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ArticleTI: IIR approximations to the fractional differentiator/integrator using Chebyshev polynomials theory
Year: 2013 Vol.vol.52 iss.4 pp.461-468; ISSN: 00190578
Imprint: 2013
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Research Article

IIR approximations to the fractional differentiator/integrator using Chebyshev polynomials theory

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ARTICLE INFO

Article history:
Received 23 October 2012
Accepted 12 February 2013
Available online 15 March 2013
This paper was recommended for publication by Dr. Y. Chen.

Keywords:
Chebyshev approximation
Discretization
Fractional calculus
IIR filters

ABSTRACT

This paper deals with the use of Chebyshev polynomials theory to achieve accurate discrete-time approximations to the fractional – order differentiator/integrator in terms of IIR filters. These filters are obtained using the Chebyshev – Padé and the Rational Chebyshev approximations, two highly accurate numerical methods that can be computed with ease using available software. They are compared against other highly accurate approximations proposed in the literature. It is also shown how the frequency response of the fractional-order integrator approximations can be easily improved at low frequencies.

1. Introduction

Fractional-order calculus deals with derivatives and integrals of arbitrary – real or complex – order [1,2]. Intuitively, fractional derivatives (integrals) interpolate between the familiar integer-order derivatives (integrals). In the last decades the importance of fractional calculus in fields different to pure theoretical mathematics has been pointed out, as new non-integer order models have been developed to describe several physical systems and engineering applications with more accuracy than their integer order counterparts [3].

Nowadays, we can find abundant applications of fractional calculus to diverse scientific disciplines where it is a focus of interest. For example and without the intention of being exhaustive, in signal processing, we can mention applications in digital image sharpening [4], in synthesizing multifractional Gaussian noise [5], in image edge detection [6], and in dielectric spectroscopy [7]; in control engineering, we find works on industrial applications of fractional PID controllers [8], CRONE control of continuous linear time periodic systems [9], practical applications of fractional predictive control [10], application of classical controller design techniques to fractional environments [11], and system model identification [12]; applications of fractional calculus to physics include chaotic systems, polymer science, rheology, and thermodynamics [13]; applications in biophysics are considered in [13,14]; and so on.

Despite their apparent simplicity, transfer functions such as (1) – the simplest fractional-order transfer function – are not easy to implement for computational purposes as simulation software normally works only with integer powers of s, and its discrete form has an unlimited number of z-terms [1]. For this reason, finding integer order approximations to fractional transfer functions is a very important task.

$$G(s) = s^{\alpha}, \quad \alpha \in \Re \tag{1}$$

In order to obtain discrete approximations many methods have been proposed. This paper focuses on discrete IIR-Infinite Impulse Response-approximation filters (2). (For an introduction to FIR-Finite Impulse Response-approximations, see [15,16] and the references therein)

$$G(z) = \frac{\sum_{i=0}^{p} b_i z^i}{\sum_{j=0}^{q} a_j z^j} \tag{2}$$

In general, there are two main ways to discretize fractional-order systems: direct methods and indirect methods. Indirect methods involve frequency domain fitting in continuous time domain and then discretizing the fit s-transfer function [17]. Direct methods are expressed in terms of a generating function, $g(z^{-1})$, such as Euler (3), Tustin (4), or Al-Alaoui (5)

$$g_E(z^{-1}) = \frac{1}{T_S}(1 - z^{-1}) \tag{3}$$

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http://dx.doi.org/10.1016/j.isatra.2013.02.002

$$g_T(z^{-1}) = \frac{2}{T_S}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{4}$$

$$g_A(z^{-1}) = \frac{8}{7T_s}\left(\frac{1-z^{-1}}{1+(z^{-1}/7)}\right) \tag{5}$$

With direct methods, the fractional-order differentiator/integrator is thus approximated as

$$s^{\alpha} \sim [g(z^{-1})]^{\alpha} \tag{6}$$

In general, direct methods proposed in literature treat (6) just as a rational function with infinite terms. A suitable method is then used to achieve a finite-order approximation in terms of z^{-1} as independent variable. For instance, one can use the well-known power series expansion (PSE) to truncate the z-series [16,18]; the application of continued fraction expansion (CFE) is discussed in [19–21]; some other direct methods can be found in [22,23] and their references. All suitable approximations share some key characteristics: their zeros and poles are all stable, simple and real and exhibit zero-pole interlacing. Thus, they lead to minimum-phase and stable realizations [24,25].

As well as digital differentiators, digital integrators are important in areas such as radar, biomedical engineering, control (to avoid non−zero steady−state error), etc., where it is necessary to guarantee "integral" slope and phase at any frequency. For this reason, the design of fractional-order integrators has been widely studied in literature [22,29]. Their design is basically the same problem as the design of fractional-order differentiators. However, it is important to point out that when fractional integrators are approximated, the integration effect is usually lost at low frequencies, as we shall see later.

