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Dynamics and Control of Initialized Fractional-Order Systems

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Abstract. Due to the importance of historical effects in fractional-order systems, this paper presents a general fractional-order system and control theory that includes the time-varying initialization response. Previous studies have not properly accounted for these historical effects. The initialization response, along with the forced response, for fractional-order systems is determined. The scalar fractional-order impulse response is determined, and is a generalization of the exponential function. Stability properties of fractional-order systems are presented in the complex w-plane, which is a transformation of the s-plane. Time responses are discussed with respect to pole positions in the complex w-plane and frequency response behavior is included. A fractional-order vector space representation, which is a generalization of the state space concept, is presented including the initialization response. Control methods for vector representations of initialized fractional-order systems are shown. Finally, the fractional-order differintegral is generalized to continuous order-distributions which have the possibility of including all fractional orders in a transfer function.

Keywords: Fractional-order systems, control systems, order-distributions, fractional calculus.

1. Introduction

1.1. Introduction to Initialization in Fractional-Order Dynamic Systems

Fractional-order systems, or systems described using fractional derivatives and integrals, have been studied by many in the engineering area [1–8]. Additionally, very readable discussions, devoted specifically to the subject, are presented by Oldham and Spanier [9], Miller and Ross [10], and Podlubny [11]. It should be noted that there are a growing number of physical systems whose behavior can be compactly described using fractional-order system theory. Of specific interest to engineers are viscoelastic materials [12–15], electrochemical processes [16, 17], long lines [1], dielectric polarization [18], colored noise [19], and chaos [20]. With the growing number of applications, it is important to establish a clear initialized system theory for these fractional-order systems, so that it may be accessible to the general engineering and scientific communities. This topic is addressed in this paper. It is also important to establish a theory of control for these initialized fractional-order systems, and for the use of initialized fractional-order systems as feedback compensators. This topic is also addressed in this paper. We first discuss the control of fractional-order systems using a vector space representation, where initialization is included in the discussion. It should be noted that a fractional state-space representation is given in [12, 21], but they do not include the general time-varying historic effects. Incorporation of these effects based on the initialized fractional calculus [22] is presented here. Input-output control is also considered, where reference is made to the previous work of Podlubny [11, 23] and of Oustaloup [24]. The input-output controllers are generalized to the concept of continuous order-distribution [25, 26].

This section will establish the definitions for the fractional-order operators to be used in this paper. Using these, fractional-order differential equations are defined. The total response is shown to be composed of a forced response and an initialization response. Requirements for a fractional-order system theory are discussed.

The fundamental definition for a fractional-order integral is based on the Riemann–Liouville definition [9]. Using our notation

$$ {}_a d_t^{-q} x(t) \equiv \frac{d^q(x(t))}{[d(t-a)]^q} \equiv \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau, \quad t \geq a, \tag{1}$$

and where the middle notation is that of Oldham and Spanier [9]. Likewise the Riemann–Liouville fractional derivative is defined as

$$ {}_a d_t^{q} x(t) \equiv \frac{d^m}{dt^m} ({}_a d_t^{q-m} x(t)), \quad t \geq a, \tag{2}$$

where m is chosen as the smallest integer such that $q - m$ is negative, and the integer-order derivatives are defined as usual. These equations define the *uninitialized* fractional integral and derivative.

In many physical systems, system components are forced into some configuration, or initialized. Using mechanical systems as an example, the initial conditions are often mass positions and velocities at time zero. Fractional-order components, however, require a more complicated initialization, as they have an inherent time-varying memory effect. Considering the fractional-order integral above, we will assume that the fractional-order integral was initialized in the past, beginning at time a, while the given problem will begin at time c. If we now wish the fractional-order integral to operate into the future from time c, then we must account for the effects of the past. We will now define the *initialized* fractional-order integration operator as

$$ {}_c D_t^{-q} x(t) \equiv {}_c d_t^{-q} x(t) + \psi(x, -q, a, c, t), \quad t \geq c, \tag{3}$$

where

$$ \psi(x, -q, a, c, t) \equiv \int_a^c \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau, \quad t \geq c. \tag{4}$$

$\psi(x, -q, a, c, t)$ is called the initialization function, and is generally a time-varying function that must be added to the fractional-order operator to account for the effects of the past. Equivalently, this is accomplished by separating the uninitialized fractional-order integral into the two time periods,

$$ {}_a d_t^{-q} x(t) = \int_a^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau = \int_a^c \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau + \int_c^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau. \tag{5}$$

The first term on the right of this equation represents the historical effect of the initialization, while the second term is a fractional-order integral beginning at time c.

Clearly then ${}_cD_t^{-q}x(t) = {}_ad_t^{-q}x(t)$, $t \geq c$. The initialization function is a time-varying function, and is required to properly bring the historical effects of the fractional-order integral into the future. The above definitions also apply for fractional-order derivatives, that is, for any real value of q. For convenience, we will usually choose $c = 0$.

With the understanding that the term 'fractional-order' component includes all of the previously understood integer-order components as a subset, connecting several fractional-order components together will give a fractional-order differential equation. As an example, a mechanical system with a spring, a mass, and viscoelastic damping, yields the following fractional-order differential equation

$$m({}_0D_t^2x(t)) + k_2({}_0D_t^qx(t)) + k_1({}_0D_t^0x(t)) = f(t), \quad t \geq 0, \tag{6}$$

where $x(t)$ represents the position of a mass in combination with the other components, $f(t)$ represents a forcing function, m, k_1, and k_2 represent mechanical constants, the second derivative term is associated with the mass, the q-derivative term is associated with the viscoelastic term, and the 0-derivative term is associated with the spring. Here, each of the derivative terms carries with it an associated initialization function based on that particular component's past history; ψ_m for the mass with displacement $x_m(t)$ for $a \leq t < 0$, ψ_v for the viscoelastic component with displacement $x_v(t)$ for $a \leq t < 0$, and ψ_k for the spring with displacement $x_k(t)$ for $a \leq t < 0$. Rewriting this using Equation (3) yields

$$\begin{aligned} m({}_0d_t^2x(t) + \psi_m(x_m, 2, a, 0, t)) + k_2({}_0d_t^qx(t) + \psi_v(x_v, q, a, 0, t)) \\ + k_1({}_0d_t^0x(t) + \psi_k(x_k, 0, a, 0, t)) = f(t), \quad t \geq 0, \end{aligned} \tag{7}$$

where the individual ψ-functions contain initial positions, velocities, and all other historical effects of that specific system component required to describe the future system behavior. It is important to understand that the individual components do not necessarily have a shared history during the initialization period and each component can be separately initialized, with the only requirement being that the components act together for $t \geq 0$. Thus we have different subscripts on the x's in the ψ-functions. These x's should be thought of as pointers to identify the history of the particular system variable, or component. Consequently, one cannot generally 'start the system earlier' since the system components need not be connected together in negative time while they are being initialized. Recognizing that the initialization functions are generally time-varying, they appear to be additional forcing functions that drive the system after time $c = 0$. As a consequence, there will be two separate system responses, a forced response due to the input function $f(t)$, and an initialization response due to all the time-varying initialization functions. This example is considered more completely in [27].

Considering this example problem, several desires for a fractional-order linear system theory can be established. An particularly important desire is that Laplace transforms can still be easily used, as they are standard tools in linear system theory. In this regard, the Laplace transform of the initialized fractional-order differintegral is shown in [28] to be

$$L\{{}_0D_t^qx(t)\} = s^qx(s) + L\{\psi(x, q, a, 0, t)\} \quad \text{for all real } q. \tag{8}$$

Following from this, the initialized differintegral is a generalization of not only the Riemann–Liouville differintegral, but also of the Caputo differintegral and the Miller–Ross sequential differintegral (see, for example, [11, pp. 104–109]).

