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How to impose physically coherent initial conditions to a fractional system?

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ABSTRACT

In this paper, it is shown that neither Riemann–Liouville nor Caputo definitions for fractional differentiation can be used to take into account initial conditions in a convenient way from a physical point of view. This demonstration is done on a counter-example. Then the paper proposes a representation for fractional order systems that lead to a physically coherent initialization for the considered systems. This representation involves a classical linear integer system and a system described by a parabolic equation. It is thus also shown that fractional order systems are halfway between these two classes of systems, and are particularly suited for diffusion phenomena modelling.

1. Introduction

Fractional calculus has found applications in assorted fields, like engineering [23], physics [21], finance [25], chemistry [24], bioengineering [22]. Fractional order systems are usually represented by fractional differential equations or pseudo state space descriptions. However, other representations of these systems exist among which the "diffusive representation", mainly studied by Montseny and Matignon [1,2]. In this paper, this representation is obtained from the impulse response of a fractional order system. Then, changes of variables are used to demonstrate that fractional order system output is the combination of a linear integer system output and a parabolic differential system output. From a physical point of view, linear fractional order systems are neither quite conventional linear systems nor quite conventional parabolic systems. They are halfway between these two classes of systems. That explains why fractional order systems are particularly suited for modelling of diffusion phenomena.

Using the representation previously introduced, the well known "time memory effect" translating that the behaviour of a fractional system takes into account the system past on an infinite time interval, is converted to "a spatial memory effect" translating that the behaviour of a fractional system results in the behaviour of an infinite number of systems spatially distributed. As a consequence, a coherent initialization can be proposed using this representation. This is to be opposed to the initialization proposed by the Riemann–Liouville or Caputo definitions [3,4] that do not permit to take into account initial condition correctly (compatible with the system physics). This last assertion is demonstrated in the paper on a counter-example.

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2. How to take into account initial conditions?

In a series of papers, Lorenzo and Hartley [5–8] have demonstrated that initial conditions are not taken into account in the same way whether Riemann–Liouville or Caputo definitions are considered. The goal of this section is to go further and to demonstrate that neither Riemann–Liouville nor Caputo definitions permit to take into account initial conditions in a coherent way with the system physics. The demonstration is done on a counter-example based on the system defined by the differential equation

$$d^{\gamma}x(t)/dt^{\gamma} = u(t), \quad 0 < \gamma < 1. \tag{1}$$

As in Lorenzo and Hartley work, the demonstration is based on the shift of the time origin. System (1) is supposed to be at rest when time $t < 0$. Input $u(t)$ is also supposed to be defined by

$$u(t) = H(t) - H(t - t_0), \tag{2}$$

where $H(.)$ denotes the Heaviside function. Time response of system (1) is thus defined by

$$x(t) = \frac{t^{\gamma}}{\Gamma(\gamma+1)}H(t) - \frac{(t-t_0)^{\gamma}}{\Gamma(\gamma+1)}H(t-t_0). \tag{3}$$

The change of variable

$$\tau = t - 2t_0 \tag{4}$$

is now used so that at time $\tau = 0$, the system is not at rest.

Using Riemann–Liouville definition, Laplace transform of the fractional derivative of $x(\tau)$ is defined by [9]

$$\mathscr{L}\{D^{\gamma}x(\tau)\} = s^{\gamma}\bar{x}(s) - \sum_{k=0}^{n-1} s^k [D^{\gamma-k-1}x(\tau)]_{\tau=0} \tag{5}$$

with $n - 1 < \gamma < n$ and where $\bar{x}(s)$ denotes the Laplace transform of $x(\tau)$ with zero initial conditions. Laplace transform applied to Eq. (1), leads to

$$s^{\gamma}\bar{x}(s) - \sum_{k=0}^{n-1} s^k [D^{\gamma-k-1}x(\tau)]_{\tau=0} = 0 \tag{6}$$

(here $n = 1$) and the corresponding time response is defined by

$$x(\tau) = \mathscr{L}^{-1}\left\{\frac{1}{s^{\gamma}}\right\}[I^{1-\gamma}x(\tau)]_{\tau=0}, \tag{7}$$

where

$$[I^{1-\gamma}x(\tau)]_{\tau=0} = \mathscr{L}^{-1}\left\{\frac{1}{s^{1-\gamma}}\left(\frac{e^{2t_0 s}}{s^{\gamma+1}} - \frac{e^{t_0 s}}{s^{\gamma+1}}\right)\right\}_{\tau=0} = \mathscr{L}^{-1}\left\{\left(\frac{e^{2t_0 s}}{s^2} - \frac{e^{t_0 s}}{s^2}\right)\right\}_{\tau=0} = 2t_0 - t_0 = t_0. \tag{8}$$