For $|\alpha| > 1$, it is recommended in the literature to split the fractional derivatives/integrals into an integer and a fractional part (7) [25, 28]. Hence, we shall focus on the case $|\alpha| < 1$

$$G(s) = s^{\alpha} = s^{r}s^{\alpha'}, \quad \alpha \in \Re, \quad r \in \mathbb{Z}, \quad \alpha' \in (-1,1) \tag{7}$$

Following the direct methodology, this work focuses on the use of numerical methods based on Chebyshev polynomials theory to obtain finite rational expressions to approximate the infinite terms expression (6). Thus, we obtain finite dimensional IIR filters that approximate the fractional behavior of (1). The Chebyshev−Padé [26] and the Rational Chebyshev [27] approximations will be considered to carry out this task due to their high accuracy; moreover, there exist easily available software implementations, and so the user can compute them without having to program the corresponding algorithms.

The paper is organized as follows: In Section 2 the fundamentals of Chebyshev methods are introduced. In Section 3 the fractional-order differentiator/integrator is discretized with both Chebyshev approximations, and the results are compared against some highly accurate methods described in the literature. Moreover, it is shown how the response of the fractional integrator approximations can be improved at low frequencies. Finally, Section 4 draws the main conclusions of this work.

2. Fundamentals

2.1. Chebyshev polynomials

It is well known that Chebyshev polynomials provide approximations very close to the true continuous functions due to their fast convergence properties. However, the theory of Chebyshev-based approximations is too vast to be described here in detail. For this reason, in this section only the most important concepts will be introduced. A complete description can be found in [30].

The first-kind Chebyshev polynomial of degree n, $T_n(x)$, is defined by

$$T_n(x) = \cos(n \cdot \arccos x) \tag{8}$$

We can deduce from (8) that this Chebyshev polynomial verifies the recurrence relationship $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$, $n \geq 1$, with the initial conditions $T_0(x)=1$ and $T_1(x)=x$.

The Chebyshev polynomials of degree $n>0$ have n zeros and $n+1$ extrema in the interval $[-1,1]$. The zeros for x of $T_n(x)$ must correspond to the zeros for θ in $[0,\pi]$ of $\cos n\theta$, so that

$$n\theta = (k-1/2)\pi \quad (k=1,2,\ldots,n) \tag{9}$$

Hence, the zeros of $T_n(x)$ are

$$x_k = \cos\frac{(k-1/2)\pi}{n} \quad (k=1,2,\ldots,n) \tag{10}$$

If $f(x)$ is an arbitrary function in the interval [−1,1] and if H coefficients c_j, $j=0, \ldots, H-1$, are defined by

$$c_j = \frac{2}{H}\sum_{k=1}^{H} f(x_k)T_j(x_k) \tag{11}$$

then the approximation

$$f(x) \approx \left[\sum_{k=0}^{H-1} c_k T_k(x)\right] - \frac{1}{2}c_0 \tag{12}$$

is exact for x equal to all the H zeros of $T(x)$. This approximation is very nearly the same *minimax* polynomial which, among all polynomials of same degree, has the smallest maximum deviation from the true function $f(x)$ [30,31].

However, the Chebyshev approximation is a series. In order to obtain IIR filters (2), rational approximations are needed. For this reason, in the following of this section we shall consider two rational approximations based on Chebyshev polynomials: the Chebyshev–Padé and the Rational Chebyshev approximations.

2.2. Chebyshev−Padé approximation

The Chebyshev−Padé approximation of a given function $f(x)$ is obtained with the Chebyshev series approximation of f (12) followed by its Padé rational approximation [31,32]. A Padé approximant is a rational function of a specified degree whose power series expansion agrees with a given power series to the highest possible degree. The rational function

$$R(x) \equiv \sum_{k=0}^{m} p_k x^k \Bigg/ \left(1 + \sum_{k=1}^{n} q_k x^k\right) \tag{13}$$

is said to be a Padé approximant to the (Chebyshev) series

$$f(x) \equiv \sum_{i=0}^{\infty} d_i x^i \tag{14}$$

if

$$R(0) = f(0) \tag{15}$$

and also

$$\left.\frac{d^k}{dx^k}R(x)\right|_{x=0} = \left.\frac{d^k}{dx^k}f(x)\right|_{x=0}, \quad k=1,2,\ldots,m+n \tag{16}$$

Expressions (15) and (16) supply $m+n+1$ equations for the unknowns $p_0,\ldots,p_m$ and $q_1,\ldots,q_n$. In order to get ps and qs, we must multiply both (13) and (14) by the denominator of Eq. (13) and equate all powers of x that have either ps and qs in their coefficients.

The Chebyshev–Padé approximation of $f(x)$ can be easily computed in MAPLE™ or MATLAB® Symbolic Math Toolbox with the function

chebpade(f,$x=a..b$,[m,n])

where a and b are numerical values specifying the interval of approximation and m and n represent the desired degree of the numerator and the denominator of the approximation, respectively.

2.3. Rational Chebyshev approximation

Let $R(x)$ be a rational function which has a numerator of degree m and a denominator of degree n

$$R(x) \equiv \frac{p_0+p_1x+\cdots+p_mx^m}{1+q_1x+\cdots+q_nx^n} \approx f(x) \tag{17}$$

where $p_0,\ldots,p_m$ and $q_1,\ldots,q_n$ are $m+n+1$ unknown quantities.

