Numerical Approximations of Fractional Derivatives with Applications*[†]

Shakoor Pooseh spooseh@ua.pt

Ricardo Almeida ricardo.almeida@ua.pt

Delfim F. M. Torres delfim@ua.pt

Center for Research and Development in Mathematics and Applications Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

Abstract

Two approximations, derived from continuous expansions of Riemann–Liouville fractional derivatives into series involving integer order derivatives, are studied. Using those series, one can formally transform any problem that contains fractional derivatives into a classical problem in which only derivatives of integer order are present. Corresponding approximations provide useful numerical tools to compute fractional derivatives of functions. Application of such approximations to fractional differential equations and fractional problems of the calculus of variations are discussed. Illustrative examples show the advantages and disadvantages of each approximation.

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1 Introduction

Fractional calculus is the study of integrals and derivatives of arbitrary real or complex order. Although the origin of fractional calculus goes back to the end of the seventeenth century, the main contributions have been made during the last few decades [31, 32]. Namely it has proven to be a useful tool when applied to engineering and optimal control problems (see, e.g., [12, 19, 29]). There are several different definitions of fractional derivatives in the literature, such as Grünwald–Letnikov, Caputo, etc. Here we consider Riemann–Liouville fractional derivatives.

Definition 1.1 (cf. [16]). Let $x(\cdot)$ be an absolutely continuous function in [a,b] and $0 < \alpha < 1$. Then,

• the left Riemann-Liouville fractional derivative of order α , ${}_aD_t^{\alpha}$, is given by

$${}_{a}D_{t}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{a}^{t}(t-\tau)^{-\alpha}x(\tau)d\tau, \quad t \in [a,b];$$

$$\tag{1}$$

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• the right Riemann-Liouville fractional derivative of order α , $_tD_b^{\alpha}$, is given by

$${}_tD_b^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)} \left(-\frac{d}{dt}\right) \int_t^b (\tau-t)^{-\alpha}x(\tau)d\tau, \quad t \in [a,b].$$

Due to the growing number of applications of fractional calculus in science and engineering (see, e.g., [8, 9, 33]), numerical methods are being developed to provide tools for solving such problems. Using the Grünwald–Letnikov approach, it is convenient to approximate the fractional differentiation operator, D^{α} , by generalized finite differences. In [24] some problems have been solved by this approximation. In [10] a predictor-corrector method is presented that converts an initial value problem into an equivalent Volterra integral equation, while [17] shows the use of numerical methods to solve such integral equations. A good survey on numerical methods for fractional differential equations can be found in [13].

A new numerical scheme to solve fractional differential equations has been recently introduced in [7] and [15], making an adaptation to cover fractional optimal control problems. The scheme is based on an expansion formula for the Riemann–Liouville fractional derivative. Here we introduce a generalized version of that expansion and, together with a different expansion formula that has been used to approximate the fractional Euler–Lagrange equation in [5], we perform an investigation of the advantages and disadvantages of approximating fractional derivatives by these expansions. The approximations transform fractional derivatives into finite sums containing only derivatives of integer order. We show the efficiency of such approximations to evaluate fractional derivatives of a given function in closed form. Moreover, we discuss the possibility of evaluating fractional derivatives of discrete tabular data. The application to fractional differential equations and the calculus of variations is also developed through some concrete examples. In each case we try to analyze problems for which the analytic solution is available. This approach gives us the ability of measuring the accuracy of each method. To this end, we need to measure how close we get to exact solutions. We use the 2-norm and define the error function $E[x(\cdot), \tilde{x}(\cdot)]$ by

$$E = \|x(\cdot) - \tilde{x}(\cdot)\|_2 = \left(\int_a^b [x(t) - \tilde{x}(t)]^2 dt\right)^{\frac{1}{2}},$$

where $x(\cdot)$ is defined on [a, b]. The results of the paper give interesting numerical procedures when applied to fractional problems of the calculus of variations.

2 Expansion formulas to approximate fractional derivatives

In this section two approximations for the left Riemann–Liouville derivative are presented. Both approximate the fractional derivatives by finite sums including only derivatives of integer order and are based on continuous expansions for the left Riemann–Liouville derivative.

2.1 Approximation by a sum of integer order derivatives

The right-hand side of (1) is expandable in a power series involving integer order derivatives [5, 28]. Let (c,d), $-\infty < c < d < +\infty$, be an open interval in \mathbb{R} , and $[a,b] \subset (c,d)$ be such that for each $t \in [a,b]$ the closed ball $B_{b-a}(t)$, with center at t and radius b-a, lies in (c,d).

For any real analytic function $x(\cdot)$ in (c,d) we can give the following expansion formula:

$${}_{a}D_{t}^{\alpha}x(t) = \sum_{n=0}^{\infty} {\alpha \choose n} \frac{(t-a)^{n-\alpha}}{\Gamma(n+1-\alpha)} x^{(n)}(t), \quad \text{where } {\alpha \choose n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.$$
 (2)

The condition $B_{b-a}(t) \subset (c,d)$ comes from the Taylor expansion of $x(t-\tau)$ at t, for $\tau \in (a,t)$ and $t \in (a,b)$. The proof of this statement that can be found in [28] uses a similar expansion for fractional integrals. Here we outline a direct proof due to our requirements in Section 2.3. Since x(t) is analytic, it can be expanded as a convergent power series, i.e.,

$$x(\tau) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(n)}(t)}{n!} (t - \tau)^n$$

and then by (1)

$${}_{a}D_{t}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{a}^{t} \left((t-\tau)^{-\alpha}\sum_{n=0}^{\infty} \frac{(-1)^{n}x^{(n)}(t)}{n!}(t-\tau)^{n}\right)d\tau, \quad t \in (a,b).$$
 (3)

Termwise integration, followed by differentiation and simplification, leads to

$${}_{a}D_{t}^{\alpha}x(t) = \frac{x(t)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)}\sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}}{(n-\alpha)(n-1)!} + \frac{(-1)^{n}}{n!}\right)x^{(n)}(t)(t-a)^{n-\alpha}$$

and finally to expansion formula (2). From the computational point of view, one can take only a finite number of terms in (2) and use the approximation

$$_{a}D_{t}^{\alpha}x(t) \simeq \sum_{n=0}^{N} C(n,\alpha)(t-a)^{n-\alpha}x^{(n)}(t), \quad \text{where } C(n,\alpha) = \binom{\alpha}{n} \frac{1}{\Gamma(n+1-\alpha)}.$$
 (4)

2.2 Approximation using moments of a function

The following lemma gives the departure point to another expansion. For a proof see [9].

