Lecture 33   § 5.7, Discrete Fourier Analysis (cont.)

Recall Given a signal for $0 \le x \le 2\pi$ we seek to write

$$(1) \quad f(x) \approx \sum_{k=0}^{n-1} c_k e^{ikx}$$

To do this, we use the sample points

$$x_0 = 0, \; x_1 = \frac{2\pi}{n} \; \ldots \; x_k = \frac{2k\pi}{n} \; \ldots \; x_{n-1} = \frac{2(n-1)\pi}{n},$$

the sample of $f(x)$

$$\vec{F} = (f(x_0), \ldots, f(x_{n-1}))^T = (f_0, \ldots, f_{n-1})^T,$$

the samples of e^{ikx}

$$\omega_k = (e^{ikx_0}, \ldots, e^{ikx_{n-1}})^T = (\zeta_n^0, \zeta_n^k, \zeta_n^{2k}, \ldots, \zeta_n^{(n-1)k})^T, \quad \zeta_n = e^{\frac{2\pi i}{n}}$$

The sampled exponentials $\omega_0 \ldots \omega_{n-1}$ form an orthonormal basis of $\mathbb{C}^n$ with respect to the averaged dot product,

$$\langle u, v \rangle = \frac{1}{n} \sum_{j=1}^{n} u_j \overline{v_j}.$$

This implies that, by the complex analogue of Thm 5.7,

$$\vec{F} = \sum_{k=0}^{n-1} c_k \omega_k \quad \text{for} \quad c_k = \langle \vec{f}, \omega_k \rangle = \frac{1}{n} \sum_{j=0}^{n-1} f_j \overline{\zeta_n^{jk}}$$

So we use these constants to approximate $f(x)$ in the sum (1).

ex Let $n=4$ sample points. Then $\zeta_4 = e^{2\pi i/4} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})$.

and

$$\omega_0 = (i^0, i^0, i^0, i^0)^T = (1, 1, 1, 1)^T$$

$$\omega_1 = (i^0, i^1, i^2, i^3) = (1, i, -1, -i)^T$$

$$\omega_2 = (i^0, i^2, i^4, i^6) = (1, -1, 1, -1)^T$$

$$\omega_3 = (i^0, i^3, i^6, i^9) = (1, -i, -1, i)^T$$

Real Valued vs Complex Valued Signals

Suppose we are seeking to understand a real valued signal. Then our sample vec for $\vec{f}$ will be realvalued, so our fourier approximation $\sum_{k=0}^{n-1} c_k e^{ikx}$ will also be realvalued on the sample points x_k, but will be $\boxed{\text{complex in between}}$! To fix this, however, we simply choose the realvalued part of our approximation $\sum_{k=0}^{n-1} c_k e^{ikx}$. However, we still need to use complex arithmatic to get there, as we see in the next example.

ex Suppose we have the $n=4$ sample values

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 1, \quad f_3 = 2.$$

Then we compute the Fourier coefficients

$$c_0 = \langle \vec{f}, \omega_0 \rangle = \frac{1}{4}\left(0\cdot\bar{1} + 1\cdot\bar{1} + 1\cdot\bar{1} + 2\cdot\bar{1}\right)$$
$$= \frac{1}{4}(0 + 1 + 1 + 2) = 1$$

$$c_1 = \langle \vec{f}, \omega_1 \rangle = \frac{1}{4}\left(0\cdot\bar{1} + 1\cdot\bar{i} + 1\cdot\overline{-1} + 2\cdot\overline{-i}\right)$$
$$= \frac{1}{4}(0 + -i + -1 + 2i) = -\frac{1}{4} + \frac{i}{4}$$

$$c_2 = \langle \vec{f}, \omega_2 \rangle = \frac{1}{4}\left(0\cdot\bar{1} + 1\cdot\overline{-1} + 1\cdot\bar{1} + 2\cdot\overline{-1}\right)$$
$$= \frac{1}{4}(0 - 1 + 1 - 2) = -\frac{1}{2}$$

$$c_3 = \langle \vec{f}, \omega_3 \rangle = \frac{1}{4}\left(0\cdot\bar{1} + 1\cdot\overline{-i} + 1\cdot\overline{-1} + 2\cdot\bar{i}\right)$$
$$= \frac{1}{4}(0 + i - 1 + -2i) = -\frac{1}{4} - \frac{i}{4}$$

We use the coefficients to compute the complex and real valued Fourier interpolants

Complex interpolant

$$f(x) \approx 1\, e^{i0x} + \left(\frac{-1}{4} + \frac{i}{4}\right) e^{ix} - \frac{1}{2} e^{i2x} + \left(-\frac{1}{4} - \frac{i}{4}\right) e^{i3x}$$

$$= 1\left(\cos(0) + i\sin(0)\right) + \left(-\frac{1}{4} + \frac{i}{4}\right)\left(\cos x + i\sin x\right) - \frac{1}{2}\left(\cos 2x + i\sin 2x\right)$$
$$+ \left(-\frac{1}{4} - \frac{i}{4}\right)\left(\cos 3x + i\sin 3x\right)$$

$$= 1 - \frac{1}{4}\cos x - \frac{1}{4}\sin x + \frac{i}{4}\cos x - \frac{i}{4}\sin x - \frac{1}{2}\cos 2x - \frac{i}{2}\sin 2x$$
$$+ -\frac{1}{4}\cos 3x + \sin 3x - \frac{i}{4}\cos 3x + \frac{i}{4}\sin 3x$$