Let $\rho(x)$ be the deviation of $R(x)$ from $f(x)$, and ρ its maximum absolute value

$$\rho(x) \equiv R(x) - f(x), \quad \rho \equiv \max_{a \le x \le b} |\rho(x)| \tag{18}$$

The ideal minimax solution would be that choice of ps and qs that minimizes ρ. Since ρ is bounded below by zero some minimax solution exists. If $R(x)$ is non-degenerate (has no common polynomial factors in numerator and denominator), then there is a unique choice of ps and qs that minimizes ρ [33,34]. However, finding this optimal solution is not an easy task [31].

Instead of making $f(x_i)$ and $R(x_i)$ equal at some $m+n+1$ points x_i, the residual $\rho(x_i)$ can be forced to any desired values y_i [31]. Remes algorithms [35], based on Chebyshev polynomials theory, indicate an iterative process that converges to these locations. Some of these algorithms are easily convertible to computer programs [36]. In this paper we shall use the implementation proposed in [31], where the source code is provided in language C and can be easily used. Its function call syntax is

*ratlsq(double(*f),double a,double b,int m, int n,double cof[],double *dev)*

where f is the function to be approximated, a and b define the interval of approximation, m and n represent the desired degree of the numerator and the denominator of the approximation, respectively. It returns in *cof* the coefficients of the rational function approximation and the maximum absolute deviation is returned as *dev.*

3. Fractional-order differentiator/integrator approximation

In this section we show how to obtain the direct discretization of the fractional differentiator/integrator of order α, $G(s)=s^{\alpha}$, by means of IIR filters obtained using the Chebyshev–Padé (CP) and the Rational Chebyshev (RC) approximations. Moreover, it is discussed how to improve the behaviour of the fractional integrator approximations at low frequencies, required by some applications.

The procedure can be summarized in the following steps:

1. Transform the s-domain fractional-order differentiator/integrator (1) into the z-domain via a generating function $g(z)$ (6).
2. Choose the degree of the numerator, m, and the denominator, n, for the approximation.
3. Define the interval of approximation [a, b]. In order to guarantee the IIR filter stability, we must always set these

Table 3
Magnitude normalized root mean square error.

α	Euler		Tustin		Al-Alaoui	
	CP	RC	CP	RC	CP	RC
0.1	0.1444	0.0897	0.3118	0.2827	0.1525	0.0943
0.3	0.1427	0.0924	0.3249	0.3277	0.1516	0.0958
0.5	0.1299	0.0810	0.3258	0.2607	0.1388	0.0810
0.7	0.1005	0.0627	0.3056	0.2524	0.1076	0.0604
0.9	0.0458	0.0403	0.2281	0.2132	0.0402	0.0270

Table 1
Numerator and denominator coefficients of the CP approximations to the fractional-order differentiator (in descending powers of z).

α	Euler	Tustin	Al-Alaoui
0.1	[1.2589–2.4915 1.4503–0.2141]	[1.3488–0.0576–1.2029 0.0369]	[1.2758–2.3158 1.1381–0.0926]
	[1.0000–1.8791 1.0092–0.1248]	[1.0000 0.1648–0.8831–0.1155]	[1.0000–1.7009 0.7403–0.0314]
0.3	[1.9952–4.1045 2.5352–0.4225]	[2.4503–0.3171–2.1748 0.2035]	[2.0769–3.9516 2.1135–0.2333]
	[1.0000–1.7572 0.8487–0.0813]	[1.0000 0.4869–0.8066–0.3178]	[1.0000–1.5601 0.5716 0.0040]
0.5	[3.1623–6.7432 4.3924–0.8086]	[4.4517–0.9774–3.9110 0.6278]	[3.3807–6.7187 3.8687–0.5260]
	[1.0000–1.6325 0.6981–0.0473]	[1.0000 0.7926–0.6427–0.4428]	[1.0000–1.4161 0.4176 0.0269]
0.7	[5.0119–11.0526 7.5547–1.5120]	[8.1067–2.5603–7.0033 1.6444]	[5.5031–11.3896 7.0015–1.1116]
	[1.0000–1.5053 0.5589–0.0224]	[1.0000 1.0856–0.3770–0.4646]	[1.0000–1.2698 0.2800 0.0387]
0.9	[7.9433–18.0814 12.9162–2.7773]	[14.7983–6.2297–12.4620 3.9938]	[8.9577–19.2610 12.5576–2.2529]
	[1.0000–1.3763 0.432482–0.0056]	[1.0000 1.3760–0.0051–0.3814]	[1.0000–1.1216 0.1602 0.0412]

Table 2
Numerator and denominator coefficients of the RC approximations to the fractional-order differentiator (in descending powers of z).

