Introduction to Fractional Calculus, Part II

How we just might help resolve some of the paradoxes of FC based on actual physical properties of materials.

The grand challenge

- Assuming would do find which definition to use under given conditions, what happens when the order changes?
  - Does a system have memory of its memory function?
  - What changes the order of the operator?

- We do know that the order does change with environmental conditions. How and why is uncertain.

- Will we be able to write anything like the Riemann, Liouville, Caputo, etc. formula? What would it look like?

$$v(t) = \frac{K}{\tau_o^{q(?)}} \int_{-\infty}^{t} \frac{(t-\tau)^{q(?)-1}}{\Gamma(q(?))} i(\tau) d\tau$$

An idea from nanotechnology

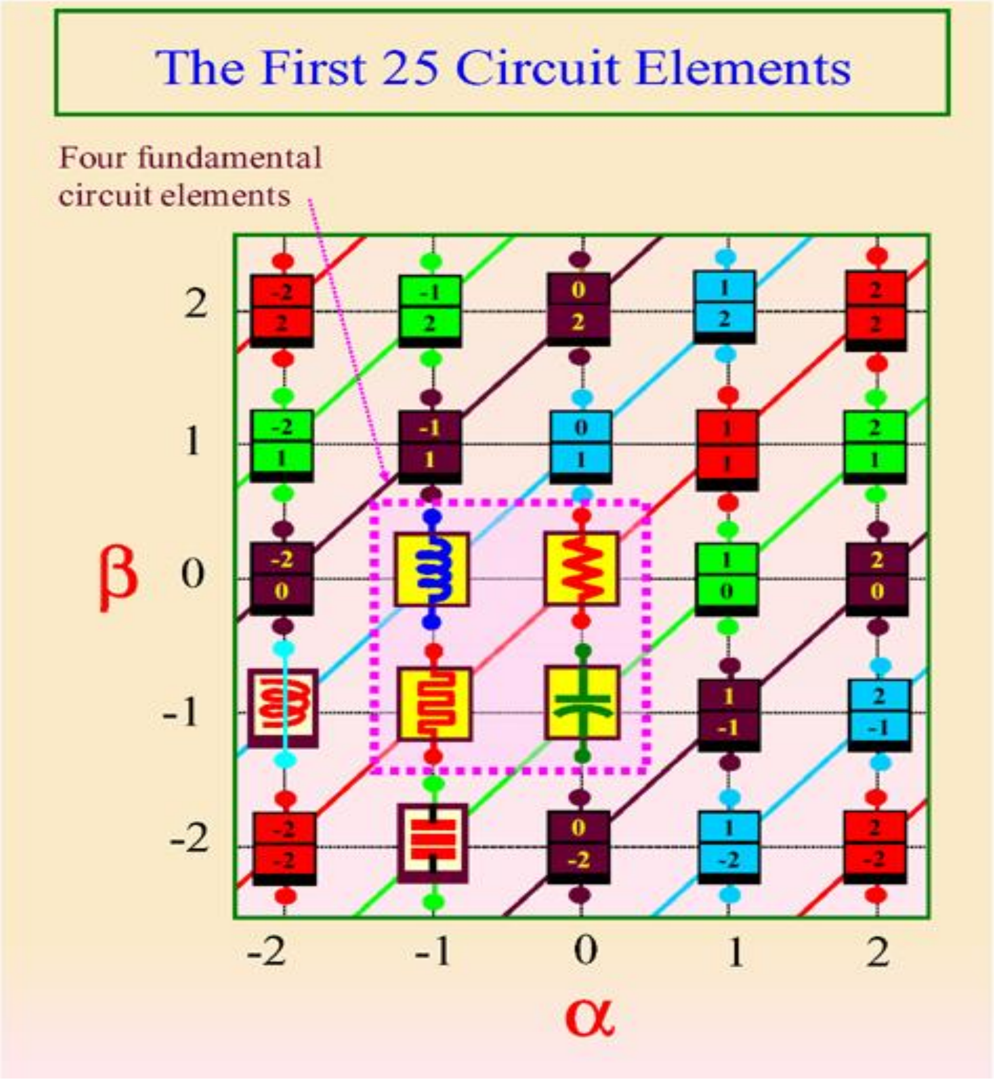
- In 1971, Leon Chua outlined the concept of a memristor; a device that can vary its impedance and remember its impedance state.
- In 2008, HP announced that they had actually built such a device some time earlier, but had just recently recognized it as such.
- With that, Chua and collaborators went on to elaborate on the idea and came up with other combinations such as the memcapacitor and meminductor. All devices that have variable, but persistent properties.
- The key insight is in the dynamics of the variation in the properties.

