Lecture 11 § 2.3 Span

Defn Let V be a vectorspace and $v_1, \dots v_k \in V$. A sum of the form

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = \sum_{i=1}^{k} c_i v_i ,$$

where c_i are scalars, is a linear combination of $v_1 - v_k$. The span is span$\{v_1, \dots, v_k\}$, the collection of all linear combinations

Prop 2.14 If V is a vectorspace and $v_1 \dots v_k \in V$, then span$\{v_1 \dots v_k\}$ is a subspace.

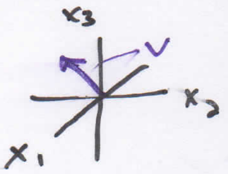
Proof By Prop 2.9, we only need to check that span$\{v_1 - v_k\}$ is closed under addition & scalar multiplication. Let $v = c_1 v_1 + \dots + c_k v_k$ and $\hat{v} = \hat{c}_1 v_1 + \dots + \hat{c}_k v_k$. Then $v + \hat{v} = c_1 v_1 + \_ + c_k v_k + \hat{c}_1 v_1 + \_ + \hat{c}_k v_k$
$= (c_1 + \hat{c}_1) v_1 + \dots + (c_k + \hat{c}_k) v_k$. So span$\{v_1 - v_k\}$ is closed under $+$. Moreover, if c is any constant, then

$$cv = c(c_1 v_1 + \_ + c_k v_k) = (cc_1) v_1 + \_ + c c_k v_k . \quad //$$

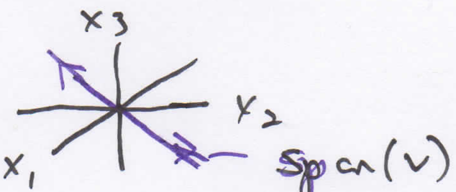
Remark In the above proof, we used axioms (a)-(g) in definition 2.1 without explicitly mentioning it. This is the usual progression of math, but it's important to know what assumptions are allowing us to draw conclusions, even when we don't list them explicitly.

ex Subspaces of $\mathbb{R}^3$.

- If $v = 0$, span$\{v\} = \{0\}$ = "trivial vector space"
- If $v \neq 0$, span$\{v\}$ = line passing though 0 & v

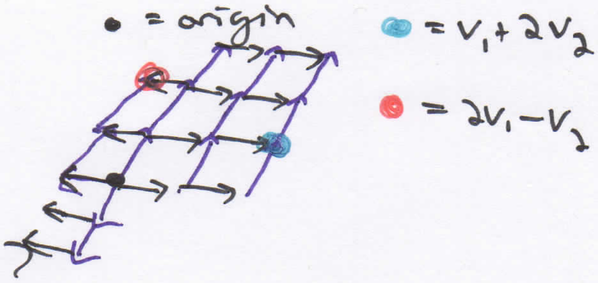


$v = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$



- If $v_1 \neq 0$, $v_2 \neq 0$, & v_1 is not parallel to v_2 (i.e., $v_1 \neq c v_2$ for any c), then span(v_1, v_2) is a plane

The vectors v_1 & v_2



ex Is $\begin{pmatrix}4\\1\\7\end{pmatrix}$ in $\operatorname{span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$?

In other words, is there a choice of $c = \begin{pmatrix}c_1\\c_2\end{pmatrix}$ such that

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}4\\1\\7\end{pmatrix}?$$

Thus, we are solving the matrix equation

$$(1) \quad \begin{pmatrix}1 & -1\\2 & 1\\3 & 0\end{pmatrix} c = \begin{pmatrix}4\\1\\7\end{pmatrix}$$

We augment and row reduce

$$\left(\begin{array}{cc|c}1 & -1 & 4\\2 & 1 & 1\\3 & 0 & 7\end{array}\right) \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left(\begin{array}{cc|c}1 & -1 & 4\\0 & 2 & -7\\0 & 3 & -5\end{array}\right) \xrightarrow{r_3 - \frac{3}{2}r_2} \left(\begin{array}{cc|c}1 & -1 & 4\\0 & 2 & -7\\0 & 0 & \frac{11}{2}\end{array}\right)$$

So we can see that it is not in the span, as

a solution to (1) implies $0 = \frac{11}{2}$.

Have them Do is $v = \begin{pmatrix}3\\3\\-\frac{5}{2}\end{pmatrix} \in \operatorname{span}\left\{\underset{v_1}{\begin{pmatrix}1\\1\\-1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}}, \underset{v_3}{\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}}\right\}$?

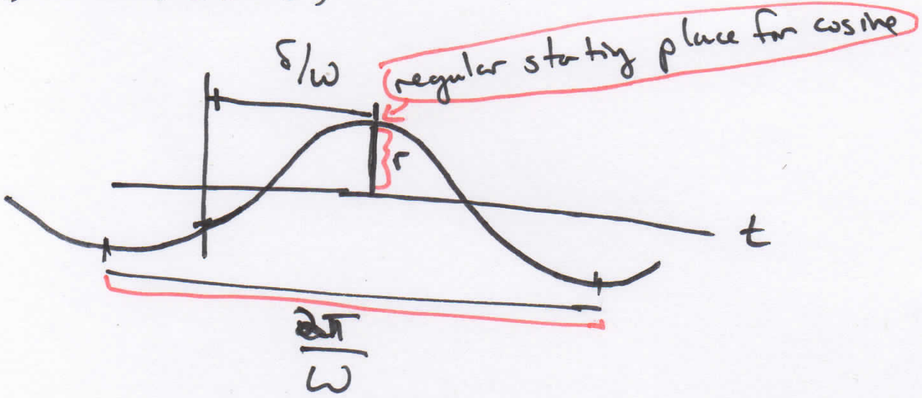
A: $v = 4v_1 - 2v_2 + v_3$. (I'd give them about 5-7 minutes & go over it)

ex $V = \mathcal{F}(\mathbb{R})$ = all functions $f: \mathbb{R} \to \mathbb{R}$.

- $\operatorname{span}\{1, x, x^2 \dots x^n\} = P^{(n)}$, poly of degree n.
- $\operatorname{span}\{1, x, y, xy, x^2, y^2\} = P^{(2,2)}$, poly of degree 2 in 2 variables

(b) To represent an oscillating system with an amplitude of r, frequency of $\frac{2\pi}{\omega}$ & phase shift of δ, we have the function

$f(t) = r\cos(\omega t - \delta)$



Using the laws of sines & cosines, we can rewrite $f(t)$ in terms of the span of two functions.

$$f(t) = r\cos(\omega t - \delta) = \underbrace{r\cos\delta}_{c_1}\cos\omega t + \underbrace{r\sin\delta}_{c_2}\sin\omega t.$$

Also, this implies any linear combination $c_1\cos\omega x + c_2\sin\omega x$ is of the form $r\cos(\omega t - \delta)$, as long as one chooses r & δ such that $c_1 = r\cos\delta$ & $c_2 = r\sin\delta$. This can be solved as

$$\frac{c_1}{c_2} = \cot(\delta), \quad c_1^2 + c_2^2 = r^2 \text{ (and choosing signs appropriately).}$$

Thus,

$$\operatorname{span}\{\cos\omega t, \sin\omega t\} = \{r\cos(\omega t - \delta) : r, \delta \in \mathbb{R}\}.$$