L10: Semantics III - First order logic/predicate calculus (FOPC)

Simon Dobnik
Department of Philosophy, Linguistics and Theory of Science

October 14, 2015

Up to now: Propositional logic to represent sentences

and $\wedge$, or $\vee$, neg $\neg$, and propositions P, Q...

(1) a. Pavarotti is hungry. P

b. Bond likes Loren. Q

c. Pavarotti is hungry and Bond likes Loren. $P \wedge Q$

d. Pavarotti is not hungry or Loren likes Loren. $\neg P \vee Q$

e. If Pavarotti is hungry, Bond likes Loren. $P \rightarrow Q$

Up to now: Propositional logic to represent sentences

and $\wedge$, or $\vee$, neg $\neg$, and propositions P, Q...

(1) a. Pavarotti is hungry. P

b. Bond likes Loren. Q

c. Pavarotti is hungry and Bond likes Loren. $P \wedge Q$

d. Pavarotti is not hungry or Loren likes Loren. $\neg P \vee Q$

e. If Pavarotti is hungry, Bond likes Loren. $P \rightarrow Q$

We cannot express internal structure of propositions.

Pronouns

(2) She likes Pavarotti but he doesn't.

(3) a. Loren likes Pavarotti and Pavarotti doesn't like Pavarotti.
    b. Loren likes Pavarotti and James Bond doesn't like Pavarotti.

Pronouns

(2) She likes Pavarotti but he doesn't.

(3) a. Loren likes Pavarotti and Pavarotti doesn't like Pavarotti.
    b. Loren likes Pavarotti and James Bond doesn't like Pavarotti.

- The denotation of "she" and "he"?

Pronouns

(2) She likes Pavarotti but he doesn't.

(3) a. Loren likes Pavarotti and Pavarotti doesn't like Pavarotti.

b. Loren likes Pavarotti and James Bond doesn't like Pavarotti.

- The denotation of "she" and "he"?
- Variables that allow alternative assignments (e.g. pointing).

Quantified NPs

(4) a. Everyone likes Loren.
 b. No one is boring.
 c. Someone is hungry.

Quantified NPs

(4) a. Everyone likes Loren.

b. No one is boring.

c. Someone is hungry.

(5) a. Loren likes Loren, and James Bond likes Loren, and Pavarotti likes Loren.

b. It is not the case that [Loren is boring or Bond is boring or Pavarotti is boring].

b′. Loren is not boring, and Bond is not boring, and Pavarotti is not boring.

c. Loren is hungry, or Bond is hungry, or Pavarotti is hungry.

Quantified NPs

(4) a. Everyone likes Loren.
    b. No one is boring.
    c. Someone is hungry.

(5) a. Loren likes Loren, and James Bond likes Loren, and Pavarotti likes Loren.
    b. It is not the case that [Loren is boring or Bond is boring or Pavarotti is boring].
    b′. Loren is not boring, and Bond is not boring, and Pavarotti is not boring.
    c. Loren is hungry, or Bond is hungry, or Pavarotti is hungry.

Finding denotations: pointing and cardinality over pointing.

Quantifiers in FOPC

- Universal quantifier: $\forall$ (every, all)
  for every assignment x
- Existential quantifier: $\exists$ (a, some)
  for some (one or more) assignment x
- Generalised quantifiers: ? (most, many, a few):
  cannot be handled in FOPC

We need to express...

(6) She likes Pavarotti but he doesn't.
$L(x_1, p) \wedge \neg L(x_2, p)$

(7) Everyone likes Loren.
$\forall x_1 [L(x_1, l)]$

- Constants: p, b, m
- Predicates: $L/2$
- Variables: x_1, x_2

Syntax of FOPC

(8) a. Variables: for any integer n, x_n is a variable.
Infinite number of variables denoting objects/individuals

Syntax of FOPC

(8) a. Variables: for any integer n, x_n is a variable.
Infinite number of variables denoting objects/individuals

b. Constants: j, m, $\ldots c_n$
denoting individuals

Syntax of FOPC

(8) a. Variables: for any integer n, x_n is a variable.
Infinite number of variables denoting objects/individuals

b. Constants: j, m,...c_n
denoting individuals

c. Terms: variables and constants

Syntax of FOPC

(8) a. Variables: for any integer n, x_n is a variable.
Infinite number of variables denoting objects/individuals

b. Constants: j, m, $\ldots c_n$
denoting individuals

c. Terms: variables and constants

d. Predicates: P, Q for one-place predicates (Pred_1), K for two-place predicates (Pred_2) and G for three-place predicates (Pred_3)