$$v^{(\alpha)} = m_Q(\vec{x}, i, t)\, i^{(\beta)}(t)$$

$$\dot{\vec{x}} = f(\vec{x}, \vec{w}, i, t)$$

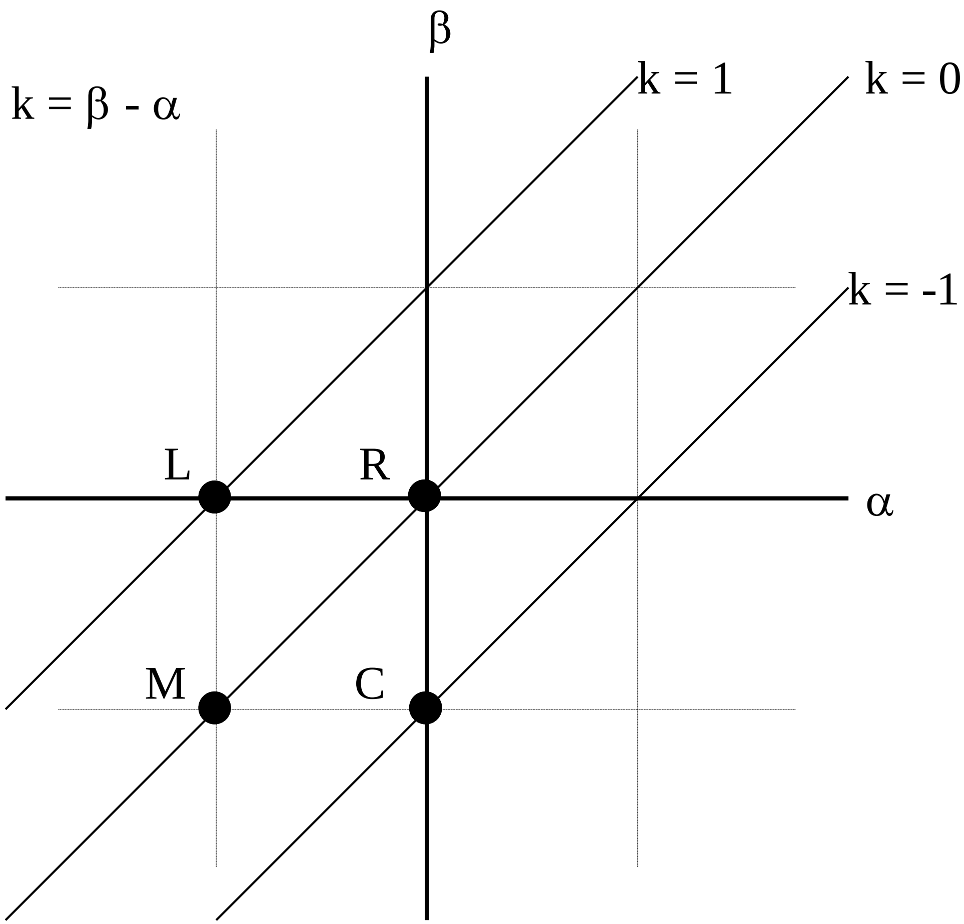
  - Where $m_Q(\vec{x}, i, t)$ represents the impedance or conductance, $\boldsymbol{x}$ represents the internal states of the system, and $\vec{w}$ represents the externally applied environment variables. (α) and (β) represent differential operations.

Chua’s basic circuit elements



Source: Leon Chua, “Resistance switching memories are memristors”, *Appl. Phys. A,* 102:765-783, 2011.

Chua's basic circuit elements



$$v^{(\alpha)}(t) = m_Q(\vec{x}, i, t)\, i^{(\beta)}(t)$$

$$\dot{\vec{x}} = f(\vec{x}, \vec{w}, i, t)$$

with $\vec{w}$ the environment varibles.

