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Variable Order and Distributed Order Fractional Operators

CARL F. LORENZO
John H. Glenn Research Center, National Aeronautics and Space Administration, 21000 Brookpark Road, Cleveland, OH 44135, U.S.A.

TOM T. HARTLEY
Department of Electrical Engineering, The University of Akron, Akron, OH 44325-3904, U.S.A.

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Abstract. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The continuum of order in the fractional calculus allows the order of the fractional operator to be considered as a variable. This paper develops the concept of variable and distributed order fractional operators. Definitions based on the Riemann–Liouville definition are introduced and the behavior of the new operators is studied. Several time domain definitions that assign different arguments to the order q in the Riemann–Liouville definition are introduced. For each of these definitions various characteristics are determined. These include: time invariance of the operator, operator initialization, physical realization, linearity, operational transforms, and memory characteristics of the defining kernels.

A measure (m_2) for memory retentiveness of the order history is introduced. A generalized linear argument for the order q allows the concept of 'tailored' variable order fractional operators whose m_2 memory may be chosen for a particular application. Memory retentiveness (m_2) and order dynamic behavior are investigated and applications are shown.

The concept of distributed order operators where the order of the time based operator depends on an additional independent (spatial) variable is also forwarded. Several definitions and their Laplace transforms are developed, analysis methods with these operators are demonstrated, and examples shown. Finally operators of multivariable and distributed order are defined and their various applications are outlined.

Keywords: Variable order fractional operator, distributed order fractional operator, order distribution, Laplace transform, tailored fractional order operator, memory measure.

1. Introduction

The fractional calculus has allowed the operations of integration and differentiation to any fractional order. The order may take on any real or imaginary value. This fact enables us to consider the order of the fractional integrals and derivatives to be a function of time or some other variable. Lorenzo and Hartley [1] first suggested the concept of variable order (or variable structure) operators. In this concept the order of the operator is allowed to vary either as a function of the independent variable of integration or differentiation (t) or as a function of some other (perhaps spatial) variable (y). In that paper a preliminary study was done on several potential variable order definitions where the order was a function of the independent variable of integration or differentiation (t), and initial properties were forwarded.

This paper explores more deeply the concept of variable order integration and differentiation and seeks to create meaningful definitions for variable order integration and differentiation and to relate the mathematical concepts to physical processes. Several candidate definitions are developed and characterized. For these definitions operational methods are provided, and example applications are shown. One of the important characteristics of these operators is the manner in which the operator accounts for the order history (or order memory). Two types of memory are attributed to the operators, first m_1, is the fading memory of the overall operator, and second is the memory related to the history of the order of the operator. This second type of memory, called m_2, is found to be an important attribute of the operator. Measures of these types of operator memory are presented.

From these candidate characterizations the concept of 'tailored' variable order integration and differentiation is evolved. By this we mean the ability to choose the form of the operator definition to control how the operator deals with its own order history.

The concept of the (continuous) order distribution, where the order is distributed over variables other than the independent variable was suggested by Hartley and Lorenzo [2] and an initial and important application of order distributions was made to the identification of fractional order systems. In a related work Bagley and Torvik [3] develop a special transform for the solution of distributed order differential equations. The present paper looks at two forms of order distributions and their applications to dynamic processes. Also, a variety of physical problems that motivate interest in these areas are briefly presented. Finally, the paper defines operators of multi-variable order and operators that combine the variable order and the order distribution concepts.

2. Motivation for Variable Fractional Order Operators

There is considerable potential physical motivation toward the creation and implementation of the concept of variable order operators. A few possibilities will be mentioned here. From the field of viscoelasticity [4], the effect of temperature on the small amplitude creep behavior (force/extension) of certain materials is to change the characteristics from elastic (spring-like, $q \simeq 0$) to viscoelastic or viscous (damper-like, $q \simeq -1$). This relates to the expression

$$ {}_cD_t^{-q} F(t) = k(x_a(t) - x_b(t)), \tag{1} $$

where F is force and x is displacement. Experience (and experiments), typically, is based on fixed temperature, but real applications may well require a time varying temperature to be analyzed. Polymer linear viscoelastic stress relaxation was studied by Bagley [5]. This process is described by fractional differential equations of order β for a given fixed temperature. The paper shows a clear dependence of β on temperature for polyisobutlene and correlates the fractional model and experiment. Further, it is indicated that β order fractal time processes lead to β order fractional derivative consitutive laws. Smit and deVries [6] studied the stress-strain behavior of viscoelastic materials (textile fibers) with fractional order differential equations of order α, with $1 \geq \alpha \geq 0$. They show, based on related experiments, α to be dependent on strain level.

Glöckle and Nonnenmacher [7] studied the relaxation processes and reaction kinetics of proteins that are described by fractional differential equations of order β. The order was found to have a temperature dependence.

Electroviscous or electrorheological fluids [8] and polymer gels [9] are known to change their properties in response to changes in imposed electric field strength. The properties of magnetorheological elastomers respond to magnetic field strength [10].

From the field of damage modeling, it is noted that as the damage accumulates (with time) in a structure the nonlinear stress/strain behavior changes. It may be that this is better described with variable order calculus. Finally, the behavior of some diffusion processes in response to temperature changes may be better described using variable order elements rather than time varying coefficients.

3. General Preliminaries for Variable Order Operators

3.1. Issues

This section presents several issues and characteristics that generally apply to the various cases that will be considered in the paper. The basic research question, relative to variable order operators, derives from the fractional differential equation

$$ {}_cD_t^q y(t) = f(t) \tag{2}$$

and the inferred integral equation

$$ {}_cD_t^{-q} f(t) = y(t). \tag{3}$$

Since q in the fractional calculus can take on any real (or complex) value, the question is asked, "What is a desirable definition for the fractional integral when q is allowed to vary either with t or y?" More specifically, what is an appropriate definition for

$$ {}_cD_t^{-q(t,y)} f(t)? \tag{4}$$

The analysis that follows is based on the Initialized Fractional Calculus of Lorenzo and Hartley [11]. The Riemann–Liouville basis for initialized fractional order integration is

$$\begin{aligned} {}_cD_t^{-q} f(t) &\equiv {}_cd_t^{-q} f(t) + \psi(f, -q, a, c, t) \\ &= \frac{1}{\Gamma(q)} \int_c^t (t-\tau)^{q-1} f(\tau)\,\mathrm{d}\tau + \psi(f, -q, a, c, t), \quad t > c, \end{aligned} \tag{5}$$

where c is typically taken as zero, and $\psi(f, -q, a, c, t)$ is the initialization function. Only variation of q with t will be considered initially. Thus the problem becomes that of defining and determining the appropriate properties. A list of general properties, G1–G5, follows.

3.2. G1-Form and Basis

With the exception of Section 4.5, consideration will be limited to fractional integrals whose forms are defined by

$$ {}_0d_t^{-q(t)} f(t) = \int_0^t \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))} f(\tau)\,\mathrm{d}\tau. \tag{6}$$

This eliminates from consideration, in this paper, such forms as

$$ {}_0d_t^{-q(t)} f(t) = \int_0^t \frac{(t-\tau)^{q(t,\tau)(1-(\mathrm{d}q(\tau)/\mathrm{d}\tau))-1}}{\Gamma\left(q(t,\tau)\left(1-\frac{\mathrm{d}q(\tau)}{\mathrm{d}\tau}\right)\right)} f(\tau)\,\mathrm{d}\tau \tag{7}$$

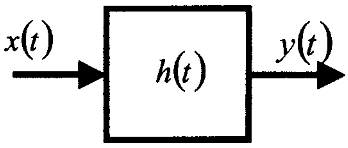

Figure 1. System block diagram.

and many other possible forms which admittedly may be valuable. Some consideration of variable order based on the Grunwald form of the fractional differintegral may be found in [1].

3.3. G2-Initialization of Variable Order Fractional Integrals

We consider an initialization period $a < t < 0$, and require that $f(t) = 0$ for all $t < a$. For terminal initialization we require that, for $t > 0$, the initialized fractional integral starting at $t = 0$ be a continuation of the fractional integral starting at $t = a$, thus

$$ {}_0D_t^{-q(t)} f(t) = {}_aD_t^{-q(t)} f(t), \quad t > 0, \tag{8}$$

or in terms of the uninitialized fractional integral

$$ {}_0d_t^{-q(t)} f(t) + \psi(f, -q, a, 0, t) = {}_ad_t^{-q(t)} f(t). \tag{9}$$

Therefore, we have

$$\int_0^t \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))} f(\tau)\,\mathrm{d}\tau + \psi(f, q(t,\tau), a, 0, t)$$

$$= \int_a^t \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))} f(\tau)\,\mathrm{d}\tau, \quad t > 0, \tag{10}$$

and

$$\psi(f, q(t,\tau), a, 0, t) = \int_a^0 \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))} f(\tau)\,\mathrm{d}\tau, \quad t > 0. \tag{11}$$

Thus it is seen that *regardless of the argument of $q(t,\tau)$, the integrand of the initialization function will always be the same as that of the operator*.

3.4. G3-Convolution Integral and Impulse Response Function

It will be useful in the discussion that follows to draw on the ideas of convolution and impulse response as they relate to linear system theory. Briefly, we consider the block diagram of a linear system (Figure 1) with input $x(t)$, and output $y(t)$. The response of this system to a unit impulse function $\delta(t)$ is $h(t)$ the impulse response function. The input, or forcing function, for this system may be written as an infinite sum of impulse functions, $\delta(t)$,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau. \tag{12}$$

Now for a linear system, superposition holds. Then, each of the input impulses in $x(t)$ will excite a response in the output $y(t)$ with a proportional strength and properly referenced in time to give the response

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau. \tag{13}$$

A causal system only responds after an input is applied, thus $h(t) = 0$, for $t < 0$. Furthermore, $h(t-\tau) = 0$ for $\tau > t$, therefore we may express the response as the convolution integral

$$y(t) = \int_{0}^{t} x(\tau)h(t-\tau)\,\mathrm{d}\tau. \tag{14}$$

Much more detailed discussion of these matters may be found in such texts as [12].

3.5. G4-Time Invariance

From conventional linear system theory, with fixed and integer order, it is well known, (and important) that when the impulse response $h(t,\tau) \neq h(t-\tau)$ the system is said to be 'time varying', that is, the coefficients of the derivatives in the associated differential equations are functions of time [12]. Fixed order operators (systems) are said to be *time invariant* if, when an input $f(t)$ produces a response $y(t)$, then the input $f(t+\sigma)$ produces the response $y(t+\sigma)$. Because the result of variable order operation (differintegration) depends on two variables, the input $f(t)$ and the order $q(t)$, this definition is inadequate for variable order operators. The concept will be generalized for the domain of variable order operators.

A variable order operator will be said to be *time invariant* if an input $f(t)$ and order profile $q(t)$ produce a response $y(t)$, then the input $f(t+\sigma)$ and an order profile $q(t+\sigma)$ will produce the response $y(t+\sigma)$.

3.6. G5-Linearity

For the form above, Equation (6), *all the operators are linear with respect to the input $f(t)$*, that is,

$$\int_{0}^{t} \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))}(af(\tau)+bg(\tau))\,\mathrm{d}\tau$$
$$= a\int_{0}^{t} \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))}f(\tau)\,\mathrm{d}\tau + b\int_{0}^{t} \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))}g(\tau)\,\mathrm{d}\tau. \tag{15}$$

It is noted that the operators are not in general linear with respect to the order $q(t,\tau)$, (i.e. $q(t)$ as an input). Hence, we will *not* in general have the composition property, i.e.

$${}_0D_t^{-u(t)}\,{}_0D_t^{-v(t)}f(t) = {}_0D_t^{-u(t)-v(t)}f(t). \tag{16}$$

We will find that this is a peculiar aspect of variable order operators; it appears they can simultaneously be both linear and non-linear!

4. Toward Variable Order Operator Definitions: Variable Order Integration

4.1. Kernels and Arguments

We start by examining several Riemann–Liouville based variable order integrator definitions. For fixed fractional order integration we have for the uninitialized fractional integral

$$ {}_c d_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_c^t (t-\tau)^{q-1} f(\tau)\,\mathrm{d}\tau. \tag{17}$$

When the order is allowed to vary with time, we write, taking $c = 0$

$$ {}_0 d_t^{-q(t)} f(t) = \int_0^t \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))} f(\tau)\,\mathrm{d}\tau, \tag{18}$$

or alternatively

$$ {}_0 d_t^{-q(t)} f(t) = \int_0^t \frac{\tau^{q^*(t,\tau)-1}}{\Gamma(q^*(t,\tau))} f(t-\tau)\,\mathrm{d}\tau, \tag{19}$$

where $q^*(t,\tau) = q(t, t-\tau)$.

In the discussion that follows we will look at the implications that result from various choices for the arguments of $q(t,\tau)$. It appears reasonable to assume that the arguments of q in the exponent of $(t-\tau)$ and in the gamma function, of Equation (18) are the same, and that assumption will be made. The following cases will be considered: $q(t,\tau) = q(t)$, $q(t,\tau) = q(\tau)$, and $q(t,\tau) = q(t-\tau)$. One approach will be to consider a step change in $q(t)$ from one constant value to another and to draw inferences relative to the character of the variable order integration operator based on the behavior of the kernel $h(t,\tau)$, where

$$ h(t,\tau) = \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))}. \tag{20}$$

4.2. Case 1: $q(t,\tau) \Rightarrow q(t)$

4.2.1. *Time Invariance*

For this case we define

$$ {}_0 d_t^{-q(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(t)-1}}{\Gamma(q(t))} f(\tau)\,\mathrm{d}\tau. \tag{21}$$

The associated impulse response here is

$$ h(t,\tau) = \frac{(t-\tau)^{q(t)-1}}{\Gamma(q(t))}. \tag{22}$$

Because the argument of q in the exponent of $(t-\tau)$ is not a function of $(t-\tau)$ it is seen that $h(t,\tau) \neq h(t-\tau)$. Therefore, the associated dynamic system (the operator) is ***not*** *time invariant.*

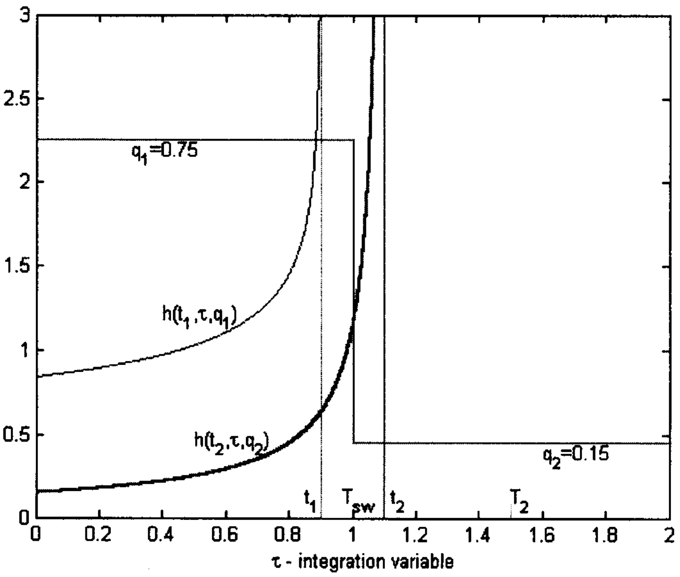

Figure 2. Kernel behavior for $q(t, \tau) = q(t)$ based definition.

