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Analogue Realizations of Fractional-Order Controllers

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Abstract. An approach to the design of analogue circuits, implementing fractional-order controllers, is presented. The suggested approach is based on the use of continued fraction expansions; in the case of negative coefficients in a continued fraction expansion, the use of negative impedance converters is proposed. Several possible methods for obtaining suitable rational appromixations and continued fraction expansions are discussed. An example of realization of a fractional-order I^{λ} controller is presented and illustrated by obtained measurements. The suggested approach can be used for the control of very fast processes, where the use of digital controllers is difficult or impossible.

Keywords: Fractional calculus, fractional differentiation, fractional integration, fractional-order controller, realization.

1. Introduction

Although digital controllers are used more and more frequently for controlling many types of complex processes, the role of analogue controllers should not be undervalued. Indeed, digital controllers have some natural limitations, coming from their discrete nature, such as the length of the sampling period and the time of computation, which should be significantly less than the length of the sampling period. This sometimes makes the use of digital controllers practically impossible, especially in case of fast processes, such as vibrations, and the alternative approach to controlling fast processes is represented by analogue controllers.

In this paper we describe an approach to the design of analogue fractional-order controllers.

The paper is organized as follows. First, we recall some basic relationships for describing fractional-order systems and fractional-order controllers. Then we discuss some uses of continued fraction expansions, including their applications in the control theory. Finally, we show how continued fraction expansions can be used for designing analogue circuits, implementing

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fractional-order systems and controllers. We also give an example of implementation of an I$^\lambda$ controller.

2. Fractional-Order Systems and Controllers

General information about various approaches to fractional-order differentiation and integration can be found in the available monographs on this subject [1–4] and in some other articles in this special issue. Because of this, we do not discuss general definitions here. Instead, we recall only the expressions for describing fractional-order systems and PI$^\lambda$D$^\mu$ controllers [3, 5], which are subjects of our interest in this paper.

2.1. Fractional Differential Equations and Transfer Functions

A fractional-order control system can be described by a fractional differential equation of the form

$$\begin{aligned} &a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \cdots + a_0 D^{\alpha_0} y(t) \\ &\quad = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \cdots + b_0 D^{\beta_0} u(t), \end{aligned} \tag{1}$$

or by a continuous transfer function of the form:

$$G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}, \tag{2}$$

where $D^\gamma \equiv {}_0D_t^\gamma$ denotes the Riemann-Liouville or Caputo fractional derivative [3]; a_k $(k = 0, \dots, n)$, b_k $(k = 0, \dots, m)$ are constant; and α_k $(k = 0, \dots, n)$, β_k $(k = 0, \dots, m)$ are arbitrary real numbers.

Without loss of generality we can assume that $\alpha_n > \alpha_{n-1} > \dots > \alpha_0$, and $\beta_m > \beta_{m-1} > \dots > \beta_0$.

2.2. PI$^\lambda$D$^\mu$ Controllers

The fractional-order PI$^\lambda$D$^\mu$ controller was proposed in [3, 5, 6] as a generalization of the PID controller with integrator of real order λ and differentiator of real order μ. The transfer function of such type the controller in Laplace domain has form:

$$G_c(s) = \frac{U(s)}{E(s)} = K + T_i\, s^{-\lambda} + T_d\, s^{\mu} \quad (\lambda, \mu > 0), \tag{3}$$

where K is the proportional constant, T_i is the integration constant and T_d is the differentiation constant. As we can see (Figure 1), the internal structure of the fractional-order controller consists of the parallel connection the proportional, integration, and derivative part [7]. Transfer function (3) corresponds in time domain with fractional differential equation (4)

$$u(t) = Ke(t) + T_i\, {}_0D_t^{-\lambda} e(t) + T_d\, {}_0D_t^{\mu}. \tag{4}$$

Taking $\lambda = 1$ and $\mu = 1$, we obtain a classical PID controller. If $\lambda = 0$ and/or $T_i = 0$, we obtain a PD$^\mu$ controller, etc. All these types of controllers are particular cases of the fractional-order controller, which is more flexible and gives an opportunity to better adjust the dynamical properties of the fractional-order control system.

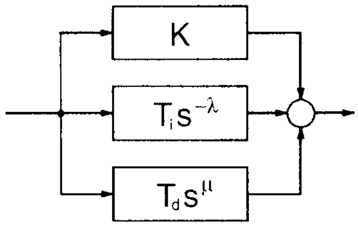

Figure 1. General structure of a $PI^{\lambda}D^{\mu}$ controller.