Syntax of FOPC

(8) a. Variables: for any integer n, x_n is a variable.
Infinite number of variables denoting objects/individuals

b. Constants: j, m, $\ldots c_n$
denoting individuals

c. Terms: variables and constants

d. Predicates: P, Q for one-place predicates (Pred_1), K for two-place predicates (Pred_2) and G for three-place predicates (Pred_3)

e. If A is an n-place predicate and $t_1, \ldots t_n$ are n terms, then $A(t_1, \ldots, t_n)$ is a formula (Form)
Terms are arguments to predicates

Syntax of FOPC

(8) a. Variables: for any integer n, x_n is a variable.
Infinite number of variables denoting objects/individuals

b. Constants: j, m, $\ldots c_n$
denoting individuals

c. Terms: variables and constants

d. Predicates: P, Q for one-place predicates (Pred_1), K for two-place predicates (Pred_2) and G for three-place predicates (Pred_3)

e. If A is an n-place predicate and $t_1, \ldots t_n$ are n terms, then $A(t_1, \ldots, t_n)$ is a formula (Form)
Terms are arguments to predicates

f. If A and B are formulae, then so are $\neg A$, $[A \wedge B]$, $[A \vee B]$, $[A \rightarrow B]$, $[A \leftrightarrow B]$, $\forall x_n A, \exists x_n A$ (recursive definition!)

Syntax of FOPC

(8) a. Variables: for any integer n, x_n is a variable.
Infinite number of variables denoting objects/individuals

b. Constants: j, m, $\ldots c_n$
denoting individuals

c. Terms: variables and constants

d. Predicates: P, Q for one-place predicates (Pred_1), K for two-place predicates (Pred_2) and G for three-place predicates (Pred_3)

e. If A is an n-place predicate and $t_1, \ldots t_n$ are n terms, then $A(t_1, \ldots, t_n)$ is a formula (Form)
Terms are arguments to predicates

f. If A and B are formulae, then so are $\neg A$, $[A \wedge B]$, $[A \vee B]$, $[A \rightarrow B]$, $[A \leftrightarrow B]$, $\forall x_n A, \exists x_n A$ (recursive definition!)

g. If t_1, t_2 are terms. then $t_1 = t_2$ is a formula.

Well-formed formula: *wff*

(9) a. $P(j)$

b. $\neg P(j)$

c. $[P(j) \wedge Q(x,y)]$

d. $\neg[(P(j) \vee Q(x,y)]$ and $[[\neg(P(j)] \vee Q(x,y)]$ (bracketing!)

e. $\exists \mathrm{x}_3[\mathrm{Q}(\mathrm{x}_3)]$ (bound variables)

f. $\mathrm{Q}(\mathrm{x}_3)$ (free variables)

(10) a. $K(j, x_4)$

b. $\forall x_2[G(j, m, x_2) \wedge P(x2)]$

c. $\exists x_3[P(x_3) \vee Q(j)]$

d. $[[\exists x_3 P(x_3)] \vee Q(j)]$

e. $\forall x_3[[\exists x_3 P(x_3)] \vee Q(x_3)]$

f. $\forall x_1[\exists x_3[P(x_3) \vee Q(x_1)]]$

Semantics of FOPC

Let $\mathcal{M}_1$ be a pair $\langle U_1, V_1 \rangle$ where U_1 is a set of individuals (domain or universe of discourse) and V_1 assigns an extension in U_1 to each constant of FOPC and an extension of *n*-tuples built from U_1 to each predicate.

$U_1 = \{\text{Bond, Pavarotti, Loren}\}$

Semantics of FOPC

Let $\mathcal{M}_1$ be a pair $\langle U_1, V_1 \rangle$ where U_1 is a set of individuals (domain or universe of discourse) and V_1 assigns an extension in U_1 to each constant of FOPC and an extension of n-tuples built from U_1 to each predicate.