The time response of system (1) is thus given by

$$x(\tau) = t_0 \frac{\tau^{\gamma-1}}{\Gamma(\gamma)}, \quad \tau \geqslant 0. \tag{9}$$

Using Caputo definition, Laplace transform of the fractional derivative of $x(\tau)$ is defined by [9]

$$\mathscr{L}\{D^{\gamma}x(t)\} = s^{\gamma}\bar{x}(s) - \sum_{k=0}^{n-1} s^{\gamma-k-1}[D^k x(\tau)]_{\tau=0}. \tag{10}$$

Laplace transform applied to Eq. (1), leads to

$$s^{\gamma}\bar{x}(s) - \sum_{k=0}^{n-1} s^{\gamma-k-1}[D^k x(\tau)]_{\tau=0} = 0 \tag{11}$$

(here $n = 1$) and the corresponding time response is given by

$$x(\tau) = \mathscr{L}^{-1}\left\{\frac{s^{\gamma-1}}{s^{\gamma}}\right\}[x(\tau)]_{\tau=0}, \tag{12}$$

where, using (3),

$$[x(\tau)]_{\tau=0} = \frac{(2t_0)^{\gamma}}{\Gamma(\gamma+1)} - \frac{(2t_0 - t_0)^{\gamma}}{\Gamma(\gamma+1)}. \tag{13}$$

Time response of system (1) is thus given by

$$x(\tau) = \frac{(2t_0)^\gamma - t_0^\gamma}{\Gamma(\gamma+1)} H(\tau), \quad \tau \geqslant 0. \tag{14}$$

In Fig. 1, system (1) response to the input given by relation (2) is compared to the response obtained using Riemann–Liouville and Caputo definitions, initial conditions being taken into account. This figure highlights that the three time responses are completely different.

Using a counter-example, it has been demonstrated in this section that neither the Riemann–Liouville nor the Caputo definitions are compatible with the real behaviour of a fractional system. Using these definitions, system response $u(t)$ does not mach system response to non-zero initial conditions. To solve this problem another representation is introduced in the following section.

3. On representation of fractional systems

Fractional systems are usually described by fractional differential equations or state space like representations. In this section another representation of fractional systems is used. It is based on the impulse response of a fractional system that is first detailed for a fractional system of the first kind. Then a generalization to a larger class of fractional system is discussed and finally a physical interpretation linked to the obtained representation is proposed.

3.1. Impulse response of a fractional system of the first kind

Consider the following fractional system of the first kind:

$$G(s) = \frac{1}{s^\gamma - a} \tag{15}$$

with $0 < \gamma < 2$ and $a > 0$ (stability condition).

Using the poles p_k definition given in Section 3.2, the impulse response of such a system is given by [10]

$$g(t) = \frac{1}{a^\gamma} \sum_k p_k e^{tp_k} + \frac{\sin(\gamma\pi)}{\pi} \int_0^\infty \frac{x^\gamma e^{-tx}}{a^2 - 2ax^\gamma \cos(\gamma\pi) + x^{2\gamma}} dx. \tag{16}$$

Response of system (1) to an input $u(t)$ is defined as the convolution product of the impulse response $g(t)$ and the input $u(t)$:

$$y(t) = \int_0^t g(t-\tau) u(\tau) d\tau, \tag{17}$$

and thus using relation (16):

$$y(t) = \int_0^t \frac{1}{a^\gamma} \sum_k p_k e^{(t-\tau)p_k} u(\tau) d\tau + \int_0^t \frac{\sin(\gamma\pi)}{\pi} \left(\int_0^\infty \frac{x^\gamma e^{-(t-\tau)x}}{a^2 - 2ax^\gamma \cos(\gamma\pi) + x^{2\gamma}} dx \right) u(\tau) d\tau \tag{18}$$