As we see from Figure 1, a $PI^{\lambda}D^{\mu}$ controller can be easily implemented in analogue form if we know how to build an analogue circuit corresponding to s^{α}, $\alpha \in R$. Below we will demonstrate how this can be done using rational approximations and continued fraction expansions.

It can also be mentioned that the other kind of fractional-order controller, which characterized by the band-limited lead effect, can be found in the available literature [8, 9]:

$$G_c(s) = C\left(\frac{1+\tau s^r}{1+\tau' s^r}\right), \quad r \in R, C \in R, \tau' < \tau. \tag{5}$$

This type of controller can be realized using a recursive distribution of poles and zeros [10].

3. Some Uses of Continued Fractions

In this section we discuss some applications of continued fractions. First we recall their use for approximating functions and investigating stability of linear systems. Then we introduce a new relationship between continued fractions and multiple nested-loop systems.

3.1. CFEs and Approximations of Functions

It is well known that the Continued Fraction Expansions (CFE) is a method for evaluation of functions, that frequently converges much more rapidly than power series expansions, and converges in a much larger domain in the complex plane [11]. The result of such approximation for an irrational function, $G(s)$, can be expressed in the form:

$$\begin{aligned} G(s) &\simeq a_0(s) + \cfrac{b_1(s)}{a_1(s) + \cfrac{b_2(s)}{a_2(s) + \cfrac{b_3(s)}{a_3(s)+\cdots}}} \\ &= a_0(s) + \frac{b_1(s)}{a_1(s)+}\,\frac{b_2(s)}{a_2(s)+}\,\frac{b_3(s)}{a_3(s)+}\,\cdots, \end{aligned} \tag{6}$$

where $a_i s$ and $b_i s$ are rational functions of the variable s, or are constant. The application of the method yields a rational function, $\widehat{G}(s)$, which is an approximation of the irrational function $G(s)$.

On the other hand, for interpolation purposes, rational functions are sometimes superior to polynomials. This is, roughly speaking, due to their ability to model functions with poles. (As it can be seen later, branch points can be considered as accumulations of interlaced poles and zeros). These techniques are based on the approximations of an irrational function, $G(s)$, by a rational function defined by the quotient of two polynomials in the variable s:

$$\begin{aligned} G(s) \;&\simeq\; R_{i(i+1)\dots(i+m)} = \frac{P_\mu(s)}{Q_\nu(s)}, \\ &= \frac{p_0 + p_1 s + \cdots + p_\mu s^\mu}{q_0 + q_1 s + \cdots + q_\nu s^\nu} \\ m+1 &= \mu + \nu + 1, \end{aligned} \tag{7}$$

passing through the points $(s_i, G(s_i)), \dots, (s_{i+m}, G(s_{i+m}))$.

3.2. CFE and Stability of Linear Systems

It is also known that continuous fraction expansions can be used for investigating stability of linear systems. For this, the characteristic polynomial $Q(s)$ of the differential equation of the system should be divided in two parts, the ‘even’ part (containing even powers of s) and the ‘odd’ part (containing odd powers of s):

$$Q(s) = m(s) + n(s).$$

Then these two parts of the characteristic polynomial are used for creating its *test function* in the form of a fraction, in which the highest power of s is contained in the denominator:

$$R(s) = \frac{m(s)}{n(s)} \quad \left(\text{or} \quad R(s) = \frac{n(s)}{m(s)}\right).$$

The rational function $R(s)$ should be written in the form of a continuous fraction:

$$R(s) = \cfrac{1}{b_1 s + \cfrac{1}{b_2 s + \cfrac{1}{\cfrac{\cdots\cdots\cdots\cdots}{b_{n-1} s + \cfrac{1}{b_n s}}}}}. \tag{8}$$

If $b_k > 0$, $k = 1, \dots, n$, then the system is stable. If some b_k is negative, then the system is unstable.

Considering the continued fraction (8) as a tool for designing a corresponding LC circuit, we can conclude that stability of a linear system is equivalent to realizability of its test function $R(s)$ with the help of only passive electric components.

3.3. CFE and Nested Multiple-Loop Control Systems

Let us now establish an interesting new relationship between continued fractions and nested multiple-loop control systems.

We first recall the known fact that the transfer function $R(s)$ of the control loop with a negative feedback shown in Figure 2 is given by [7]

$$R(s) = \frac{G(s)}{1 + G(s)H(s)}. \tag{9}$$

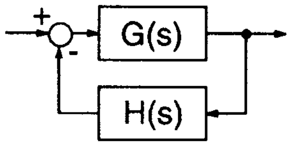

Figure 2. A control loop with a negative feedback.

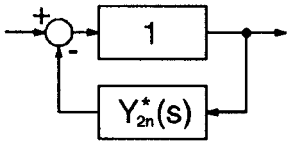

Figure 3. Nested multiple-loop control system – level 1.

