FRACTIONAL ORDER NUMERICAL DIFFERENTIATION WITH B-SPLINE FUNCTIONS

D.Y. Liu[1], T.M. Laleg-Kirati[1], O. Gibaru[2,4] and W. Perruquetti[3,4]

[1] *King Abdullah University of Science and Technology (KAUST), CEMSE Division, Thuwal, Saudi Arabia*
[2] *Arts et Métiers ParisTech, LSIS (CNRS, UMR 7296), Lille, France*
[3] *Ecole Centrale de Lille, LAGIS (CNRS, UMR 8146), Villeneuve d'Ascq, France*
[4] *INRIA Lille-Nord Europe, Non-A Team, Villeneuve d'Ascq, France*

***Abstract* — Smoothing noisy data with spline functions is well known in approximation theory. Smoothing splines have been used to deal with the problem of numerical differentiation. In this paper, we extend this method to estimate the fractional derivatives of a smooth signal from its discrete noisy data. We begin with finding a smoothing spline by solving the Tikhonov regularization problem. Then, we propose a fractional order differentiator by calculating the fractional derivative of the obtained smoothing spline. Numerical results are given to show the efficiency of the proposed method in comparison with some existing methods.**

***Keywords:* Fractional order numerical differentiation, B-Spline, Ill-posed problem, Tikhonov regularization, L-curve method, Generalized cross-validation method**

I – Introduction

Fractional calculus were introduced in many fields of science and engineering long time ago [1, 2]. The recent work of A. Oustaloup showed that using fractional derivatives in control design can improve the performances and robustness properties [3, 4, 5, 6]. This motivated the interest in using fractional derivatives in signal processing applications, such as edge detection [7], electrocardiogram signal detection [8], biological signal processing [9], and image signal processing [10]. For these applications, we need to apply a fractional order differentiator which estimates the fractional derivatives of an unknown signal from its discrete noisy observed data, which is the scope of this paper.

The problem of integer order numerical differentiation is a well known ill-posed problem in the sense that a small error in noisy observed data can induce a large error in the approximated derivatives. Various numerical methods have been developed to obtain stable integer order differentiators more or less sensitive to additive noises. One immediate idea is to smooth noisy data by a filter and then to use the derivative of the filter as a differentiator. Bearing this idea in mind, different filters have been used in the integer order derivative case, such as the Savitzky-Golay filter [11, 12], Jacobi polynomial filter [13, 14], and splines filter [15, 16]. Recently, some of these differentiators have been generalized from the integer order to the fractional order by calculating the fractional derivative of the filters. The **D**igital **F**ractional **O**rder **S**avitzky-**G**olay **D**ifferentiator (DFOSGD) was introduced in [17], where it has been shown that the DFOSGD is better than some existing fractional order differentiators. The integer order differentiation by integration method based on the Jacobi polynomial filter [18, 19, 20, 21, 22] has been generalized to the fractional case in [23, 24]. Let us recall that the integer order Jacobi differentiator obtained by the differentiation by integration method has also been given using a recent algebraic parametric method [25, 26] which exhibits good robustness properties with respect to corrupting noises without the need of knowing their statistical properties [27, 28]. Moreover, it has been shown in [23] that the fractional order Jacobi differentiator is better than the DFOSGD both in the noisy and noise-free cases.

Unlike the classical integer order derivative which can be estimated using a sliding window, the fractional derivative is an hereditary operator and needs a total memory of past states [1]. Hence, it was mentioned in [24] that when the length of the interval, where we estimate the fractional derivative, increases, the degree of the smoothing Jacobi polynomial in the fractional order Jacobi differentiator must be increased so as to decrease the truncated term error. While as the spline is a piecewise polynomial, we can avoid such a problem using a smoothing spline. Let us recall that smoothing noisy data with spline functions is well known in approximation theory. Many researchers proposed to use smoothing splines to deal with the problem of numerical differentiation (see e.g., [15, 16, 29]). Very recently, Cubic B-Splines have been used to solve fractional differential equations [30]. However, to the best of our knowledge, B-Splines of an arbitrary order have not been used to solve the fractional order numerical differentiation problem.

