# On Dynamic-order Fractional Dynamic System

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(Dated: March 15, 2011)

## Abstract

Motivated by the experimental results of an electronic circuit element "Fractor", we introduce the concept of dynamic-order fractional dynamic system, in which the variable-order of a fractional dynamic system is determined by the output signal of another dynamic system. The new concept offers a comprehensible explanation of physical mechanism of variable-order dynamic systems. The properties and potential applications of dynamic-order dynamic systems are further explored by analysis of anomalous relaxation process and anomalous diffusion process.

PACS numbers: 66.10.C-, 05.10.Gg, 02.60.Cb

Fractional dynamic system has been focused by physicists and mathematicians over the last decades, and has received great success in the analysis of anomalous diffusion process [1–3], viscoelastic rheology [4, 5], control systems [6–8], complex networks [9, 10], wave dissipation in human tissue and electrochemical corrosion process [11], etc. [12–15].

In the past several years, the variable-order operator, an advanced form of fractional calculus, has been applied in various physical fields [16–18]. In the comprehensive analysis of the variable-order fractional dynamic system and its applications, the following situations are usually encountered. For example, to exploit the physical mechanism of variable-order in a considered system, another dynamic system must be involved . Meanwhile, when we study some multi-field physical processes, the behavior of one dynamic system may change with the evolution of other dynamic systems. How to characterize these dynamic systems? Establishing a system of equations which includes intricate interaction terms, will cause great difficulties in modeling and computation. Even worse, the established model may produce inaccurate results which greatly deviate from experimental results or field measurement data. The reason is they miss capturing the physical mechanism of the considered problems. In addition, researchers have confirmed that the differential orders in the some fractional dynamic systems are non-constant and are often functions of other variables or system outputs [18, 19]. For instance, in [20], the authors have found that the differential order of fractional relaxation system is a function of temperature. Therefore, in order to exploit the physical mechanism of variable-order fractional dynamic system, the variable-order dynamics system in which the variable-order may be an output signal from other dynamic systems, should be considered. For simplicity, we name this type of variable-order fractional dynamic system as dynamic-order fractional dynamic system.

The purpose of this letter is to make an innovative study of dynamic-order fractional dynamic systems. As we have known, one of the biggest obstacles for applications of variable-order fractional dynamic models is obtaining the variable-order in the considered system. The optimal idea is to analyze the physical mechanism about how environmental factors or system variables change the system's differential order. From the analysis of fractional dynamic system, especially the variable-order dynamic system, we found the fractional differential order of one dynamic system stems from another dynamic system in most cases. Hence, in many fractional systems, the differential order should be called dynamic-order. We believe, the comprehensive study of dynamic-order system will give us insights about multi-field coupling processes.

Next, we introduce the motivation of dynamic-order fractional system via an experiment of Fractor. This experiment will give us a basic understanding of dynamic-order fractional dynamic systems.

"A Fractor is a two lead passive electronic circuit element similar to a resistor or capacitor, but exhibiting a non-integer order power-law impedance versus frequency." [21]. The Fractor is proven effective as a feedback element in a control system for real-world applications such as temperature control, robotics, etc. [22]. The prototype Fractor is made by hand. It is not much larger than typical through-hole capacitors and the typical unit is 3.5 cm on a side and about 1.0 cm thick, as shown in Fig. 1. The impedance behavior of Fractor can be accurately modeled by the following form which is achieved by Laplace transform of the fractional order operator $Z_{Frac}(\omega) = K/(j\omega\tau)^{\lambda}$, where $K$ is the impedance magnitude at calibration frequency ($\omega_c = 1/\tau$); $\lambda \in (0,1)$ is the fractional exponent and $\omega$ is the frequency [21, 23]. From the previous study of Fractor, it has been confirmed that the order of the Fractor may change with the temperature and the internal material structure. Several experimental results have implied that, temperature can influence the derivative order or integral order which will determine the capacity of the equipment [20, 21]. In our experiment, we intend to give a clear relationship between the temperature and the fractional order of the Fractor. In this experiment, the temperature is controlled by the equipment of Quanser Heatflow Experiment (HFE) [21]. The evolution process of the temperature is a typical heat transfer process. In this heat transfer system, the instantaneous temperature value of the Fractor is the output signal we intend to obtain. Based on the previous experimental observation, we believe that the order of the Fractor is a function of the temperature, i.e.,