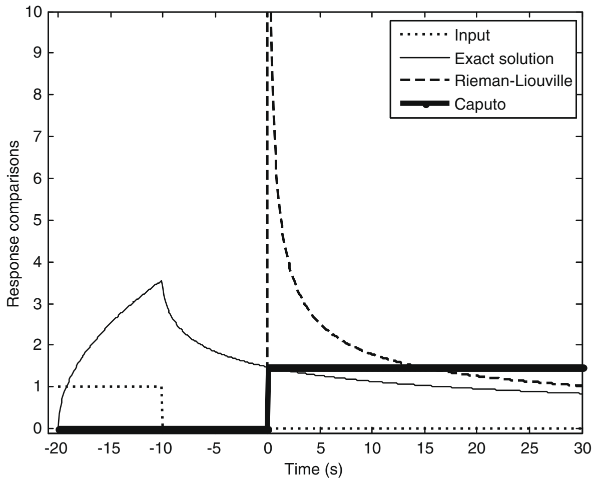

Fig. 1. Comparison of the exact response of system (1) with the responses obtained with Riemann–Liouville and Caputo definitions ($t_0 = 10$ s).

or, through an integral permutation:

$$y(t) = \int_0^t \frac{1}{a^\gamma} \sum_k p_k e^{(t-\tau)p_k} u(\tau) d\tau + \int_0^\infty \frac{\sin(\gamma\pi)}{\pi} \frac{x^\gamma}{a^2 - 2ax^\gamma \cos(\gamma\pi) + x^{2\gamma}} \left(\int_0^t e^{-(t-\tau)x} u(\tau) d\tau \right) dx. \tag{19}$$

Let

$$w(t,x) = \int_0^t e^{-(t-\tau)x} u(\tau) d\tau. \tag{20}$$

$w(t,x)$ is thus solution of the differential equation

$$\dot{w}(t,x) = -xw(t,x) + u(t). \tag{21}$$

Using relations (19) and (21), the following state space representation can be obtained for the system (15):

$$\begin{aligned} \begin{bmatrix} \dot{w}_1(t) \\ \vdots \\ \dot{w}_k(t) \\ \dot{w}(t,x) \end{bmatrix} &= \begin{bmatrix} p_1 & & & (0) \\ & \ddots & & \\ & & p_k & \\ (0) & & & -x \end{bmatrix} \begin{bmatrix} w_1(t) \\ \vdots \\ w_k(t) \\ w(t,x) \end{bmatrix} + \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = y_1(t) + y_2(t) &= \begin{bmatrix} \frac{p_1}{a^\gamma} & \cdots & \frac{p_k}{a^\gamma} \end{bmatrix} \begin{bmatrix} w_1(t) \\ \vdots \\ w_k(t) \end{bmatrix} + \frac{\sin(\gamma\pi)}{\pi} \int_0^\infty \frac{x^\gamma w(t,x)}{a^\gamma - 2ax^\gamma \cos(\gamma\pi) + x^{2\gamma}} dx. \end{aligned} \tag{22}$$

Such a representation is connected to the diffusive representation introduced by Montseny and Matignon [1,11,2].

3.2. Generalization to a larger class of fractional systems

Representation (22) defined for system (15) can be generalized to a larger class of fractional systems defined by

$$G(s) = \frac{B(s)}{A(s)} \tag{23}$$

with $B(s) = \sum_{l=0}^{r} q_l s^{\beta_l}$ and $A(s) = \sum_{k=0}^{m} r_k s^{\alpha_k}$ where $\beta_{l+1} \geqslant \beta_l \geqslant 0$ and $\alpha_{k+1} \geqslant \alpha_k \geqslant 0$.

Let $\{p_1, \ldots, p_k\}$ be the poles of the transfer function $G(s)$. Computation of these poles permits to define output $y_1(t)$ of (22). Under commensurate order hypothesis and through a partial fraction decomposition, transfer function $G(s)$ appears as a linear combination of terms

$$(1/(s^n - \lambda_l))^{q_i}, \tag{24}$$

where λ_l and q_i represent respectively the s^n poles of $G(s)$ and their order.

Poles p_k of sub-system (24) are then defined by $p_k = |p_k| e^{j\theta_k}$, with

$$\begin{cases} |p_k| = (|\lambda_l|)^{1/n}, \\ \theta_k = \frac{\arg(\lambda_l)}{n} + \frac{2k\pi}{n}, \\ -\frac{n}{2} - \frac{\arg(\lambda_l)}{2\pi} < k < \frac{n}{2} - \frac{\arg(\lambda_l)}{2\pi}. \end{cases} \tag{25}$$

