Lecture 20 §3.2 Inequalities Related to Inner Products

In this section, we study some fundamental inequalities related to inner products. First, we give some definitions to make precise some "shakey" statements in our text.

Defn Let $v, w \in V$, a vector space. We say v is parallel to w if $v = cw$ for some $c > 0$. We say v is anti-parallel to w if $v = cw$ for some $c < 0$.

Thm 3.5 (The Cauchy Schwarz Inequality) Let $(V, \langle , \rangle)$ be an innerproduct space. Then

(1) $$|\langle v, w\rangle| \leq \|v\| \, \|w\| .$$

Moreover, $|\langle v, w \rangle| = \|v\| \, \|w\|$ if and only if v is ~~prall~~ parallel or anti parallel to w.

Pf If $w = 0$, then equation (1) holds trivially, as it needs simply $0 \leq 0$.

Suppose $w \neq 0$. Let $t \in \mathbb{R}$. Then

$$0 \leq \|v + tw\|^2 = \langle v+tw, v+tw\rangle \overset{1}{=} \langle v, v+tw\rangle + t\langle w, v+tw\rangle$$

(2) $$\overset{2}{=} \langle v, v\rangle + \cancel{2}\, t\langle v, w\rangle + t\langle w, v\rangle + t^2\langle w, w\rangle$$

$$\overset{3}{=} \|v\|^2 + 2t\langle v, w\rangle + t^2\|w\|^2.$$

(Note: 1 & 2 hold by bilinearity and 3 holds by symmetry). Let $p(t) := \|v\|^2 + 2t\langle v, w\rangle + t^2\|w\|^2$. Then p is a parabola opening up. Its minimum is acheived at

$$t_0 := -\frac{\langle v, w\rangle}{\|w\|^2} .$$

Plugging in t_0, we find

(3) $$p(t_0) = \|v\|^2 - 2\frac{\langle v, w\rangle^2}{\|w\|^2} + \frac{\langle v, w\rangle^2}{\|w\|^2} = \|v\|^2 - \frac{\langle v, w\rangle^2}{\|w\|^2} .$$

By equation (2), we have that

$$0 \leq p(t_0).$$

Combining equations (2) & (3), we have that

$$0 \leq \|v\|^2 - \frac{\langle v, w\rangle^2}{\|w\|^2},$$

whence it follows that

$$\frac{\langle v, w\rangle^2}{\|w\|^2} \leq \|v\|^2.$$

Because $\|w\|^2 > 0$, we multiply by $\|w\|^2$ to get

$$\langle v, w\rangle^2 \leq \|v\|^2 \|w\|^2.$$

Extracting square roots, we get

$$|\langle v, w\rangle| \leq \|v\| \|w\|,$$

establishing (1).

Moreover, from (3), we can find that $p(t_0) = 0$ if and only v & w are parallel or antiparallel ($v = cw$). //

Angles in an Inner product Space

In $\mathbb{R}^n$, the angle θ between two vectors can be computed as

$$v \cdot w = |v||w| \cos\theta,$$

or in other words,

$$\theta = \cos^{-1}\left(\frac{v \cdot w}{\|v\|\|w\|}\right).$$

By the Cauchy Schwarz inequality,

$$-1 \leq \frac{\langle v, w\rangle}{|v||w|} \leq 1$$

So in any inner product space, we can define the angle between two vectors as

$$\theta = \cos^{-1}\left(\frac{\langle v, w\rangle}{\|v\|\|w\|}\right).$$

The most important application of this is the idea of orthogonal vectors. That is, $\theta = \frac{\pi}{2}$, or equivalently, $\langle v, w\rangle = 0$.

<u>ex</u> In $L^2([0,1])$, the functions

$1, \cos(\pi x), \cos(2\pi x), \ldots, \sin(\pi x), \sin(2\pi x), \ldots$

are all orthogonal.

The triangle Inequality

Thm 3.9 Let $(V, \langle , \rangle)$ be an inner product space, and $\| \ \|$ be the associated norm. Then

$$\|v+w\| \leq \|v\| + \|w\| \quad \text{for all } v, w \in V.$$

Moreover, equality holds if and only if v is parallel to w.

Pf We compute

$$\|v+w\|^2 = \langle v+w, v+w\rangle = \|v\|^2 + 2\langle v, w\rangle + \|w\|^2.$$

By the Cauchy Schwarz formula, $\langle v, w\rangle \leq \|v\|\|w\|$, so

$$\|v+w\|^2 \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2.$$

Extracting square roots,

$$\|v+w\| \leq \|v\| + \|w\|.$$

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Interpreting the Cauchy Schwarz and Triangle Inequalities in $L^2([a,b])$, we have

$$\int_a^b fg \leq \int f^2 \int g^2$$

$$\left(\int_a^b (f+g)^2\right)^{1/2} \leq \left(\int_a^b f^2\right)^{1/2} + \left(\int_a^b g^2\right)^{1/2}.$$