The aim of this paper is to extend to fractional orders the method of numerical differentiation via the use of B-Spline functions. In Section II, we recall the Riemann-Liouville fractional derivative and some B-Spline properties. After introducing an important result from the approximation theory with spline functions, we give a fractional order differentiator by solving the Tikhonov regularization problem in Section III. Then, we give a numerical algorithm that applies the proposed differentiator. In Section IV, we compare the proposed fractional order differentiator to the DFOSGD and the fractional order Jacobi differentiator in some numerical simulations, where the L-curve method and the Generalized cross-validation method are used to find the regularization parameter respectively. Finally in Section V, we give some conclusions and perspectives for future works.

II – Preliminary

In this section, we recall the Riemann-Liouville fractional derivative and some properties of B-Splines.

A. Riemann-Liouville fractional derivative

This study only considers the Riemann-Liouville fractional derivative. Similar results can be obtained using the other fractional derivatives' definitions.

Let $l \in \mathbb{N}^*$, $a \in \mathbb{R}$, and $f \in \mathscr{C}^l(\mathbb{R})$ where $\mathscr{C}^l(\mathbb{R})$ refers to the set of functions being l-times continuously differentiable on $\mathbb{R}$. Then, the Riemann-Liouville fractional derivative (see [1] p. 62) of f is defined as follows: $\forall t > a$,

$$\mathrm{D}^{\alpha}_{a,t} f(\cdot) := \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_a^t (t-\tau)^{l-\alpha-1} f(\tau)\, d\tau, \quad (1)$$

where $l-1 \leq \alpha < l$, and $\Gamma(\cdot)$ is the Gamma function (see [31] p. 255).

As an example, if we take $f(t) = (t-a)^n$ with $\alpha \leq n \in \mathbb{N}$ and $a \leq t \in \mathbb{R}$, then using (1) we obtain (see [1] p. 72):

$$\mathrm{D}^{\alpha}_{a,t} f(\cdot) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} (t-a)^{n-\alpha}. \quad (2)$$

Let us recall the linearity property of the fractional derivative (see [1] p. 91): $\forall t > a$,

$$\begin{aligned} &\mathrm{D}^{\alpha}_{a,t} \{\lambda_1 f_1(\cdot) + \lambda_2 f_2(\cdot)\} = \\ &\qquad \lambda_1 \mathrm{D}^{\alpha}_{a,t} f_1(\cdot) + \lambda_2 \mathrm{D}^{\alpha}_{a,t} f_2(\cdot), \end{aligned} \quad (3)$$

where $l-1 \leq \alpha < l$ with $l \in \mathbb{N}^*$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $f_1, f_2 \in \mathscr{C}^l(\mathbb{R})$.

Consequently, based on (2) and (3) we can calculate the fractional derivatives of any polynomial. Moreover, we can also calculate the fractional derivatives of a truncated power function.

Lemma 1 *Let $(\cdot - b)^n_+$ ($b \in \mathbb{R}$, $n \in \mathbb{N}$) be a truncated power function defined as follows (see [32] p. 46): $\forall t \in \mathbb{R}_+$,*

$$(t-b)^n_+ := \begin{cases} (t-b)^n, & \text{if } t \geq b, \\ 0, & \text{else.} \end{cases} \quad (4)$$

Then, the α^{th} ($n \geq \alpha \in \mathbb{R}_+$) order derivative of the truncated power function defined in (4) is given as follows: $\forall t \in \mathbb{R}_+$,

$$\mathrm{D}^{\alpha}_{0,t} (\cdot - b)^n_+ = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} (t-b)^{n-\alpha}_+. \quad (5)$$