From (9) it immediately follows that the transfer function of the circuit shown in Figure 3 is

$$P_{2n}(s) = \frac{1}{1 + 1 \cdot Y^*_{2n}(s)} = \frac{1}{Y_{2n}(s)}, \tag{10}$$

where $Y_{2n}(s) = Y^*_{2n}(s) + 1$.

Using Equations (9) and (10) we obtain the transfer function of the system shown in Figure 4:

$$Q_{2n-1}(s) = Z_{2n-1}(s) + P_{2n}(s) = Z_{2n-1}(s) + \frac{1}{Y_{2n}(s)}. \tag{11}$$

Combining Equations (9) and (10) we find the transfer function of the nested multiple-loop system shown in Figure 5:

$$\begin{aligned} P_{2n-2}(s) &= \frac{Q_{2n-1}(s)}{1 + Q_{2n-1}(s)Y_{2n-2}(s)} = \frac{1}{Y_{2n-2}(s) + \cfrac{1}{Q_{2n-1}(s)}} \\ &= \frac{1}{Y_{2n-2}(s) + \cfrac{1}{Z_{2n-1}(s) + \cfrac{1}{Y_{2n}(s)}}}. \end{aligned} \tag{12}$$

The transfer function of the system shown in Figure 6 is then given by the relationship

$$\begin{aligned} Q_{2n-3}(s) &= Z_{2n-3}(s) + P_{2n-2}(s) \\ &= Z_{2n-3}(s) + \frac{1}{Y_{2n-2}(s) + \cfrac{1}{Z_{2n-1}(s) + \cfrac{1}{Y_{2n}(s)}}}. \end{aligned} \tag{13}$$

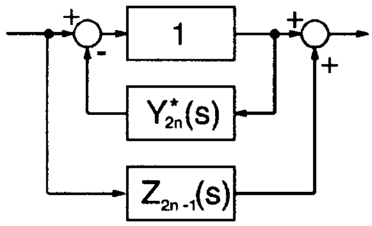

Figure 4. Nested multiple-loop control system – level 2.

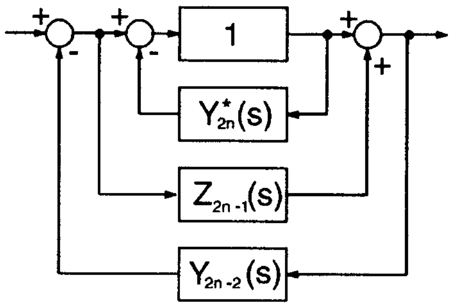

Figure 5. Nested multiple-loop control system – level 3.

Continuing this process, we obtain the transfer function of the nested multiple-loop control system shown in Figure 7 in the form of a continued fraction expansion, which is identical with the Equation (24):

$$Z(s) = Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_3(s) + \cfrac{1}{Y_4(s) + \cfrac{1}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \atop \cfrac{1}{Y_{2n-2}(s) + \cfrac{1}{Z_{2n-1}(s) + \cfrac{1}{Y_{2n}(s)}}}}}}}.$$

Similarly to the above considerations, we can obtain a continued fraction expansion of the transfer function of the other interesting type of a nested multiple-loop control system, depicted in Figure 8:

$$Z(s) = \cfrac{1}{Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_3(s) + \cfrac{1}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \atop \cfrac{1}{Y_{2n-2}(s) + \cfrac{1}{Z_{2n-1}(s) + \cfrac{1}{Y_{2n}(s)}}}}}}}. \tag{14}$$

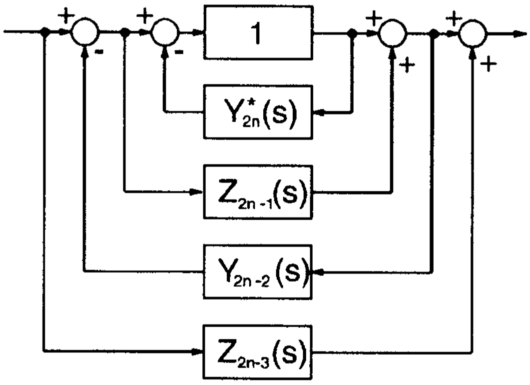

Figure 6. Nested multiple-loop control system – level 4.