We now limit consideration to $0 \le q \le 1$, and write

$$h(t, \tau) = \frac{1}{\Gamma(q(t))(t - \tau)^{1-q(t)}}. \tag{23}$$

To examine the behavior of this definition we consider a step in $q(t)$ at $t = T_{sw}$ (Figure 2), and examine $h(t, \tau)$ in the vicinity of the step. From this figure it can be seen that ${}_0d_t^{-q(t)} f(t)$ under this definition immediately 'forgets' that, from time $t = 0$ to time $t = t_1$ that the impulse response (kernel) was

$$h(t_1, \tau) = \frac{(t_1 - \tau)^{q_1-1}}{\Gamma(q_1)}$$

and switches with no 'memory' of q_1 to

$$h(t_2, \tau) = \frac{(t_2 - \tau)^{q_2-1}}{\Gamma(q_2)} \quad \text{for } t_2 > T_{sw}.$$

Specifically all 'memory' of q_1 is lost after a change in q.

4.2.2. *Initialized Operator*

Now for starting time T_2 with $t \ge T_2 \ge T_{sw} > 0$, and with $f(t) = 0$, $\forall t < 0$ we have from the general property G2

$$ {}_{T_2}D_t^{-q(t)} f(t) = \int_{T_2}^{t} \frac{(t - \tau)^{q(t)-1}}{\Gamma(q(t))} f(\tau)\, \mathrm{d}\tau + \psi(f, -q(t), 0, T_2, t), \quad t > T_2, \tag{24}$$

or

$$ {}_{T_2}D_t^{-q(t)} f(t) = \int_{T_2}^{t} \frac{(t - \tau)^{q(t)-1}}{\Gamma(q(t))} f(\tau)\, \mathrm{d}\tau + \int_{0}^{T_2} \frac{(t - \tau)^{q(t)-1}}{\Gamma(q(t))} f(\tau)\, \mathrm{d}\tau, \quad t > T_2, \tag{25}$$

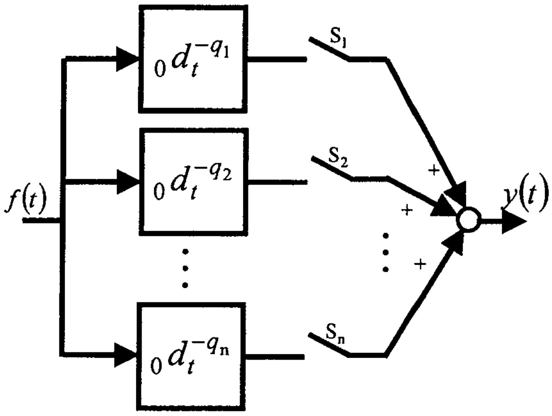

Figure 3. Block diagram for physical realization of $q(t, \tau) = q(t)$ based definition.

For the $q(t)$ profile chosen this becomes

$$_{T_2}D_t^{-q(t)} f(t) = \int_{T_2}^{t} \frac{(t-\tau)^{q_2-1}}{\Gamma(q_2)} f(\tau)\,\mathrm{d}\tau + \int_{0}^{T_2} \frac{(t-\tau)^{q_2-1}}{\Gamma(q_2)} f(\tau)\,\mathrm{d}\tau, \quad t > T_2. \tag{26}$$

Clearly after the step change in $q(t)$ all 'memory' of q_1 is lost. That is, even the initialization function, which is the last integral in Equation (26), does not 'remember' q_1. In fact, all initialization is as if the operator (system) was always operating at the current value of q. Specifically, the properties of the operator shift instantaneously with $q(t)$ and behave as though it has always been operating with the current value of q.

4.2.3. *Physical Realization*

We consider q to be a general function of time, namely $q = q(t)$. We start by allowing q to change in a step-wise manner, more specifically, in a piece-wise constant manner that approximates $q(t)$. The block diagram in Figure 3 may be used to represent this form of time varying fractional integration. Here, the switches, S_i, associated with each constant q_i integration, are normally open and are sequentially closed for duration T (initially finite), then opened again. In this realization, all of the fractional integrators start at time $t = 0$, run simultaneously and in isolation of the others.

The following then applies:

Order	Switch	Interval closed
q_1	S_1	$0 < t < T$
q_2	S_2	$T < t < 2T$
q_3	S_3	$T < t < 3T$
$\vdots$	$\vdots$	$\vdots$
q_n	S_n	$(n-1)T < t < nT$

(27)

The order of fractional integration proceeds as $q_1, q_2, q_3, \ldots$, that is, as a sequence of piece-wise constant order fractional integrations. The q_i approximate $q(t)$ as indicated in Figure 4. The approach therefore is to fractionally integrate first $f(t)$ at order q_1, then at order q_2, and so on.

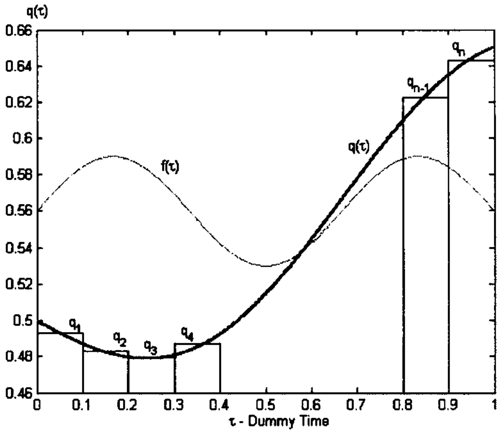

Figure 4. Approximation of $q(\tau)$, $t = \tau = 1$.

The output of the process, $y(t)$, then is given by

$$y(t) = \begin{cases} {}_0d_t^{-q_1} f(t), & 0 < t < T, \\ {}_0d_t^{-q_2} f(t), & T < t < 2T, \\ {}_0d_t^{-q_3} f(t), & 2T < t < 3T, \\ \vdots & \vdots \\ {}_0d_t^{-q_n} f(t), & (n-1) < t < nT. \end{cases} \tag{28}$$

Thus for any particular increment we have

$$y(t) = {}_0d_t^{-q_k} f(t), \quad (k-1)T < t < kT, \tag{29}$$

or

$$y(t) = \int_0^t \frac{(t-\tau)^{q_k - 1}}{\Gamma(q_k)} f(\tau)\,\mathrm{d}\tau, \quad (k-1)T < t < kT. \tag{30}$$

It can be seen from Equation (28) that the final value of q is used at any point in time, t. Also, from the block diagram (Figure 3), it is apparent that during the time that any switch S_i is closed, the output $y(t)$ is as if q_i had been in effect from the beginning. That is, $q = q_k$, for $(k-1)T < t < kT$, $\forall k$. We now take the limit of Equation (30) as $T \to 0$, then the stepped $q_k \to q(t)$, approach the original function $q(t)$, and on the left-hand side of the equation, the output of the process $y(t) \to {}_0d_t^{q(t)} f(t)$. On the right-hand side as $T \to 0$, $n \to \infty$ such that for $k = n$, $nT = t$ and we have as the definition for variable order integration for this type of process

$$ {}_0d_t^{-q(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(t)-1}}{\Gamma(q(t))} f(\tau)\,\mathrm{d}\tau, \quad t > 0. \tag{31}$$

With the initialization included we have

$$ {}_0D_t^{-q(t)} f(t) \equiv {}_0d_t^{-q(t)} f(t) + \psi(f, -q(t), a, 0, t), \quad t > 0, \tag{32}$$

where

$$\psi(f, -q(t), a, 0, t) = \int_a^0 \frac{(t-\tau)^{q(t)-1}}{\Gamma(q(t))} f(\tau)\, \mathrm{d}\tau, \quad t > 0. \tag{33}$$

The principal characteristic of this definition is its distorted ‘memory’, it always remembers the past as if the order of the operator is q at the current time, that is, it has no q memory.

4.2.4. *Operational Method:* $q(t,\tau) \Rightarrow q(t)$

Our interest here is to present a Laplace transform based procedure for dealing with variable order fractional operators based on the definition using $q(t,\tau) \to q(t)$, i.e. those based on Equation (31). All integrators (Figure 3) operate continuously on the common input $f(t)$. The result of the individual (fixed order) integrations we will call $x(q_i, t)$. We observe

$$x(q_i, t) = {}_0d_t^{-q_i(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q_i-1}}{\Gamma(q_i)} f(\tau)\, \mathrm{d}\tau, \quad 0 \le q(t) \le m \tag{34}$$

and

$$L\{x(q_i, t)\} = s^{-q_i} F(s). \tag{35}$$

In the limiting process the number of integrations becomes an infinite continuum and the combined result may be viewed as a solution surface $x(q(t), t)$ that is progressively sampled as $q(t)$ increases. The solution then for the variable $q(t)$ case is a path defined on this surface by $q(t)$ which may be considered as a function on a new dimension t_1. Because this definition has no memory of past values of q we consider two time bases; t as applies in Equation (34), and t_1 for the time varying order $q(t_1)$. This allows us to describe the behavior of Equation (34) using an iterated Laplace transform. Specifically, we have

$$L_{s_1}\{L_s\{{}_0d_t^{-q(t_1)} f(t)\}\} = L_{s_1}\{s^{-q(t_1)} F(s)\}, \tag{36}$$

$$L_{s_1}\{L_s\{{}_0d_t^{-q(t_1)} f(t)\}\} = L\{s^{-q(t_1)}\} F(s), \quad 0 \le q(t) \le m. \tag{37}$$

In this formulation the iterated transform serves to order the procedure and after inverse transforming we will set $t_1 = t$.

EXAMPLE 1. Consider the variable order fractional differential equation

$${}_0d_t^{(a+bt)} h^*(t) = U(t), \quad a > 0, b > 0, \tag{38}$$

where it is desired to find the function $h^*(t)$ which when fractionally differentiated to the variable order $a + bt$ yields the unit step function. Neglecting initialization, and remembering that *composition* does not apply we will instead solve a related integral equation. Using the dual time base, we write

$$h(t) = {}_0d_t^{-(a+bt_1)} U(t). \tag{39}$$

Then transforming we have

$$L_{s_1} L_s\{h(t)\} = L_{s_1}[L_s\{{}_0d_t^{-(a+bt_1)}\} U(t)]. \tag{40}$$

Using Equation (37) above

$$L_{s_1}L_s\{h(t)\} = \frac{1}{s}L_{s_1}[s^{-(a+bt_1)}] = \frac{1}{s^{1+a}}L_{s_1}[\mathrm{e}^{-b[\ln(s)]t_1}], \tag{41}$$

$$L_{s_1}L_s\{h(t)\} = \frac{1}{s^{1+a}}\frac{1}{s_1 + b\ln(s)}, \quad \operatorname{Re} s_1 > -\operatorname{Re}(b\ln(s)). \tag{42}$$

Inverse transforming first with respect to s_1 yields

$$L_s\{h(t)\} = \frac{1}{s^{1+a}}\mathrm{e}^{-b(\ln s)t_1} = \frac{s^{-bt_1}}{s^{1+a}} = s^{-(1+a+bt_1)}, \quad \operatorname{Re} s_1 > -\operatorname{Re}(b\ln(s)). \tag{43}$$

Inverse transforming with respect to s, and replacing t_1 by t, yields

$$h(t) = \frac{t^{a+bt_1}}{\Gamma(1+a+bt_1)} \to \frac{t^{a+bt}}{\Gamma(1+a+bt)}, \quad 1+a+bt > 0. \tag{44}$$

4.3. Case 2: $q(t,\tau) = q(\tau)$

4.3.1. *Time Invariance*

For this case the variable order fractional integral is defined as

$${}_0d_t^{-q(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} f(\tau)\,\mathrm{d}\tau. \tag{45}$$

The associated impulse response here is

$$h(t,\tau) = \frac{(t-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))}. \tag{46}$$

Because of the argument of q in both the exponent of $(t-\tau)$ and in the gamma function it is seen that $h(t,\tau) \neq h(t-\tau)$. Therefore, the associated dynamic system (the operator) is ***not*** *time invariant*. Limiting consideration to $0 \le q \le 1$, we write

$$h(t,\tau) = \frac{1}{\Gamma(q(\tau))(t-\tau)^{1-q(\tau)}}. \tag{47}$$

As in the previous case the behavior of this definition is studied by consideration of a step in $q(t)$ (Figure 5), and examination of $h(t,\tau)$ in the vicinity of the step at $t = T_{sw}$. For $q(t)$ as shown in the figure, we have for $t = t_1 < T_{sw}$

$$h_1(t,\tau) = \frac{1}{\Gamma(q_1)(t_1-\tau)^{1-q_1}}, \quad 0 < \tau < t = t_1, \tag{48}$$

and for $t = t_2 > T_{sw}$

$$h_2(t,\tau) = \begin{cases} \dfrac{1}{\Gamma(q_1)(t_2-\tau)^{1-q_1}}, & 0 < \tau < T_{sw_2}, \\ \dfrac{1}{\Gamma(q_2)(t_2-\tau)^{1-q_2}}, & T_{sw} < \tau < t = t_2. \end{cases} \tag{49}$$

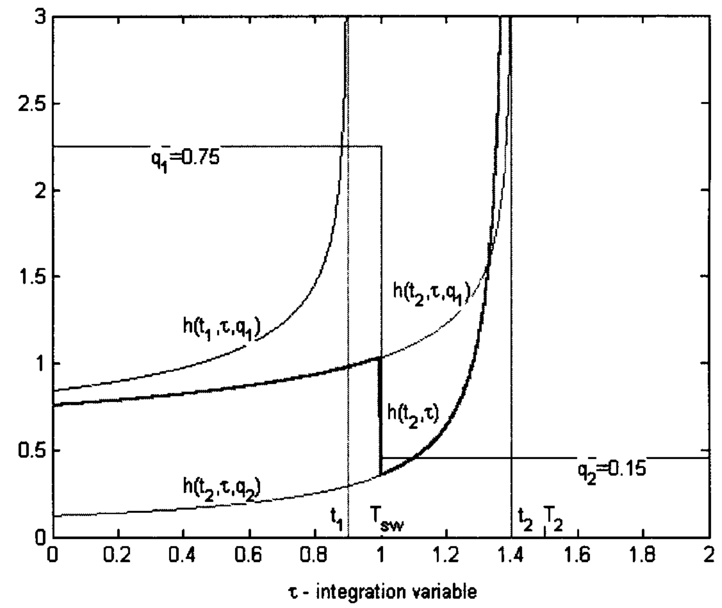

Figure 5. Kernel behavior for $q(t, \tau) = q(\tau)$ based definition.

