# Math 217.003 F25
# Quiz 29 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) An *inner product* on a vector space $V$ is ...

   **Solution:** (Over $\mathbb{R}$.) A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that for all $\vec{u}, \vec{v}, \vec{w} \in V$ and all $a, b \in \mathbb{R}$:

   - **Bilinearity:**
   $$\langle a\vec{u} + b\vec{v},\, \vec{w} \rangle = a\langle \vec{u}, \vec{w} \rangle + b\langle \vec{v}, \vec{w} \rangle, \quad \langle \vec{u},\, a\vec{v} + b\vec{w} \rangle = a\langle \vec{u}, \vec{v} \rangle + b\langle \vec{u}, \vec{w} \rangle;$$
   - **Symmetry:** $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$;
   - **Positive–definiteness:** $\langle \vec{v}, \vec{v} \rangle \geq 0$ for all $\vec{v}$, and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

   (b) An *inner product space* is ...

   **Solution:** A vector space $V$ together with a specified inner product $\langle \cdot, \cdot \rangle$ on $V$. We usually denote it by $(V, \langle \cdot, \cdot \rangle)$.

2. Prove that if $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ is an orthonormal basis of the inner product space $V$, then
$$\langle \vec{x}, \vec{y} \rangle \;=\; [\vec{x}]_{\mathcal{U}} \cdot [\vec{y}]_{\mathcal{U}}$$
for all $\vec{x}, \vec{y} \in V$.

**Solution:** Because $\mathcal{U}$ is an orthonormal basis, every vector $\vec{x} \in V$ has an expansion
$$\vec{x} = \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i \rangle \, \vec{u}_i,$$
so its coordinate vector in this basis is
$$[\vec{x}]_{\mathcal{U}} = \begin{bmatrix} \langle \vec{x}, \vec{u}_1 \rangle \\ \vdots \\ \langle \vec{x}, \vec{u}_n \rangle \end{bmatrix}.$$
Similarly,
$$\vec{y} = \sum_{j=1}^{n} \langle \vec{y}, \vec{u}_j \rangle \, \vec{u}_j, \qquad [\vec{y}]_{\mathcal{U}} = \begin{bmatrix} \langle \vec{y}, \vec{u}_1 \rangle \\ \vdots \\ \langle \vec{y}, \vec{u}_n \rangle \end{bmatrix}.$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Then

$$\begin{aligned}\langle \vec{x}, \vec{y}\rangle &= \left\langle \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i\rangle\, \vec{u}_i,\ \sum_{j=1}^{n} \langle \vec{y}, \vec{u}_j\rangle\, \vec{u}_j \right\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle \vec{x}, \vec{u}_i\rangle\, \langle \vec{y}, \vec{u}_j\rangle\, \langle \vec{u}_i, \vec{u}_j\rangle.\end{aligned}$$

Since the basis is orthonormal, $\langle \vec{u}_i, \vec{u}_j\rangle = \delta_{ij}$, so the double sum collapses:

$$\langle \vec{x}, \vec{y}\rangle = \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i\rangle\, \langle \vec{y}, \vec{u}_i\rangle.$$

But this is exactly the dot product of the coordinate vectors:

$$[\vec{x}]_{\mathcal{U}} \cdot [\vec{y}]_{\mathcal{U}} = \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i\rangle\, \langle \vec{y}, \vec{u}_i\rangle.$$

Hence $\langle \vec{x}, \vec{y}\rangle = [\vec{x}]_{\mathcal{U}} \cdot [\vec{y}]_{\mathcal{U}}$ as claimed.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Suppose $n \in \mathbb{N}$. The map $f\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ that sends $A$ to $\det(A)$ is linear.

**Solution: FALSE.** For linearity we would need, in particular,

$$\det(A + B) = \det(A) + \det(B)$$

for all $A, B$. Take $n = 2$ and

$$A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = I_2.$$

Then

$$\det(A) = 1, \quad \det(B) = 1, \quad A + B = 2I_2, \quad \det(A + B) = \det(2I_2) = 2^2 = 4.$$

But $4 \neq 1 + 1 = 2$, so $\det(A + B) \neq \det(A) + \det(B)$. Thus det is not linear.

(b) Suppose $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ is a basis of the inner product space $V$, then

$$\langle \vec{x}, \vec{y}\rangle \ = \ [\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}}.$$

**Solution: FALSE.** The formula

$$\langle \vec{x}, \vec{y}\rangle = [\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}}$$

holds exactly when the basis $\mathcal{B}$ is *orthonormal*. For a general basis one has instead

$$\langle \vec{x}, \vec{y}\rangle = [\vec{x}]_{\mathcal{B}}^{\top} G_{\mathcal{B}} [\vec{y}]_{\mathcal{B}},$$

where $G_{\mathcal{B}}$ is the Gram matrix $G_{\mathcal{B}} = (\langle \vec{v}_i, \vec{v}_j \rangle)_{i,j}$. If $G_{\mathcal{B}} \neq I$, the dot product of coordinates will not equal the inner product.

*Concrete counterexample in* $\mathbb{R}^2$*:* Consider the standard inner product and the basis

$$\mathcal{B} = (\vec{v}_1, \vec{v}_2) = (\vec{e}_1,\ \vec{e}_1 + \vec{e}_2),$$

which is not orthonormal. Let $\vec{x} = \vec{y} = \vec{e}_2$. Write $\vec{e}_2 = a\vec{e}_1 + b(\vec{e}_1 + \vec{e}_2)$. Then

$$\vec{e}_2 = (a+b)\vec{e}_1 + b\vec{e}_2,$$

so $b = 1$ and $a = -1$. Hence $[\vec{e}_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and

$$[\vec{e}_2]_{\mathcal{B}} \cdot [\vec{e}_2]_{\mathcal{B}} = (-1)^2 + 1^2 = 2.$$

But $\langle \vec{e}_2, \vec{e}_2 \rangle = 1$. Therefore the claimed equality fails.

(c) If $A$ is a square matrix such that $\det(A) = -1$, then $A$ is orthogonal.

**Solution: FALSE.** An orthogonal matrix $A$ must satisfy $A^\top A = I$; having determinant $\pm 1$ is necessary but not sufficient. We can find a matrix with determinant $-1$ that is not orthogonal.

Consider

$$A = \begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Then

$$\det(A) = (-2) \cdot \tfrac{1}{2} = -1,$$

but

$$A^\top A = \begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \neq I.$$

Thus $A$ is not orthogonal, even though $\det(A) = -1$.