Another desire for a fractional-order linear system theory is that we can determine the system's impulse response, Green's function, or the generalization of the exponential function, as it is needed in the convolutions associated with both the forced response and the initialization response. This is done first in the next section by considering the fundamental linear fractional-order differential equation. Once the impulse response is found, it is used to find the forced response and the initialization response. We also would like to have some analysis tools, such as a time response–pole location association, and some type of frequency response tools. These necessities are addressed later in this paper. As much of modern system theory uses a vector space approach, fractional-order systems are expressed in a vector space representation. The vector space approach and the associated control methods are presented in the second section. The third section presents an input-output control approach for initialized fractional-order systems. Following this, the next section generalizes the proportional-plus-integral-control (PI-control) and PID-control (PI-plus-derivative) concepts using fractional integrals. This is then further generalized using general fractional-order compensators. Finally the compensator concept is generalized by the use of a continuum of fractions in the compensator via the concept of order-distributions.

1.2. The Fundamental Fractional-Order Differential Equation

The problem to be addressed in this section is the solution of the fundamental linear fractional-order differential equation [29]

$$ {}_cD_t^q x(t) \equiv {}_cd_t^q x(t) + \psi(x, q, a, c, t) = -ax(t) + bu(t), \quad q > 0, \tag{9}$$

where the left side should be interpreted to be the q-th derivative of $x(t)$ starting at time c and continuing until time t. This system is considered to be fundamental because its solution is the fundamental time response, whose combinations provide the solution of more complicated systems, analogous to the exponential function for integer-order systems. Here it will be assumed for clarity that the problem starts at $c = 0$. We also assume temporarily in this section that the initialization function $\psi(x, q, a, c, t)$, is zero. Thus we will be concerned only with the forced response for now. The initialization response is discussed later in this paper. Rewriting Equation (9) with these assumptions gives

$$ {}_0d_t^q x(t) = -ax(t) + bu(t), \quad q > 0. \tag{10}$$

We will use Laplace transform techniques to simplify the solution of this differential equation. In order to do so for this problem, the Laplace transform of the fractional differential is required. Using the results given in [28], and ignoring initialization terms, Equation (10) can be Laplace transformed as

$$ s^q X(s) = -aX(s) + bU(s), \quad q > 0. \tag{11}$$

This equation can be rearranged to obtain the system transfer function

$$ \frac{X(s)}{U(s)} = F(s) = \frac{b}{s^q + a}, \quad q > 0. \tag{12}$$

This is the transfer function of the fundamental linear fractional-order differential equation. As such, it contains the fundamental 'fractional' pole (to be discussed later) and is a fundamental building block for more complicated fractional-order systems. As the constant b in Equation (12) is a constant multiplier, it can be assumed, with no loss of generality, to be unity.

Typically, transfer functions are used to study various properties of a particular system. Specifically, they can be inverse transformed to obtain the system impulse response, which can then be used with the convolution approach to the problem. Generally, if $U(s)$ is given, then the product $F(s)U(s)$ can be expanded using partial fractions, and the forced response obtained by inverse transforming each term separately. To accomplish these tasks, it is necessary to obtain the inverse transform of Equation (12), which is the impulse response, or generalized exponential function, of the fundamental fractional-order system. This is done in the next section.

1.3. The Generalized Impulse Response Function

Although the Laplace transform of Equation (12) is not contained in standard Laplace transform tables, the following transform pair is available:

$$\frac{1}{s^q} = L\left\{\frac{t^{q-1}}{\Gamma(q)}\right\}, \quad q > 0. \tag{13}$$

If we expand the right side of Equation (12) in descending powers of s, we can then inverse transform the series term-by-term and obtain the generalized impulse response. Then expanding the right side of Equation (12) about $s = \infty$ using long division gives

$$F(s) = \frac{1}{s^q + a} = \frac{1}{s^q} - \frac{a}{s^{2q}} + \frac{a^2}{s^{3q}} - \cdots = \frac{1}{s^q}\sum_{n=0}^{\infty}\frac{(-a)^n}{s^{nq}}, \quad q > 0. \tag{14}$$

This series can now be inverse transformed term-by-term using Equation (13). The result is

$$\begin{aligned} L^{-1}\{F(s)\} &= L^{-1}\left\{\frac{1}{s^q} - \frac{a}{s^{2q}} + \frac{a^2}{s^{3q}} - \cdots\right\} \\ &= \frac{t^{q-1}}{\Gamma(q)} - \frac{at^{2q-1}}{\Gamma(2q)} + \frac{a^2 t^{3q-1}}{\Gamma(3q)} - \cdots, \quad q > 0. \end{aligned} \tag{15}$$

The right side can now be collected into a summation and used as the definition of the generalized impulse response function [29]

$$F_q[-a, t] \equiv t^{q-1}\sum_{n=0}^{\infty}\frac{(-a)^n t^{nq}}{\Gamma((n+1)q)}, \quad q > 0. \tag{16}$$

Thus, we have the important Laplace transform identity

$$L\{F_q[a, t]\} = \frac{1}{s^q - a}, \quad q > 0. \tag{17}$$

It is seen that $F_q[-a, t]$ is a generalization of the exponential function, since for $q = 1$,

$$F_1[-a, t] = \sum_{n=0}^{\infty}\frac{(-at)^n}{\Gamma(n+1)} \equiv \mathrm{e}^{-at}. \tag{18}$$

This generalization is the basis for the solution of most linear fractional-order differential equations.

This section has established the F-function as the impulse response of the fundamental linear fractional differential equation. This function is important because it will allow the creation of a concise system theory for fractional-order systems, which is a generalization of

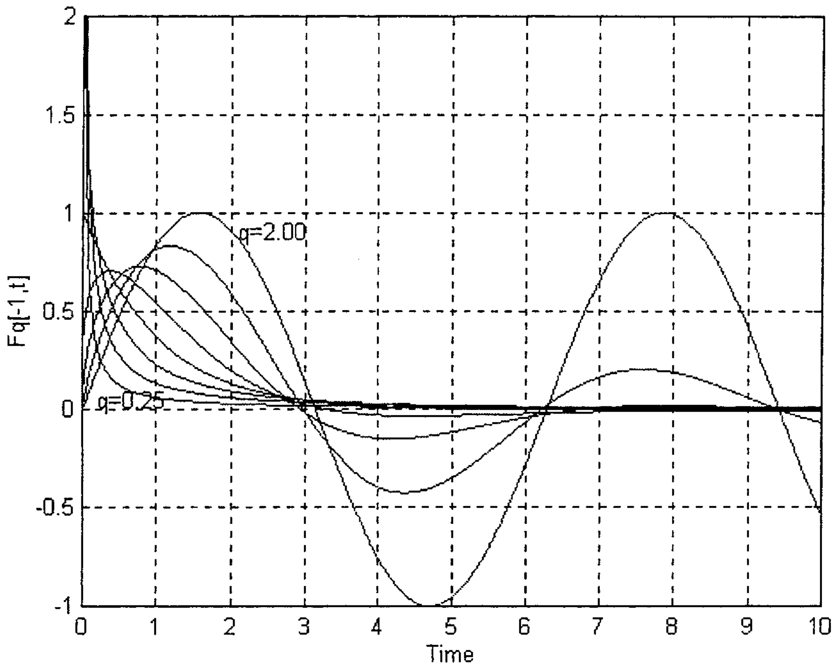

Figure 1. The $F_q[-1, t]$-function *versus* time as q varies from 0.25 to 2.0 in 0.25 increments.

that for integer-order systems, and where the F-function generalizes and replaces the usual exponential function. Plots of the F-function for various values of q are given in Figure 1. This and other function plots can be found in [29].