$U_1 = \{\text{Bond, Pavarotti, Loren}\}$
$V_1(j) = \text{Bond}$
$V_1(m) = \text{Loren}$

Semantics of FOPC

Let $\mathcal{M}_1$ be a pair $\langle U_1, V_1 \rangle$ where U_1 is a set of individuals (domain or universe of discourse) and V_1 assigns an extension in U_1 to each constant of FOPC and an extension of n-tuples built from U_1 to each predicate.

$U_1 = \{\text{Bond, Pavarotti, Loren}\}$
$V_1(j) = \text{Bond}$
$V_1(m) = \text{Loren}$
$V_1(P) = \{\text{Loren, Pavarotti}\}$
$V_1(Q) = \{\text{Loren, Bond}\}$
$V_1(K) = \{\langle\text{Bond,Bond}\rangle, \langle\text{Bond,Loren}\rangle, \langle\text{Loren,Pavarotti}\rangle,$
$\langle\text{Pavarotti,Loren}\rangle\}$
$V_1(G) = \{\langle\text{Bond,Loren,Pavarotti}\rangle, \langle\text{Loren,Loren,Bond}\rangle,$
$\langle\text{Loren,Bond,Pavarotti}\rangle, \langle\text{Pavarotti,Pavarotti,Loren}\rangle$

Semantics of FOPC

- We need to assume some way of interpreting variables by associating them with elements in U_1.
- If g is an assignment of values to variables, v a variable and a an individual in U_1, then $g[v/a]$ is an assignment exactly like g except that it assigns the value a to v. (If g already assigned a to v then there is no change.)
- g helps us to keep track of assignments of members of U_1 to variables.

Assigning $u_1 \in U_1$ to g_1

E.g. we start with g_1 like this...

$$g_1 = \begin{bmatrix} x_1 & \rightarrow & \text{Bond} \\ x_2 & \rightarrow & \text{Loren} \\ x_n & \rightarrow & \text{Pavarotti; where } n \geq 3 \end{bmatrix}$$

Assigning Bond to x_3 gives us...

$$g_1[Bond/x_3] = \begin{bmatrix} x_1 & \rightarrow & \text{Bond} \\ x_2 & \rightarrow & \text{Loren} \\ x_3 & \rightarrow & \text{Bond} \\ x_n & \rightarrow & \text{Pavarotti; where } n \geq 4 \end{bmatrix}$$

$$g_1[Bond/x_1] = g_1$$

Assigning $u_1 \in U_1$ to g_1

Assignments may be nested: we can modify already a modified assignment, c.f. expressions with multiple quantifiers

$$g_1[[Bond/x_3]/Loren/x_1] = \begin{bmatrix} x_1 & \rightarrow & \text{Loren} \\ x_2 & \rightarrow & \text{Loren} \\ x_3 & \rightarrow & \text{Bond} \\ x_n & \rightarrow & \text{Pavarotti; where } n \geq 4 \end{bmatrix}$$

Checking the value in the assignment:

- $g_1[[Bond/x_3]/Loren/x_1](x_1) = \text{Loren}$
- $g_1[[Bond/x_3]/Loren/x_1](x_6) = \text{Pavarotti}$

Interpreting FOPC

If A is either a predicate or a constant, then $[\![A]\!]^{\mathcal{M}_1,g_1} = V_1(A)$.
If A is a variable, $[\![A]\!]^{\mathcal{M}_1,g_1} = g_1(A)$.

Interpreting FOPC

For any formulas A, B, any $\mathrm{Pred}_n R$, and any terms $t_1, \ldots, t_n$,

a. $[\![R(t_1, \ldots, t_n)]\!]^{\mathcal{M}_1, g_1} = 1$ iff $\langle [\![t_1]\!]^{\mathcal{M}_1, g_1}, \ldots, [\![t_n]\!]^{\mathcal{M}_1, g_1} \rangle \in [\![R]\!]^{\mathcal{M}_1, g_1}$

Interpreting FOPC

For any formulas A, B, any $\text{Pred}_n R$, and any terms $t_1, \ldots, t_n$,

a. $[\![R(t_1, \ldots, t_n)]\!]^{\mathcal{M}_1,g_1} = 1$ iff $\langle [\![t_1]\!]^{\mathcal{M}_1,g_1}, \ldots, [\![t_n]\!]^{\mathcal{M}_1,g_1} \rangle \in [\![R]\!]^{\mathcal{M}_1,g_1}$

b. $[\![A \wedge B]\!]^{\mathcal{M}_1,g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1,g_1} = 1$ and $[\![B]\!]^{\mathcal{M}_1,g_1} = 1$

