Fractional calculus: basic theory and applications

(Part III)

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5.b) Applications to transport in fusion plasmas

Here we present phenomenological models of plasma transport based on the use of fractional diffusion operators. In particular, we extend the fractional diffusion models discussed before, by incorporating finite-size domain effects, boundary conditions, sources, spatially dependent diffusivities, and general asymmetric fractional operators. These additions are critical in order to order to go beyond tracers transport calculations. We show that the extended fractional model is able to reproduce within a unified framework some of the phenomenology of non-local, non-diffusive transport processes observed in fusion plasmas, including anomalous confinement time scaling, “up-hill” transport, pinch effects, and on-axis peaking with off-axis fuelling. The discussion presented here is based on [del-Castillo-Negrete-etal,2005a] and [del-Castillo-Negrete-etal,2005b] where further details can be found.

Beyond the standard diffusive transport paradigm

•Experimental and theoretical evidence suggests that transport in fusion plasmas deviates from the diffusion paradigm:

$$\partial_t T = \partial_x \left[\chi(x,t)\, \partial_x T \right] + S(x,t)$$

Examples: •Anomalous confinement time scaling
• Fast propagation and non-local transport phenomena
• Inward transport observed in off-axis fueling experiments

•To overcome the limitations of the diffusion paradigm, we have proposed transport models based on fractional derivative operators that incorporate in a unified way non-locality, memory effects, and non-diffusive scaling.

Towards an effective transport model for non-diffusive turbulent transport

•Individual tracers follow the turbulent field $\frac{d\vec{r}}{dt} = \tilde{V} = \frac{1}{B^2} \nabla \tilde{\Phi} \times \vec{B}$

•The distribution of tracers P evolves according to $\frac{\partial P}{\partial t} + \tilde{V} \cdot \nabla P = 0$

•The idea is to construct a model that "encapsulates" the complexity of the turbulence field $\tilde{V}$ in an effective flux Γ, and reproduces the observed pdf $\frac{\partial P}{\partial t} = -\frac{\partial \Gamma}{\partial x}$

homogeneous isotropic turbulence → diffusive transport → Gaussian closures → $\Gamma = -\chi\, \partial_x P$

transport in pressure driven plasma turbulence → Non-Gaussian distributions → $\boxed{\Gamma = -\chi\, \partial_x^\alpha P}$

Riemann-Liouville derivatives

$n = [\alpha] + 1$

[α]=integral part of α

Left derivative $$ {}_{a}D_{x}^{\alpha}\,\phi = \frac{1}{\Gamma(n-\alpha)}\,\frac{\partial^{n}}{\partial x^{n}} \int_{a}^{x} \frac{\phi(u)}{(x-u)^{\alpha-n+1}}\,du $$

Right derivative $$ {}_{x}D_{b}^{\alpha}\,\phi = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\,\frac{\partial^{n}}{\partial x^{n}} \int_{x}^{b} \frac{\phi(u)}{(u-x)^{\alpha-n+1}}\,du $$

Some limitations of the RL definition

Although a well-defined mathematical object, the Riemman-Liouville definition of the fractional derivative has some problems when it comes to apply it in physical problems. In particular:

The derivative of a constant is not zero $$ {}_{a}D_{x}^{\alpha}\,A = \frac{A}{\Gamma(1-\alpha)}\,\frac{1}{(x-a)^{\alpha}} $$ This might be an issue when using RL operators for writing evolution equations

The RL derivative is in general singular at the lower limit $$ \lim_{x\to a}\ {}_{a}D_{x}^{\alpha}\,\phi = \infty $$ unless $\phi^{(k)}(a) = 0$ This might be an issue when applying boundary conditions

The Laplace transform of the RL derivative depends on the fractional derivative at zero $$ L\left[{}_{0}D_{t}^{\alpha}\,\phi\right] = s^{\alpha}\,\hat{\phi}(s) - \sum_{k=0}^{n-1} s^{k}\left[{}_{0}D_{t}^{\alpha-k-1}\,\phi\right]_{t=0} $$ This might be an Issue when solving Initial value problems

Behavior near the lower terminal

Let $\phi(x) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(a)}{k!}(x-a)^k$ $\quad {}_aD_x^\alpha\ \phi = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(a)}{k!}\ {}_aD_x^\alpha\,(x-a)^{k-\alpha}$

using

$${}_aD_x^\alpha\,(x-a)^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}(x-a)^{k-\alpha} \qquad {}_aD_x^\alpha\ \phi = \sum_{k=0}^{\infty} \phi^{(k)}(a)\frac{(x-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}$$

$$\lim_{x\to a}\ {}_aD_x^\alpha\ \phi = \lim_{x\to a}\sum_{k=0}^{m-1} \frac{\phi^{(k)}(a)}{\Gamma(k+1-\alpha)}\frac{1}{(x-a)^{\alpha-k}} \qquad m-1 \le \alpha < m$$

$$\boxed{\lim_{x\to a}\ {}_aD_x^\alpha\ \phi = \infty \quad \text{unless} \quad \phi^{(k)}(a) = 0 \quad k = 1,\cdots m-1}$$

