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ORIGINALARBEITEN · ORIGINALS

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Calculation of electrical circuits with fractional characteristics of construction elements

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Abstract Based on the memristor and applications in the field of filter technique, the elements of higher order have been developed at Technische Universität Ilmenau as an extension of the classical bipolar construction elements. The elements of higher order form a set of points in the α, β-range. For these points only integer coordinates are allowed. Elements along a straight line with $\alpha - \beta = const.$ show the same characteristic behaviour. To close the gap between the points of integer coordinates Riemann–Liouville fractional calculus is applied. So the class of construction elements of higher, integer order ist extended to the one of higher, real order. The differences are shown by calculations of the voltage-current behaviour of electrical circuits.

Zusammenfassung Die Erweiterung der klassischen Zweipolelemente sind ausgehend vom Memristor und Anwendung in der Filtertechnik an der TU Ilmenau die Elemente höherer Ordnung erfunden und definiert worden. Die Elemente auf einer Geraden im Winkel von 45° verfügen über dieselben Grundeigenschaften. Um die Bereiche zwischen den ganzzahligen Elementen in der α, β-Ebene zu schließen, bedient man sich der fraktionalen Differentiation nach Riemann–Liouville. Dadurch erweitert sich die Klasse von Bauelementen ganzzahliger höherer Ordnung auf die allgemeine Klasse reeller höherer Ordnung. Die Unterschiede sind durch die Berechnung des Strom-Spannungs-Verhaltens elektrischer Schaltungen aufgezeigt.

1 Introduction

The construction elements of higher order were defined and applied in the department of "Theoretical Electrical Engineering" at the former Technische Hochschule Ilmenau [1–5]. So there are countable infinitly many new elements with the main properties of closure and independence of the α-β-elements. Their practical usability was shown in the field of filter technique in modelling of technical construction elements and approximation of equivalent circuits of transistors.

In the 70s of the last century an theory of fractional calculus was developed. In 1695 Gottfried Wilhelm Leibniz was the first one who thought about non-integer derivatives. Other famous mathematicians like L´Hospital, Euler, Laplace and Lacroix [6–9] contributed to the development of calculus of non-integer order.

In the last decades a lot of scientific works on physical and technical problems which apply fractional differential equations have been published [10–13]. In contrast to differential equations of integer orders differential equations of arbitrary orders take non-local conditions like causality into consideration. In spite of several applications the usage of fractional calculus is often complicated and not fully developed. This fact becomes obvious by looking at the numerous definitions of fractional derivatives (e.g. Riemann–Liouville, Caputo and others). They are also different with regard to their basic properties (semigroup property, linearity, and translation invariance).

2 Definition of fractional integrals and derivatives

Various mathematicians developed and examined several approaches to the generalization of the notion of differentiation and integration. Most of them base on integral and series representations.

It is impossible to regard every definition within the framework of this article. The starting point of the following applications is the Riemann–Liouville definition of derivatives and integrals of arbitrary order. The integration of an arbitrary function $f(x)$ with respect to x can be considered as a differentiation of negative order:

$$\left.\frac{d^{-1} f(y)}{dy^{-1}}\right|_{y=a}^{y=x} = \int_a^x f(y)\, dy$$

$$\left.\frac{d^{-1} f(y)}{dy^{-1}}\right|_{y=a}^{y=x} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}} \int_a^x (x-y)^{n-1} f(y)\, dy \tag{1}$$

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with $x \geq a$, $n \in \mathbb{N}$. The function $f(x)$ has to satisfy the condition that it is integrable in the interval $[a, x]$. Then the integral in Eq. 1 exists for $x \geq a$ and can be differentiated $(n-1)$ times. Functions applied in the field of electrical engineering fulfill these conditions. In Eq. 1 the function $f(x)$ is weighted by the term $(x-y)^{n-1}$. So, for real n with $0 < n < 1$ the near "past" of the function $f(x)$ is much more important than the distant "past". The present moment x ist he most weighted one. By differentiating Eq. 1 n times we obtain the Riemann–Liouville definition for the integration of arbitrary order:

$$
\begin{aligned}
{}_aI_x^q\,\{f(x)\} &= \frac{d^{-q}f}{dx^{-q}} \\
{}_aI_x^q\,\{f(x)\} &= \frac{1}{\Gamma(q)}\int_a^x (x-y)^{q-1}\,f(y)\,dy \\
&\qquad (q \in \mathbb{R}^+, x \geq a, a > -\infty)\,.
\end{aligned}
\tag{2}
$$