With :

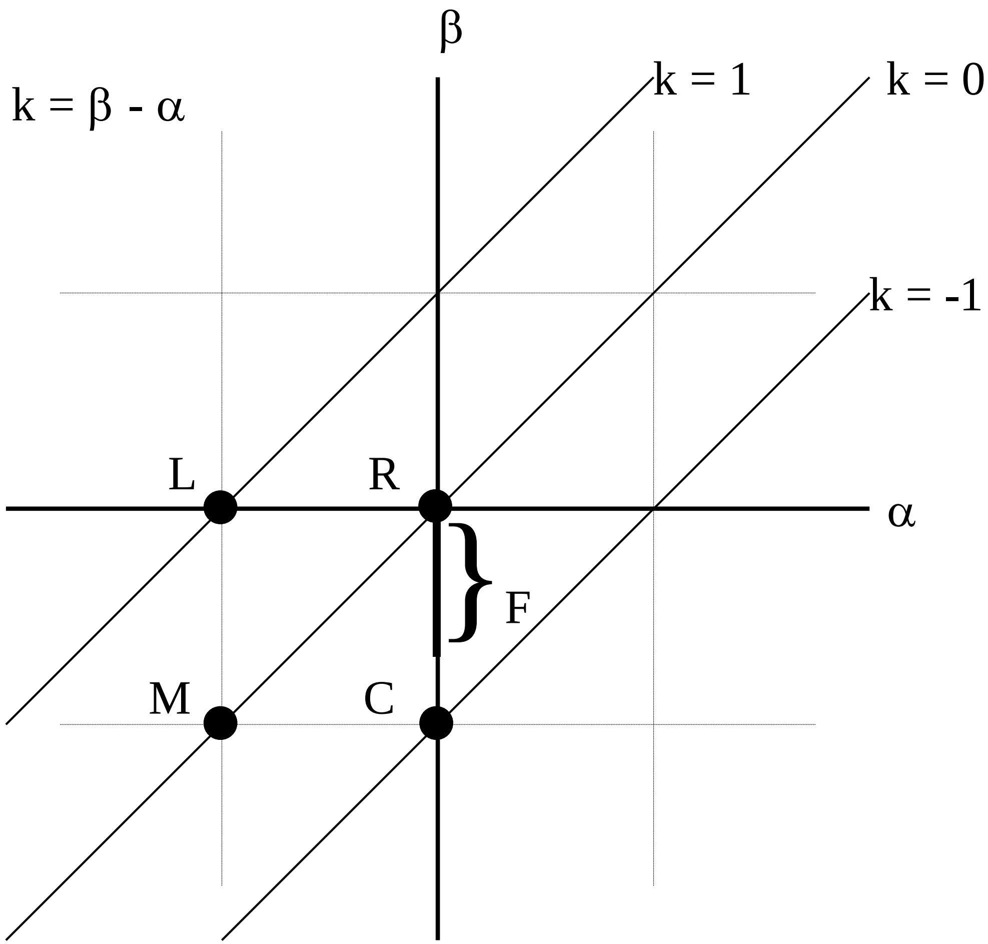
$$q \equiv \int i(t')dt'$$

$$\varphi \equiv \int v(t')dt'$$

Suppose α and β aren't integers

- We have successfully built fractors with non-integer order impedance relationship.
- What if we could find the "ingredients" for a "recipe" that would allow us to design fractors with arbitrary order?
- Further, what if we could figure out how to adjust the order dynamically? What change in environment produces what effect?
- If we could achieve this, then we have the potential to redefine the calculus as it applies to physics, engineering, and just about everything else.
- The desire for such a device is already documented.