Caputo fractional derivative

These problems can be resolved by defining the fractional operators in the Caputo sense. Consider the case $1 < \alpha < 2$

$$\underbrace{{}_aD_x^\alpha\ \phi - \frac{\phi(a)}{\Gamma(1-\alpha)}\frac{1}{(x-\alpha)^\alpha} - \frac{\phi'(a)}{\Gamma(2-\alpha)}\frac{1}{(x-\alpha)^{\alpha-1}}}_{\text{singular terms}} = \underbrace{\sum_{k=0}^{\infty}\frac{\phi^{(k+2)}(a)(x-a)^{k+2-\alpha}}{\Gamma(k+3-\alpha)}}_{\text{regular terms}}$$

$${}_aD_x^\alpha\left[\phi(x) - \phi(a) - \phi^{(1)}(a)\right] = \sum_{k=0}^{\infty}\frac{\phi^{(k+2)}(a)(x-a)^{k+2-\alpha}}{\Gamma(k+3-\alpha)}$$

Define the Caputo derivative by subtracting the singular terms

$$\boxed{{}_0^CD_x^\alpha\phi = {}_aD_x^\alpha\left[\phi(x) - \phi(a) - \phi^{(1)}(a)\right]}$$

using $\displaystyle\int_a^x \frac{\phi''(u)}{(x-u)^{\alpha-1}}du = \sum_{k=0}^{\infty}\frac{\phi^{(k+2)}(a)}{k!}\int_a^x \frac{(x-a)^k}{(x-u)^{\alpha-1}}du$

$$\boxed{{}_a^CD_x^\alpha\ \phi = \frac{1}{\Gamma(2-\alpha)}\int_a^x \frac{\phi''(u)}{(x-u)^{\alpha-1}}du}$$

In general

$$ {}_{a}^{C}D_{x}^{\alpha}\,\phi = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \frac{\partial_u^n \phi}{(x-u)^{\alpha-n+1}}\, du $$

$$ {}_{x}^{C}D_{b}^{\alpha}\,\phi = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{a}^{x} \frac{\partial_u^n \phi}{(x-u)^{\alpha-n+1}}\, du $$

and as expected: $\quad {}_{a}^{C}D_{x}^{\alpha}\, A = 0$

$$ \lim_{x\to a}\ {}_{a}D_{x}^{\alpha}\,\phi = 0 $$

$$ L\left[{}_{0}D_{t}^{\alpha}\,\phi\right] = s^{\alpha}\,\hat{\phi}(s) - \sum_{k=0}^{n-1} s^{k}\left[{}_{0}\,\phi^{(\alpha-k-1)}\right]_{t=0} $$

Note that in an infinite domain

$$ {}_{-\infty}^{C}D_{x}^{\alpha}\,\phi = {}_{-\infty}D_{x}^{\alpha}\,\phi \qquad\qquad {}_{x}^{C}D_{\infty}^{\alpha}\,\phi = {}_{x}D_{\infty}^{\alpha}\,\phi $$

Nonlocal transport model

$$ \frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}\left[\, q_\ell + q_r + q_G \right] + P(x,t) $$

Diffusive flux $\qquad q_G = -\chi_G\,\partial_x T$

Left fractional flux

$$ q_\ell = -l\,(2-\alpha)\,\chi_l\ {}_{0}D_{x}^{\alpha-1}\,T $$

Regularized finite-size left derivative

$$ {}_{0}D_{x}^{\alpha-1}\,T = \frac{1}{\Gamma(3-\alpha)} \int_{0}^{x} (x-y)^{2-\alpha}\, T''\, dy $$

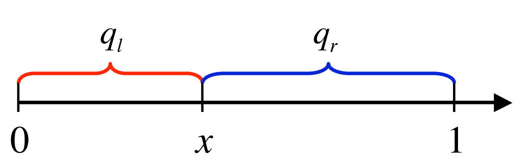
Right fractional flux

$$ q_r = r\,(2-\alpha)\,\chi_r\ {}_{x}D_{1}^{\alpha-1}\,T $$

Regularized finite-size right derivative

$$ {}_{x}D_{1}^{\alpha-1}\,T = \frac{1}{\Gamma(3-\alpha)} \int_{x}^{1} (y-x)^{2-\alpha}\, T''\, dy $$

Nonlocal fluxes



Nonlocal transport model

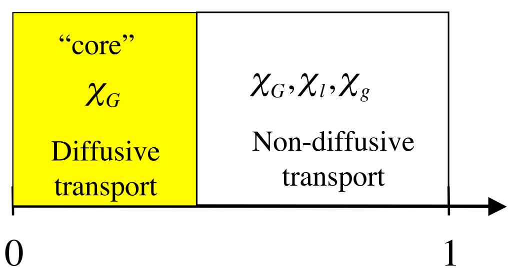
$$\frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}\left[q_\ell + q_r + q_G \right] + P(x,t)$$

Diffusive flux $\quad q_G = -\chi_G\, \partial_x T$

Nondiffusive fluxes
$$\begin{cases} q_\ell = -l\,(2-\alpha)\chi_l \; {}_0D_x^{\alpha-1}\, T \\ q_r = r\,(2-\alpha)\chi_r \; {}_xD_1^{\alpha-1}\, T \end{cases}$$

Finite size domain

$x \in (0,1)$

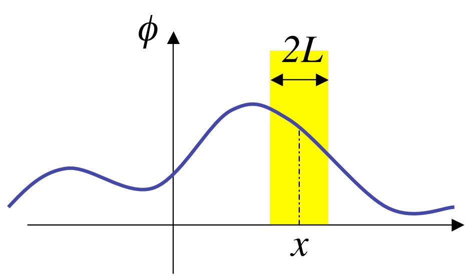


Boundary conditions

$$q_l(0) + q_r(0) + q_G(0) = 0$$

$$T'(0) = T(1) = 0$$

Local and non-local transport



$$\langle \phi \rangle_L (x) = \frac{1}{2L} \int_L \phi(x+s)\, ds$$

L=“radius of influence”