It is represented by the symbol ${}_aI_x^q\,\{f(x)\}$. For the particular case $q=1$ and $a=0$ Eq. 2 results in the classical rule of integration:

$$
{}_0I_x^1\,\{f(x)\} = \frac{1}{\Gamma(1)}\int_0^x (x-y)^{1-1}\,f(y)\,dy = \int_0^x f(y)\,dy\,. \tag{3}
$$

The definition of the fractional Riemann–Liouville derivative gives the k-fold differentiation of Eq. 2.

$$
\frac{d^k}{dx^k}\,{}_aI_x^q\,\{f(x)\} = \frac{d^k}{dx^k}\frac{d^{-q}f}{dx^{-q}} = \frac{d^{k-q}f}{dx^{k-q}}\,. \tag{4}
$$

Introducing the new order $p = k - q$ we arrive at the differentiation of arbitrary order p according to Riemann–Liouville:

$$
\begin{aligned}
{}_aD_x^p\,\{f(x)\} &= \frac{1}{\Gamma(k-p)}\frac{d^k}{dx^k}\int_a^x (x-y)^{k-p-1}\,f(y)\,dy \\
&\qquad (k-1 \leq p < k, k \in \mathbb{N}, p \in \mathbb{R})\,.
\end{aligned}
\tag{5}
$$

A second imortant approach was developed by Caputo. It is defined as:

$$
\begin{aligned}
{}_a^CD_x^p\{f(x)\} &= \frac{1}{\Gamma(k-p)}\int_a^x \frac{f^{(k)}(y)}{(x-y)^{p-k+1}}\,dy \\
&\qquad (k-1 \leq p < k, k \in \mathbb{N}, p \in \mathbb{R})\,.
\end{aligned}
$$

3 Elements of higher, integer order

Elements of higher, integer order help to describe new or already known complicated, electrical characteristics by means of single construction elements. Examples are the bipolar transistor and the super capacity and the super inductance in the field of filter technique. It should be mentioned here that it is difficult to realize elements of higher order. However, they prove to be very interesting in theoretical analyses of complicated circuits since they often help to simplify such circuits.

The elements of higher, integer order are defined by the normed equation

$$
p^{\,\beta}\,y \;=\; f(p^{\,\alpha}\,x)\,, \qquad \alpha,\ \beta \in \mathbb{Z}\,. \tag{6}
$$

x and y are normed state variables, which are current and voltage in the field of electromagnetics and power and velocity in the field of mechanics. So, there are two useful standardizations, the charge formulation ($x = \frac{i}{I_0}$, $y = \frac{u}{U_0}$) and the flux formulation ($x = \frac{u}{U_0}$, $y = \frac{i}{I_0}$).

Furthermore, the function f is n times continuously differentiable. The differential operator p of integer order n contained in Eq. 6 is given by:

$$
p^n(\ldots) = \frac{d^n(\ldots)}{d\tau^n} = {T_0}^n\,\frac{d^n(\ldots)}{dt^n}\,. \tag{7}
$$

If the values for α and β run from $-n$ to $+n$, we obtain numerous new elements, which are described by:

$$
y = p^{-\beta}\,f(p^{\alpha}\,x)\,. \tag{8}
$$

where $p^{-\beta}$ with $\beta > 0$ means a β-fold integration. Equation 8 is valid for linear and non-linear elements. To simplify the following considerations we assume linear properties for the elements of higher order.

$$
f(p^{\alpha}\,x) = K\,p^{\alpha}\,x\,, \qquad K = const.\,. \tag{9}
$$

So, Eq. 8 results in:

$$
y = p^{-\beta}\,K\,p^{\alpha}\,x = K\,p^{\alpha-\beta}\,x = K\,p^k\,x\,. \tag{10}
$$

Obviously, elements along a straight line with $k = \alpha - \beta = constant$ have the same characteristic. This is shown in Fig. 1 by the inclined lines running through the points which represent the elements.