It is noted that other authors have obtained a solution to Equation (10), but they are less direct. Bagley and Calico [12] obtain a solution as a series of Mittag–Leffler functions. Miller and Ross [10] obtain a solution as a series of the fractional derivatives of the exponential function. They use the function

$$E_t(v, a) \equiv {}_0d_t^{-v}\, \mathrm{e}^{at}, \tag{19}$$

whose Laplace transform is

$$L\{E_t(v, a)\} = \frac{s^{-v}}{s-a}. \tag{20}$$

Glockle and Nonnenmacher [30] obtain a solution in terms of the even more complicated Fox functions. Clearly, all of these functions are useful for this problem (Equation (12)), but the F-function presented here appears to most properly generalize the exponential function for use with fractional differential equations. It should be noted that the F-function is also mentioned by other authors as well. Oldham and Spanier [9, p. 122] mention such a function in passing in a footnote discussing eigenfunctions. Robotnov [31] studied the F-function with respect to hereditary integrals (he calls it the Cyrillic backwards E, or 'eh'-function), and even created an extensive set of tables [32]. Padovan and Sawicki [21] also discuss a similar function in the context of viscoelasticity and constitutive equations. Even more recently, Matignon [33] derives the same function and calls it the script-E function. He goes on to generalize it further and discuss some stability properties. Finally, Podlubny [11] derives a generalization of the

F-function for transfer functions whose denominators contain relatively prime powers of the Laplace variable s. We have chosen to call the impulse response function the F-function, as it is a different generalization of the exponential function than the Mittag–Leffler E-function, and is more easily referred to than many of the other more 'exotic' symbols. In the next section we will consider various properties of the fractional impulse response function.

Another function of considerable utility is given below along with its Laplace transform,

$$R_{q,v}[a,t] \equiv \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1-v}}{\Gamma((n+1)q-v)} \Leftrightarrow \frac{s^v}{s^q-a}, \quad q-v>0. \tag{21}$$

This function generalizes the F-function, and is useful because the fractional derivatives and integrals of the F-function are built directly into it. This is discussed at length in [34, 35]. There, the even more general function given below is also presented:

$$\begin{aligned} G_{q,v,r}[a,t] &\equiv \sum_{j=0}^{\infty} \frac{\{(-r)(-r-1)\cdots(1-j-r)\}(-a)^j (t)^{(r+j)q-1-v}}{\Gamma(1+j)\Gamma((r+j)q-v)} \\ &\Leftrightarrow \frac{s^v}{(s^q-a)^r}, \quad qr-v>0. \end{aligned} \tag{22}$$

This function allows multiple roots in the denominator, and further, r is not restricted to be an integer.

1.4. Stability Analysis in the w-Plane

To understand the dynamic behavior and stability properties of the system of Equation (10), it is necessary to analyze the pole location of the system transfer function in Equation (12). For integer-order linear system theory ($q = 1$), the pole locations are studied in the complex Laplace s-plane. The stability boundary in the s-plane is the imaginary axis; any poles lying to the left of the imaginary axis represent a stable time response, while the poles lying to the right of the imaginary axis represent an unstable time response. Examining Equation (12), however, indicates that the poles of the transfer function must now be evaluated in what would appear to be the s^q-plane. Rather than dealing with the fractional power of s, the analysis is simplified if a change of variables is used. We will define $w = s^q$, and then the pole location analysis will be performed in the new complex w-plane, which is a mapping of the s-plane.

To accomplish this, it is necessary to map the s-plane, along with the time-domain function properties associated with each point, into the complex w-plane. To simplify discussion we will limit the order of the fractional operator to $0 < q \leq 1$. Let

$$w = \rho\, \mathrm{e}^{j\phi} = \alpha + j\beta \quad \text{and} \quad s = r\, \mathrm{e}^{j\theta}. \tag{23}$$

Then referring to the definition of w,

$$w = s^q = (r\, \mathrm{e}^{j\theta})^q = r^q\, \mathrm{e}^{jq\theta} = \rho\, \mathrm{e}^{j\phi}, \tag{24}$$

which gives

$$\rho = r^q \quad \text{and} \quad \phi = q\theta. \tag{25}$$

With this equation, it is possible to map either lines of constant radius, or lines of constant angle from the s-plane into the w-plane. Of particular interest is the image of the s-plane stability boundary (the imaginary axis), that is $s = r\,\mathrm{e}^{\pm j\pi/2}$. The image of this line in the w-plane is

$$w = r^q\,\mathrm{e}^{\pm jq\pi/2}, \tag{26}$$

which is the pair of lines at $\phi = \pm q\pi/2$. Thus, the right half of the s-plane maps into a wedge in the w-plane of angle less than $\pm 90q$ degrees, that is, the right half s-plane maps into

$$|\phi| < \frac{q\pi}{2}. \tag{27}$$

For example, with $q = 1/2$, the right half of the s-plane maps into the wedge bounded by $|\phi| < \pi/4$. A half-order system with its w-plane poles in this wedge, that is with $|\phi| < \pi/4$, would be unstable, and the corresponding F-function response would grow without bound.

It is also important to consider the mapping of the negative real s-plane axis, $s = r\,\mathrm{e}^{\pm j\pi}$. The image is

$$w = r^q\,\mathrm{e}^{\pm jq\pi}. \tag{28}$$

Thus the entire primary sheet of the s-plane maps into a w-plane wedge of angle less than $\pm 180q$ degrees, while all of the secondary s-domain sheets map into the remainder of the w-plane. For example, if $q = 1/2$, then the negative real s-plane axis maps into the w-plane lines at ± 90 degrees.

To summarize the above, the shape of the F-function time response, $F_q[a, t]$, depends upon both q, and the parameter a, which is the pole location of the system transfer function of Equation (12). This is shown in Figure 2. For a fixed value of q, the angle ϕ of the parameter a, as measured from the positive real w-axis, determines the type of response to expect. For small angles, $|\phi| < q\pi/2$, the time response will be *unstable* and oscillatory, corresponding to poles in the right half s-plane (see plots in [29]). For larger angles, $q\pi/2 < |\phi| < q\pi$, the time response will be stable and oscillatory (*underdamped*), corresponding to poles in the left half s-plane. For $|\phi| = q\pi$, the time response will be *overdamped*. For even larger angles, $q\pi < |\phi| < \pi$, the time response will be called *hyperdamped*, corresponding to poles on secondary Riemann sheets. Finally, if $|\phi| = \pi$, the time response will be called *ultradamped* (which could otherwise be called over-hyperdamped). These regions are discussed further in Section 3.7, and shown in Figure 3.

All of the usual control techniques concerning poles, or eigenvalues, can be used in the w-plane, remembering that the stability boundary is now the image of the s-plane imaginary axis. In the discussion that follows, pole placement using state feedback will place the closed-loop poles in the w-plane, and all the root-locus analysis is performed in the w-plane as well.

1.5. The Scalar Initialization Problem

The initialization response is now considered. Although initial conditions are considered in many of the outstanding works in the area of fractional-order systems [9, 11], the initializations presented in these references are not sufficiently general to solve many realistic problems. Rather than the usual initial conditions, fractional-order systems require initialization functions to adequately represent the historical effects of past inputs. The scalar problem is considered first. Then the more general vector problem is considered.