Then according to [2], output $y_2(t)$ in (22) is defined by

$$y_2(t) = \int_0^\infty \mu(x) w(t,x) dx \tag{26}$$

with

$$\mu(x) = \frac{1}{2i\pi} \left[G((-x)^-) - G((-x)^+) \right] = \frac{1}{\pi} \frac{\sum_{k=0}^{m} \sum_{l=0}^{q} a_k q_l \sin((\alpha_k - \beta_l)\pi) x^{\alpha_k + \beta_l}}{\sum_{k=0}^{m} a_k^2 x^{2\alpha_k} + \sum_{0 \leqslant k < l < m} 2a_k a_l \cos((\alpha_k - \alpha_l)\pi) x^{\alpha_k + \alpha_l}}. \tag{27}$$

3.3. Physical interpretation

Several mathematical and physical interpretations of fractional differentiation and of fractional systems exist in the literature [12–18]. A demonstration of Montseny [1] for a fractional integrator is now adapted to deduce a physical interpretation of a fractional system. Consider a fractional system described by (exponential part omitted)

$$\begin{cases} \dot{w}(t) = -xw(t,x) + u(t), \\ y_2(t) = \int_0^\infty \mu(x) w(t,x) dx, \end{cases} \quad x \in R^+. \tag{28}$$

Initial condition are defined for this system by $w(0,x) = \rho(x)$.

System (28) can also be written as

$$\begin{cases} \dot{w}(t,x) = -xw(t,x) + u(t), \\ y_2(t) = \frac{1}{2}\int_0^\infty \mu(x)w(t,x)dx + \frac{1}{2}\int_0^\infty \mu(x)w(t,x)dx. \end{cases} \tag{29}$$

Using the following changes of variable $z = \sqrt{x}/(2\pi)$ and $z = -\sqrt{x}/(2\pi)$ applied respectively to the first and the second integral of (29), system (28) is also equivalent to

$$\begin{cases} \dot{\psi}(t,z) = -4\pi^2 z^2 \psi(t,z) + u(t) \\ y_2(t) = \int_{-\infty}^{\infty} 4\pi^2 z\mu(4\pi z^2)\psi(t,z)dz \end{cases} \tag{30}$$

with $\psi(t,z) = w(t, 4\pi^2 z^2)$ and $\psi(0,z) = \rho(4\pi^2 z^2),\ z \in R$.

If $\Psi(t,z)$ denotes spatial Fourier transform of function $\phi(t,\zeta)$ such that

$$\psi(t,z) = \mathscr{F}\{\phi(t,\zeta)\} = \int_{-\infty}^{\infty} \phi(t,\zeta)e^{-iz\zeta}d\zeta, \tag{31}$$

system (30) is also given by

$$\begin{cases} \mathscr{F}\left\{\frac{\partial\phi(t,\zeta)}{\partial t}\right\} = -4\pi^2 z^2 \mathscr{F}\{\phi(t,\zeta)\} + u(t), \\ y_2(t) = \int_{-\infty}^{\infty} 4\pi^2 z\mu(4\pi z^2)\mathscr{F}\{\phi(t,\zeta)\}dz. \end{cases} \tag{32}$$

Given that

$$y_2(t) = \int_{-\infty}^{\infty} 4\pi^2 z\mu(4\pi^2 z^2)\left(\int_{-\infty}^{\infty} \phi(t,\zeta)e^{-iz\zeta}d\zeta\right)dz = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} 4\pi^2 z\mu(4\pi^2 z^2)e^{-iz\zeta}dz\right)\phi(t,\zeta)d\zeta \tag{33}$$

inverse spatial Fourier transform applied to (32), leads to

$$\begin{cases} \frac{\partial\phi(t,\zeta)}{\partial t} = \frac{\partial^2\phi(t,\zeta)}{\partial\zeta^2} + u(t)\delta(\zeta) \\ y_2(t) = \int_{-\infty}^{\infty} m(\zeta)\phi(t,\zeta)d\zeta \end{cases} \tag{34}$$

with $m(\zeta) = \mathscr{F}^{-1}\{4\pi^2\zeta\mu(4\pi^2\zeta^2)\},\ \phi(\zeta,0) = \mathscr{F}^{-1}\{\rho(4\pi^2 z^2)\},\ \zeta \in R$.

