Lecture 7 §1.8 General & Homogeneous Linear Equations

Generalized Permuted LU factorization

square-regular matrices $A \leadsto$ Use ERO1 $\leadsto A = \underset{\Delta}{L}\,\underset{\Delta}{U}$ factorization

L lower triangular

U upper triangular

square-nonsingular matrices $A \leadsto$ Use ERO1 & 2 $\leadsto \underset{\Delta}{P}A = \underset{\Delta}{L}\,\underset{\Delta}{U}$ factorization

P permutation matrix

rectangular matrices $A \leadsto \begin{cases} \text{use ERO1} \leadsto A = \underset{\Delta}{L}\,\underset{\Delta}{U} \text{ factorization} \\ \quad U \text{ row echelon form} \\ \text{use ERO 1 \& 2} \leadsto \underset{\Delta}{P}A = \underset{\Delta}{L}\,\underset{\Delta}{U} \text{ factorization} \end{cases}$

ex

$$\begin{pmatrix} 2 & 3 & -1 \\ -4 & -6 & 3 \\ 2 & 5 & 9 \\ 3 & 0 & 2 \end{pmatrix} \begin{matrix} r_2 + 2r_1 \\ r_3 - r_1 \\ r_4 - \frac{3}{2}r_1 \end{matrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 2 & 10 \\ 0 & -\frac{9}{2} & \frac{7}{2} \end{pmatrix} \xrightarrow{(23)} \begin{pmatrix} 2 & 3 & -1 \\ 0 & 2 & 10 \\ 0 & 0 & 1 \\ 0 & -\frac{9}{2} & \frac{7}{2} \end{pmatrix}$$

$$r_4 + \frac{9}{4}r_2 \begin{pmatrix} 2 & 3 & -1 \\ 0 & 2 & 10 \\ 0 & 0 & 1 \\ 0 & 0 & 26 \end{pmatrix} \quad r_4 - 26r_3 \begin{pmatrix} 2 & 3 & -1 \\ 0 & 2 & 10 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = U$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & 0 & 1 & \\ 3/2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -2 & 0 & 1 & \\ 3/2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 3/2 & -9/4 & 26 & 1 \end{pmatrix}$$

Homogeneous Equations

An equation of the form $Ax = 0$ is called <u>homogeneous</u>.

The solutions to $Ax = 0$ tell us a lot about the solutions to $Ax = b$.

See remark at end of these notes

Thm Let A be an $m \times n$ matrix. Let* $b \in \mathbb{R}^n$. Define

*$S_0 = \{ x \in \mathbb{R}^m : Ax = 0 \}$ and $S_b = \{ x \in \mathbb{R}^m : Ax = b \}$.

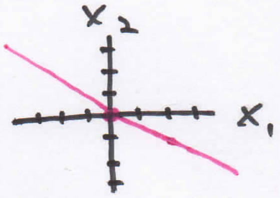
That is, S_0 is the set of solutions to $Ax = 0$ and S_b is the set of solutions to $Ax = b$. Let $x^{(1)}$ be any solution to $Ax = b$. Then

*$$S_b = \{ x^{(1)} + x : x \in S_0 \}.$$

What does this say? It says that if we know the solutions to $Ax = 0$ and any one solution to $Ax = b$, then we get any other solution to $Ax = b$ by adding a solution of $Ax = 0$.

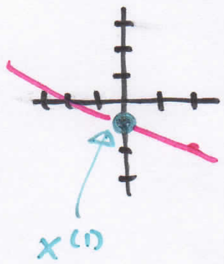
Picture ex Consider the simple matrix $A = (2 \;\; 1)$.