Proof. According to (1), we have: $\forall t \geq b$,

$$\begin{aligned} \mathrm{D}^{\alpha}_{0,t} (\cdot - b)^n_+ &= \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_0^t \frac{(\tau-b)^n_+}{(t-\tau)^{\alpha+1-l}} d\tau \\ &= \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_b^t \frac{(\tau-b)^n}{(t-\tau)^{\alpha+1-l}} d\tau. \end{aligned} \quad (6)$$

Consequently, this proof can be completed using (2). □

B. Uniform B-Splines

Let $I = [0,h]$ be an interval of $\mathbb{R}_+$. Then, we take M real values T_i, called *knots*, with $T_i = \frac{h}{M-1} i$ and $2 \leq M \in \mathbb{N}$, for $i = 0, \cdots, M-1$. Hence, using these equidistant knots and the truncated power functions, we can define the N^{th} ($N \in \mathbb{N}$) degree uniform B-Splines as follows [33]: $\forall t \in I$,

$$b_{j,N}(t) = b_N(t - T_j), \quad j = 0, \cdots, M-N-2, \quad (7)$$

where

$$b_N(t) := (T_{N+1} - T_0) \sum_{i=0}^{N+1} w_{i,N} (t - T_i)^N_+, \quad (8)$$

with

$$w_{i,N} := \prod_{k=0, k\neq i}^{N+1} \frac{1}{T_k - T_i}. \quad (9)$$

Consequently, the uniform B-Splines for a given degree N are just shifted copies of each others. Moreover, using Lemma 1 and the linearity of the fractional derivative, we can obtain the following lemma.

Lemma 2 *Let $b_{j,N}$, for $j = 0, \cdots, M-N-2$, be the uniform B-Spline defined in (7). Then, the fractional derivative of $b_{j,N}$ is given as follows: $\forall t \in I$,*

$$\begin{aligned} &\mathrm{D}^{\alpha}_{0,t} b_{j,N}(\cdot) = \\ &(T_{N+1} - T_0) \sum_{i=0}^{N+1} \frac{w_{i,N}\, \Gamma(N+1)}{\Gamma(N+1-\alpha)} (t - T_j - T_i)^{N-\alpha}_+, \end{aligned} \quad (10)$$

where $\alpha \leq N$.

III – Fractional order differentiator with B-Spline functions

Let us consider the following discrete noisy signal observed on an interval $I = [0,h] \subset \mathbb{R}$:

$$y^{\varpi}(t_i) = y(t_i) + \delta\varpi(t_i), \quad (11)$$

where $t_i = \frac{h}{m-1} i$ for $i = 0, \cdots, m-1$ with $2 \leq m \in \mathbb{N}$, the noise $\{\delta\varpi(t_i)\}$ is a sequence of random variables with a zero-mean and an unknown variance δ^2, and $y \in \mathscr{W}_2^{(n)}(I) := \left\{ f : f^{(i)} \text{ abs. cont.}, i = 0, \cdots, n-1, f^{(n)} \in \mathscr{L}_2(I) \right\}$.

We want to estimate the fractional derivative of y using its noisy observation y^{ϖ}. For this purpose, we first find a smoothing spline to approximate the original signal y.

A. Smoothing with spline functions

The problem of numerical differentiation is a well known ill-posed problem in Hadamard's sense [34]. Indeed, a small error in noisy observed data can induce a large error in the approximated derivatives [35]. In order to tackle this problem, various regularization methods have been used to smooth a signal from its discrete noisy data [15, 11, 12, 13, 14]. One of the famous regularization criteria widely considered in numerical analysis and statistic, which can be used to find a smoothing function of y^{ϖ} on I, is the Tikhonov regularization defined as follows (see [15, 36]):

$$\min_{f\in \mathscr{W}_2^{(n)}(I)} \left\{ \frac{1}{m}\sum_{i=0}^{m-1} \left(f(t_i)-y^{\varpi}(t_i)\right)^2 + \lambda \left\| f^{(n)} \right\|^2_{\mathscr{L}_2(I)} \right\}, \tag{12}$$

where $\lambda \in \mathbb{R}_+$. Let us recall that the solution is a smoothing spline of degree $2n-1$ ([15]), and the regularization parameter λ controls the tradeoff between the robustness against the corrupting noise and the accuracy of approximation of the original signal. In particular, if λ is equal to zero, then the minimization in (12) refers to the classical least-squares approximation by a spline of degree $2n-1$.