The initialized linear scalar fractional differential equation, for $q > 0$, is given in Equation (9) as

$$_0D_t^q x(t) \equiv {_0d_t^q} x(t) + \psi(x, q, a, 0, t) = -ax(t) + bu(t), \tag{29}$$

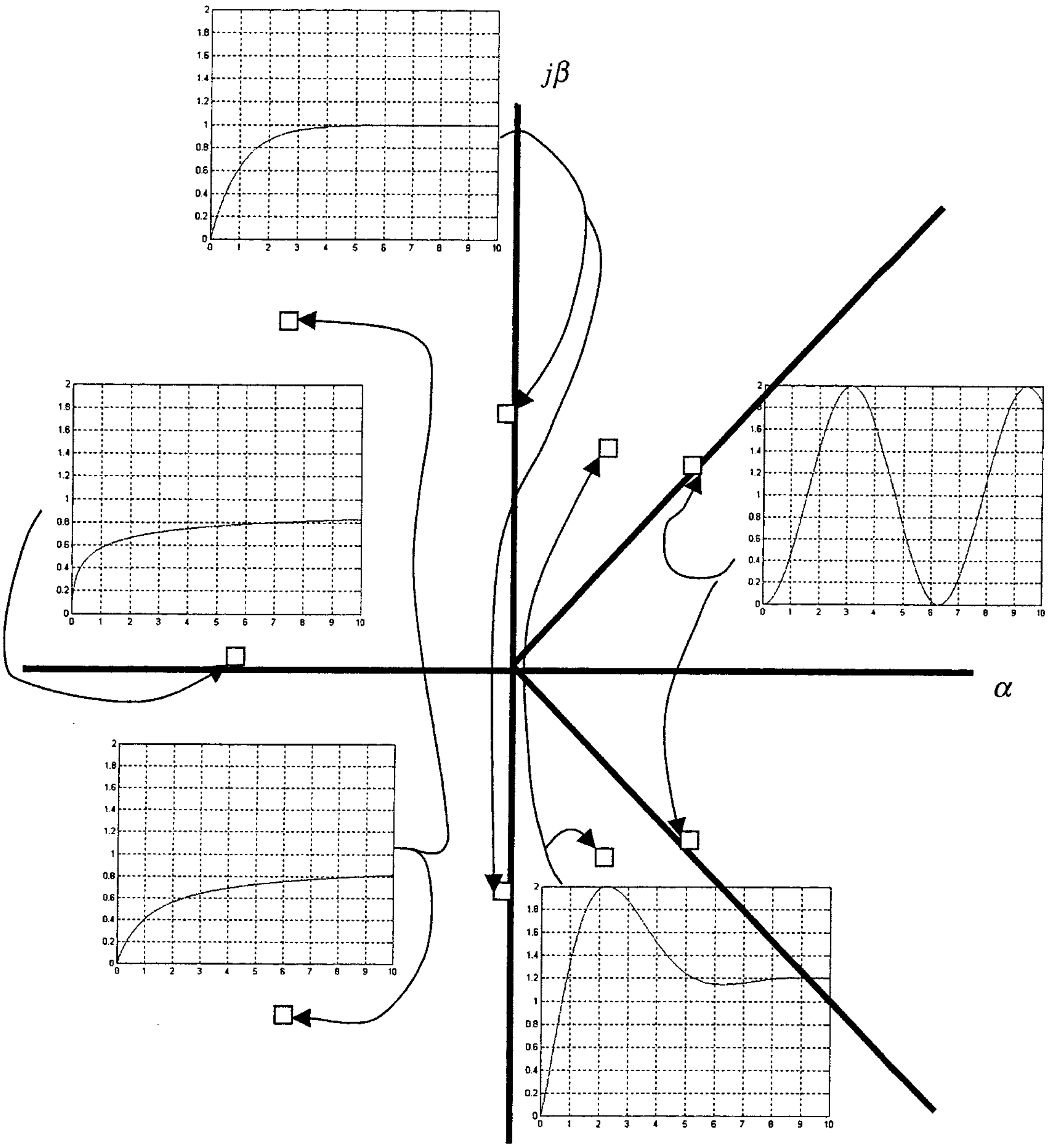

Figure 2. Step responses corresponding to various pole locations in the w-plane, for $q = 1/2$.

where it is assumed that the system is quiescent at time a, that is $x(t) = 0$ for all $t < a$ (and the reader should not confuse the initial time $t = a$ with the system pole at $w = a$). Here $\psi(x, -q, a, c, t)$ is called the initialization function, and is a time-varying function that must be added to the fractional-order operator to account for the effects of the past. It is important to point out that the fractional derivative is generally not a local operator, as the integer derivative normally is, but is an operator with a fading memory. The ψ-term on the left side of Equation (29) represents the historical effects due to the behavior of the differintegral before $t = 0$, and it is assumed that the initialization of the x-term on the right side of the equation is zero, which is usually the case. The history of each system component must be individually known for as long as the component has been in use to obtain the correct initialization response. Only 'terminal initialization' is considered in this paper, although more general initializations are discussed in [28].

The initialization problem is now solved using Laplace transforms. Equation (29) can be Laplace transformed as

$$s^q X(s) + L\{\psi(x, q, a, 0, t\} = -aX(s) - bU(s). \tag{30}$$

From Equations (4) and (8), the Laplace transform of the initialization function is

$$L\{\psi(x, q, a, 0, t)\} = L\left\{\frac{d}{dt}\left[\frac{1}{\Gamma(1-q)}\int_a^0 \frac{x(\tau)}{(t-\tau)^q}\,d\tau\right]\right\}. \tag{31}$$

Rearranging Equation (30), and with $\psi(s) \equiv L\{\psi(x, q, a, 0, t)\}$, gives

$$X(s) = \frac{b}{s^q + a}U(s) - \frac{1}{s^q + a}\psi(s), \tag{32}$$

which can be inverse transformed, using the Laplace transform convolution theorem, as

$$x(t) = \int_0^t F_q[-a, \tau]Bu(t-\tau)\,d\tau - \int_0^t F_q[-a, \tau]\psi(x, q, a, 0, t-\tau)\,d\tau. \tag{33}$$

Clearly, the first term in Equation (33) represents any forced response due to $u(t)$, and the second term represents the initialization response of the system due to the past history of $x(t)$. Traditionally, for integer-order systems ($q = 1$) this initialization term is the Laplace transform of a constant, and becomes $\psi(s) = k/s$. Alternatively, for fractional-order systems, this term is time-varying into the future; that is, the history of $x(t)$ has the appearance of a time dependent forcing term into the infinite future.

2. Vector Space Methods for Fractional-Order Control

2.1. General Vector Representation

A useful representation for systems of fractional-order differential equations is the vector space of fractional dynamic variables. This representation is a generalization of the state equations of usual integer-order system theory. It will be important to notice that for fractional-order systems, the 'state' of the system is not given by the dynamic variable vector, because the initialization vector, traditionally a vector of constants, has been shown to be time varying [22]. This section summarizes the multivariable initialized fractional-order system theory.

For a chosen q, typically $0 < q \leq 1$ which implies a fractional-order system, the vector representation can be written as

$$_cD_t^q \underline{x}(t) = A\underline{x}(t) + B\underline{u}(t), \quad \underline{\psi}(\underline{x}, q, a, c, t) \quad \text{given for } t > c, \tag{34}$$

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t), \tag{35}$$

where $\underline{x}(t)$, $\underline{y}(t)$, $\underline{u}(t)$, and $\underline{\psi}(\underline{x}, q, a, 0, t)$ can generally be considered to be vectors of the appropriate dimension, with A, B, C, and D now being matrices of the appropriate dimension, and the starting time, c, will usually be chosen as zero. The time period $a \leq t < c$ is called the initialization period. It is assumed that all $x(t) = 0$ for all $t < a$. The choice of q is problem dependent and is discussed further below. The derivative notation is

$$ {}_0D_t^q \underline{x}(t) \equiv {}_0d_t^q \underline{x}(t) + \underline{\psi}(\underline{x}, q, a, 0, t), \tag{36} $$

where ${}_0d_t^q$ is the uninitialized fractional derivative starting at $t = 0$, and $\underline{\psi}$, the vector initialization function, is determined as

$$ \psi_i(x_i, q, a, 0, t) \equiv {}_ad_0^q x_i(t) = \frac{d}{dt}\left[\frac{1}{\Gamma(1-q)}\int_a^0 \frac{x_i(\tau)}{(t-\tau)^q}d\tau\right]. \tag{37} $$

Otherwise stated, the initialization functions ψ_i depend upon the particular history of the corresponding fractional-order component x_i for $t < 0$, and these particular histories do not need to be the same for all the components. Note that only terminal initialization to each differintegral is considered in this paper, although more general initializations are discussed in [28]. It is also assumed that the initializations of the $Ax + Bu$ terms on the right side of Equation (34) are zero, which is usually the case, and that the initializations of all the terms in Equation (35) are zero.