Local diffusive transport

$$\partial_t \phi = D\partial_x^2 \phi = 6D \frac{\langle \phi \rangle_L - \phi}{L^2} \quad \text{as} \quad L \to 0$$

$$\partial_t \phi + \partial_x Q = 0 \qquad Q = -D\, \partial_x \phi$$

Non-local diffusive transport

$$Q\,(x,t) = -D \int G(x-x')\, \partial_{x'} \phi(x',t)\, dx'$$

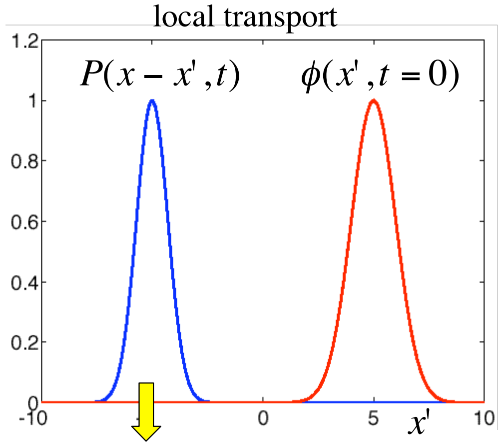
Perturbative transport experiments have shown evidence of non-local diffusion in fusion plasmas

Nonlocal transport

$$\phi(x,t) = \int_{-\infty}^{\infty} dx' \; P(x-x';t)\, \phi(x',t=0)$$

$\lim_{x\to\infty} P \sim e^{-x^2}$

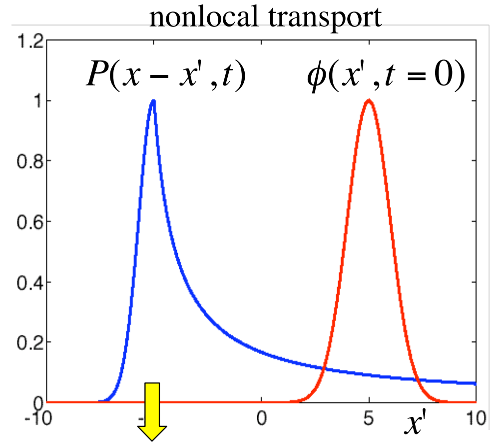
local transport



$\phi(x,t) \approx 0$

$\lim_{x\to\infty} P \sim x^{-(1+\alpha)}$

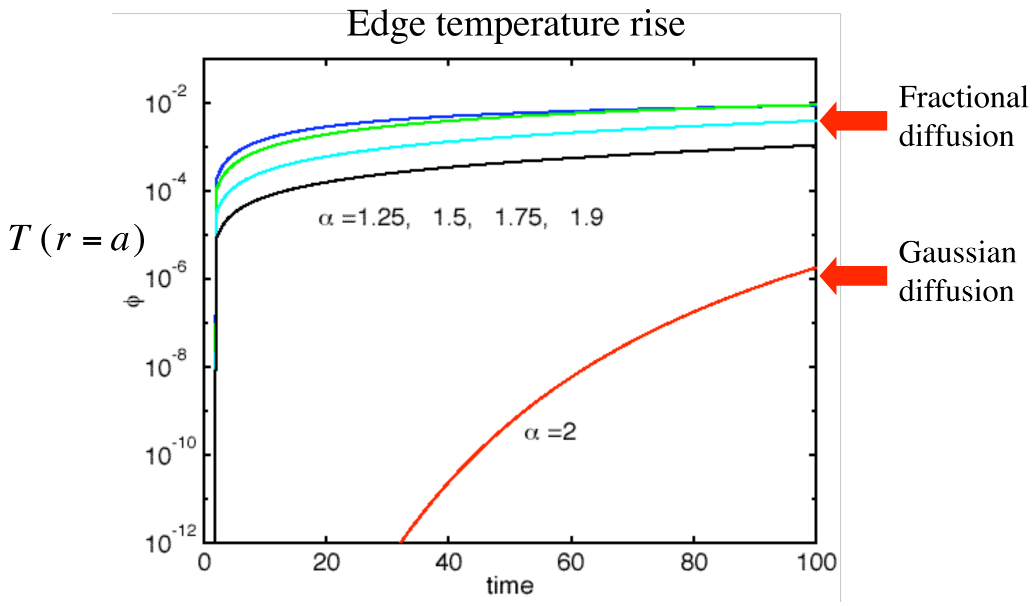
nonlocal transport



$\phi(x,t) \neq 0$

Due to the algebraic tail of the propagator, fractional diffusion leads to nonlocal transport

Nonlocal transport



Nonlocal transport and global diffusive coupling

Grunwald-Letnikov definition $\qquad \partial_x^\alpha \phi = \lim_{h \to 0} \frac{\Delta_h^\alpha \phi}{h^\alpha}$

$$\Delta_h^\alpha \phi(x) = \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} \phi(x - j\,h)$$

$$\partial_x^\alpha \phi_k \approx \frac{\left(\Delta_h^\alpha \phi\right)_k}{h^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^{k} w_j^\alpha \; \phi_{k-j}$$