Adding the fractor to the mix



$$v^{(\alpha)}(t) = m_Q(\vec{x}, i, t)\, i^{(\beta)}(t)$$

$$\dot{\vec{x}} = f(\vec{x}, \vec{w}, i, t)$$

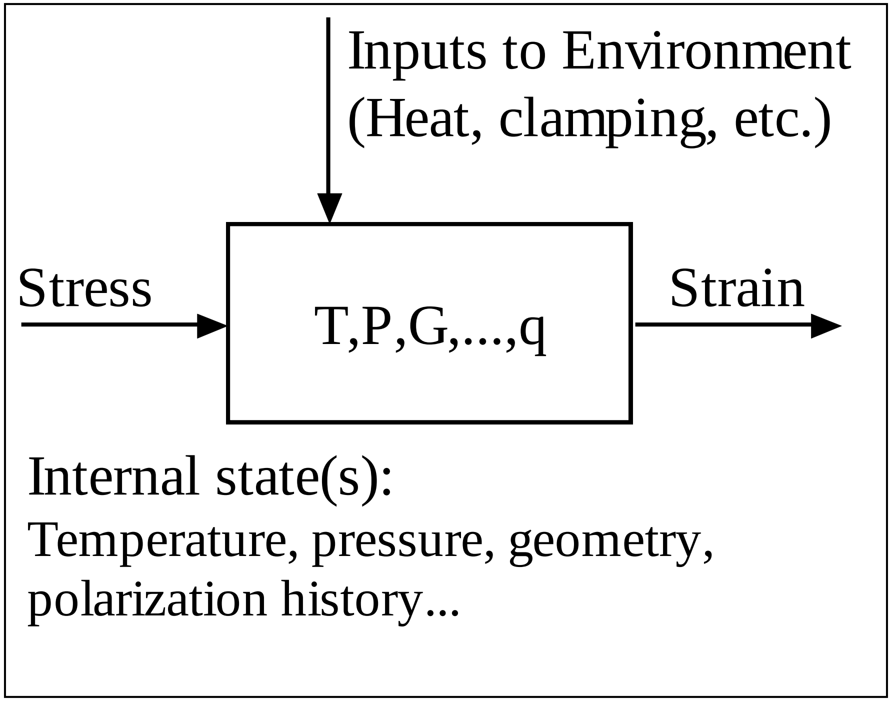
With:

$$q \equiv \int i(t')dt'$$

$$\varphi \equiv \int v(t')dt'$$

External Input vs. Internal State

- You do not have direct access to internal states.
- You can affect the states by applying external stresses.
- With near certainty, the resultant change in the stress/strain relation will be non-integer and nonlinear.



Intriguing…

- If you also include fractional order inductance, with -1 > $\alpha > 0$, how much of the α,β plane represents possible physical implementations?
- Maybe we can modulate the α and β parameters…

What hints do we get?

- The fractor depends on a distribution of ion trapping sites to get the power-law time scaling.
- It uses a number of dopants to generate the mobile carriers and the trapping sites.
- Temperature and pressure also affect the trapping statistics.

