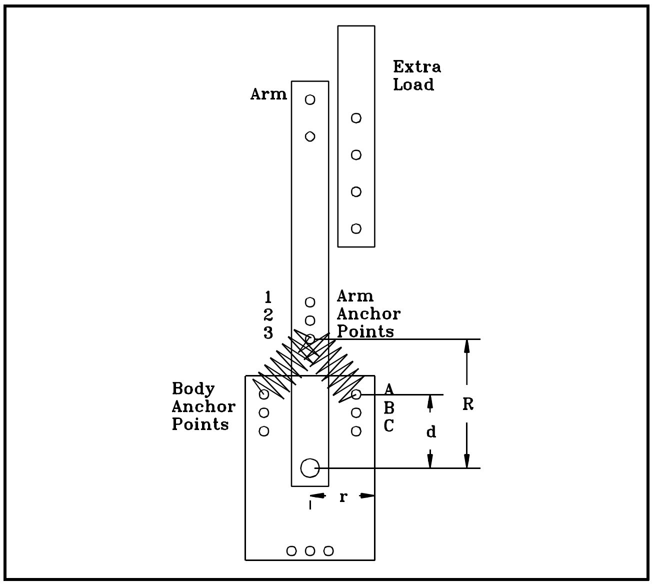
2.4.1 DESCRIPTION

The rotary flexible joint module (RFJ) shown in Figure RFJ1 is designed as an attachment to the plant SRV-02. The RFJ consist of a body and a load. A potentiometer is attached to the body on which axis the load is attached. Joint flexibility is attained via two identical springs which are anchored to the body and to the load. Joint stiffness can be varied by changing the springs or the anchor points. A small arm can be attached to the end of the main arm thus allowing you to change the load inertia.

The purpose of the experiment is to design a controller which allows you to command a desired tip angle position. The output is the tip angle of the load w.r.t. the fixed inertial frame (the table on which the plant is placed). Thus the tip angle is the motor output angle (θ) plus the joint twist (α).



RFJ 1 Rotational Flexible Joint Module

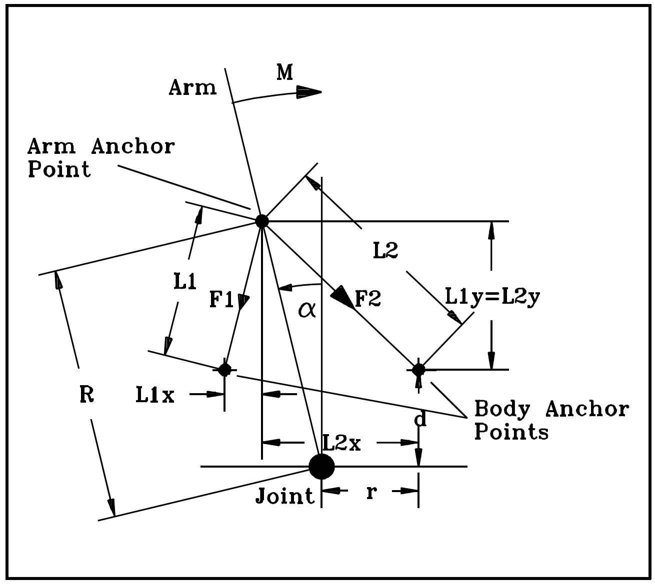
2.4.2 MATHEMATICAL MODELLING

2.4.2.1 *Deriving the Stiffness of the Joint*

Consider the diagram in Figure RFJ2. The arm is moved away from zero such that spring#1 is stretched to length L1 and spring#2 is stretched to length L2.

2.4.2.1.1 *Derive the length of each spring*

$$L_{1_x} = r - R\sin(\theta)$$
$$L_{1_y} = R\cos(\theta) - d$$
$$L_{2_x} = r + R\sin(\theta)$$
$$L_{2_y} = R\cos(\theta) - d$$
$$L_1 = \sqrt{(L_{1_x}^2 + L_{1_y}^2)}$$
$$L_2 = \sqrt{(L_{2_x}^2 + L_{2_y}^2)}$$



RFJ 2 System geometry when joint is not at rest.