Then $Ax = 0$ has solutions
$$x_1 = -\tfrac{1}{2} x_2$$
$$x = \begin{pmatrix} -\tfrac{1}{2} x_2 \\ x_2 \end{pmatrix}$$



The equation $Ax = -1$ has, for example, $x^{(1)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ as a solution, so the solutions to $Ax = b$ are

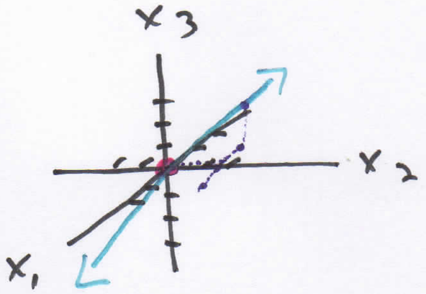
$$x + x^{(1)} = \begin{pmatrix} -\tfrac{1}{2} x_2 + 0 \\ x_2 \;\; -1 \end{pmatrix}$$



So one simply "shifts" the solutions of $Ax = 0$ to get the solutions of $Ax = b$.

3d-ex $\quad A = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 0 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

$Ax = 0$ $\quad \begin{pmatrix} 3 & 2 & -1 \\ 5 & 0 & 4 \end{pmatrix} \xrightarrow{r_2 - \frac{5}{3} r_1} \begin{pmatrix} 3 & 2 & -1 \\ 0 & -\frac{10}{3} & \frac{17}{3} \end{pmatrix} \Rightarrow$ Free variable x_3



$$\boxed{x_2 = \frac{17}{10} x_3}$$

$$3x_1 = -2\left(\frac{17}{10} x_3\right) + x_3$$

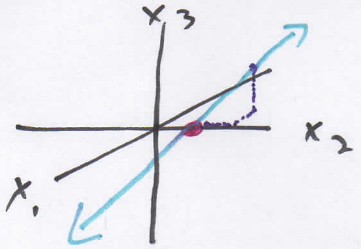
$$3x_1 = -\frac{12}{5} x_3$$

$$\boxed{x_1 = -\frac{4}{5} x_3}$$

$x_3 = 0 \rightsquigarrow x = \vec{0}$

$x_3 = 1 \rightsquigarrow x = \begin{pmatrix} -4/5 \\ 17/10 \\ 1 \end{pmatrix}$

$Ax = b$ One solution can be eyeballed as $x = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}$



general solution is

$$x = \begin{pmatrix} -4/5\, x_3 \\ 1/2 + \frac{17}{10} x_3 \\ x_3 \end{pmatrix}$$

Proof of Thm Let A be an $m \times n$ matrix, $b \in \mathbb{R}^n$, and S_0 & S_b be defined as above. ~~Suppose~~ Let $x^{(1)} \in S_b$; that is, $Ax^{(1)} = b$.

We need to show 2 things:

1. If $x^{(2)} \in S_b$, $x^{(2)} = x^{(1)} + x$, for some $x \in S_0$.
2. If $x \in S_0$, then $x^{(2)} = x^{(1)} + x \in S_b$.

Pf of 1 $\boxed{\text{Suppose } x^{(2)} \in S_b}$ hypothesis. We compute that

$A(x^{(2)} - x^{(1)}) = Ax^{(2)} - Ax^{(1)} = b - b = 0$. $\boxed{\text{So } x := x^{(2)} - x^{(1)} \text{ is in } S_0.}$ conclusion

Pf of 2 | $\boxed{\text{Suppose } x \in S_0}$ —hypothesis. Set $x^{(2)} := x^{(1)} + x$. Then

$A x^{(2)} = A(x^{(1)} + x) = Ax^{(1)} + Ax = b + 0 = b$. Thus $\boxed{x^{(2)} \in S_b}$. //
conclusion

* Remark $\mathbb{R}^m$ means the space of m-dimensional vectors.

* Remark A set is (more or less) any collection of objects. to write a set of m dimensional vectors, we may write

$$S = \{x \in \mathbb{R}^m : \text{"property"}\}.$$

This is read " the set of all vectors x in $\mathbb{R}^m$ such that "property" holds.

* Remark to say $x \in \mathbb{R}^n$ or $x \in \mathbb{R}^m$ or $x \in S$ is read "x in $\mathbb{R}^n$", "x in $\mathbb{R}^m$" or "x in S". The reason is due to Greek mathematicians. $\in$, "epsilon," is the first letter in the Greek word for "is." Thus, $x \in \{\text{blue things}\}$ means x is blue, or $x \in S_0$ means x is a solution to $Ax = 0$.