Interpreting FOPC

For any formulas A, B, any $\mathrm{Pred}_n R$, and any terms $t_1, \ldots, t_n$,

a. $[\![R(t_1, \ldots, t_n)]\!]^{\mathcal{M}_1, g_1} = 1$ iff $\langle [\![t_1]\!]^{\mathcal{M}_1, g_1}, \ldots, [\![t_n]\!]^{\mathcal{M}_1, g_1} \rangle \in [\![R]\!]^{\mathcal{M}_1, g_1}$

b. $[\![A \wedge B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ and $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

c. $[\![A \vee B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ or $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

Interpreting FOPC

For any formulas A, B, any $\text{Pred}_n R$, and any terms $t_1, \ldots, t_n$,

a. $[\![R(t_1, \ldots, t_n)]\!]^{\mathcal{M}_1, g_1} = 1$ iff $\langle [\![t_1]\!]^{\mathcal{M}_1, g_1}, \ldots, [\![t_n]\!]^{\mathcal{M}_1, g_1} \rangle \in [\![R]\!]^{\mathcal{M}_1, g_1}$

b. $[\![A \wedge B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ and $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

c. $[\![A \vee B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ or $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

d. $[\![A \to B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 0$ or $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

Interpreting FOPC

For any formulas A, B, any $\text{Pred}_n R$, and any terms $t_1, \ldots, t_n$,

a. $[\![R(t_1, \ldots, t_n)]\!]^{\mathcal{M}_1, g_1} = 1$ iff $\langle [\![t_1]\!]^{\mathcal{M}_1, g_1}, \ldots, [\![t_n]\!]^{\mathcal{M}_1, g_1} \rangle \in [\![R]\!]^{\mathcal{M}_1, g_1}$

b. $[\![A \wedge B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ and $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

c. $[\![A \vee B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ or $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

d. $[\![A \rightarrow B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 0$ or $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

e. $[\![A \leftrightarrow B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = [\![B]\!]^{\mathcal{M}_1, g_1}$

Interpreting FOPC

For any formulas A, B, any $\text{Pred}_n R$, and any terms $t_1, \ldots, t_n$,

a. $[\![R(t_1, \ldots, t_n)]\!]^{\mathcal{M}_1, g_1} = 1$ iff $\langle [\![t_1]\!]^{\mathcal{M}_1, g_1}, \ldots, [\![t_n]\!]^{\mathcal{M}_1, g_1} \rangle \in [\![R]\!]^{\mathcal{M}_1, g_1}$

b. $[\![A \wedge B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ and $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

c. $[\![A \vee B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ or $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

d. $[\![A \rightarrow B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 0$ or $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

e. $[\![A \leftrightarrow B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = [\![B]\!]^{\mathcal{M}_1, g_1}$

f. $[\![\neg A]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 0$

Interpreting FOPC

For any formulas A, B, any $\text{Pred}_n R$, and any terms $t_1, \ldots, t_n$,

a. $[\![R(t_1, \ldots, t_n)]\!]^{\mathcal{M}_1, g_1} = 1$ iff $\langle [\![t_1]\!]^{\mathcal{M}_1, g_1}, \ldots, [\![t_n]\!]^{\mathcal{M}_1, g_1} \rangle \in [\![R]\!]^{\mathcal{M}_1, g_1}$

b. $[\![A \wedge B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ and $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

c. $[\![A \vee B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 1$ or $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

d. $[\![A \rightarrow B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 0$ or $[\![B]\!]^{\mathcal{M}_1, g_1} = 1$

e. $[\![A \leftrightarrow B]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = [\![B]\!]^{\mathcal{M}_1, g_1}$

f. $[\![\neg A]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![A]\!]^{\mathcal{M}_1, g_1} = 0$

g. $[\![t_1 = t_j]\!]^{\mathcal{M}_1, g_1} = 1$ iff $[\![t_1]\!]^{\mathcal{M}_1, g_1}$ is the same as $[\![t_j]\!]^{\mathcal{M}_1, g_1}$

Interpreting FOPC: Quantifiers

h. $[\![A]\!]^{\mathcal{M}_i,g_i[u/x_n]}$ stands for a donation of A where u is assigned to every occurrence of x_n in A.