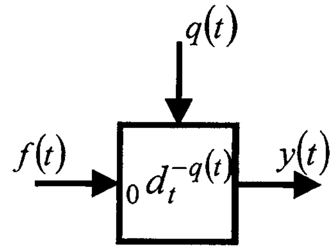

Figure 6. Variable order operator block diagram.

It can be seen that for $t > T_{sw}$ the impulse response function is segmented, i.e., it is discontinuous at T_{sw}, as is $q(t)$. This definition 'remembers' the past q, namely q_1 in this case. It is also apparent that memory of q_1 decreases (relatively) as $t > T_{sw}$ increases. This behavior is a key difference when this definition is contrasted with the definition of Case 1.

It is important to observe that under this definition both f and q are functions of τ, i.e. $f(\tau)$ and $q(\tau)$. When the operator is viewed as a dynamic system (block diagram of Figure 6) it is not unreasonable to expect the 'inputs' to have the same argument. It is interesting that we have here, and for all variable order operators, three terminal block diagram elements that explicitly contain neither an addition nor a multiplication.

4.3.2. *Initialized Operator*

From property G2, and for a starting time $T_2 > T_{sw} > 0$ and for $f(t) = 0, \forall t < 0$, we have

$$ {}_{T_2}D_t^{-q(t)} f(t) = \int_{T_2}^{t} \frac{(t-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} f(\tau)\, d\tau + \psi(f, -q(t), 0, T_2, t), \quad t > T_2, \tag{50} $$

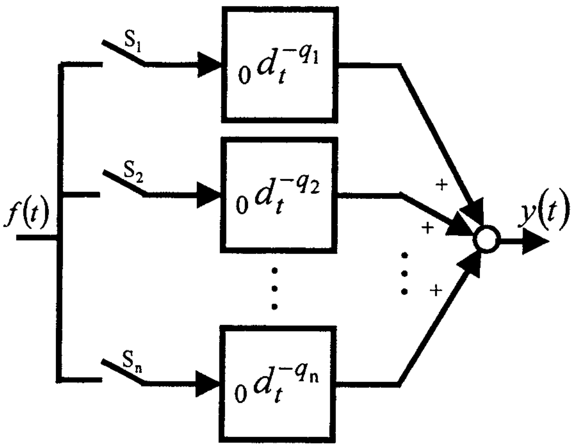

Figure 7. Block diagram for physical realization of $q(t, \tau) = q(\tau)$ based definition.

where

$$\psi(f, -q(t), 0, T_2, t) = \int_0^{T_2} \frac{(t-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} f(\tau)\,\mathrm{d}\tau.$$

For the $q(t)$ profile chosen this becomes

$$\begin{aligned} {}_{T_2}D_t^{-q(t)} f(t) &= \int_{T_2}^{t} \frac{(t-\tau)^{q_2-1}}{\Gamma(q_2)} f(\tau)\,\mathrm{d}\tau + \int_0^{T_{sw}} \frac{(t-\tau)^{q_1-1}}{\Gamma(q_1)} f(\tau)\,\mathrm{d}\tau \\ &\quad + \int_{T_{sw}}^{T_2} \frac{(t-\tau)^{q_2-1}}{\Gamma(q_2)} f(\tau)\,\mathrm{d}\tau, \quad t > T_2. \end{aligned} \tag{51}$$

The sum of the last two integrals here is the initialization function for ${}_{T_2}D_t^{-q(t)} f(t)$.

4.3.3. *Physical Realization*

We again consider q to be a general function of time, namely $q = q(t)$. Again, we start by allowing q to change in a step-wise manner, more specifically, in a piece-wise constant manner that approximates $q(t)$. The block diagram (Figure 7) is used to represent this form of time varying fractional integration. The switches, S_i, associated with each constant q_i integration, are normally open and are sequentially closed for duration T (initially finite) then opened again. The switches in this case are on the input to the individual operators, and have the effect of parsing the dynamic input $f(t)$. In this realization, all of the fractional integrators operate continuously from time $t = 0$ until time $t = t$, that is, the end of the integration period. It is also noted that the fractional integrators continue to generate output even after the parsed input function $f(t)$ becomes zero. The outputs are summed to form $y(t)$.

The following then applies:

Order	Switch	Interval closed
q_1	S_1	$0 < t < T$
q_2	S_2	$T < t < 2T$
q_3	S_3	$2T < t < 3T$
$\vdots$	$\vdots$	$\vdots$
q_n	S_n	$(n-1)T < t < nT$

(52)

The order of fractional integration proceeds as $q_1, q_2, q_3, \ldots$, that is, as a sequence of piecewise constant order fractional integrations. The q_i approximate $q(t)$ as indicated in Figure 4. The approach, therefore, is to fractionally integrate first $f(t)[U(t) - U(t-T)]$ at order q_1, then $f(t)[U(t-T) - U(t-2T)]$ at order q_2, and so on.

The output of the process, $y(t)$, then is given by

$$\begin{aligned} y(t) \;=\; & {}_0d_t^{-q(T)}[f(t)((u(t) - u(t-T)))] \\ & + {}_0d_t^{-q(2T)}[f(t)((u(t-T) - u(t-2T)))] + \cdots \\ & + {}_0d_t^{-q(nT)}[f(t)((u(t-(n-1)T) - u(t-nT)))], \end{aligned} \tag{53}$$

More briefly

$$y(t) = \sum_{i=1}^{n} {}_0dt^{-q(iT)}[f(t)\{u(t-(i-1)T) - u(t-iT)\}], \tag{54}$$

or

$$y(t) = \sum_{i=1}^{n} \int_0^t \frac{(t-\tau)^{q(iT)-1}}{\Gamma(q(iT))}[f(\tau)\{u(\tau-(i-1)T) - u(\tau-iT)\}]\,\mathrm{d}\tau. \tag{55}$$

Interchanging integration and summation gives

$$y(t) = \int_0^t \sum_{i=1}^{n} \frac{(t-\tau)^{q(iT)-1}}{\Gamma(q(iT))}\left[f(\tau)\frac{\{u(\tau-(i-1)T) - u(\tau-iT)\}}{T}T\right]\mathrm{d}\tau. \tag{56}$$

We now take the limit of this equation as $n \to \infty$. On the right-hand side $T \to 0$ such that $t = nT$. For clarity we introduce the dummy variable σ, and $iT \to \sigma$, $T \to \mathrm{d}\sigma$ and $f(\tau) \to f(iT) = f(\sigma)$, $q(iT) \to q(\sigma)$ and the quantity

$$\frac{\{u(\tau-(i-1)T) - u(\tau-iT)\}}{T} \to \delta(\tau-\sigma)$$

the unit impulse function and we have

$$y(t) = \int_0^t \int_0^t \frac{(t-\tau)^{q(\sigma)-1}}{\Gamma(q(\sigma))} f(\sigma)\delta(\tau-\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{57}$$

Through the limiting process the left-hand side becomes the desired variable order operator and we have the definition for this case as

$${}_0d_t^{-q(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} f(\tau)\,\mathrm{d}\tau. \tag{58}$$

4.3.4. *Operational Method*

Here we seek an operational method for solving equations (systems) with variable order fractional integrals based on the definition

$$ {}_0d_t^{q(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(\tau)-1}}{\Gamma(q(\tau))} f(\tau)\,\mathrm{d}\tau. \tag{59}$$

It is useful to refer to the block diagram in Figure 7. Conceptually we will take the finite array of fixed order integrators of order q_i, $i = 1, 2, \ldots, n$ to the limit to become an infinite continuum of integrators. That is, an infinite array of fixed order fractional integrators of order $q_\lambda = q(\lambda)$ where λ is allowed to take on all real values $0 \le \lambda \le m < \infty$. We now convert $f(t)$ into a λ distributed input, thus we define

$$ x(\lambda, t) = f(\lambda)\delta(t-\lambda), \tag{60}$$

as the input directly into the integrator (after the switches), and observe

$$ x(\lambda, t) = \begin{cases} f(t), & \lambda = t, \\ 0, & \lambda = 0. \end{cases} \tag{61}$$

We further note that

$$ f(t) = \int_{\lambda_0}^{\lambda_m} x(\lambda, t)\delta(\lambda - t)\,\mathrm{d}\lambda. \tag{62}$$

The output of the individual integrations, $\Theta(\lambda, t)$, is

$$ \Theta(\lambda, t) = {}_0d_t^{-q(\lambda)} x(\lambda, t), \tag{63}$$

and the integration over λ, of these outputs, will be

$$ {}_0d_t^{-q(t)} f(t) = \int_{\lambda_0}^{\lambda_m} \Theta(\lambda, t)\,\mathrm{d}\lambda = \int_{\lambda_0}^{\lambda_m} {}_0d_t^{-q(\lambda)} x(\lambda, t)\,\mathrm{d}\lambda, \tag{64}$$

$$ {}_0d_t^{-q(t)} f(t) = \int_{\lambda_0}^{\lambda_m} {}_0d_t^{-q(\lambda)} f(\lambda)\delta(t-\lambda)\,\mathrm{d}\lambda. \tag{65}$$

The Laplace transform with respect to t, with Laplace variable s, then is

$$ \begin{aligned} L_s\{{}_0d_t^{-q(t)} f(t)\} &= L_s\left\{ \int_{\lambda_0}^{\lambda_m} {}_0d_t^{-q(\lambda)} f(\lambda)\delta(t-\lambda)\,\mathrm{d}\lambda \right\} \\ &= \int_0^\infty \mathrm{e}^{-st} \int_{\lambda_0}^{\lambda_m} {}_0d_t^{-q(\lambda)} f(\lambda)\delta(t-\lambda)\,\mathrm{d}\lambda\,\mathrm{d}t. \end{aligned} \tag{66}$$

Under appropriate conditions the order of integration is interchanged giving

$$L_s\{{}_0d_t^{-q(t)} f(t)\} = \int_{\lambda_0}^{\lambda_m} f(\lambda) \int_0^{\infty} \mathrm{e}^{-st}\ {}_0d_t^{-q(\lambda)} \delta(t-\lambda)\, \mathrm{d}t\, \mathrm{d}\lambda, \tag{67}$$

$$L_s\{{}_0d_t^{-q(t)} f(t)\} = \int_{\lambda_0}^{\lambda_m} f(\lambda) s^{-q(\lambda)}\, \mathrm{e}^{-\lambda s}\, \mathrm{d}\lambda. \tag{68}$$

At this point there are two possible paths, the first is to recognize that λ is a secondary time base that we will call t_1 and write the time varying Laplace transform

$$L_s\{{}_0d_t^{-q(t)} f(t)\} = \int_0^{t_1} f(\lambda) s^{-q(\lambda)}\, \mathrm{e}^{-\lambda s}\, \mathrm{d}\lambda. \tag{69}$$

In application of Equation (69) the order of solution is: inverse transform with respect to s, integrate over λ, and replace t_1 by t.

The second approach, is to take the Laplace transform with respect to λ of Equation (68), taking $\lambda_0 = 0$, this gives

$$L_{s_1}\{L_s\{{}_0d_t^{-q(t)} f(t)\}\} = \frac{1}{s_1} L_{s_1}\{f(\lambda) s^{-q(\lambda)}\, \mathrm{e}^{-\lambda s}\}, \tag{70}$$

$$L_{s_1}\{L_s\{{}_0d_t^{-q(t)} f(t)\}\} = \frac{1}{s_1} \int_0^{\infty} f(\lambda) s^{-q(\lambda)}\, \mathrm{e}^{-\lambda(s+s_1)}\, \mathrm{d}\lambda. \tag{71}$$

In application of Equation (71) the order of solution is: inverse transform with respect to s, integrate over λ, inverse transform with respect to s_1 and replace t_1 by t.