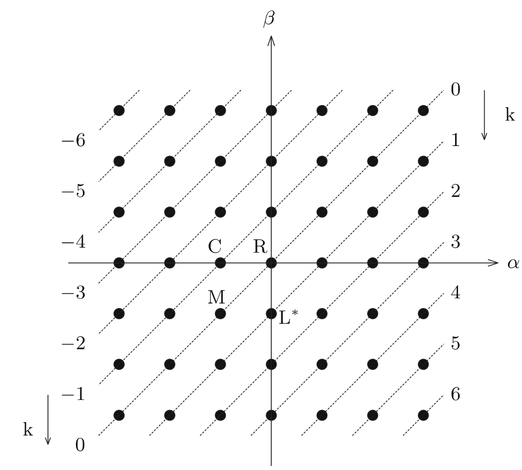

Fig. 1. Scheme of elements for elements of electromagnetics with integer order $k = \alpha - \beta$

For instance, all elements along the straight line with $k = 0$ are positive resistors. The most simple one is the commonly-known ohmic resistor. It is represented by the coordinate origin $\alpha = \beta = 0$. Therefore, the memristor M which has storage properties is represented by the point $\alpha = \beta = -1$.

If we consider the properties of elements along the straight lines with $k = -n, \ldots, n$ $(n \in \mathbb{Z})$, we find out that there are four different characteristics which alternate and depend on $k = \alpha - \beta$. For $k = 0$ they are positive, frequency-independent resistors, for $k = 4n$ positive, frequency-dependent resistors, for $k = 4n + 1$ positive, frequency-dependent inductances, for $k = 4n + 2$ negative, frequency-dependent resistors and for $k = 4n + 3$ positive, frequency-dependent capacities.

4 Elements of higher, real order

The fractional calculus is the generalization and unification of integration and differentiation of integer order. That is the reason why the Eqs. 6 and 8 are also valid for elements of higher, real order, if the integer values for α and β are replaced by real ones [3]. As a result of replacing p^α and p^β by ${}_aD_x^\alpha$ and ${}_aD_x^\beta$ Eq. 8 changes into:

$$y = {}_aD_\tau^{-\beta} \left\{ f \left({}_aD_\tau^{\alpha}\{x\} \right) \right\} . \tag{11}$$

For a linear construction element with the real orders $\alpha = 0$ and $\beta = -0,4$ we obtain the denormed equation:

$$u(t) = \xi \, \frac{d^{0,4} i(t)}{dt^{0,4}} . \tag{12}$$

Obviously, ξ is of the dimension $[\xi] = 1 \frac{\mathrm{V}}{\mathrm{As}^{0.4}}$.

4.1 Elements of the real α-β-range

We are able to define arbitrary elements of the α-β-range, the coordinates of which consist of real numbers. The Riemann–Liouville operator is one possible operator to obtain the equation that describes the behaviour of elements of higher, real order.

So, now not only points of integer coordinates but every arbitrary one in Fig. 1. represents an element of higher (real) order. The behaviour of magnitude and phase of elements of higher, real order is describable with equations of the same form like those for elements of integer order [3].

