Variable order systems

A discussion of what we've learned and where we need to go.

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Motivation

- Lots of systems show fractional order behavior.
- Most of those exhibit changes in the order over time or due to changes in the environment.
- We would like to construct variable order fractance devices by design for tunable control purposes.

From Lorenzo & Hartley

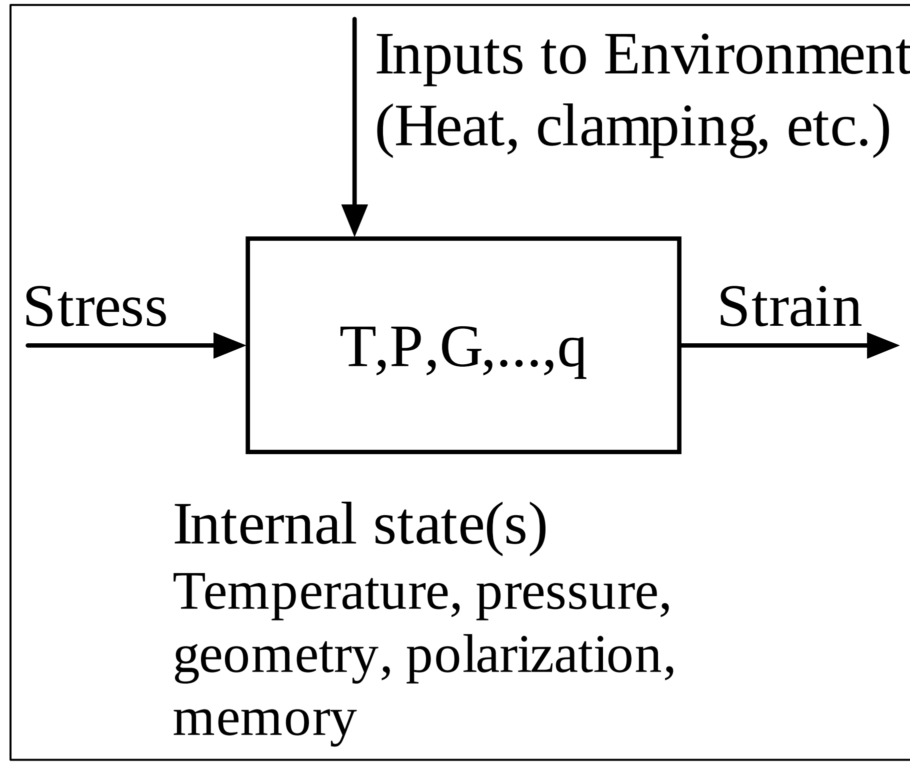
Variable Order and Distributed Order Fractional Operators

Nonlinear Dynamics **29:** 57–98, 2002

- Glöckle and Nonnenmacher studied the relaxation processes and reaction kinetics of proteins that are described by fractional differential equations of order β. The order was found to have a temperature dependence.
- Electroviscous or electrorheological fluids and polymer gels are known to change their properties in response to changes in imposed electric field strength. The properties of magnetorheological elastomers respond to magnetic field strength.
- From the field of damage modeling, it is noted that as the damage accumulates (with time) in a structure the nonlinear stress/strain behavior changes. It may be that this is better described with variable order calculus. Finally, the behavior of some diffusion processes in response to temperature changes may be better described using variable order elements rather than time varying coefficients.

Input vs. Internal State

- You do not have direct access to internal states.
- You can affect the states by applying an external stress.



The fractional constitutive equation

$$y(t) = \left(\frac{K}{\tau_0^q} \frac{1}{\Gamma(q)} \right) \int_a^t \frac{x(\tau)}{(t-\tau)^{1-q}} d\tau$$

Integer order examples:

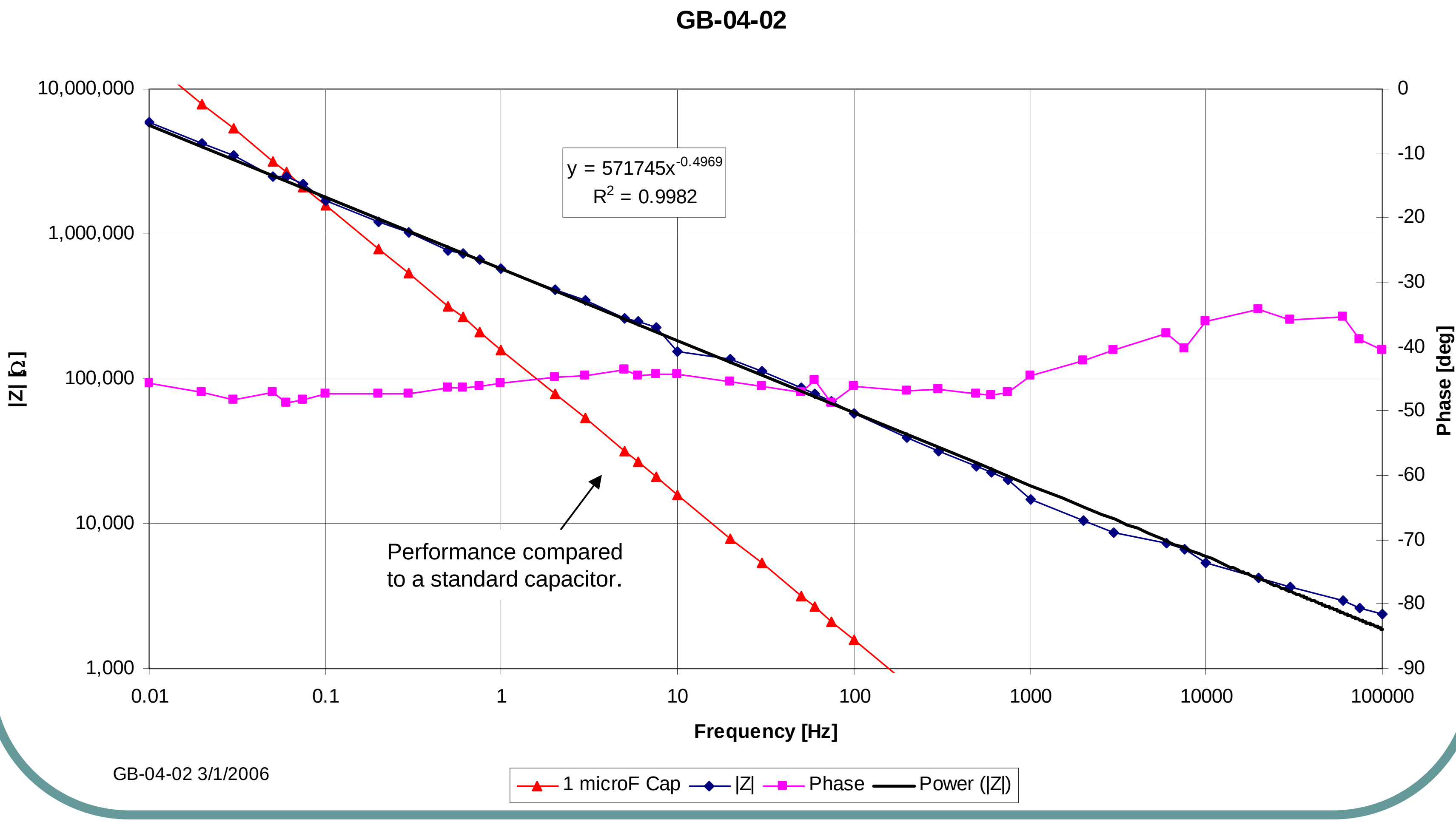
$v(t) = Ri(t)$ where $q = 0$. R has dimensions of Ohms.