Equation (34) can be rewritten using Equation (36) as

$$ {}_0d_t^q \underline{x}(t) + \underline{\psi}(\underline{x}, q, a, 0, t) = A\underline{x}(t) + B\underline{u}(t), \tag{38} $$

where the initialization function is the vector of functions

$$ \underline{\psi}(\underline{x}, q, a, 0, t) = \begin{bmatrix} \psi(x_1, q, a, 0, t) \\ \psi(x_2, q, a, 0, t) \\ \vdots \\ \psi(x_n, q, a, 0, t) \end{bmatrix}. \tag{39} $$

In what follows, this will, without loss of generality, be written as

$$ \underline{\psi}(\underline{x}, q, a, 0, t) = \underline{\psi}(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \\ \psi_n(t) \end{bmatrix}. \tag{40} $$

An alternative system representation that can be useful for determining the initialization response is to rewrite the system dynamic equations into the fractional integral form, which can significantly simplify the determination of the initialization functions.

The fractional dynamic variables in the system of Equations (34) and (35) are not states in the true sense of the name 'state' space. In the usual integer-order system theory, the set of states of the system, known at any given point in time, along with the system equations, are sufficient to predict the response of the system both forward or backward in time. Otherwise stated, for integer-order systems, the collection of numbers $\underline{x}(t)$, at any time t, specify the complete 'state' of the system at that time, and with the differential equations, for all future times. Mathematically stated, for integer-order systems, the system will have a unique time response given its initial state. It should be clear, however, that the fractional dynamic variables do not represent the 'state' of a system at any given time due to the requirement of the initialization function vector, which carries information about the history of the elements of the system. Consequently, as the initialization function vector is generally required, the set of elements of the vector $\underline{x}(t)$, evaluated at any point in time, does not specify the entire 'state' of the system. Thus for fractional-order systems, the ability to predict the future response of a system requires the set of fractional differential equations along with their initialization functions, that is, Equation (34).

The availability of the fractional vector formulation allows many choices for the basis q. Thus, the total number of fractional dynamic variables can increase if the basis q is chosen to be smaller. The least number of fractional dynamic variables is obtained by *choosing the basis q as the largest common fraction of the differential orders*. For example, a system with two elements of order $q_1 = 1/2$, and $q_2 = 1/3$, would require a basis q of at most $q = 1/6$. Clearly, such a system could also use a basis q of 1/12, 1/18, 1/24, or any other fraction of the form $1/(6n)$, $n = 1, 2, 3, \ldots$. If the original differential equations have non-rationally related derivatives, such as ${}_0d_t^{1/\sqrt{2}}x(t) + {}_0d_t^{1/\pi}x(t) + x(t) = u(t)$, then this approach will require an approximation of the value of q, as integer sub-multiples of the orders are not obtainable.

As in the case with integer-order systems, a particular input-output representation can have an infinite number of vector space representations. We can change vector variables in Equations (34) and (35) using

$$\underline{x}(t) = T\underline{z}(t), \tag{41}$$

where T is a square matrix of the same size as the vectors $\underline{x}(t)$ and $\underline{z}(t)$. With this definition, Equations (34) and (35) become

$${}_0d_t^q T\underline{z}(t) + \underline{\psi}(T\underline{z}, q, a, 0, t) = AT\underline{z}(t) + B\underline{u}(t), \tag{42}$$

$$\underline{y}(t) = CT\underline{z}(t) + D\underline{u}(t). \tag{43}$$

Rearranging the first of these gives the set to be

$${}_0d_t^q \underline{z}(t) + T^{-1}\underline{\psi}(T\underline{z}, q, a, 0, t) = T^{-1}AT\underline{z}(t) + T^{-1}B\underline{u}(t), \tag{44}$$

$$\underline{y}(t) = CT\underline{z}(t) + D\underline{u}(t). \tag{45}$$

Thus we have changed the set of vector variables from $\underline{x}(t)$ to $\underline{z}(t)$. Notice that the new system matrix is $T^{-1}AT$, the new input matrix is $T^{-1}B$, the new output matrix is CT, and the direct feedthrough matrix is unchanged as D. The w-plane eigenvalues of the system matrix are the system poles, and these must be unchanged with the change of variable. Thus the original system matrix A and the new system matrix $T^{-1}AT$ must have the same w-plane eigenvalues. Thus, as in integer-order systems, the system eigenvalues are unchanged via the similarity transformation.

2.2. The Vector Initialization Problem

The initialization problem can be solved for $\underline{x}(t)$ by using Laplace transforms generalized by the proper inclusion of the vector initialization function. As is shown in [28], the Laplace transforms of Equations (38) and (35) are

$$s^q\underline{X}(s) + \underline{\psi}(s) = A\underline{X}(s) + B\underline{U}(s), \tag{46}$$

$$\underline{Y}(s) = C\underline{X}(s) + D\underline{U}(s). \tag{47}$$

Rearranging Equation (46) gives

$$(Is^q - A)\underline{X}(s) = B\underline{U}(s) - \underline{\psi}(s), \tag{48}$$

or equivalently

$$\underline{X}(s) = (Is^q - A)^{-1}B\underline{U}(s) - (Is^q - A)^{-1}\underline{\psi}(s). \tag{49}$$

Inserting Equation (49) into Equation (47) gives the Laplace transform of the system output response

$$\underline{Y}(s) = \{C(Is^q - A)^{-1}B + D\}\underline{U}(s) - C(Is^q - A)^{-1}\underline{\psi}(s). \tag{50}$$

Using the system's impulse response, the matrix F-function (which is the generalized matrix exponential, see [29]),

$$F_q[A,t] \equiv t^{q-1}\sum_{n=0}^{\infty}\frac{A^n t^{nq}}{\Gamma(n+1)q)} = L^{-1}\{(Is^q - A)^{-1}\}, \quad q > 0, \tag{51}$$

Equation (50) can be inverse Laplace transformed to give the overall time response

$$\underline{y}(t) = C\int_0^t F_q[A,\tau]B\underline{u}(t-\tau)\,d\tau + D\underline{u}(t) - C\int_0^t F_q[A,t]\underline{\psi}(t-\tau)\,d\tau. \tag{52}$$

This can also be represented as

$$\underline{y}(t) = C\int_0^t F_q[A,\tau]\{B\underline{u}(t-\tau) - \underline{\psi}(t-\tau)\}\,d\tau + D\underline{u}(t), \tag{53}$$

or equivalently as

$$\underline{y}(t) = C\int_0^t F_q[A,t-\tau]\{B\underline{u}(\tau) - \underline{\psi}(\tau)\}\,d\tau + D\underline{u}(t). \tag{54}$$

2.3. Fractional Vector Feedback

This section considers the use of fractional vector feedback for control design. Related information can be found in [36]. The system is the vector space of fractional-order elements from Equations (46) and (47) and has the form

$${}_0d_t^q\underline{x}(t) + \underline{\psi}(\underline{x},q,a,0,t) = A\underline{x}(t) + B\underline{u}(t), \quad \underline{y}(t) = C\underline{x}(t) + D\underline{u}(t). \tag{55}$$

Typically, vector feedback is implemented in the form

$$\underline{u}(t) = -K\underline{x}(t) + \underline{r}(t), \tag{56}$$

where $\underline{r}$ is a vector reference input, and K is the feedback gain matrix to be determined. The closed-loop system then becomes

$$
\begin{aligned}
{}_0d_t^q \underline{x}(t) &= [A - BK]\underline{x}(t) - \underline{\psi}(\underline{x}, q, a, 0, t) + B\underline{r}(t),\\
\underline{y}(t) &= [C - DK]\underline{x}(t) + D\underline{r}(t).
\end{aligned} \tag{57}
$$

By choosing K appropriately and using any standard pole placement tools, such as the Bass–Gura method or Ackerman's method [37], it is possible to place the system eigenvalues anywhere in the w-plane. In doing this, the eigenvalues should now be placed to the left of the instability wedge which is bounded by the mapping of the s-plane imaginary axis. It should be remembered that the Bass–Gura approach is based on similarity transformations of the vector space. As shown above, similarity transformations can be applied directly to fractional-order systems given in vector space, exactly as in the integer-order situation.