α	Euler	Tustin	Al-Alaoui
0.1	[1.2588–2.8141 1.9532–0.3972]	[1.3444 0.0738–1.2555–0.0682]	[1.2756–2.6570 1.6378–0.2552]
	[1.0000–2.1346 1.3813–0.2455]	[1.0000 0.2728–0.9183–0.2229]	[1.0000–1.9678 1.0990–0.1292]
0.3	[1.9949–4.5219 3.2063–0.6784]	[2.4263 0.5254–2.2496–0.4723]	[2.0762–4.3991 2.7931–0.4687]
	[1.0000–1.9657 1.1192–0.1504]	[1.0000 0.7943–0.7282–0.5288]	[1.0000–1.7755 0.8210–0.0405]
0.5	[3.1615–7.3676 5.4315–1.2247]	[4.3463 0.0719–4.1254–0.1690]	[3.3794–7.4627 5.0538–0.9695]
	[1.0000–1.8298 0.9247–0.0889]	[1.0000 0.9645–0.6523–0.6187]	[1.0000–1.6369 0.6380 0.0082]
0.7	[5.0111–11.9229 9.0433–2.1310]	[8.0785–2.3542–7.1341 1.4459]	[5.5021–12.4143 8.6873–1.7742]
	[1.0000–1.6792 0.7324–0.0409]	[1.0000 1.0822–0.4296–0.5141]	[1.0000–1.4567 0.4350 0.0418]
0.9	[7.9430–19.2578 14.9797–3.6646]	[14.8130–6.6050–11.7642 3.6474]	[8.9574–20.6299 14.8773–3.2044]
	[1.0000–1.5245 0.5584–0.0099]	[1.0000 1.3461–0.0061–0.3530]	[1.0000–1.2747 0.2615 0.0538]

Table 4
Phase normalized root mean square error.

α	Euler		Tustin		Al-Alaoui	
	CP	RC	CP	RC	CP	RC
0.1	0.4412	0.3593	0.5104	0.4854	0.4388	0.3549
0.3	0.4390	0.3646	0.5151	0.5356	0.4371	0.3607
0.5	0.4238	0.3496	0.5035	0.4518	0.4217	0.3396
0.7	0.3881	0.3197	0.4701	0.2807	0.3847	0.3102
0.9	0.3024	0.2645	0.3635	0.3505	0.2910	0.2476

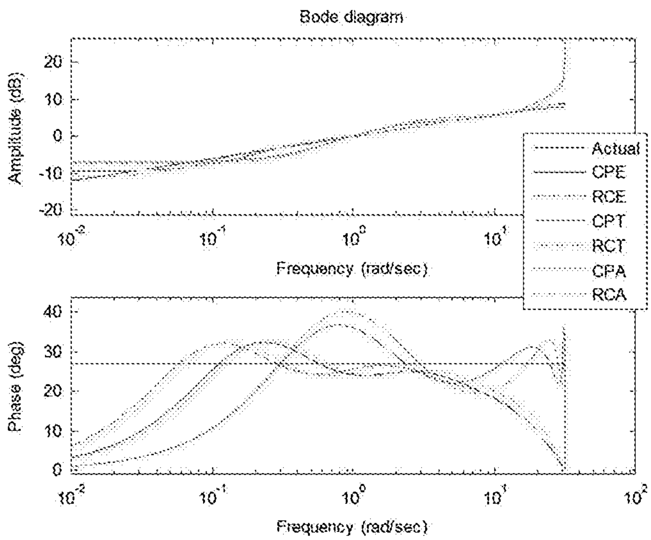

Fig. 1. Bode magnitude and phase diagrams for $\alpha=0.3$. The legend reads: Chebyshev–Padé with Euler (CPE); Rational Chebyshev with Euler (RCE); Chebyshev–Padé with Tustin (CPT); Rational Chebyshev with Tustin (RCT); Chebyshev–Padé with Al-Alaoui (CPA); and Rational Chebyshev with Alaoui (RCA).

values within the interval (−1,1). Values close to 0.995 yield good results.

4. Apply either Chebyshev method (CP or RC) to (6) to obtain a feasible IIR approximation to (1).

In practice, we shall only consider the case $m=n$; the case $m<n$ obtains inferior results [22,37] and, obviously, does not lead to zero-pole interlacing (there are not the same number of zeros and poles).

In order to illustrate the method, let us consider the approximations to the fractional-order differentiator, with $n=m=3$, Euler (E) (3), Tustin (T) (4), and Al-Alaoui (A) (5) generating functions, $\alpha\in(0,1)$, and sampling time $T_S=0.1$ s. The approximation interval is $[a,\ b]=[-0.995,\ 0.995]$. Tables 1 and 2 show the coefficients of the numerator and denominator of the corresponding CP and RC approximations, respectively. Coefficients are given in descending powers of z. Tables 3 and 4 summarize the results in terms of the normalized root mean square error (NRMS) (19), both in magnitude and phase, within the frequency interval $[10^{-2},\ \pi/T_S]$

$$NRMS=\sqrt{\frac{\sum_{j=1}^{F}\left|\tilde{G}(z)-G(z)\right|^2}{\sum_{j=1}^{F}\left|G(z)\right|^2}} \tag{19}$$

where F is the number of samples taken from the interval $[10^{-2},\ \pi/T_S]$. $\tilde{G}$ means approximated values and G actual response, either for magnitude or phase.

