L12: Semantics II - Model-theoretic semantics and propositional logic

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Model theory

The idea of model theory

- linguistic expressions $\rightsquigarrow$ formal language $\rightsquigarrow$ objects in a model
- the model provides
  - a set of basic objects
  - an assignment of objects to the basic expressions of the (formal) language
- for each rule combining expressions in the (formal) language there is a corresponding rule operating on the corresponding model-theoretic objects expressing truth conditions

Propositional logic

Propositional logic I

- the syntax
  - a set of basic propositions: $\{p, q, r, p_1, p_2, ...\}$
  - the set of expressions of propositional logic:
    - any basic proposition is an expression of propositional logic
    - if α, β are expressions of propositional logic then $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$ and $\alpha \rightarrow \beta$ are expressions of propositional logic
- the model theory (semantics)
  - the set of objects provided by a model are a set of two truth values $\{0, 1\}$
  - a model M also provides an assignment F_M of truth-values to the basic propositions, e.g. $F(p) = 0, F(q) = 1, F(r) = 1$ and so on
  - if α is an expression of propositional logic the interpretation of α with respect to a model M is represented by $[\![\ \alpha\ ]\!]^M$
    - If α is a basic proposition then $[\![\ \alpha\ ]\!]^M$ is $F_M(\alpha)$
    - $[\![\ \neg\alpha\ ]\!]^M$=0 if $[\![\ \alpha\ ]\!]^M$=1; $[\![\ \neg\alpha\ ]\!]^M$=1 otherwise
    - $[\![\ \alpha \wedge \beta\ ]\!]^M$=1 if $[\![\ \alpha\ ]\!]^M$=$[\![\ \beta\ ]\!]^M$=1; $[\![\ \alpha \wedge \beta\ ]\!]^M$=0 otherwise
    - $[\![\ \alpha \vee \beta\ ]\!]^M$=0 if $[\![\ \alpha\ ]\!]^M$=$[\![\ \beta\ ]\!]^M$=0; $[\![\ \alpha \vee \beta\ ]\!]^M$=1 otherwise

Propositional logic II

- $[\![\alpha \rightarrow \beta]\!]^M$=0 if $[\![\alpha]\!]^M$=1 and $[\![\beta]\!]^M$=0; $[\![\alpha \rightarrow \beta]\!]^M$=1 otherwise

Truth tables

Conjunction

p	$\wedge$	q
1	1	1
1	0	0
0	0	1
0	0	0

Disjunction

p	$\vee$	q
1	1	1
1	1	0
0	1	1
0	0	0

Implication

p	$\rightarrow$	q
1	1	1
1	0	0
0	1	1
0	1	0

Bidirectional implica-tion/equivalence

p p	$\leftrightarrow$ $\equiv$	q q
1	1	1
1	0	0
0	0	1
0	1	0

Exclusive disjunction

p	$\oplus$	q
1	0	1
1	1	0
0	1	1
0	0	0

Equivalent to $\neg(p \leftrightarrow q)$. Check the values in the tables!

Negation

$\neg$	p
1	0
0	1

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$[\![\ (\mathrm{p} \wedge \mathrm{q}) \to \mathrm{r}\ ]\!]^M$

(	p	$\wedge$	q	)	$\to$	r
	1		0			1
		0				1
					1	

Logic and inference

- Logic is the study of valid arguments or inferences.
- Validity of arguments is different to truth and falsity of their components
- But if an argument is valid and the premises are true, then the conclusion must also be true.
- ... if an argument is valid and the premises are false, then the conclusion can be either true or false.

(2) a. Pavarotti was Italian. All Italians are European. Therefore, Pavarotti was European.
(Valid, premises and conclusions true).

b. Pavarotti was French. All French are European. Therefore, Pavarotti was European
(Valid, premises false, conclusion true).

c. Pavarotti was French. All French are European. Therefore, Pavarotti was European.
(Valid, premises and conclusion false).

d. Pavarotti was Italian. All Italians are European. Therefore, Loren was Italian.
(Invalid, premises and conclusions true).

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- A and B are logically equivalent ($A \equiv B$) if $A \leftrightarrow B$ is a tautology.