From now on, we are going to consider a spline function of degree $2n-1$ as the solution of (12). Based on this smoothing spline function, we can define a fractional order differentiator in the following proposition.

Proposition 1 *Let y^{ϖ} be the discrete noisy observed data of $y \in \mathscr{W}_2^{(n)}(I)$, which is defined in (11), and $b_{j,N}$, for $j = 0,\cdots,M-N-2$, be the N^{th} degree uniform B-Spline defined by (7) with $N = 2n-1$. Then, the values of the α^{th} ($\alpha \le n-1$) order derivative of y can be estimated as follows: for $i = 0,\cdots,m-1$,*

$$\mathrm{D}^{\alpha}_{0,t_i} y(\cdot) \approx \sum_{j=0}^{M-N-2} \beta_j \, \mathrm{D}^{\alpha}_{0,t_i} b_{j,N}(\cdot), \tag{13}$$

where $\mathrm{D}^{\alpha}_{0,t_i} b_{j,N}(\cdot)$ is given by Lemma 2 with $t = t_i$, and $\beta = [\beta_0,\cdots,\beta_{M-N-2}]^T$ can be obtained by solving the Tikhonov regularization problem:

$$\min_{\beta\in\mathbb{R}^{M-N-1}} \left\{ \frac{1}{m} \left\| Y^{\varpi} - B\beta \right\|_2^2 + \lambda \left\| H_n B \beta \right\|_2^2 \right\}, \tag{14}$$

with $Y^{\varpi} = [y^{\varpi}(t_0),\cdots,y^{\varpi}(t_{m-1})]^T$,

$$B(i,j) = b_{j-1,N}(t_{i-1}), \tag{15}$$

for $i = 1,\cdots,m$ and $j = 1,\cdots,M-N-1$, and H_n is the differentiation matrix given by:

$$H_n(i,j) = \begin{cases} (-1)^{n+j-i}\binom{n}{j-i}\left(\frac{m-1}{h}\right)^n, & \text{if } i \le k \le i+n, \\ 0, & \text{else}, \end{cases} \tag{16}$$

for $i = 1,\cdots,m-n$, and $j = 1,\cdots,m$.

Proof. Let the spline $\hat{y}(\cdot) = \sum_{j=0}^{M-N-2} \beta_j b_{j,N}(\cdot)$ be the solution of the Tikhonov regularization problem (14), where we approximate the norm $\|\cdot\|^2_{\mathscr{L}_2(I)}$ in (12) by $\|\cdot\|_2^2$ using the finite difference scheme for the n^{th} order derivative [16]. Then, we use the fractional derivative value of $\hat{y}$ at $t = t_i$ to estimate $\mathrm{D}^{\alpha}_{0,t_i} y(\cdot)$. Hence, this proof can be completed by using Lemma 2 and the linearity of the fractional derivative. □

One classical method to choose the regularization parameter λ is the Generalized cross-validation method introduced in [15]. In this paper, we propose both the L-curve method and the Generalized cross-validation method to choose λ. Once λ is given, we can solve the Tikhonov regularization problem (14) to find the smoothing spline $\hat{y}$. Then, the coefficients vector β can be obtained by solving the following least-square problem:

$$\beta = (B^T B)^{-1} B^T \hat{Y}, \tag{17}$$

where $\hat{Y} = [\hat{y}(t_0),\cdots,\hat{y}(t_{m-1})]^T$.

In the next subsection, we are going to give the numerical algorithm for our fractional order differentiator.

B. Numerical algorithm

In this subsection, we summarize the estimation procedure in the following steps:

Step 1 Define M uniform knots for a given interval $I = [0,h]$.

Step 2 Construct the N^{th} order uniform B-Splines using (7) and the knots defined in Step 1.