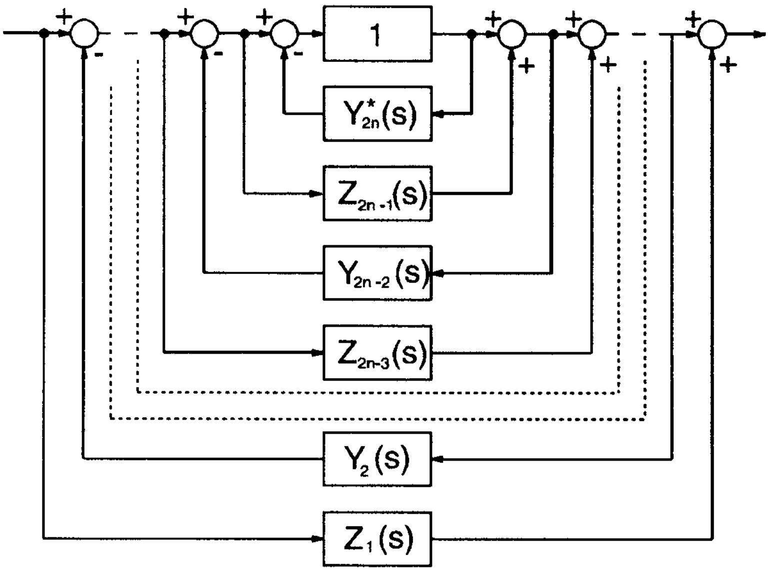

Figure 7. Nested multiple-loop control system of the first type.

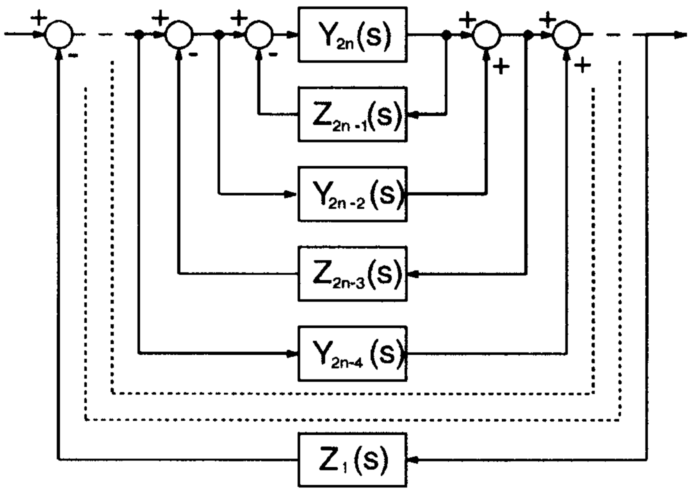

Figure 8. Nested multiple-loop control system of the second type.

Both types of nested multiple-loop systems, presented in this section, can be used for simulations and realizations of arbitrary transcendental transfer functions. For this, the transfer function should be developed in a continued fraction, which after truncation can be represented by a nested multiple-loop system shown in Figures 7 or 8.

4. CFE and Rational Approximations of s^{α}

In general [12], a rational approximation of the function $G(s) = s^{-\alpha}$, $0 < \alpha < 1$ (the fractional integral operator in the Laplace domain) can be obtained by performing the CFE of the functions:

$$G_h(s) = \frac{1}{(1+sT)^{\alpha}}, \tag{15}$$

$$G_l(s) = \left(1+\frac{1}{s}\right)^{\alpha}, \tag{16}$$

where $G_h(s)$ is the approximation for high frequencies ($\omega T \gg 1$), and $G_l(s)$ the approximation for low frequencies ($\omega \ll 1$).

EXAMPLE 1. Performing the CFE of the function (15), with $T = 1$, $\alpha = 0.5$, we obtain

$$H_1(s) = \frac{0.3513s^4 + 1.405s^3 + 0.8433s^2 + 0.1574s + 0.008995}{s^4 + 1.333s^3 + 0.478s^2 + 0.064s + 0.002844}.$$

EXAMPLE 2. Performing the CFE of the function (16), with $T = 1$, $\alpha = 0.5$, we obtain

$$H_2(s) = \frac{s^4 + 4s^3 + 2.4s^2 + 0.448s + 0.0256}{9s^4 + 12s^3 + 4.32s^2 + 0.576s + 0.0256}.$$

5. Other Rational Approximations for s^{α}

Besides using continued fractions, there are also other methods [13] for obtaining rational approximations of fractional-order systems. However, since a ratio of two polynomials can be expressed in the form of a finite continued fraction, any rational approximation is equivalent to a certain finite continued fraction.