Interpreting FOPC: Quantifiers

h. $[\![A]\!]^{\mathcal{M}_i, g_i[u/x_n]}$ stands for a donation of A where u is assigned to every occurrence of x_n in A.

i. $[\![\forall x_n A]\!]^{\mathcal{M}_1, g_1} = 1$ iff for all $u \in U$, $[\![A]\!]^{\mathcal{M}_1, g_1[u/x_n]} = 1$, where $g_1[u/x_n] = g_1$, except that $g_1[u/x_n](x_n) = u$
The function $g_1[u/x_n]$ is the same as g_1 except that x_n is assigned to u, for every u, other assignments are the same

Interpreting FOPC: Quantifiers

h. $[\![A]\!]^{\mathcal{M}_i,g_i[u/x_n]}$ stands for a donation of A where u is assigned to every occurrence of x_n in A.

i. $[\![\forall x_n A]\!]^{\mathcal{M}_1,g_1} = 1$ iff for all $u \in U$, $[\![A]\!]^{\mathcal{M}_1,g_1[u/x_n]} = 1$, where $g_1[u/x_n] = g_1$, except that $g_1[u/x_n](x_n) = u$
The function $g_1[u/x_n]$ is the same as g_1 except that x_n is assigned to u, for every u, other assignments are the same

j. $[\![\exists x_n A]\!]^{\mathcal{M}_1,g_1} = 1$ iff for some $u \in U$, $[\![A]\!]^{\mathcal{M}_1,g_1[u/x_n]} = 1$

Interpreting FOPC: Quantifiers

h. $[\![A]\!]^{\mathcal{M}_i, g_i[u/x_n]}$ stands for a donation of A where u is assigned to every occurrence of x_n in A.

i. $[\![\forall x_n A]\!]^{\mathcal{M}_1, g_1} = 1$ iff for all $u \in U$, $[\![A]\!]^{\mathcal{M}_1, g_1[u/x_n]} = 1$, where $g_1[u/x_n] = g_1$, except that $g_1[u/x_n](x_n) = u$
The function $g_1[u/x_n]$ is the same as g_1 except that x_n is assigned to u, for every u, other assignments are the same

j. $[\![\exists x_n A]\!]^{\mathcal{M}_1, g_1} = 1$ iff for some $u \in U$, $[\![A]\!]^{\mathcal{M}_1, g_1[u/x_n]} = 1$

- For $\forall$ stop assignment if some u results in falsehood.

Interpreting FOPC: Quantifiers

h. $[\![A]\!]^{\mathcal{M}_i, g_i[u/x_n]}$ stands for a donation of A where u is assigned to every occurrence of x_n in A.

i. $[\![\forall x_n A]\!]^{\mathcal{M}_1, g_1} = 1$ iff for all $u \in U$, $[\![A]\!]^{\mathcal{M}_1, g_1[u/x_n]} = 1$, where $g_1[u/x_n] = g_1$, except that $g_1[u/x_n](x_n) = u$
The function $g_1[u/x_n]$ is the same as g_1 except that x_n is assigned to u, for every u, other assignments are the same

j. $[\![\exists x_n A]\!]^{\mathcal{M}_1, g_1} = 1$ iff for some $u \in U$, $[\![A]\!]^{\mathcal{M}_1, g_1[u/x_n]} = 1$

- For $\forall$ stop assignment if some u results in falsehood.
- For $\exists$ stop assignment if some u results in truth.

An example interpretation I

Evaluate $\exists x_1 P(x_1)$ in $\mathcal{M}_1$ with respect to g_1.