Lemma 2.1 (Lemma 2.12 of [9]). Let $x(\cdot) \in AC[a,b]$ and $0 < \alpha < 1$. Then the left Riemann–Liouville fractional derivative ${}_aD_t^{\alpha}x(\cdot)$ exists almost everywhere in [a,b]. Moreover, ${}_aD_t^{\alpha}x(\cdot) \in L_p[a,b]$ for $1 \le p < \frac{1}{\alpha}$ and

$${}_{a}D_{t}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{x(a)}{(t-a)^{\alpha}} + \int_{a}^{t} (t-\tau)^{-\alpha} \dot{x}(\tau) d\tau \right], \qquad t \in (a,b).$$
 (5)

Let $V_p(x(\cdot)), p \in \mathbb{N}$, denote the (p-2)th moment of a function $x(\cdot) \in AC^2[a,b]$ (cf. [7]):

$$V_p(t) := V_p(x(t)) = (1-p) \int_a^t (\tau - a)^{p-2} x(\tau) d\tau, \quad p \in \mathbb{N}, \ t \ge a.$$
 (6)

Following [7], it is easy to show that, by successive integrating by parts, (5) is reduced to

$${}_{0}D_{t}^{\alpha}x(t) = \frac{x(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + \frac{\dot{x}(a)}{\Gamma(2-\alpha)}(t-a)^{1-\alpha} + \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)} \int_{a}^{t} \left(1 - \frac{\tau - a}{t-a}\right)^{1-\alpha} \ddot{x}(\tau)d\tau.$$
(7)

Using the binomial theorem we conclude that

$${}_{a}D_{t}^{\alpha}x(t) = \frac{x(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)}\dot{x}(a) + \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)}\int_{a}^{t} \left(\sum_{p=0}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \left(\frac{\tau-a}{t-a}\right)^{p}\right) \ddot{x}(\tau)d\tau, \quad t > a.$$
 (8)

Further integration by parts and simplification in (8) gives

$${}_{a}D_{t}^{\alpha}x(t) = A(\alpha)(t-a)^{-\alpha}x(t) + B(\alpha)(t-a)^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{\infty}C(\alpha,p)(t-a)^{1-p-\alpha}V_{p}(t), \quad (9)$$

where $V_p(t)$ is defined by (6) and

$$A(\alpha) = \frac{1}{\Gamma(1-\alpha)} \left[1 + \sum_{p=2}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!} \right],$$

$$B(\alpha) = \frac{1}{\Gamma(2-\alpha)} \left[1 + \sum_{p=1}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \right],$$

$$C(\alpha,p) = \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \frac{\Gamma(p-1+\alpha)}{(p-1)!}.$$

The moments $V_p(t)$, p=2,3,..., are regarded as the solutions to the following system of differential equations:

$$\begin{cases}
\dot{V}_p(t) = (1-p)(t-a)^{p-2}x(t) \\
V_p(a) = 0, \quad p = 2, 3, \dots
\end{cases}$$
(10)

For numerical purposes, only a finite number of terms in the series (9) are used. We approximate the fractional derivative as

$$_{a}D_{t}^{\alpha}x(t) \simeq A(\alpha, N)(t-a)^{-\alpha}x(t) + B(\alpha, N)(t-a)^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{N}C(\alpha, p)(t-a)^{1-p-\alpha}V_{p}(t),$$
 (11)

where $A(\alpha, N)$ and $B(\alpha, N)$ are given by

$$A(\alpha, N) = \frac{1}{\Gamma(1-\alpha)} \left[1 + \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!} \right], \tag{12}$$

$$B(\alpha, N) = \frac{1}{\Gamma(2-\alpha)} \left| 1 + \sum_{p=1}^{N} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \right|. \tag{13}$$

Remark 2.2. Our approximation (11) is different from the one presented in [7]: since the infinite series $\sum_{p=1}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}$ tends to -1, $B(\alpha)=0$ and thus

$${}_{0}D_{t}^{\alpha}x(t) \simeq A(\alpha, N)t^{-\alpha}x(t) - \sum_{p=2}^{N} C(\alpha, p)t^{1-p-\alpha}V_{p}(t). \tag{14}$$

However, regarding the fact that we use a finite sum, in practice one has

$$1 + \sum_{p=1}^{N} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \neq 0.$$

Therefore, and similarly to [11, 27], we keep here the approximation in the form (11). The value of $B(\alpha, N)$ for some values of N and for different choices of α is given in Table 1. It

N	4	7	15	30	70	120	170
B(0.1, N)	0.0310	0.0188	0.0095	0.0051	0.0024	0.0015	0.0011
B(0.3, N)	0.1357	0.0928	0.0549	0.0339	0.0188	0.0129	0.0101
B(0.5, N)	0.3085	0.2364	0.1630	0.1157	0.0760	0.0581	0.0488
B(0.7, N)	0.5519	0.4717	0.3783	0.3083	0.2396	0.2040	0.1838
B(0.9, N)	0.8470	0.8046	0.7481	0.6990	0.6428	0.6092	0.5884
B(0.99, N)	0.9849	0.9799	0.9728	0.9662	0.9582	0.9531	0.9498

Table 1: $B(\alpha, N)$ for different values of α and N.

shows that even for a large N, when α tends to one, $B(\alpha, N)$ cannot be ignored. In Figure 1 we plot $B(\alpha, N)$ as a function of N for different values of α . In Section 3 we compare both

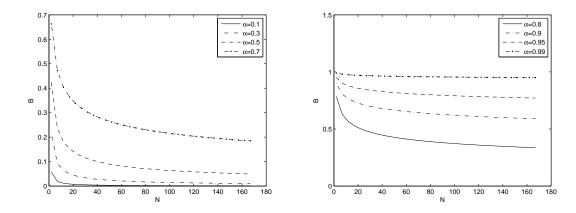


Figure 1: $B(\alpha, N)$ for different values of α and N.

approximations with some examples. We also refer to [6, 26], where such type of expansion formulas are studied.

A similar argument gives the expansion formula for ${}_tD_b^{\alpha}$, the right Riemann–Liouville fractional derivative. We propose the following approximation:

$$_{t}D_{b}^{\alpha}x(t) \simeq A(\alpha, N)(b-t)^{-\alpha}x(t) - B(\alpha, N)(b-t)^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{N}C(\alpha, p)(b-t)^{1-p-\alpha}W_{p}(t),$$

where $W_p(t) = (1-p) \int_t^b (b-\tau)^{p-2} x(\tau) d\tau$. Here $A(\alpha, N)$ and $B(\alpha, N)$ are the same as (12) and (13), respectively.

Formula (9) consists of two parts: an infinite series and two terms including the first derivative and the function itself. It can be generalized to contain derivatives of higher-order.

Theorem 2.3. Fix $n \in \mathbb{N}$ and let $x(\cdot) \in C^n[a,b]$. Then,

$${}_{a}D_{t}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}(t-a)^{-\alpha}x(t) + \sum_{i=1}^{n-1}A(\alpha,i)(t-a)^{i-\alpha}x^{(i)}(t) + \sum_{p=n}^{\infty} \left[\frac{-\Gamma(p-n+1+\alpha)}{\Gamma(-\alpha)\Gamma(1+\alpha)(p-n+1)!}(t-a)^{-\alpha}x(t) + B(\alpha,p)(t-a)^{n-1-p-\alpha}V_{p}(t)\right], \quad (15)$$

where

$$A(\alpha, i) = \frac{1}{\Gamma(i+1-\alpha)} \left[1 + \sum_{p=n-i}^{\infty} \frac{\Gamma(p-n+1+\alpha)}{\Gamma(\alpha-i)(p-n+i+1)!} \right], \quad i = 1, \dots, n-1,$$

$$B(\alpha, p) = \frac{\Gamma(p-n+1+\alpha)}{\Gamma(-\alpha)\Gamma(1+\alpha)(p-n+1)!},$$

$$V_p(t) = (p-n+1) \int_a^t (\tau-a)^{p-n} x(\tau) d\tau.$$

Proof. Successive integrating by parts in (5) gives

$${}_{a}D_{t}^{\alpha}x(t) = \frac{x(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + \frac{\dot{x}(a)}{\Gamma(2-\alpha)}(t-a)^{1-\alpha} + \dots + \frac{x^{(n-1)}(a)}{\Gamma(n-\alpha)}(t-a)^{n-1-\alpha} + \frac{1}{\Gamma(1-\alpha)}\int_{a}^{t}(t-\tau)^{n-1-\alpha}x^{(n)}(\tau)d\tau.$$

