Nonlinear Dynamics **38:** 155–170, 2004.

Continued Fraction Expansion Approaches to Discretizing Fractional Order Derivatives – an Expository Review

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(Received: 3 November 2003; accepted: 8 March 2004)

Abstract. This paper attempts to present an expository review of continued fraction expansion (CFE) based discretization schemes for fractional order differentiators defined in continuous time domain. The schemes reviewed are limited to infinite impulse response (IIR) type generating functions of first and second orders, although high-order IIR type generating functions are possible. For the first-order IIR case, the widely used Tustin operator and Al-Alaoui operator are considered. For the second order IIR case, the generating function is obtained by the stable inversion of the weighted sum of Simpson integration formula and the trapezoidal integration formula, which includes many previous discretization schemes as special cases. Numerical examples and sample codes are included for illustrations.

Key words: Al-Alaoui operator, fractional differentiator, fractional-order dynamic systems, fractional-order differentiator, discretization, Tustin operator

1. Introduction

Although the fractional order calculus is a 300-years-old topic, the theory of fractional-order derivative was developed mainly in the nineteenth century. Recent books [14, 15, 25, 28] provide a good source of references on fractional calculus. However, applying fractional-order calculus to dynamic systems control is just a recent focus of interest [10, 21, 22, 26, 27]. For pioneering works, we cite [4, 12, 13, 17]. For recent developments, we cite [8, 16, 35]. In most cases, our objective is to apply fractional order control to enhance the system control performance. For example, as in the CRONE[1] [18, 21, 22], *fractal robustness* is pursued. The desired frequency template leads to fractional transmittance [20, 23] on which the CRONE controller synthesis is based. In CRONE controllers, the major ingredient is the fractional-order derivative s^r, where r is a real number and s is the Laplace transform symbol of differentiation. Another example is the $PI^\lambda D^\mu$ controller [24, 26], an extension of PID controller. In general form, the transfer function of $PI^\lambda D^\mu$ is given by $K_p + T_i s^{-\lambda} + T_d s^\mu$, where λ and μ are positive real numbers; K_p is the proportional gain, T_i the integration constant and T_d the differentiation constant. Clearly, taking $\lambda = 1$ and $\mu = 1$, we obtain a classical PID controller. If $T_i = 0$ we obtain a PD^μ controller, etc. All these types of controllers are particular cases of the $PI^\lambda D^\mu$ controller. It can be expected that the $PI^\lambda D^\mu$ controller may enhance the systems control performance due to more tuning knobs introduced. Actually, in theory, $PI^\lambda D^\mu$ itself is an infinite dimensional linear filter due to the fractional order in the differentiator or integrator. It should be pointed out that a band-limit implementation of FOC is important in practice, i.e., the finite dimensional approximation of the FOC should be done in a proper range of

[1] CRONE is a French abbreviation for "*Commande robuste d' ordre non-entier*" (which means non-integer order robust control).

frequencies of practical interest [19, 20]. Moreover, the fractional order can be a complex number as discussed in [19]. In this paper, we focus on the case where the fractional order is a real number.

The key step in digital implementation of an FOC is the numerical evaluation or discretization of the fractional-order differentiator s^r. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods [19], two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit s-transfer function. Other frequency-domain fitting methods can also be used but without guaranteeing the stable minimum-phase discretization. Existing *direct discretization* methods include the application of the direct power series expansion (PSE) of the Euler operator [11, 31–33], continuous fractional expansion (CFE) of the Tustin operator [6, 31–34], and numerical integration based method [5, 6, 11]. However, as pointed out in [1–3], the Tustin operator based discretization scheme exhibits large errors in high frequency range. A new mixed scheme of Euler and Tustin operators is proposed in [6] which yields the so-called Al-Alaoui operator [1]. These discretization methods for s^r are in infinite impulse response (IIR) form.

Recently, there are some reported methods to directly obtain the digital fractional order differentiators in finite impulse response (FIR) form [29, 30]. However, using an FIR filter to approximate s^r may be less efficient due to very high order of the FIR filter. In this paper, we focus on discretizing fractional differentiators in IIR forms.

Based on some recent results [5–7, 34], this paper tries to present an expository overview of CFE-based discretization schemes for fractional order differentiators defined in continuous time domain. The schemes reviewed are limited to IIR type generating functions of first and second orders, although high-order IIR type generating functions are possible. For the first-order IIR case, the widely used Tustin operator and Al-Alaoui operator are considered. For the second order IIR case, the generating function is obtained by the stable inversion of the weighted sum of Simpson integration formula and the trapezoidal integration formula, which includes many previous discretization schemes as special cases. Numerical examples and sample codes are included for illustrations.

This paper is organized as follows: in Section 2, the fractional order derivative and its discretization are briefly introduced. Section 3 introduces the basic concept of generating function when performing the discretization. In Section 4, two first-order IIR type generating functions are considered: Tustin and Al-Alaoui operators, with illustrative examples. Section 5 presents a general second-order IIR type generating function family. This family of generating functions are obtained by first deriving IIR-type (integer) first-order digital integrator from mixed weighted Simpson and Tustin schemes and then stably inverting this integrator. Some illustrative examples are also presented in Section 5. Section 7 is for the special case of using Tustin operator alone. Finally, Section 6 concludes this paper with some additional remarks.

2. Fractional-Order Derivative and Its Discretization

Fractional calculus generalizes the integration and differentiation to the non-integer (fractional) order fundamental operator ${}_a\mathrm{D}_t^r$, where a and t are the limits and r, ($r \in \mathbb{R}$) the order of the operation. Among different definitions, two definitions commonly used for the general fractional integrodifferential are the Grünwald–Letnikov (GL) definition and the Riemann–Liouville (RL) definition [15, 25]. The GL definition is that

$$ {}_a\mathrm{D}_t^r f(t) = \lim_{h \to 0} h^{-r} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{r}{j} f(t - jh) \tag{1}$$

where $[\cdot]$ means the integer part while the RL definition

$$ {}_a\mathrm{D}_t^r f(t) = \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{r-n+1}} \mathrm{d}\tau \tag{2} $$

for $(n-1 < r < n)$ and where $\Gamma(\cdot)$ is the Euler's *gamma* function.

For convenience, Laplace domain notion is usually used to describe the fractional integro-differential operation [25]. The Laplace transform of the RL fractional derivative/integral (2) under zero initial conditions for order r $(0 < r < 1)$ is given by [15]:

$$ \pounds\left\{ {}_a\mathrm{D}_t^{\pm r} f(t); s \right\} = s^{\pm r} F(s). \tag{3} $$

The key point in digital implementation of a FOC is the numerical evaluation or discretization of the fractional-order differentiator. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods, two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit s-transfer function. In this paper, we focus on the *direct discretization* method.