2.4.2.1.2 *Derive the force in each spring*

Let **L** be the initial unstretched length of the spring, then each spring is generating a force pulling from the arm towards the body given by:

$$F_1 = K\,(L_1 - L) + Fr$$
$$F_2 = K\,(L_2 - L) + Fr$$

where Fr is the restoring force on each spring. This means the spring will not stretch before the force Fr is applied. Note it is assumed that the two springs have identical stiffness K and restoring force Fr.

2.4.2.1.3 *Derive the force components*

The forces generated by each spring can be decomposed to their x and y components as shown in Figure RFJ 2.

$$F_{1_x} = (F_1 \quad L_{1_x}/L_1)$$
$$F_{1_y} = (F_1 \quad L_{1_y}/L_1)$$
$$F_{2_x} = (F_2 \quad L_{2_x}/L_2)$$
$$F_{2_y} = (F_2 \quad L_{2_y}/L_2)$$

2.4.2.1.4 *Derive the moment*

The restoring moment due to these components is given by:

$$M = (R\cos(\theta)\ (F_{2_x} - F_{1_x}) - R\sin(\theta)\ (F_{1_y} + F_{2_y}));$$

2.4.2.1.5 *Linearize*

The above equation is nonlinear and can be linearized about the zero angle to obtain a linear estimate of the joint stiffness:

$$K_{STIFF} = \frac{\delta M}{\delta\theta}\Big|_{\theta=0}$$

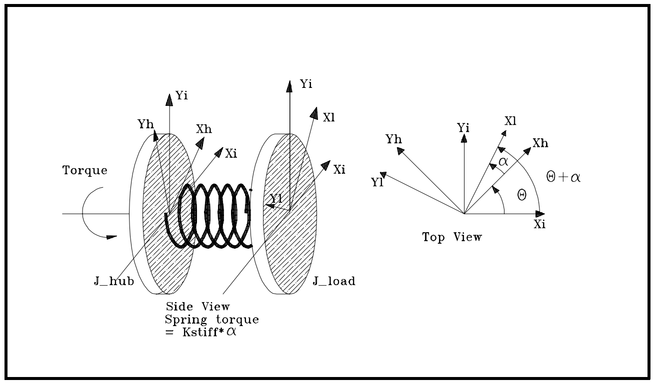
A MAPLE program is written that derives the above equations and evaluates the joint stiffness for a selected configuration. The complete expression for K_{STIFF} is given below:

$$K_{STIFF} = [2\frac{R}{D^{3/2}}]\ [(Dd - Rr^2)\ Fr + (D^{3/2}d - DLd + Rr^2L)\ K$$
$$\textit{with}$$
$$D = r^2 + (R-d)^2$$

2.4.2.2 Deriving the System Dynamic Equations

The dynamic equations are derived using the Euler Lagrange formulation as follows:

Consider the simplified diagram in Figure RFJ3 above.



RFJ 3 Simplified model for system dynamics

θ is the servo plant SRV-02 output angle while α is the relative angle of the arm to the plant output. This means α is the measurement of the angular deflection of the arm. The total output angle is $\alpha + \theta$. The total inertia at the motor output is given by J_{hub} and the total inertia of the arm is given as J_{load}. The spring stiffness is K_{STIFF}.

The kinetic and potential energies in the system are given by:

$$PE_{spring} = \frac{1}{2} K_{STIFF}\ \alpha^2$$
$$KE_{hub} = \frac{1}{2} J_{hub}\ \dot{\theta}^2$$
$$KE_{load} = \frac{1}{2} J_{load}\ (\dot{\theta} + \dot{\alpha})^2$$

The total kinetic energy and potential energies are:

$$T = KE_{hub} + KE_{load}$$

$$L = PE_{spring}$$

and the Lagrangian is given by:

$$L = T - V$$

We can now proceed to develop the equations of motion using the generalized Lagrangian formulation:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} = 0$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \tau$$

resulting in:

$$(J_{hub} + J_{load})\,\ddot{\theta} + J_{load}\,\ddot{\alpha} = \tau$$
$$J_{load}\,\ddot{\alpha} + J_{load}\,\ddot{\theta} + K_{stiff}\,\alpha = 0$$