If now the exponential part (whose output is $y_1(t)$ in (22)) is taken into account, then any fractional order system can be described by

$$\begin{gathered} \begin{bmatrix} \dot{w}_1(t) \\ \vdots \\ \dot{w}_k(t) \end{bmatrix} = \begin{bmatrix} p_1 & & (0) \\ & \ddots & \\ (0) & & p_k \end{bmatrix} \begin{bmatrix} w_1(t) \\ \vdots \\ w_k(t) \end{bmatrix} + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} u(t), \quad \frac{\partial\phi(t,\zeta)}{\partial t} = \frac{\partial^2\phi(t,\zeta)}{\partial\zeta^2} + u(t)\delta(\zeta), \\ y_1(t) + \begin{bmatrix} \frac{p_1}{a_1^{\gamma}} & \cdots & \frac{p_k}{a_k^{\gamma}} \end{bmatrix} \begin{bmatrix} w_1(t) \\ \vdots \\ w_k(t) \end{bmatrix}, \quad y_2(t) = \int_{-\infty}^{\infty} m(\zeta)\phi(t,\zeta)d\zeta, \quad y(t) = y_1(t) + y_2(t). \end{gathered} \tag{35}$$

Any fractional system can thus be seen as an infinite dimensional system described by a diffusion equation (parabolic differential equation [19]) associated to a classical linear (exponential) system.

Fig. 2 is a representation of the interpretation previously given for the system whose output is $y_2(t)$. $\phi(\zeta,t)$ is the system state (of infinite dimension), and $u(t)$ is both the input of a classical linear system and of the infinite dimensional system applied at $\zeta = 0$.

To conclude, it can be said that a fractional order system is at the boundary of two classes of systems: classical linear "exponential" rational systems and distributed parameters systems.

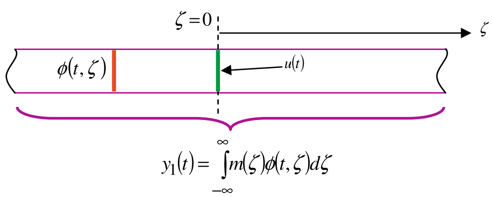

Fig. 2. Representation of system whose output is $y_2(t)$.

4. Interest for the initialization problem

The representation introduced in the previous section permits to take into account initial condition in a coherent way with system physics.

4.1. Initial conditions compatible with a physical system

In Section 2, it was demonstrated that neither Riemann–Liouville nor Caputo definitions, allow taking into account correctly the initial conditions. Using representation (22) it is proposed now to take into account the initial conditions in the following way:

$$y(t) = \sum_k \frac{p_k}{a^\gamma}\left(w_k(0)e^{p_k t} + \int_0^t e^{p_k(t-\tau)}u(\tau)\right) + \int_0^\infty \mu(x)\left(w(0,x)e^{-xt} + \int_0^t e^{-x(t-\tau)}u(\tau)d\tau\right)dx, \tag{36}$$

In the case of system (1) and given that no pole is generated, relation (36) becomes

$$y(t) = \int_0^\infty \mu(x)\left(w(0,x)e^{-xt} + \int_0^t e^{-x(t-\tau)}u(\tau)d\tau\right)dx \tag{37}$$

with $\mu(x) = \sin(\gamma\pi)/\pi x^\gamma$. Given (21), $w(0,x)$ is defined by

$$w(0,x) = (1 - e^{-2xt_0}) - (1 - e^{-xt_0}) = e^{-xt_0} - e^{-2xt_0} \tag{38}$$

and $y(\tau)$ (see (4) for a definition of τ) is

$$y(\tau) = \frac{\sin(\gamma\pi)}{\pi}\int_0^\infty \frac{(e^{-xt_0} - e^{-2xt_0})}{x^\gamma}e^{-xt}\,dx. \tag{39}$$

Fig. 3 compares the responses given by (39) and by (3). Such a comparison reveals that (39) permits a correct initialization of the system.

4.2. Initial conditions estimation

Obviously, an infinite number of initial conditions $w(0,x)$ (or $\phi(\xi,0)$ in the interpretation of Fig. 2), is required for initializing the system. Suppose that the problem is to find an approximation of these initial conditions. Using relation (36), the system output is defined by

$$y(t) = \sum_k \frac{p_k}{a^\gamma}w_k(0)e^{p_k t} + \int_0^\infty \mu(x)w(0,x)e^{-xt}\,dx + \sum_k \frac{p_k}{a^\gamma}\int_0^t e^{p_k(t-\tau)}u(\tau)d\tau + \int_0^\infty \mu(x)\int_0^t e^{-x(t-\tau)}u(\tau)d\tau\,dx. \tag{40}$$

