Lecture 23 §4.2 Minimization of Quadratic Functions

Defn A quadratic function in the variable $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ is of the form

$$(1) \quad p(x) = \sum_{i,j=1}^{n} q_{ij} x_i x_j + \sum_{i=1}^{n} l_i x_i + c .$$

Our goal in this section is to understand how to minimize a quadratic function, as well as understand ~~how~~ when it has a global minimum.

Observation 1 ~~By using~~ Because $x_i x_j = x_j x_i$, we can manipulate the expression (1) above by setting

$$K_{ij} = \frac{q_{ij} + q_{ji}}{2} .$$

Doing this, $K_{ij} = K_{ji}$, and $\sum_{i,j=1}^{n} q_{ij} x_i x_j = \sum_{i,j=1}^{n} K_{ij} x_i x_j$.

Additionally, for convenience, we will set $f_i = -\frac{1}{2} l_i$, so that $\sum_{i=1}^{n} l_i x_i = -2 \sum_{i=1}^{n} f_i x_i$.

Doing this, we acheive the representation

$$(2) \quad p(x) = \sum_{i,j=1}^{n} K_{ij} x_i x_j - 2 \sum_{i=1}^{n} f_i x_i + c$$

Observation 2 Rewriting equation (2) in matrix form, we see that

$$p(x) = x^T K x - 2 x^T f + c ,$$

where K is a symmetric matrix.

ex If $p(x) = x_1^2 + 2x_1x_2 - 3x_2^2 + x_1 - 2x_2 + 1$, we can write p in symmetric form

$$p(x) = x_1^2 + x_1x_2 + x_2x_1 - 3x_2^2 - 2\left(-\frac{x_1}{2} + x_2\right) + 1 ,$$

and we see

$$K = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \quad f = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \quad c = 1 .$$

Using this representation, we now find the global minimum for a ~~path~~ quadratic polynomial where K is positive definite.

Thm 4.1 Assume that K is a positive definite matrix. Then the quadratic polynomial $p(x) = x^T K x - 2x^T f + c$ has a unique global minimizer

$$x^* = K^{-1} f,$$

and the minimum value is

$$p(x^*) = c - f^T K^{-1} f = c - f^T x^* = c - (x^*)^T K x^*$$

Pf By prop 3.25, ~~K is pos~~ $K > 0$ implies that K is invertible. Thus, we may define $x^* = K^{-1} f$. In particular, $Kx^* = f$. We compute, for any $x \in \mathbb{R}^n$,

$$p(x) = x^T K x - 2x^T f + c = x^T K x - 2x^T K x^* + c$$

$$(3) \qquad = x^T K x - x^T K x^* - x^T K x^* + c.$$

$$= x^T K (x - x^*) - x^T K x^* + c.$$

Next, we observe that $-x^T K x^* = -(x^*)^T K x$ (for example, by viewing K as an inner product matrix). ~~Thus~~ Further,

$$-(x^*)^T K x = -(x^*)^T K (x - x^*) - (x^*)^T K x^*.$$

Substituting this into (3), we get

$$p(x) = x^T K (x - x^*) - (x^*)^T K (x - x^*) - (x^*)^T K x^* + c$$

$$= (x - x^*)^T K (x - x^*) - (x^*)^T K x^* + c.$$

Because K is positive definite, the minimum of

$$(x - x^*)^T K (x - x^*)$$

is 0, and is ~~always~~ achieved only when $x = x^*$. Moreover, $-(x^*)^T K x^* + c$ is a constant.

Thus the unique minimizer of $p(x)$ is x^*, and the minimum value is

$$p(x^*) = -(x^*)^T K x^* + c = -f^T x^* + c = -f^T K^{-1} f + c.$$

ex Let $K = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$, $f = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $c = -2$.

Then we recall K is positive definite if and only if it is regular with positive pivots.

We row reduce to find

$$\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow{r_2 - \frac{1}{4} r_1} \begin{pmatrix} 4 & 1 \\ 0 & 7/4 \end{pmatrix},$$ thus

$K > 0$. Solving $Kx^* = f$, we see $f \xrightarrow{r_2 - \frac{1}{4} r_1} \begin{pmatrix} 1 \\ 7/4 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 4 & 1 & 1 \\ 0 & 7/4 & 7/4 \end{array}\right),\quad x_2^* = 1,\ x_1^* = 0,$$

So the minimizer is $x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.