The simplest and most straightforward method is the direct discretization using finite memory length expansion from the GL definition (1). This approach is based on the fact that, for a wide class of functions, the two definitions – GL (1) and RL (2) are equivalent [25]. In general, the discretization of fractional-order differentiator/integrator $s^{\pm r}$, $(r \in \mathbb{R})$ can be expressed by the so-called generating function $s = \omega(z^{-1})$. This generating function and its expansion determine both the form of the approximation and the coefficients [9]. For example, when a backward difference rule is used, i.e., $\omega(z^{-1}) = (1 - z^{-1})/T$, performing the PSE of $(1 - z^{-1})^{\pm r}$ gives the discretization formula for GL formula (1). By using the short memory principle [25], the discrete equivalent of the fractional-order integro-differential operator, $(\omega(z^{-1}))^{\pm r}$, is given by

$$ (\omega(z^{-1}))^{\pm r} = T^{\mp r} z^{-[L/T]} \sum_{j=0}^{[L/T]} (-1)^j \binom{\pm r}{j} z^{[L/T]-j} \tag{4} $$

where T is the sampling period, L is the memory length and $(-1)^j \binom{\pm r}{j}$ are binomial coefficients $c_j^{(r)}$, $(j = 0, 1, \ldots)$ where

$$ c_0^{(r)} = 1, \qquad c_j^{(r)} = \left(1 - \frac{1 + (\pm r)}{j}\right) c_{j-1}^{(r)}. \tag{5} $$

It is very important to note that PSE scheme leads to approximations in the form of polynomials, that is, the discretized fractional order derivative is in the form of an FIR filter. Taking into account that our aim is to obtain discrete equivalents to the fractional integrodifferential operators in the Laplace domain, $s^{\pm r}$, the following considerations have to be observed:

1. s^r, $(0 < r < 1)$, viewed as an operator, has a branch cut along the negative real axis for arguments of s on $(-\pi, \pi)$ but is free of poles and zeros.
2. A dense interlacing of simple poles and zeros along a line in the s plane is, in some way, equivalent to a branch cut.

3. It is well known that, for interpolation or evaluation purposes, rational functions are sometimes superior to polynomials, roughly speaking, because of their ability to model functions with zeros and poles. In other words, for evaluation purposes, rational approximations frequently converge much more rapidly than PSE and have a wider domain of convergence in the complex plane.
4. Trapezoidal rule maps adequately the stability regions of the s plane on the z plane, and maps the points $s = 0,\ s = -\infty$ to the points $z = 1$ and $z = -1$, respectively.

3. The Concept of Generating Function

In general, the discretization of the fractional-order differentiator s^r (r is a real number) can be expressed by the so-called generating function $s = \omega(z^{-1})$. This generating function and its expansion determine both the form of the approximation and the coefficients [9]. For example, as shown in the last section, when a backward difference rule is used, i.e., $\omega(z^{-1}) = (1 - z^{-1})/T$ with T the sampling period, performing the power series expansion (PSE) of $(1 - z^{-1})^{\pm r}$ gives the discretization formula which is actually in FIR filter form [11, 31]. In [6, 33], the trapezoidal (Tustin) rule is used as a generating function

$$(\omega(z^{-1}))^{\pm r} = \left(\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right)^{\pm r}. \tag{6}$$

The digital fractional order differentiator can then be obtained by using the CFE [33] or a new recursive expansion formula [6]. It is interesting to note that in [6], the so-called Al-Alaoui operator is used which is a mixed scheme of Euler and Tustin operators [1]. Correspondingly, the generating function for discretization is

$$(\omega(z^{-1}))^{\pm r} = \left(\frac{8}{7T}\frac{1 - z^{-1}}{1 + z^{-1}/7}\right)^{\pm r}. \tag{7}$$

Clearly, both (6) and (7) are rational discrete-time transfer functions of infinite orders. To approximate it with a finite order rational one, CFE is an efficient way. In general, any well-behaved function $G(z)$ can be represented by continued fractions in the form of

$$G(z) \simeq a_0(z) + \cfrac{b_1(z)}{a_1(z) + \cfrac{b_2(z)}{a_2(z) + \cfrac{b_3(z)}{a_3(z) + \dots}}} \tag{8}$$

where the coefficients a_i and b_i are either rational functions of the variable z or constants. By truncation, an approximate rational function, $\hat{G}(z)$, can be obtained.

In this review paper, the following two classes of IIR-type generating functions are considered:

– The first-order IIR-type generating functions
  - Tustin operator (6);
  - Al-Alaoui operator (7);

– The second-order IIR-type generating function family (see Section 5).

4. The First-Order IIR-Type Generating Functions

4.1. Tustin Operator

The CFE of Tustin operator (6) is explained in this subsection. Let the resulting discrete transfer function, approximating fractional-order operators, be expressed by

$$
\begin{aligned}
D^{\pm r}(z) &= \frac{Y(z)}{F(z)} = \left(\frac{2}{T}\right)^{\pm r} \mathrm{CFE}\left\{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm r}\right\}_{p,q} \\
&= \left(\frac{2}{T}\right)^{\pm r} \frac{P_p(z^{-1})}{Q_q(z^{-1})}
\end{aligned}
\tag{9}
$$

where T is the sample period, CFE$\{u\}$ denotes the function resulting from applying the continued fraction expansion to the function u, $Y(z)$ is the Z transform of the output sequence $y(nT)$, $F(z)$ is the Z transform of the input sequence $f(nT)$, p and q are the orders of the approximation, and P and Q are polynomials of degrees p and q, correspondingly, in the variable z^{-1}.

By using the MAPLE call

```
Drp:=cfrac(((1-x)/(1+x))^r,x,p)
```

where $x = z^{-1}$, the obtained symbolic approximation has the following form:

$$
D^r(z) = 1 + \cfrac{z^{-1}}{-\frac{1}{2}\,\frac{1}{r} + \cfrac{z^{-1}}{-2 + \cfrac{z^{-1}}{\frac{3}{2}\,\frac{r}{r^2-1} + \cfrac{z^{-1}}{2 + \cfrac{z^{-1}}{-\frac{5}{2}\,\frac{r^2-1}{r(-4+r^2)} + \cfrac{z^{-1}}{-2+\cdots}}}}}}.
\tag{10}
$$

In MATLAB Symbolic Math Toolbox, we can get the same result by the following script:

```
syms x r;maple('with(numtheory)');
f = ((1-x)/(1+x))^r;
maple(['cf := cfrac(' char(f) ',x,10);'])
maple('nd5 := nthconver','cf',10)
maple('num5 := nthnumer','cf',10)
maple('den5 := nthdenom','cf',10)
```

In Table 1, the general expressions for numerator and denominator of $D^r(z)$ in (9) are listed for $p = q = 1, 3, 5, 7, 9$.

With $r = 0.5$ and $T = 0.001$ sec the approximate models for $p = q = 1, 3, 7, 9$ are:

$$
\begin{aligned}
&G_1(z) = 44.72\frac{z-0.5}{z+0.5}, \quad G_3(z) = 44.72\frac{z^3-0.5z^2-0.5z+0.125}{z^3+0.5z^2-0.5z-0.125}, \\
&G_7(z) = 44.72\frac{z^7-0.5z^6-1.5z^5+0.625z^4+0.625z^3-0.1875z^2-0.0625z+0.007813}{z^7+0.5z^6-1.5z^5-0.625z^4+0.625z^3+0.1875z^2-0.0625z-0.007813}, \\
&G_9(z) = 44.72\frac{z^9-0.5z^8-2z^7+0.875z^6+1.313z^5-0.4688z^4-0.3125z^3+0.07813z^2+0.01953z-0.001953}{z^9+0.5z^8-2z^7-0.875z^6+1.313z^5+0.4688z^4-0.3125z^3-0.07813z^2+0.01953z+0.001953}.
\end{aligned}
$$

Table 1. Expressions of $D^r(z)$ in (10) for different orders.