Finite difference approximation

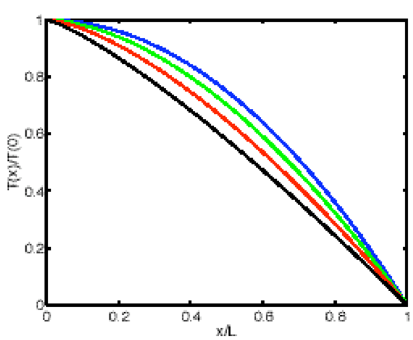
$$w_0^\alpha = 1 \qquad w_j^\alpha = \left(1 - \frac{\alpha + 1}{j}\right) w_{j-1}^\alpha$$

Global coupling coefficients

For α =1 and 2 usual nearest neighbor coupling of Laplacian operator
For 1 < α < 2 fractional diffusion implies global coupling

Steady state and anomalous confinement time scaling

Uniform heating

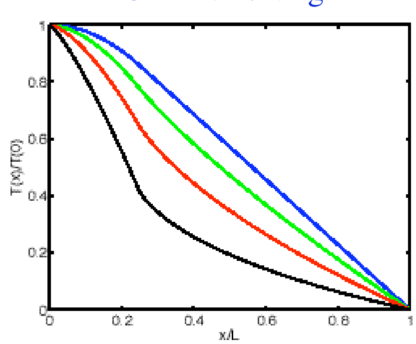


$\chi_r = 0$

$\chi_l \neq 0$

Blue α=2
Green α =1.75
Red α =1.5
Black α =1.25

On axis heating



Steady state solution

$$T = P_0 \left(\frac{L^\alpha}{\chi_l}\right) \frac{1}{\Gamma(\alpha+1)} \left[1 - \left(\frac{x}{L}\right)^\alpha\right]$$