The amplitude increase ot the impedance function:

$$\frac{d \dfrac{|z|}{|z_0|}}{d \lg \left(\dfrac{\omega}{\omega_0} \right)} = 20 \cdot k , \qquad k \in \mathbb{R} \tag{13}$$

is discontinuous if the order k is chosen from integer numbers. For real order k the increase becomes continuous. The continuous increase is shown in Fig. 2 where the order k rises by $\Delta k = 0.2$ each time starting from $k = 0.1$ to $k = 6.1$

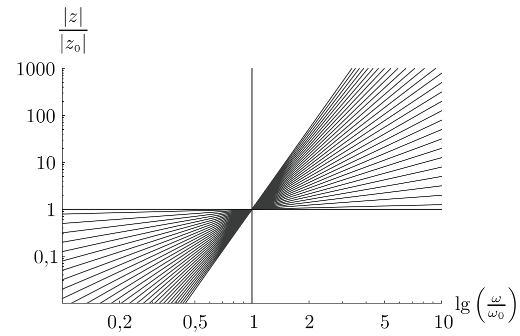

Fig. 2. Behaviour of the impedance function for positive orders k

The increases for the limits $k = 0$ and $k \to \infty$ are plotted additionally.

The value of Δ can be chosen arbitrarily small. Therefore, the difference between the single increases becomes arbitrarily small, too. For the phase angle φ the equation:

$$\varphi(k) = 90^\circ \cdot k = k \cdot \frac{\pi}{2} , \qquad k \in \mathbb{R} \tag{14}$$

is valid. So, the spectrum of phase angles becomes continuous, too.

4.2 Intermediate cases between elements of order $k = \pm 1$

The dimension of the constant K follows from the extension of Eq. 6.

$$[K] = \frac{[y]}{[x]} \cdot [p^k] = \frac{\mathrm{V}}{\mathrm{A}} \cdot \mathrm{s}^k \tag{15}$$

However, for example for $k = 1$ valids:

$$[K] = \frac{[y]}{[x]} \cdot [p^1] = \frac{\mathrm{V}}{\mathrm{A}} \cdot \mathrm{s} = \frac{\mathrm{Vs}}{\mathrm{A}} = \mathrm{H} = [L] , \tag{16}$$

for $k = 0$:

$$[K] = \frac{[y]}{[x]} \cdot [p^0] = \frac{\mathrm{V}}{\mathrm{A}} \cdot \mathrm{s}^0 = \frac{\mathrm{V}}{\mathrm{A}} = \Omega = [R] , \tag{17}$$

and for $k = -1$:

$$[K] = \frac{[y]}{[x]} \cdot [p^{-1}] = \frac{\mathrm{V}}{\mathrm{A}} \cdot \mathrm{s}^{-1} = \frac{\mathrm{V}}{\mathrm{As}} = \frac{1}{\mathrm{F}} = \left[\frac{1}{C} \right] . \tag{18}$$

So, the exponent of time can become non-integer for $k \in \mathbb{R}$. Now, let us discuss the intermediate cases between $k = 0$ and $k = \pm 1$. A jump function with an amplitude value of one serves as input signal:

$$f(x) = \begin{cases} 1 & : \quad \text{if } a < x \\ 0 & : \quad \text{otherwise} \end{cases} . \tag{19}$$

For the case of $0 \leq k \leq 1$ we write:
$\underline{x \leq a}$:

$${}_0I_x^k \{ f(x) \} = \frac{1}{\Gamma(k)} \int\limits_0^x (x - y)^{k-1} \cdot 0 \, dy = C = 0 , \tag{20}$$

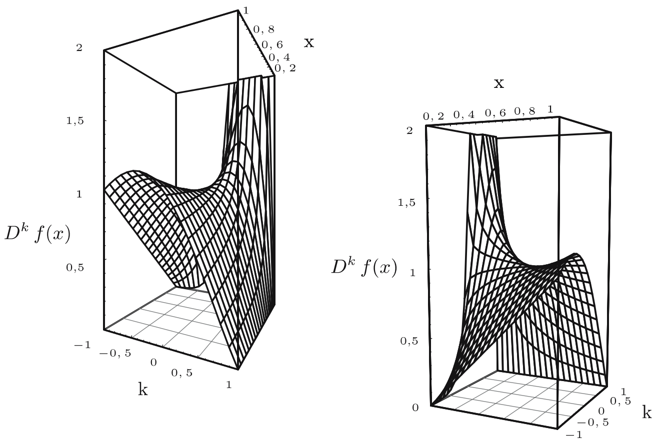

Fig. 3. Voltage responses of elements of order $-1 \leq k \leq 1$

$\underline{x < a}$:

$$ {}_0I_x^k\{f(x)\} = \frac{1}{\Gamma(k)} \int_0^x (x-y)^{k-1} \cdot 1\, dy = \frac{(x-a)^k}{\Gamma(k+1)} . \tag{21}$$