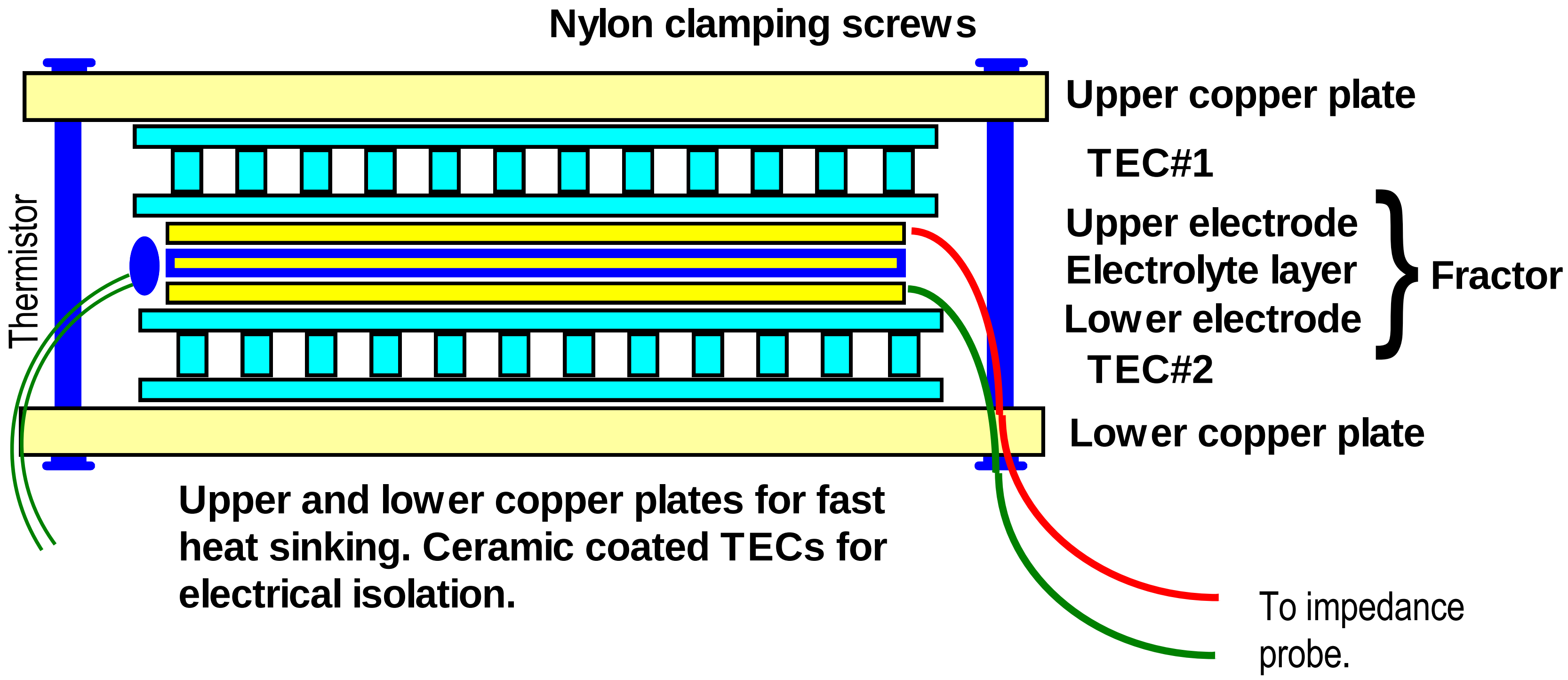
Things to try

- Modulate the chemistry
- Pro: If we can modulate the concentration and distribution of the dopants selectively, we may modulate $\boldsymbol{K}$ and $\boldsymbol{\beta}$.
- Con: Modulation of dopants best done in fabrication. Dynamics usually requires slow, large signals. We need this regime for control of mechatronics.

Things to try

- Modulate temperature to modulate phase
- Pro: Changing temperature will likely affect the phase more than has been shown so far.
- Con: Peltier devices can be used to control temperature locally, but they require their own controllers and are expensive.