$$Z_{Frac}(\omega) = \frac{K}{(j\omega\tau)^{\lambda(T)}}, \tag{1}$$

where $\lambda(T) \in (0,1)$ is the fractional exponent, $T$ is the temperature.

In this experiment, we firstly placed the Fractor in the HFE unit, then the temperature of the Fractor is measured by three temperature sensor. Next, the order of the Fractor is measured by an HP DSA and fitted via the expression (1). Finally, The relationship between the temperature and the order of the Fractor is obtained as shown in Fig. 2. It is observed that the order of the Fractor is an approximately linear function of the temperature.

From this experiment, we can claim that the order of the Fractor changes with the

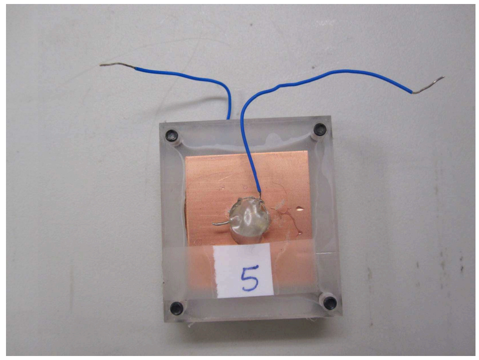


FIG. 1: Photo of a sample hand-made prototype Fractor. The detailed preparation and description of Fractor can be found in [21].

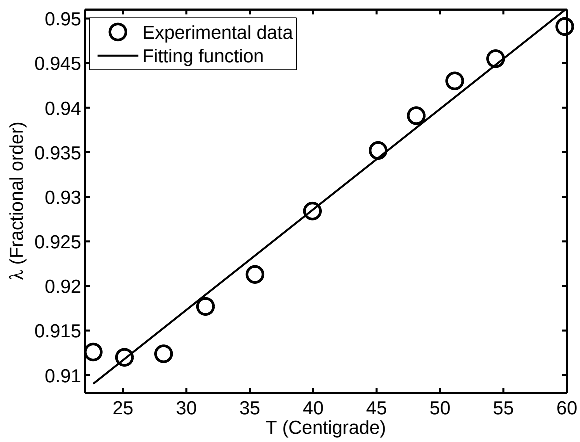


FIG. 2: The evolution data of fractional order of the Fractor. In this temperature range, the linear fitting function is $\lambda(T) = p_1 T + p_2$, where $p_1 = 0.001127$, $p_2 = 0.8835$.

temperature which can be recognized as the output of the heat transfer system in the Fractor. Therefore, this system is a typical dynamic-order system.

Next, we discuss the dynamic-order fractional dynamic system from the viewpoint of variable-order fractional dynamic system. A representative definition of variable-order fractional derivative in Caputo sense can be stated as follows [24, 25]

$$ {}^{C}D_{0+}^{\alpha(Z)} f(t) = \frac{1}{\Gamma(1-\alpha(Z))} \int_0^t \frac{f'(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha(Z)}}, \tag{2} $$

where $Z$ denotes certain system variables or other independent variables, which govern the differential-order of interested dynamic-order system. $0 < \alpha(Z) < 1$ is a function of the independent variable of $Z$ and $\Gamma()$ is the Gamma function.

In a variable-order fractional dynamic system, the variable-order is often a function of certain variable, such as temperature and concentration. In many realistic applications, these order-influencing variables usually vary with time, in other words, it usually can be regarded as output signals of other dynamic systems.