5.1. Carlson's Method

The method proposed by Carlson in [14], derived from a regular Newton process used for iterative approximation of the α-th root, can be considered as belonging to this group. The starting point of the method is the statement of the following relationships:

$$(H(s))^{1/\alpha} - (G(s)) = 0; \qquad H(s) = (G(s))^{\alpha}. \tag{17}$$

Defining $\alpha = 1/q$, $m = q/2$, in each iteration, starting from the initial value $H_0(s) = 1$, an approximated rational function is obtained in the form:

$$H_i(s) = H_{i-1}(s)\frac{(q-m)\,(H_{i-1}(s))^2 + (q+m)G(s)}{(q+m)\,(H_{i-1}(s))^2 + (q-m)G(s)}. \tag{18}$$

EXAMPLE 3. Starting from $H(s) = (1/s)^{1/2}$, $H_0(s) = 1$, after two iterations, we obtain

$$H_3(s) = \frac{s^4 + 36s^3 + 126s^2 + 84s + 9}{9s^4 + 84s^3 + 126s^2 + 36s + 1}.$$

5.2. Matsuda's Method

The method suggested in [15] is based on the approximation of an irrational function by a rational one, obtained by CFE and fitting the original function in a set of logarithmically spaced points. Assuming that the selected points are s_k, $k = 0, 1, 2, \ldots$, the approximation takes on the form:

$$H(s) = a_0 + \frac{s - s_0}{a_1 +} \, \frac{s - s_1}{a_2 +} \, \frac{s - s_2}{a_3 +} \, \cdots, \tag{19}$$

where

$$a_i = v_i(s_i), \quad v_0(s) = H(s), \quad v_{i+1}(s) = \frac{s - s_i}{v_i(s) - a_i}. \tag{20}$$

EXAMPLE 4. With $G(s) = (1/s)^{1/2}$, $f_{\text{initial}} = 1$, $f_{\text{final}} = 100$, $f_k = \{1, 1.7783, 3.1623, 5.6234, 10, 17.783, 31.623, 56.234, 100\}$, we obtain

$$H_4(s) = \frac{0.08549s^4 + 4.877s^3 + 20.84s^2 + 12.995s + 1}{s^4 + 13s^3 + 20.84s^2 + 4.876s + 0.08551}.$$

5.3. Oustaloup's Method

The method [8–10] is based on the approximation of a function of the form:

$$H(s) = s^{\delta}, \quad \delta \in R^{+} \tag{21}$$

by a rational function

$$\widehat{H}(s) = C \prod_{k=-N}^{N} \frac{1 + s/\omega_k}{1 + s/\omega_k'}, \tag{22}$$

using the following set of synthesis formulas:

$$\begin{aligned}
\omega_0' &= \alpha^{-0.5}\omega_u, \quad \omega_0 = \alpha^{0.5}\omega_u, \quad \frac{\omega_{k+1}'}{\omega_k'} = \frac{\omega_{k+1}}{\omega_k} = \alpha\eta > 1, \\
\frac{\omega_{k+1}'}{\omega_k} &= \eta > 0, \quad \frac{\omega_k}{\omega_k'} = \alpha > 0, \quad N = \frac{\log(\omega_N/\omega_0)}{\log(\alpha\eta)}, \quad \delta = \frac{\log\alpha}{\log(\alpha\eta)},
\end{aligned} \tag{23}$$

with ω_u being the unit gain frequency and the central frequency of a band of frequencies geometrically distributed around it. That is, $\omega_u = \sqrt{\omega_h \omega_b}$, ω_h, ω_b are the high and low transitional frequencies.

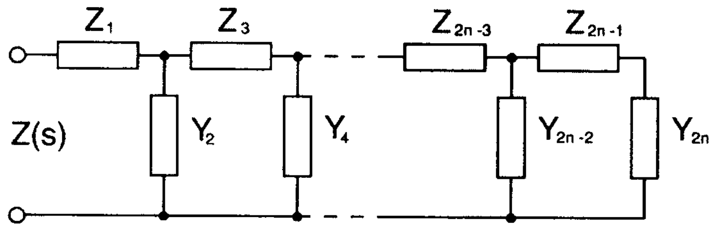

Figure 9. Finite ladder circuit.

EXAMPLE 5. Using the Oustaloup's method with

$$\omega_h = 10^2, \quad \omega_b = 10^{-2},$$

from which we have $\alpha = \eta = 2.5119$, the obtained approximation for $s^{-1/2}$ is

$$H_5(s) = \frac{s^5 + 74.97s^4 + 768.5s^3 + 1218s^2 + 298.5s + 10}{10s^5 + 298.5s^4 + 1218s^3 + 768.5s^2 + 74.97s + 1}.$$

6. Design of Fractances Based on Rational Approximations and CFEs

A circuit exhibiting fractional-order behaviour is called a *fractance* [3].