Using the binomial theorem, we expand the integral term as

$$\int_{a}^{t} (t-\tau)^{n-1-\alpha} x^{(n)}(\tau) d\tau = (t-a)^{n-1-\alpha} \sum_{p=0}^{\infty} \frac{\Gamma(p-n+1+\alpha)}{\Gamma(1-n+\alpha)p!(t-a)^p} \int_{a}^{t} (\tau-a)^p x^{(n)}(\tau) d\tau.$$

Splitting the sum into p = 0 and $p = 1...\infty$, and integrating by parts the last integral, we get

$$_{0}D_{t}^{\alpha}x(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}x(a) + \dots + \frac{(t-a)^{n-2-\alpha}}{\Gamma(n-1-\alpha)}x^{(n-2)}(a)$$

$$+ \frac{(t-a)^{n-1-\alpha}}{\Gamma(n-\alpha)}x^{(n-2)}(t) \left[1 + \sum_{p=1}^{\infty} \frac{\Gamma(p-n+1+\alpha)}{\Gamma(-n+1+\alpha)p!}\right]$$

$$+ \frac{(t-a)^{n-1-\alpha}}{\Gamma(n-1-\alpha)} \sum_{n=1}^{\infty} \frac{\Gamma(p-n+1+\alpha)}{\Gamma(-n+2+\alpha)(p-1)!(t-a)^{p}} \int_{a}^{t} (\tau-a)^{p-1}x^{(n-1)}(\tau)d\tau.$$

The rest of the proof follows a similar routine, i.e., by splitting the sum into two parts, the first term and the rest, and integrating by parts the last integral until $x(\cdot)$ appears in the integrand.

Remark 2.4. The series that appear in $A(\alpha, i)$ are convergent for all $i \in \{1, ..., n-1\}$. Fix an i and observe that

$$\sum_{p=n-i}^{\infty} \frac{\Gamma(p-n+1+\alpha)}{\Gamma(\alpha-i)(p-n+i+1)!} = \sum_{p=1}^{\infty} \frac{\Gamma(p+\alpha-i)}{\Gamma(\alpha-i)p!} = {}_1F_0(\alpha-i,1) - 1.$$

Since $i > \alpha$, ${}_1F_0(\alpha - i, 1)$ converges by Theorem 2.1.1 of [2]. In practice we only use finite sums and for $A(\alpha, i)$ we can easily compute the truncation error. Although this is a partial error, it gives a good intuition of why this approximation works well. Using the fact that ${}_1F_0(a, 1) = 0$ if a < 0 (cf. Eq. (2.1.6) in [2]), we have

$$\frac{1}{\Gamma(i+1-\alpha)} \sum_{p=N+1}^{\infty} \frac{\Gamma(p-n+1+\alpha)}{\Gamma(\alpha-i)(p-n+i+1)!}$$

$$= \frac{1}{\Gamma(i+1-\alpha)} \left({}_{1}F_{0}(\alpha-i,1) - \sum_{p=0}^{N-n+i+1} \frac{\Gamma(p+\alpha-i)}{\Gamma(\alpha-i)p!} \right)$$

$$= \frac{-1}{\Gamma(i+1-\alpha)} \sum_{p=0}^{N-n+i+1} \frac{\Gamma(p+\alpha-i)}{\Gamma(\alpha-i)p!}.$$

In Table 2 we give some values for this error, with $\alpha = 0.5$ and different values for i and N - n.

N-n	0	5	10	15	20
1	-0.4231	-0.2364	-0.1819	-0.1533	-0.1350
2	0.04702	0.009849	0.004663	0.002838	0.001956
3	-0.007052	-0.0006566	-0.0001999	-0.00008963	-0.00004890
4	0.001007	0.00004690	0.000009517	0.000003201	0.000001397

Table 2: Truncation errors of $A(\alpha, i, N)$ for $\alpha = 0.5$.

Remark 2.5. Using Euler's reflection formula, one can define $B(\alpha, p)$ of Theorem 2.3 as

$$B(\alpha, p) = \frac{-\sin(\pi\alpha)\Gamma(p - n + 1 + \alpha)}{\pi(p - n + 1)!}.$$

For numerical purposes, only finite sums are taken to approximate fractional derivatives. Therefore, for a fixed $n \in \mathbb{N}$ and $N \ge n$, one has

$$_{a}D_{t}^{\alpha}x(t) \approx \sum_{i=0}^{n-1} A(\alpha, i, N)(t-a)^{i-\alpha}x^{(i)}(t) + \sum_{p=n}^{N} B(\alpha, p)(t-a)^{n-1-p-\alpha}V_{p}(t),$$
 (16)

where

$$A(\alpha, i, N) = \frac{1}{\Gamma(i+1-\alpha)} \left[1 + \sum_{p=2}^{N} \frac{\Gamma(p-n+1+\alpha)}{\Gamma(\alpha-i)(p-n+i+1)!} \right], \quad i = 0, \dots, n-1,$$

$$B(\alpha, p) = \frac{\Gamma(p-n+1+\alpha)}{\Gamma(-\alpha)\Gamma(1+\alpha)(p-n+1)!},$$

$$V_{p}(t) = (p-n+1) \int_{a}^{t} (\tau-a)^{p-n} x(\tau) d\tau.$$

Similarly, we can deduce an expansion formula for the right fractional derivative.

Theorem 2.6. Fix $n \in \mathbb{N}$ and $x(\cdot) \in C^n[a, b]$. Then,

$${}_{t}D_{b}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}(b-t)^{-\alpha}x(t) + \sum_{i=1}^{n-1}A(\alpha,i)(b-t)^{i-\alpha}x^{(i)}(t) + \sum_{p=n}^{\infty} \left[\frac{-\Gamma(p-n+1+\alpha)}{\Gamma(-\alpha)\Gamma(1+\alpha)(p-n+1)!}(b-t)^{-\alpha}x(t) + B(\alpha,p)(b-t)^{n-1-\alpha-p}W_{p}(t)\right],$$

where

$$A(\alpha, i) = \frac{(-1)^{i}}{\Gamma(i+1-\alpha)} \left[1 + \sum_{p=n-i}^{\infty} \frac{\Gamma(p-n+1+\alpha)}{\Gamma(-i+\alpha)(p-n+1+i)!} \right], \quad i = 1, \dots, n-1,$$

$$B(\alpha, p) = \frac{(-1)^{n} \Gamma(p-n+1+\alpha)}{\Gamma(-\alpha)\Gamma(1+\alpha)(p-n+1)!},$$

$$W_{p}(t) = (p-n+1) \int_{t}^{b} (b-\tau)^{p-n} x(\tau) d\tau.$$

Proof. Analogous to the proof of Theorem 2.3.