In engineering practice, we usually encounter the phenomena of the multi-field coupling in physics, such as temperature field, stress field and electromagnetic field coupled together with the concerned dynamic system under investigation. In the traditional approach, when considering the coupling effect of different fields or some multi-scale physical processes, we are accustomed to employing a system of differential equations and have received great successes in the past decades [26]. For example, In [27], when modeling the tracer transport in a fractured granite, the authors established a convection-diffusion equation in fracture scale and a diffusion equation in matrix scale, in which the intricate interaction is reflected by an additional term in the convection-diffusion equation in fracture scale. However, in the recent decades, experimental and theoretical studies have implied that the dynamic-order system approach can be employed to reflect the coupling effect in some real-world multi-field or multi-scale problems [18, 21]. The link between these coupled systems may be better characterized by variable differential order of dynamic systems, which can be visually illustrated by Fig. 3.

From the viewpoint of dynamic-order system, when we consider some multi-scale or multi-field problems, we can establish the following generalized form of dynamic-order fractional dynamic systems

$$\begin{cases} \dfrac{d^{\alpha_1(X_n,X_{n-1},\ldots,X_1,t)}X_1}{dt^{\alpha_1(X_n,X_{n-1},\ldots,X_1,t)}} = g_1(X_1,t), \\ \dfrac{d^{\alpha_2(X_n,X_{n-1},\ldots,X_1,t)}X_2}{dt^{\alpha_2(X_n,X_{n-1},\ldots,X_1,t)}} = g_2(X_2,t), \\ \qquad\qquad\vdots \\ \dfrac{d^{\alpha_n(X_n,X_{n-1},\ldots,X_1,t)}X_n}{dt^{\alpha_n(X_n,X_{n-1},\ldots,X_1,t)}} = g_n(X_n,t). \end{cases} \tag{3}$$

The most important feature of the above generalized form is that the intricate interaction between dynamic systems has been represented in the differential order of each sub-system.

In the following, we illustrate the dynamic-order fractional dynamic system by two cases.

*Case 1.* We firstly consider the relaxation system, which has been widely applied in energy dissipation, viscoelasticity and rheology, etc. [24, 28–30]. When we characterize the relaxation process, in which the relaxation pattern changes with control parameters or other

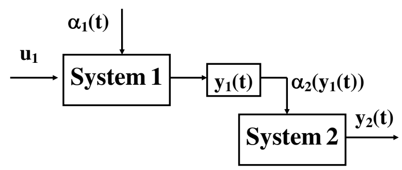


FIG. 3: An illustration of a dynamic-order fractional dynamic system. In this schematic, $u_1$ is the input of system 1 and $\alpha_1(t)$ is the differential order of system 1, $y_1(t)$ is the output signal of system 1. $\alpha_2(y_1(t))$ denotes that the differential-order of system 2, $u_2$ and $y_2(t)$ represent the input and output signals of system 2.

variables, the following variable-order fractional differential equation should be employed

$$\begin{cases} {}^{C}D_t^{\alpha(Z)}x(t) = -Bx(t) + f(t),\, 0 < \alpha(Z) < 1, \\ x(0) = 1, \end{cases} \tag{4}$$

where $B$ is the relaxation coefficient and $Z$ is an independent variable.

In engineering fields, fuzzy systems have extensive applications in control and system parameter calibration [31–33]. Fuzzy systems have become an efficient tool to exploit clear conclusion from uncertain information and incomplete data [34, 35]. Therefore, we consider the dynamic system (4) in which the variable-order is governed by the following fuzzy dynamic system. Here, we select the Takagi-Sugeno (T-S) fuzzy system as an example for illustration purpose. The T-S fuzzy system obeys the following rule [36]:

Plant Rule $i$: If $s_1(t)$ is $\mu_{i1}$ and $\cdots$ and $s_p(t)$ is $\mu_{ip}$, then

$$y(t) = A_i y_0(i), i = 1, 2, \ldots, N, \tag{5}$$

where $\mu_{ij}$ is the fuzzy set and $N$ is the number of IF-THEN rules; $s_1(t), \ldots, s_p(t)$ are the premise variables. Then the final output of T-S fuzzy system is inferred as follows:

$$\begin{cases} \dot{y}(t) = \sum_{i=1}^{N} h_i(y(t)) A_i y(t), \\ y(0) = 1.0. \end{cases} \tag{6}$$

In this illustrative case, we assume $N = 2$ and the fuzzy membership $h_1(y(t)) = \frac{1}{2} - \frac{1}{2}y(t)$, $h_2(y(t)) = \frac{1}{2} + \frac{1}{2}y(t)$.

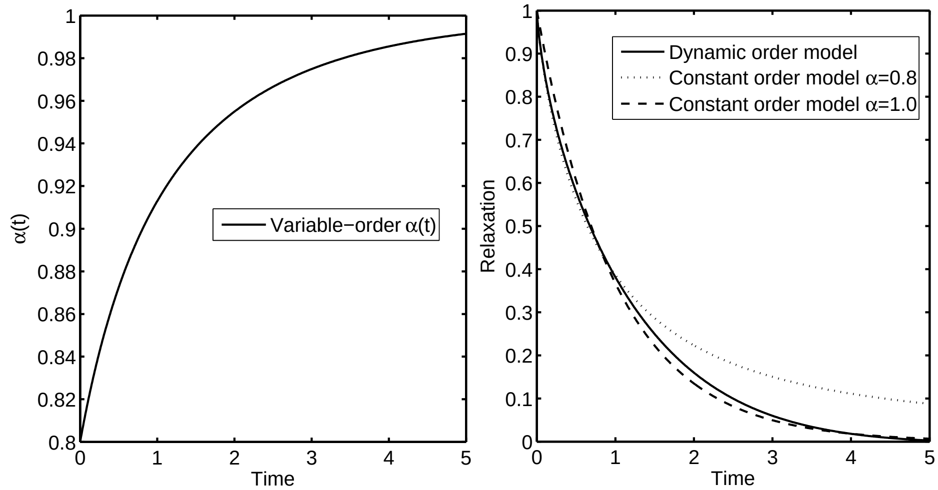


FIG. 4: Left: The curve of the variable-order $\alpha(t)$ (8) which is originated from the fuzzy system (6). Right: The relaxation curve of the system (4) with the dynamic-order $\alpha(t)$ given in (8)and $B = 1.0$.

If $A_1 = 0, A_2 = -1$, then the exact solution of (6) is

$$y(t) = \frac{1}{2e^{t/2} - 1}. \tag{7}$$

For numerical simulation, we assume that the relationship between the output of fuzzy system (6) and variable-order of fractional dynamic system (4) can be stated as follows

$$\alpha(Z) = \alpha(t) = 1.0 - 0.2y(t). \tag{8}$$

The evolution curve of variable-order $\alpha(t)$ and the corresponding numerical result of the relaxation system (4) are shown in Fig. 4.

From Fig. 4, we can observe that, the system (4) exhibits accelerating relaxation behavior with the variable-order (8). Since fuzzy systems have been regarded as efficient tools to tackle real-world engineering problems, this case implies that, fuzzy systems may be an important method to characterize the evolution behavior of variable-order in fractional dynamic system.

*Case 2.* We further consider the variable-order fractional diffusion system, in which the differential order is determined by a fractional order dynamic system. It means that the differential order in the considered fractional diffusion system is related with the output of a