Design of fractances can be done easily using any of the aforementioned rational approximations or a truncated CFE, which also gives a rational approximation (see, for example, [16]). Truncated CFE does not require any further transformation; a rational approximation based on any other methods must be transformed to the form of a continued fraction. The values of the electric elements, which are necessary for building a fractance, are then determined from the obtained finite continued fraction. If all coefficients of the obtained finite continued fraction are positive, then the fractance can be made of classical passive elements (resistors and capacitors). If some of the coefficients are negative, then the fractance can be made with the help of negative impedance converters (Section 6.2).

6.1. Domino Ladder Circuit

Let us consider the circuit depicted in Figure 9, where $Z_{2k-1}(s)$ and $Y_{2k}(s)$, $k = 1, \ldots, n$, are given impedances of the circuit elements. The resulting impedance $Z(s)$ of the entire circuit can be found easily, if we consider it in the right-to-left direction:

$$Z(s) = Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_3(s) + \cfrac{1}{Y_4(s) + \cfrac{1}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \atop \cfrac{1}{Y_{2n-2}(s) + \cfrac{1}{Z_{2n-1}(s) + \cfrac{1}{Y_{2n}(s)}}}}}}}. \tag{24}$$

The relationship between the finite domino ladder network, shown in Figure 9, and the continued fraction (24) provides an easy method for designing a circuit with a given impedance

$Z(s)$. For this one has to obtain a continued fraction expansion for $Z(s)$. Then the obtained particular expressions for $Z_{2k-1}(s)$ and $Y_{2k}(s)$, $k = 1, \ldots, n$, will give the types of necessary components of the circuit and their nominal values.

EXAMPLE 6. To design a circuit with the impedance

$$Z(s) = \frac{s^4 + 4s^2 + 1}{s^3 + s}, \tag{25}$$

we have to develop $Z(s)$ in continued fraction

$$Z(s) = \frac{s^4 + 4s^2 + 1}{s^3 + s} = s + \cfrac{1}{\frac{1}{3}s + \cfrac{1}{\frac{9}{2}s + \cfrac{1}{\frac{2}{3}s.}}} \tag{26}$$

From this expansion it follows that

$$Z_1(s) = s, \quad Z_3(s) = \frac{9}{2}s, \quad Y_2(s) = \frac{1}{3}s, \quad Y_4(s) = \frac{2}{3}s.$$

Therefore, for the analogue realization in the form of the first Cauer's canonic LC circuit [17] we have to choose the following values of coils and capacitors:

$$L_1 = 1\,[H], \quad L_3 = \frac{9}{2}\,[H], \quad C_2 = \frac{1}{3}\,[F], \quad C_4 = \frac{2}{3}\,[F].$$

EXAMPLE 7. The function $Z(s)$ given by Equation (25) can be written also in the form

$$Z(s) = \frac{s^4 + 4s^2 + 1}{s^3 + s} = \frac{1}{s} + \cfrac{1}{\frac{1}{3s} + \cfrac{1}{\frac{9}{2s} + \cfrac{1}{\frac{2}{3s}.}}} \tag{27}$$

From this expansion it follows that

$$Z_1(s) = \frac{1}{s}, \quad Z_3(s) = \frac{9}{2s}, \quad Y_2(s) = \frac{1}{3s}, \quad Y_4(s) = \frac{2}{3s}.$$

Therefore, for the analogue realization in the form of the second Cauer's canonic LC circuit [17] we have to choose the following values of coils and capacitors:

$$C_1 = 1\,[F], \quad C_3 = \frac{2}{9}\,[F], \quad L_2 = 3\,[H], \quad L_4 = \frac{3}{2}\,[H].$$

EXAMPLE 8. To design a circuit with the impedance

$$Z(s) = \frac{s^4 + 3s^2 + 8}{2s^3 + 4s}, \tag{28}$$

one has to obtain a continuous fraction representation of the function $Z(s)$,

$$Z(s) = \frac{s^4 + 3s^2 + 8}{2s^3 + 4s} = \frac{1}{2}s + \cfrac{1}{2s + \cfrac{1}{-\frac{1}{12}s + \cfrac{1}{-\frac{3}{2}s}}}. \tag{29}$$

From this expansion it follows that

$$Z_1(s) = \frac{1}{2}s, \quad Z_3(s) = -\frac{1}{12}s, \quad Y_2(s) = 2s, \quad Y_4(s) = -\frac{3}{2}s.$$

Therefore, for the analogue realization in the form of the first Cauer's canonic LC circuit [17] we have to choose the following values of coils and capacitors:

$$L_1 = \frac{1}{2}\,[H], \quad L_3 = -\frac{1}{12}\,[H], \quad C_2 = 2\,[F], \quad C_4 = -\frac{3}{2}\,[F].$$

Here we see negative inductances and capacitance. Such elements cannot be realized using passive electric components. However, they can be realized with the help of active components, namely operating amplifiers.