Possible set up for Temperature Modulation

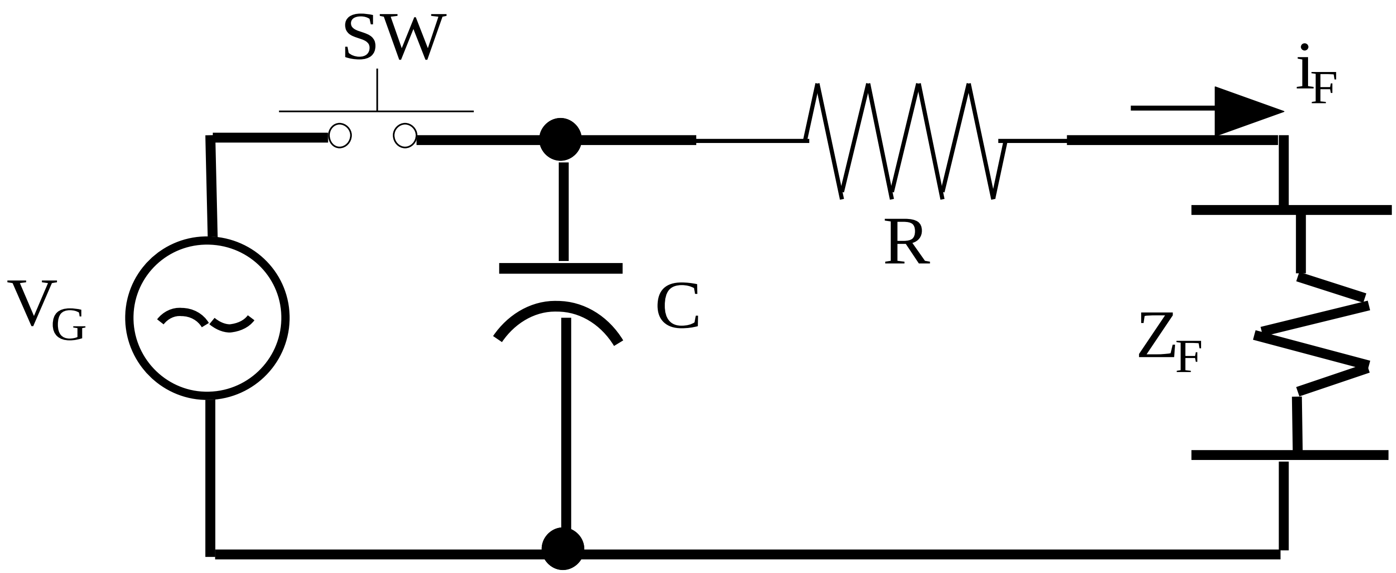


Upper and lower copper plates for fast heat sinking. Ceramic coated TECs for electrical isolation.

Capable of +/- 40°C temperature change.

Drive Circuit

- Various waveforms plus relaxation modes possible.
  - Record v_F, i_F as function of time.

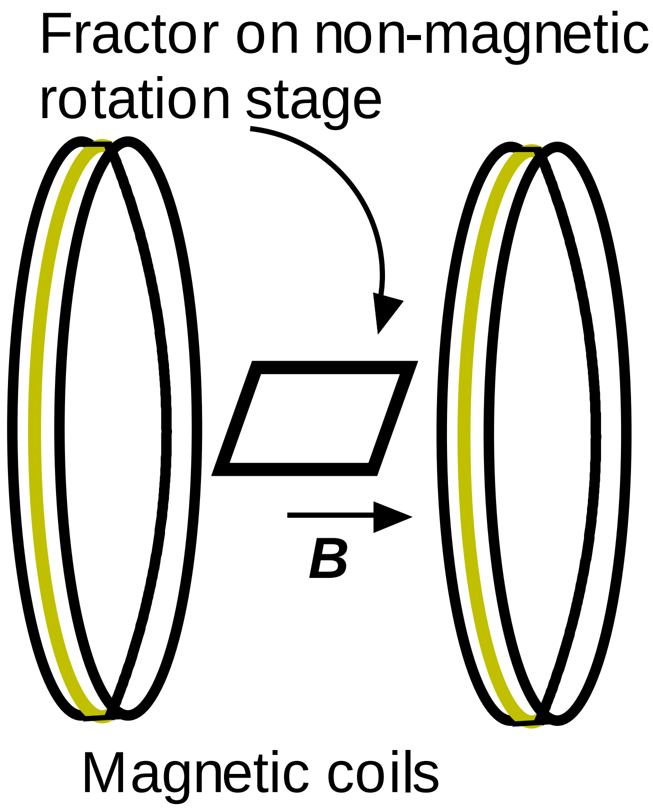


Things to try

- Apply magnetic field to modulate current.
- Pro: Magnetic fields may affect the flow of the charge carriers due to Hall effect, possibly significantly. Coils on FR4 board are easy to construct.
- Con: Coils require controlled current sources that can be expensive.

Magnetic field test

- Insert a modified* fractor in a Helmholtz coil; allowing various orientation
  - * modifications to the ionic current paths.