$v(t) = \left(\frac{1}{C} \right) \int_a^t i(\tau) d\tau$ where $q = 1.0$ and $1/C$ has dimension Ohms/sec

We model the current as the stress and voltage as the strain.

All of the other stresses on the system are kept constant with the effect that the parameters K, τ_0, and q model their overall impact on the impedance.

Example impedance spectrum



Impedance Description

$$Z(\omega) \cong \frac{K}{\left(i\dfrac{\omega}{\omega_C}\right)^q} \quad or \quad Z(s) \cong \frac{K}{\left(s\tau_0\right)^q}$$

Where $\boldsymbol{K}$ is the impedance magnitude at calibration frequency $\boldsymbol{\omega_c} = 1/\tau_0$ and $\boldsymbol{q}$ is a real number $0 < \boldsymbol{q} < 1$. The phase is $\phi = -90° \times \boldsymbol{q}$.
This describes “fractance”. It is in units of Ohms.

Resistance is the limiting case $\boldsymbol{q} \rightarrow 0$,
Capacitance is the limiting case $\boldsymbol{q} \rightarrow 1$.

Fractional order integral

- The Laplace transform of that impedance leads to the fractional order integral:

$$v(t) = \left(\frac{K}{\tau_0{}^q}\right) {}_aI_t^q i(t) \equiv \underbrace{\left(\frac{K}{\tau_0{}^q}\right)}_{\substack{\text{scaling\&}\\ \text{dimensions}}} \int_a^t \underbrace{\left(\frac{(t-\tau)^{q-1}}{\Gamma(q)}\right)}_{\text{memory kernel}} i(\tau)\,d\tau$$

The limit form by Grünwald

- The Grünwald definition of the fractional derivative:

$$ {}_aD_t^q f(t) = \lim_{N\to\infty} \left\{ \frac{[dt]^{-q}}{\Gamma(-q)} \left[\sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f(t - j[dt]) \right] \right\} $$

$$ \text{where}\, dt = {(t-a)}\big/{N} $$

Notes:

1. The order q can be positive, negative, or even complex.
2. Even derivatives are computed over a finite interval.

A constitutive equation

$$v_F(t) = \left(\frac{K}{\tau_0{}^q}\right) {}_aI_t^q i(t) \equiv \int_a^t \left(\frac{K}{\tau_0{}^q}\right) \underbrace{\left(\frac{(t-\tau)^{q-1}}{\Gamma(q)}\right)}_{\text{memory kernel}} i(\tau) d\tau$$

Or, rearranging the equivalent Grünwald form:

$$v_F(t) = \lim_{N\to\infty} \sum_{j=0}^{N-1} K \frac{\Gamma(j+q)}{\Gamma(q)\Gamma(j+1)} i(t - j[d\tau]) \left[\frac{d\tau}{\tau_0}\right]^q$$

where $d\tau = [t-a]/N$

The big question

- What if the “environment” parameters are not constant?
- We expect that the coefficient $\boldsymbol{K}$, the time scaling $\boldsymbol{\tau_0}$, and the order $\boldsymbol{q}$ will probably all change together in some way.
- We’ve measured such a cumulative effect with the fractors and the results confirm this expectation.

Variable order description

$$v_F(t) \overset{?}{=} {}_aI_t^{q(t)}\left(\frac{K(t,\tau)}{\tau_0(t,\tau)^{q(t,\tau)}}\right)i(t) \equiv \int_a^t \left(\frac{K(t,\tau)}{\tau_0{}^{q(t,\tau)}}\right)\underbrace{\left(\frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))}\right)}_{memory?} i(\tau)\,d\tau$$

$$v_F(t) \overset{?}{=} \lim_{N\to\infty}\sum_{j=0}^{N-1} K(t,\tau)\frac{\Gamma(j+q(t,\tau))}{\Gamma(q(t,\tau))\Gamma(j+1)} i(t-j[d\tau])\left[\frac{d\tau}{\tau_0(t,\tau)}\right]^{q(t,\tau)}$$

Lorenzo and Hartley* suggest a number of possible forms, for example:

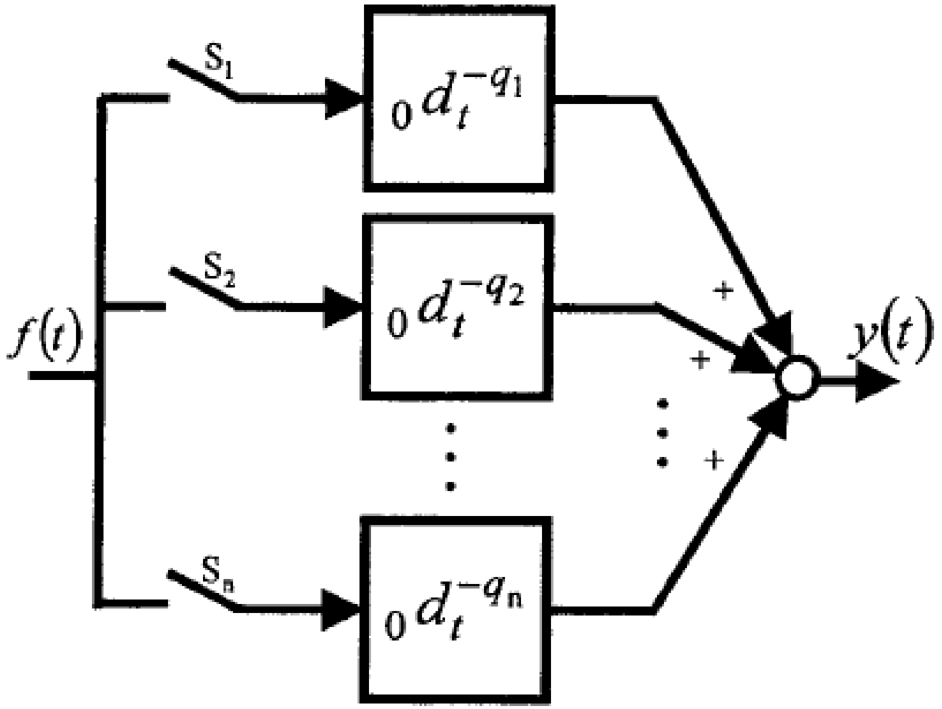


Figure 7. Block diagram for physical realization of $q(t,\tau) = q(\tau)$ based definition.

* Lorenzo & Hartley, Variable Order and Distributed Order Fractional Operators, *Nonlinear Dynamics* 29: 57–98, 2002

Comments on their forms

- While their math is consistent, they don't use dimensionally correct forms for physical implementation.
  - They ignore the scaling
- Their forms violate conservation of energy or causality or both.

For example

- The most popular form is

$$v_F(t) = \underbrace{\left(\frac{K}{\tau^{q(t)}}\right)}_{\substack{\text{scaling}\\ \text{almost}\\ \text{always}\\ \text{ignored}}} \int_a^t \underbrace{\left(\frac{(t-\tau)^{q(t)-1}}{\Gamma(q(t))}\right)}_{\text{memory kernel}} i(\tau)\, d\tau$$

But a close look at the resulting predictions suggest that it will violate conservation of energy. If $\boldsymbol{q}$ is reduced, energy will be dissipated. Increase $\boldsymbol{q}$ again and the memory and internal energy are instantly regenerated. (See section 7 of the Lorenzo & Hartley paper, figures 14 – 18.)

Case 1: $q(t,\tau)\rightarrow q(t)$

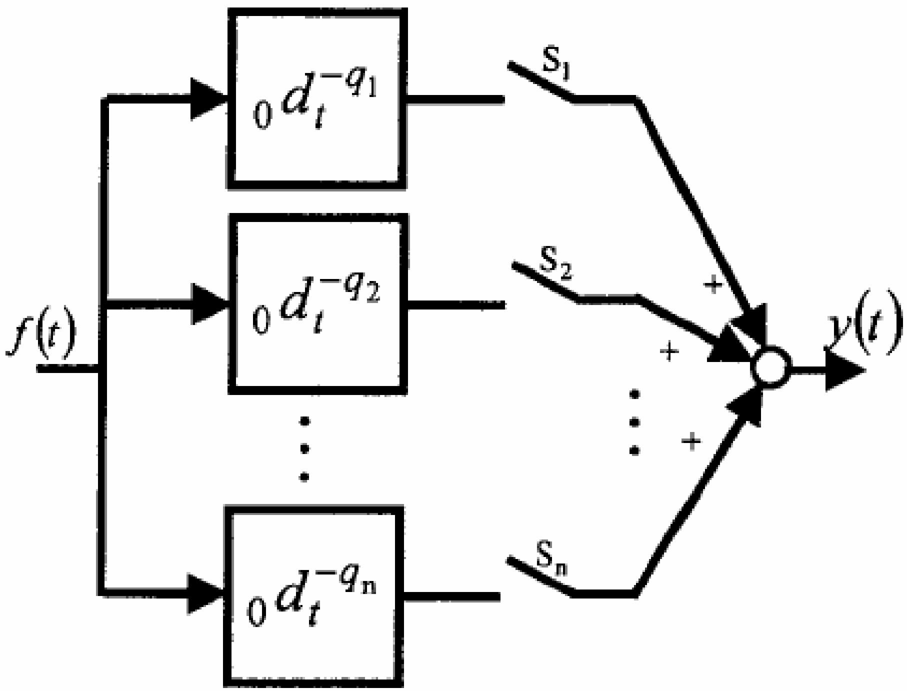


Figure 3. Block diagram for physical realization of $q(t,\tau) = q(t)$ based definition.

* Lorenzo & Hartley, Variable Order and Distributed Order Fractional Operators, *Nonlinear Dynamics* 29: 57–98, 2002