A particular problem associated with fractional-order systems is the presence of the initialization term on the right side of Equation (57). In the usual integer-order equation ($q = 1$), the initialization term is the familiar initial condition vector for all the states. Now, however, the initialization function is no longer a constant function of time, but is a time varying function. It is important to realize that the initialization term will always be a decaying function of time (for $0 < q < 1$). To understand this, it will be assumed that the initialization begins at some time a less than zero but not equal to negative infinity, that the input function $u(t)$ is of finite size except for the possibility of delta functions, and that the order of the fractional derivative is $0 < q < 1$. With these assumptions, and remembering that the initialization function is the future response of only the fractional derivative term due to all past inputs to the system, it must necessarily decay to zero as time goes to infinity. This can be seen from the fact that for $0 < q < 1$, the fractional derivative is effectively a lossy element, and even if the system response is unstable, the finite negative a assumption requires that the energy put into the fractional element is finite. Thus the energy coming back from the fractional element in positive time is necessarily finite. Thus we may proceed with the implementation of vector feedback controllers without the worry of an unstable plant in negative time generating an unstable initialization response in positive time, implying that any vector feedback controller would never be able to stabilize the response.

It is not clear at this time how the linear optimal quadratic regulator theory (LQR) may be applied to fractional-order systems. It can, however, still be used directly, using A and B and any positive semi-definite Q and R matrices, to give a guaranteed stable, hyperdamped system, for $q < 1$.

2.4. Observers for Fractional-order Systems

Just as in integer-order system theory, it is important to create observers, or vector estimators, for fractional-order systems [38]. This section will present the theory necessary for designing initialized fractional-order system observers. The fractional-order vector estimator has the form

$$
\begin{aligned}
{}_0d_t^q \underline{\hat{x}}(t) + \underline{\hat{\psi}}(\underline{x}, q, a, 0, t) &= A\underline{\hat{x}}(t) + B\underline{u}(t) - L[\underline{y}(t) - \underline{\hat{y}}(t)],\\
\underline{\hat{y}}(t) &= C\underline{\hat{x}}(t) + D\underline{u},
\end{aligned} \tag{58}
$$

where a non-zero initialization function, $\underline{\hat{\psi}}$, has been assumed for the observer. The vector error $\underline{e}(t)$ is defined as the difference between the real system output $\underline{x}(t)$, and the estimated observer output $\underline{\hat{x}}(t)$

$$\underline{e}(t) = \underline{x}(t) - \underline{\hat{x}}(t). \tag{59}$$

The observer error gain L is determined so as to force the error between the two plant vectors to go to zero. The dynamics of the error are obtained by fractionally differentiating Equation (59),

$$_0d_t^q \underline{e}(t) = {}_0d_t^q \underline{x}(t) - {}_0d_t^q \underline{\hat{x}}(t). \tag{60}$$

Substituting the system equations from Equations (58) and (55) yields

$$\begin{aligned} _0d_t^q \underline{e}(t) \;=\; & \left[A\underline{x}(t) + B\underline{u}(t) - \underline{\psi}(\underline{x}, q, a, 0, t)\right] \\ & - \left[A\underline{\hat{x}}(t) + B\underline{u}(t) - \underline{\hat{\psi}}(\underline{x}, q, a, 0, t) - L[\underline{y}(t) - \underline{\hat{y}}(t)]\right] \end{aligned} \tag{61}$$

Now replacing the sensed system outputs, $\underline{y}(t)$ and $\underline{\hat{y}}(t)$, with the vector variables using Equations (55) and (58), yields

$$\begin{aligned} _0d_t^q \underline{e}(t) \;=\; & \left[A\underline{x}(t) + B\underline{u}(t) - \underline{\psi}(\underline{x}, q, a, 0, t)\right] \\ & - \left[A\underline{\hat{x}}(t) + B\underline{u}(t) - \underline{\hat{\psi}}(\underline{x}, q, a, 0, t)\right. \\ & \left. - L[(C\underline{x}(t) + D\underline{u}(t)) - (C\underline{\hat{x}}(t) + D\underline{u}(t))]\right]. \end{aligned} \tag{62}$$

Eliminating the terms that subtract out, replacing $\underline{e}(t) = \underline{x}(t) - \underline{\hat{x}}(t)$, and combining terms, gives

$$_0d_t^q \underline{e}(t) = (A - LC)\underline{e}(t) - (\underline{\psi}(\underline{x}, q, a, 0, t) - \underline{\hat{\psi}}(\underline{x}, q, a, 0, t)). \tag{63}$$

The matrix L is determined to force the observer error to zero by placing the eigenvalues of $A - LC$ in a stable region of the w-plane using standard methods. As discussed previously, the initialization response eventually decays to zero for any system for $0 < q < 1$, and only has a transient effect on the observation error, however a proper choice of $\underline{\hat{\psi}}$ will help drive the error to go to zero sooner than if $\hat{\psi}$ was simply zero.

3. Input-Output Methods for Control

3.1. Essence of the Approach

This section presents a discussion of fractional-order systems from a classical input-output control approach. The important results here are that the classical control tools, frequency response and root locus, are still applicable with the appropriate modifications. The frequency response approach applies directly to fractional-order systems as long as the primary roots are used in evaluating the individual fractional elements. Likewise, the root locus approach applies directly to fractional-order systems as long as the root locus analysis is performed in the w-plane. Throughout the discussion, the time-varying initialization function is considered to be a disturbance entering the system, and may be accommodated with disturbance rejection techniques. Alternatively, if the plant initialization is well known, the compensator initialization can be designed to eliminate the disturbance due to the plant initialization (see Section 3.8).

3.2. Conversion of Vector Equations to Transfer Matrix Form

Input-output representations can be obtained in several ways, and they provide significant insight into system behavior that compliment the vector space approach. The term input-output representation usually implies a transfer function, or transfer matrix, but we will include the initialization response for completeness. One way to obtain an input-output representation is to replace any physical system elements by their Laplace domain impedance equivalents, and then do the necessary algebra to obtain a transfer function. This approach works equally well with both integer-order systems and systems containing fractional-order elements.

The other approach to obtaining an input-output representation is to convert an already existing vector space representation into input-output form. Fortunately, this is easily done in a manner similar to that in integer-order system theory. As presented in Equation (50), the conversion is

$$\underline{Y}(s) = \{C(Is^q - A)^{-1}B + D\}\underline{U}(s) - C(Is^q - A)^{-1}\underline{\psi}(s), \tag{64}$$

where the system transfer matrix is

$$G(s) = \{C(Is^q - A)^{-1}B + D\} \tag{65}$$

and

$$Y(s) = G(s)U(s) - C(Is^q - A)^{-1}\underline{\psi}(s). \tag{66}$$

It is just as easy to go from a transfer function representation containing fractional elements to a vector space representation using any of the standard canonical forms from integer-order theory, and a chosen value of q (see [37] for a discussion of canonical forms).

3.3. Sinusoidal Response of Fractional-order Operators in the Time Domain

This section presents the time response of a fractional order system to a sinusoidal input. The purpose of this section, and the next, is to show how we can replace $s^q \to (j\omega)^q$ to obtain a frequency response for a transfer function. The results will be used in the next section to clarify the multivalued nature of the frequency response of fractional operators. It should be remembered that the frequency domain approach assumes that the time responses are in sinusoidal steady state. Whenever an input is applied to a system, the response will always consist of a transient part plus a steady state part. The frequency response approach assumes that the transient has decayed away, and that the response is in sinusoidal steady state. For fractional-order systems, sinusoidal steady state also implies that the initialization response has decayed to near zero.