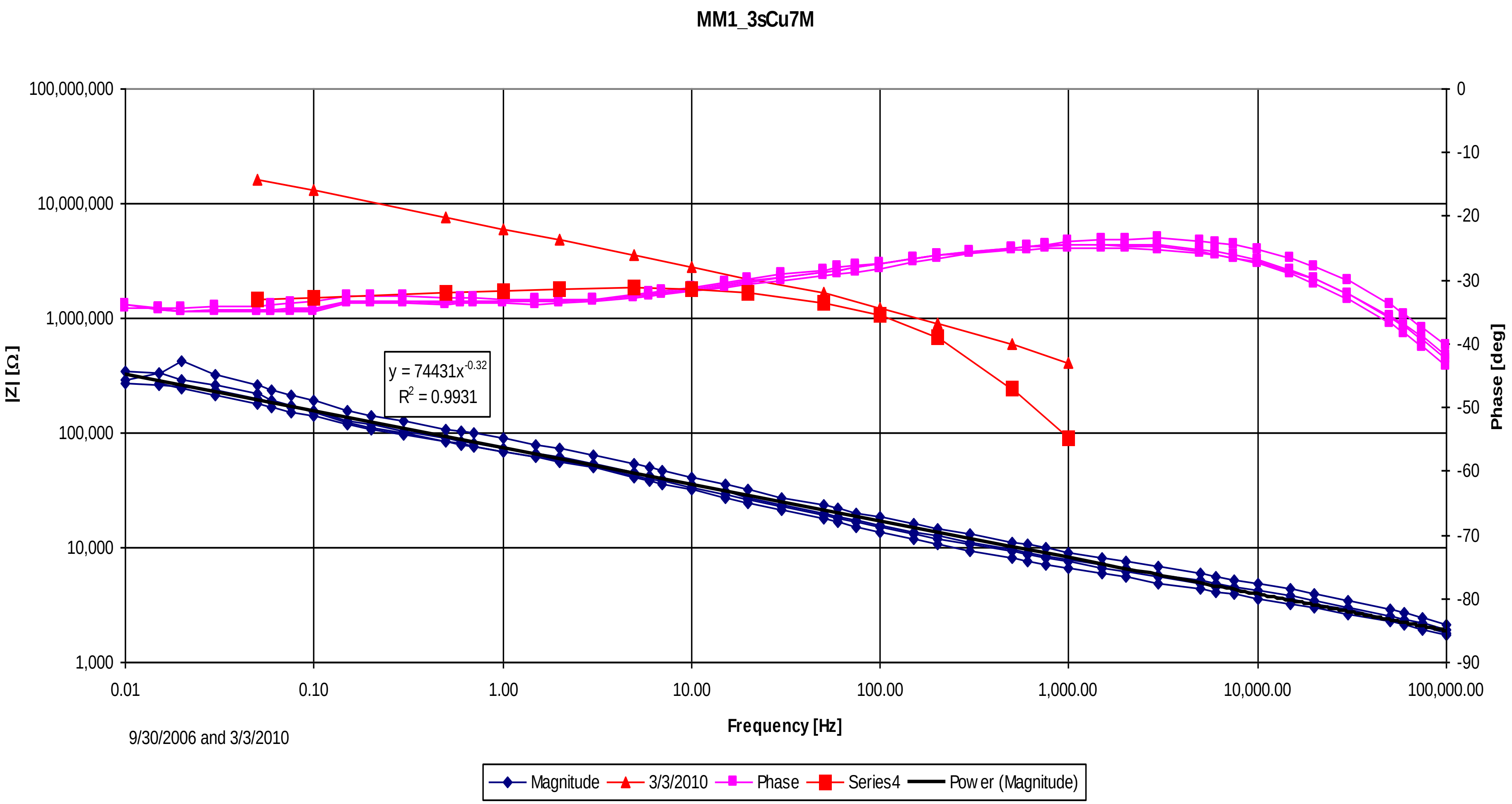
Things to try

- Vary pressure on the plates.
- Pro: Initial coarse tests indicate that the phase could be sensitive to pressure.
- Con: How to build a controlled pressure?

Things to try

- Add different phases
- Pro: We might use an adder circuit to smoothly switch between fractional order circuits with different phases.
- Con: Extra op-amps and other circuitry, rather “sloppy”, but maybe good starting point.

Also need to address aging



The aging problem

- It seems that polymer based electronic systems exhibit aging.
  - Even seen in polymer based lasers.
- Can we use something other than a polymer in the fractor recipe?
- Can we figure out what causes the aging?

What’s all this come to?

- The point is that there are a lot of possible avenues of investigation open to new-comers.
  - Basic mathematics
  - Numerical methods in C++ as well as Maple, etc.
  - System modeling, in time and s-domain
  - Circuit design and simulation
  - Test fixture design and construction
  - Chemistry and material and device creation and measurement
  - Statistical analysis
  - As well as literature reviews…

You get the idea

- There are a lot of possible projects to work on.
- The real problem is that there are too many possibilities.
- So, we'll spend some time on the math and its current applications while we discuss interest areas.
  - setting up fractional order differential equations.
  - use of fractional order operators in control systems.
  - use of fractional order dynamics in modeling biological systems.
- We'll have a look at a weird case of ultra-anomalous diffusion.