3.1. Comparative study

Here we compare the approximations described above against other discrete IIR approximations widely used in fractional calculus. In the first place, we consider the fifth-order approximation of the differentiator $s^{0.5}$. We have chosen three accurate

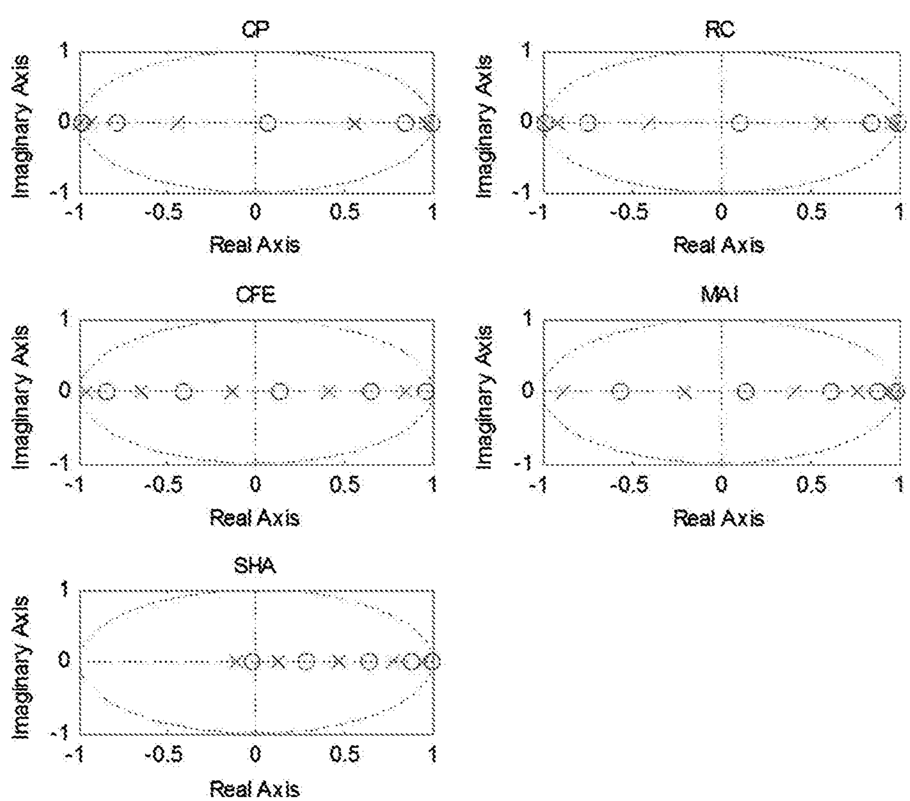

Fig. 2. Pole-zero maps for the different approximations to the fractional-order differentiator ($\alpha=0.5$).

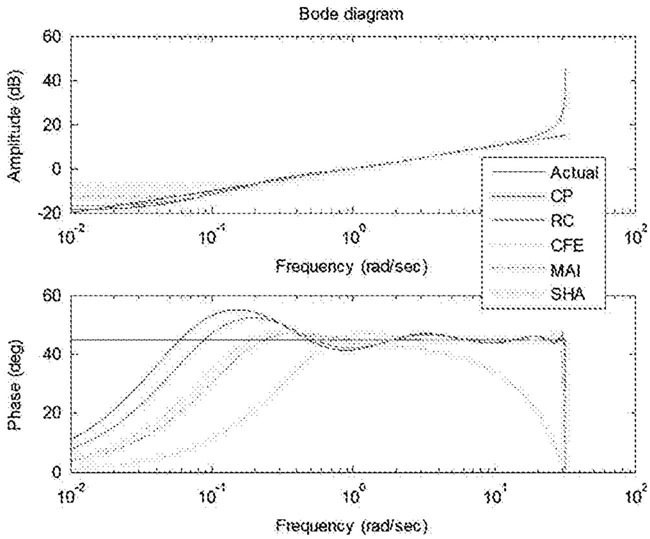

Fig. 3. Bode magnitude and phase diagrams for the different approximations to the fractional-order differentiator ($\alpha=0.5$).

Table 5
Normalized root mean square errors of different approximations to $s^{0.5}$.