EXAMPLE 2. We now consider the same variable order fractional integral Equation (39) of the previous case, however now with the definition based on $q(t,\tau) = q(\tau)$. Thus we seek to find the function $h(t)$, i.e., which is the variable $(a + bt)$ order fractional integral of the unit step function. We neglect initialization, and using the dual time base, we write the integral equation

$$h(t) = {}_0d_t^{-(a+bt_1)} U(t). \tag{72}$$

Then applying Equation (69) we have

$$L_s\{{}_0dt^{-(a+bt)} U(t)\} = L_s\{h(t)\} = \int_0^{t_1} U(\lambda) s^{-(a+b\lambda)}\, \mathrm{e}^{-\lambda s}\, \mathrm{d}\lambda, \tag{73}$$

$$L_s\{h(t)\} = \frac{1}{s^a} \int_0^{t_1} \mathrm{e}^{-(b\ln(s)+s)\lambda}\, \mathrm{d}\lambda, \tag{74}$$

$$L_s\{h(t)\} = \frac{1}{s^a}\left[\frac{1 - \mathrm{e}^{-t_1(s+b\ln(s))}}{s + b\ln(s)}\right]. \tag{75}$$

This result should be compared with that of the previous case Equation (43), the forms are found to be substantially different. Then the solution is given by

$$h(t) = \left[L_s^{-1}\left\{\frac{1 - \mathrm{e}^{-t_1(s+b\ln(s))}}{s^a(s + b\ln(s))}\right\}\right]\Bigg|_{t_1\to t}. \tag{76}$$

4.4. Case 3: $q(t,\tau) = q(t-\tau)$

4.4.1. *Time Invariance*

For this case we consider the definition

$$_0d_t^{-q(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(t-\tau)-1}}{\Gamma(q(t-\tau))} f(\tau)\,\mathrm{d}\tau. \tag{77}$$

The associated impulse response here is

$$h(t,\tau) = \frac{(t-\tau)^{q(t-\tau)-1}}{\Gamma(q(t-\tau))}. \tag{78}$$

Because the argument of q in the exponent of $(t-\tau)$ and in the gamma function is of the form $(t-\tau)$ it is seen that $h(t,\tau) = h(\tau-\tau)$. Therefore, the associated dynamic system (the operator) for this case ***is*** *time invariant*, as defined in property G4 earlier.

We now limit consideration to $0 \le q \le 1$, and we write

$$h(t,\tau) = \frac{1}{\Gamma(q(t-\tau))(t-\tau)^{1-q(t-\tau)}}. \tag{79}$$

We again examine the behavior of this definition by considering a step in $q(t)$ (Figure 8) and examining $h(t,\tau)$ in the vicinity of the step at $t = T_{sw}$. For $q(t)$ as shown in the figure, we have for $t = t_1 < T_{sw}$

$$h_1(t,\tau) = \frac{1}{\Gamma(q_1)(t_1-\tau)^{1-q_1}}, \quad 0 < \tau < t = t_1 \tag{80}$$

and for $t = t_2 > T_{sw}$

$$h_2(t,\tau) = \begin{cases} \dfrac{1}{\Gamma(q_2)(t_2-\tau)^{1-q_2}}, & 0 < \tau < t - T_{sw}, \\ \dfrac{1}{\Gamma(q_1)(t_2-\tau)^{1-q_1}}, & T_{sw} < \tau < t = t_2. \end{cases} \tag{81}$$

As in Case 2, it is seen that the impulse response function is segmented, i.e., it is discontinuous at $t_2 - T_{sw}$. This definition very strongly ‘remembers’ the past q, namely q_1 in this case. This effect is so strong that if we consider the integral at time $t = t_2$ where $t_2 = T_{sw} + \varepsilon$, for small ε, we have the situation where even though q has switched from q_1 to q_2 the variable order integral is almost completely evaluated at q_1! This is a key difference when this definition is contrasted with the definitions of the previous cases.

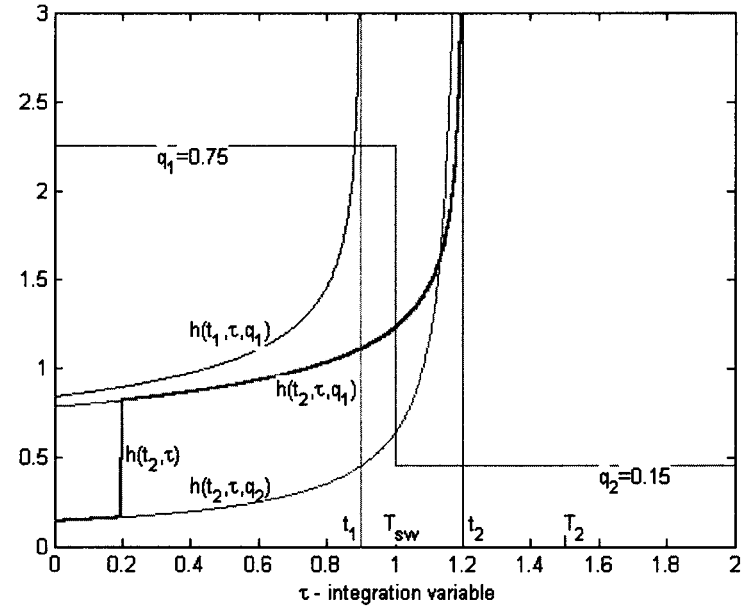

Figure 8. Kernel behavior for $q(t, \tau) = q(t - \tau)$ based definition.

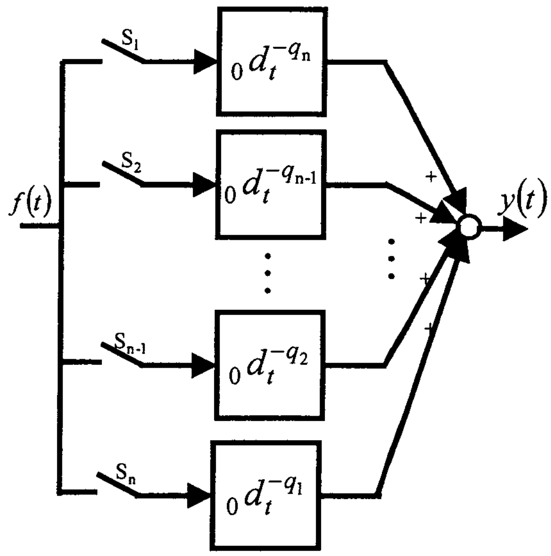

Figure 9. Block diagram for physical realization of $q(t, \tau) = q(t - \tau)$ based definition.

4.4.2. *Physical Realization*

We again consider q to be a general function of time, namely $q = q(t)$. Again, we start by allowing q to change in a step-wise manner, more specifically, in a piece-wise constant manner that approximates $q(t)$. The block diagram (Figure 9) is used to represent this form of time varying fractional integration. The switches, S_i, are now each associated with a constant q_{n-i} integration. It is seen that the sequence of integration now runs backward relative to the switching of the input function $f(t)$. This necessitates knowing the entire $q(t)$ profile in advance. The switches are normally open and are sequentially closed for duration T (initially finite), then opened again.

The switches in this case are on the input to the individual operators, and have the effect of parsing the dynamic input $f(t)$. In this realization, all of the fractional integrators operate continuously from time $t = 0$ until time $t = t$, that is, the end of the integration period. It is also noted that the fractional integrators continue to generate output even after the parsed input function $f(t)$ becomes zero. The outputs are summed to form $y(t)$.

The following then applies:

Order	Switch	Interval closed
q_n	S_1	$0 < t < T$
q_{n-1}	S_2	$T < t < 2T$
q_{n-2}	S_3	$2T < t < 3T$
$\vdots$	$\vdots$	$\vdots$
q_2	S_{n-1}	$(n-2)T < t < (n-1)T$
q_1	S_n	$(n-1)T < t < nT$

(82)

The order of fractional integration proceeds as $q_n, q_{n-1}, q_{n-2}, \ldots$, that is, as a reversed sequence of piecewise constant order fractional integrations. The q_i approximate $q(t)$. The approach therefore is to fractionally integrate first $f(t)[U(t) - U(t-T)]$ at order q_n, then $f(t)[U(t-T) - U(t-2T)]$ at order q_{n-1}, and so on.

The output of the process, $y(t)$, then is given by

$$\begin{aligned} y(t) \;=\; & {}_0d_t^{-q(nT)}[f(t)((u(t) - u(t-T)))] \\ & + {}_0d_t^{-q((n-1)T)}[f(t)((u(t-T) - u(t-2T)))] + \cdots \\ & + {}_0d_t^{-q(T)}[f(t)((u(t-(n-1)T) - u(t-nT)))]. \end{aligned} \tag{83}$$

More briefly

$$y(t) = \sum_{i=0}^{n-1} {}_0d_t^{-q((n-i)T)}[f(t)\{u(t-iT) - u(t-(i+1)T)\}], \tag{84}$$

or

$$y(t) = \sum_{i=1}^{n} \int_0^t \frac{(t-\tau)^{q((n-i)T)-1}}{\Gamma(q((n-i)T))}[f(\tau)\{u(\tau - iT) - u(\tau-(i+1)T)\}]\,\mathrm{d}\tau. \tag{85}$$

Interchanging integration and summation gives

$$y(t) = \int_0^t \sum_{i=1}^{t} \frac{(t-\tau)^{q((n-i)T)-1}}{\Gamma(q((n-i)T))}\left[f(\tau)\frac{\{u(\tau - iT) - u(\tau-(i+1)T)\}}{T}T\right]\mathrm{d}\tau. \tag{86}$$

We now take the limit of this equation as $n \to \infty$. On the right-hand side $T \to 0$ such that $t = nT$, for clarity we introduce the dummy variable σ, and $iT \to \sigma$, $T \to \mathrm{d}\sigma$, and $f(\tau) \to f(iT) = f(\sigma)$, $q(iT) \to q(\sigma)$ and the quantity

$$\frac{\{u(\tau - iT) - u(\tau-(i+1)T)\}}{T} \to \delta(\tau - \sigma)$$

the unit impulse function and we have

$$y(t) = \int_0^t \int_0^t \frac{(t-\tau)^{q(t-\sigma)-1}}{\Gamma(q(t-\sigma))} f(\sigma)\delta(\tau-\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{87}$$

Through the limiting process the left-hand side becomes the desired variable order operator and we have the definition for this case as

$$_0d_t^{-q(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(t-\tau)-1}}{\Gamma(q(t-\tau))} f(\tau)\,\mathrm{d}\tau. \tag{88}$$

It is apparent from this realization that the $(t-\tau)$ argument of q effectively causes q to run backwards in τ; that is, the entire $q(t)$ profile must be known *a priori*. Thus, for this definition, $q(t)$ appears to behave as an integral part of the kernel not as an 'input' to the operator.

4.4.3. *Initialized Operator*

Directly applying the G2 property, for the starting time $t = 0 > a$, and for $f(t) = 0, \forall t < a$ the initialized operator is given by

$$_0D_t^{-q(t)} f(t) \equiv {}_0d_t^{-q(t)} f(t) + \psi(f, -q(t), a, 0, t), \quad t > 0, \tag{89}$$

or

$$_0D_t^{-q(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(t-\tau)-1}}{\Gamma(q(t-\tau))} f(\tau)\,\mathrm{d}\tau + \int_a^0 \frac{(t-\tau)^{q(t-\tau)-1}}{\Gamma(q(t-\tau))} f(\tau)\,\mathrm{d}\tau, \quad t < 0. \tag{90}$$

Then for the test condition associated with Figure 8, with the starting point of the integration at $T_2 \geq T_{sw} > 0$, and for $f(t) = 0, \forall t < 0$, we have for $t \geq T_2$

$$\begin{aligned} {}_{T_2}D_t^{-q(t)} f(t) \equiv & \int_{T_2}^t \frac{(t-\tau)^{q(t-\tau)-1}}{\Gamma(q(t-\tau))} f(\tau)\,\mathrm{d}\tau + \int_0^{t+T_{sw}} \frac{(t-\tau)^{q_2-1}}{\Gamma(q_2)} f(\tau)\,\mathrm{d}\tau \\ & + \int_{t-T_{sw}}^{T_2} \frac{(t-\tau)^{q_1-1}}{\Gamma(q_1)} f(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{91}$$

where the sum of the last two integrals is the initialization function.

4.4.4. *Laplace Transform*

The derivation of the Laplace transform of the variable order integral follows that for the fixed order case, since the convolution theorem can be applied. Then, considering the uninitialized case, i.e., $_0d_t^{-q(t)} f(t) = {}_0D_t^{-q(t)} f(t)$,

$$L\{_0D_t^{-q(t)} f(t)\} = \int_0^\infty \mathrm{e}^{-st} \left(\int_0^t \frac{(t-\tau)^{q(t-\tau)-1}}{\Gamma(q(t-\tau))} f(\tau)\,\mathrm{d}\tau \right) \mathrm{d}t, \quad q(t) > 0, t > 0, \tag{92}$$

then the Laplace convolution theorem is given by

$$L\{u(t) * v(t)\} = U(s)V(s) = L\left(\int_0^t v(t-\tau)u(\tau)\,\mathrm{d}\tau\right). \tag{93}$$

Now taking $u(t) = f(t)$ and $v(t) = t^{q(t)-1}/\Gamma(q(t)) = h(t)$ the convolution theorem yields

$$L\{{}_0D_t^{-q(t)} f(t)\} = U(s)V(s) = L\{f(t)\}L\left\{\frac{t^{q(t)-1}}{\Gamma(q(t))}\right\}. \tag{94}$$

It is important to observe that under this definition q is fully associated with the kernel impulse function i.e., $q = q(t-\tau)$, that is, the argument of q is the same as the $(t-\tau)$ term that it exponentiates. When the operator is viewed as a dynamic system (block diagram of Figure 6) it is not unreasonable to expect the 'inputs' to have the same argument. This is the case when $q(t,\tau) = q(\tau)$ but not so here.

4.5. Alternate Approach

4.5.1. *Variable Order Operator Based on Laplace Transform*

The following approach uses an extension of the Laplace transform for fixed order operators to infer a possible definition for the variable order operator. Neglecting initialization (or initial condition terms), and assuming functions of exponential order, we note the following

$$\begin{aligned}
L\left\{\int_0^t \ldots \int_0^{t_{n-1}} f(t_n)\,\mathrm{d}t_n \ldots \mathrm{d}t\right\} &= s^{-n}F(s) \\
&\vdots \\
L\left\{\int_0^t f(t)\,\mathrm{d}t\right\} &= s^{-1}F(s) \\
L\left\{\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right\} &= sF(s) \\
L\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t)\right\} &= s^2F(s) \\
&\vdots \\
L\left\{\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right\} &= s^nF(s).
\end{aligned} \tag{95}$$

Thus, in general we have

$$L\{{}_0d_t^q f(t)\} = s^q F(s), \quad \forall q. \tag{96}$$

Now since $L\{cU(t)\} = c/s$, Equation (96) may also be written as

$$L\{{}_0d_t^q f(t)\} = s^{sL\{q\}} F(s), \quad \forall q. \tag{97}$$

Table 1. Variable order integral characteristics.