2.3 Error estimation

This section is devoted to the study of the error caused by choosing a finite number of terms in the expansions. For the expansion (2), we separate the error term in (3) and rewrite it as

$${}_{a}D_{t}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{a}^{t} \left((t-\tau)^{-\alpha}\sum_{n=0}^{N}\frac{(-1)^{n}x^{(n)}(t)}{n!}(t-\tau)^{n}\right)d\tau + \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{a}^{t} \left((t-\tau)^{-\alpha}\sum_{n=N+1}^{\infty}\frac{(-1)^{n}x^{(n)}(t)}{n!}(t-\tau)^{n}\right)d\tau.$$
(17)

The first term in (17) gives (4) directly and the second term is the error caused by truncation. The next step is to give a local upper bound for this error, $E_{tr}(t)$. The series

$$\sum_{n=N+1}^{\infty} \frac{(-1)^n x^{(n)}(t)}{n!} (t-\tau)^n, \quad \tau \in (a,t), \quad t \in (a,b),$$

is the remainder of the Taylor expansion of $x(\tau)$ and thus bounded by $\left|\frac{M}{(N+1)!}(t-\tau)^{N+1}\right|$ in which $M = \max_{\tau \in [a,t]} \left|x^{(N+1)}(\tau)\right|$. Then,

$$E_{tr}(t) \le \left| \frac{M}{\Gamma(1-\alpha)(N+1)!} \frac{d}{dt} \int_a^t (t-\tau)^{N+1-\alpha} d\tau \right| = \frac{M}{\Gamma(1-\alpha)(N+1)!} (t-a)^{N+1-\alpha}.$$

For approximation (11), we observe that the integrand in (7) can be expanded, by the binomial theorem, as

$$\left(1 - \frac{\tau - a}{t - a}\right)^{1 - \alpha} = \sum_{p=0}^{\infty} \frac{\Gamma(p - 1 + \alpha)}{\Gamma(\alpha - 1)p!} \left(\frac{\tau - a}{t - a}\right)^{p}$$

$$= \sum_{p=0}^{N} \frac{\Gamma(p - 1 + \alpha)}{\Gamma(\alpha - 1)p!} \left(\frac{\tau - a}{t - a}\right)^{p} + R_{N}(\tau), \tag{18}$$

where

$$R_N(\tau) = \sum_{p=N+1}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \left(\frac{\tau-a}{t-a}\right)^p.$$

Substituting (18) into (7), we get

$${}_{0}D_{t}^{\alpha}x(t) = \frac{x(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + \frac{\dot{x}(a)}{\Gamma(2-\alpha)}(t-a)^{1-\alpha}$$

$$+ \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)} \int_{a}^{t} \left(\sum_{p=0}^{N} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \left(\frac{\tau-a}{t-a} \right)^{p} + R_{N}(\tau) \right) \ddot{x}(\tau) d\tau$$

$$= \frac{x(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} + \frac{\dot{x}(a)}{\Gamma(2-\alpha)}(t-a)^{1-\alpha}$$

$$+ \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)} \int_{a}^{t} \left(\sum_{p=0}^{N} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \left(\frac{\tau-a}{t-a} \right)^{p} \right) \ddot{x}(\tau) d\tau$$

$$+ \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)} \int_{a}^{t} R_{N}(\tau) \ddot{x}(\tau) d\tau.$$

At this point, we apply the techniques of [7] to the first three terms with finite sums. Then, we receive (11) with an extra term of truncation error:

$$E_{tr}(t) = \frac{(t-a)^{1-\alpha}}{\Gamma(2-\alpha)} \int_{a}^{t} R_{N}(\tau) \ddot{x}(\tau) d\tau.$$

Since $0 \le \frac{\tau - a}{t - a} \le 1$ for $\tau \in [a, t]$, one has

$$|R_{N}(\tau)| \leq \sum_{p=N+1}^{\infty} \left| \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \right| = \sum_{p=N+1}^{\infty} \left| \binom{1-\alpha}{p} \right| \leq \sum_{p=N+1}^{\infty} \frac{e^{(1-\alpha)^{2}+1-\alpha}}{p^{2-\alpha}}$$

$$\leq \int_{p=N}^{\infty} \frac{e^{(1-\alpha)^{2}+1-\alpha}}{p^{2-\alpha}} dp = \frac{e^{(1-\alpha)^{2}+1-\alpha}}{(1-\alpha)N^{1-\alpha}}.$$

Finally, assuming $L_2 = \max_{\tau \in [a,t]} |\ddot{x}(\tau)|$, we conclude that

$$|E_{tr}(t)| \le L_2 \frac{e^{(1-\alpha)^2 + 1 - \alpha}}{\Gamma(2-\alpha)(1-\alpha)N^{1-\alpha}} (t-a)^{2-\alpha}.$$

In the general case, the error is given by the following result.

Theorem 2.7. If we approximate the left Riemann-Liouville fractional derivative by the finite sum (16), then the error $E_{tr}(\cdot)$ is bounded by

$$|E_{tr}(t)| \le L_n \frac{e^{(n-1-\alpha)^2 + n - 1 - \alpha}}{\Gamma(n-\alpha)(n-1-\alpha)N^{n-1-\alpha}} (t-a)^{n-\alpha},$$
 (19)

where

$$L_n = \max_{\tau \in [a,t]} \left| x^{(n)}(\tau) \right|.$$

From (19) we see that if the test function grows very fast or the point t is far from a, then the value of N should also increase in order to have a good approximation. Clearly, if we increase the value of n, then we need also to increase the value of N to control the error.

3 Numerical evaluation of fractional derivatives

In [24] a numerical method to evaluate fractional derivatives is given based on the Grünwald–Letnikov definition of fractional derivatives. It uses the fact that for a large class of functions, the Riemann–Liouville and the Grünwald–Letnikov definitions are equivalent. We claim that the approximations discussed so far provide a good tool to compute numerically the fractional derivatives of given functions. For functions whose higher-order derivatives are easily available, we can freely choose between approximations (4) or (11). But in the case that difficulties arise in computing higher-order derivatives, we choose the approximation (11) that needs only the values of the first derivative and function itself. Even if the first derivative is not easily computable, we can use the approximation given by (14) with large values for N and α not so close to one. As an example, we compute ${}_{0}D_{t}^{\alpha}x(t)$, with $\alpha = \frac{1}{2}$, for $x(t) = t^{4}$ and $x(t) = e^{2t}$. The exact formulas of the derivatives are derived from

$$_{0}D_{t}^{0.5}(t^{n}) = \frac{\Gamma(n+1)}{\Gamma(n+1-0.5)}t^{n-0.5}$$
 and $_{0}D_{t}^{0.5}(e^{\lambda t}) = t^{-0.5}E_{1,1-0.5}(\lambda t),$

where $E_{\alpha,\beta}$ is the two parameter Mittag-Leffler function [24]. Figure 2 shows the results using approximation (4). As we can see, the third approximations are reasonably accurate for both cases. Indeed, for $x(t) = t^4$, the approximation with N = 4 coincides with the exact solution because the derivatives of order five and more vanish. The same computations are carried out

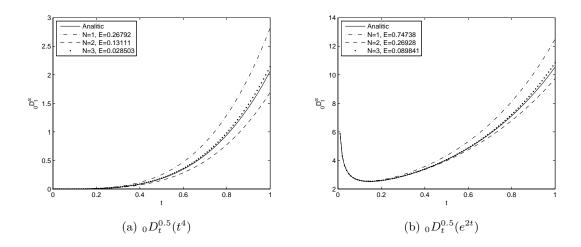


Figure 2: Analytic (solid line) versus numerical approximation (4).

using approximation (11). In this case, given a function $x(\cdot)$, we can compute V_p by definition or integrate the system (10) analytically or by any numerical integrator. As it is clear from Figure 3, one can get better results by using larger values of N. Comparing Figures 2 and 3, we find out that the approximation (4) shows a faster convergence. Observe that both functions are analytic and it is easy to compute higher-order derivatives. The approximation (4) fails for non-analytic functions as stated in [7].

Remark 3.1. A closer look to (4) and (11) reveals that in both cases the approximations are not computable at a and b for the left and right fractional derivatives, respectively. At these points we assume that it is possible to extend them continuously to the closed interval [a,b].