p = q	$\mathbf{P}_p(z^{-1})(k=1)$, **and** $\mathbf{Q}_q(z^{-1})(k=0)$
1	$(-1)^k z^{-1} r + 1$
3	$(-1)^k (r^3 - 4r) z^{-3} + (6r^2 - 9) z^{-2} + (-1)^k 15 z^{-1} r + 15$
5	$(-1)^k (r^5 - 20r^3 + 64r) z^{-5} + (-195r^2 + 15r^4 + 225) z^{-4} + (-1)^k (105r^3 - 735r) z^{-3} + (420r^2 - 1050) z^{-2}$ $+ (-1)^k 945 z^{-1} r + 945$
7	$(-1)^k (784r^3 + r^7 - 56r^5 - 2304r) z^{-7} + (10612r^2 - 1190r^4 - 11025 + 28r^6) z^{-6}$ $+ (-1)^k (53487r + 378r^5 - 11340r^3) z^{-5} + (99225 - 59850r^2 + 3150r^4) z^{-4} + (-1)^k (17325r^3 - 173250r) z^{-3}$ $+ (-218295 + 62370r^2) z^{-2} + (-1)^k 135135 z^{-1} r + 135135$
9	$(-1)^k (-52480r^3 + 147456r + r^9 - 120r^7 + 4368r^5) z^{-9} + (45r^8 + 120330r^4 - 909765r^2 - 4410r^6 + 893025) z^{-8}$ $+ (-1)^k (-5742495r - 76230r^5 + 1451835r^3 + 990r^7) z^{-7} + (-13097700 + 9514890r^2 - 796950r^4 + 13860r^6) z^{-6}$ $+ (-1)^k (33648615r - 5405400r^3 + 135135r^5) z^{-5} + (-23648625r^2 + 51081030 + 945945r^4) z^{-4}$ $+ (-1)^k (-61486425r + 4729725r^3) z^{-3} + (16216200r^2 - 72972900) z^{-2} + (-1)^k 34459425 z^{-1} r + 34459425$

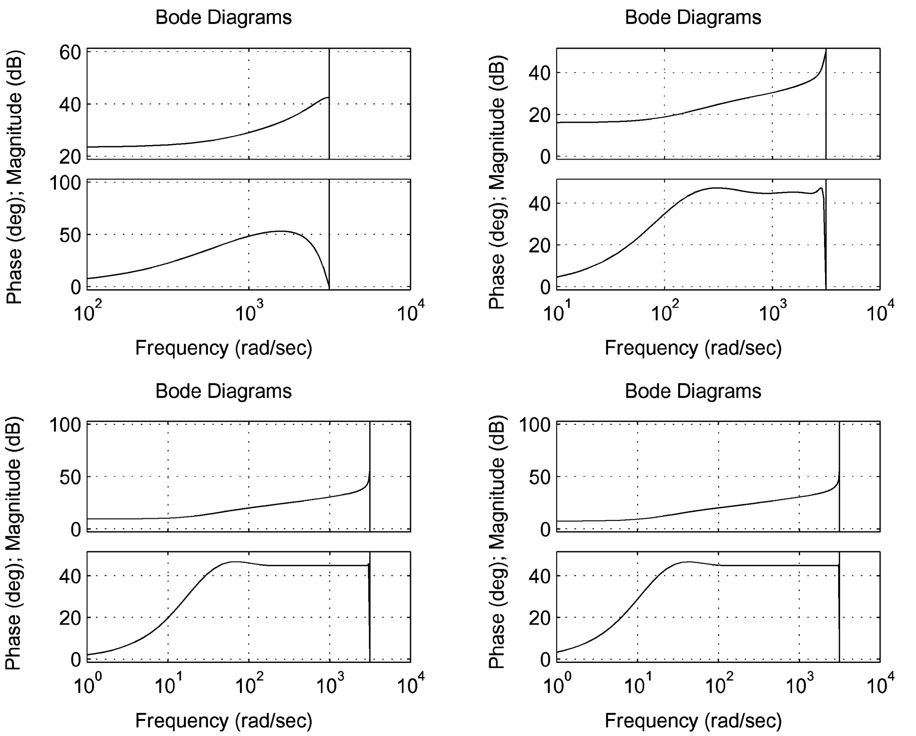

Figure 1. Bode plots (approximation orders 1, 3, 7, 9) by Tustin CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ sec.

In Figure 1, the Bode plots and the distributions of zeros and poles of the approximations are presented. In Figure 1, the effectiveness of the approximations fitting the ideal responses in a wide range of frequencies, in both magnitude and phase, can be observed. In Figure 2, it can be observed that the approximations fulfill the two desired properties: (i) all the poles and zeros lie inside the unit circle, and (ii) the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1, 1)$.

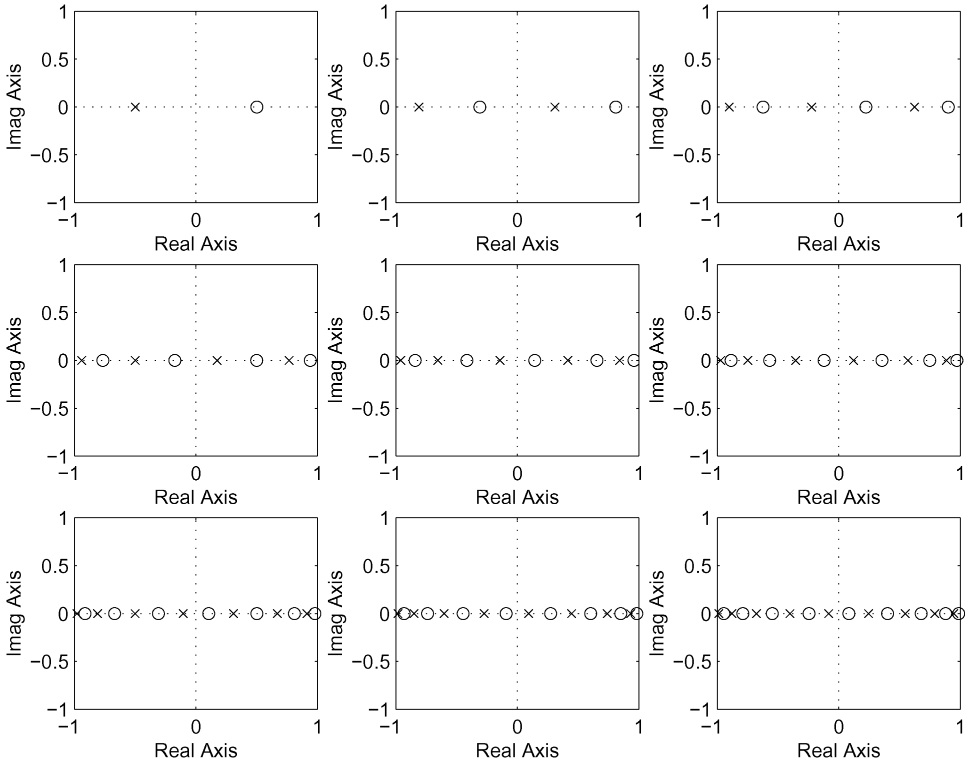

Figure 2. Zero-pole distribution (approximation orders $1, 2, \ldots, 9$) by Tustin CFE approximate discretization of $s^{0.5}$ at $T = 0.001$ sec.