For $-1 \leq k \leq 0$ we arrive at:
$\underline{x \leq a}$:

$$ {}_0D_x^k\{f(x)\} = \frac{1}{\Gamma(m-k)} \frac{d^m}{dx^m} \int_0^x (x-y)^{m-k-1} \cdot 0\, dy = 0 , \tag{22}$$

$\underline{x < a}$:

$$ {}_0D_x^k\{f(x)\} = \frac{1}{\Gamma(m-k)} \frac{d^m}{dx^m} \int_0^x (x-y)^{m-k-1} \cdot 1\, dy = \frac{(x-a)^{-k}}{\Gamma(1-k)} . \tag{23}$$

In Fig. 3 we see the responses of all elements of orders $-1 \leq k \leq +1$ to a voltage jump at the moment $t = a = 0$. So, we realize a continuous change from one characteristic behaviour to the next one. This is shown from two different points of view.

For $k = -1$ we recognize the behaviour of an ideal integrator of first order. In the case of an ideal capacity this curve corresponds with the electrical charge shape $q(t)$ if $f(x)$ is equal to the electrical current $i(t)$. For $k = +1$ the characteristic of an ideal inductance results. However, with the help of fractional calculus we are able to show that the intermediate cases between the known characteristic behaviours of the electrical basic construction elements resistor R, inductance L^* and capacity C change continuously.

5 Calculation on electrical circuits

Apart from the resistor, the most frequently used elements are coils and capacitors. That is why we choose two examples that are intermediate cases between resistor and inductance, and resistor and capacity respectively.

5.1 Elements of fractional order and ohmic-inductive characteristic

The first example considered is a series circuit consisting of an element ξ with α and β and a real voltage source with the internal resistance R_i in Fig. 4.

The element ξ of real order can generally be described with the help of the parameter ξ and the real orders α and β. Let us assume the theoretical case where $\alpha = 0$ and $\beta = -0.6$. Therefore we expect a characteristic behaviour between $k = 0$ and $k = 1$. An explicit value for ξ is not necessary for this calculation. The source voltage $u_q(t) = \hat{U}_q \sin(\omega t + \varphi_{u_q})$ is representable in the complex plane by $\underline{u}_q = \hat{U}_q\, e^{j(\omega t + \varphi_{u_q})}$. For subsequent considerations we use the effective value vectors:

$$\underline{U}_q = U_q\, e^{j\varphi_{u_q}} . \tag{24}$$

The mesh-network equation is:

$$\underline{U}_q = \underline{U}_{R_i} + \underline{U}_\xi = (R_i + \underline{Z}_\xi)\, \underline{I}_\xi = \underline{Z} \cdot \underline{I}_\xi . \tag{25}$$

The complex resistance $\underline{Z}_\xi$ of a fractional element is generally described by:

$$\underline{Z}_\xi = (j\omega)^{\alpha-\beta}\, \xi . \tag{26}$$

The expression:

$$\underline{Z} = R_i + \underline{Z}_\xi = R_i + (j\omega)^{0.6}\, \xi \tag{27}$$

gives the total impedance of this circuit. Separation into a real part and an imaginary part gives:

$$\underline{Z} = R_i + \frac{1}{2}\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}\, \omega^{0.6}\, \xi + j\, \frac{1}{4}\left(1+\sqrt{5}\right)\, \omega^{0,6}\, \xi \quad (\omega > 0) . \tag{28}$$