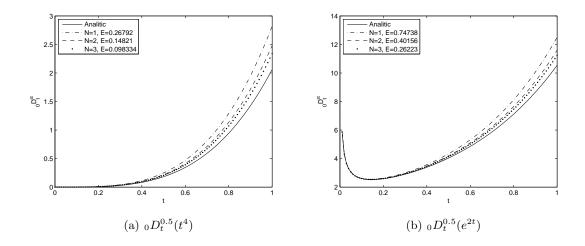


Figure 3: Analytic (solid line) versus numerical approximation (11).

In what follows, we show that by omitting the first derivative from the expansion, as done in [7], one may loose a considerable accuracy in computation. Once again, we compute the fractional derivatives of $x(t) = t^4$ and $x(t) = e^{2t}$, but this time we use the approximation given by (14). Figure 4 summarizes the results. Our expansion gives a more realistic approximation using quite small N, 3 in this case. To show how the appearance of higher-order derivatives

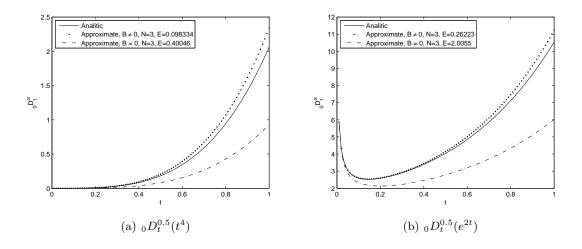
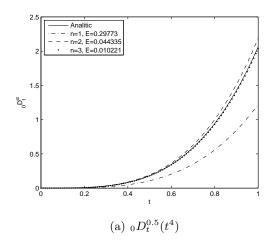


Figure 4: Comparison of approximation (11) proposed here and approximation (14) of [7]

in generalization (15) gives better results, we evaluate fractional derivatives of $x(t) = t^4$ and $x(t) = e^{2t}$ for different values of n. We consider n = 1, 2, 3, N = 6 for $x(t) = t^4$ (Figure 5(a)) and N = 4 for $x(t) = e^{2t}$ (Figure 5(b)).

3.1 Fractional derivatives of tabular data

In many applications (see Section 5), the function itself is not accessible in a closed form, but as a tabular data for discrete values of the independent variable. Thus, we cannot use the



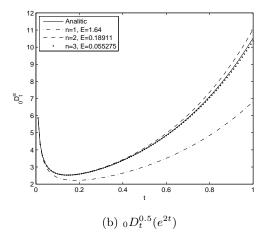


Figure 5: Analytic (solid line) versus numerical approximation (15).

definition to compute the fractional derivative directly. Our approximation (11), that uses the function and its first derivative to evaluate the fractional derivative, seems to be a good candidate in those cases. Suppose that we know the values of $x(t_i)$ on n+1 distinct points in a given interval [a, b], i.e., for t_i , i = 0, 1, ..., n, with $t_0 = a$ and $t_n = b$. According to formula (11), the value of the fractional derivative of $x(\cdot)$ at each point t_i is given approximately by

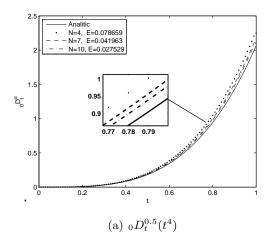
$$_{a}D_{t}^{\alpha}x(t_{i}) \simeq A(\alpha, N)(t_{i} - a)^{-\alpha}x(t_{i}) + B(\alpha, N)(t_{i} - a)^{1-\alpha}\dot{x}(t_{i}) - \sum_{p=2}^{N}C(p, \alpha)(t_{i} - a)^{1-p-\alpha}V_{p}(t_{i}).$$

The values of $x(t_i)$, $i=0,1,\ldots,n$, are given. A good approximation for $\dot{x}(t_i)$ can be obtained using the forward, centered, or backward difference approximation of the first-order derivative [30]. For $V_p(t_i)$ one can either use the definition and compute the integral numerically, i.e., $V_p(t_i) = \int_a^{t_i} (1-p)(\tau-a)^{p-2} x(\tau) d\tau$, or it is possible to solve (10) as an initial value problem. All required computations are straightforward and only need to be implemented with the desired accuracy. The only thing to take care is the way of choosing a good order, N, in the formula (11). Because no value of N, guaranteeing the error to be smaller than a certain preassigned number, is known a priori, we start with some prescribed value for N and increase it step by step. In each step we compare, using an appropriate norm, the result with the one of previous step. For instance, one can use the Euclidean norm $\|(aD_t^{\alpha})^{new} - (aD_t^{\alpha})^{old}\|_2$ and terminate the procedure when it's value is smaller than a predefined ϵ . For illustrative purposes, we compute the fractional derivatives of order $\alpha=0.5$ for tabular data extracted from $x(t)=t^4$ and $x(t)=e^{2t}$. The results are given in Figure 6.

4 Numerical solution to fractional differential equations

The classical theory of ordinary differential equations is a well developed field with many tools available for numerical purposes. Using the approximations (4) and (11), one can transform a fractional ordinary differential equation into a classical ODE.

We should mention here that, using (4), derivatives of higher-order appear in the resulting ODE, while we only have a limited number of initial or boundary conditions available. In



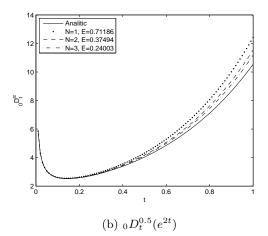


Figure 6: Fractional derivatives of tabular data

this case the value of N, the order of approximation, should be equal to the number of given conditions. If we choose a larger N, we will encounter lack of initial or boundary conditions. This problem is not present in the case in which we use the approximation (11), because the initial values for the auxiliary variables V_p , $p = 2, 3, \ldots$, are known and we don't need any extra information.

Consider, as an example, the following initial value problem:

$$\begin{cases} {}_{0}D_{t}^{0.5}x(t) + x(t) = t^{2} + \frac{2}{\Gamma(2.5)}t^{\frac{3}{2}}, \\ x(0) = 0. \end{cases}$$
 (20)

We know that ${}_{0}D_{t}^{0.5}(t^{2}) = \frac{2}{\Gamma(2.5)}t^{\frac{3}{2}}$. Therefore, the analytic solution for system (20) is $x(t) = t^{2}$. Because only one initial condition is available, we can only expand the fractional derivative up to the first derivative in (4). One has

$$\begin{cases} 1.5642 \ t^{-0.5}x(t) + 0.5642 \ t^{0.5}\dot{x}(t) = t^2 + 1.5045 \ t^{1.5}, \\ x(0) = 0. \end{cases}$$
 (21)

This is a classical initial value problem and can be easily treated numerically. The solution is drawn in Figure 7(a). As expected, the result is not satisfactory. Let us now use the approximation given by (11). The system in (20) becomes

$$\begin{cases}
A(N)t^{-0.5}x(t) + B(N)t^{0.5}\dot{x}(t) - \sum_{p=2}^{N} C(p)t^{0.5-p}V_p + x(t) = t^2 + \frac{2}{\Gamma(2.5)}t^{1.5}, \\
\dot{V}_p(t) = (1-p)(t-a)^{p-2}x(t), \quad p = 2, 3, \dots, N, \\
x(0) = 0, \\
V_p(0) = 0, \quad p = 2, 3, \dots, N.
\end{cases}$$
(22)

We solve this initial value problem for N=7. The Matlab ode45 built-in function is used to integrate system (22). The solution is given in Figure 7(b) and shows a better approximation when compared with (21).