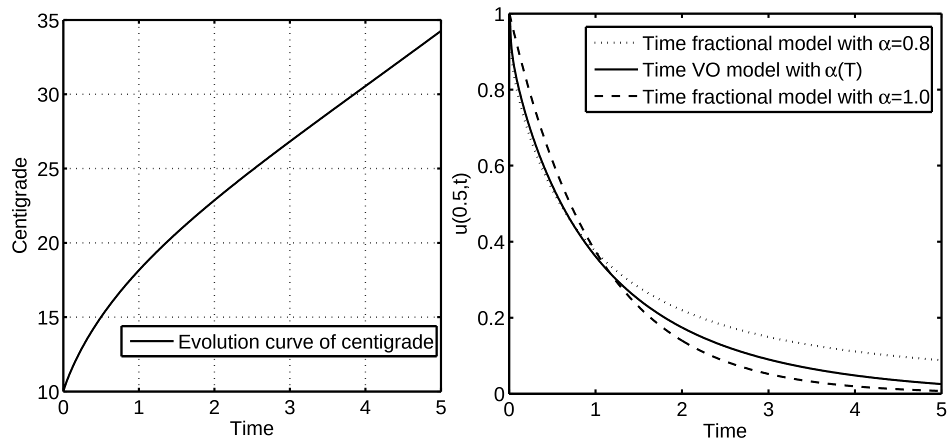


FIG. 5: Left: The evolution curve of the temperature in system (10) with $\beta = 0.9$. In numerical simulation, the time step is 0.01. Right: The diffusion curve of the system (9) with $L = 1.0$, $M = 5.0$, $x = 0.5$, $q(x,t) = 0$ and $K = 0.1$. The differential order follows the fractional relaxation system (10).

fractional dynamic system. It is a more general form to investigate the generalized process of dynamic-order system. The variable-order fractional diffusion system is stated as

$$\begin{cases} D_t^{\alpha(T)} u(x,t) = K \dfrac{\partial^2 u(x,t)}{\partial x^2} + q(x,t), \\ u(x,0) = \sin(x),\ x \in [0,L], \\ u(0,t) = 0; c(L,t) = 0,\ t \in [0,M], \end{cases} \tag{9}$$

where $K > 0$ is the generalized diffusion coefficient, $u(x,t)$ is concentration, mass or other quantities of interests, $q(x,t)$ is a source term and $\alpha(T)$ is the variable-order of the Caputo-type fractional derivative, which is a function of the temperature.

We suppose the evolution behavior of the temperature in the porous medium can be characterized by

$$\begin{cases} {}^C D_t^{\beta} T(t) = 0.1T(t) + 10/(1.3t+1), \\ T(0) = 10, \end{cases} \tag{10}$$

where $\beta \in (0,1]$ is factional order, $T$ is temperature. Then, we suppose the relationship between the differential order of (9) and the temperature of the porous medium is

$$\alpha(T) = 0.8 + 0.005T. \tag{11}$$

It indicates that the diffusion process characterized by (9) is an accelerating subdiffusion process. The trajectory of $\alpha(t)$ and the diffusion curve of system (9) at $x = 5.0$ are drawn in Fig. 5.

From Fig. 5, we observe that the considered diffusion is indeed an accelerating subdiffusion process. This example has shown that the heat transfer process with result of temperature increasing, has caused the accelerating behavior of considered diffusion process. In engineering situations, this model can offer an effective tool to explore physical mechanism of real-world diffusion related dynamic processes.

Finally, we should note that, through we have made a first attempt to consider such type of variable-order fractional dynamic system, in which the differential order is governed by the output signal of other dynamic systems. The further investigations about physical mechanism and application potentials is deserved.

The authors thank G. W. Bohannan, X. N. Song and the AFC reading group meeting in CSOIS, Utah State University for discussions. The work is partial supported by 2010CB832702, 10774038 and 2010B19114.

[1] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).

[2] G. M. Zaslavsky, Phys. Rep. **371**, 461 (2002).

[3] R. L. Magin, O. Abdullah, D. Baleanu, and X. J. Zhou, J. Magn. Reson. **190(2)**, 255 (2008).

[4] A. Hernandez-Jimenez, J. Hernandez-Santiago, A. Macias-Garcia, and J. Sanchez-Gonzalez, Polym. Test. **21(3)**, 325 (2002).

