Math 217 Fall 2025
Quiz 34 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:
   (a) The linear transformation $T : V \to V$ of a finite dimensional vector space V is *diagonalizable* if ...

   Solution: The linear transformation $T : V \to V$ is *diagonalizable* if there exists a basis $\mathcal{B}$ of V such that the matrix of T with respect to $\mathcal{B}$, denoted $[T]_{\mathcal{B}}$, is a diagonal matrix.

   (b) Suppose $\lambda \in \mathbb{R}$ and $T : V \to V$ is a linear transformation of a finite dimensional vector space V. The *geometric multiplicity* "gemu(λ)" of λ is ...

   Solution: The *geometric multiplicity* of the eigenvalue λ is the dimension of its eigenspace

   $$E_\lambda \;=\; \{\, v \in V : T(v) = \lambda v \,\}.$$

   In other words,

   $$\operatorname{gemu}(\lambda) \;=\; \dim(E_\lambda).$$

   (c) Suppose $n \in \mathbb{N}$. An $n \times n$ matrix A is *diagonalizable* if ...

   Solution: An $n \times n$ matrix A is *diagonalizable* if A is similar to a diagonal matrix.

   (d) Suppose $m \in \mathbb{N}$. An $m \times m$ matrix A is *symmetric* if ...

   Solution: An $m \times m$ matrix A is *symmetric* if it is equal to its transpose:

   $$A^\top = A.$$

   Equivalently, in terms of entries, A is symmetric if $a_{ij} = a_{ji}$ for all $1 \le i, j \le m$.

2. (a) Let $V \xrightarrow{T} V$ be a linear transformation, where V is a finite dimensional vector space.
   (i) Define the determinant of T.
   (ii) Show the determinant of T you defined above is independent of any choices you made.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Solution: (i) Definition.
Let $\dim(V) = n$, and let $\mathcal{B} = (v_1, \ldots, v_n)$ be any basis of V. Consider the matrix of T with respect to $\mathcal{B}$,

$$[T]_{\mathcal{B}} \in M_{n\times n}(\mathbb{R}).$$

We define the *determinant of* T to be

$$\det(T) \ := \ \det\big([T]_{\mathcal{B}}\big).$$

(ii) Independence of basis.
We must show that this number does not depend on the choice of basis $\mathcal{B}$.
Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of V, and let $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ denote the matrices of T in these bases. There exists an invertible change-of-basis matrix S such that

$$[T]_{\mathcal{C}} = S\,[T]_{\mathcal{B}}\,S^{-1}.$$

(In words: the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ are similar.)
Taking determinants and using the multiplicative property of determinants, we obtain

$$\det\big([T]_{\mathcal{C}}\big) = \det\big(S[T]_{\mathcal{B}}S^{-1}\big) = \det(S)\ \det\big([T]_{\mathcal{B}}\big)\ \det(S^{-1}).$$

Since $\det(S^{-1}) = 1/\det(S)$, this simplifies to

$$\det\big([T]_{\mathcal{C}}\big) = \det\big([T]_{\mathcal{B}}\big).$$

Thus the determinant of T computed in basis $\mathcal{C}$ equals the determinant computed in basis $\mathcal{B}$. Because any two choices of basis are related by an invertible change-of-basis matrix, the value $\det(T)$ is well-defined and independent of the basis used.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   Suppose V is a vector space and $T\colon V \to V$ is a linear transformation. For all $\lambda \in \mathbb{R}$ we have $T[E_\lambda] \subset E_\lambda$ with equality if and only if $\lambda \neq 0$.

Solution: TRUE.

First recall that

$$E_\lambda = \{\, v \in V : T(v) = \lambda v \,\}$$

is the eigenspace corresponding to λ.