Oldham and Spanier [9, pp. 108–110] give the fractional differintegral of a periodic function, and we will follow that discussion. We can write any periodic function with period T, as follows:

$$f(t) = \sum_{k=1}^{\infty}(c_k\, \mathrm{e}^{j2\pi kt/T} + \bar{c}_k\, \mathrm{e}^{-j2\pi kt/T}), \tag{67}$$

where the coefficients c_k can be determined from the corresponding Fourier integrals

$$c_k = \frac{1}{T} \int_0^T f(t)\, \mathrm{e}^{-j2\pi kt/T}\, dt \tag{68}$$

and $\bar{c}_k$ is the complex conjugate of c_k. The fractional differintegral of $f(t)$ then requires the differintegral of each of the separate exponentials. Oldham and Spanier [9] show that

$${}_0d_t^q (\mathrm{e}^{\pm j2\pi k/T} = \left(\frac{\pm j2\pi k}{T}\right)^q \gamma^* \left(-q, \frac{\pm j2\pi kt}{T}\right), \tag{69}$$

where γ^* is the incomplete gamma function. These two differintegrals contain both the transient and steady state responses. To obtain the sinusoidal steady state fractional derivative of the periodic function in Equation (67), an asymptotic expansion for large values of t of the γ^* terms gives

$${}_0d_t^q f(t) = \sum_{k=1}^{\infty} \left(\frac{2\pi k}{T}\right)^q \left(c_k\, \mathrm{e}^{j2\pi[(kt/T)+(q/4)]} + \bar{c}_k\, \mathrm{e}^{-j2\pi[(kt/T)+(q/4)]}\right). \tag{70}$$

Defining the radian frequency to be $\omega_0 = 2\pi/T$, gives the equivalent response

$${}_0d_t^q f(t) = \sum_{k=1}^{\infty} (k\omega_0)^q \left(c_k\, \mathrm{e}^{j[k\omega_0 t+(\pi q/2)]} + \bar{c}_k\, \mathrm{e}^{-j[k\omega_0 t+(\pi q/2)]}\right). \tag{71}$$

For any given input frequency $k\omega_0$, it can clearly be seen that the magnitude of the corresponding fractionally differintegrated steady-state output sinusoid has its magnitude scaled by $(k\omega_0)^q$, and is phase shifted by $\pi q/2$. For example, after the decay of the transient, ${}_0d_t^q \sin(\omega t) = (\omega)^q \sin(\omega t+(\pi q/2))$. This result generalizes the response obtained for integer-order systems. Here it is important to understand that frequency response results require sinusoidal steady state, and that the initialization functions play a less important role in the analysis.

3.4. Sinusoidal Response of Fractional-Order Operators in the Frequency Domain

The transfer functions for integer-order systems are usually given in the Laplace domain. To obtain the frequency response from such transfer functions, the terms containing the Laplace variable s, are replaced by the radian frequency ω, that is $s^n \to (j\omega)^n$, where n is any integer exponent. When this is done for fractional-order transfer functions however, the question of multiple solutions exists, as the substitution becomes $s^{nq} \to (j\omega)^{nq}$, where q is fractional and nq is not necessarily an integer. The question here is actually, which of the roots of j^{nq} do you use, as there are generally many roots, which lie on the unit circle in the complex plane. The primary root is considered to be the one with the smallest angle from the positive real axis, with the remaining roots being the secondary roots. The answer to this question is given by the time domain result of the previous section. That is, using the primary roots given by the frequency domain substitution $s^{nq} \to (j\omega)^{nq}$ will give the frequency response corresponding to the correct time domain response. For example, for $s^{0.5}$, substituting $s = j\omega$ gives $(j\omega)^{0.5} = j^{0.5}\omega^{0.5}$. Recognizing that $j^{0.5}$ has two roots, $\mathrm{e}^{j\pi/4}$ and $\mathrm{e}^{j5\pi/4}$, the primary root is always chosen for the frequency response, that is $e^{j\pi/4}$. This observation then allows the use of the standard frequency domain analysis tools such as the Bode plot and the Nyquist plot.

3.5. Frequency Response of Fractional-Order Differintegrals

To demonstrate the frequency domain approach, this section considers the frequency response of simple differintegrals. The uninitialized Laplace transform of a fractional integral ($q < 0$) or derivative ($q > 0$) operation is

$$L\{{}_0d_t^q f(t)\} = s^q f(s), \quad \text{for } q \text{ real.} \tag{72}$$

Thus the transfer function of the operator is

$$H(s) = \frac{L\{{}_0d_t^q f(t)\}}{f(s)} = \frac{s^q f(s)}{f(s)} = s^q. \tag{73}$$

To obtain the frequency response, let $s^q \to (j\omega)^q$, thus

$$H(j\omega) = (j\omega)^q. \tag{74}$$

The magnitude response is simply

$$|H(j\omega)| = \omega^q, \tag{75}$$

which rolls off at $20q$ dB/decade on the Bode plot, and the phase shift is given by the angle of the primary root of j^q

$$\angle H(j\omega) = \frac{q\pi}{2}. \tag{76}$$

3.6. Frequency Response of Ultradamped Systems

The time domain behavior of fractional-order systems was discussed in Section 2.2 with respect to their w-plane pole locations. There it was shown that any poles lying to the left of the wedge, given by lines with angles $q\pi$ in the w-plane, are on the secondary Riemann sheet of the s-plane, and these poles were termed *hyperdamped*, as they were damped more than the usual integer-order overdamped poles. Now, with respect to the w-plane, it is with some necessity that we distinguish between poles that are on the negative real w-plane axis, which we will call *ultradamped* (rather than over-hyperdamped), and complex conjugate poles that are still in the hyperdamped region (Figure 3).

An ultradamped system will consist of parallel combinations of systems of the form

$$H(s) = \frac{Y(s)}{U(s)} = \frac{k}{s^q + a}, \quad a > 0 \text{ and real,} \tag{77}$$

where Y is the system output, U is the system input, k is the system gain, $-a$ is the ultradamped system pole in the w-plane, and $q > 0$. For insight, we let $a = 1$, and $k = 1$, and analyze the frequency response of this transfer function,

$$H(j\omega) = \frac{1}{(j\omega)^q + 1}. \tag{78}$$

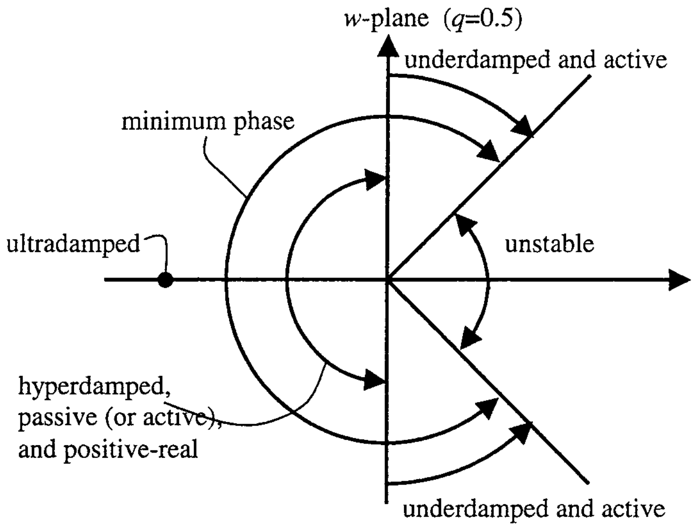

Figure 3. w-plane behavior regions for $q = 0.5$.