Case	a	b	Relative q memory	Response to q change
1	1	0	none	immediate
2	0	1	weak	intermediate
3	1	–1	strong	very slow

In this form q is of course considered as a constant. However, it is now a simple matter to generalize q by considering it to be a function of time. Therefore, a Laplace transform based definition for non-initialized variable order integration and differentiation is

$$ {}_0d_t^{q(t)} f(t) \equiv L^{-1}\{s^{sL\{q(t)\}}F(s)\}, \quad \forall q. \tag{98}$$

If it is assumed that the order requires no initialization then the definition for the initialized operator is given by

$$ {}_0D_t^{q(t)} f(t) \equiv L^{-1}\{s^{sL\{q(t)\}}F(s)\} + \psi(f(t), q(t), 0, a, s), \quad \forall q, t > 0, \tag{99}$$

where we require ${}_0d_t^{q(t)} f(t)$ and $q(t)$ to be of exponential order and piece-wise regular. While this unorthodox definition lacks a clear time domain meaning and realization, it is operationally compelling and may find important application.

5. Characterization of Variable Order Fractional Integrals

5.1. Generalizing the q-Argument

Through the study of the previous cases we have seen the behavior of three time-based possible definitions for the variable order fractional integral operator. We have seen that the operator based on $q(t, \tau) = q(t)$ has no memory of past q, the operator based on $q(t, \tau) = q(\tau)$ has weak memory of past q, and the operator based on $q(t, \tau) = q(t - \tau)$ has very strong memory of past q. It is clear that the response of the operator to changes in q will be inversely related to its memory of past q. These qualitative results are summarized in Table 1, where a generalized operator is defined based on

$$ q(t, \tau) = q(at + b\tau), \tag{100}$$

that is

$$ {}_0d_t^{-q(t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(at+b\tau)-1}}{\Gamma(q(at+b\tau))} f(\tau)\,\mathrm{d}\tau. \tag{101}$$

The summary in Table 1 suggests that the behavior of a variable order fractional integral operator might be ‘tailored’ to a particular application by suitable selection of the constants a and b in the generalized form, Equation (101). The rest of this section will create some tools that will be used to evaluate such definitions and to characterize their behavior. The first task is to determine the useable range for selection of a and b.

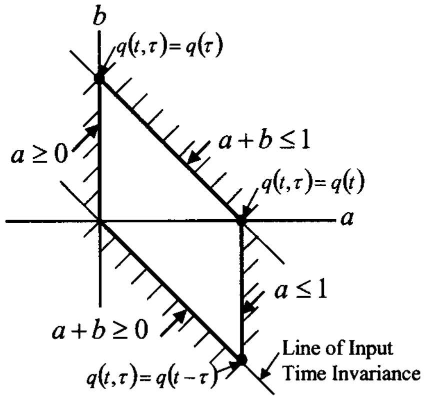

Figure 10. Allowable range of $q(at + b\tau)$ argument.

For the defining uninitialized operator, Equation (101) above, the variable of integration τ, ranges as $0 \le \tau \le t$. Therefore, we require that the argument of q also fall in this range, i.e.,

$$0 \le at + b\tau \le t. \tag{102}$$

If the lower limit is violated q is not defined, violating the upper limit the operator is non-causal.

Examining the boundaries, from $t = \tau$ we have $0 \le (a + b)t \le t$ or $0 \le a + b \le 1$. From the condition $\tau = 0$, we have $0 \le at + 0 \le t$ or $0 \le a \le 1$ which together with the previous condition yields $-1 \le b \le 1$, constraints that are preempted. The acceptable range of a and b is shown in Figure 10. Also shown in this figure are the three cases analyzed previously. It is noted that for certain non-real time applications, such as digital filtering, it may be desirable to relax the upper constraint, i.e., causality, and allow the use of future values of q. This has the effect of removing the upper and right constraints in Figure 10.

5.2. Memories

Two types of memory are considered. The first type, m_1, is the memory associated with the integral operator itself. This is the so-called fading memory effect of the fractional integral when compared to the integer order, $q = 1$, integral. A second type of memory, m_2, is that associated with the change of order as we have indicated above. In the material that follows, the focus will be on variable integrators of order $q(t)$ bounded as $0 \le q(t) \le 1$.

5.2.1. m_1 *Memory*

This type of memory is the well-understood fading memory of the fractional operation and will be discussed only briefly. We first consider the fixed order fractional integral

$$_0d_t^{-q} f(t) = \int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} f(\tau)\,\mathrm{d}\tau. \tag{103}$$

In this integral the past is described mathematically by $(t - \tau)$ in the kernel. When τ is small or zero we are at or near the present, i.e., $t - \tau \to t$, conversely when τ is large or approaches t in magnitude we are in the distant past, i.e., $t - \tau \to 0$. Thus we integrate, fractionally, from the present, $\tau = 0$ or $t - \tau = t$, to some point in the past, $\tau = t$ or $t - \tau = 0$. The form

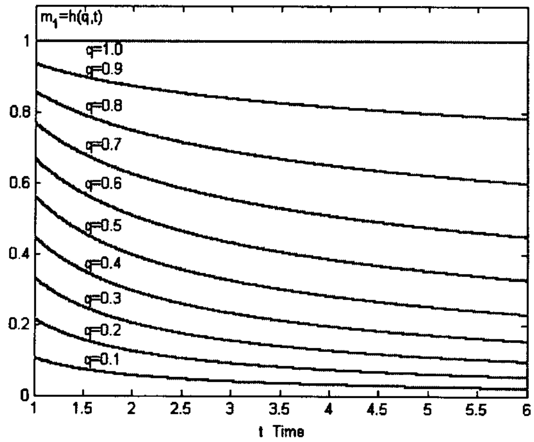

Figure 11. m_1 memory measure *versus* time for fixed order fractional integrals.

of the kernel, through q in the exponent and in the gamma function, determines how the past information is weighted. The response of the integral for a unit impulse function at time $t = 0$ is

$$ {}_0d_t^{-q} f(t) = \int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} \delta(\tau)\,\mathrm{d}\tau = \frac{t^{q-1}}{\Gamma(q)} = h(q, t, \tau = 0), \tag{104}$$

which is the operator impulse response function. The effect of the fading memory may be observed by comparing the response for fractional q, say after $t = 1$, to the integer order case $q = 1$, which remains constant at a value of unity.

The impulse response function may, therefore, be used as an indicator of m_1 type memory. Thus, a primitive measure of the memory of the fixed order integral operator is the shifted response

$$ m_1(t) \equiv \left[\frac{(t)^{q-1}}{\Gamma(q)}\right]_{t\geq 1} = h(q,t)_{t\geq 1}, \quad 0 \leq q \leq 1, t \geq 1. \tag{105}$$

This measure is shown for the fixed order fractional integral in Figure 11. When the order is variable with t (time), this measure shows a variation with $q(t)$. The following is an alternative measure for the m_1 type of memory.

Here we observe, for $f(t) = U(t)$ a unit step at $t = 0$, that the ratio

$$ m_1(t) \equiv \frac{\int_0^t \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))} U(\tau)\,\mathrm{d}\tau}{\int_0^t U(\tau)\,\mathrm{d}\tau} = \frac{1}{t} \int_0^t \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))} \mathrm{d}\tau, \quad 0 \leq q(t,\tau) \leq 1, \tag{106}$$

is essentially the same as Equation (104) above when q is of fixed order. This measure also shows variation when q is a function of time. These modified measures will show the fading memory effect with variable q. Better measures of are certainly possible, but are outside the scope of this paper.

5.2.2. m_2 *Memory*

A more important memory type for the study of the 'tailored' variable order fractional integral is the m_2 type that is associated with memory of the order history. While the m_1 memory is

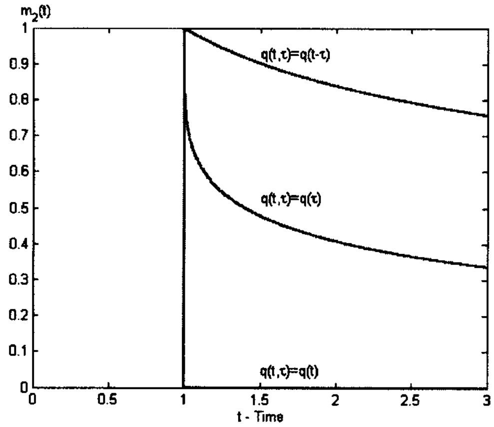

Figure 12. $m_2(t)$ *versus* t time for $T_{sw} = 1, q(t, \tau) = q(t), = q(\tau), = q(t - \tau)$.

meant to be a measure of the retentiveness of the entire fractional integral, here we seek a measure of the retentiveness of the order within the fractional integral after a change in order. Thus we consider a step change in order from q_1 to q_2 occurring at time $t = T_{sw}$ as shown in Figures 2, 5, and 8. Setting $f(t) = 1$ eliminates the effects of changes in the input function. We define m_2 memory retentiveness as

$$m_2(t) \equiv \frac{\int_{\tau_1}^{\tau_2} \frac{(t-\tau)^{q_1-1}}{\Gamma(q_1)} \mathrm{d}\tau}{\int_0^t \frac{(t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))} \mathrm{d}\tau} = \frac{\text{area under } q_1 \text{ portion of } h(t,\tau)}{\text{area under } h(t,\tau)}, \quad 0 \le q(t,\tau) \le 1, \tag{107}$$

where τ_1 and τ_2 are the lower and upper bounds respectively associated with area under the q_1 portion of $h(t, \tau)$. Thus m_2 measures the strength or retentiveness of the memory of q_1 at time t when $t > T_{sw}$.

This measure may be readily applied to the variable order definitions of Cases 1 to 3. It is readily seen that for Case 1, $q(t, \tau) = q(t)$ (refer to Figure 2)

$$m_2(t) = \begin{cases} 1, & t < T_{sw}, \\ 0, & t > T_{sw}. \end{cases} \tag{108}$$

Then for Case 2 above, $q(t, \tau) = q(\tau)$ (refer to Figure 5)

$$m_2(t) = \frac{\int_0^{T_{sw}} \frac{(t-\tau)^{q_1-1}}{\Gamma(q_1)} \mathrm{d}\tau}{\int_0^{T_{sw}} \frac{(t-\tau)^{q_1-1}}{\Gamma(q_1)} \mathrm{d}\tau + \int_{T_{sw}}^{t} \frac{(t-\tau)^{q_2-1}}{\Gamma(q_2)} \mathrm{d}\tau}, \quad t > T_{sw}, \tag{109}$$

and for Case 3, $q(t, \tau) = q(t - \tau)$ (refer to Figure 8)

$$m_2(t) = \frac{\int_{t-T_{sw}}^{T_{sw}} \frac{(t-\tau)^{q_1-1}}{\Gamma(q_1)} \mathrm{d}\tau}{\int_{t-T_{sw}}^{T_{sw}} \frac{(t-\tau)^{q_1-1}}{\Gamma(q_1)} \mathrm{d}\tau + \int_0^{T_{sw}} \frac{(t-\tau)^{q_2-1}}{\Gamma(q_2)} \mathrm{d}\tau}, \quad t > T_{sw}. \tag{110}$$

The behavior of the m_2-measure for these three cases is shown graphically in Figure 12, and thus the results discussed subjectively above, for these cases, have been quantified.

To examine the behavior for the ‘tailored’ integrals, Equation (101), we select a particular value of t, say $t = 3$, in the m_2-measure. This result $m_2(3)$ *versus* a and b is shown in

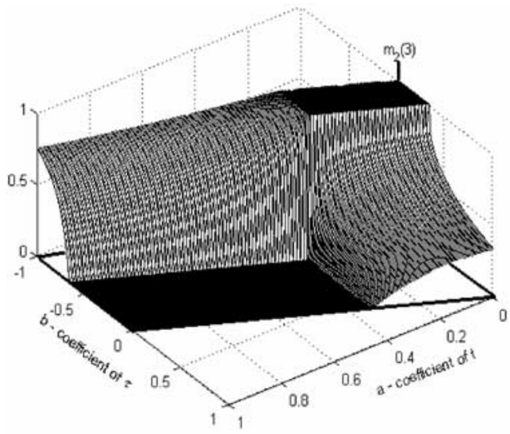

Figure 13a. $m_2(3)$ memory behavior for 'tailored' variable order fractional integrals.

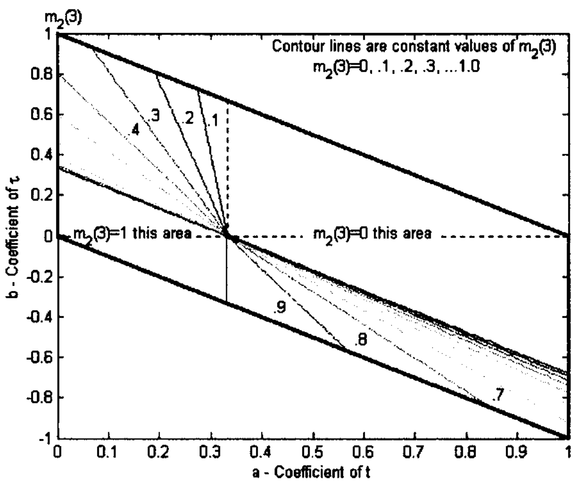

Figure 13b. Contours of constant $m_2(3)$ memory for 'tailored' variable order fractional integrals.

Figure 13a. This figure shows the variation in order retention (m_2 memory) as the a and b parameters of the order $q(at + b\tau)$ of the variable order fractional integral vary over the allowable range. Figure 13b presents the same data as Figure 13a but now as contours indicating loci of constant values of $m_2(3)$. It is important to note that the contours are straight lines emanating from the point $(1/t, 0) = (1/3, 0)$. From such data one may select the desired properties for the variable order integrals (and derivatives) for a specific application.

5.3. Dynamic Behavior: Response to q Changes

The responsiveness of the variable order fractional integral to changes in the order q is an important consideration in the application of these operators. In general, the longer the memory of past q, i.e. the greater its m_2 retentiveness, the slower will be the response of a particular definition to a change in q. This effect as well as the responsiveness of the variable order fractional integral to changes in the input function $f(t)$ will be illustrated in the application section.