Step 3 Construct the matrices B and H_n ($n = \frac{N+1}{2}$) given in (15) and (16) respectively.

Step 4 Choose the regularization parameter $\hat{\lambda}$ using the algorithms given in [37], such as the L-curve method or the Generalized cross-validation method.

Step 5 Solve the Tikhonov regularization problem (14) to find the smoothing spline $\hat{Y}$ using the parameter $\hat{\lambda}$ obtained in Step 4 and the algorithm of the Tikhonov regularization given in [37].

Step 6 Find the coefficients vector β by solving the least-square problem (17) using the smoothing spline $\hat{Y}$ and B.

Step 7 Using (13), calculate the matrix product $B^{\alpha}\beta$ with

$$B^{\alpha}(i,j) = \mathrm{D}^{\alpha}_{0,t_{i-1}} b_{j-1,N}(\cdot), \tag{18}$$

for $i = 1,\cdots,m$ and $j = 1,\cdots,M-N-1$.

IV – Simulation results

Some comparisons among some exiting fractional order differentiators have been given in [23, 17]. In this section, we show the accuracy and the robustness with respect to corrupting noises of the proposed fractional order differentiator by comparing with the DFOSGD method and the fractional order Jacobi differentiator method introduced in [17] and [23, 24] respectively.

In the following examples, we assume that $y^{\varpi}(t_i) = y(t_i) + \delta\varpi(t_i)$ is a discrete noisy observation of a sinusoidal signal $y(\cdot) = \sin(5\cdot)$ on $I = [0,h]$, with $t_i = T_s i$, for $i = 0, \cdots, m-1$, where $T_s = \frac{h}{m-1}$ is an equidistant sampling period. The variance δ^2 is adjusted such that the signal-to-noise ratio $SNR = 10\log_{10}\left(\frac{\sum|y^{\varpi}(t_i)|^2}{\sum|\delta\varpi(t_i)|_2^2}\right)$ is equal to $SNR = 10$dB (see [38] for this well known concept in signal processing). Let us recall the classical Riemann-Liouville fractional derivative of $\sin(5\cdot)$ (see [39] p. 83):

$$\mathrm{D}_{0,t}^{\alpha}\sin(5\cdot) = \frac{5t^{1-\alpha}}{\Gamma(2-\alpha)}\,{}_1\mathrm{F}_2\left(1;\frac{1}{2}(2-\alpha),\frac{1}{2}(3-\alpha);-\frac{1}{4}5^2t^2\right), \tag{19}$$

where ${}_1\mathrm{F}_2$ is the generalized hypergeometric function (see [39] p. 303).

We estimate the 0.5^{th} and 1.5^{th} order derivatives of y using the DFOSGD, the fractional order Jacobi differentiator, and the proposed fractional order B-Spline differentiator on different intervals. In order to find the regularization parameter in our differentiator, we propose to use both the L-curve method and the Generalized cross-validation method. Since each differentiator depends on several parameters, it is difficult to do the comparison. In the following examples, the parameters used for each differentiators are the optimal ones that we can find out.

Example 1. In this example, we take $h = 4$ and $m = 2001$ with $T_s = 0.002$. The value of δ is equal to 0.22. We can see the noisy signal in Figure 1. We take $M = 1001$ and $N = 3$ ($n = 2$) in our differentiator. For the DFOSGD and the fractional order Jacobi differentiator, the degree of the approximation polynomial is set to 14. Moreover, we take $\theta = 1$ for the the DFOSGD, and $\kappa = \mu = 0$ for the fractional order Jacobi differentiator. The obtained absolute estimation errors are shown in Figure 2 and Figure 3 respectively. Consequently, we can observe that all these three fractional order differentiators give small estimation errors in the interval except near the boundaries.

