1. For each part, state whether the statement is true or false. If it is true, prove it. If it is false, provide a counter example.

(a) (3 points) The set $V = \{x \in \mathbb{R}^3 : x_1 = 2x_2\}$ is a subspace of $\mathbb{R}^3$.

True.

Proof 1 Let $A = (1\ -2\ 0)$. Then $V = \ker(A)$, which is a subspace of $\mathbb{R}^3$.

Proof 2 Let $x, y \in V$. Then $x + y = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix}$.

Then $x_1 + y_1 = 2x_2 + 2y_2 = 2(x_2 + y_2)$, so $x + y \in V$. ✓ So V is closed under +. For $c \in \mathbb{R}$, $x \in V$, $cx = \begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \end{pmatrix}$, and $cx_1 = c(2x_2) = 2(cx_2)$, so $cx \in V$.

(b) (3 points) Let A be a square matrix. If U is the row echelon form of A, $\det(U) = \det(A)$.

False. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) (4 points) Let A be an $m \times n$ matrix. If $m > n$, then $\text{rng}(A)$ is not all of $\mathbb{R}^m$.

True. By the FTLA,

$\dim(\text{rng}(A)) = \text{rank}(A) \leq \min(n,m) = n < m$.

So $\text{rng}(A)$ cannot be $\mathbb{R}^m$.

2. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -2 & 2 \\ -2 & 0 & 1 \end{pmatrix}$.

(a) (7 points) Compute a permuted LU factorization for A, $PA = LU$.

$$A \xrightarrow[r_3+2r_1]{r_2+r_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 1 \end{pmatrix} \xrightarrow{(23)} \boxed{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix} = U}$$

$$P: \boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = P}$$

$$L: \begin{pmatrix} 1 & & \\ -1 & 1 & \\ -2 & 0 & 1 \end{pmatrix} \to \boxed{\begin{pmatrix} 1 & & \\ -2 & 1 & \\ -1 & 0 & 1 \end{pmatrix} = L}$$

(b) (3 points) Use the permuted LU factorization above to solve for x in the equation

$$Ax = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}.$$

Solve $Lc = Pb$

$$\left(\begin{array}{ccc|c} 1 & & & 0 \\ -2 & 1 & & -1 \\ -1 & 0 & 1 & 3 \end{array}\right) \rightsquigarrow \begin{array}{l} c_1 = 0 \\ c_2 = -1 \\ c_3 = 3 \end{array}$$

Solve $Ux = c$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 4 & 1 & -1 \\ 0 & 0 & 2 & 3 \end{array}\right)$$

$\boxed{x_3 = \frac{3}{2}}$

$4x_2 = -1 - x_3$

$x_2 = \frac{1}{4}\left(-1 - \frac{3}{2}\right) = \boxed{-\frac{5}{8} = x_2}$

$x_1 = 0 - 2x_2 = \boxed{\frac{5}{4} = x_1}$

3. Let $A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 4 & 1 \\ 1 & 2 & 0 \end{pmatrix}$.

(a) (6 points) Find rng(A) and ker(A). Identify a basis of each.

$$(A|b) \underset{r_1 - r_4}{\overset{r_1+2r_4}{\sim}} \left(\begin{array}{ccc|c} 0 & 5 & 0 & b_1+2b_4 \\ 0 & 1 & -1 & b_2-b_4 \\ 0 & 4 & 1 & b_3 \\ 1 & 2 & 0 & b_4 \end{array}\right) \overset{r_1-r_2-r_3}{\sim} \left(\begin{array}{ccc|c} 0 & 0 & 0 & b_1+2b_4-(b_2-b_4)-b_3 \\ 0 & 1 & -1 & b_2-b_4 \\ 0 & 4 & 1 & b_3 \\ 1 & 2 & 0 & b_4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & b_4 \\ 0 & 1 & -1 & b_2-b_4 \\ 0 & 0 & 5 & b_3-4b_2+4b_4 \\ 0 & 0 & 0 & b_1-b_2-b_3+3b_4 \end{array}\right).$$

So $\boxed{\mathrm{rng}(A) = \{b \in \mathbb{R}^4 : b_1 - b_2 - b_3 + 3b_4 = 0\}}$

No free variables $\Rightarrow$ $\boxed{\ker(A) = \{0\}}$

~~The~~ Because there are no free variables, the columns of A form a basis of rng(A). ~~Ker(A) = {0} implies~~ Because Ker(A) = $\{0\}$, the empty set is a basis.

(b) (4 points) Let $b = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$. Is $b \in \mathrm{rng}(A)$?

$b_1 - b_2 - b_3 + 3b_4 = 1 - 0 - 2 - 3 = -4 \neq 0$, so $b \notin \mathrm{rng}(A)$.

4. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

(a) (5 points) Show that this set spans $\mathbb{R}^3$.

We now reduce

$$A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 2 \\ 1 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{r_3 - r_1} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 2 \\ 0 & -2 & -4 & 0 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

So $\text{rank}(A) = 3$, and hence $\text{rng}(A) = \mathbb{R}^3$. ✓

(b) (5 points) Find a set of 3 of the above vectors which forms a basis.

We choose the columns of A corresponding to the basic ~~vectors~~ variables,

v_1, v_2, v_4.

5. For this problem, let A be an invertible $n \times n$ matrix.

and $v_1 \dots v_k$ are linearly independent

(a) (4 points) Let $v_1, v_2, \dots, v_k \in \mathbb{R}^n$. Prove that if the vectors $v_1, \dots, v_k$ are linearly independent, then $Av_1, Av_2, \dots, Av_k$ are also linearly independent

Suppose $\sum_{i=1}^{K} c_i A v_i = 0$. Because A is invertible, we know A^{-1} exists (by definition). Multiplying by A^{-1} on the left, we get

$$A^{-1} \sum_{i=1}^{k} c_i A v_i = A^{-1} \cdot 0 \Rightarrow \sum_{i=1}^{k} c_i A^{-1} A v_i = \sum_{i=1}^{k} c_i v_i = 0.$$

By assumption, $v_1 \dots v_k$ are linearly independent, and hence $c_1 = \dots = c_k = 0$. Thus, $Av_1 \dots Av_k$ are linearly independent.

(b) (4 points) Use the result of part (a) to prove that $v_1, v_2, \dots, v_n$ is a basis of $\mathbb{R}^n$, then $Av_1, Av_2, \dots, Av_n$ is a basis of $\mathbb{R}^n$.

Suppose $v_1 \dots v_n$ is a basis of $\mathbb{R}^n$. Then $v_1 \dots v_n$ are linearly independent. By part (a), $Av_1 \dots Av_n$ are linearly independent. Hence, because $\dim \mathbb{R}^n = n$, $Av_1 \dots Av_n$ forms a basis.