4.2. Al-Alaoui Operator

This subsection shows how to perform CFE of Al-Alaoui operator (7). Now, the resulting discrete transfer function, approximating fractional-order operators, can be expressed as:

$$
\begin{aligned}
D^{\pm r}(z) &\approx \left(\frac{8}{7T}\right)^{\pm r} \mathrm{CFE}\left\{\left(\frac{1-z^{-1}}{1+z^{-1}/7}\right)^{\pm r}\right\}_{p,q} \\
&= \left(\frac{8}{7T}\right)^{\pm r} \frac{P_p(z^{-1})}{Q_q(z^{-1})}.
\end{aligned}
\tag{11}
$$

Normally, we can set $p = q = n$. In MATLAB Symbolic Math Toolbox, we can easily get the approximate direct discretization of the fractional order derivative by the following script, for a given n (replace 14 by $2n$):

```
clear all;close all; syms x z r
%Al-Alouoi's scheme
x=((1-z)/(1+z/7))^r;
[RESULT,STATUS] = maple('with(numtheory)')
%7-th order; put 2*7 here.
h7=maple('cfrac',x,z,14);
h7n=maple('nthnumer(%%,14)');
```

```
h7d=maple('nthdenom(%%%,14)');
h7ns=sym(h7n);h7ds=sym(h7d);
num7=collect(h7ns,z);den7=collect(h7ds,z);
fn7=subs(num7,z,1/z),fd7=subs(den7,z,1/z)
```

The CFE scheme presented in the above (Tustin and Al-Alaoui) contains two tuning parameters, namely p and q. The optimal choice of these two parameters is possibly based on a quantitative measure. One possibility is the use of the least squares (LS) error between the continuous frequency response and discretized frequency response. Note that in practice, p and q can usually be set to be equal.

The discretization of the half-differentiator $s^{0.5}$ sampled at 0.001 sec is studied numerically, and the approximate models are

$$G_1(z) = \frac{236.6z - 169}{7z - 1}, \quad G_3(z) = \frac{1657z^3 - 2603z^2 + 1048z - 62.78}{49z^3 - 49z^2 + 7z + 1},$$
$$G_5(z) = \frac{2.47e04z^5 - 5.999e04z^4 + 4.941e04z^3 - 1.512e04z^2 + 956.9z + 98.48}{730.7z^5 - 1357z^4 + 745.7z^3 - 89.48z^2 - 15.52z + 1},$$
$$G_7(z) = \frac{3.128e05z^7 - 1.028e06z^6 + 1.283e06z^5 - 7.433e05z^4 + 1.87e05z^3 - 9772z^2 - 2140z + 104.5}{9253z^7 - 2.512e004z^6 + 2.436e004z^5 - 9577z^4 + 905.7z^3 + 219.7z^2 - 23.67z - 1}.$$

We present four plots, shown in Figure 3, to demonstrate the effectiveness of the approximate discretization. We can observe from Figure 3 that this scheme is much better than the Tustin scheme in magnitude fit to the original s^r. After the linear phase compensation, the maximum phase error of the Al-Alaoui operator-based discretization scheme is around $r \times 8.25^\circ$ at 55% of the Nyquist frequency (around 275 Hz in this example) as shown in Figure 3. To compensate the linear phase drop, a half sample phase advance is used which means that we should cascade $z^{0.5r}$ to the obtained approximately discretized transfer function $G(z)$. However, in this example, the phase compensator is $z^{0.25}$ which is noncausal. In implementation, we can simply use $z^{-0.75}/z^{-1}$ instead.

5. The Second-Order IIR-Type Generating Functions

5.1. Integer Order IIR-Type Digital Integrator by Weighted Simpson and Tustin Schemes

It was pointed out in [1, 3] that the magnitude of the frequency response of the ideal integrator $1/s$ lies between that of the Simpson and trapezoidal digital integrators. It is reasonable to "interpolate" the Simpson and trapezoidal digital integrators to compromise the high frequency accuracy in frequency response. This leads to the following hybrid digital integrator

$$H(z) = aH_S(z) + (1 - a)H_T(z), \quad a \in [0, 1] \tag{12}$$

where a is actually a weighting factor or tuning knob. $H_S(z)$ and $H_T(z)$ are the z-transfer functions of the Simpson's and the trapezoidal integrators given respectively as follows:

$$H_S(z) = \frac{T(z^2 + 4z + 1)}{3(z^2 - 1)} \tag{13}$$

and

$$H_T(z) = \frac{T(z + 1)}{2(z - 1)}. \tag{14}$$

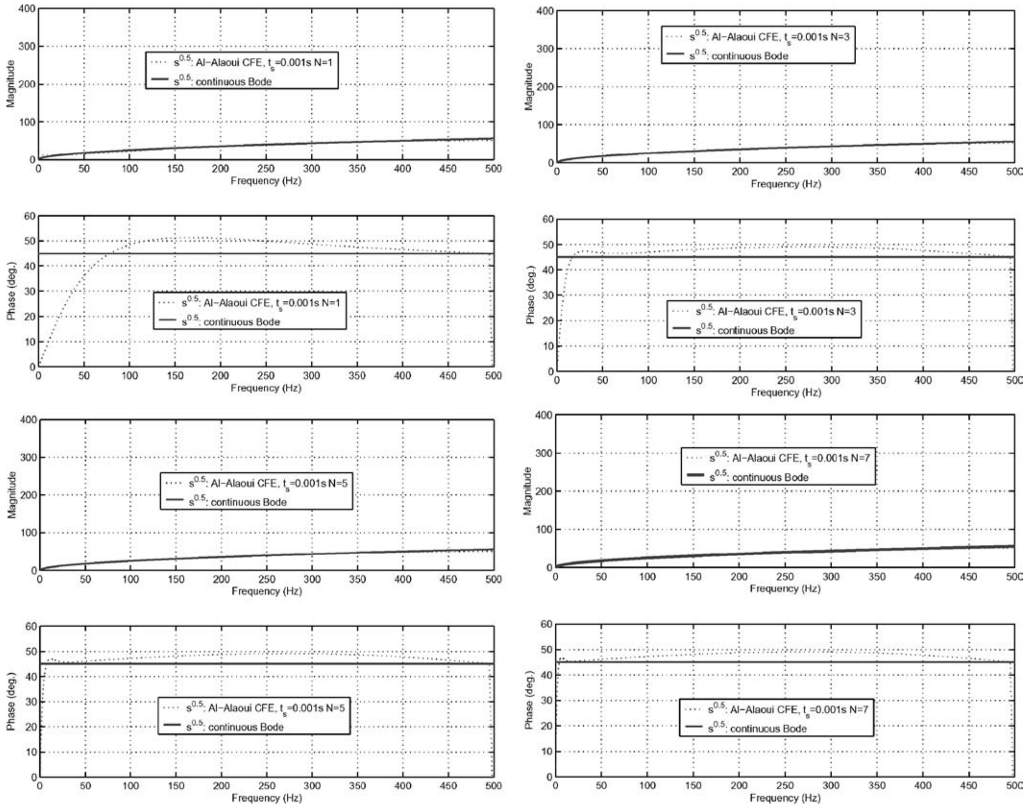

Figure 3. CFE (Al-Alaoui) discretization of $s^{0.5}$ at $T = 0.001$ sec (Bode plots of top left: $G_1(z)$; top right: $G_3(z)$; bottom left: $G_5(z)$; bottom right: $G_7(z)$).

The overall weighted digital integrator with the tuning parameter a is hence given by

$$\begin{aligned} H(z) &= \frac{T(3-a)\{z^2 + [2(3+a)/(3-a)]z + 1\}}{6(z^2-1)} \\ &= \frac{T(3-a)(z+r_1)(z+r_2)}{6(z^2-1)} \end{aligned} \tag{15}$$

where

$$r_1 = \frac{3+a+2\sqrt{3a}}{3-a}, \quad r_2 = \frac{3+a-2\sqrt{3a}}{3-a}.$$

It is interesting to note the fact that $r_1 = \frac{1}{r_2}$ and $r_1 = r_2 = 1$ only when $a = 0$ (trapezoidal). For $a \neq 0$, $H(z)$ must have one non-minimum phase (NMP) zero.