Filter	Magnitude Error	Phase Error
CP	0.1937	0.2424
RC	0.1886	0.2870
CFE	0.4309	0.5350
Maione	0.2353	0.3942
Shanks	0.1543	0.4274

approximations proposed in the literature: 1. CFE with Tustin (20) [19], for it is widely used; 2. Maione approximation (MAI) (21) [38], a method optimized for, and limited to, the particular case $s^{0.5}$; and 3. Shanks method (SHA) (22) [39] that uses an advanced recursion technique in order to perform filter operations rapidly and efficiently

$$G_{CFE}(z)=\frac{4.4721z^5-2.2360z^4-4.4721z^3+1.6770z^2+0.8385z-0.1397}{z^5+0.5z^4-1.0z^3-0.375z^2+0.1875z+0.0312} \tag{20}$$

$$G_{MAI}(z)=\frac{4.4721z^5-9.1932z^4+3.9256z^3+2.3518z^2-1.7414z+0.1887}{z^5-1.0557z^4-0.6776z^3+0.8743z^2-0.0725z-0.0526} \tag{21}$$

$$G_{SHA}(z)=\frac{3.3806z^5-9.3913z^4+9.2489z^3-3.7169z^2+0.4695z+0.0103}{z^5-2.2066z^4+1.5566z^3-0.3318z^2-0.0173z+0.0046} \tag{22}$$

Since these three approximations, as proposed in their respective references, use the Tustin generating function (4) with sampling time $T_S=0.1$ s, for comparison purposes we shall design the Chebyshev-based approximations with the same generating function and sampling time. Thus, the corresponding CP (23) and RC (24) approximations are as follows:

$$G_{CP}(z)=\frac{4.4713z^5-0.6566z^4-7.3299z^3+0.8468z^2+2.8903z-0.2117}{z^5+0.8535z^4-1.2906z^3-1.0321z^2+0.3355z+0.2232} \tag{23}$$

$$G_{RC}(z)=\frac{4.4716z^5-0.9565z^4-7.1068z^3+1.1940z^2+2.6924z-0.2810}{z^5+0.7865z^4-1.3063z^3-0.9371z^2+0.3545z+0.1982} \tag{24}$$

for an approximation interval $[a, b]=[-0.995, 0.995]$.

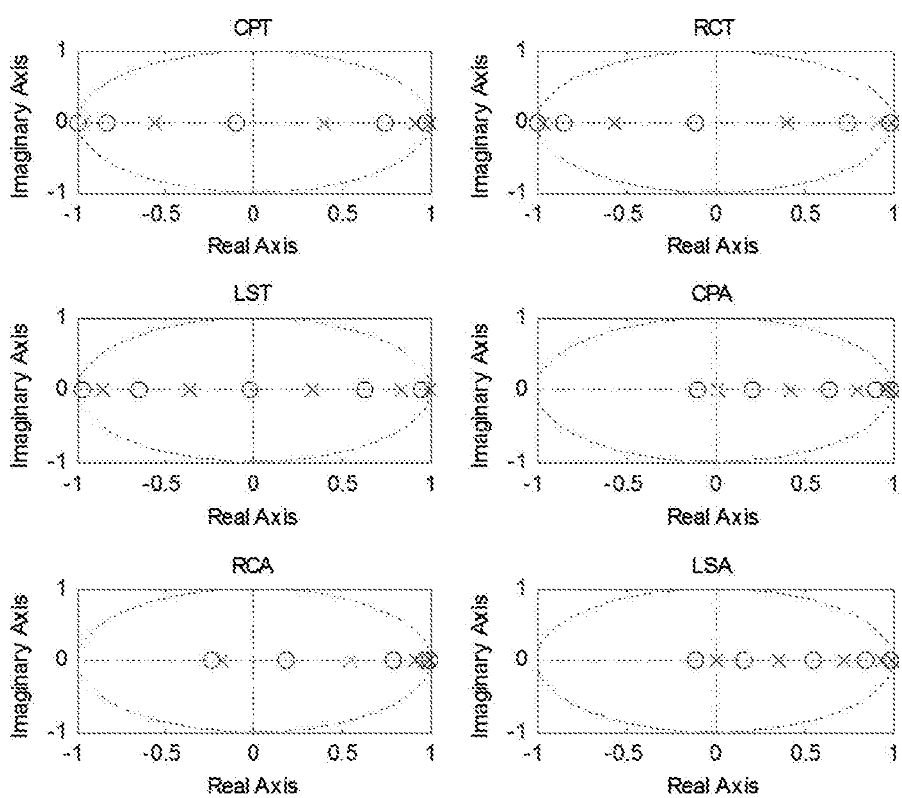

Fig. 4. Pole-zero maps for the different approximations to the fractional-order integrator ($\alpha=-0.5$).

Fig. 2 shows their pole-zero maps that are always inside the unit circle. Moreover, all of them have pole-zero interlacing. Hence, all of them are minimum-phase and stable approximations.

Fig. 3 shows the corresponding frequency responses in the interval $[10^{-2}, \pi/T_S]$. It is easily seen that both Chebyshev approximations have a good behaviour at low frequencies, both in magnitude and phase. A better measurement of the approximations accuracy is shown in Table 5, where the corresponding NRMS are given: in phase, both Chebyshev approximations are closer to the actual fractional differentiator in a wider range of frequencies; in magnitude, they have good behaviour, although the best approximation in terms of error is SHA.