Using the change of variable $x = e^{-z}$, and thus $dx = -e^{-z}dz$, Eq. (40) becomes

$$y(t) = \sum_k \frac{p_k}{a^\gamma}w_k(0)e^{p_k t} + \int_{-\infty}^\infty \mu(e^{-z})w(0,e^{-z})e^{-e^{-z}t}e^{-z}\,dz + \sum_k \frac{p_k}{a^\gamma}\int_0^t e^{p_k(t-\tau)}u(\tau)d\tau + \int_{-\infty}^\infty \mu(e^{-z})\int_0^t e^{-e^{-z}(t-\tau)}u(\tau)d\tau\,e^{-z}\,dz. \tag{41}$$

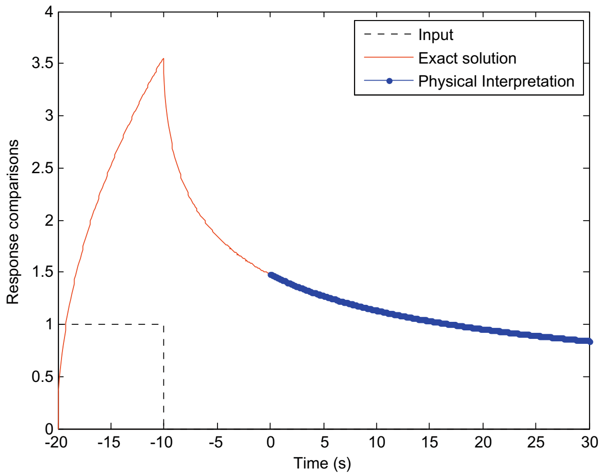

Fig. 3. Comparison of the exact response of system (1) with the responses obtained with relation (39).

An approximation of (41) is thus

$$
\begin{aligned}
y(t) \approx & \sum_k \frac{p_k}{a^\gamma} w_k(0) e^{p_k t} + \sum_{j=-N_1}^{N_2} \mu(e^{-j\Delta z}) w(0, e^{-j\Delta z}) e^{-e^{-j\Delta z} t} e^{-j\Delta z} \Delta z + \sum_k \frac{p_k}{a^\gamma} \int_0^t e^{p_k(t-\tau)} u(\tau) d\tau \\
& + \sum_{j=-N_1}^{N_2} \int_0^t \mu(e^{-j\Delta z}) e^{-e^{-j\Delta z}(t-\tau)} e^{-j\Delta z} \Delta z \, u(\tau) d\tau
\end{aligned} \tag{42}
$$

In the interpretation of Fig. 2, such an approximation of the initialization consists in considering a limited number of areas in which the initial conditions are constant (initial conditions being considered null elsewhere). This initialization is represented in Fig. 4.

If the system model, its input and its output are known, a least square problem can thus be defined for the computation of $w_k(0)$ and $w(0, e^{-j\Delta z})$.

4.3. Approximation of the initialization function

Suppose that the problem is now to compute an initialization function in order to restore the system state if initial conditions defined in Section 3.1 are taken into account.

Let $y(t)$ denotes the response of the system (at rest before $t = 0$) to the input $u(t)$. Let $y_{t0}(t)$ be the response of the system to the input $u(t)H(t - t_0)$. The problem now is to find an initialization function as in [6] denoted $\psi(t)$ such that $y(t) = y_{t0}(t) + \psi(t)$ with $t > t_0$ (Fig. 5).

An approximation of $\psi(t)$ was proposed for a fractional order integrator by [20] using a recursive RC network.

For a general fractional system, the approximation of such an initialization function can be obtained using the initialization introduced in Section 3.1.

Using relation (40) and the change of variable introduced before (41), the system output with zero initial conditions can be approximated by

$$
y(t) \approx \sum_k \frac{p_k}{a^\gamma} \int_0^t e^{p_k(t-\tau)} u(\tau) d\tau + \sum_{j=-N_1}^{N_2} \int_0^t \mu(e^{-j\Delta z}) e^{-e^{-j\Delta z}(t-\tau)} e^{-j\Delta z} \Delta z \, u(\tau) d\tau. \tag{43}
$$

Using Laplace transform relation (43) becomes

$$
Y(s) \approx \sum_k \frac{p_k}{a^\gamma} \frac{U(s)}{s + p_k} + \sum_{j=-N_1}^{N_2} \frac{\mu(e^{-j\Delta z}) \Delta z}{\frac{s}{\omega_k} + 1} U(s) \tag{44}
$$

with $\omega_k = e^{-j\Delta z}$. One can note that the change of variable defined before relation (41) permits an approximation of $y(t)$ involving a recursive distribution of poles. A recursive ratio $e^{-\Delta z}$ indeed exists between two consecutive poles.