Logical equivalences

1. $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ (De Morgan's law)
2. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ (De Morgan's law)
3. $P \to Q \equiv \neg P \vee Q$
4. $P \to Q \equiv \neg Q \to \neg P$ (Contrapositive)
5. $P \wedge Q \equiv Q \wedge P$
6. $P \vee Q \equiv Q \vee P$
7. $P \equiv \neg\neg P$
8. $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ (Distributivity)
9. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ (Distributivity)

Inference rules

- Modus Ponendo Ponens:
  From A, $A \to B$, infer B
- Modus Tollendo Tollens:
  From $\neg B$, $A \to B$, infer $\neg A$
- Conjunction introduction:
  Given A, B, infer $A \wedge B$
- Conjunction elimination:
  Given $A \wedge B$, infer A.

Inference

- If it is snowing (P), you will get wet (Q) and cold (R).
- If you get cold, you will get flu (X) or frostbite (Y).
- If you get wet, you won't get frostbite.
- It is snowing.
- Will you get flu?

Inference

1. $P \rightarrow Q \wedge R$
2. $R \rightarrow X \vee Y$
3. $Q \rightarrow \neg Y$
4. P
5. X?
6. $Q \wedge R$ (from 1 and 4 by MPP)
7. R (from 6 by conjunction elimination)
8. $X \vee Y$ (from 7 and 2 by MPP)
9. Q (from 6 by conjunction elimination)
10. $\neg Y$ (from 3 and 9 by MPP)
11. $(\neg(\neg X)) \vee Y$ (from 8 by equivalence 7)
12. $(\neg X) \rightarrow Y$ (from 11 by equivalence 3)
13. $\neg(\neg X)$ (from 12 and 10 by MTT)
14. X (from 12 by equivalence 3)

Contradiction and inconsistency

- If $P \to Q$ is tautology, then $P \wedge \neg Q$ will be a contradiction.
- $P \to Q$ should be true for every assignment of truth values to P and Q and there should be no assignment of truth values to P and Q that makes $P \wedge \neg Q$ true (cf. definitions on s.11)
- A set of propositions is consistent (inconsistent) if there is AN assignment (no assignment) of truth values to the conjunction of them that gives truth overall.
- If premises are inconsistent, we can deduce anything:
  - $(P \wedge \neg P) \to Q$: will be tautology
  - $(P \wedge \neg P) \to \neg Q$: will also be a tautology.

Checking of valid arguments

- EITHER create a conjunction of premises and construct an implication to the conclusion. Using truth tables check whether it is logically valid, i.e. a tautology.
  $P_1 \wedge P_2 \ldots P_n \rightarrow Q$
- OR check whether the conjunction of the premises and the negation of the conclusion is consistent:
  $P_1 \wedge P_2 \ldots P_n \wedge \neg Q$
  If so, the conclusion does not follow logically from the premises; it is possible for the premises to be true and the conclusion false.

Examples

1. If George plays in the garden (P), Lydia will be happy. If George sings an aria, Lydia will be happy (Q). Either George will play in the garden or sing an aria. So Lydia will be happy.

$$((P \to R) \wedge (Q \to R) \wedge (P \vee Q)) \to R$$

2. George likes catching mice (P). George has a long tail (Q). If George is a cat, then both of these facts are explained (P and Q). Hence, George is a cat (R).

$$((R \to P) \wedge (R \to Q) \wedge P \wedge Q) \to R$$

Exercises

1. Demonstrate the validity of the Propositional Calculus equivalences 1 and 2 on s.12 (De Morgan's laws), using truth tables.

2. Formalise the following argument and determine by the truth table method whether or not it is valid:

If the firemen go on strike, lives will be lost. If the government give the firemen more money, there will be less money for hospitals, but the firemen will not go on strike. If there is less money for hospitals, lives will be lost. So whether the firemen go on strike or not, lives will be lost.

3. Check if you get the same result by applying your formalisation to a theorem prover in NLTK.

Further reading

- On semantics of natural language: (Chierchia and McConnell-Ginet, 2000), Chapter 2
- On logic: (Bird, Klein, and Loper, 2009): Chapter 10, Section 1 and 2 and (Allwood, Andersson, and Dahl, 1977), Chapters 4 and 6

Acknowledgements

- Slides 5–6 based on slides by Robin Cooper
- Slides 9–19 based on slides by Stephen Pulman

References I

Allwood, Jens, Lars-Gunnar Andersson, and Osten Dahl. 1977. *Logic in linguistics*. Cambridge University Press.

Bird, Steven, Ewan Klein, and Edward Loper. 2009. *Natural language processing with Python*. O'Reilly, Beijing, Cambridge, Farnham, Köln, Sebastopol and Tokyo, 1st ed edition.

Chierchia, Gennaro and Sally McConnell-Ginet. 2000. *Meaning and grammar: an introduction to semantics*. MIT Press, Cambridge, Mass, 2nd ed edition.