5.2. IIR-Type Fractional Order Digital Differentiator

Firstly, we can obtain a family of new integer order digital differentiators from the digital integrators introduced in the last section. Direct inversion of $H(z)$ will give an unstable filter since $H(z)$ has a NMP zero r_1. By reflecting the NMP r_1 to $1/r_1$, i.e., r_2, we have

$$\tilde{H}(z) = K\frac{T(3-a)(z+r_2)^2}{6(z^2-1)}.$$

To determine K, let the final values of the impulse responses of $H(z)$ and $\tilde{H}(z)$ be the same, i.e., $\lim_{z\to 1}(z-1)H(z) = \lim_{z\to 1}(z-1)\tilde{H}(z)$, which gives $K = r_1$. Therefore, the new family of first-order digital differentiators are given by

$$\omega(z) = \frac{1}{\tilde{H}(z)} = \frac{6(z^2-1)}{r_1 T(3-a)(z+r_2)^2} = \frac{6r_2(z^2-1)}{T(3-a)(z+r_2)^2}. \tag{16}$$

We can regard $\omega(z)$ in (16) as the generating function introduced in Section 3. Finally, we can obtain the expression for a family of digital fractional order differentiator as

$$G(z^{-1}) = (\omega(z^{-1}))^r = k_0 \left(\frac{1-z^{-2}}{(1+bz^{-1})^2} \right)^r \tag{17}$$

where $r \in [0, 1]$, $k_0 = (\frac{6r_2}{T(3-a)})^r$ and $b = r_2$.

Using CFE, an approximation for an irrational function $G(z^{-1})$ can be expressed in the form of (8). Similar to (7), here, the irrational transfer function $G(z^{-1})$ in (17) can be expressed by an infinite order of rational discrete-time transfer function by CFE method as shown in (8).

The CFE expansion can be automated by using a symbolic computation tool such as the MATLAB Symbolic Math Toolbox. For illustrations, let us denote $x = z^{-1}$. Referring to (17), the task is to perform the following expansion:

$$\mathrm{CFE} \left(\frac{1-x^2}{(1+bx)^2} \right)^r$$

to the desired order n. The following MATLAB script will generate the above CFE with `p1` and `q1` containing, respectively, the numerator and denominator polynomials in `x` or z^{-1} with their coefficients being functions of `b` and `r`.

```
clear all;close all;syms x r b;maple('with(numtheory)');
aas = ((1-x*x)/(1+b*x)^2)^r; n=3; n2=2*n;
maple(['cfe := cfrac(' char(aas) ',x,n2);']);
pq=maple('P_over_Q := nthconver','cfe',n2);
p0=maple('P := nthnumer','cfe',n2);
q0=maple('Q := nthdenom','cfe',n2);
p=(p0(5:length(p0))); q=(q0(5:length(q0)));
p1=collect(sym(p),x); q1=collect(sym(q),x);
```

5.3. Illustrative Examples

Here we present some results for $r = 0.5$. The values of the truncation order n and the weighting factor a are denoted as subscripts of $G_{(n,a)}(z)$. Let $T = 0.001$ sec. We have the following:

$$G_{(2,0.00)}(z^{-1}) = \frac{178.9 - 89.44z^{-1} - 44.72z^{-2}}{4 + 2z^{-1} - z^{-2}}$$

$$G_{(2,0.25)}(z^{-1}) = \frac{138.8 + 98.07z^{-1} - 158.2z^{-2}}{4 + 5.034z^{-1} - z^{-2}}$$

$$
\begin{aligned}
G_{(2,0.50)}(z^{-1}) &= \frac{127 + 41.26z^{-1} - 112.6z^{-2}}{4 + 2.98z^{-1} - z^{-2}} \\
G_{(2,0.75)}(z^{-1}) &= \frac{119.3 + 25.56z^{-1} - 97.96z^{-2}}{4 + 2.19z^{-1} - z^{-2}} \\
G_{(2,1.00)}(z^{-1}) &= \frac{113.4 + 17.74z^{-1} - 89.81z^{-2}}{4 + 1.698z^{-1} - z^{-2}}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
G_{(3,0.00)}(z^{-1}) &= \frac{357.8 - 178.9z^{-1} - 178.9z^{-2} + 44.72z^{-3}}{8 + 4z^{-1} - 4z^{-2} - z^{-3}} \\
G_{(3,0.25)}(z^{-1}) &= \frac{392.9 - 78.04z^{-1} - 349.8z^{-2} + 88.97z^{-3}}{11.32 + 4z^{-1} - 5.66z^{-2} - z^{-3}} \\
G_{(3,0.50)}(z^{-1}) &= \frac{1501 - 503.6z^{-1} - 1289z^{-2} + 446.5z^{-3}}{47.26 + 4z^{-1} - 23.63z^{-2} - z^{-3}} \\
G_{(3,0.75)}(z^{-1}) &= \frac{968.1 - 442z^{-1} - 820.8z^{-2} + 363z^{-3}}{32.47 - 4z^{-1} - 16.24z^{-2} + z^{-3}} \\
G_{(3,1.00)}(z^{-1}) &= \frac{353.1 - 208z^{-1} - 297.4z^{-2} + 164.7z^{-3}}{12.46 - 4z^{-1} - 6.228z^{-2} + z^{-3}}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
G_{(4,0.00)}(z^{-1}) &= \frac{715.5 - 357.8z^{-1} - 536.7z^{-2} + 178.9z^{-3} + 44.72z^{-4}}{16 + 8z^{-1} - 12z^{-2} - 4z^{-3} + z^{-4}} \\
G_{(4,0.25)}(z^{-1}) &= \frac{555.3 - 392.9z^{-1} - 477.2z^{-2} + 349.8z^{-3} - 19.56z^{-4}}{16 - 2.489z^{-1} - 12z^{-2} + 1.245z^{-3} + z^{-4}} \\
G_{(4,0.50)}(z^{-1}) &= \frac{508.1 - 1501z^{-1} - 4.478z^{-2} + 1289z^{-3} - 382.9z^{-4}}{16 - 40.54z^{-1} - 12z^{-2} + 20.27z^{-3} + z^{-4}} \\
G_{(4,0.75)}(z^{-1}) &= \frac{477 + 968.1z^{-1} - 919z^{-2} - 820.8z^{-3} + 422.7z^{-4}}{16 + 37.8z^{-1} - 12z^{-2} - 18.9z^{-3} + z^{-4}} \\
G_{(4,1.00)}(z^{-1}) &= \frac{453.6 + 353.1z^{-1} - 661.7z^{-2} - 297.4z^{-3} + 221.5z^{-4}}{16 + 16.74z^{-1} - 12z^{-2} - 8.371z^{-3} + z^{-4}}.
\end{aligned} \tag{20}
$$

The Bode plot comparisons for the above three groups of approximate fractional order digital differentiators are summarized in Figures 4–6, respectively. We can observe the improvement in high frequency magnitude response. If trapezoidal scheme is used, the high frequency magnitude response is far from the ideal one. The role of the tuning knob a is obviously useful in some applications. MATLAB code for this new digital fractional-order differentiator is available upon request.