$$\tau = \frac{\int_0^L T\,dx}{\int_0^L P\,dx}$$

Confinement time

$$\tau = \frac{\alpha}{\Gamma(2+\alpha)} \frac{L^\alpha}{\chi_l}$$

Anomalous scaling

$1 < \alpha < 2$

Confinement time scaling

τ = Confinement time

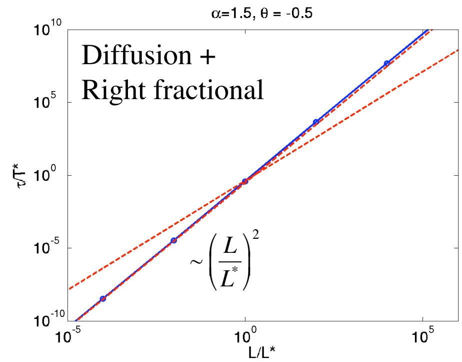
L = Domain size

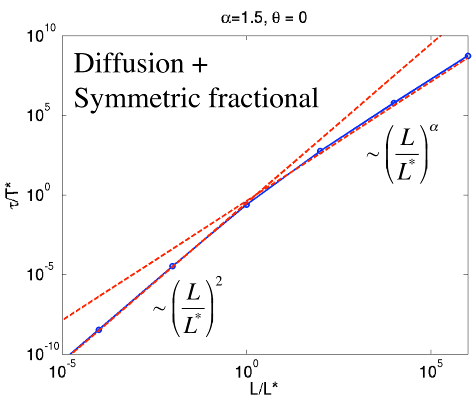
χ_d = Standard diffusivity

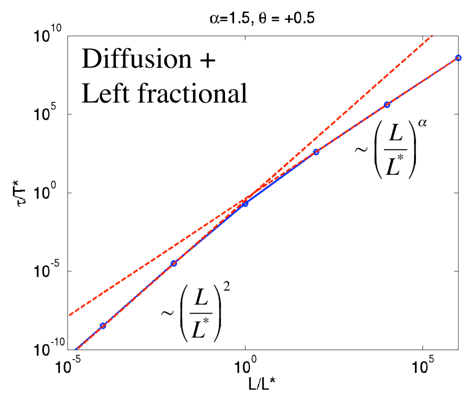
χ_a = Anomalous diffusivity

$$L^* = \left(\frac{\chi_d}{\chi_a}\right)^{\frac{1}{2-\alpha}}$$

$$T^* = \left(\frac{\chi_d^{\alpha}}{\chi_a^{2}}\right)^{\frac{1}{2-\alpha}}$$

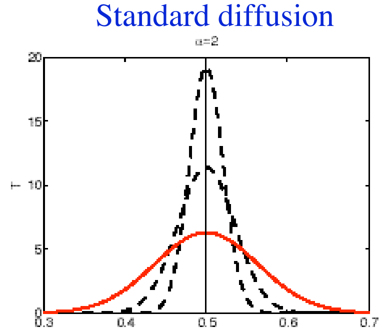




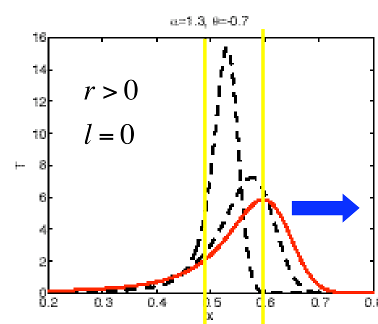


"Up-hill" transport and pinch velocity

Standard diffusion



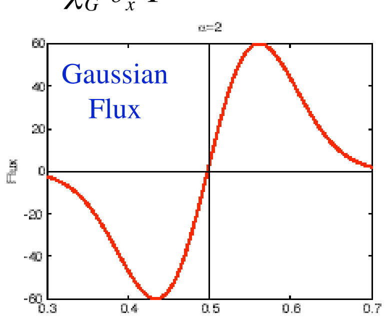
Fractional diffusion

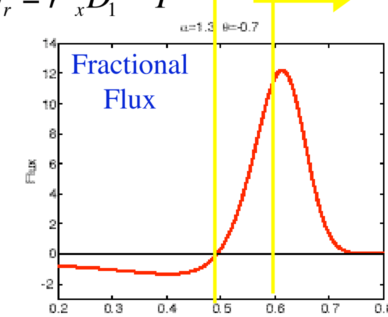


$$q_G = -\chi_G\, \partial_x\, T$$

$$q_r = r\ {}_x D_1^{\alpha-1}\, T$$

Up-hill Transport zone

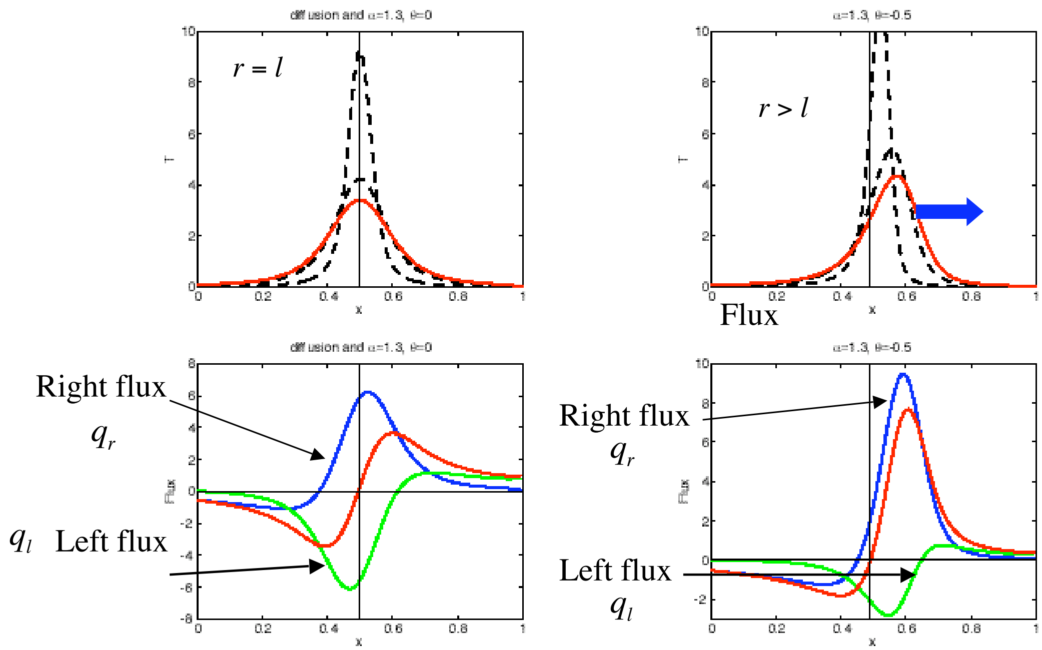




Fractional flux asymmetry and pinch velocity

Symmetric fractional diffusion

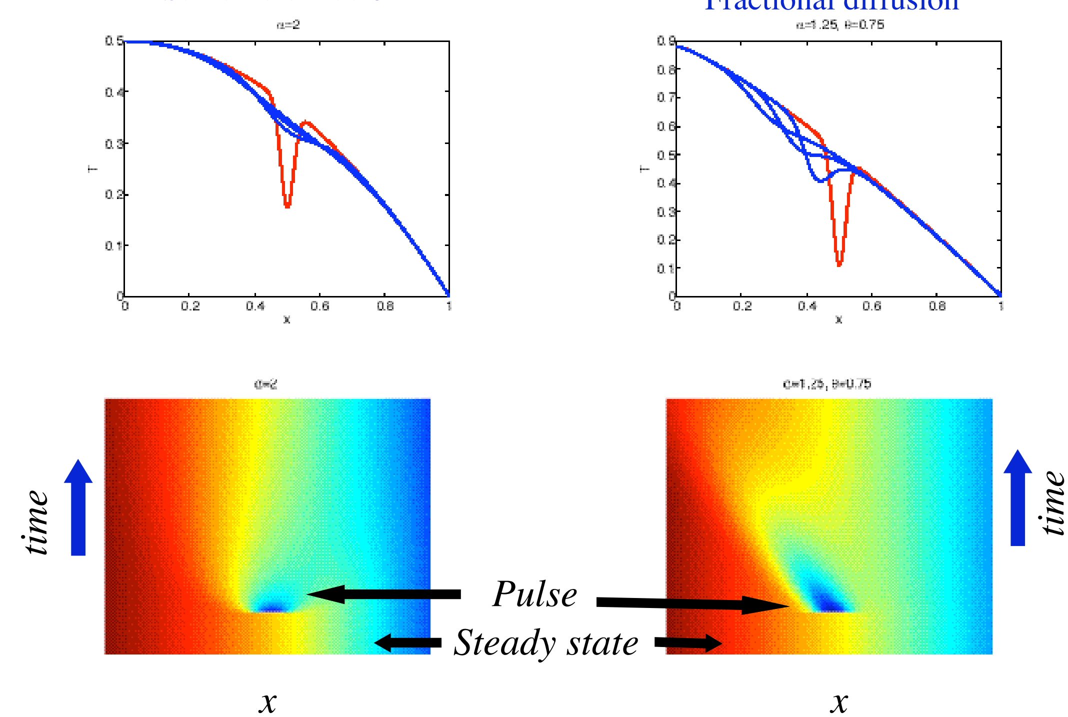
Asymmetric fractional diffusion



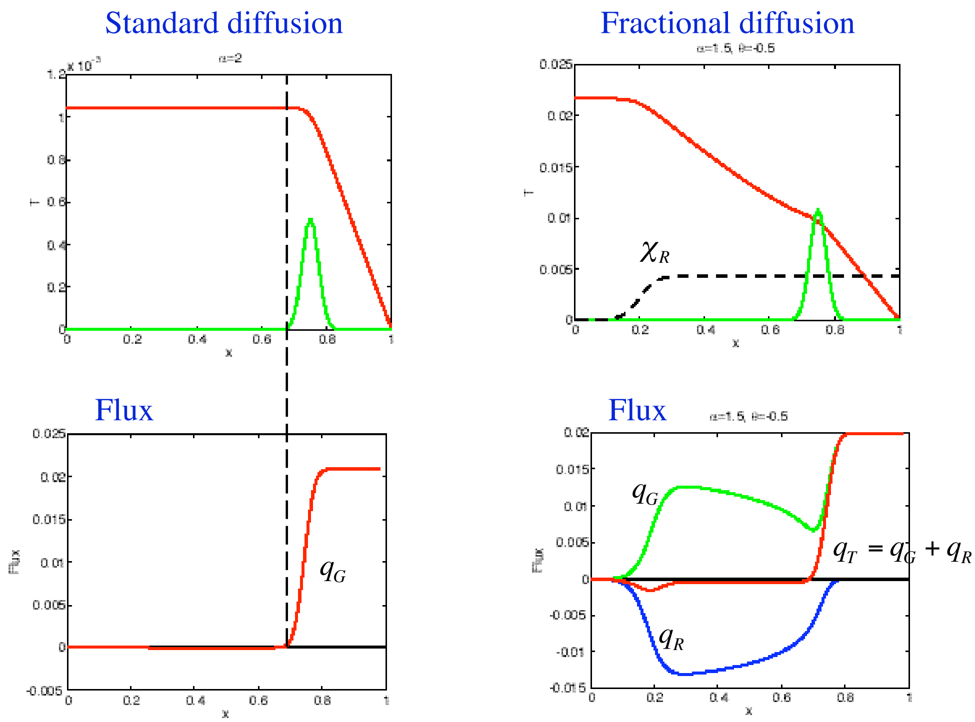
"Cold-pulse" fast propagation