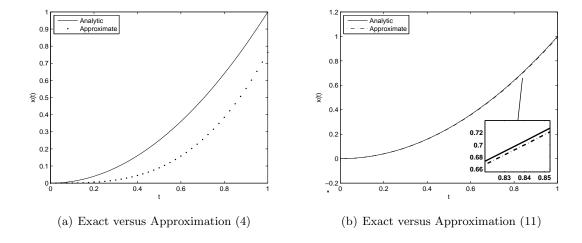


Figure 7: Two approximations applied to fractional differential equation (20)

Remark 4.1. To show the difference caused by the appearance of the first derivative in formula (11), we solve the initial value problem (20) with $B(\alpha, N) = 0$. Since the original fractional differential equation does not depend on integer order derivatives of function $x(\cdot)$, i.e., it has the form

$$_{a}D_{t}^{\alpha}x(t) + f(x,t) = 0,$$

by (14) the dependence to derivatives of $x(\cdot)$ vanishes. In this case one needs to apply the operator ${}_aD_t^{1-\alpha}$ to the above equation and obtain

$$\dot{x}(t) +_a D_t^{1-\alpha}[f(x,t)] = 0.$$

Nevertheless, we can use (11) directly without any trouble. Figure 8 shows that at least for a moderate accurate method, like the Matlab routine ode45, taking $B(\alpha, N) \neq 0$ into account gives a better approximation.

5 Application to the fractional calculus of variations

The fractional calculus of variations consists in the study of dynamic optimization problems in which the objective functional or constraints depend on derivatives and/or integrals of non-integer order. This is a recent and promising research subject, under strong current research (see, e.g., [14, 20, 21] and references therein). Here we show how to use expansions to transform a fractional problem into a classical one, where we can benefit from the vast number of techniques available in the field. Consider the following fractional variational problem including a left Riemann–Liouville fractional derivative, ${}_{a}D_{t}^{\alpha}$, of order $\alpha \in (0,1)$:

min
$$J[x(\cdot)] = \int_a^b L(t, x(t), \dot{x}(t), aD_t^{\alpha} x(t)) dt$$
$$x(a) = x_a, \quad x(b) = x_b.$$
 (23)

One can deduce fractional necessary optimality equations to problem (23) of Euler-Lagrange type [22, 23]: if $x(\cdot)$ is a solution to problem (23), then it satisfies the fractional Euler-

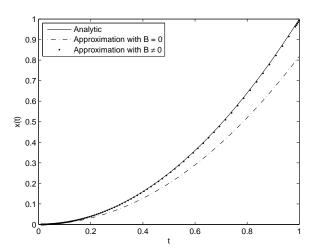


Figure 8: Comparison of our approach to that of [7]

Lagrange equation

$$\frac{\partial L}{\partial x} + {}_{t}D_{b}^{\alpha} \frac{\partial L}{\partial_{a}D_{t}^{\alpha}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0. \tag{24}$$

There are a few attempts in the literature to present analytic solutions to fractional variational problems. Simple problems have been treated in [1]; some other examples are presented in [4]. In this work we use the two expansions discussed in Section 2 to reduce a fractional problem to a problem with derivatives of integer order. In order to illustrate the usefulness of our ideas in the area of the calculus of variations, we need to consider problems (23) with a known exact solution. Examples 5.1 and 5.2 below are suitable for our purposes, since the analytic solutions can be easily obtained. Knowing the exact solutions, we compare the effectiveness of different approximation methods.

Example 5.1. Let $\alpha \in (0,1)$. Consider the following minimization problem:

min
$$J[x(\cdot)] = \int_0^1 [{}_0D_t^{\alpha}x(t) - \dot{x}^2(t)]dt$$

 $x(0) = 0, \quad x(1) = 1.$ (25)

In this case the Euler-Lagrange equation (24) gives

$$_{t}D_{1}^{\alpha}1 + 2\ddot{x}(t) = 0$$
, or $\ddot{x}(t) = -\frac{1}{2\Gamma(1-\alpha)}(1-t)^{-\alpha}$,

which subject to the given boundary conditions has solution

$$x(t) = -\frac{1}{2\Gamma(3-\alpha)}(1-t)^{2-\alpha} + \left(1 - \frac{1}{2\Gamma(3-\alpha)}\right)t + \frac{1}{2\Gamma(3-\alpha)}.$$
 (26)

The Lagrangian in Example 5.1 is linear with respect to the fractional derivative. This linearity makes the fractional Euler-Lagrange equation easy to solve. In a slightly different situation, e.g. Example 5.2, there is no well-known methods to solve the Euler-Lagrange equation (24).

Example 5.2. Given $\alpha \in (0,1)$, consider now the functional

$$J[x(\cdot)] = \int_0^1 ({}_0D_t^{\alpha}x(t) - 1)^2 dt, \tag{27}$$

to be minimized subject to the boundary conditions x(0) = 0 and $x(1) = \frac{1}{\Gamma(\alpha+1)}$. Since the integrand in (27) is non-negative, the functional attains its minimum when ${}_{0}D_{t}^{\alpha}x(t) = 1$, i.e., for $x(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$.

5.1 Numerical solutions to Example 5.1

We use two different approaches.

5.1.1 Expansion to integer orders

Using approximation (4) for the fractional derivative in (25), we get the approximated problem

min
$$\tilde{J}[x(\cdot)] = \int_0^1 \left[\sum_{n=0}^N C(n,\alpha) t^{n-\alpha} x^{(n)}(t) - \dot{x}^2(t) \right] dt$$
 (28)
 $x(0) = 0, \quad x(1) = 1,$

which is a classical higher-order problem of the calculus of variations that depends on derivatives up to order N. The corresponding necessary optimality condition is a well-known result.

Theorem 5.3 (cf., e.g., [18]). Suppose that $x(\cdot) \in C^{2N}[a,b]$ minimizes

$$\int_{a}^{b} L(t, x(t), x^{(1)}(t), x^{(2)}(t), \dots, x^{(N)}(t)) dt$$

with given boundary conditions

$$x(a) = a_0,$$
 $x(b) = b_0,$ $x^{(1)}(a) = a_1,$ $x^{(1)}(b) = b_1,$ \vdots $x^{(N-1)}(a) = a_{N-1},$ $x^{(N-1)}(b) = b_{N-1}.$

Then $x(\cdot)$ satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial x^{(1)}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial x^{(2)}} \right) - \dots + (-1)^N \frac{d^N}{dt^N} \left(\frac{\partial L}{\partial x^{(N)}} \right) = 0. \tag{29}$$

In general (29) is an ODE of order 2N, depending on the order N of the approximation we choose, and the method leaves 2N-2 parameters unknown. In our example, however, the Lagrangian in (28) is linear with respect to all derivatives of order higher than two. The resulting Euler-Lagrange equation is the second order ODE

$$\sum_{n=0}^{N} (-1)^{n} C(n,\alpha) \frac{d^{n}}{dt^{n}} (t^{n-\alpha}) - \frac{d}{dt} [-2\dot{x}(t)] = 0$$

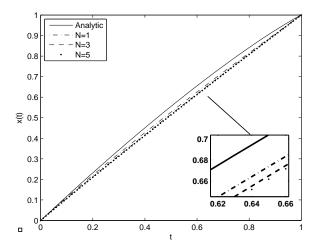


Figure 9: Analytic vs. approximate solutions to Example 5.1 using approximation (4).

that has solution

$$x(t) = -\frac{1}{2\Gamma(3-\alpha)} \left[\sum_{n=0}^{N} (-1)^n \Gamma(n+1-\alpha) C(n,\alpha) \right] t^{2-\alpha}$$

$$+ \left[1 + \frac{1}{2\Gamma(3-\alpha)} \sum_{n=0}^{N} (-1)^n \Gamma(n+1-\alpha) C(n,\alpha) \right] t.$$

Figure 9 shows the analytic solution together with several approximations. It reveals that by increasing N, approximate solutions do not converge to the analytic one. The reason is the fact that the solution (26) to Example 5.1 is not an analytic function. We conclude that (4) may not be a good choice to approximate fractional variational problems. In contrast, as we shall see, the approximation (11) introduced in this paper leads to good results.

