Lecture 8 § 1.9 Determinants

The determinant of a square matrix is a quantity which determines whether ~~it~~ it is singular or nonsingular.

It can be defined in several ways, and they are all pretty messy.

<u>Method 1</u> Inductively: Let A be a square matrix. Let A_{ij} = the matrix given by removing the i^{th} row & j^{th} column.

<u>ex</u> $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $A_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$, $A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$, ...

<u>$n=1$</u> A 1×1 matrix, $A = (a_{11})$, the determinant is a_{11}

<u>$n>1$</u> An $n\times n$ matrix has determinant given by fixing a row i and computing $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$.

This formula gives a method for computing the determinant of an $n\times n$ matrix in terms of smaller ~~mat~~ $(n-1)\times(n-1)$ matrices.

<u>ex</u> $A = (-3)$. $\det A = -3$

$A = \begin{pmatrix} 1 & -1 \\ 2 & 7 \end{pmatrix}$. $\det A = 1\cdot 7 - (-1)2 = 9$.

In general, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Pick the first row to expand, $i=1$ above

$$\det A = (-1)^{1+1} \cdot a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} + (-1)^{1+2} \cdot b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + (-1)^{1+3} \cdot c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Harribly computationally inefficient for large matrices.

Why? to compute an $n\times n$ determinant, you must do n $(n-1)\times(n-1)$ determinants. Thus, an $n\times n$ determinant takes

$$n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(1)=n! \text{ steps!}$$

Thm 1.50 The determinant of a matrix A satisfies the following:

(i) Adding a multiple of one row to another (ERO1) does not change the determinant.

(ii) Swapping two rows (ERO2) changes the sign of the determinant.

(iii) Multiplying a row by c (ERO3) multiplies the determinant by c.

(iv) The determinant of an upper triangular matrix U (or a lower triangular matrix L) is $u_{11}\cdot u_{22}\cdot\ldots\cdot u_{nn}$ (respectively $l_{11}\cdot l_{22}\cdot\ldots\cdot l_{nn}$).

What does this say? We can compute determinants by GE!

ex $A=\begin{pmatrix}3&5&1&7\\0&1&-2&4\\-9&-5&1&0\\1&-1&0&2\end{pmatrix}$. Compute $\det A$ using GE.

$$A \xrightarrow{(14)} \begin{pmatrix}1&-1&0&2\\0&1&-2&4\\-9&-5&0&0\\3&5&1&7\end{pmatrix} \xrightarrow[r_4-3r_1]{r_3+9r_1} \begin{pmatrix}1&-1&0&2\\0&1&-2&4\\0&-14&0&18\\0&8&1&1\end{pmatrix}$$

$$\xrightarrow[r_4-8r_2]{r_3+14r_2} \begin{pmatrix}1&-1&0&2\\0&1&-2&4\\0&0&-28&74\\0&0&17&-31\end{pmatrix}$$

$$r_3 \cdot \frac{1}{28} \begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 37/14 \\ 0 & 0 & 17 & -31 \end{pmatrix} \quad r_4 - 17 r_1 \begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 37/14 \\ 0 & 0 & 0 & \frac{-1063}{14} \end{pmatrix} = U$$

$-31 = \frac{-434}{14}$

$17 \cdot \frac{37}{14} = \frac{629}{14}$

$$\begin{array}{r} 434 \\ 629 \\ \hline 1063 \end{array}$$

Thus, the determinant of U, the upper triangular matrix, is $-\frac{1063}{14}$. Thm 1.50 tells us that

$$\det A = (-1)^1 \cdot 28 \cdot \det U = -1 \cdot 28 \cdot \frac{-1063}{14} = 2126.$$

Fact A matrix A is nonsingular if and only if $\det A \neq 0$.

However Determinants, while providing interesting theoretical tools, are rarely of high interest in applications with lots of variables.