- $[\![\exists x_1 P(x_1)]\!]^{\mathcal{M}_1,g_1} = 1$ iff for some u $\in U_1$, $[\![P(x_1)]\!]^{\mathcal{M}_1,g_1[u/x_1]} = 1$
- Assign Bond to x_1; i.e., consider $g_1[Bond/x_1]$
- $[\![P(x_1)]\!]^{\mathcal{M}_1,g_1[u/x_1]} = 1$ iff $[\![x_1]\!]^{\mathcal{M}_1,g_1[u/x_1]} \in [\![P]\!]^{\mathcal{M}_1,g_1[u/x_1]}$
- $[\![x_1]\!]^{\mathcal{M}_1,g_1[u/x_1]} \in [\![P]\!]^{\mathcal{M}_1,g_1[u/x_1]}$ iff $g_1[Bond/x_1](x_1) \in V_1(P)$ $=$ {Loren, Pavarotti}
- $g_1[Bond/x_1](x_1) =$ Bond
- $[\![P(x_1)]\!]^{\mathcal{M}_1,g_1[u/x_1]} = 1$ iff Bond $\in$ {Loren, Pavarotti}
- $[\![P(x_1)]\!]^{\mathcal{M}_1,g_1[u/x_1]} = 0$
- Assign Loren to x_1; i.e., consider $g_1[Loren/x_1]$
- $[\![P(x_1)]\!]^{\mathcal{M}_1,g_1[Loren/x_1]} = 1$ iff $[\![x_1]\!]^{\mathcal{M}_1,g_1[Loren/x_1]} \in [\![P]\!]^{\mathcal{M}_1,g_1[Loren/x_1]}$

An example interpretation II

- $[\![x_1]\!]^{\mathcal{M}_1, g_1[Loren/x_1]} \in [\![P]\!]^{\mathcal{M}_1, g_1[Loren/x_1]}$ iff $g_1^{[Loren/x_1]}(x_1) \in V_1(P) = \{\text{Loren, Pavarotti}\}$
- $g_1[Loren/x_1](x_1) = \text{Loren}$
- $[\![P(x_1)]\!]^{\mathcal{M}_1, g_1[Loren/x_1]} = 1$ iff Loren $\in \{\text{Loren, Pavarotti}\}$
- $[\![P(x_1)]\!]^{\mathcal{M}_1, g_1[Loren/x_1]} = 1$
- $[\![\exists x_1 P(x_1)]\!]^{\mathcal{M}_1, g_1} = 1$

Leaving out interpretation steps...

$\mathcal{M}_2\langle U_2, V_2\rangle$
$U_2 = \{l, a, g, f\}$
$V_2(\text{Lydia}) = l$; $V_2(\text{Alex}) = a$; $V_2(\text{Goldy}) = g$; $V_2(\text{Fido}) = f$
$V_2(\text{likes}) = \{\langle l, a\rangle, \langle l, f\rangle, \langle a, g\rangle, \langle l, g\rangle\}$
$V_2(\text{runs}) = \{l, f\}$
$V_2(\text{human}) = \{l, a\}$
$V_2(\text{dog}) = \{g, f\}$
$V_2(\text{owns}) = \{\langle l, f\rangle, \langle a, g\rangle\}$
$V_2(\text{sybling\_of}) = \{\langle l, a\rangle, \langle a, l\rangle, \langle g, f\rangle, \langle f, g\rangle\}$

$[\![\ \forall x.\ (\exists y.\ (\text{likes}(x, y)))\ ]\!]^{\mathcal{M}_2, g}$

$\forall x.$	$(\exists y.$	$(\text{likes}(x, y)))$	x	y
		0	l	l
	1	1	l	a
		1	l	g
		1	l	f
		0	a	l
	1	0	a	a
		1	a	g
0		0	a	f
		0	g	l
	0	0	g	a
		0	g	g
		0	g	f
		0	f	l
	0	0	f	a
		0	f	g
		0	f	f

So **FALSE**.

Representing natural language in first order logic, I

(11) Every cat has a secret.

Representing natural language in first order logic, I

(11) Every cat has a secret.
$\forall x.(\mathrm{cat}(x) \rightarrow \exists y.(\mathrm{secret}(y) \wedge \mathrm{have}(x, y)))$

Representing natural language in first order logic, I

(11) Every cat has a secret.

$\forall x.(\text{cat}(x) \rightarrow \exists y.(\text{secret}(y) \wedge \text{have}(x, y)))$

$\exists y.(\text{secret}(y) \wedge \forall x.(\text{cat}(x) \rightarrow \text{have}(x, y)))$

Representing natural language in first order logic, I

(11) Every cat has a secret.

$\forall x.(\text{cat}(x) \rightarrow \exists y.(\text{secret}(y) \wedge \text{have}(x, y)))$

$\exists y.(\text{secret}(y) \wedge \forall x.(\text{cat}(x) \rightarrow \text{have}(x, y)))$

(12) George owns a dog and his brother owns a dog.