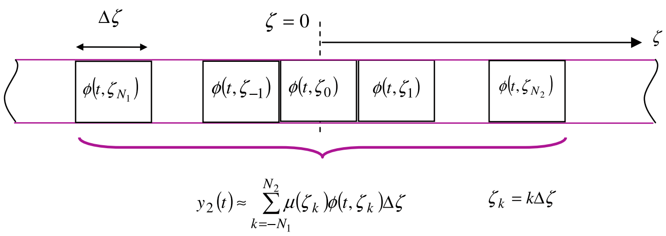

Fig. 4. Approximation of the initialization.

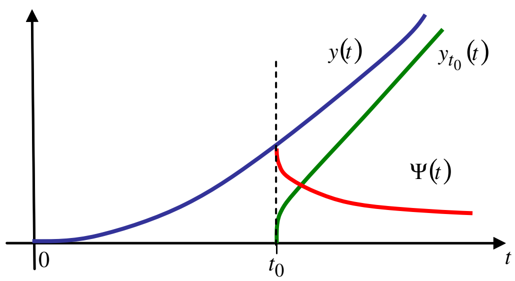

Fig. 5. Initialization function definition.

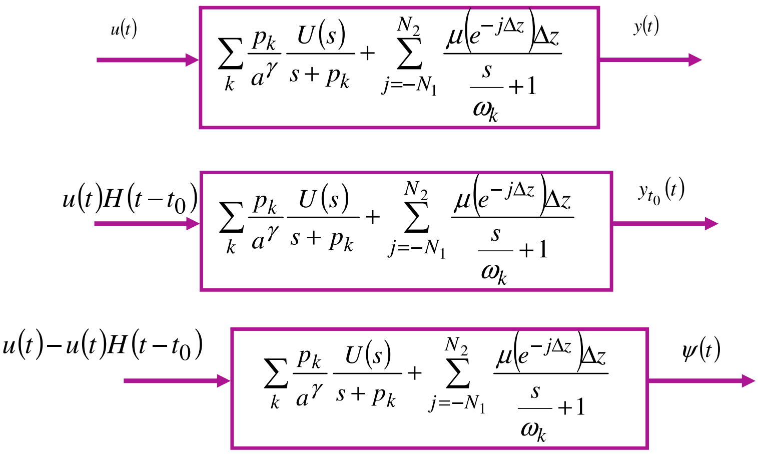

Fig. 6. Approximation of $y(t), y_{t0}(t)$ and of the initialization function $\psi(t)$.

Now consider a time t_0 such that $0 < t_0 < t$, then Eq. (44) can be written as

$$y(t) \approx \sum_k \frac{p_k}{a^\gamma} \int_0^{t_0} e^{p_k(t-\tau)} u(\tau)d\tau + \sum_{j=-N_1}^{N_2} \int_0^{t_0} \mu(e^{-j\Delta z}) e^{-e^{-j\Delta z}(t-\tau)} e^{-j\Delta z} \Delta z\, u(\tau)d\tau + \sum_k \frac{p_k}{a^\gamma} \int_{t_0}^{t} e^{p_k(t-\tau)} u(\tau)d\tau$$
$$+ \sum_{j=-N_1}^{N_2} \int_{t_0}^{t} \mu(e^{-j\Delta z}) e^{-e^{-j\Delta z}(t-\tau)} e^{-j\Delta z} \Delta z\, u(\tau)d\tau. \tag{45}$$

Thus it can be written that

$$y_{t_0} \approx \sum_k \frac{p_k}{a^\gamma} \int_{t_0}^{t} e^{p_k(t-\tau)} u(\tau)d\tau + \sum_{j=-N_1}^{N_2} \int_{t_0}^{t} \mu(e^{-j\Delta z}) e^{-e^{-j\Delta z}(t-\tau)} e^{-j\Delta z} \Delta z\, u(\tau)d\tau \tag{46}$$

and that

$$\psi(t) \approx \sum_k \frac{p_k}{a^\gamma} \int_0^{t_0} e^{p_k(t-\tau)} u(\tau)d\tau + \sum_{j=-N}^{N} \int_0^{t_0} \mu(e^{-j\Delta z}) e^{-e^{-j\Delta z}(t-\tau)} e^{-j\Delta z} \Delta z\, u(\tau)d\tau. \tag{47}$$

Using a software such as MATLAB/SIMULINK, an approximation of $y(t),\ y_{t0}(t)$ and $\psi(t)$ can be obtained with the diagrams represented by Fig. 6.

