process produces orthonormal polynomials q_0, q_1, q_2, q_3.

<u>Rk</u> We can continue to apply this procedure to produce an orthonormal basis $q_0, \dots, q_n$ of $\mathcal{P}^{(n)}$ for any n.