In the second place, we shall consider the fifth-order approximation of the integrator $s^{-0.5}$. We shall compare both Chebyshev approximations against an effective method based on least-squares (LS) [22]. Tustin (4) ("T" in the legend) and Al-Alaoui (5) ("A" in the legend) generating functions will be used, with sampling time $T_S = 0.01$ s. The corresponding transfer functions are as follows:

$$G_{CPT}(z) = \frac{0.0707z^5+0.0150z^4-0.1117z^3-0.0186z^2+0.0419z+0.0043}{z^5-0.7871z^4-1.2959z^3+0.9296z^2+0.3470z-0.1930} \tag{25}$$

$$G_{RCT}(z) = \frac{0.0707z^5+0.0151z^4-0.1123z^3-0.0188z^2+0.0425z+0.0044}{z^5-0.7865z^4-1.3063z^3+0.9371z^2+0.3545z-0.1982} \tag{26}$$

$$G_{LST}(z) = \frac{0.0707z^5+0.0047z^4-0.0939z^3-0.0043z^2+0.0265z+0.0005}{z^5-0.9331z^4-0.8957z^3+0.8006z^2+0.1144z-0.0846} \tag{27}$$

$$G_{CPA}(z) = \frac{0.0935z^5-0.2471z^4+0.2206z^3-0.0696z^2+0.0015z+0.0011}{z^5-3.2136z^4+3.7868z^3-1.9347z^2+0.3679z-0.0064} \tag{28}$$

$$G_{RCA}(z) = \frac{0.0935z^5-0.2514z^4+0.2145z^3-0.0448z^2-0.0148z+0.0031}{z^5-3.2604z^4+3.7460z^3-1.6196z^2+0.0433z+0.0906} \tag{29}$$

$$G_{LSA}(z) = \frac{0.0935z^5-0.2293z^4+0.1860z^3-0.0506z^2-0.0002z+0.0007}{z^5-3.0234z^4+3.3084z^3-1.5363z^2+0.2527z-0.0014} \tag{30}$$

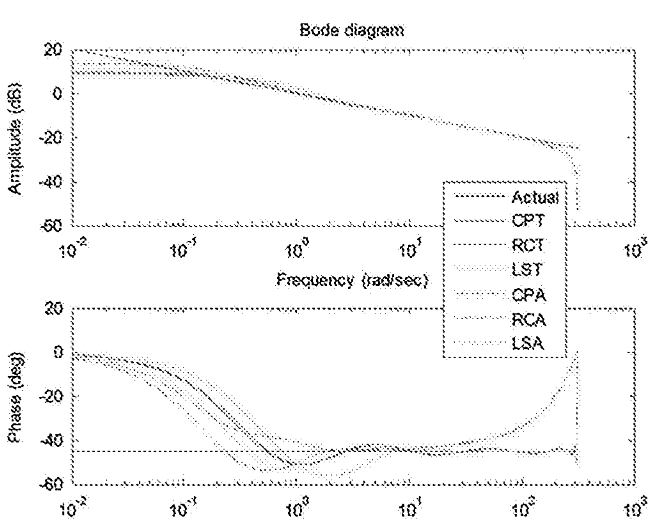

Fig. 5. Bode magnitude and phase diagrams for the different approximations to the fractional-order integrator ($\alpha = -0.5$).

The pole-zero maps are shown in Fig. 4. We observe that all the poles and zeros are real, interlaced, and lie inside the unit circle. Fig. 5 depicts the corresponding Bode plots. All the approximations fit the actual response well. However, the Chebyshev-based approximations are slightly closer to the actual response, as we can see numerically in Table 6. Moreover, approximations which have been discretized using Al-Alaoui formula present better values than the Tustin-based ones.

To sum up, results show that:

- Euler and Al-Alauoi generating functions lead to quite similar approximations. They are better than Tustin-based approximations in terms of NRMS.
- The best results are obtained using the Al-Alaoui generating function. With it, the RC approximations are closer to the actual fractional response than the CP ones.
- In general, the approximation tends to be more accurate when the fractional order α tends to 1.

Fig. 1 depicts the corresponding Bode diagram for $\alpha = 0.3$. This plot shows that:

- All these approximations have a quite similar behaviour in magnitude.
- Al-Alaoui and Euler approximations exhibit a flatter phase response than the Tustin ones, although the latter are more accurate in phase at high frequencies.

Table 6
Normalized root mean square errors of different approximations to $s^{-0.5}$.

Filter	Magnitude Error	Phase Error
CPT	0.2729	0.4593
RCT	0.2658	0.4537
LST	0.2946	0.4997
CPA	0.1357	0.4201
RCA	0.2256	0.4517
LSA	0.1875	0.4657

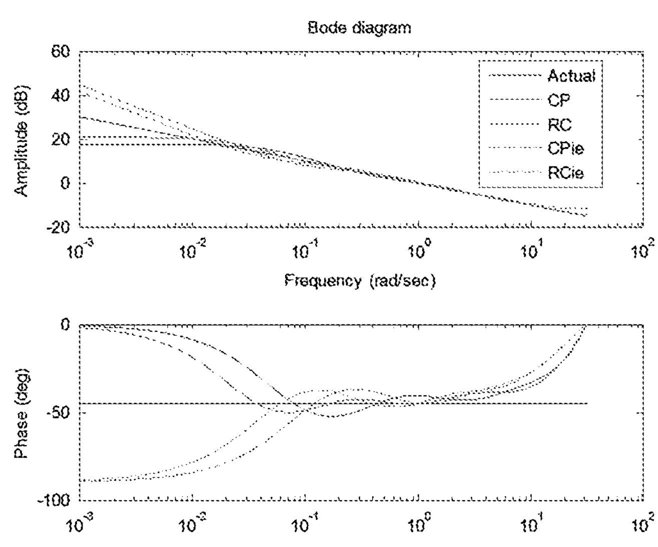

Fig. 6. Magnitude and phase comparison between different approximations to the fractional-order integrator $s^{-0.5}$ ("ie" stands for improved integration effect at low frequencies).