The torque is generated by a DC motor with the following equations:

$$V = IR_m + K_m\,\omega_m = IR + K_m K_g\,\omega\ ;\ since\ \omega_m = K_g\omega$$
$$Thus \quad I = \frac{V}{R} - \frac{K_m K_g}{R}\omega$$
$$But \quad \tau = K_g\,\tau_m = K_g\,K_m\,I$$
$$Then \quad \tau = \frac{K_m K_g}{R}V - \frac{K_m^{\,2} K_g^{\,2}}{R}\omega$$

substituting for τ and solving for the accelerations we obtain the state space representation:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{K_{STIFF}}{J_{hub}} & -\dfrac{Km^2\ Kg^2}{RJ_{hub}} & 0 \\ 0 & -\dfrac{K_{STIFF}\,(J_{load} + J_{hub})}{J_{hub}\ J_{load}} & \dfrac{Km^2\ Kg^2}{RJ_{hub}} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{Km\ Kg}{RJ_{hub}} \\ -\dfrac{KmKg}{RJ_{hub}} \end{bmatrix}$$

$$X = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}'$$

Substituting the system parameters and using J_load = .0059 and K_STIFF = 1.61 Nm/rad (ie spring #2 at anchors [A,3]) we obtain:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 766 & -53 & 0 \\ 0 & -1040 & 53 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 99 \\ -99 \end{bmatrix}$$

2.4.3 CONTROL SYSTEM DESIGN

Using MATLAB LQR design, with

Q = diag([1000 4500 10 0]) and r = 10

we obtain the optimal feedback gain

k = [10 -13.7 1.2 0.42]

for units in volts per rad and volts/(rad/sec)

or

k = [.17 -.24 .012 .0074]

Note that feedback from the joint twist (α) is POSITIVE!.

The closed loop eigenvalues are at :

-103
-10.9
-8.8 +/- j12.7.

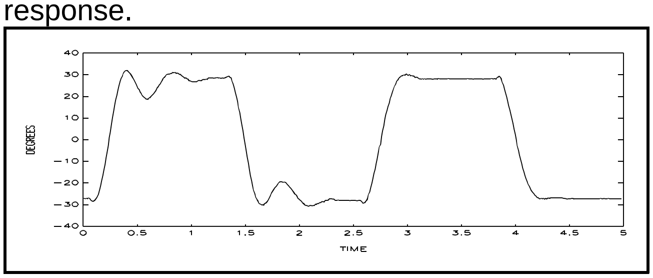
The new natural frequency of the system is at:

12.7/2/3.14 = 2.0 Hz.

The system can be dampened a little more by increasing the magnitude of k(4). The actual implementation uses a value of k(4) = .01.

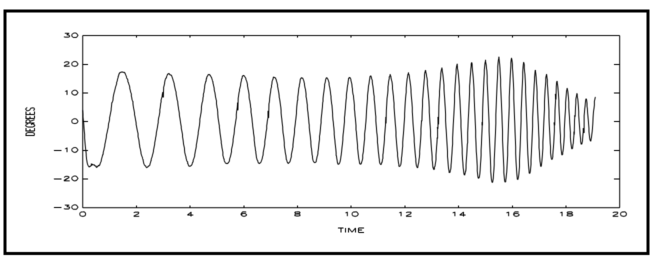
2.4.4 RESULTS

Figure RFJ4 shows the step response of the system using two types of controllers. The first two steps are tip angle (α+θ) when the controller has k2 and k4 set to zero (no deflection feedback) while the next two steps show the response when the controller is the full state feedback controller. Clearly joint twist feedback results in considerable improvement in the response.

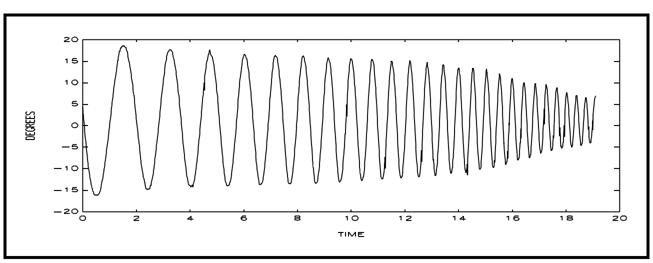


RFJ 4 Tip position step response using two types of control

The sine sweep response is also compared for the two types of control as shown in Figure RFJ5 (K2=K4=0) and Figure RFJ6 (Full state feedback). Note how the resonant peak is removed using full state feedback.



RFJ 5 Sine sweep response without deflection feedback



RFJ 6 Sine sweep response with deflection feedback