Example 2. In this example, we take $h = 10$ and $m = 2001$ with $T_s = 0.005$. The value of δ is equal to 0.22. We take the same parameter values as in Example 1 for the three differentiators, except that we increase the degree of the approximation polynomial to 26 for the DFOSGD and the fractional order Jacobi differentiator. The obtained absolute estimation errors are shown in Figure 4 and Figure 5 respectively. Consequently, we can observe that the estimation errors obtained in our differentiator almost remain the same level as in Example 1. However, the DFOSGD gives large estimation error, and the fractional order Jacobi differentiator becomes divergent when the time t increases.

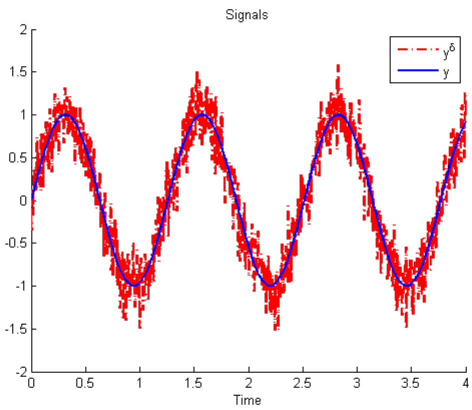

Figure 1: *Signal y and noisy signal* y^{δ}.

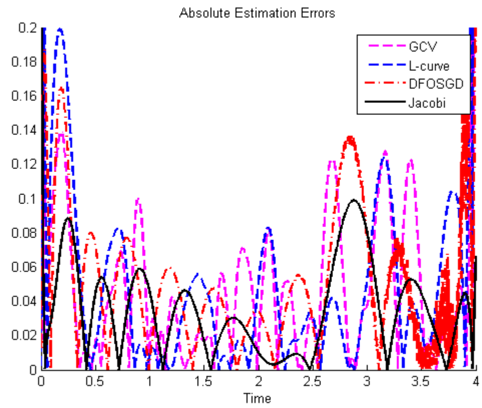

Figure 2: *Absolute estimation errors in the case where* $h = 4$ *and* $\alpha = 0.5$.

V – Conclusion

In this paper, we have proposed a fractional order differentiator which is deduced from a smoothing spline function and the the Riemann-Liouville fractional derivative definition. The smoothing spline was obtained by solving the Tikhonov regularization problem using the L-curve method or the Generalized cross-validation method. Numerical examples have shown that this differentiator can accurately estimate the fractional order derivatives of noisy signals even defined on a long interval. However, the proposed numerical algorithm can only be used in off-line applications. In a future work, we will improve the proposed differentiator for on-line applications.

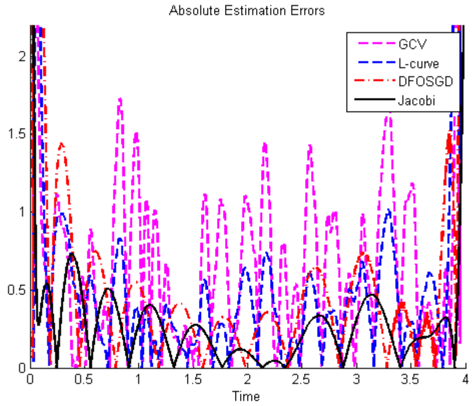

Figure 3: *Absolute estimation errors in the case where* $h = 4$ *and* $\alpha = 1.5$.

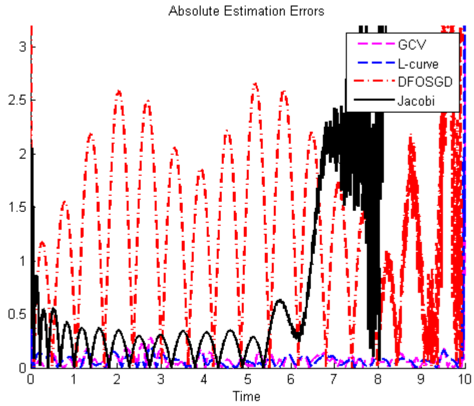

Figure 4: *Absolute estimation errors in the case where* $h = 10$ *and* $\alpha = 0.5$.

