# Math 217.002 F25
## Solutions to Quiz 27

1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) If $A$ is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$, the vector $\vec{x}^* \in \mathbb{R}^n$ is a *least-squares solution* of the linear system $A\vec{x} = \vec{b}$ if ...

   **Solution:** $\vec{x}^*$ minimizes the residual norm:
   $$\|A\vec{x}^* - \vec{b}\| \;\leq\; \|A\vec{x} - \vec{b}\| \quad \text{for all } \vec{x} \in \mathbb{R}^n.$$
   Equivalently, $\vec{x}^*$ satisfies the normal equations $A^\top A\,\vec{x}^* = A^\top \vec{b}$.

   (b) Suppose $A$ is an $m \times n$ matrix, the *transpose* of $A$ is ...

   **Solution:** The $n \times m$ matrix $A^\top = [a_{ji}]$ obtained by interchanging rows and columns of $A = [a_{ij}]$; i.e., $(A^\top)_{ij} = a_{ji}$.

2. The functions $\mathfrak{B} = (e^{2x}, \sin(3x), \cos(3x))$ in $C(\mathbb{R})$ are linearly independent. Let $V = \operatorname{Span}(\mathfrak{B})$.

   (a) True or False, no justification needed: The list $\mathfrak{B}$ is a basis for $V$.

   **Solution:** **TRUE.** It spans $V$ by definition and is given to be linearly independent.

   (b) Show: if $f \in V$, then $f' + 7f$ is also in $V$.

   **Solution:** If $f(x) = \alpha e^{2x} + \beta\sin(3x) + \gamma\cos(3x)$, then
   $$f'(x) + 7f(x) = 9\alpha\, e^{2x} + (7\beta - 3\gamma)\sin(3x) + (3\beta + 7\gamma)\cos(3x) \in V.$$

   (c) Compute $[T]_{\mathfrak{B}}$ where $T\colon V \to V$ is the linear transformation defined by
   $$T(g) = g' + 7g$$
   for $g \in V$.

   **Solution:**
   $$T(e^{2x}) = 9e^{2x}, \quad T(\sin 3x) = 7\sin 3x + 3\cos 3x, \quad T(\cos 3x) = -3\sin 3x + 7\cos 3x.$$
   Thus, relative to $\mathfrak{B} = (e^{2x}, \sin 3x, \cos 3x)$,
   $$[T]_{\mathfrak{B}} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 7 & -3 \\ 0 & 3 & 7 \end{bmatrix}.$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

(d) The list $\mathcal{A} = (e^{2x}, e^{2x} + \cos(3x) - \sin(3x), e^{2x} + \cos(3x) + \sin(3x))$ is another basis for $V$. Compute both $S_{\mathcal{A}\to\mathfrak{B}}$ and $S_{\mathfrak{B}\to\mathcal{A}}$.

**Solution:** Coordinates of $\mathcal{A}$-basis vectors in $\mathfrak{B}$:

$$[e^{2x}]_{\mathcal{B}} = (1,0,0)^\top, \quad [e^{2x}+\cos 3x-\sin 3x]_{\mathcal{B}} = (1,-1,1)^\top, \quad [e^{2x}+\cos 3x+\sin 3x]_{\mathcal{B}} = (1,1,1)^\top.$$

Hence

$$S_{\mathcal{A}\to\mathfrak{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad S_{\mathfrak{B}\to\mathcal{A}} = (S_{\mathcal{A}\to\mathfrak{B}})^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(e) Compute $[T]_{\mathcal{A}}$.

**Solution:**

$$[T]_{\mathcal{A}} = S_{\mathfrak{B}\to\mathcal{A}}\,[T]_{\mathfrak{B}}\,S_{\mathcal{A}\to\mathfrak{B}} = \begin{bmatrix} 9 & 5 & -1 \\ 0 & 7 & 3 \\ 0 & -3 & 7 \end{bmatrix}.$$

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) The standard matrix of an orthogonal projection of $\mathbb{R}^n$ onto a subspace $V$ is symmetric.

**Solution: TRUE.** The orthogonal projection $P$ (w.r.t. the standard inner product) satisfies $P = P^\top$. For example, if $A$ has columns forming a basis of $V$, then

$$P = A\,(A^\top A)^{-1}A^\top,$$

and $P^\top = A\,(A^\top A)^{-1}A^\top = P$.