Representing natural language in first order logic, I

(11) Every cat has a secret.
$\forall x.(\text{cat}(x) \rightarrow \exists y.(\text{secret}(y) \wedge \text{have}(x, y)))$
$\exists y.(\text{secret}(y) \wedge \forall x.(\text{cat}(x) \rightarrow \text{have}(x, y)))$

(12) George owns a dog and his brother owns a dog.
$\exists x.(\text{dog}(x) \wedge \text{own}(\text{George}, x)) \wedge \exists y.(\text{dog}(y) \wedge \text{own}(\text{brother\_of}(\text{George}), y))$

Representing natural language in first order logic, I

(11) Every cat has a secret.

$\forall x.(\mathrm{cat}(x) \rightarrow \exists y.(\mathrm{secret}(y) \wedge \mathrm{have}(x, y)))$

$\exists y.(\mathrm{secret}(y) \wedge \forall x.(\mathrm{cat}(x) \rightarrow \mathrm{have}(x, y)))$

(12) George owns a dog and his brother owns a dog.

$\exists x.(\mathrm{dog}(x) \wedge \mathrm{own}(\mathrm{George}, x)) \wedge \exists y.(\mathrm{dog}(y) \wedge \mathrm{own}(\mathrm{brother\_of}(\mathrm{George}), y))$

$(?)\exists x.(\mathrm{dog}(x) \wedge \mathrm{own}(\mathrm{George}, x) \wedge \exists y.(\mathrm{dog}(y) \wedge \mathrm{own}(\mathrm{brother\_of}(x), y)))$

Representing natural language in first order logic, I

(11) Every cat has a secret.

$\forall x.(\mathrm{cat}(x) \rightarrow \exists y.(\mathrm{secret}(y) \wedge \mathrm{have}(x, y)))$

$\exists y.(\mathrm{secret}(y) \wedge \forall x.(\mathrm{cat}(x) \rightarrow \mathrm{have}(x, y)))$

(12) George owns a dog and his brother owns a dog.

$\exists x.(\mathrm{dog}(x) \wedge \mathrm{own}(\mathrm{George}, x)) \wedge \exists y.(\mathrm{dog}(y) \wedge \mathrm{own}(\mathrm{brother\_of}(\mathrm{George}), y))$

$(?)\exists x.(\mathrm{dog}(x) \wedge \mathrm{own}(\mathrm{George}, x) \wedge \exists y.(\mathrm{dog}(y) \wedge \mathrm{own}(\mathrm{brother\_of}(x), y)))$

$(??)\exists x.(\mathrm{dog}(x) \wedge \mathrm{own}(\mathrm{George}, x) \wedge \mathrm{own}(\mathrm{brother\_of}(\mathrm{George}), x))$

Representing natural language in first order logic, I

(13) No cat meowed in the night.

Representing natural language in first order logic, I

(13) No cat meowed in the night.
$\neg \exists x.(\mathrm{cat}(x) \wedge \mathrm{meowed\_in\_the\_night}(x))$

Representing natural language in first order logic, I

(13) No cat meowed in the night.
$\neg \exists x.(\text{cat}(x) \wedge \text{meowed\_in\_the\_night}(x))$
$\forall x.(\text{cat}(x) \rightarrow \neg \text{meowed\_in\_the\_night}(x))$

Representing natural language in first order logic, I

(13) No cat meowed in the night.
$\neg \exists x.(\text{cat}(x) \wedge \text{meowed\_in\_the\_night}(x))$
$\forall x.(\text{cat}(x) \rightarrow \neg \text{meowed\_in\_the\_night}(x))$
$(?) \neg \exists t.(\text{time}(t) \wedge \text{in\_the\_night}(t) \wedge \exists d.(\text{cat}(d) \wedge \text{meow\_at\_time}(d, t)))$

Representing natural language in first order logic, I

(13) No cat meowed in the night.
$\neg \exists x.(\mathrm{cat}(x) \wedge \mathrm{meowed\_in\_the\_night}(x))$
$\forall x.(\mathrm{cat}(x) \rightarrow \neg \mathrm{meowed\_in\_the\_night}(x))$
$(?)\neg \exists t.(\mathrm{time}(t) \wedge \mathrm{in\_the\_night}(t) \wedge \exists d.(\mathrm{cat}(d) \wedge \mathrm{meow\_at\_time}(d, t)))$
$(?)\neg \exists e.([\mathrm{event}(e) \wedge] \exists d.(\mathrm{cat}(d) \wedge \mathrm{meow}(e, d) \wedge \mathrm{in}(e, \mathrm{the\_night})))$

Inference: definitions

- A wff is logically valid iff it is true for every interpretation.
- A wff is contradictory iff ¬wff is logically valid.
- A logically implies B (B is a logical consequence of A) iff in every interpretation, when A is true, B is true (i.e. $A \to B$ is logically valid).
- A and B are logically equivalent iff they logically imply each other.