The equation:

$$\underline{Z} = |\underline{Z}| \cdot e^{j\,\varphi_z} \tag{29}$$

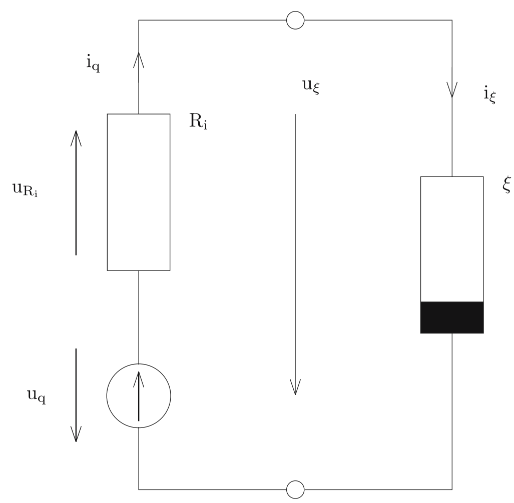

Fig. 4. Series circuit with a real voltage source and an element ξ with fractional characteristic

describes the total impedance with respect to magnitude and phase. So, it is possible to determine the current $\underline{I}_\xi$ and, in addition, both voltages $\underline{U}_{R_i}$ and $\underline{U}_\xi$:

$$\underline{I}_\xi = \frac{\underline{U}_q}{\underline{Z}} = \frac{\underline{U}_q}{R_i + \underline{Z}_\xi} \, . \tag{30}$$

For magnitude and phase of current $\underline{I}_\xi$ we obtain:

$$|\underline{I}_\xi| = \frac{U_q}{|\underline{Z}|} \, , \tag{31}$$

$$\varphi_{i_\xi} = \varphi_{u_q} - \varphi_z \, . \tag{32}$$

Finally, the voltage over the internal resistance R is:

$$\underline{U}_{R_i} = R_i \cdot \underline{I}_\xi \tag{33}$$

and over the element ξ of real order:

$$\underline{U}_\xi = \underline{Z}_\xi \cdot \underline{I}_\xi \, . \tag{34}$$

These results are shown in the vector diagram in Fig. 5.

So, we arrived at the expected intermediate case. The phase shift between the current and the voltage over the element is $\varphi_y = k \cdot 90° = (\alpha - \beta) \cdot 90° = 54°$. These phase shifts for $0 < k < 1$ can be realized by a series circuit of a resistor and an inductance where the source voltage has a certain frequency.

In Fig. 6 the three voltage shapes $u_q(t)$, $u_{R_i}(t)$ and $u_\xi(t)$ are plotted. We recognize the change of the phase angle by the phase shifted voltages.

For example the input voltage has a frequency of 50 Hz, so the phase shift of 54° between $\underline{U}_\xi$ and $\underline{U}_{R_i}$ corresponds to a time difference of $\Delta t = 2.7$ ms.

5.2 Element of fractional order with ohmic-capacitive characteristic

Now let us consider the circuit in Fig. 4 where ξ is an element with $\alpha = -0.4$ and $\beta = 0$. So, an intermediate characteristic between resistor and capacity is expected. The total impedance $\underline{Z}$ is given by:

$$\underline{Z} = R_i + (j\omega)^{-0,4}\,\xi = R_i + \frac{\xi}{(j\omega)^{0,4}} \, . \tag{35}$$

Separation into a real and an imaginary part gives:

$$\underline{Z} = R_i + \frac{1+\sqrt{5}}{4}\frac{\xi}{\omega^{0,4}} - j\,\frac{1}{2}\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}\frac{\xi}{\omega^{0,4}} \, . \tag{36}$$

With the help of Eq. 36 the current $\underline{I}_\xi$ and both voltages $\underline{U}_{R_i}$ and $\underline{U}_\xi$ can be determined by the rules of the algebra of complex quantities. After that it is possible to plot a vector diagram.