5.1.2 Expansion through the moments of a function

If we use (11) to approximate the optimization problem (25), we have

$$\tilde{J}[x(\cdot)] = \int_{0}^{1} \left[A(\alpha, N)t^{-\alpha}x(t) + B(\alpha, N)t^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{N} C(\alpha, p)t^{1-p-\alpha}V_{p}(t) - \dot{x}^{2}(t) \right] dt,
\dot{V}_{p}(t) = (1-p)t^{p-2}x(t), \quad p = 2, 3, \dots, N,
V_{p}(0) = 0, \quad p = 2, 3, \dots, N,
x(0) = 0, \quad x(1) = 1.$$
(30)

Problem (30) is constrained with a set of ordinary differential equations and is natural to look to it as an optimal control problem [25]. For that we introduce the control variable $u(t) = \dot{x}(t)$.

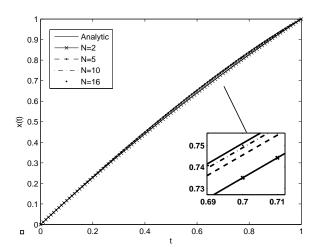


Figure 10: Analytic vs. approximate solutions to Example 5.1 using approximation (11).

Then, using the Lagrange multipliers $\lambda_1, \lambda_2, \dots, \lambda_N$, and the Hamiltonian system, one can reduce (30) to the study of the two point boundary value problem

$$\begin{cases}
\dot{x}(t) &= \frac{1}{2}B(\alpha, N)t^{1-\alpha} - \frac{1}{2}\lambda_{1}(t), \\
\dot{V}_{p}(t) &= (1-p)t^{p-2}x(t), \quad p = 2, 3, \dots, N, \\
\dot{\lambda}_{1}(t) &= A(\alpha, N)t^{-\alpha} - \sum_{p=2}^{N}(1-p)t^{p-2}\lambda_{p}(t), \\
\dot{\lambda}_{p}(t) &= -C(\alpha, p)t^{(1-p-\alpha)}, \quad p = 2, 3, \dots, N,
\end{cases}$$
(31)

with boundary conditions

$$\begin{cases} x(0) = 0, \\ V_p(0) = 0, \quad p = 2, 3, \dots, N, \end{cases} \begin{cases} x(1) = 1, \\ \lambda_p(1) = 0, \quad p = 2, 3, \dots, N, \end{cases}$$

where x(0) = 0 and x(1) = 1 are given. We have $V_p(0) = 0$, p = 2, 3, ..., N, due to (10) and $\lambda_p(1) = 0$, p = 2, 3, ..., N, because V_p is free at final time for p = 2, 3, ..., N [25]. In general, the Hamiltonian system is a nonlinear, hard to solve, two point boundary value problem that needs special numerical methods. In this case, however, (31) is a non-coupled system of ordinary differential equations and is easily solved to give

$$x(t) = M(\alpha, N)t^{2-\alpha} - \sum_{p=2}^{N} \frac{C(\alpha, p)}{2p(2-p-\alpha)} t^p + \left[1 - M(\alpha, N) + \sum_{p=2}^{N} \frac{C(\alpha, p)}{2p(2-p-\alpha)} \right] t,$$

where

$$M(\alpha, N) = \frac{1}{2(2-\alpha)} \left[B(\alpha, N) - \frac{A(\alpha, N)}{1-\alpha} - \sum_{p=2}^{N} \frac{C(\alpha, p)(1-p)}{(1-\alpha)(2-p-\alpha)} \right].$$

Figure 10 shows the graph of $x(\cdot)$ for different values of N.

5.2 Numerical solutions to Example 5.2

Similarly to Example 5.1, it turns out that expansion (4) does not provide a good method while (11) leads to good results.

5.2.1 Expansion to integer orders

Using (4) as an approximation for the fractional derivative in (27) gives

min
$$\tilde{J}[x(\cdot)] = \int_0^1 \left(\sum_{n=0}^N C(n,\alpha) t^{n-\alpha} x^{(n)}(t) - 1 \right)^2 dt,$$

 $x(0) = 0, \quad x(1) = \frac{1}{\Gamma(\alpha+1)}.$

The Euler-Lagrange equation (29) gives a 2N order ODE. For $N \geq 2$ this approach is inappropriate since the two given boundary conditions x(0) = 0 and $x(1) = \frac{1}{\Gamma(\alpha+1)}$ are not enough to determine the 2N constants of integration.

5.2.2 Expansion through the moments of a function

Let us approximate Example 5.2 using (11). The resulting minimization problem has the following form:

$$\min \quad \tilde{J}[x(\cdot)] = \int_0^1 \left[A(\alpha, N) t^{-\alpha} x(t) + B(\alpha, N) t^{1-\alpha} \dot{x}(t) - \sum_{p=2}^N C(\alpha, p) t^{1-p-\alpha} V_p(t) - 1 \right]^2 dt,
\dot{V}_p(t) = (1-p) t^{p-2} x(t), \quad p = 2, 3, \dots, N,
V_p(0) = 0, \quad p = 2, 3, \dots, N,
x(0) = 0, \quad x(1) = \frac{1}{\Gamma(\alpha+1)}.$$
(32)

Following the classical optimal control approach of Pontryagin [25] as in Example 5.1, this time with

$$u(t) = A(\alpha, N)t^{-\alpha}x(t) + B(\alpha, N)t^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{N} C(\alpha, p)t^{1-p-\alpha}V_p(t),$$

we conclude that the solution to (32) satisfies the system of differential equations

$$\begin{cases}
\dot{x}(t) &= -AB^{-1}t^{-1}x(t) + \sum_{p=2}^{N} B^{-1}C_{p}t^{-p}V_{p}(t) + \frac{1}{2}B^{-2}t^{2\alpha-2}\lambda_{1}(t) + B^{-1}t^{\alpha-1}, \\
\dot{V}_{p}(t) &= (1-p)t^{p-2}x(t), \quad p = 2, 3, \dots, N, \\
\dot{\lambda}_{1}(t) &= AB^{-1}t^{-1}\lambda_{1} - \sum_{p=2}^{N}(1-p)t^{p-2}\lambda_{p}(t), \\
\dot{\lambda}_{p}(t) &= -B^{-1}C(\alpha, p)t^{-p}\lambda_{1}, \quad p = 2, 3, \dots, N,
\end{cases} (33)$$

where $A = A(\alpha, N)$, $B = B(\alpha, N)$ and $C_p = C(\alpha, p)$ are defined according to Section 2, subject to the boundary conditions

$$\begin{cases} x(0) = 0, \\ V_p(0) = 0, \quad p = 2, 3, \dots, N, \end{cases} \begin{cases} x(1) = \frac{1}{\Gamma(\alpha + 1)}, \\ \lambda_p(1) = 0, \quad p = 2, 3, \dots, N. \end{cases}$$
(34)

The solution to system (33)–(34), with N=2, is shown in Figure 11.