[5] P. Yang, Y. Lam, and K. Q. Zhu, J. Non-Newtonian Fluid Mech. **165**, 88 (2010).

[6] C. P. Li, C. X. Xu, and e. a. W. G. Sun, Chaos **19(1)**, 013106 (2009).

[7] Y. Q. Chen and K. L. Moore, IEEE Trans. on Circuits and Systems (II) **49(3)**, 363 (2002).

[8] I. Podlubny, IEEE Trans. Automat. Contr. **44(1)**, 208 (1999).

[9] B. J. West, E. L. Geneston, and P. Grigolini, Phys. Rep. **468**, 1 (2008).

[10] Y. Li, Y. Q. Chen, and I. Podlubny, Automatica **45(9-10)**, 1965 (2009).

[11] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (North-Holland, Boston, 2006).

[12] D. Baleanu and S. I. Muslih, Phys. Scripta **72**, 119 (2005).

[13] I. Petráš, Nonlinear Dyn. **57**, 157 (2009).

[14] M. Magdziarz, A. Weron, and J. Klafter, Phys. Rev. Lett. **101**, 210601 (2008).

[15] A. I. Saichev and G. M. Zaslavsky, Chaos **7(4)**, 753 (1997).

[16] S. G. Samko, J. Anal. Math. **21**, 213 (1995).

[17] C. F. Lorenzo and T. T. Hartley, Nonlinear Dyn. **29**, 57 (2002).

[18] H. G. Sun, W. Chen, and Y. Q. Chen, Phys. A **388**, 4586 (2009).

[19] D. Ingman, J. Suzdalnitsky, and M. Zeifman, J. Appl. Mech. **67**, 383 (2000).

[20] W. G. Glöckle and T. F. Nonnenmacher, Biophys. J. **46**, 787 (1995).

[21] G. W. Bohannan, J. Vib. Control **14**, 1487 (2008).

[22] H. Sheng, H. G. Sun, C. Coopmans, Y. Q. Chen, and G. W. Bohannan, Eur. Phys. J. Special Topics (Submitted) (2008).

[23] I. S. Jesus, J. A. T. Machado, and J. B. Cunha, J. Vib. Control **14**, 1389 (2008).

[24] C. F. M. Coimbra, Ann. Phys. (Leipzig) **12(11-12)**, 692 (2003).

[25] H. G. Sun, W. Chen, H. Sheng, and Y. Q. Chen, Phys. Lett. A **374**, 906 (2010).

[26] S. A. Meier, M. A. Peter, and M. Böhm, Comput. Mater. Sci. **39**, 29 (2007).

[27] P. Reimus, G. Pohll, T. Mihevc, J. Chapman, M. Haga, S. K. B.Lyles, R. Niswonger, and P. Sanders, Water Resour. Res. **39(12)**, 1356 (2003).

[28] R. L. Bagley and P. J. Torvik, J. Rheol. **27(3)**, 201 (1983).

[29] D. Craiem and R. L. Magin, Phys. Biol. **7**, 013001 (2010).

[30] R. Metzler and J. Klafter, J. of Non-Crystalline Solids **305**, 81 (2002).

[31] H. O. Wang, K. Tanaka, and M. F. Griffin, IEEE Trans. on Fuzzy Systems **4(1)**, 14 (1996).

[32] G. Aronica, B. Hankin, and K. Beven, Adv. Water Resour. **22(4)**, 349 (1998).

[33] S. W. Franks, P. Gineste, and e. a. K. J. Beven, Water Resour. Res. **34(4)**, 787 (1998).

[34] B. Kosko and J. C. Burgess, J. Acoust. Soc. Am. **103(6)**, 3131 (1998).

[35] H. L. Wang, S. Kwong, Y. C. Jin, W. Wei, and K. F. Man, Fuzzy Sets and Systems **149(1)**, 149 (2005).

[36] X. N. Song, S. Y. Xu, and H. Shen, Informat. Sci. **178**, 4341 (2008).