6. Variable Order Differentiation

The above definitions for variable order fractional integration may be formally extended to variable order fractional differentiation in a manner similar to that done for constant q. For the uninitialized case we define

$$ {}_0d_t^{q(t)} f(t) \equiv {}_0d_t^m \, {}_0d_t^{-u(t)} f(t), \quad t > 0,\ q(t) \geq 0,\ \forall t, \tag{111} $$

where m is an integer such that $q(t) = m - u(t)$ and $m \geq q(t) \geq 0$, $\forall t$. Similarly for the initialized case we formally define

$$ {}_0D_t^{q(t)} f(t) \equiv {}_0D_t^m D_t^{-u(t)} f(t), \quad t > 0, \quad q(t) > 0, \tag{112} $$

where $q(t) = m - u(t)$, and m is taken as the least integer greater than the maximum value of $q(t)$.

Composition is inferred in the above definitions. However, the determination of a general composition law for the variable order fractional integral may subject these definitions to later reconsideration. Lack of a composition law increases the difficulty of analytical studies using the definitions for both the variable order fractional integral and derivative, however, numerical approaches are viable. Another complication is the fact that it may be desirable to allow $q(t)$ to range over both positive and negative values. This places an analytical 'seam' at $q = 0$, requiring the use of different definitions at each crossing, which may cause difficulties when using an approach based on the Riemann–Liouville definition. Thus an approach based on the Grunwald definition may be required. A consideration of the variable order differintegral based on the Grunwald definition is presented in [1].

The character of the variable order fractional differentiator will depend on the character of the basis definition used for variable order fractional integration. That is, the memory and response behaviors of the variable order fractional differentiator will mimic those of the basis variable order fractional integrator definition. Thus for example, using the Case 1 definition $q(t, \tau) \to q(t)$, the fractional variable order derivative will have no memory of past order q and will behave in a manner similar to the integer order derivative.

7. Application of Variable Order Operators

The motivation section of this paper presents a number of potential application areas for the variable order operators. The introduction and consideration of the 'tailored' variable order operators also allows application to a variety of adaptive filtering, detection, and modeling applications. The scope of this paper does not allow detailed study of these. This section will examine some simple applications to show some of the behavior of these definitions. To study the response of the variable order fractional integral to a change in $q(t)$ we start by holding $f(t) = 1$. Figure 14 shows numerical results for the three basic cases studied earlier to a square wave 'input' of $q(t)$. The m_2 memory retention of past q is readily observed by comparison of the three responses. For example, for the $q(t, \tau) \to q(t)$ transient the switching between the constant $q = 0.75$ and $q = 0.25$ responses is obvious. A more detailed view of the response to a step change in $q(t)$ is shown in Figure 15. Here attention is drawn to the response of the integral after time $t = 0.1$, the effects of q the argument in the kernel become very apparent for the three cases.

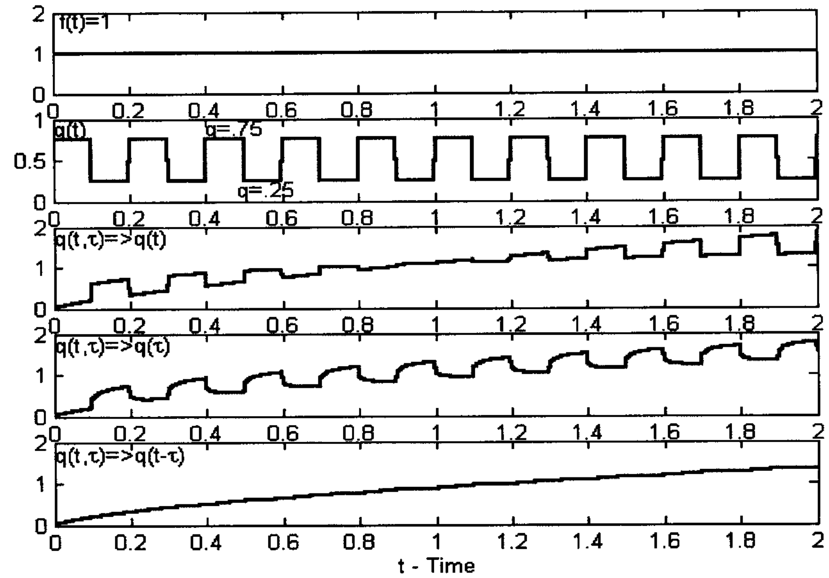

Figure 14. Variable order fractional integral responses for arguments $q(t, \tau) = q(t), = q(\tau), = q(t - \tau), q(t)$ input is a square wave.

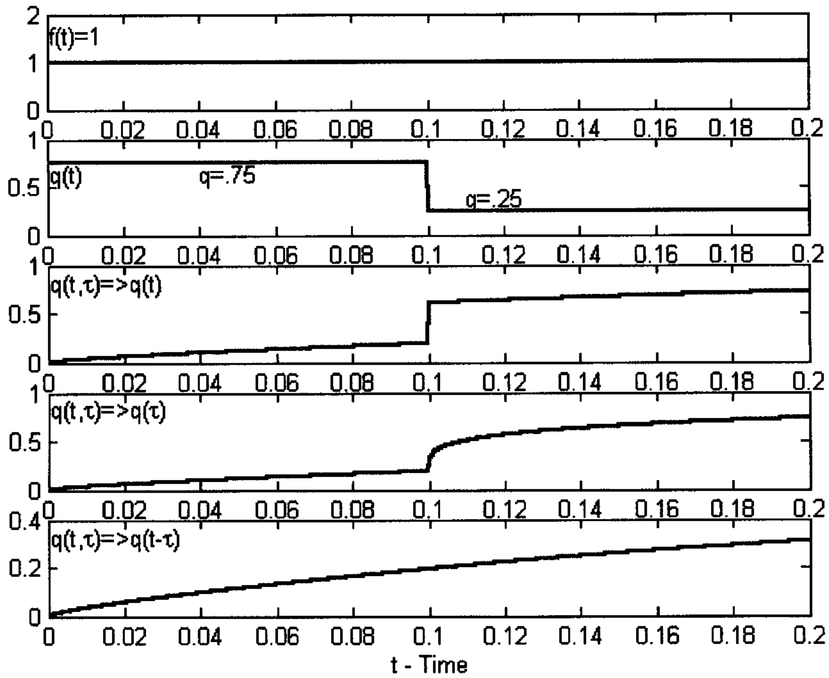

Figure 15. Variable order fractional integral responses for arguments $q(t, \tau) = q(t), = q(\tau), = q(t - \tau), q(t)$ input is a step at $t = 0.1$.

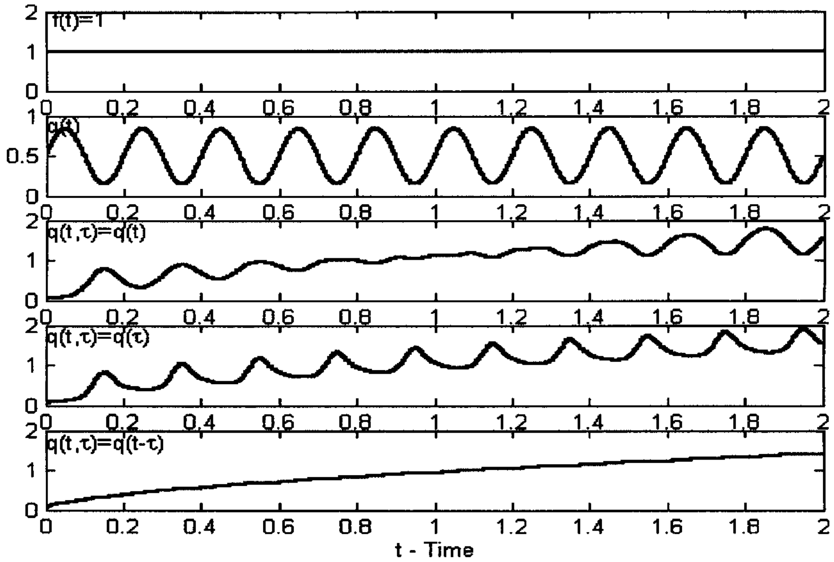

Figure 16. Variable order fractional integral responses for arguments $q(t,\tau) = q(t)$, $= q(\tau)$, $= q(t-\tau)$, $q(t) = 0.5 + 0.35\sin(10\pi t)$.

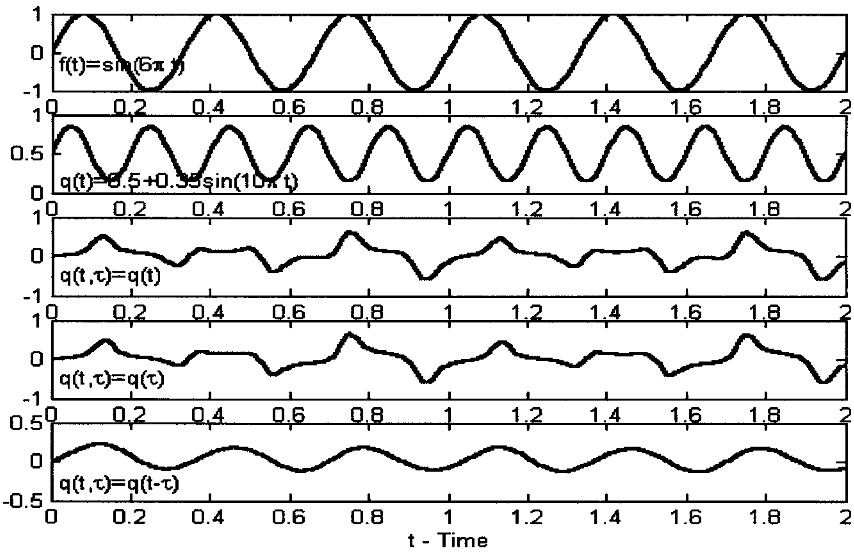

Figure 17. Variable order fractional integral responses for arguments $q(t,\tau) = q(t)$, $= q(\tau)$, $= q(t-\tau)$, $q(t) = 0.5 + 0.35\sin(10\pi t)$, $f(t) = \sin(6\pi t)$.

We now consider a variable order integration with $q(t) = 0.5 + 0.35\sin(10\pi t)$; then we have

$$ {}_0d_t^{-(0.5+0.35\sin(10\pi t))}U(t) = \int_0^t \frac{(t-\tau)^{q(t,\tau)-1)}}{\Gamma(q,t,\tau))}\mathrm{d}\tau, \tag{113} $$

where again the three definitions of $q(t,\tau)$ are applied. Figure 16 shows the results of these variable order fractional integrations. It is observed that the Case 1 definition $q(t,\tau) \to q(t)$

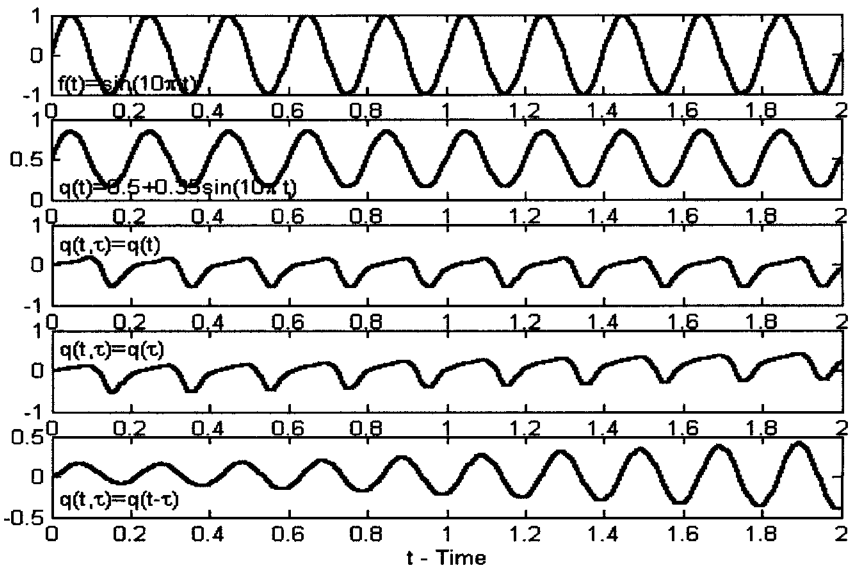

Figure 18. Variable order fractional integral responses for arguments $q(t, \tau) = q(t), = q(\tau), = q(t - \tau)$, $q(t) = 0.5 + 0.35 \sin(10\pi t)$, $f(t) = \sin(10\pi t)$.

result starts out 180 degrees out of phase with the 'input' of $q(t)$ but then shifts to an in-phase response, the Case 2 definition $q(t, \tau) \rightarrow q(\tau)$ result remains 180 degrees out of phase with the 'input' of $q(t)$ for the entire transient, while for the Case 3 definition $q(t, \tau) \rightarrow q(t - \tau)$ the sinusoidal effect of $q(t)$ is barely perceptible in the result.

The effect of the integrand is now considered. Figure 17 shows the results for $f(t) = \sin(6\pi t)$ while maintaining $q(t) = 0.5 + 0.35 \sin(10\pi t)$. The Case 1 definition $q(t, \tau) \rightarrow q(t)$ and the Case 2 definition $q(t, \tau) \rightarrow q(\tau)$ results appear to be quite similar and have apparent frequency content at the sum frequency $\sin(16\pi t)$. Close examination of the difference of these two results however shows content at the product frequency (not shown). For the Case 3 definition $q(t, \tau) \rightarrow q(t - \tau)$, the result follows the integrand with little, if any, phase shift.

That the integrand can interact with the order 'input' is demonstrated in Figure 18. Here again the Case 1 definition $q(t, \tau) \rightarrow q(t)$ and the Case 2 definition $q(t, \tau) \rightarrow q(\tau)$ results appear to be quite similar, both having significant harmonic content. The startling feature of this figure however is the amplitude growth observed for the Case 3 definition $q(t, \tau) \rightarrow q(t - \tau)$ result.

The matching of the frequencies, apparently, allows energy to accumulate in the resultant output of the integration. This is a resonant-like condition. It is also observed in Figure 18 that the $q(t, \tau) = q(\tau)$ result is slowly ramping up, also an indication of energy accumulation. Thus, the physical inference may be drawn that: *varying the order requires (or yields) external energy*.