Case 2: $q(t,\tau)\rightarrow q(\tau)$

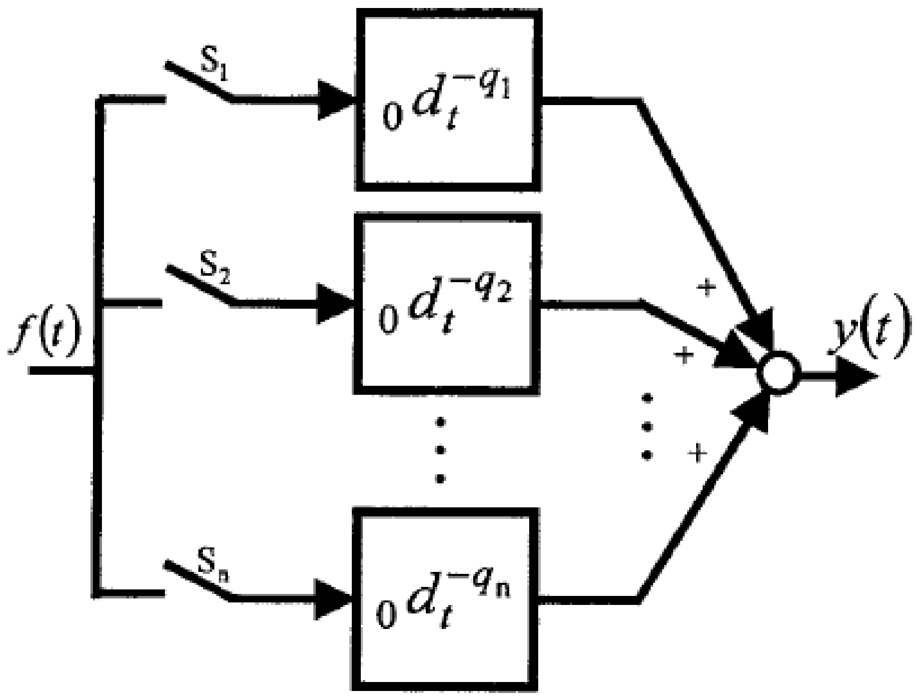


Figure 7. Block diagram for physical realization of $q(t, \tau) = q(\tau)$ based definition.

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Case 3: $q(t,\tau) \to q(t-\tau)$

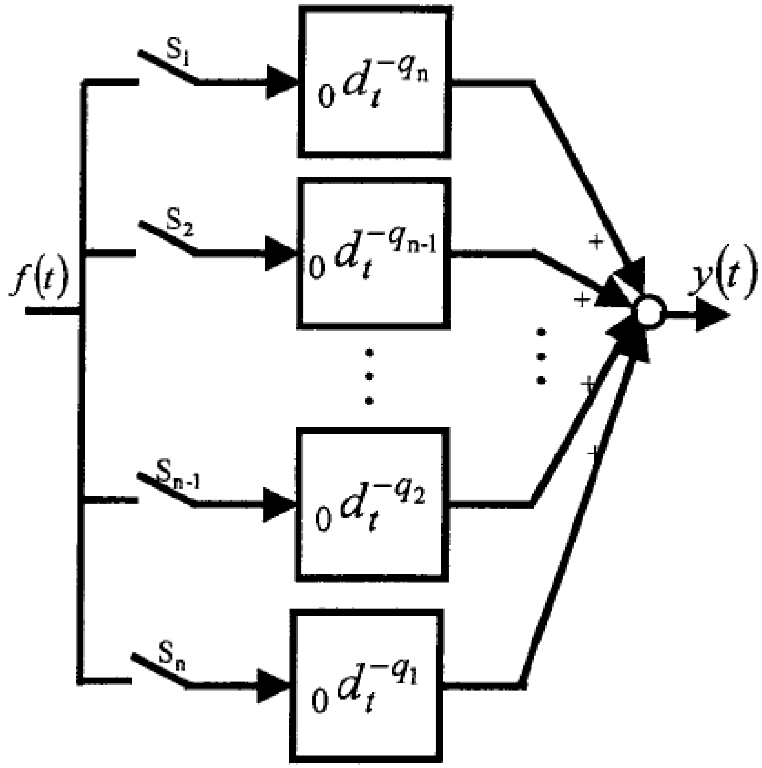

Figure 9. Block diagram for physical realization of $q(t,\tau) = q(t-\tau)$ based definition.

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Step response from L&H.

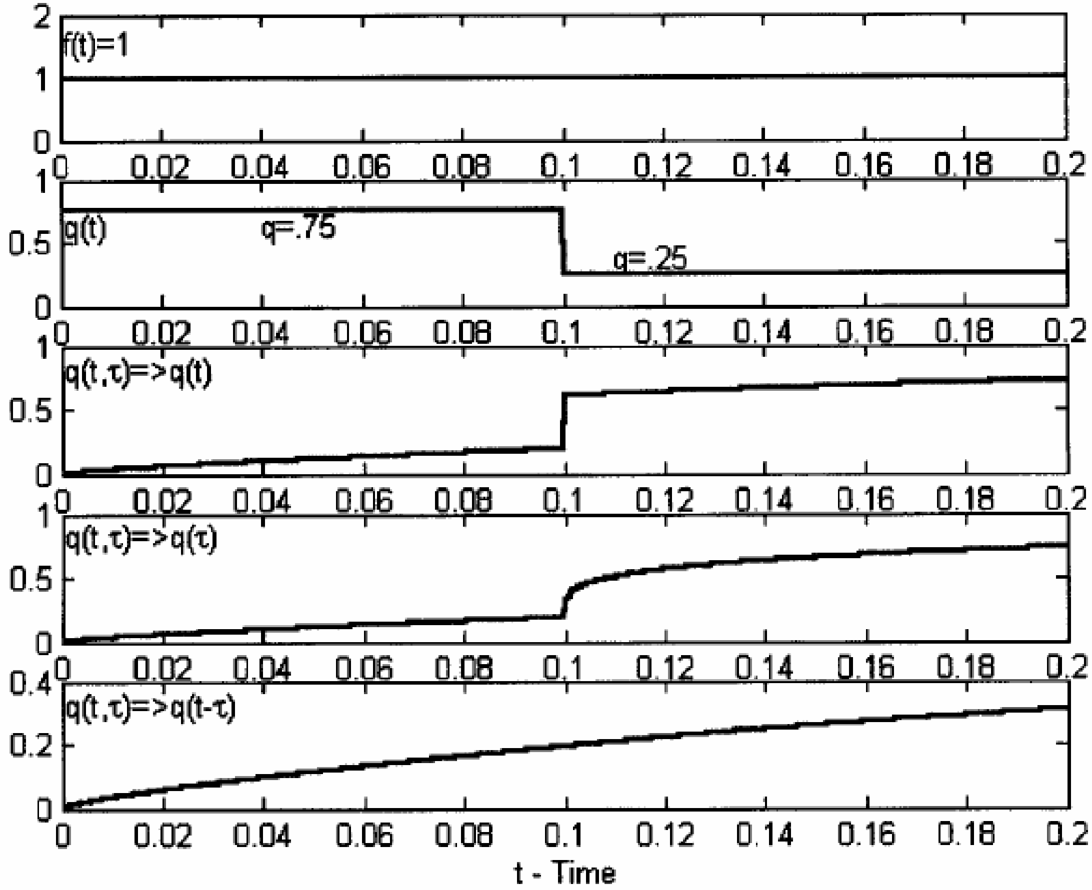


Figure 15. Variable order fractional integral responses for arguments $q(t,\tau) = q(t), = q(\tau), = q(t-\tau)$, $q(t)$ input is a step at $t = 0.1$.