Step 1: Show $T[E_\lambda] \subset E_\lambda$ for all λ.
Let $v \in E_\lambda$. Then $T(v) = \lambda v$. Apply T again:

$$T\big(T(v)\big) = T(\lambda v) = \lambda T(v).$$

Hence $T(v)$ is an eigenvector for the same eigenvalue λ. Therefore $T(v) \in E_\lambda$ (or $T(v) = 0$, which is in E_λ as well). Therefore $T[E_\lambda] \subset E_\lambda$.

Step 2: Equality when $\lambda \neq 0$.
Assume $\lambda \neq 0$. Let $v \in E_\lambda$. Then

$$T(v) = \lambda v \quad \Longrightarrow \quad v = \frac{1}{\lambda} T(v).$$

This shows that every $v \in E_\lambda$ comes from applying T to another vector in E_λ. Therefore,

$$T[E_\lambda] = E_\lambda \quad \text{when } \lambda \neq 0.$$

Step 3: Why equality may fail when $\lambda = 0$.
If $\lambda = 0$, then $E_0 = \ker(T)$, and

$$T[E_0] = T[\ker(T)] = \{\, T(v) : v \in \ker(T) \,\} = \{0\}.$$

So $T[E_0]$ is *always* contained in $\{0\}$, while E_0 may be larger than $\{0\}$ (for example, when T is not injective). In that case $T[E_0] \subsetneq E_0$, so equality fails.

Thus we have $T[E_\lambda] \subset E_\lambda$ for all λ, and equality $T[E_\lambda] = E_\lambda$ holds exactly when $\lambda \neq 0$.

4. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

Suppose $\vec{v}_1, \vec{v}_2$ are linearly independent vectors in $\mathbb{R}^3$. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by orthogonal projection onto the 2-dimensional subspace $V = \operatorname{span}(\vec{v}_1, \vec{v}_2)$.

(a) The geometric multiplicity of 1 is 1.

Solution: FALSE.
Since T is orthogonal projection onto V, we have

$$T(\vec{v}) = \vec{v} \quad \text{for all } \vec{v} \in V.$$

Thus every nonzero vector in V is an eigenvector with eigenvalue 1, so

$$E_1 = V.$$

Because V is a 2-dimensional subspace of $\mathbb{R}^3$, we have

$$\operatorname{gemu}(1) = \dim(E_1) = \dim(V) = 2.$$

Therefore the geometric multiplicity of 1 is 2, not 1, so the statement is false.

(b) The geometric multiplicity of 0 is 1.

Solution: TRUE.
For orthogonal projection T onto the plane V, every vector orthogonal to V is mapped to $\vec{0}$. Let $V^\perp$ denote the orthogonal complement of V in $\mathbb{R}^3$. Then

$$E_0 = \{\vec{w} \in \mathbb{R}^3 : T(\vec{w}) = 0\} = V^\perp.$$

Since V is a 2-dimensional subspace of $\mathbb{R}^3$, its orthogonal complement $V^\perp$ is 1-dimensional. Therefore

$$\text{gemu}(0) = \dim(E_0) = \dim(V^\perp) = 1.$$

So the statement is true.

(c) The geometric multiplicity of -1 is 1.

Solution: FALSE.

An eigenvalue λ of T must satisfy

$$T(\vec{x}) = \lambda \vec{x}.$$

But T is an orthogonal projection onto V, so its only possible eigenvalues are 1 and 0:

- If $\vec{x} \in V$ and $\vec{x} \neq 0$, then $T(\vec{x}) = \vec{x}$, so $\lambda = 1$.
- If $\vec{x} \in V^\perp$ and $\vec{x} \neq 0$, then $T(\vec{x}) = 0$, so $\lambda = 0$.

There is no nonzero vector $\vec{x}$ such that $T(\vec{x}) = -\vec{x}$, so -1 is not an eigenvalue of T. Thus

$$E_{-1} = \{\vec{0}\},$$

and the geometric multiplicity of -1 is

$$\text{gemu}(-1) = \dim(E_{-1}) = 0,$$

not 1. Hence the statement is false.