Inference: rules

- All the inference rules of propositional calculus, plus:
  - Universal instantiation:
    $\forall x.P(x) \vdash P(a)$ where a is a constant
  - Existential generalisation:
    $P(a) \vdash \exists x.P(x)$

Inference: rules

- All the inference rules of propositional calculus, plus:
  - Universal instantiation:
    $\forall x.P(x) \vdash P(a)$ where a is a constant
  - Existential generalisation:
    $P(a) \vdash \exists x.P(x)$
- FOPC is consistent – no inference rule will go from a logically valid statement to an invalid one, and
- complete – every logically valid statement can be derived by some sequence of application of the inference rules.

Inference

We can show the validity of our inference pattern either via the semantics of the expressions, or by the application of inference rules:

Inference

We can show the validity of our inference pattern either via the semantics of the expressions, or by the application of inference rules:

- Pavarotti was Italian; all Italians are Europeans; therefore Pavarotti was European.
- Italian(Pavarotti)
  $\forall x.$ Italian(x) $\rightarrow$ European(x)
  European(Pavarotti)

Lewis Carroll's puzzle

1. All honest industrious men are healthy.
   $\forall x.\text{honest}(x) \wedge \text{industrious}(x) \rightarrow \text{healthy}(x)$
2. No grocers are healthy.
   $\neg\exists x.\text{grocer}(x) \wedge \text{healthy}(x)$
3. All industrious grocers are honest.
   $\forall x.\text{industrious}(x) \wedge \text{grocer}(x) \rightarrow \text{honest}(x)$
4. All cyclists are industrious.
   $\forall x.\text{cyclist}(x) \rightarrow \text{industrious}(x)$
5. All unhealthy cyclists are dishonest.
   $\forall x.\text{cyclist}(x) \wedge \neg\text{healthy}(x) \rightarrow \neg\text{honest}(x)$
6. We have to show that it follows that:
   No grocer is a cyclist
   $\neg\exists x.\text{grocer}(x) \wedge \text{cyclist}(x)$

Proof by refutation

Assume the contrary and try to derive a contradiction.

A. Assume $\exists x.grocer(x) \wedge cyclist(x)$
B. $\exists x.grocer(x) \wedge cyclist(x) \wedge industrious(x)$ (from A and 4)
C. $\exists x.grocer(x) \wedge cyclist(x) \wedge industrious(x) \wedge honest(x)$ (from B and 3)
D. $\exists x.grocer(x) \wedge cyclist(x) \wedge industrious(x) \wedge honest(x) \wedge healthy(x)$ (from C and 1)
E. $\exists x.grocer(x) \wedge healthy(x)$ (from D by conjunction elimination)

- E contradicts 2. QED

Examples from NLTK

- ex04.py
- ex05.py

Further reading

- On semantics of natural language: (Chierchia and McConnell-Ginet, 2000): Chapter 3
- On logic: (Bird, Klein, and Loper, 2009): Chapter 10, Section 3 and (Allwood, Andersson, and Dahl, 1977), Chapters 5 and 6

Acknowledgements

Slides 22–26 based on slides by Stephen Pulman

References I

Allwood, Jens, Lars-Gunnar Andersson, and Osten Dahl. 1977. *Logic in linguistics*. Cambridge University Press.

Bird, Steven, Ewan Klein, and Edward Loper. 2009. *Natural language processing with Python*. O'Reilly, Beijing, Cambridge, Farnham, Köln, Sebastopol and Tokyo, 1st ed edition.

Chierchia, Gennaro and Sally McConnell-Ginet. 2000. *Meaning and grammar: an introduction to semantics*. MIT Press, Cambridge, Mass, 2nd ed edition.