6.2. Negative-Impedance Converters

The previous example shows that the use of CFE for analogue realization of arbitrary transfer functions may lead to the appearance of negative impedances. This observation is not unknown. For example, in the paper [12], Dutta Roy recalls Khovanskii's continued fraction expansion for $x^{1/2}$ found in [18] and makes a remark that

> ... if x is replaced by the complex frequency variable s, then the realization would require a negative resistance. Thus, the [Khovanskii's] CFEs do not seem to be useful for realization of fractional inductor or capacitor.

Then he describes a method for circumventing this difficulty, which gives a continued fraction expansion with positive coefficients.

However, the possibility of realization of negative impedances in electric circuits has been pointed out by Bode [19, chapter IX]. Later, in 1970s, operational amplifiers appeared, which significantly simplified creation of circuits exhibiting negative resistances, negative capacitances, and negative inductances. Such circuits are called *negative-impedance converters* [20].

The simplest scheme of a negative-impedance converter (or current inverter) is shown in Figure 10. The circuit consists of an operational amplifier, two resistors of equal resistance R, and a component with the impedance Z. The entire circuit, considered as a single element, has negative impedance $-Z$. This means that $I_{\text{in}} = V_{\text{in}}/(-Z)$).

For example, taking a resistor of resistance R_Z instead of the element Z, we obtain a circuit, which behaves like a negative resistance $-R_Z$. The negative resistance means that if such an element of negative resistance, for instance, -10 kΩ is connected in series with a classical 20 kΩ resistor, then the resistance of the resulting connection is 10 kΩ.

Let us now recall Example 8. Using negative-impedance converters, it is possible to design a circuit with the required impedance $Z(s)$, which will contain a negative capacitance C_4 and a negative inductance L_3.

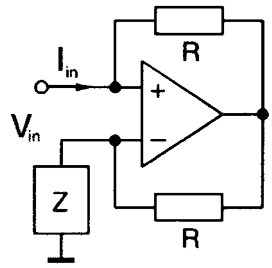

Figure 10. Negative-impedance converter.

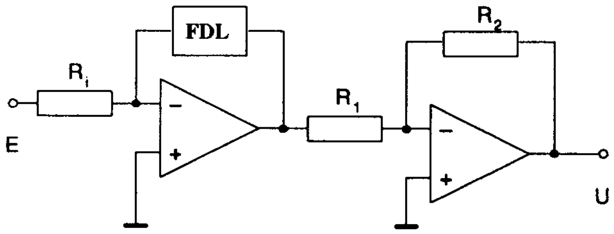

Figure 11. Analogue fractional-order I^λ controller.

7. Example: Fractional-Order I^λ Controller

For experimental measurement we built a fractional-order I^λ controller which is a particular case of the $PI^\lambda D^\mu$ controller, (if $K = 0$ and $T_d = 0$). The controller was realized in the form of the finite domino ladder ($n = 12$), connected to feedback in operational amplifier (Figure 11). It should be noted that the described methods work for arbitrary orders, but the circuit elements with computed values are not usually available. Because of this, in our experiment we proposed and realized the integrator with order $\lambda = 0.5$. It should be mentioned that this simple case of the controller order can be realized also using the methods described in [21–24], which do not involve explicit rational approximations.

In the case, if we will use identical resistors (R-series) and identical capacitors (C-shunt) in the FDL, then the behaviour of the circuit will be as a half-order integrator/differentiator. We used the resistor values $R = 1k\Omega$ ($R_j = R,\ j = 1, \dots, n$) and the capacitor values $C = 1\mu F$ ($C_j = C,\ j = 1, \dots, n$). For better measurement results we used two operational amplifiers TL081CN in inverting connection.

A block diagram of the analogue fractional-order I^λ controller realization is shown in Figure 11.

The resistors R_1 and R_2 are $R_1 = R_2 = 22$ kΩ. The integration constant T_i can be computed from relationship $T_i = 1/\sqrt{R/(R_i^2 * C)}$, and for $R_i = 22$ kΩ we have $T_i = 1.4374$. The transfer function of the realized analogue fractional-order I^λ controller is

$$G_c(s) = 1.4374\ \mathrm{s}^{-0.5}. \tag{30}$$

Adjustment of the integration constant T_i of the fractional-order I^λ controller depicted in Figure 11 was done by resistor R_i. If we change the resistor R_i, the integration constant changes the value in the required interval.