Real interpolant We choose the real valued part of the above expression,

$$f(x) \approx 1 - \frac{1}{4}\cos x - \frac{1}{4}\sin x - \frac{1}{2}\cos 2x - \frac{1}{4}\cos 3x + \sin 3x.$$

The choise of a complex or real interpolant depends on the application. Moreover, if we only want the real valued interpolant, we don't actually need to carry out the multiplications which lead to an imaginary. So, for example, in computing the real part of

$$\left(-\frac{1}{4} + \frac{i}{4}\right)\left(\cos x + i\sin x\right)$$

We simply compute $-\frac{1}{4}\cdot\cos x$ and $\frac{i}{4}\cdot i\sin x$ to get

$$-\frac{1}{4}\cos x - \frac{1}{4}\sin x$$

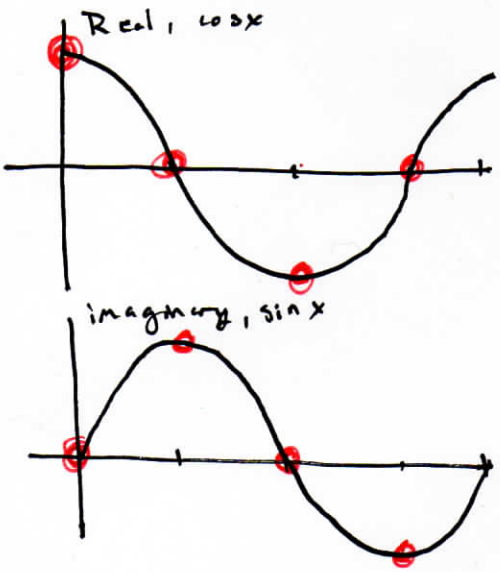
and ignore the other terms.

Aliasing and the Low frequency Fourier Alternative

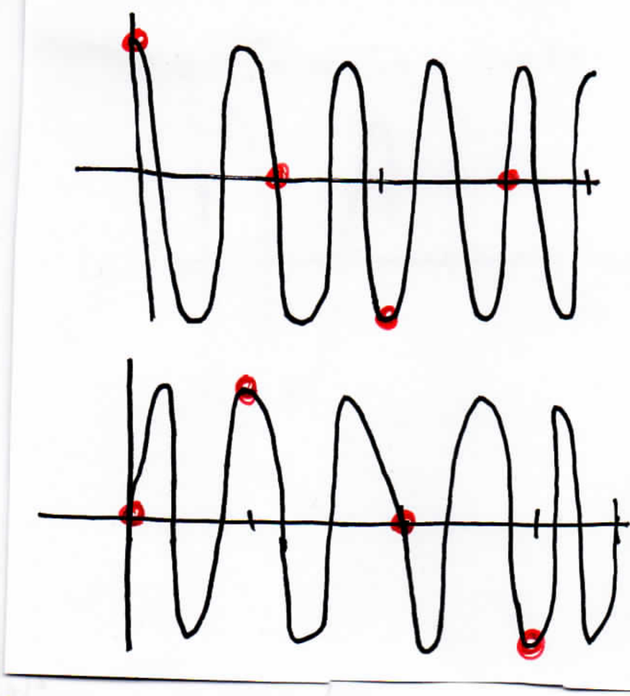
The phenomenon of aliasing is the effect that, given n sample points, frequencies e^{ikx} and $e^{i(k+n\cdot\ell)x}$ $(\ell = 0, \pm1, \pm2, \ldots)$ cannot be distinguished. For example, given $n=4$ sample points, we plot the graphs of the real and imaginary parts of e^{ix} and e^{i5x} below

o = sample values

$e^{ix} = \cos x + i \sin x$



$e^{i5x} = \cos 5x + i \sin 5x$



Aliasing, or the fact that frequencies varying by the number of sample points, cannot be distinguished, is why helicopter blades and wheels often appear to be moving slowly, backwards, or standing still; there are only finitely many frames, and your brain fills in the gaps as best it can.

Because e^{ikx} and $e^{i(k+nl)x}$ are indistinguishable on the sample points $x_0, \dots, x_{n-1}$, we have that

$$\omega_k = \omega_{k+n} = \omega_{k-n} = \omega_{k+2n} = \dots = \omega_{k+ln} \qquad l = \text{any integer.}$$

Mathematically, we can harness the fact that $\omega_k = \omega_{k-n}$ to see that

$$\omega_k = \omega_{k-n} \quad \text{for } k \geq \tfrac{n}{2}, \text{ and } c_k = c_{k-n}.$$

That allows us to replace the terms ω_k, $k \geq \frac{n}{2}$ with ω_{k-n} (since $\omega_{-\frac{n}{2}} \dots \omega_{\frac{n}{2}-1}$ form an o.n. basis) and with the low frequency alternative fourier transforms

$$f(x) \approx \sum_{k=-\frac{n}{2}}^{\frac{n}{2}-1} c_k e^{ikx} \qquad n \text{ even}$$

$$f(x) \approx \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} c_k e^{ikx} \qquad n \text{ odd}$$

This version provides a more accurate interpolation (see text pp 283-284 for an example).

The Noise Reducing Alternative.

Typically, noise in signals (noise meaning innaccuracy in data) surfaces in the high frequencies. So to de noise a signal, we take the low frequencies and omit the high ones. How to balance this depends on the application. So we have the noise reducing alternatives,

$$f(x) \approx \sum_{k=-l}^{l} c_k e^{ikx} \qquad \text{for } l \ll \tfrac{n}{2}$$

$$f(x) \approx \sum_{k=-l}^{l} c_k e^{ikx} \qquad \text{for } l \ll \tfrac{n}{2}$$