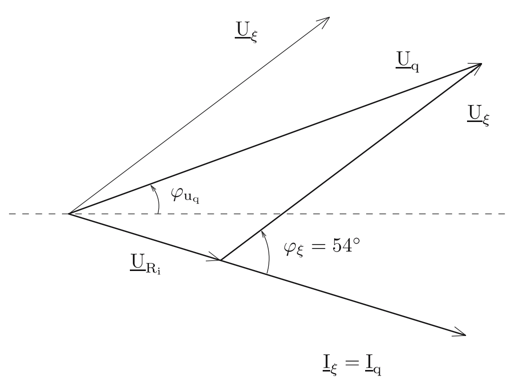

Fig. 5. Vector diagram for a real voltage source and an element of real order $k = \alpha - \beta = 0.6$

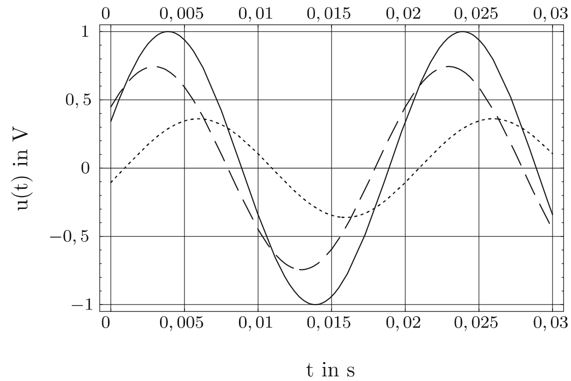

Fig. 6. Shape of the voltages $u_q(t)$ (—), $u_{R_i}(t)$ (···) and $u_\xi(t)$ (– –) as a function of time

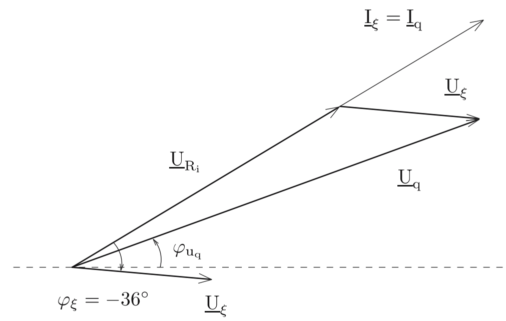

Fig. 7. Vector diagram for a real source and an element of real order $k = \alpha - \beta = -0.4$

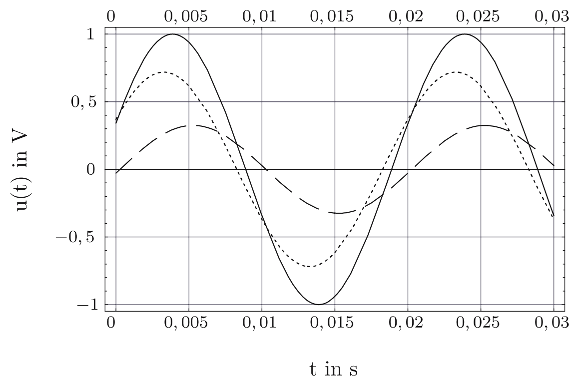

Fig. 8. Shape of the voltages $u_q(t)$ (—), $u_{R_i}(t)$ (···) and $u_\xi(t)$ (– –) as a function of time

In this case we notice a phase shift of $-36°$ between the current and the voltage over the element ξ. The relation 14 is valid for all $k \in \mathbb{R}$. Phase shifts for $-1 < k < 0$ can be realized by means of a series circuit of a resistor and a capacitor where the source voltage has a certain frequency.

In Fig. 8 the voltage shapes $u_q(t)$, $u_{R_i}(t)$ and $u_\xi(t)$ are shown.

If the input voltage $u_q(t)$ has a frequency of 50 Hz, the phase shift of $-36°$ between $\underline{U}_\xi$ and $\underline{U}_{R_i}$ corresponds to a time difference of $\Delta t = 2$ ms.

6 Summary

This contribution extends the theory of the elements of higher, integer order to higher, real order. For this we propose a procedure based on the definition of the Riemann–Liouville derivative to close this gap. Finally, it is possible to determine a characteristic behaviour for every arbitrary point. Each point is defined by a pair of real valued coordinates α, β, which represents a certain element. This behaviour is independent from the characteristic curve of the construction element. The examples considered fulfill the expectations of intermediate cases between the integer ones. It is shown that the phase shift between the current and the voltage over the considered element is directly dependent on the order of the element $k = \alpha - \beta$.

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