5. Conclusion

This paper demonstrates that neither the Riemann–Liouville nor the Caputo definitions permit to obtain a physically acceptable initialization of a fractional system. The diffusive representation introduced in [1,11,6] is then used to propose a physically acceptable solution for the initialization problem. Methods are also proposed to obtain an approximation of the initialization function defined in [6] and to estimate an approximation of the system initial state if the system input and output are known. A physical interpretation of a fractional system is also proposed that demonstrates that any fractional system can be viewed as an infinite dimensional system described by a diffusion equation (parabolic differential equation) associated to a classical rational linear (exponential) system.

References

[1] Montseny G. Diffusive representation of pseudo-differential time-operators. ESAIM: proceedings, vol. 5; 1998. p. 159–75.
[2] Matignon D. Stability properties for generalized fractional differential systems. ESAIM: proceedings, vol. 5; 1998. p. 145–58.
[3] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives. New York: Gordon & Breach; 1993.
[4] Podlubny I. Fractional differential equations, Mathematics in sciences and engineering, vol. 198. New York: Academic Press; 1999.
[5] Lorenzo CF, Hartley TT. Initialization in fractional order systems. In: Proceedings of the European control conference, Porto, Portugal; 2001. p. 1471–76.
[6] Lorenzo CF, Hartley TT. Initialization of fractional differential equations: theory and application. In: Proceedings of the ASME 2007 international design engineering technical conferences, DETC2007-34814 Las Vegas, USA; 2007.
[7] Hartley TT, Lorenzo CF. Dynamics and control of initialized fractional-order systems. Nonlinear Dynam 2002;29(1–4):201–33.
[8] Hartley TT, Lorenzo CF. Application of incomplete gamma functions to the initialization of fractional order systems. In: Proceedings of the ASME 2007 international design engineering technical conferences, DETC2007-34814 Las Vegas, USA; 2007.
[9] Debnath L. Recent applications of fractional calculus to science and engineering. Int J Math Math Sci 2003;54:3413–42.
[10] Oustaloup A. Systèmes asservis linéaires d'ordre fractionnaire. Paris: Masson; 1983.

[11] Montseny G. Représentation diffusive. France: Lavoisier; 2005.
[12] Nigmatullin RR. A fractional integral and its physical interpretation. Theoret Math Phys 1992;90(3).
[13] Rutman RS. On physical interpretations of fractional integration and differentiation. Theoret Math Phys 1995;105(3):393–404.
[14] Ben Adda F. Geometric interpretation of the fractional derivative. J Fract Calc 1997;11:21–52.
[15] Gorenflo R. Afterthoughts on interpretation of fractional derivatives and integrals. In: Rusev P, Dimovski I, Kiryakova V, editors. Transform methods and special functions. Varna'96, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia; 1998.
[16] Mainardi F. Considerations on fractional calculus: interpretations and applications. In: Rusev P, Dimovski I, Kiryakova V, editors. Transform methods and special fucntions. Varna'96, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia; 1998.
[17] Podlubny I. Geometric and physical interpretation of fractional integration and fractional differentiation. J Fract Calc Appl Anal 2002;5(4):357–66.
[18] Tenreiro Machado JA. A probabilistic interpretation of the fractional-order differentiation. J Fract Calc Appl Anal 2003;6(1):73–80.
[19] Weinberger HF. Partial differential equations. Waltham (MA): Blaisdel; 1965.
[20] Orjuela R, Malti R, Moze M, Oustaloup A. Prise en compte des conditions initiales lors de la simulation de fonctions de transfert non entières. In: Conférence Internationale Francophone d'Automatique. CIFA'2006, Bordeaux, France; 2006.
[21] Hilfer R, editor. Applications of fractional calculus in physics. Singapore: World Scientific; 2000.
[22] Magin RL. Fractional calculus in bioengineering. Connecticut: Begell House; 2006.
[23] Sabatier J, Agrawal O, Tenreiro Machado J. Advances in fractional calculus: theoretical developments and application in physics and engineering. Berlin: Springer; 2007.
[24] Nigmatullin RR, Le Mehaute A. Is there a physical/geometrical meaning of the fractional integral with complex exponent? J Non-Crystalline Solids 2005;351:2888–99.
[25] Scalas E. Applications of continuous-time random walks in finance and economics. Physica A 2006;362(2):225–39.