Remark 5.1. The phase approximations in Figures 4–6 did not consider the linear phase lag compensation as is done in [6]. For a given a and r, a pure linear phase lead compensation can be added without affecting the magnitude approximation. For example, when $a = r = 0.5$, a pure phase lead $z^{0.5}$ can be cascaded to $G_{(4,0.50)}(z^{-1})$ and the phase approximation can be improved as shown in Figure 7. Note that $z^{0.5}$ can be realized by $z^{-0.5}/z^{-1}$ which is causally realizable.

For $n = 3$ and $n = 4$, the pole-zero maps are shown respectively in Figures 8 and 9 for some different values of a. First of all, we observe that there are no complex conjugate poles or zeros. We can further observe that for odd order of CFE ($n = 3$), the pole-zero maps are nicely behaved, that is, all the poles and zeros lie inside the unit circle and the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1, 1)$. However, when n is even, and when a is near 1, there may have one

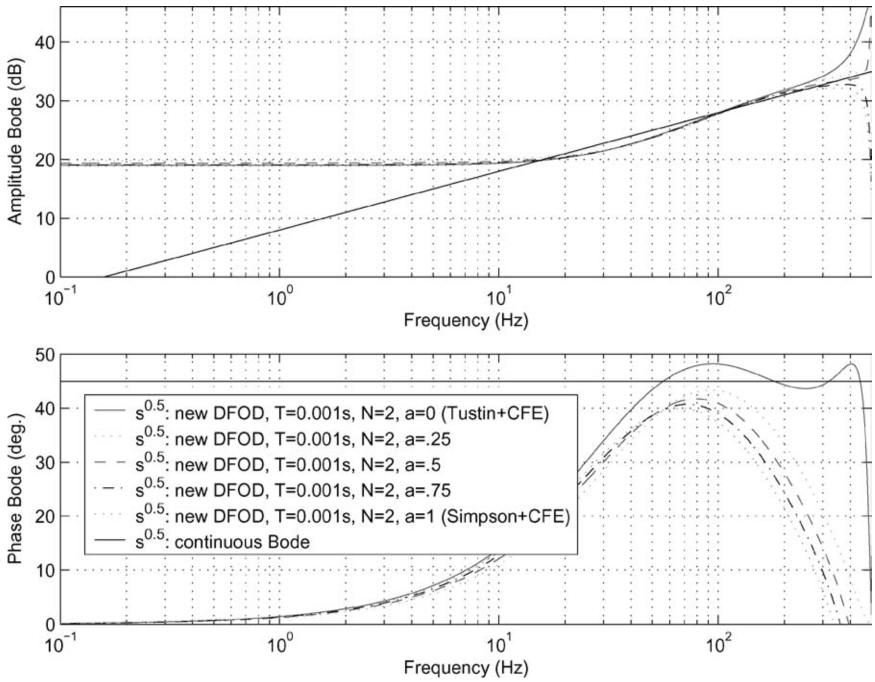

Figure 4. Bode plot comparison for $r = 0.5$, $n = 2$ and $a = 0, 0.25, 0.5, 0.75, 1$.

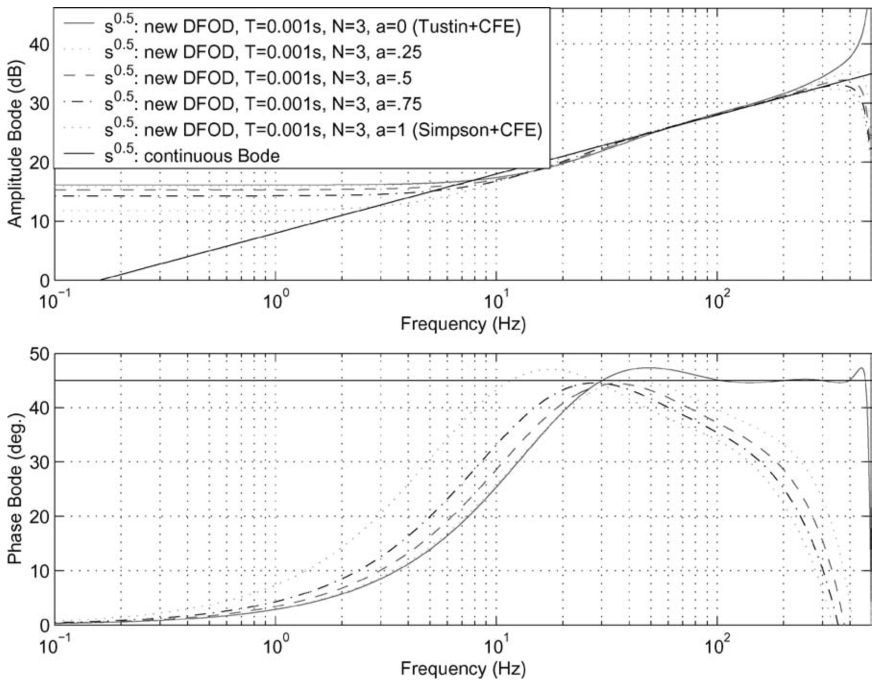

Figure 5. Bode plot comparison for $r = 0.5$, $n = 3$ and $a = 0, 0.25, 0.5, 0.75, 1$.

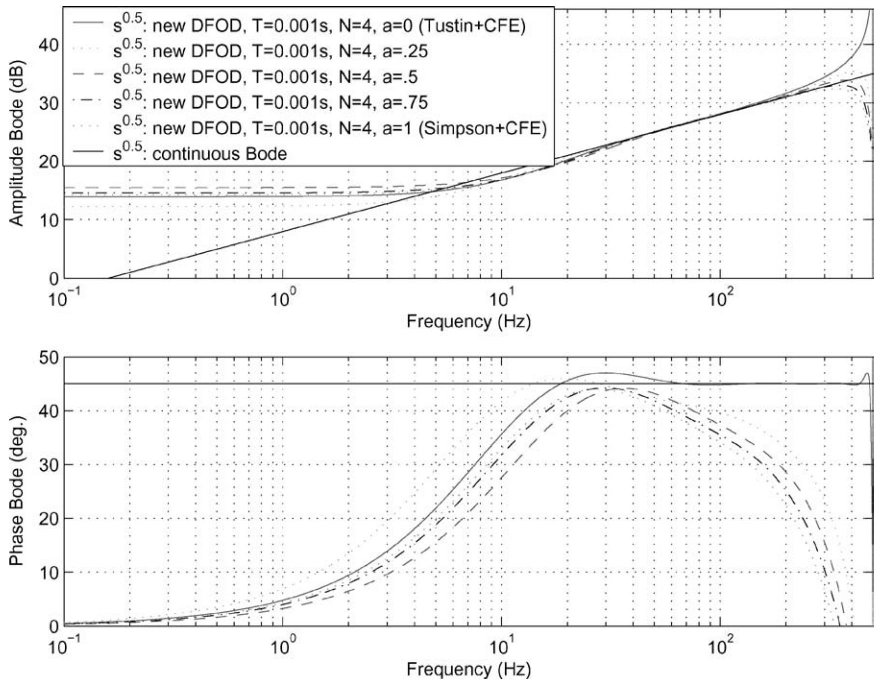

Figure 6. Bode plot comparison for $r = 0.5$, $n = 4$ and $a = 0, 0.25, 0.5, 0.75, 1$.