Let's start from a different perspective

- A possible form that includes all of the relevant parameters:

$$v_F(t) = \sum_{j=0}^{N-1} W_j K_{N-j} \left[\frac{d\tau}{\tau_{0,N-j}} \right]^{q_{N-j}} i(t - j[d\tau])$$

where:

$$W_0 = 1$$

$$W_j = \prod_{k=1}^{j} \frac{(k - 1 + q_{N-k})}{k}$$

The weighting terms – products of the columns

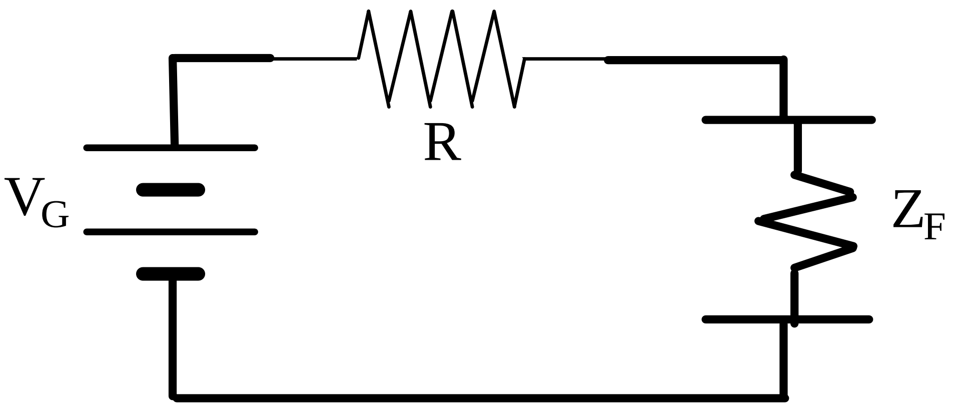
$$
\begin{array}{ccccccccc}
f_0 & f_1 & \cdots & f_k & \cdots & f_{N-2} & f_{N-1} & f_N \\
\hline
W_N & W_{N-1} & \cdots & W_{N-k} & \cdots & W_2 & W_1 & W_0 \\
\hline
1 & 1 & \cdots & 1 & \cdots & 1 & 1 & 1 \\
q_0 & q_1 & & q_{N-k} & & q_{N-2} & q_{N-1} & \\
\frac{1+q_1}{2} & \frac{1+q_2}{2} & & \frac{1+q_{N-k+1}}{2} & & \frac{1+q_{N-1}}{2} & & \\
\frac{2+q_2}{3} & \frac{2+q_3}{3} & & \frac{2+q_{N-k+2}}{3} & & & & \\
\frac{3+q_3}{4} & \frac{3+q_4}{4} & & \vdots & & & & \\
\vdots & \vdots & & \frac{N-k-1+q_{N-1}}{N-k} & & & & \\
 & \frac{N-2+q_{N-1}}{N-1} & & & & & &
\end{array}
$$

Properties

- Reduces to the constant order Grünwald form.
- Couples the scaling coefficients with the time the memory is “laid down”.
- Does not completely erase the past, but does attenuate consistent with energy conservation.
- Can be coded relatively efficiently, with an update of past weights each time step.

