Workshop 1 / Lecture 12

Today focuses on proofs, in the mathematical sense. Other fields have familiar methods of proof. In science, proof ultimately means experimental verification. In mathematics, it means unassailable logic.

We focus on two kinds of proof:

(1) High power proofs: These are proofs that turn big theorems into other results.

Ex/ (Cor 1.55) Determinants obey the following law for invertible matrices A:

$$\det(A^{-1}) = (\det(A))^{-1}.$$

Pf

Suppose that A is an invertible matrix. By proposition 1.54,

$$\det(I) = \det(A^{-1} \cdot A) = \det(A^{-1}) \cdot \det(A).$$

By theorem 1.50, $\det(I) = 1$, so we have that

$$1 = \det(A^{-1}) \cdot \det(A).$$

Hence, we conclude that

$$\det(A^{-1}) = (\det(A))^{-1}.$$

$\square$

Rk The above proves that Corollary 1.55 holds once we know that Proposition 1.54 holds. Proposition 1.54, in addition, relies on many other results; mainly, Theorem 1.50. Thus, the above is a "high power proof."

Example question Using Proposition 1.54, prove that $\det(AB) = \det(BA)$. State what assumptions on the sizes of A & B are necessary for this question to make sense.

Class Example

Problem 1.9.12 Prove the determinantal product formula, (1.82); $\det(AB) = \det(A)\det(B)$.

(a) Let E be an $n \times n$ elementary matrix and B be any $n \times n$ matrix. Prove that
$\det(EB) = \det(E)\det(B)$.

PF By Theorem 1.50, we can analyze the cases that E is type 1, 2, or 3.

Suppose E is a type 1 elementary matrix. Then because E is lower triangular ~~with~~ with 1's on the main diagonal,

$$\det(E) = 1.$$

Moreover, multiplying on the left ~~[illegible]~~ by a type 1 matrix corresponds to the appropriate ERO 1. Thus by Theorem 1.50,

$$\det(EB) = \det(B) = \det(E)\det(B).$$

Suppose E is type 2. Then $\det(E) = -1$ by Thm 1.50 (ii) (noting that E is obtained from I by one row swap). Moreover, multiplying B on the left by E ~~[illegible]~~ corresponds to an ERO 2, so by another application of Thm 1.50 (ii),

$$\det(B) = -\det(EB) = \det(E)\det(B).$$

Finally, suppose E is type 3; specifically, $E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$.
Then $\det(E) = c$, and because multiplying by E is the same as applying an ERO 3, so by Theorem 1.50(iii),

$$\det(EB) = c \cdot \det(B) = \det(E)\det(B).$$

(b) Prove that if $A = E_1 \cdots E_N$ where E_i is an elementary matrix for $1 \le i \le N$, then for any B,

(3) $\det(AB) = \det(A)\det(B)$.

<u>Pf</u> Let $A = E_1 \cdots E_N$, where E_i are elementary matrices. Then

$$\det(AB) = \det(E_1 E_2 \cdot E_3 \cdots E_N \cdot B) = \det(E_1)\cdot\det(E_2 \cdot E_3 \cdots E_N \cdot B)$$
$$= \cdots = \det(E_1)\det(E_2)\cdots\det(E_N)\cdot\det(B).$$

Taking $B = I$, we get

$$\det(A) = \det(E_1)\cdots\det(E_N).$$

Thus,

$$\det(AB) = \det(E_1)\cdots\det(E_N)\cdot\det(B) = \det(A)\det(B). \quad \square$$

Remark The above proof is not quite rigorous. To prove a property of this form completely, one must use proof by induction. A complete proof by induction appears below

<u>Pf (w/ induction)</u> Induction proceeds by proving (3) when $N=1$, and then proves that if (3) holds for N, then it holds for $N+1$.

<u>Base case; $N=1$</u> When $N=1$, ~~the~~ equation (3) states

$$\det(E_1 B) = \det(E_1)\det(B) \text{ for any } B.$$

This is proven in part (a).

<u>$N \Rightarrow N+1$; inductive step</u> Suppose that (3) holds for N. We prove this implies that (3) holds for $N+1$. By hypothesis,

$$\det(E_1 \cdots E_N \cdot (E_{N+1}\cdot B)) = \det(E_1)\cdots\det(E_N)\cdot\det(E_{N+1}\cdot B).$$

By part (a), $\det(E_{N+1}\cdot B) = \det(E_{N+1})\cdot\det(B)$. Thus, for any B,

$$\det(A\cdot B) = \det(E_1)\cdots\det(E_{N+1})\cdot\det(B)$$

Taking $B = I$,

$$\det(A) = \det(E_1)\cdots\det(E_{N+1}).$$

Thus, $\det(A\cdot B) = \det(E_1)\cdots\det(E_{N+1})\cdot\det(B) = \det(A)\det(B)$. $\square$

(b) Continued Explain why this proves

$$\det(AB) = \det(A)\det(B)$$

for any nonsingular matrix A and any matrix B.

Pf Let A be a nonsingular matrix. Then by definition, A can be row reduced to an upper triangular matrix with nonzero entries on the diagonal. It follows that A can be row reduced to I (using ERO 1, 2, & 3). Thus, by the correspondence of row operations and multiplication by elementary matrices,

$$A = E_1 \cdot \ldots \cdot E_N \cdot I = E_1 \cdot \ldots \cdot E_N,$$

for some elementary matrices $E_1, \ldots, E_N$. $\square$