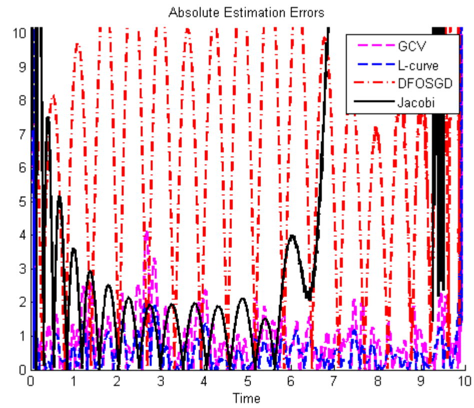

Figure 5: *Absolute estimation errors in the case where* $h = 10$ *and* $\alpha = 1.5$.

References

[1] I. Podlubny, *Fractional Differential Equations*, vol. 198 of Mathematics in Science and Engineering, Academic Press, New York, NY, USA, 1999.

[2] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[3] A. Oustaloup, B. Mathieu and P. Lanusse, The CRONE control of resonant plants: application to a flexible transmission, *European Journal of Control*, vol. 1, no. 2, pp. 113-121, 1995.

[4] A. Oustaloup, J. Sabatier and X. Moreau, From fractal robustness to the CRONE approach, in *Proc. ESAIM: Proceedings*, pp. 177-192, Dec. 1998.

[5] A. Oustaloup, J. Sabatier and P. Lanusse, From fractal robustness to the CRONE control, *FCAA*, vol. 1, no. 2, pp. 1-30, Jan. 1999.

[6] V. Pommier, J. Sabatier, P. Lanusse and A. Oustaloup, CRONE control of a nonlinear hydraulic actuator, *Control Engineering Practice*, vol. 10, pp. 391-402, Jan. 2002.

[7] B. Mathieu, P. Melchior, A. Oustaloup, and Ch. Ceyral, Fractional differentiation for edge detection, *Signal Process.*, vol. 83, no. 11, pp. 2421-2432, 2003.

[8] M. Benmalek, and A. Charef, Digital fractional order operators for R-wave detection in electrocardiogram signal, *IET Signal Processing*, vol. 3, no. 5, pp. 381-391, 2009.

[9] B. Guo, J. Li, and H. Zmuda, A new FDTD formulation for wave propagation in biological media with colecole model, *IEEE Microwave and Wireless Components Letters*, vol. 16, no. 12, pp. 633-635, 2006.

[10] J. Bai, and X.C. Feng, Fractional-order anisotropic diffusion for image denoising, *IEEE Trans. on Image Processing*, vol. 16, no. 10, pp. 2492-2502, 2007.

[11] A. Savitzky and M.J.E. Golay, Smoothing and differentiation of data by simplified least squares procedures, *Anal. Chem.*, vol. 36, no. 8, pp. 1627-1639, 1964.

[12] R.W. Schafer, What is a Savitzky-Golay filter, *IEEE Signal Process. Mag.*, vol. 28, no. 4, pp. 111-117, Jul. 2011.

[13] P.O. Persson and G. Strang, Smoothing by Savitzky-Golay and Legendre filters, *IMA Volume on Math. Systems Theory in Biology, Comm., Comp., and Finance*, vol. 134, pp. 301-316, 2003.

[14] P. Meer and I. Weiss, Smoothed diferentiation filters for images, *J. Visual Comm. Image Repr.*, vol. 3, pp. 58-72, 1992.

[15] P. Craven and G. Wahba, Smoothing noisy data with spline functions, *Numer. Math.*, vol. 31, pp. 377-403, 1979.

[16] S. Ibrir and S. Diop, A numerical procedure for filtering and efficient high-order signal differentiation, *Internat. J. Appl. Math. Comput. Sci.*, vol. 14, pp. 201-208, 2004.

[17] D.L. Chen, Y.Q. Chen and D.Y. Xue, Digital Fractional Order Savitzky-Golay Differentiator, *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 58, no. 11, pp. 758-762, 2011.

