Math 217 Fall 2025

Quiz 27 – Solutions

Dr. Samir Donmazov

1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose $m \in \mathbb{N}$. An $m \times m$ matrix A is *orthogonal* if

   Solution: its transpose is its inverse:
   $$A^\top A \;=\; I_m \qquad \text{(equivalently } AA^\top = I_m\text{)}.$$

   (b) A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_r\}$ in $\mathbb{R}^n$ is *orthonormal* provided that

   Solution: each vector has unit length and distinct vectors are orthogonal:
   $$\vec{v}_i \cdot \vec{v}_j \;=\; \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

   (c) Let $T : V \to V$ be linear and $\mathfrak{B} = (v_1, \ldots, v_n)$ an ordered basis of V. The *matrix of T with respect to $\mathfrak{B}$* is

   Solution: the $n \times n$ matrix $[T]_{\mathfrak{B}}$ whose j-th column is the $\mathfrak{B}$-coordinate column vector of $T(v_j)$:
   $$[T]_{\mathfrak{B}} \;=\; \big[\, [T(v_1)]_{\mathfrak{B}} \quad [T(v_2)]_{\mathfrak{B}} \quad \cdots \quad [T(v_n)]_{\mathfrak{B}} \,\big].$$
   Equivalently, for every $v \in V$ we have $[T(v)]_{\mathfrak{B}} = [T]_{\mathfrak{B}}\, [v]_{\mathfrak{B}}$.

2. (a) Let V be a vector space and $\mathcal{S} = (w_1, \ldots, w_m)$ a spanning list for V. Suppose w_j is in the span of $\mathcal{S}' = (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_m)$. Prove that $\mathcal{S}'$ also spans V.

   Solution: Because $\mathcal{S}$ spans V, every $v \in V$ can be written as
   $$v \;=\; a_1 w_1 + \cdots + a_{j-1} w_{j-1} + a_j w_j + a_{j+1} w_{j+1} + \cdots + a_m w_m.$$
   By hypothesis, $w_j = \sum_{i \neq j} c_i w_i$ for some scalars c_i. Substituting,
   $$v \;=\; \sum_{i \neq j} a_i w_i \;+\; a_j \sum_{i \neq j} c_i w_i \;=\; \sum_{i \neq j} (a_i + a_j c_i) w_i,$$
   which is a linear combination of vectors in $\mathcal{S}'$. Thus every $v \in V$ lies in $\mathrm{Span}(\mathcal{S}')$, and $\mathcal{S}'$ spans V.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

(b) Suppose $m \in \mathbb{N}$ and $O : \mathbb{R}^m \to \mathbb{R}^m$ is an orthogonal transformation. Show that O is surjective.

Solution: Orthogonality means $\langle O(x), O(y)\rangle = \langle x, y\rangle$ for all x, y. In particular, if $O(x) = 0$ then

$$\|x\|^2 = \langle x, x\rangle = \langle O(x), O(x)\rangle = \|0\|^2 = 0,$$

so $x = 0$. Hence O is injective. Any injective linear map $\mathbb{R}^m \to \mathbb{R}^m$ is automatically surjective (rank–nullity). Alternatively, in matrix form $Q^\top Q = I_m$, so Q is invertible and O^{-1} exists; thus O is bijective and hence surjective.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$ with

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(a) Let $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$. The matrix $[T]_\mathcal{E}$ is orthogonal.

Solution: TRUE. Here $[T]_\mathcal{E} = A$. Compute

$$A^\top A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

So A is orthogonal (indeed a $90°$ counterclockwise rotation).

(b) Let $\mathcal{B} = (\vec{e}_1, \vec{e}_2 + 5\vec{e}_1)$. The matrix $[T]_\mathcal{B}$ is orthogonal.

Solution: FALSE. With $P = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ whose columns are $\mathcal{B}$, we have

$$[T]_\mathcal{B} = P^{-1}AP = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -5 & -26 \\ 1 & 5 \end{bmatrix}.$$

Then

$$[T]_\mathcal{B}^\top [T]_\mathcal{B} = \begin{bmatrix} 26 & 135 \\ 135 & 701 \end{bmatrix} \neq I_2.$$

So it is not orthogonal. (In general, the matrix of an orthogonal map is orthogonal *iff* the chosen basis is orthonormal.)

(c) Let $\mathcal{C} = (\vec{u}_1, \vec{u}_2)$ be any orthonormal basis of $\mathbb{R}^2$. The matrix $[T]_\mathcal{C}$ is orthogonal.

Solution: TRUE. If Q is the orthogonal change-of-basis matrix with columns $\vec{u}_1, \vec{u}_2$, then

$$[T]_\mathcal{C} = Q^\top AQ.$$

Hence

$$[T]_\mathcal{C}^\top [T]_\mathcal{C} = (Q^\top AQ)^\top (Q^\top AQ) = Q^\top A^\top AQ = Q^\top IQ = I,$$

because A is orthogonal and $Q^\top Q = I$. Thus $[T]_\mathcal{C}$ is orthogonal.