Standard diffusion

Fractional diffusion



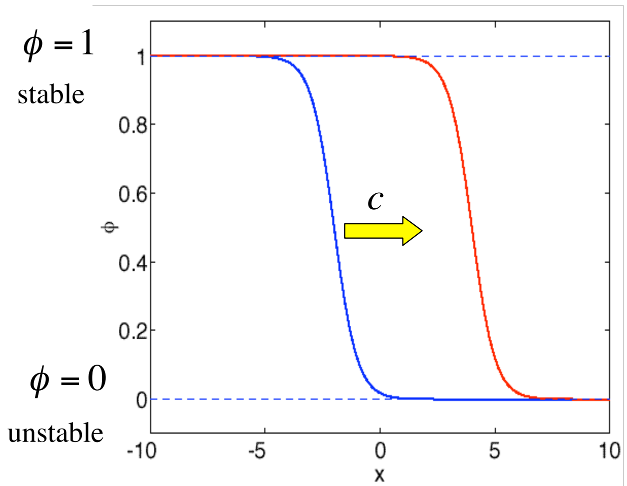
Off-axis heating



5.c) Application to reaction-diffusion systems

Here we discuss the role of fractional diffusion in reaction-diffusion systems. In particular, we present a numerical and analytical study of front propagation in the fractional Fisher-Kolmogorov equation. The discussion presented here is based on [del-Castillo-Negrete-etal,2003] where further details can be found.

Reaction diffusion models of front dynamics



Fisher-Kolmogorov equation

$$\partial_t \phi = \chi \partial_x^2 \phi + \gamma \phi (1 - \phi)$$

diffusion reaction

$$c = 2\sqrt{\gamma \chi}$$

- Plasma physics
- Model of genes dynamics
- Model of logistic population growth with dispersion
- Other applications include combustion and chemistry.

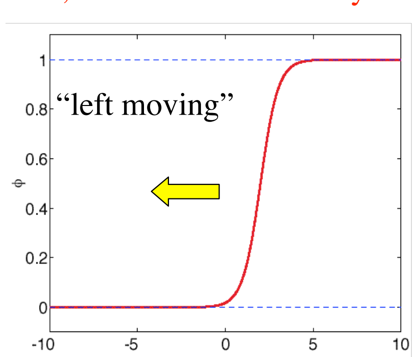
The fractional Fisher-Kolmogorov equation

$$\partial_t \phi = \chi \, {}_{-\infty}D_x^{\alpha} \phi + \gamma \phi (1 - \phi) \qquad 1 < \alpha < 2$$

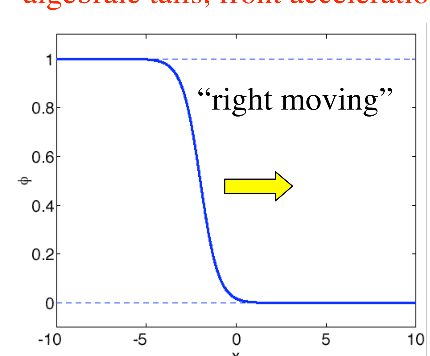
$${}_{-\infty}D_x^{\alpha} \phi = \frac{1}{\Gamma(2-\alpha)} \partial_x^2 \int_{-\infty}^{x} \frac{\phi(y)}{(x-y)^{\alpha-1}} dy$$