In Figures 12 and 13 the measured characteristics of realized analogue fractional-order I^λ controller are presented. In Figure 12 Bode plots a shown, and in Figure 13 is the time response

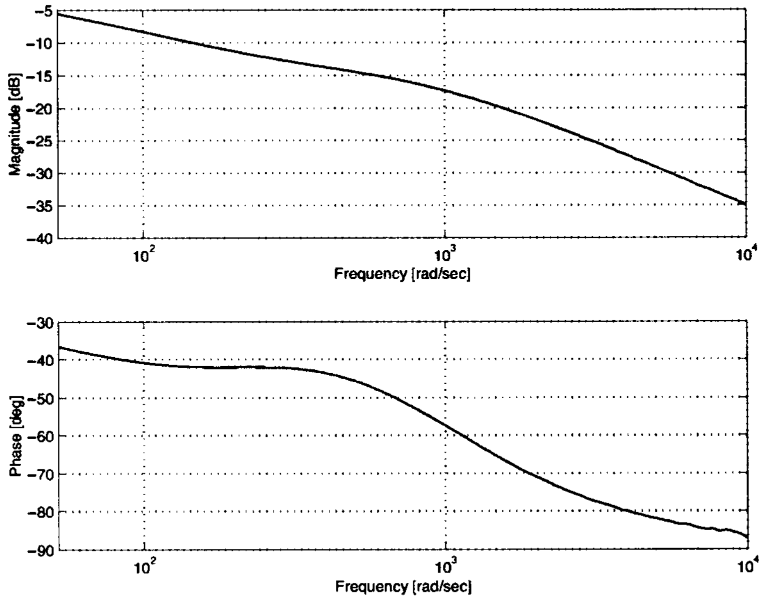

Figure 12. Bode plots of the $I^{1/2}$ controller (measurements).

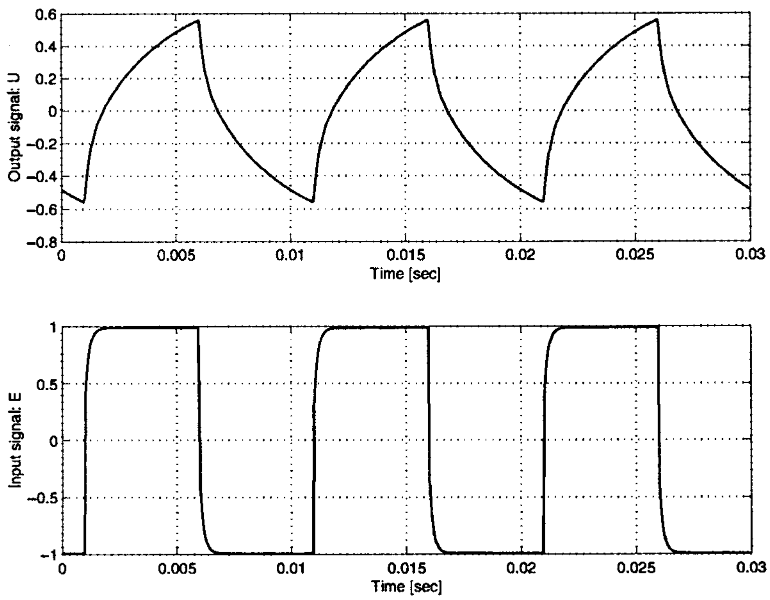

Figure 13. Time response of the $I^{1/2}$ controller to unit step input (measurements).

to the square input signal (unit step). We used frequency 100 Hz and amplitude ± 10 V. It can be seen from Figure 12 that the realized analogue of fractional-order I^{λ} controller provides a good approximation in the frequency range $[10^2 \text{ rad/sec}, 5 \cdot 10^2 \text{ rad/sec}]$.

Measurements were done using IWATSU Digital Storagescope DS-8617 100 MHz, Hewlett Packard 35670A dynamic signal analyzer, Hewlett Packard 33120A 15 MHz function/arbitrary waveform generator, power supply Thurlby-Thandar PL320QMD.

8. Conclusion

In this paper we have demonstrated that the suggested use of continued fraction expansions is a good general method for obtaining analogue devices (fractances) described by fractional differential equations or by fractional-order transfer functions. Moreover, this approach can be used for realization of other types of systems with transcendental transfer functions, which can be developed in continued fractions. Furthermore, it has been shown that any rational approximation of the transfer function can be used for designing the corresponding analogue circuit, even if some of the coefficients of the resulting continued fraction are negative.

We have also introduced two types of nested multiple-loop systems, which can be easily used for modelling, simulation, and realization of fractional-order systems and controllers, and more generally for modelling, simulation and realization of systems, for which a rational approximation of the transfer function can be obtained.

The exposition has been illustrated with several examples, including analogue realization of an I^{λ} controller, for which experimental results were presented.

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