[18] M. Mboup, C. Join and M. Fliess, Numerical differentiation with annihilators in noisy environment, *Numerical Algorithms*, vol. 50, no. 4, pp. 439-467, 2009.

[19] M. Mboup, C. Join and M. Fliess, A revised look at numerical differentiation with an application to nonlinear feedback control, in *Proc. 15th Mediterranean conference on Control and automation (MED'07)*, Athenes, Greece, 2007.

[20] D.Y. Liu, O. Gibaru and W. Perruquetti, Differentiation by integration with Jacobi polynomials, *J. Comput. Appl. Math.*, vol. 235, no. 9, pp. 3015-3032, 2011.

[21] D.Y. Liu, O. Gibaru and W. Perruquetti, Error analysis of Jacobi derivative estimators for noisy signals, *Numerical Algorithms*, vol. 58, no. 1, pp. 53-83, 2011.

[22] D.Y. Liu, O. Gibaru and W. Perruquetti, Convergence Rate of the Causal Jacobi Derivative Estimator, *Curves and Surfaces 2011, LNCS 6920 proceedings*, pp. 45-55, 2011.

[23] D.Y. Liu, O. Gibaru, W. Perruquetti and T.M. Laleg-Kirati, Fractional order differentiation by integration with Jacobi polynomials, in *Proc. 51st IEEE Conference on Decision and Control*, Hawaii, USA, 2012.

[24] D.Y. Liu, O. Gibaru, W. Perruquetti and T.M. Laleg-Kirati, Fractional order differentiation by integration and error analysis in noisy environment, submitted to *IEEE Transactions on Automatic Control*.

[25] M. Fliess, C. Join, M. Mboup and H. Sira-Ramrez, Compression diffrentielle de transitoires bruits, *Comptes Rendus Mathematique*, vol. 339, no. 11, pp. 821-826, 2004.

[26] M. Fliess and H. Sira-Ramrez, An algebraic framework for linear identification, *ESAIM Control Optim. Calc. Variat.*, vol. 9, pp. 151-168, 2003.

[27] M. Fliess, Analyse non standard du bruit, *Comptes Rendus Mathematique*, vol. 342, no. 15, pp. 797-802, 2006.

[28] M. Fliess, Critique du rapport signal à bruit en communications numériques – Questioning the signal to noise ratio in digital communications, in *Proc. International Conference in Honor of Claude Lobry*, *Revue africaine d'informatique et de Mathématiques appliquées*, vol. 9, pp. 419-429, 2008.

[29] D.N. Hao, L.H. Chuong and L. Desnic, Heuristic regularisation methods for numerical differentiation, *Comput. Math. Appl.*, vol. 63, pp. 816-826, 2012.

[30] X. Li, Numerical solution of fractional differential equations using cubic B-spline wavelet collocation method, *Commun. Nonlinear Sci. Numer. Simulat.*, vol. 17, no. 10, pp. 3934-3946, 2012.

[31] M. Abramowitz and I.A. Stegun, editeurs, *Handbook of mathematical functions*, GPO, 1965.

[32] P. Massopust, *Interpolation and Approximation with Splines and Fractals*, Oxford University Press, USA, 2010.

[33] C. de Boor, *A Practical Guide to Splines*, Springer-Verlag, 1978.

[34] J. Hadamard, Sur les problèmes aux dérivées partielles et leur signification physique, *Princeton University Bulletin*, pp. 49-52, 1902.

[35] S. Diop, J.W. Grizzle and F. Chaplais,On numerical differentiation algorithms for nonlinear estimation, in *Proc. 39th IEEE Conference on Decision and Control*, Sydney, 2000.

[36] A.N. Tikhonov and V.Y. Arsenin, *Solution of Ill-posed Problems*, Washington: Winston & Sons, 1977.

[37] P.C. Hansen, Regularization tools, a matlab package for analysis and solution of discrete ill-posed problems, Version 4.1 for Matlab 7.3, 2008.

[38] S. Haykin and B. Van Veen, *Signals and Systems*, 2nd edn. John Wiley & Sons, 2002.

[39] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.