8. Distributed Order Operators

8.1. Approaches to Distributed Order Operators

As in the case of time variable operators, because the order of the fractional integral and derivative may take on any real value (or imaginary) we are allowed to consider variations of the order with some variables other than time, say for example, space. This section explores the distributed order operator. Two approaches are considered: a direct approach that does not assign a new variable for the variation in order and an independent variable approach that considers to be a function of some independent (perhaps spatial) variable.

8.2. Direct Approach

The approach to be discussed here is a general mathematical method. However, to aid understanding we will develop the concept around a reference mechanical system. The general form for an element in such a system is

$$k\ {}_0d_t^q\, y(t) = f(t), \tag{114}$$

where $q = 2$ for an inertial mass, $q = 1$ for a dashpot, $q = 0$ for a spring, $y(t)$ is the displacement (response), and $f(t)$ is the time varying force on the component. Other possible elements may include $0 \leq q \leq 1$ for various viscoelastic materials, and $1 \leq q \leq 2$ for viscoinertial elements. Thus an assembly of such elements in parallel would be represented by

$$\sum_i k_i^* ({}_0d_t^{q_i}\, y(t) + \psi_i) = \sum_i f_i(t), \tag{115}$$

where $\psi_i = \psi(y, q_i, a, 0, t)$ is the associated initialization for each element.

Some materials have rheological properties that depend on temperature, electrostatic field strength [9] or magnetic field strength [10]. Bagley [5] discusses materials that display complex thermorheological behavior. This means that the order of the viscoelastic element, q_n, depends upon the temperature of the material. If a sample of this material is subject to a temperature distribution, a corresponding order-distribution will exist throughout the material, as each individual element will have its own order. In the limit as the elements approach differential size this leads to the concept of the order-distribution. While this concept has been introduced using a mechanical example, it is conceptually simple to jump from the summation in Equation (115) to the integral in Equation (116) below. Assuming elements with orders that vary from zero to two, the general mechanical system of Equation (115), is replaced by an integral over the system order,

$$\int_0^2 k(q)\ {}_0d_t^q\, y(t)\, \mathrm{d}q + \int_0^2 k(q)\psi(y, q, a, 0, t)\, \mathrm{d}q = f(t), \tag{116}$$

where the first integral is named the cumulative order distribution over the range $0 \leq q \leq 2$. With $k(q)$ being called the order-weighting distribution or the order strength distribution. In general we will use the notation

$${}_{q_1,q_2}\Omega_{c,t}^q(k(q), y(t), q) \equiv \int_{q_1}^{q_2} k(q)\ {}_cd_t^q\, y(t)\, \mathrm{d}q, \tag{117}$$

for the cumulative order distribution, where the q in the argument of Ω is the integration variable. For the special limits this will be written as

$$\Omega^{q}_{c,f}(k(q), y(t), q) \equiv \int_0^\infty k(q)\ {}_c d_t^q\, y(t)\, \mathrm{d}q. \tag{118}$$

For convergence of the integral, Equation (116), we shall require that the integrand be bounded and that $k(q)$ be non-zero only over a finite range. The second integral of Equation (116) is the cumulative order distribution initialization, and the forces of Equation (115) have been combined into a single term. The general notation for the cumulative order distribution initialization is

$$_{q_1,q_2}\Psi^{q}_{a,c}(k(q), y(t), q) \equiv \int_{q_1}^{q_2} k(q)\psi(y, q, a, c, t)\, \mathrm{d}q. \tag{119}$$

The Laplace transform of Equation (116), assuming the integral converges and is of exponential order, is given by

$$\left(\int_0^2 k(q)s^q\, \mathrm{d}q\right) Y(s) + \left(\int_0^2 k(q)L(\psi(y, q, a, 0, t))\, \mathrm{d}q\right) = F(s). \tag{120}$$

Thus we now have a very general formulation for representing dynamic systems. Indeed the sum of all orders of constant coefficient linear derivatives can be expressed as a single term!

To demonstrate that familiar equations can be written in this form, the common second order mechanical system may be readily expressed using Equation (120). Setting the initialization integral to zero (or including it with the force), gives

$$\left(\int_0^2 [m\delta(q-2) + b\delta(q-1) + k\delta(q)]s^q\, \mathrm{d}q\right) Y(s) = F(s), \tag{121}$$

or

$$(ms^2 + bs + k)Y(s) = F(s). \tag{122}$$

The range of system order may be extended in both directions, then the general system representation becomes

$$\left(\int_{-\infty}^{\infty} k(q)s^q\, \mathrm{d}q\right) Y(s) + \left(\int_{-\infty}^{\infty} k(q)L(\psi(y, q, a, 0, t))\, \mathrm{d}q\right) = F(s), \tag{123}$$

which has the time domain representation

$$\int_{-\infty}^{\infty} k(q)\ {}_0D_t^q\, y(t)\, \mathrm{d}q = f(t), \quad -\infty < q < \infty, \tag{124}$$

and $k(q)$ is constrained to be non-zero only over a finite range to assure convergence of Equation (124). The forcing term on the right-hand side of this equation may also be a cumulative order distribution, giving the more general form

$$\left(\int_{-\infty}^{\infty} k(q)s^q \,\mathrm{d}q\right) Y(s) = \left(\int_{-\infty}^{\infty} g(p)s^p \,\mathrm{d}p\right) F(s) - \left(\int_{-\infty}^{\infty} k(q)L(\psi(y,q,a,0,t))\,\mathrm{d}q\right). \tag{125}$$

This form contains all non-partial constant coefficient linear differential equations (both integer order and fractional) when the lower integral limits are set to zero! With the limits as written, it also includes all possible integral terms. Further extension to include time varying coefficients may be obtained by replacing $k(q)$ with $k(q,t)$ in Equation (124) and transforming appropriately. The time domain solution is given by

$$y(t) = L^{-1}\left\{\frac{\int_{-\infty}^{\infty} g(p)s^p \,\mathrm{d}p}{\int_{-\infty}^{\infty} k(q)s^q \,\mathrm{d}q} F(s)\right\} - L^{-1}\left\{\frac{\int_{-\infty}^{\infty} k(q)L(\psi(y,q,a,0,t))\,\mathrm{d}q}{\int_{-\infty}^{\infty} k(q)s^q \,\mathrm{d}q}\right\}. \tag{126}$$

The analysis section considers some specific systems that can be represented by Equation (123).

8.3. Independent Variable Approach

8.3.1. *Definition*

This approach to distributed order operators considers q as a function of some independent variable x (here considered to be spatial). Thus we have $q = q(x)$ and $k = k(x)$ and we consider operators of the form

$${}_0d_t^{q(x)} f(t). \tag{127}$$

For this differintegration $q(x)$ is considered to be constant in the fractional operator (relative to the t (time) based fractional integration). Operators of this form will be explored in more detail in Section 10. The cumulative *spatial* order distribution then is defined as

$${}_{A,B}\Omega_{c,t}^{q(x)}(k(q), f(t), x) \equiv \int_A^B k(x)\, {}_cd_t^{q(x)} f(t)\,\mathrm{d}x. \tag{128}$$

Normally we shall take the limits of the x integration as 0 and ∞, and assure convergence of the integral by limiting $k(x)$ to be non-zero only over a finite range of x. Taking $c = 0$, we will write for the cumulative *spatial* order distribution

$$\Omega_t^{q(x)}(k(q), f(t), x) \equiv \int_0^{\infty} k(x)\, {}_0d_t^{q(x)} f(t)\,\mathrm{d}x. \tag{129}$$

In this definition $q(x)$ is considered to be constant in the fractional operator (relative to the time based fractional differintegration). The general notation for the cumulative *spatial* order distribution initialization is

$${}_{A,B}\Psi_{a,c}^{q(x)}(k(q), f(t), x) \equiv \int_A^B k(x)\psi(f,q,a,c,t,x)\,\mathrm{d}x, \tag{130}$$

where the limits of the integral may be modified to suit the problem to be solved. From an engineering or scientific view x might be the distance along some thermorheologically complex material that is experiencing a temperature gradient. Thus at each point x the material responds with a different order $q(x)$ and spatial strength spectrum $k(x)$. The application (Figure 19) of the next section shows such a situation. Extension to multi-dimensional cumulative (spatial) order distributions is obvious

$$\Omega_t^{q(x_1,x_2,\ldots,x_3)}(k(x_1,x_2,\ldots,x_n), f(t), x_1,\ldots,x_n)$$
$$\equiv \int_0^\infty \cdots \int_0^\infty \int_0^\infty k(x_1,x_2,\ldots,x_n)\, {}_0d_t^{q(x_1,x_2,\ldots,x_n)} f(t)\,\mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_n, \tag{131}$$

where again the limits may modified to suit the problem to be solved, and k is constrained to assure convergence of the multiple integrations.

8.3.2. *Laplace Transform*

The Laplace transform of the cumulative order distribution, Equation (129), assuming that it is bounded and of exponential order, is

$$\begin{aligned} L\{\Omega_t^{q(x)}(k(q), f(t), x)\} &= L\left\{\int_0^\infty k(x)\, {}_0d_t^{q(x)} f(t)\,\mathrm{d}x\right\} \\ &= \int_0^\infty \mathrm{e}^{-st} \int_0^\infty k(x)\, {}_0d_t^{q(x)} f(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \Omega_s^{q(x)}(k(q), f(s), x). \end{aligned} \tag{132}$$

Under conditions allowing the interchanging the order of integration, we have

$$L\left\{\int_0^\infty k(x)\, {}_0d_t^{q(x)} f(t)\,\mathrm{d}x\right\} = \int_0^\infty k(x) \int_0^\infty \mathrm{e}^{-st}\, {}_0d_t^{q(x)} f(t)\,\mathrm{d}t\,\mathrm{d}x, \tag{133}$$

or

$$L\left\{\int_0^\infty k(x)\, {}_0d_t^{q(x)} f(t)\,\mathrm{d}x\right\} = \int_0^\infty k(x) L\{{}_0d_t^{q(x)} f(t)\}\,\mathrm{d}x. \tag{134}$$

Since $q(x)$ is considered to be constant in the fractional operator, this becomes

$$L\{\Omega_t^{q(x)}(k(q), f(t), x)\} = L\left\{\int_0^\infty k(x)\, {}_0d_t^{q(x)} f(t)\,\mathrm{d}x\right\} = \int_0^\infty k(x) s^{q(x)}\,\mathrm{d}x\, F(s). \tag{135}$$

In application it will be often desired to also transform in the x dimension, thus using s_1 as the Laplace parameter with respect to x we have, limiting the integration range to x

$$L_{s_1}\{L_s\{\Omega_t^q(k, f, x)\}\} = L_{s_1}\left\{L_s\left\{\int_0^x k(x)\, {}_0d_t^{q(x)} f(t)\,\mathrm{d}x\right\}\right\}$$

$$= L_{s_1}\left\{\int_0^x k(x)s^{q(x)}\,\mathrm{d}x\,F(s)\right\}, \tag{136}$$

$$= \frac{1}{s_1}L_{s_1}\{k(x)s^{q(x)}\}F(s). \tag{137}$$

It is interesting to compare this result to the variable order results Equations (37) and (70).

9. Analysis with Distributed Order Operators

It is useful to reconsider the transform of the cumulative order distribution, Equation (118), by rewriting the exponential in s. That is

$$L\{\Omega^q_{c,t}(k(q), y(t), q)\} \equiv L\left\{\int_0^\infty k(q)_c d^q_t\, y(t)\,\mathrm{d}q\right\} = \int_0^\infty k(q)s^{\,q}Y(s)\,\mathrm{d}q$$

$$= \int_0^\infty k(q)\mathrm{e}^{q\ln(s)}\,\mathrm{d}q\,Y(s). \tag{138}$$

This cumulative order distribution transform is effectively a Laplace transform of the function $k(q)$ with the new Laplace variable being $r = -\ln(s)$. As long as the order-distribution $k(q)$ is such that the integral converges, the integral is easy to calculate by treating it as a Laplace transform with Laplace variable $(-\ln(s))$.

Consider the direct (approach) order distribution $k(q) = U(q) - U(q-2)$, that is, a uniform distribution of magnitude 1, between $q = 0$ and $q = 2$. The Laplace transform of from the tables is

$$\frac{1}{r} - \frac{\mathrm{e}^{-2q}}{r}. \tag{139}$$

Then replacing r by $-\ln(s)$ gives

$$\int_0^\infty k(q)\,\mathrm{e}^{q\ln(s)}\,\mathrm{d}q = \int_0^2 \mathrm{e}^{q\ln(s)}\,\mathrm{d}q = \frac{1-\mathrm{e}^{2\ln(s)}}{\ln(s)} = \frac{1-s^2}{\ln(s)}, \tag{140}$$

for the transform of the distribution.

The tables of Laplace transforms can also assist in evaluating the integral on the right hand side of Equation (135). That is, if $q(x)$ can be written as $q(x) = -x + v(x)$ then we may write the transform as

$$L\left\{\int_0^\infty k(x)\,{}_0d^{q(x)}_t f(t)\,\mathrm{d}x\right\} = F(s)\int_0^\infty k(x)s^{q(x)}\,\mathrm{d}x = F(s)\int_0^\infty \mathrm{e}^{q(x)\ln(s)}k(x)\,\mathrm{d}x$$

$$= F(s)\int_0^\infty \mathrm{e}^{-x\ln(s)}\{\mathrm{e}^{v(x)\ln(s)}k(x)\}\,\mathrm{d}x$$

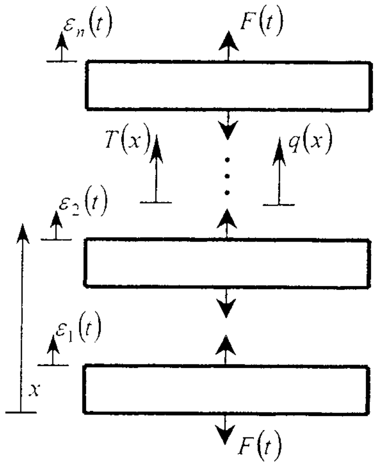

Figure 19. Free body diagram for viscoelastic plate.

$$= F(s) \int_0^\infty \mathrm{e}^{-x\ln(s)}\{s^{(x+q(x))}k(x)\}\,\mathrm{d}x. \tag{141}$$

Then, provided $\{\mathrm{e}^{v(x)\ln(s)}k(x)\} = \{s^{(x+q(x))}k(x)\}$ is bounded and is of exponential order, the integral may be evaluated as an x based Laplace transform which may be determined from the tables. The Laplace variable is then replaced by $\ln(s)$.