Simulated circuit

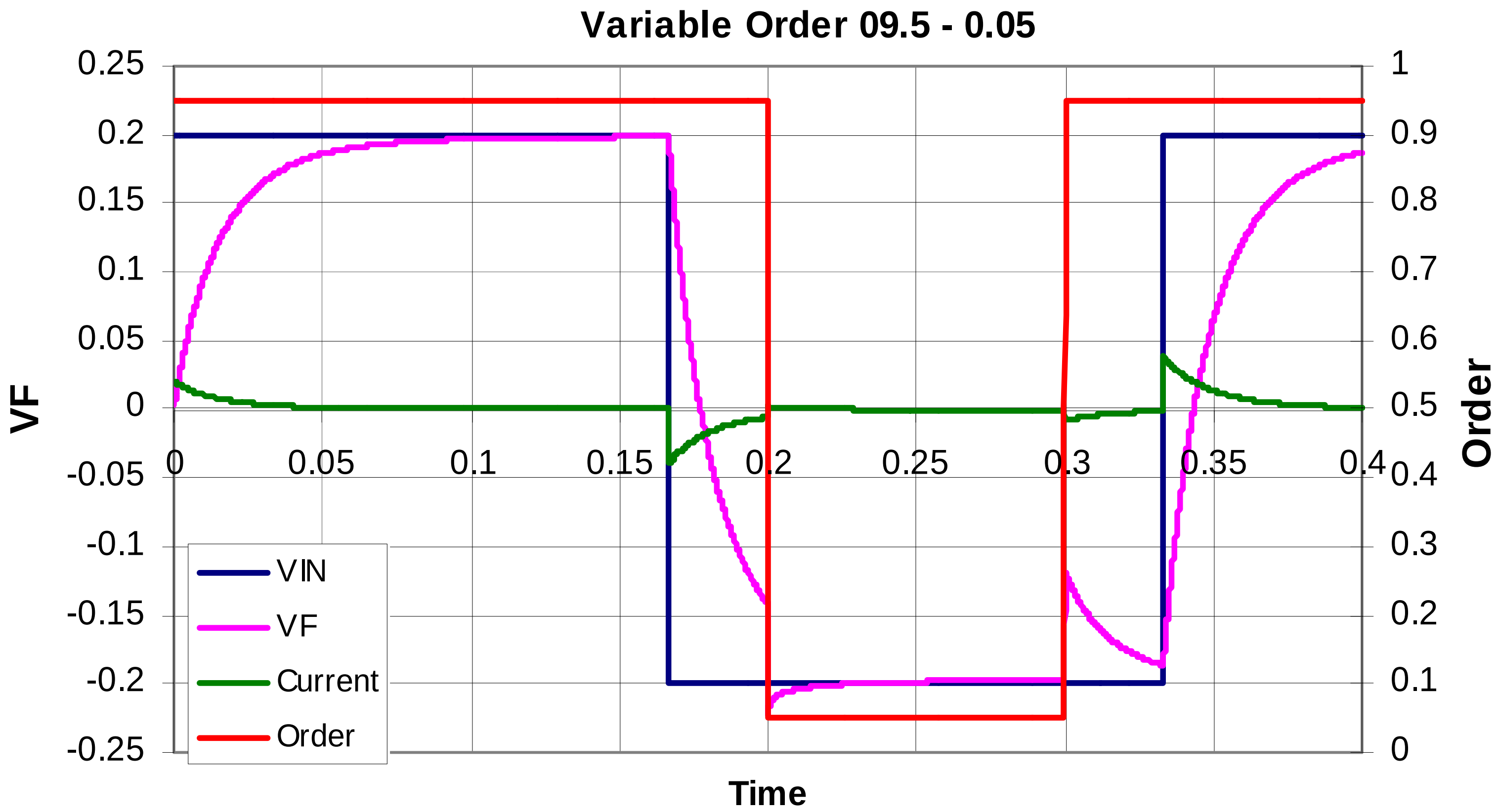
- Resistor-Fractor simple circuit



- We'll model changes to Z_F.

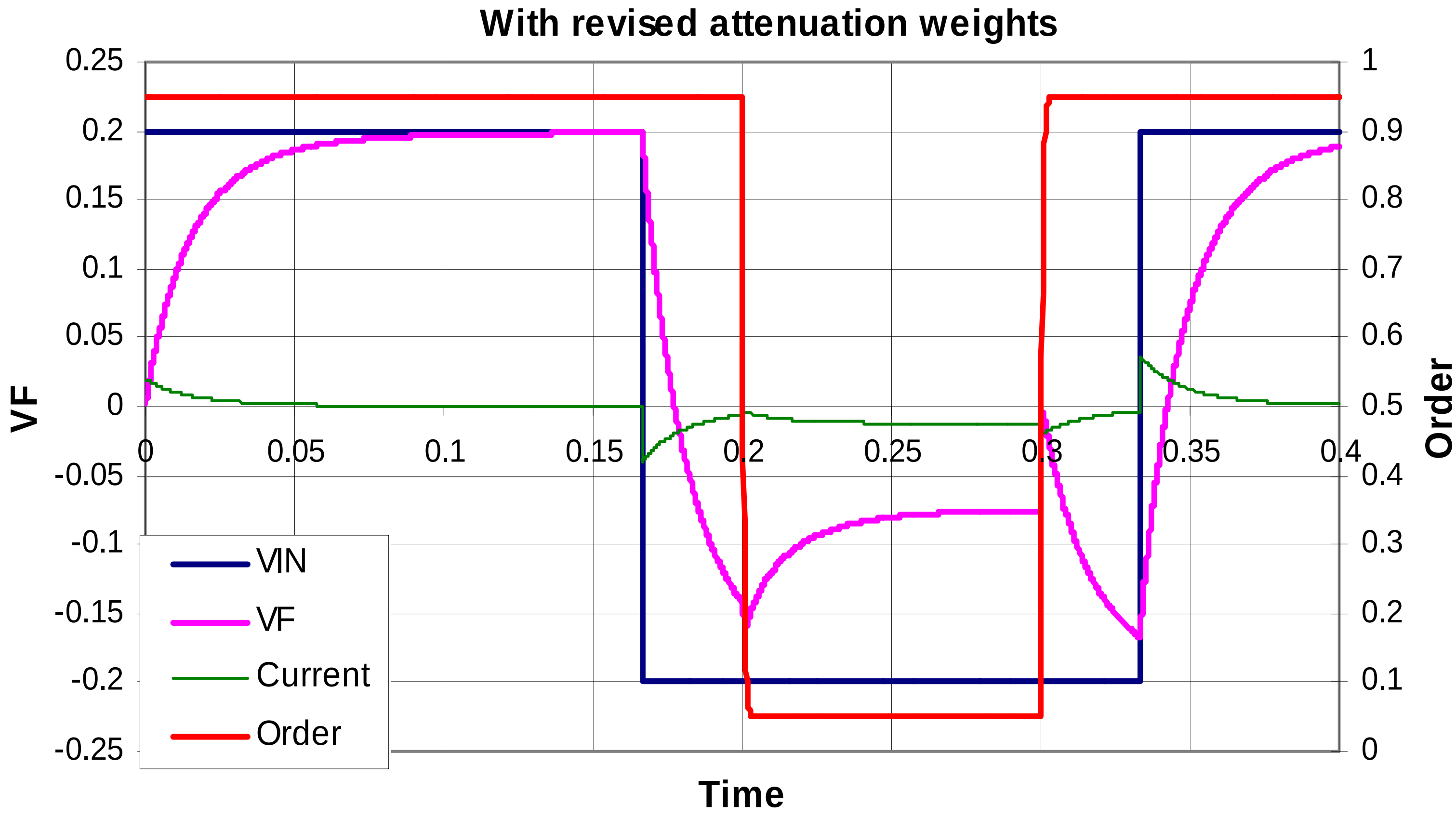
Simulation: L&H Case 1

- $q(t,\tau) \to q(t)$



Simulation results: New Model

- Including the scaling in my suggestion



How to write an integral form?

$$v_F(t) = \int_a^t \underbrace{\left(\frac{K(t,\tau)}{{\tau_0}^{q(t,\tau)}}\right)}_{\text{scaling}} \underbrace{\left(\frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))}\right)}_{\text{memory kernel}} i(\tau)\,d\tau$$

- Can we write such an integral form that matches the Grünwald variant above?
- Do we need to replace the memory kernel itself?