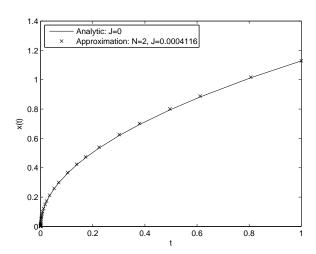


Figure 11: Analytic versus approximate solution to Example 5.2 using approximation (11).

6 Conclusion

During the last three decades, several numerical methods have been developed in the field of fractional calculus. Some of their advantages, disadvantages, and improvements, are given in [3]. Based on two continuous expansion formulas (2) and (9) for the left Riemann–Liouville fractional derivative, we studied two approximations (4) and (11) and their applications in the computation of fractional derivatives. Despite the fact that the approximation (4) encounters some difficulties from the presence of higher-order derivatives, it exhibits better results regarding convergence. Approximation (11) can also be generalized to include higher-order derivatives in the form of (15). The possibility of using (11) to compute fractional derivatives for a set of tabular data was discussed. Fractional differential equations are also treated successfully. In this case the lack of initial conditions makes (4) less useful. In contrast, one can freely increase N, the order of approximation (11), and find better approximations. Comparing with (14), our modification provides better results. We finished by discussing the solution of fractional variational problems using the introduced approximations. Similar methods can also be applied to fractional optimal control problems.

For fractional variational problems, the proposed expansions may be used at two different stages during the solution procedure. The first approach, the one considered in Sections 5.1 and 5.2, consists in a direct approximation of the problem, and then treating it as a classical problem, using standard methods to solve it. The second approach would be to apply the fractional Euler–Lagrange equation and then to use the approximations in order to obtain a classical differential equation. However, the Euler–Lagrange equations (24) involve right Riemann–Liouville derivatives, which introduces undesirable difficulties.

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References

- [1] R. Almeida and D. F. M. Torres, Leitmann's direct method for fractional optimization problems, Appl. Math. Comput. **217** (2010), no 3, 956–962. arXiv:1003.3088
- [2] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, 71, Cambridge Univ. Press, Cambridge, 1999.
- [3] M. Aoun, R. Malti, F. Levron and A. Oustaloup, Numerical simulations of fractional systems: An overview of existing methods and improvements, Nonlinear Dynam. **38** (2004), no. 1-4, 117–131.
- [4] T. M. Atanacković, S. Konjik, Lj. Oparnica and S. Pilipović, Generalized Hamilton's principle with fractional derivatives, J. Phys. A 43 (2010), no. 25, 255203, 12 pp. arXiv:1101.2963
- [5] T. M. Atanacković, S. Konjik and S. Pilipović, Variational problems with fractional derivatives: Euler-Lagrange equations, J. Phys. A 41 (2008), no. 9, 095201, 12 pp. arXiv:1101.2961
- [6] T. M. Atanackovic and B. Stankovic, An expansion formula for fractional derivatives and its application, Fract. Calc. Appl. Anal. 7 (2004), no. 3, 365–378.
- [7] T. M. Atanacković and B. Stankovic, On a numerical scheme for solving differential equations of fractional order, Mech. Res. Comm. **35** (2008), no. 7, 429–438.
- [8] S. Das, Functional fractional calculus for system identification and controls, Springer, Berlin, 2008.
- [9] K. Diethelm, *The analysis of fractional differential equations*, Lecture Notes in Mathematics, 2004, Springer, Berlin, 2010.
- [10] K. Diethelm, N. J. Ford and A. D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dynam. **29** (2002), no. 1-4, 3–22.
- [11] V. D. Djordjevic and T. M. Atanackovic, Similarity solutions to nonlinear heat conduction and Burgers/Korteweg-de Vries fractional equations, J. Comput. Appl. Math. 222 (2008), no. 2, 701– 714
- [12] M. Ö Efe, Battery power loss compensated fractional order sliding mode control of a quadrotor UAV, Asian J. Control 14 (2012), no. 2, 413-425.
- [13] N. J. Ford and J. A. Connolly, Comparison of numerical methods for fractional differential equations, Commun. Pure Appl. Anal. 5 (2006), no. 2, 289–306.
- [14] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory, Nonlinear Dynam. **53** (2008), no. 3, 215–222. arXiv:0711.0609
- [15] Z. D. Jelicic and N. Petrovacki, Optimality conditions and a solution scheme for fractional optimal control problems, Struct. Multidiscip. Optim. **38** (2009), no. 6, 571–581.
- [16] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.
- [17] P. Kumar and O. P. Agrawal, An approximate method for numerical solution of fractional differential equations, Signal Process. 86 (2006), 2602-2610.

- [18] L. P. Lebedev and M. J. Cloud, *The calculus of variations and functional analysis*, World Sci. Publishing, River Edge, NJ, 2003.
- [19] Y. Li, Y. Chen and H.-S. Ahn, Fractional-order iterative learning control for fractional-order linear systems, Asian J. Control 13 (2011), no. 1, 54–63.
- [20] A. B. Malinowska and D. F. M. Torres, Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative, Comput. Math. Appl. **59** (2010), no. 9, 3110–3116. arXiv:1002.3790
- [21] D. Mozyrska and D. F. M. Torres, Modified optimal energy and initial memory of fractional continuous-time linear systems, Signal Process. 91 (2011), no. 3, 379–385. arXiv:1007.3946
- [22] T. Odzijewicz, A. B. Malinowska and D.F.M. Torres, Fractional variational calculus with classical and combined Caputo derivatives, Nonlinear Anal. 75 (2012), no. 3, 1507–1515. arXiv:1101.2932
- [23] T. Odzijewicz and D.F.M. Torres, Calculus of variations with classical and fractional derivatives Math. Balkanica 26 (2012), no 1-2, 191–202. arXiv:1007.0567
- [24] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering, 198, Academic Press, San Diego, CA, 1999.
- [25] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, The mathematical theory of optimal processes, Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt Interscience Publishers John Wiley & Sons, Inc. New York, 1962.
- [26] S. Pooseh, R. Almeida and D. F. M. Torres, Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative, Numer. Funct. Anal. Optim. 33 (2012), no. 3, 301–319. arXiv:1112.0693
- [27] S. Pooseh, R. Almeida and D. F. M. Torres, Approximation of fractional integrals by means of derivatives Comput. Math. Appl. (2012), DOI: 10.1016/j.camwa.2012.01.068 arXiv:1201.5224
- [28] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives, translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
- [29] J. Shen and J. Cao, Necessary and sufficient conditions for consensus of delayed fractional-order systems, Asian J. Control (2012), DOI: 10.1002/asjc.492
- [30] J. Stoer and R. Bulirsch, Introduction to numerical analysis, translated from the German by R. Bartels, W. Gautschi and C. Witzgall, third edition, Texts in Applied Mathematics, 12, Springer, New York, 2002.
- [31] W. Sun, Y. Li, C. Li and Y. Chen, Convergence speed of a fractional order consensus algorithm over undirected scale-free networks, Asian J. Control 13 (2011), no. 6, 936–946.
- [32] J. A. Tenreiro Machado, V. Kiryakova and F. Mainardi, Recent history of fractional calculus, Commun. Nonlinear Sci. Numer. Simul. 16 (2011), no. 3, 1140–1153.
- [33] J. A. Tenreiro Machado, M. F. Silva, R. S. Barbosa, I. S. Jesus, C. M. Reis, M. G. Marcos and A. F. Galhano, Some applications of fractional calculus in engineering, Math. Probl. Eng., Art. ID 639801, (2010) 34 pp.