Math 217.003 F25
Quiz 34 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) The linear transformation $T : V \to V$ of a finite dimensional vector space V is *diagonalizable* if ...

   > **Solution:** The linear transformation $T : V \to V$ is *diagonalizable* if there exists a basis $\mathcal{B}$ of V such that the matrix of T with respect to $\mathcal{B}$, denoted $[T]_{\mathcal{B}}$, is a diagonal matrix.

   (b) Suppose $\lambda \in \mathbb{R}$ and $T : V \to V$ is a linear transformation of a finite dimensional vector space V. The *geometric multiplicity* "gemu(λ)" of λ is ...

   > **Solution:** The *geometric multiplicity* of the eigenvalue λ is the dimension of its eigenspace
   >
   > $$E_\lambda = \{\, v \in V : T(v) = \lambda v \,\}.$$
   >
   > In other words,
   > $$\mathrm{gemu}(\lambda) = \dim(E_\lambda).$$

   (c) Suppose $n \in \mathbb{N}$. An $n \times n$ matrix A is *diagonalizable* if ...

   > **Solution:** An $n \times n$ matrix A is *diagonalizable* if A is similar to a diagonal matrix.

   (d) Suppose $m \in \mathbb{N}$. An $m \times m$ matrix A is *symmetric* if ...

   > **Solution:** An $m \times m$ matrix A is *symmetric* if it is equal to its transpose:
   >
   > $$A^\top = A.$$
   >
   > Equivalently, in terms of entries, A is symmetric if $a_{ij} = a_{ji}$ for all $1 \le i, j \le m$.

2. (a) Let $V \xrightarrow{T} V$ be a linear transformation, where V is a finite dimensional vector space.

   (i) Define the determinant of T.

   (ii) Show the determinant of T you defined above is independent of any choices you made.

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*For full credit, please write out fully what you mean instead of using shorthand phrases.

Solution: (i) Definition.

Let $\dim(V) = n$, and let $\mathcal{B} = (v_1, \ldots, v_n)$ be any basis of V. Consider the matrix of T with respect to $\mathcal{B}$,

$$[T]_{\mathcal{B}} \in M_{n \times n}(\mathbb{R}).$$

We define the *determinant of T* to be

$$\det(T) := \det\left([T]_{\mathcal{B}}\right).$$

(ii) Independence of basis.

We must show that this number does not depend on the choice of basis $\mathcal{B}$.

Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of V, and let $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ denote the matrices of T in these bases. There exists an invertible change-of-basis matrix S such that

$$[T]_{\mathcal{C}} = S\,[T]_{\mathcal{B}}\,S^{-1}.$$

(In words: the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ are similar.)

Taking determinants and using the multiplicative property of determinants, we obtain

$$\det\left([T]_{\mathcal{C}}\right) = \det\left(S[T]_{\mathcal{B}}S^{-1}\right) = \det(S)\,\det\left([T]_{\mathcal{B}}\right)\,\det(S^{-1}).$$

Since $\det(S^{-1}) = 1/\det(S)$, this simplifies to

$$\det\left([T]_{\mathcal{C}}\right) = \det\left([T]_{\mathcal{B}}\right).$$

Thus the determinant of T computed in basis $\mathcal{C}$ equals the determinant computed in basis $\mathcal{B}$. Because any two choices of basis are related by an invertible change-of-basis matrix, the value $\det(T)$ is well-defined and independent of the basis used.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   Suppose V is a vector space and $T\colon V \to V$ is a linear transformation. For all $\lambda \in \mathbb{R}$ we have $T[E_\lambda] \subset E_\lambda$ with equality if and only if $\lambda \neq 0$.

Solution: False.

The inclusion $T[E_\lambda] \subset E_\lambda$ is true for every λ, because if $v \in E_\lambda$ then $T(v) = \lambda v$, and hence

$$T(T(v)) = T(\lambda v) = \lambda T(v),$$

so $T(v) \in E_\lambda$.

However, the additional claim "with equality if and only if $\lambda \neq 0$" is false. For example, let $T = I$ on $\mathbb{R}^n$. Then the 0-eigenspace is

$$E_0 = \ker(I) = \{0\}.$$

Thus

$$T[E_0] = \{0\} = E_0,$$

so equality holds even though $\lambda = 0$. This contradicts the statement's "if and only if" condition.

Therefore, the statement is **false**.

4. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

Suppose $\vec{v}_1, \vec{v}_2$ are linearly independent vectors in $\mathbb{R}^3$. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by orthogonal projection onto the 2-dimensional subspace $V = \text{span}(\vec{v}_1, \vec{v}_2)$.

(a) The geometric multiplicity of 1 is 1.

> **Solution: FALSE.**
>
> Since T is orthogonal projection onto V, we have
>
> $$T(\vec{v}) = \vec{v} \quad \text{for all } \vec{v} \in V.$$
>
> Thus every nonzero vector in V is an eigenvector with eigenvalue 1, so
>
> $$E_1 = V.$$
>
> Because V is a 2-dimensional subspace of $\mathbb{R}^3$, we have
>
> $$\text{gemu}(1) = \dim(E_1) = \dim(V) = 2.$$
>
> Therefore the geometric multiplicity of 1 is 2, not 1, so the statement is false.

(b) The geometric multiplicity of 0 is 1.

> **Solution: TRUE.**
>
> For orthogonal projection T onto the plane V, every vector orthogonal to V is mapped to $\vec{0}$. Let $V^\perp$ denote the orthogonal complement of V in $\mathbb{R}^3$. Then
>
> $$E_0 = \{\vec{w} \in \mathbb{R}^3 : T(\vec{w}) = 0\} = V^\perp.$$
>
> Since V is a 2-dimensional subspace of $\mathbb{R}^3$, its orthogonal complement $V^\perp$ is 1-dimensional. Therefore
>
> $$\text{gemu}(0) = \dim(E_0) = \dim(V^\perp) = 1.$$
>
> So the statement is true.

(c) The geometric multiplicity of -1 is 1.

> **Solution: FALSE.**
>
> An eigenvalue λ of T must satisfy
>
> $$T(\vec{x}) = \lambda \vec{x}.$$
>
> But T is an orthogonal projection onto V, so its only possible eigenvalues are 1 and 0:
>
> - If $\vec{x} \in V$ and $\vec{x} \neq 0$, then $T(\vec{x}) = \vec{x}$, so $\lambda = 1$.

- If $\vec{x} \in V^{\perp}$ and $\vec{x} \neq 0$, then $T(\vec{x}) = 0$, so $\lambda = 0$.

There is no nonzero vector $\vec{x}$ such that $T(\vec{x}) = -\vec{x}$, so -1 is not an eigenvalue of T. Thus

$$E_{-1} = \{\vec{0}\},$$

and the geometric multiplicity of -1 is

$$\text{gemu}(-1) = \dim(E_{-1}) = 0,$$

not 1. Hence the statement is false.