What we have

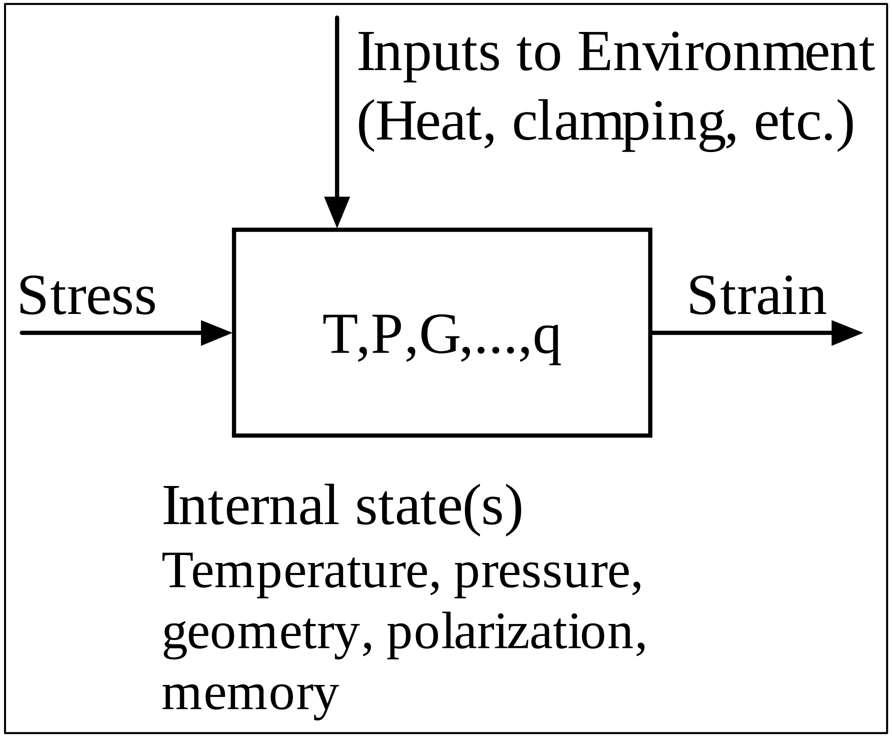
- A set of criteria for evaluating possible models for variable order
  - Don’t violate energy conservation
  - Don’t violate causality
  - Be dimensionally correct
- Suggestions for real experiments
  - Don’t suggest changing internal states without an actuating input

What we might do

- Revisit the fractors
  - Release the pressure
    - Loosen the four screws to allow mechanical relaxation. (They won’t come apart)
  - Adjust pressure and temperature via screws and heating/cooling
  - Look at impedance magnitude as well as phase
- Obtain other samples from L&H list.

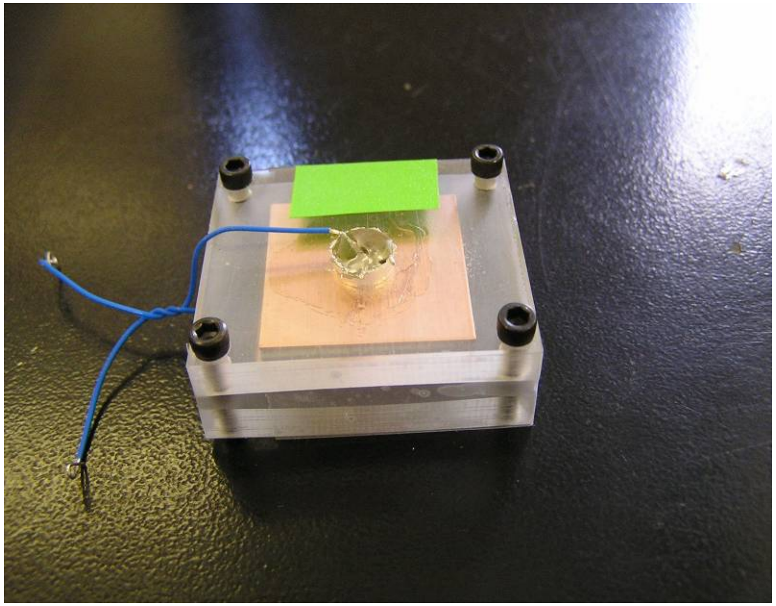
Input vs. Internal State

- You do not have direct access to internal states.
- You can affect the states by applying an external stress.