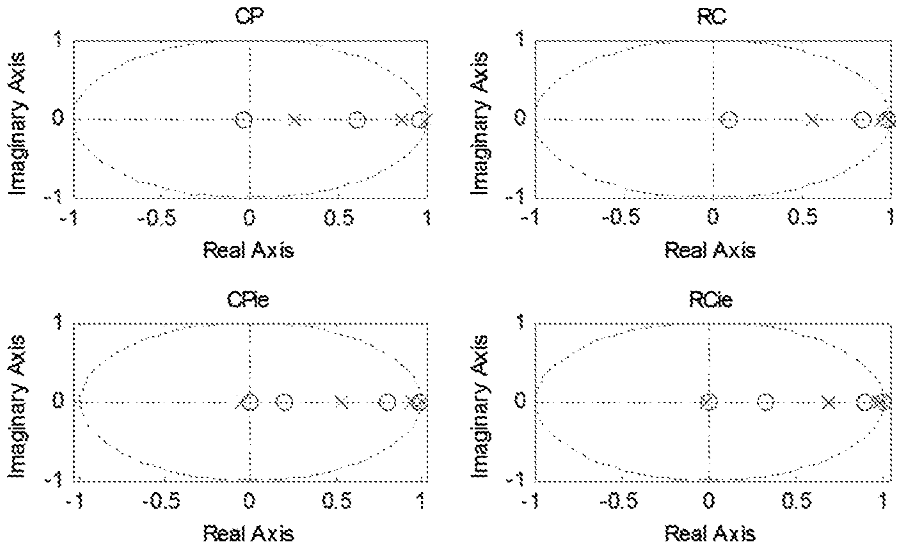

Fig. 7. Pole-zero maps of the approximations to the fractional-order integrator $s^{-0.5}$ (**"ie"** stands for improved integration effect at low frequencies).

Table 7
Normalized root mean square errors of different approximations to $s^{-0.5}$ ("ie" stands for integrator effect).

Filter	Magnitude Error	Phase Error
CP	0.0881	0.3476
RC	0.0615	0.3476
CPie	0.1356	0.4352
RCie	0.0981	0.3613

3.2. *Improving the integrator effect at low frequencies*

The integration effect of all the integrator approximations described above is lost: as it can be easily seen in Fig. 5, the magnitude curve is flat and the phase tends to 0° at low frequencies.

In order to keep the integration effect at low frequencies, the α-order integrator can be expressed as

$$G(s) = s^{-\alpha} = \frac{1}{s}s^{1-\alpha} \approx [g(z)]^{-1}[g(z)]^{1-\alpha} \tag{31}$$

i.e., the product of an integer-order integrator and a fractional differentiator of order $1-\alpha$. Consequently, the integration effect is guaranteed by a pole at the origin (in the s-domain) introduced by the conventional integrator. On the one hand, the integer integrator is discretized using one of the generating functions (3), (4), or (5); on the other hand, the fractional differentiator is obtained with one of the Chebyshev-based approximations described above.

In Fig. 6, the fractional integrator of order 0.5, $s^{-0.5}$ in (31), is considered for illustrative purposes. Both Chebyshev approximations, with and without improved integration effect at low frequencies ("ie" in the legend), are compared. All the approximations are obtained with the Al-Alaoui generating function. It is easily seen that both improved response approximations have a slope of −20 dB/dec and a phase of −90° at low frequencies, as corresponds to the improved integrator effect. This is done by placing a pole at $z=1$ (Fig. 7).

Table 7 shows the corresponding NRMS. Numerical results are better for approximations with no improved integration effect; however, only the improved approximations exhibit the desired magnitude and phase responses at low frequencies.

4. Conclusions

In this paper, two methods to design IIR approximations to the fractional-order differentiators/integrators, based on Chebyshev polynomials theory (the Chebyshev–Padé and the Rational Chebyshev approximations), have been presented. Both of them are easy to compute as there exist widely available software implementations. Together with the Al-Alaoui generating function, they are much more accurate, in terms of NRMS errors, than other approximations widely cited in the literature.

It has been also shown how to improve the frequency response of the fractional-order integrator approximations, guaranteeing "integral" slopes (−20 dB/dec) and phases (−90°) at low frequencies.

Acknowledgments

The authors wish to acknowledge the economical support of the Spanish Distance Education University (UNED), under project reference PROY29.

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