The asymmetry in the fractional derivative leads to an asymmetry in the fronts

Self-similar dynamics, exponential tails, constant front velocity



Quasi-self-similar dynamics, algebraic tails, front acceleration

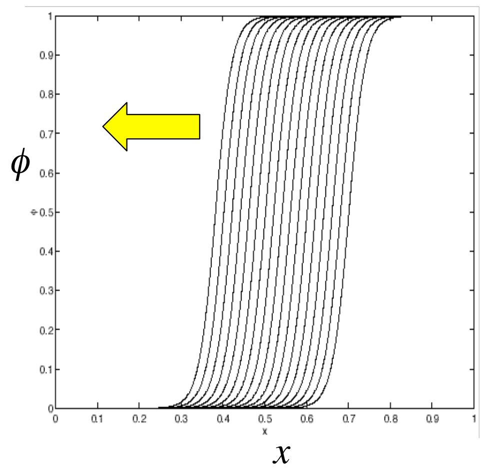


Asymmetric front dynamics

$\alpha = 1.25$

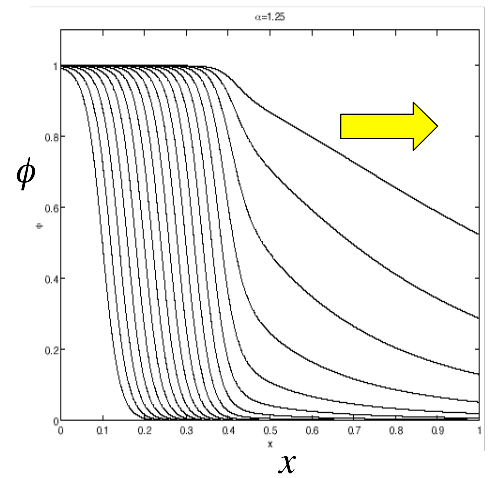
Exponential decaying constant speed fronts

$$\phi(x,0) = \frac{1}{2}\left[1 + \tanh\left(\frac{x - x_0}{2W}\right)\right]$$



Algebraic decaying accelerated fronts

$$\phi(x,0) = 1 - \frac{1}{2}\left[1 + \tanh\left(\frac{x - x_0}{2W}\right)\right]$$



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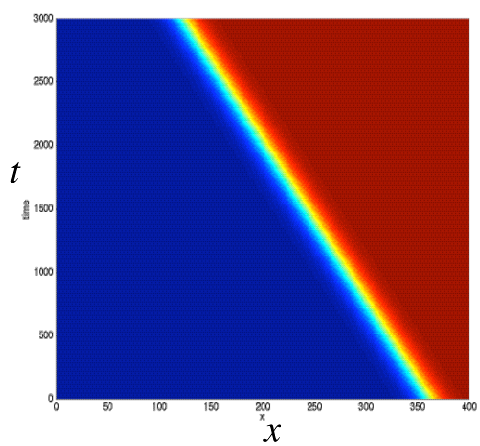

Algebraic decaying accelerated fronts

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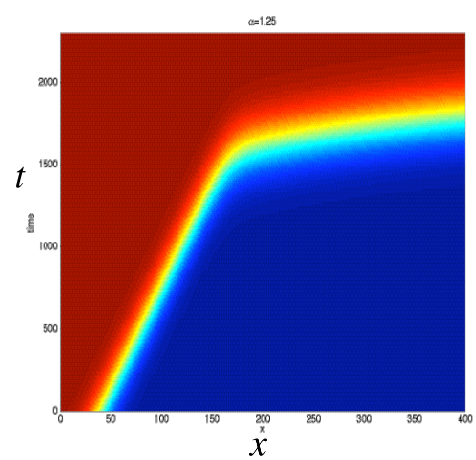


Leading edge calculation for algebraic fronts

$$\partial_t \phi = \chi\, {}_{-\infty}D_x^{\alpha}\phi + \gamma\phi(1-\phi) \qquad \phi(x,0) = \begin{cases} 1 & x<0 \\ e^{-\lambda x} & x>0 \end{cases}$$

At large x, $\phi << 1$ implies $\qquad \partial_t \phi = \chi D_x^{\alpha}\phi + \gamma\phi$

$$\phi = e^{\gamma t}\psi(x,t) \qquad \partial_t \psi = \chi D_x^{\alpha}\psi$$

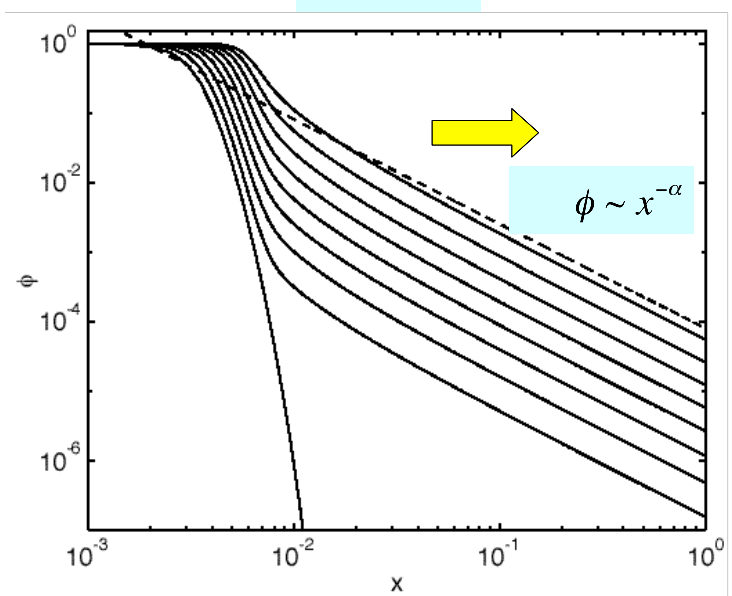
$$\psi(x,t) = \int_{-\infty}^{\infty} L_{\alpha,-1}(\eta)\,\psi_0\left[x-(\chi t)^{1/\alpha}\eta\right]d\eta$$