9.1. Example Application

This example demonstrates the application of distributed order operators to obtain the analytical solution of the behavior of a viscoelastic material with spatially varying properties. We consider a plate constructed of thermorheologically complex material subjected to a temperature gradient $T(x)$ and to a force $F(t)$. Thus, the order q varies with temperature and therefore with the spatial dimension x. A free body diagram showing the distributed parameters is presented in Figure 19. For any constant temperature element, the elemental stress-strain relationship [13] is given by

$$\sigma(t) = E\tau^q\,{}_0d_t^q\varepsilon(t), \tag{142}$$

where $\sigma(t)$ is the applied stress and $\varepsilon(t)$ is the resulting strain, q is the fractional order, with $0 < q < 1$, and E and τ are material properties. Setting $k_q = E\tau^q$ we write for each incremental element and for a unit area

$$\varepsilon_i(t) = \frac{1}{k_{q_i}}\,{}_0D_t^{-q_i}F(t). \tag{143}$$

Then the total strain for the plate will be

$$\varepsilon(t) = \sum_{i=1}^{n}\varepsilon_i(t) = \sum_{i=1}^{n}\frac{1}{k_{q_i}}\,{}_0D_t^{-q_i}F(t). \tag{144}$$

We will assume that the spatial temperature variation affects the order q such that $q = c + dx$, and in the limit as $n \to \infty$, we have

$$\varepsilon(t) = \int_0^l \frac{1}{k_q(x)}\, {}_0D_t^{-(c+dx)} F(t)\, \mathrm{d}x. \tag{145}$$

This is an initialized cumulative order distribution in terms of the independent variable x. For simplicity and clarity in this example, we will assume an initialization $\psi = 0$, and $l = 1$. Then taking the Laplace transform we have

$$\varepsilon(s) = \int_0^1 \frac{1}{k_q(x)} L\{{}_0d_t^{-(c+dx)} F(t)\}\, \mathrm{d}x. \tag{146}$$

Again for simplicity, we take $k_q(x) = 1$, and thus assume for this example that only the integral order, $q = c + dx$, is affected by the temperature gradient and that E and τ^q are not. Evaluating the transform gives

$$\varepsilon(s) = \int_0^1 s^{-(c+dx)} F(s)\, \mathrm{d}x = \int_0^1 s^{-(c+dx)}\, \mathrm{d}x\, F(s). \tag{147}$$

Performing the integration with $F(t)$ taken as a unit step function, i.e. $F(s) = 1/s$ yields after some algebra

$$\varepsilon(s) = \frac{1}{d}\left[\frac{1}{s^{c+1}\ln(s)} - \frac{1}{s^{c+1+d}\ln(s)}\right]. \tag{148}$$

The inverse transform may be found from the tables of transforms [14] and the time domain solution is the sum of two Volterra type functions, thus

$$\varepsilon(t) = \frac{1}{d}\left\{\int_{c+1}^{\infty} \frac{t^{p+c}}{\Gamma(p+c+1)}\mathrm{d}p - \int_{c+d+1}^{\infty} \frac{t^{p+c+d}}{\Gamma(p+c+d+1)}\mathrm{d}p\right\}. \tag{149}$$

10. Operators of Multivariable Order and Distributed Variable Order

10.1. Multivariable Order Operators

In the past sections we have seen the order of fractional order operators vary with time and distributed over a (spatial) dimension . We will now assign meaning to operators of the form ${}_0d_t^{q(t,x)} f(t)$, with order a function of multiple variables and combine the variable order and order distribution operators. We start by limiting consideration to uninitialized operators for clarity of discussion. Thus we define

$${}_0d_t^{-q(x)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(x)-1}}{\Gamma(q(x))} f(\tau)\, \mathrm{d}\tau, \quad q > 0, \tag{150}$$

where the integral is to be evaluated with the variable x treated as a constant. This definition would be used in applications where the order is an independent function of the (spatial) x dimension. An example might be to describe the local properties of a plate composed of different viscoelastic materials stacked along the x dimension, where each material has different (spatial) order properties. This definition is readily extended to multiple independent dimensions as

$$ {}_0d_t^{-q(x_1,x_2,\ldots,x_n)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(x_1,x_2,\ldots,x_n)-1}}{\Gamma(q(x_1,x_2,\ldots,x_n))} f(\tau)\,\mathrm{d}\tau, \quad q > 0, \tag{151} $$

with each dimension treated as a constant. Application here generalizes the example above. That is to describe the local material properties for a viscoelastic solid composed of a three dimensional lay-up of thermorheologically complex materials and subjected to a temperature gradient $T(x, y, z)$. Thus, the order q varies with temperature and therefore with the x, y, z spatial dimensions along which each material has different order properties.

For the above definitions the fractionally integrated function $f(t)$ may also be generalized in an obvious way to be a function of $x_1, x_2, \ldots, x_n, t$ by use of the concept of the partial fractional integral. In this concept the function variables $x_1, x_2, \ldots, x_n$ are treated as a constants in $f(x_1, x_2, \ldots, x_n, t)$. The interested reader is referred to [15] for discussion of the fixed order partial fractional integral.

The (time) variable order fractional integral has been defined as (Equation (18))

$$ {}_0d_t^{-q(t)} f(t) \equiv \int_0^t \frac{t-\tau)^{q(t,\tau)-1}}{\Gamma(q(t,\tau))} f(\tau)\,\mathrm{d}\tau, \quad q > 0, \tag{152} $$

where for the ‘tailored’ case we take $q(t, \tau) = q(at + b\tau)$ with a and b set by the application requirements.

We now combine the t (time) and x (space) variable order operators and define

$$ {}_0d_t^{-q(x,t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(x,(t,\tau))-1}}{\Gamma(q(x,(t,\tau)))} f(\tau)\,\mathrm{d}\tau, \quad q > 0, \tag{153} $$

where again x is taken as constant under the integration, and the notation (t, τ) refers to the t (time), variable order relationship as developed in Section 4, e.g. $(t, \tau) \Rightarrow (at + b\tau)$. Application here might be to a viscoelastic solid composed of different complex thermo-viscoelastic materials along the x dimension, where each material has different order properties, and those properties vary with a time varying temperature. For example let us assume the transient behavior relates as functions of $(t-\tau)$, temperature is linear with time and baseline properties are linear (and independent) along the x dimension then $q(x, (t-\tau))$ might be written as $q(x, (t-\tau)) = q(cx(t-\tau))$. Extension of Equation (153) to multiple dimensions yields

$$ {}_0d_t^{-q(x_1,x_2,\ldots,x_n,t)} f(t) \equiv \int_0^t \frac{(t-\tau)^{q(x_1,x_2,\ldots,x_n,(t,\tau))-1}}{\Gamma(q(x_1,x_2,\ldots,x_n,(t,\tau)))} f(\tau)\,\mathrm{d}\tau \tag{154} $$

as the general uninitialized definition. The initialized definition is a direct application of property G2, thus for locally independent histories

$$
\begin{aligned}
{}_0D_t^{-q(x_1,x_2,\ldots,x_n,t)} f(t) \equiv\ & {}_0d_t^{-q(x_1,x_2,\ldots,x_n,t)} f(t) \\
& + \psi(f, q(x_1, x_2, \ldots, x_n, t), a, 0, t), \quad t > 0, q > 0, \qquad (155)
\end{aligned}
$$

where

$$
\begin{aligned}
& \psi(f, q(x_1, x_2, \ldots, x_n, t), a, 0, t) \\
& \quad \equiv \int_a^0 \frac{(t-\tau)^{q(x_1,x_2,\ldots,x_n,(t,\tau))-1}}{\Gamma(q(x_1, x_2, \ldots, x_n, (t, \tau)))} f(\tau)\,\mathrm{d}\tau, \quad t > 0, q > 0. \qquad (156)
\end{aligned}
$$

The generalized variable order fractional derivative is defined as a generalization of Equation (112), thus we define

$$
{}_0D_t^{q(x_1,x_2,\ldots,x_n,t)} f(t) \equiv {}_0D_t^m\, {}_0D_t^{-u(x_1,x_2,\ldots,x_n,t)} f(t), \quad t > 0, q > 0, \qquad (157)
$$

where $q(x_1, x_2, \ldots, t) = m - u(x_1, x_2, \ldots, t)$, and m is usually taken as the least integer greater than the maximum value of $q(x_1, x_2, \ldots, t)$.

10.2. Distributed Variable Order Operators

The variable order and the order distribution operators may be combined to allow order distributions based on variable order operators. In a physical sense we are typically discussing time varying order distributions over some spatial extent. We limit discussion here to uninitialized operators for clarity. Then combining the results of Equations (111) and (117) it is reasonable to write for the variable order direct approach cumulative distribution

$$
{}_{q_1,q_2}\Omega_{c,t}^{q(t)}(k(q), y(t), q) \equiv \int_{q_1}^{q_2} k(q)\, {}_cd_t^{q(t)} y(t)\,\mathrm{d}q, \qquad (158)
$$

or

$$
{}_{q_1,q_2}\Omega_{c,t}^{q(t)}(k(q), y(t), q) \equiv \int_{q_1}^{q_2} k(q)\, {}_0d_t^m \int_c^t \frac{(t-\tau)^{u(t,\tau)-1}}{\Gamma(u(t,\tau))} y(\tau)\,\mathrm{d}\tau\,\mathrm{d}q, \qquad (159)
$$

where $q(t) = m - u(t)$, and the argument of u of the inner integral is selected as discussed in earlier sections. In similar manner the variable order independent variable (spatial) cumulative distribution is obtained by combining Equations (111) and (128) and is given as

$$
{}_{A,B}\Omega_{c,t}^{q(x,t)}(k(q), f(t), x) \equiv \int_A^B k(x)\, {}_cd_t^{q(x,t)} f(t)\,\mathrm{d}x. \qquad (160)
$$

Alternatively using the form of Equation (129) we have

$$
\Omega_t^{q(x,t)} k(q), f(t), x) \equiv \int_0^\infty k(x)\, {}_0d_t^{q(x,t)} f(t)\,\mathrm{d}x. \qquad (161)
$$

In terms of the Riemann–Liouville formulation this is written

$$\Omega_t^{q(x,t)}(k(q), f(t), x) = \int_0^\infty k(x)\, {}_0d_t^m \int_0^t \frac{(t-\tau)^{u(x,(t,\tau))-1}}{\Gamma(u(x,(t,\tau)))} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}x, \tag{162}$$

where $q(t,x) = m - u(x,t)$ and the argument of u of the inner integral is selected as discussed in earlier sections. Then, for example using the Case 3 definition, for variable order fractional integration, i.e. $q(t,\tau) = q(t-\tau)$, Equation (162) becomes

$$\Omega_t^{q(x,t)}(k(q), f(t), x) = \int_0^\infty k(x)\, {}_0d_t^m \int_0^t \frac{(t-\tau)^{u(x,(t-\tau))-1}}{\Gamma(u(x,(t-\tau)))} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}x. \tag{163}$$

In this case the Laplace transform may be written, using the result of Equation (94), as

$$L\{\Omega_t^{q(x,t)}(k(x), f(t), x\} = \int_0^\infty k(x) s^m L\left\{\frac{t^{q(x,t)}}{\Gamma(q(x,t))}\right\} F(s)\,\mathrm{d}x, \tag{164}$$

where all initializations are assumed to be zero. Such formulations, as Equations (162) or (164) may be applied to advantage. For example, the viscoelastic problem associated with Figure 19 could be subject to a *time varying* temperature, the form of Equation (162) or its transform may be applied directly to determine the time varying cumulative strain for this complex system. To be determined for such a problem however, is which time variable order argument $q(t,\tau)$ best describes the transient physics.

The spatial strength spectrum may be time varying, i.e., $k(x) \to k(x,t)$ also the integrand (input force) may be spatially distributed, i.e. $f(t) \to f(t,x)$ then Equation (162) generalizes to

$$\Omega_t^{q(x,t)}(k(x,t), f(x,t), x) = \int_0^\infty k(x,t)\, {}_0d_t^{q(t,x)} f(x,t)\,\mathrm{d}x. \tag{165}$$

This form requires treating the fractional derivative (or fractional integral) as partial fractional derivative (or fractional integral); that is, taking x as a constant under the fractional operation.

11. Summary

This paper introduces the concepts of variable order fractional integration and differentiation where the order is a function of both time, t, and space, x. The variable order fractional integral is shown to allow considerable freedom in the argument of q, the order, which may be utilized to advantage in a variety of applications. The argument of q is also shown to control the manner in which the operator 'remembers' the order history. Two types of memory measures have been shown. The first relates to the well understood, fading memory of the fractional operator, while the second type gives insight to the manner in which the variations in order within the operator are remembered. Measures have been introduced to quantify the memory effects.

Physical models for the three primary cases of variable integration have been shown. More significantly operational methods have been determined for the three primary cases and some applications of the methods have been shown. Extension from the primary cases to 'tailored' operators was developed, and some properties of these tailored operators have been shown. Further research in this area is needed to derive a general operational method that applies to all tailored variable order differintegrals.

Initialization of the variable order operators has been shown.

The fact that the variable order fractional operator is linear with respect to the input, $f(t)$, and nonlinear with respect to q when order is considered as an input presents both problems and opportunity. The problem is the loss of the conventional composition law, which increases analytical difficulty. Until generalized composition laws are evolved for the variable order differintegral, the analyst is left primarily with numerical approaches. On the positive side, the dual linear/nonlinear nature of the operator may enable progress in the area of nonlinear systems.

Two approaches to distributed order fractional operators have been presented. This distributed operator effectively combines all linear constant coefficient derivatives into a single operator. Laplace transform operations with distributed order operators have been derived and a viscoelastic strain application with a temperature gradient is shown.

Finally, operators of multivariable order have been defined generalizing the variable order operators. Further, the effect of (time) variable fractional differentiation has been introduced into the order distribution, increasing its applicability. It is the authors' belief that very many applications will be found for the new variable and distributed order operator definitions, forwarded here, in engineering and the sciences.

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