Math 217 Fall 2025

Quiz 20 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Let $\mathfrak{B} = (v_1, \dots, v_d)$ be a basis for the vector space V. Let $v \in V$. The $\mathfrak{B}$-*coordinates* of v are ...

   Solution: The unique scalars $a_1, \dots, a_d \in \mathbb{R}$ such that
   $$v = a_1 v_1 + \cdots + a_d v_d.$$

   (b) Let $\mathfrak{B} = (v_1, \dots, v_d)$ be a basis for the vector space V. Let $v \in V$. The $\mathfrak{B}$-*coordinate column vector* of v is ...

   Solution: The column vector formed from the $\mathfrak{B}$-coordinates of v:
   $$[v]_{\mathfrak{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix} \in \mathbb{R}^d, \quad \text{where } v = a_1 v_1 + \cdots + a_d v_d.$$

   (c) Let $T : V \to V$ be linear and $\mathfrak{B} = (v_1, \dots, v_n)$ an ordered basis of V. The *matrix of T with respect to $\mathfrak{B}$* is ...

   Solution: The $n \times n$ matrix $[T]_{\mathfrak{B}}$ whose j-th column is $[T(v_j)]_{\mathfrak{B}}$; i.e.
   $$[T]_{\mathfrak{B}} = \begin{bmatrix} | & & | \\ [T(v_1)]_{\mathfrak{B}} & \cdots & [T(v_n)]_{\mathfrak{B}} \\ | & & | \end{bmatrix}.$$

2. Suppose V is a vector space and $v_1, \dots, v_m$ are linearly independent in V. Show that if $v \in V \setminus \operatorname{Span}(v_1, \dots, v_m)$, then the vectors $v, v_1, \dots, v_m$ are linearly independent.

   Solution: Suppose $a\,v + \sum_{i=1}^m b_i v_i = 0$. If $a \neq 0$, then
   $$v = -\frac{1}{a} \sum_{i=1}^m b_i v_i \in \operatorname{Span}(v_1, \dots, v_m),$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

contradiction. Hence $a = 0$, so $\sum b_i v_i = 0$. By linear independence of $v_1, \ldots, v_m$, we get $b_1 = \cdots = b_m = 0$. Thus $v, v_1, \ldots, v_m$ are linearly independent.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   (a) Suppose $V = \mathbb{R}^{2\times 2}$ with basis $\mathfrak{B} = \left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)$. Consider $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by $T(X) = AX - XA$ where $A = \begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix}$. The matrix of T with respect to $\mathfrak{B}$ is

$$\begin{bmatrix} 0 & 0 & 4 & 0\\ -4 & 0 & 0 & 4\\ 0 & 0 & 0 & 0\\ 0 & 0 & -4 & 0 \end{bmatrix}.$$

Solution: FALSE. Let $E_{11}, E_{12}, E_{21}, E_{22}$ be the listed basis. Compute:

$$\begin{aligned} T(E_{11}) &= AE_{11} - E_{11}A = \begin{bmatrix}0 & -3\\ 0 & 0\end{bmatrix} = -3E_{12},\\ T(E_{12}) &= AE_{12} - E_{12}A = \mathbf{0},\\ T(E_{21}) &= AE_{21} - E_{21}A = 3E_{11} - 3E_{22},\\ T(E_{22}) &= AE_{22} - E_{22}A = 3E_{12}. \end{aligned}$$

Thus the columns of $[T]_{\mathfrak{B}}$ are

$$\begin{bmatrix}0\\ -3\\ 0\\ 0\end{bmatrix}, \quad \begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}, \quad \begin{bmatrix}3\\ 0\\ 0\\ -3\end{bmatrix}, \quad \begin{bmatrix}0\\ 3\\ 0\\ 0\end{bmatrix},$$

so

$$[T]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 3 & 0\\ -3 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & -3 & 0 \end{bmatrix},$$

not the given matrix with 4's.

4. Let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ with $\vec{v}_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}-1\\ 1\end{bmatrix}$.

   (a) Let $\vec{w}$ be the vector whose $\mathfrak{B}$-coordinates are $\begin{bmatrix}2\\ -1\end{bmatrix}$. What is $[\vec{w}]_{\mathcal{E}}$?

Solution:

$$\vec{w} = 2\vec{v}_1 + (-1)\vec{v}_2 = 2\begin{bmatrix}1\\ 1\end{bmatrix} - \begin{bmatrix}-1\\ 1\end{bmatrix} = \begin{bmatrix}3\\ 1\end{bmatrix}.$$

Hence $[\vec{w}]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b) Let $\vec{v}$ satisfy $[\vec{v}]_{\mathfrak{B}} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. Represent $\vec{v}$ in standard coordinates. Find $[\vec{e}_1]_{\mathfrak{B}}$ and $[\vec{e}_2]_{\mathfrak{B}}$.

Solution:

$$\vec{v} = \tfrac{1}{2}\vec{v}_1 + \tfrac{1}{2}\vec{v}_2 = \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{e}_2.$$

Let $P = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $[\cdot]_{\mathfrak{B}} = P^{-1}[\cdot]_{\mathcal{E}}$ with

$$P^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Hence

$$[\vec{e}_1]_{\mathfrak{B}} = P^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad [\vec{e}_2]_{\mathfrak{B}} = P^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$