$$\phi = e^{-\lambda x+\gamma t}\int_{-\infty}^{x(\chi t)^{-1/\alpha}} e^{\lambda(\chi t)^{1/\alpha}\eta} L_{-1,\alpha}(\eta)\,d\eta + e^{\gamma t}\int_{x(\chi t)^{-1/\alpha}}^{\infty} L_{-1,\alpha}(\eta)\,d\eta$$

$$L_{-1,\alpha}(\eta) \sim \eta^{-(1+\alpha)} \Rightarrow \phi \sim t e^{t} x^{-\alpha} \text{ for } x/(\chi t)^{1/\alpha} >> 1$$

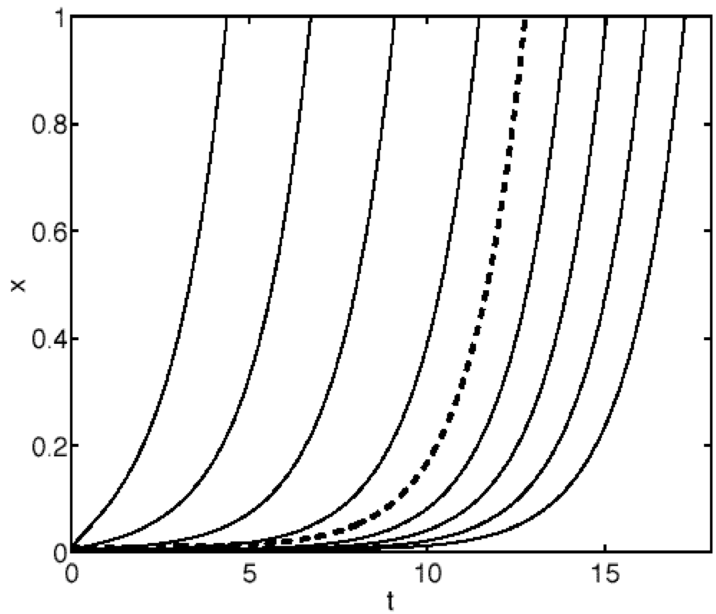
Right moving algebraic decaying, accelerated front

$$c \sim e^{\gamma t/\alpha}$$



Right moving algebraic decaying, accelerated front

$$\phi \sim x^{-\alpha}\, e^{\gamma\, t}$$



Leading edge calculation for exponential fronts

At large x, $\phi << 1$ implies

$$\partial_t \phi = D\, \partial_x^{\alpha} \phi \;+\; \gamma\, \phi$$

$$\phi(x,0) = \begin{cases} e^{\lambda x} & x < 0 \\ 1 & x > 0 \end{cases}$$

In this case the calculation is simple because the front is self-similar and retains the exponential decaying shape.

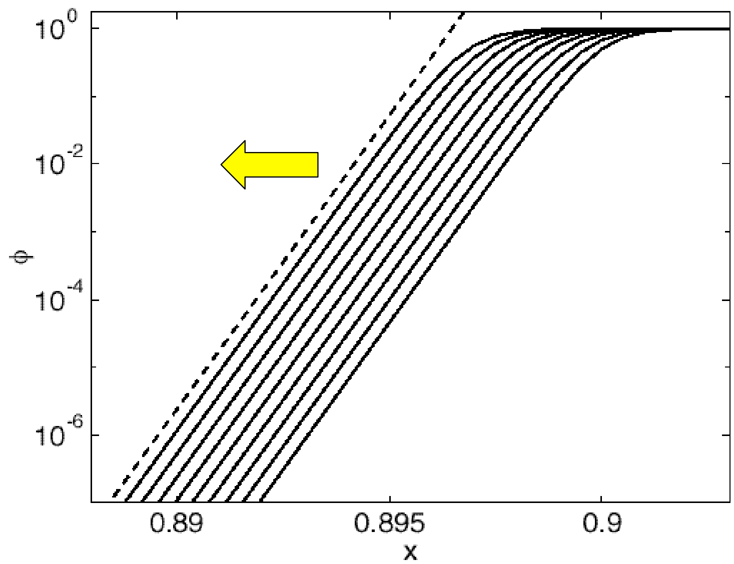
Using the fact that $\partial_x^{\alpha}\, e^{\lambda x} = \lambda^{\alpha}\, e^{\lambda x}$ we conclude

$$c = \begin{cases} \left(\dfrac{\alpha\,\gamma}{\alpha - 1}\right)\left[\dfrac{(\alpha-1)D}{\gamma}\right]^{1/\alpha} & \text{if} \quad 1/\lambda \le W_c \\[2ex] \dfrac{\gamma}{\lambda} + \dfrac{D}{\lambda^{1-\alpha}} & \text{if} \quad 1/\lambda \ge W_c \end{cases}$$

Left moving exponential decaying, constant velocity front

$$c = \left(\frac{\alpha\,\gamma}{\alpha - 1}\right)\left[\frac{(\alpha - 1)D}{\gamma}\right]^{1/\alpha}$$

Fractional generalization of Kolmogorov front speed.



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