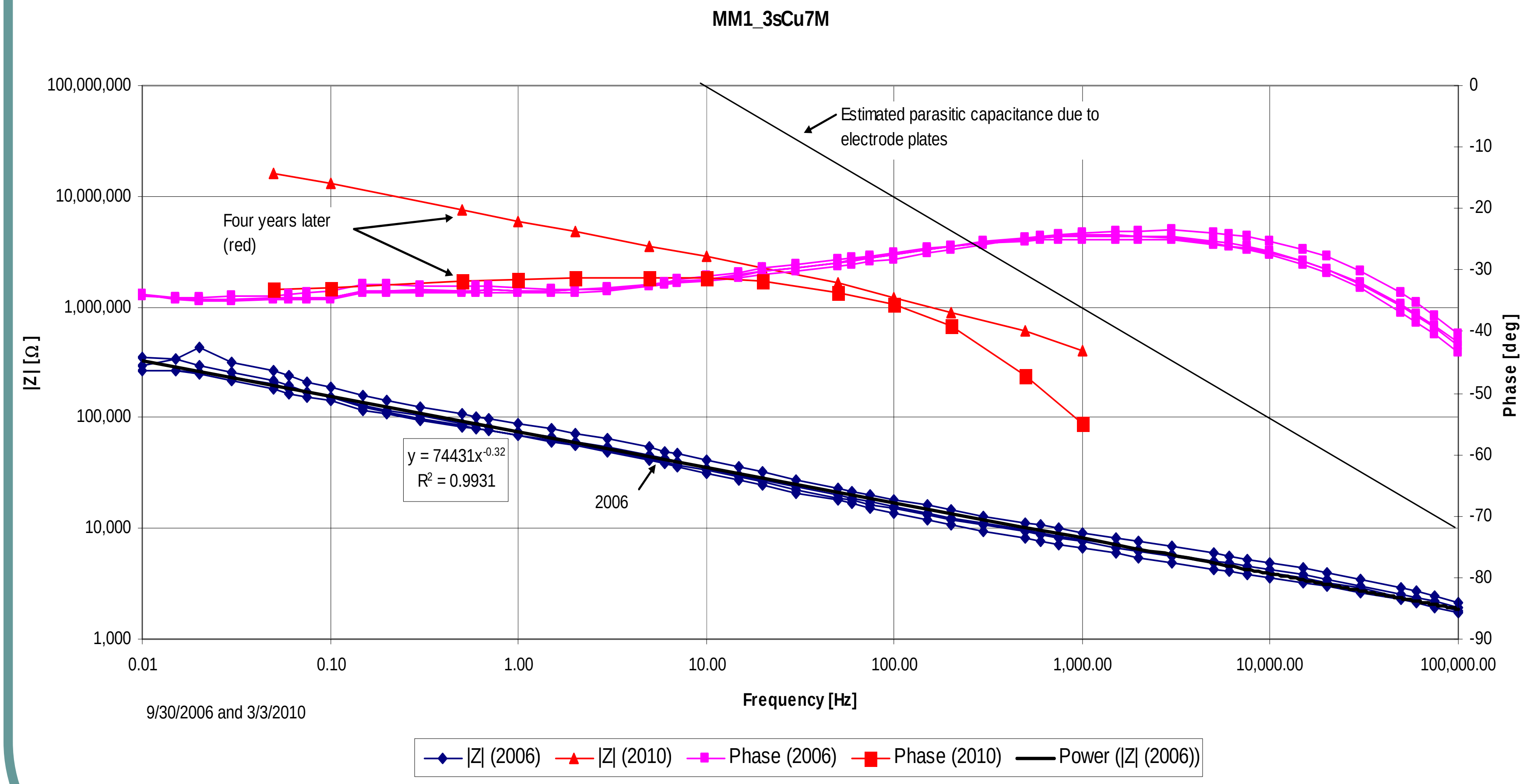
The fractor

- Known to age over time
- Changes phase with temperature and pressure
- Sensitive to humidity



<-   3.5 cm on a side ->

Fractor aging



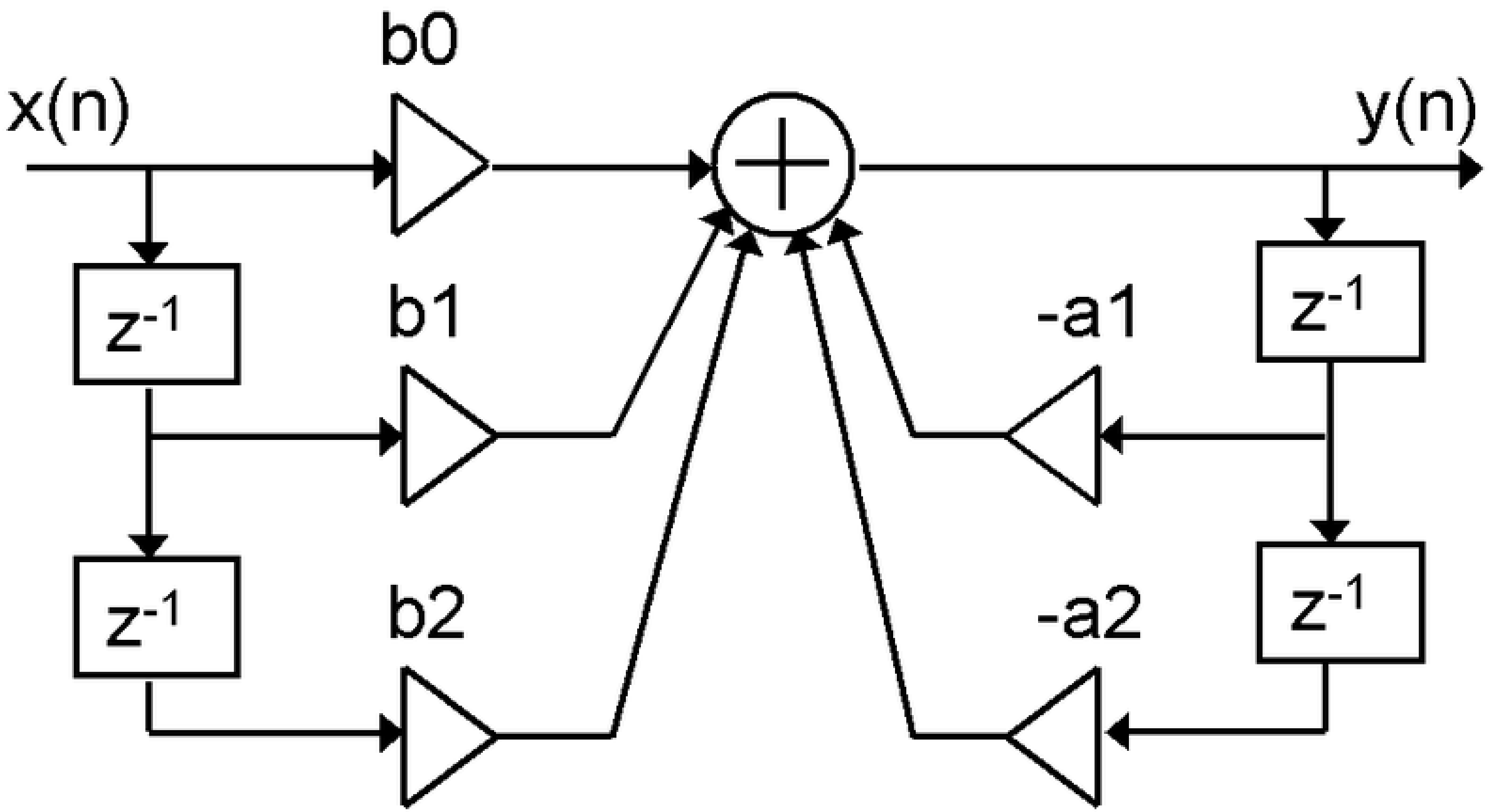
Conclusion

- There are still plenty of open questions with regard to physical models for dynamic order systems.
- Experiments are not that difficult or expensive.
- Experimental results should be compared with model predictions, both in magnitude and phase.

Appendix

- How to implement a change of order in IIR or FIR digital filter approximation?
- Note the delayed response no matter whether the coefficients are updated simultaneously or staggered.
- Do the delayed memory terms continue to have meaning?

Digital IIR filter: Form 1



Digital IIR filter: Form 2