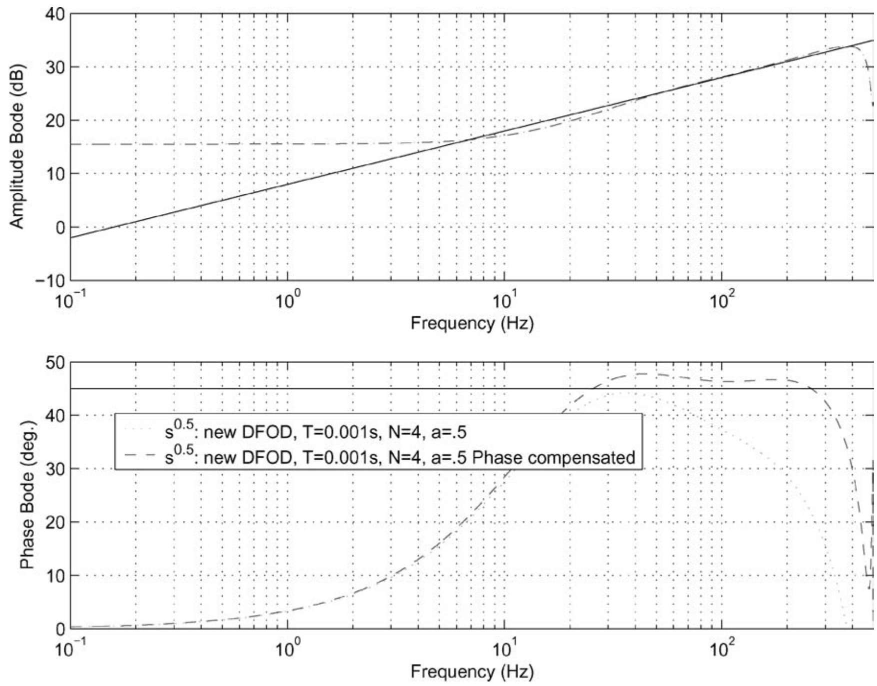

Figure 7. Effect of linear phase compensation for $r = 0.5$, $n = 4$, and $a = 0.5$.

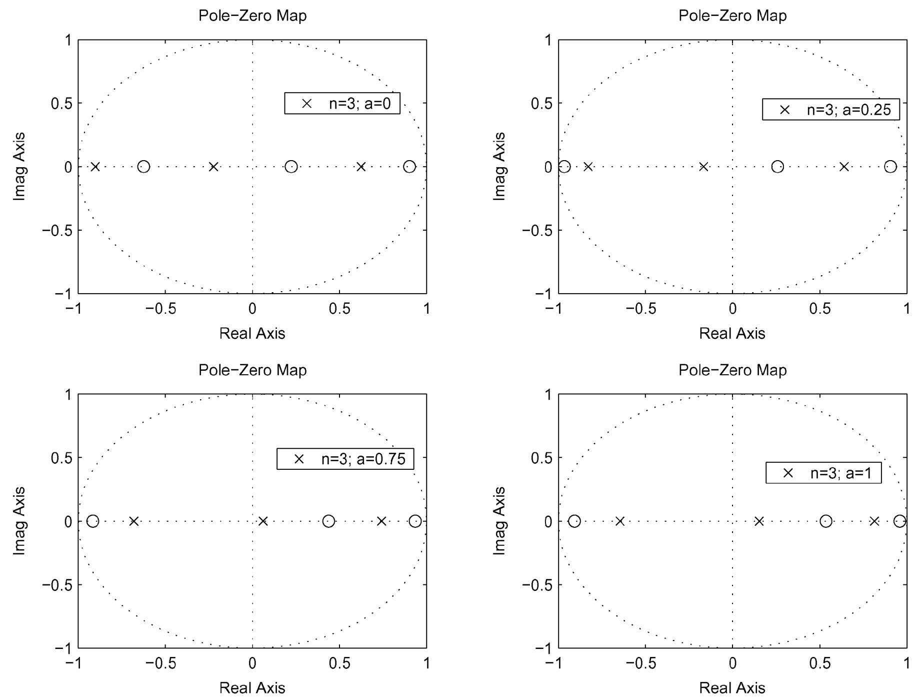

Figure 8. Pole-zero maps for $r = 0.5$, $n = 3$ and $a = 0, 0.25, 0.75, 1$.

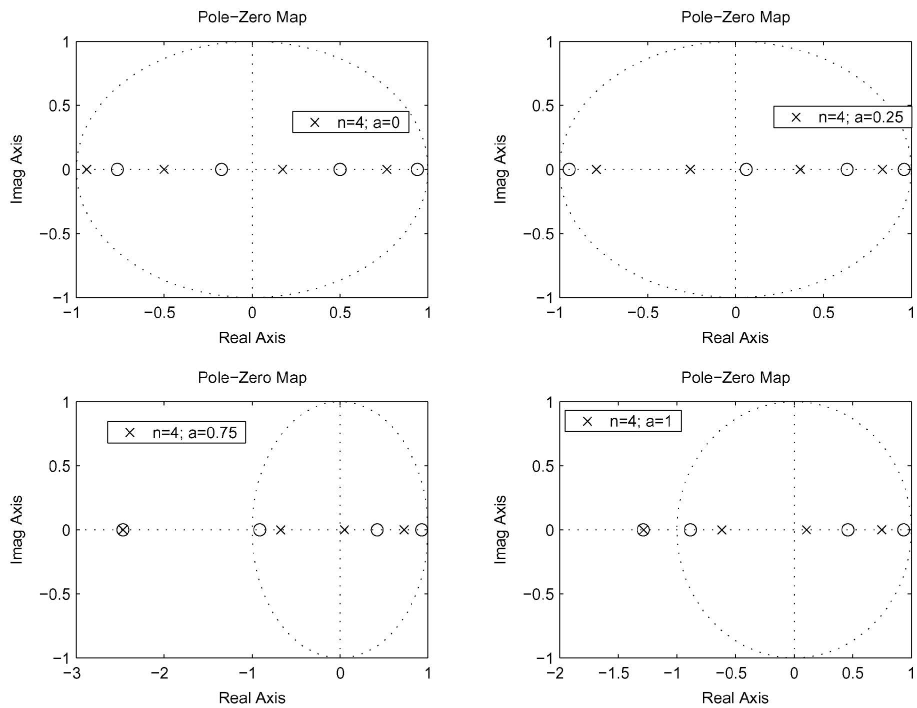

Figure 9. Pole-zero maps for $r = 0.5$, $n = 4$, and $a = 0, 0.25, 0.75, 1$.

cancelling pole-zero pair as seen in Figure 9 which may not be desirable. We suggest to use an odd n when applying this discretization scheme.

6. Concluding Remarks

The purpose of this paper is to present an expository review of CFE method based discretization schemes for fractional order differentiators defined in continuous time domain. The schemes reviewed are limited to IIR-type generating functions of first and second orders. For the first-order IIR case, the widely used Tustin and Al-Alaoui operators are considered. For the second order IIR case, the generating function is obtained by the stable inversion of the weighted sum of Simpson integration formula and the trapezoidal integration formula, which includes many previous discretization schemes as special cases. Numerical examples and sample codes are included for illustrations throughout this paper.

Clearly, although only first- and second-order IIR-type generating functions are reviewed in this paper, using high-order IIR-type generating functions for discretizing fractional order differentiators is totally possible. The question is if it is worthwhile to consider the high-order case. Another question is, among all possible second-order IIR-type generating functions, what the best generating function is to given a balanced frequency- and time-domain approximation of fractional order differentiators.

Acknowledgements

This project has been funded in part by the National Academy of Sciences under the Collaboration in Basic Science and Engineering Program/Twinning Program supported by Contract No. INT-0002341 from the National Science Foundation. The contents of this publication do not necessarily reflect the views or policies of the National Academy of Sciences or the National Science Foundation, nor does it mention trade names, commercial products or organizations that imply endorsement by the National Academy of Sciences or the National Science Foundation.

Blas M. Vinagre is partially supported by the Research Grant (Junta de Extremadura and FEDER) 2PR02A024.

The authors are grateful to the comments from anonymous reviewers, in particular, Reviewer No. 1, which improved the structure and the accuracy of the presentation.

References

1. Al-Alaoui, M. A., 'Novel digital integrator and differentiator', *Electronics Letters* **29**(4), 1993, 376–378.
2. Al-Alaoui, M. A., 'A class of second-order integrators and low-pass differentiators', *IEEE Transactions on Circuit and Systems I: Fundamental Theory and Applications* **42**(4), 1995, 220–223.
3. Al-Alaoui, M. A., 'Filling the gap between the bilinear and the backward difference transforms: An interactive design approach', *International Journal of Electrical Engineering Education* **34**(4), 1997, 331–337.
4. Axtell, M. and Bise, E. M., 'Fractional calculus applications in control systems', in *Proceedings of the IEEE 1990 National Aerospace and Electronics Conference*, New York, 1990, pp. 563–566.
5. Chen, Y. Q. and Vinagre, B. M., 'A new IIR-type digital fractional order differentiator', *Signal Processing* **83**(11), 2003, 2359–2365.
6. Chen, Y. Q. and Moore, K. L., 'Discretization schemes for fractional order differentiators and integrators', *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* **49**(3), 2002, 363–367.
7. Chen, Y. Q., Vinagre, B. M., and Podlubny, I., 'A new discretization method for fractional order differentiators via continued fraction expansion', in *Proceedings of the First Symposium on Fractional Derivatives and Their Applications at The 19th*

Biennial Conference on Mechanical Vibration and Noise, the ASME International Design Engineering Technical Conferences & Computers and Information in Engineering Conference (ASME DETC2003), DETC2003/VIB-48391, Chicago, Illinois, 2003, pp. 1–8,

8. Machado, J. A. T. (guest editor), 'Special issue on fractional calculus and applications', *Nonlinear Dynamics* **29**, 2002, 1–385.
9. Lubich, C. H., 'Discretized fractional calculus', *SIAM Journal on Mathematical Analysis* **17**(3), 1986, 704–719.
10. Lurie, Boris J., 'Three-parameter tunable tilt-integral-derivative (TID) controller', *US Patent US5371670*, 1994.
11. Machado, J. A. T., 'Analysis and design of fractional-order digital control systems', *Journal of Systems Analysis, Modelling and Simulation* **27**, 1997, 107–122.
12. Manabe, S., 'The non-integer integral and its application to control systems', *JIEE (Japanese Institute of Electrical Engineers) Journal* **80**(860), 1960, 589–597.
13. Manabe, S., 'The non-integer integral and its application to control systems', *ETJ of Japan* **6**(3/4), 1961, 83–87.
14. Miller, K. S. and Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
15. Oldham, K. B. and Spanier, J., *The Fractional Calculus*, Academic Press, New York, 1974.
16. Ortigueira, M. D. and Machado, J. A. T. (guest editors), 'Special issue on fractional signal processing and applications' *Signal Processing* **83**(11), 2003, 2285–2480.
17. Oustaloup, A., 'Fractional order sinusoidal oscilators: Optimization and their use in highly linear FM modulators', *IEEE Transactions on Circuits and Systems* **28**(10), 1981, 1007–1009.
18. Oustaloup, A., *La dérivation non entière*, HERMES, Paris, 1995.
19. Oustaloup, A., Levron, F., Nanot, F., and Mathieu, B., 'Frequency band complex non integer differentiator: Characterization and synthesis', *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* **47**(1), 2000, 25–40.
20. Oustaloup, A. and Mathieu, B., *La commande CRONE: du scalaire au multivariable*, HERMES, Paris, 1999.
21. Oustaloup, A., Mathieu, B., and Lanusse, P., 'The CRONE control of resonant plants: Application to a flexible transmission', *European Journal of Control* **1**(2), 1995, pp. 113–121.
22. Oustaloup, A., Moreau, X., and Nouillant, M., 'The CRONE suspension', *Control Engineering Practice* **4**(8), 1996, 1101–1108.
23. Oustaloup, A., Sabatier, J., and Lanusse, P., 'From fractal robustness to CRONE control', *Fractionnal Calculus and Applied Analysis* **2**(1), 1999, 1–30.
24. Petráš, I., 'The fractional-order controllers: Methods for their synthesis and application', *Journal of Electrical Engineering* **50**(9–10), 1999, 284–288.
25. Podlubny, I., 'Fractional-order systems and fractional-order controllers', Technical Report UEF-03-94, Slovak Academy of Sciences. Institute of Experimental Physics, Department of Control Engineering. Faculty of Mining, University of Technology, Kosice, Slovak Republic, November 1994.
26. Podlubny, I., 'Fractional-order systems and $PI^{\lambda}D^{\mu}$-Controllers', *IEEE Transactions Automatic Control* **44**(1), 1999, 208–214.
27. Raynaud, H.-F. and Zergaïnoh, A., 'State-space representation for fractional order controllers', *Automatica* **36**, 2000, 1017–1021.
28. Samko, S. G., Kilbas, A. A., and Marichev, O. I., *Fractional Integrals and Derivatives and Some of Their Applications*. Nauka i Technika, Minsk, Russia, 1987.
29. Tseng, C.-C., 'Design of fractional order digital FIR differentiator', *IEEE Signal Processing Letters* **8**(3), 2001, 77–79.
30. Tseng, C.-C., Pei, S.-C., and Hsia, S.-C., 'Computation of fractional derivatives using fourier transform and digital FIR differentiator', *Signal Processing* **80**, 2000, 151–159.
31. Vinagre, B. M., Petras, I., Merchan, P., and Dorcak, L., 'Two digital realisation of fractional controllers: Application to temperature control of a solid', in *Proceedings of the European Control Conference (ECC2001)*, Porto, Portugal, September 2001, pp. 1764–1767.
32. Vinagre, B. M., Podlubny, I., Hernandez, A., and Feliu, V., 'On realization of fractional-order controllers', in *Proceedings of the Conference Internationale Francophone d'Automatique*, Lille, France, July 2000.
33. Vinagre, B. M., Podlubny, I., Hernandez, A., and Feliu, V., 'Some approximations of fractional order operators used in control theory and applications', *Fractional Calculus and Applied Analysis* **3**(3), 2000, 231–248.
34. Vinagre, B. M., Chen, Y. Q., and Petras, I., 'Two direct tustin discretization methods for fractional-order differentiator/integrator', *The Journal of Franklin Institute* **340**(5) 2003, 349–362.
35. Vinagre, B. M. and Chen, Y. Q., 'Lecture notes on fractional calculus applications in automatic control and robotics' in *The 41st IEEE CDC2002 Tutorial Workshop No. 2*, B. M. Vinagre and Y. Q. Chen (eds.), retrieved from http://mechatronics.ece.usu.edu/foc/cdc02_tw2_ln.pdf, Las Vegas, Nevada, 2002, pp. 1–310.