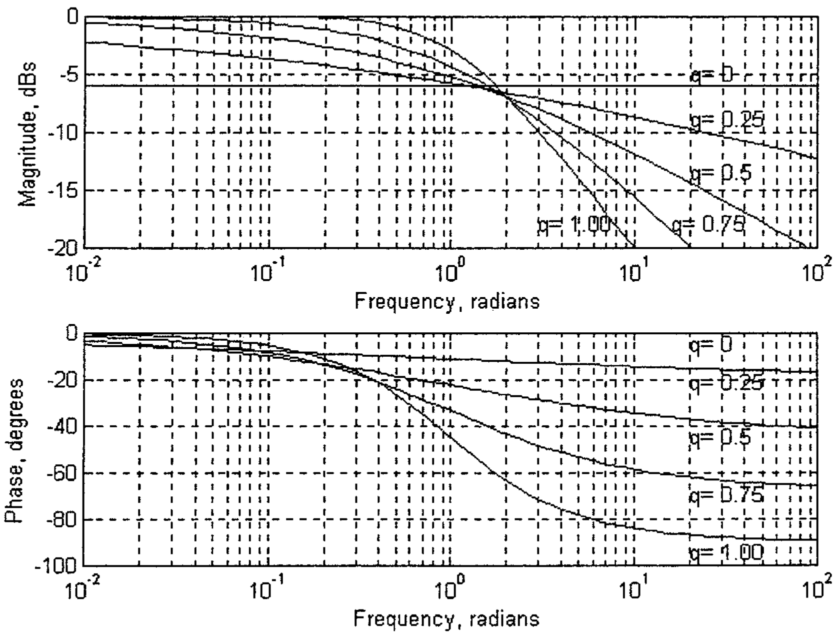

Figure 4. Bode frequency response for ultradamped systems.

For small values of the frequency, the transfer function is $H(j\omega) = 1$, and the magnitude is thus 1, and the phase lag is 0. For large values of the frequency, the transfer function becomes $H(j\omega) = (j\omega)^{-q}$, and the frequency response reverts to that of the simple fractional-order operator discussed in the previous section. Bode and Nyquist plots of this behavior are presented in Figures 4 and 5 respectively. With poles and zeros in these locations, it is simple to construct both phase lead and phase lag systems. These will be discussed later with regard to feedback to control.

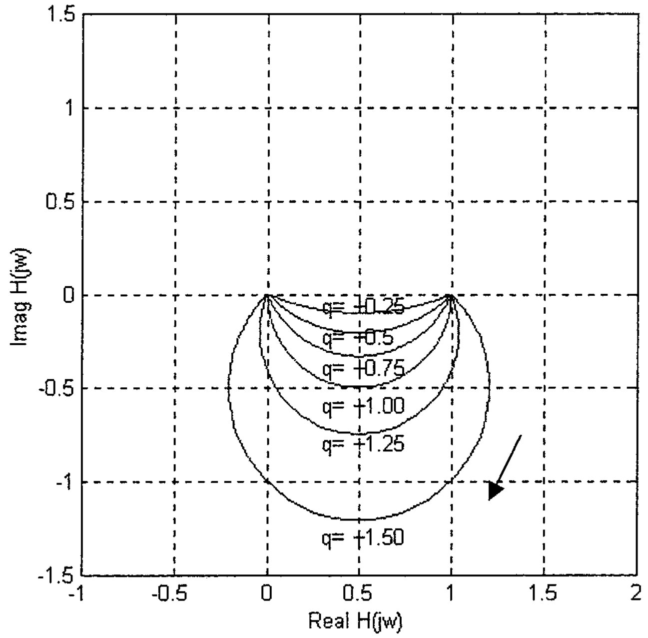

Figure 5. Nyquist frequency response for ultradamped systems. The arrow shows the direction of increasing frequency.

3.7. Frequency Response of Hyperdamped Systems

Hyperdamped systems have a pair of complex conjugate poles off the negative real axis of the w-plane, but are farther to the left than the underdamped region (Figure 3). Here there are several types of system behavior depending upon the specific locations of these poles in the w-plane. These behaviors are addressed in this section. Several definitions are given first along with corresponding pole locations in the w-plane. A related discussion can be found in [39].

We will first address systems that can be realized with passive energy storage elements. A passive energy storage element is one that cannot return more energy to a system than was placed into the element by the system in the past. An active element is one that can return more energy to a system than the system placed into it in the past, and obviously can behave as a passive element if designed so. Typically, an active element will have associated with it either a large gain or a negative gain. Necessary, but not sufficient, conditions for a fractional-order system to be *passive* are that its minimal transfer function denominator have all positive coefficients and that all of its poles lie to the left of the w-plane stability boundary. Another concept traditionally associated with passivity is the positive-real concept. To be a *positive-real system*, the frequency response of its transfer function must always lie in the right-half Nyquist plane. Stated otherwise, the maximum phase shift of a positive-real system is bounded by plus, or minus, ninety degrees. Passive fractional-order systems are not necessarily positive-real (Figure 5 shows passive system frequency responses some of which are not positive-real).

A *minimum phase system* has the smallest possible phase shift for a given magnitude response. An implication of this for integer-order systems is that all of the system's poles and zeros must lie in the left half s-plane. For fractional-order systems, the implication of

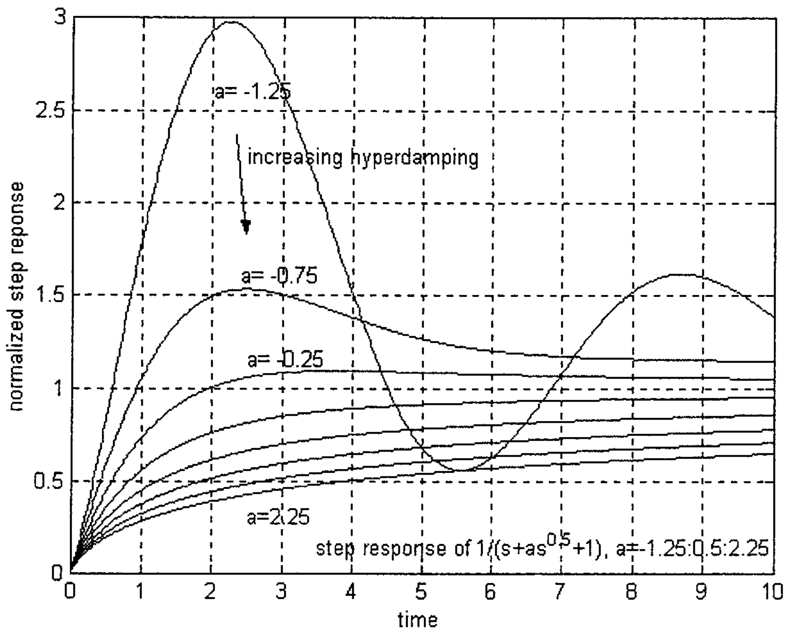

Figure 6. Step responses of a first-order system with different half-order damping.

being minimum phase is that all of the poles and zeros of a given system must lie to the left of the instability wedge in the w-plane. Considering all-pole systems, passive and positive-real systems are clearly minimum phase, however, a fractional-order system can be minimum phase without being passive or positive-real.

A simple example of this is the transfer function

$$H(s) = \frac{1}{s + as^{0.5} + 1}. \tag{79}$$

Referring to Figures 3 and 6, the w-plane is shown for Equation (79) with identification of the regions of passivity, positive realness, minimum phase, various damping, and stability for this particular system, along with corresponding step responses for various values of a. For $a > 2$, the system is ultradamped. For $2 > a > 0$, the system is hyperdamped, passive, and positive-real. For $0 > a > -\sqrt{2}$, the system is underdamped and minimum phase. For $a < -\sqrt{2}$, the system is unstable.

The underdamped active region bears further consideration. Specifically, the negative value of a indicates an active system, while the w-plane poles remain in the underdamped region. The fact that there exist underdamped poles implies that this system has a resonance, and a resonant peak should appear in the system frequency response. Figure 7 shows the Bode frequency response for several values of a. A resonant peak is clear in this response. The phase response behavior is more interesting. For integer-order systems, the phase will decrease by 180 degrees at a resonance, independent of the amount of damping. This also occurs for fractional-order systems, however, as Figure 7 shows, the phase increases before the resonant frequency, then the phase decreases at its maximum rate at the resonant frequency, and finally increases again back to the high frequency asymptote, which is -90 degrees for this system. The phase decrease at a marginally stable resonance remains 180 degrees, although the phase decrease gets smaller with more damping in the system (see the $a = -1.4$ and $a = -1$

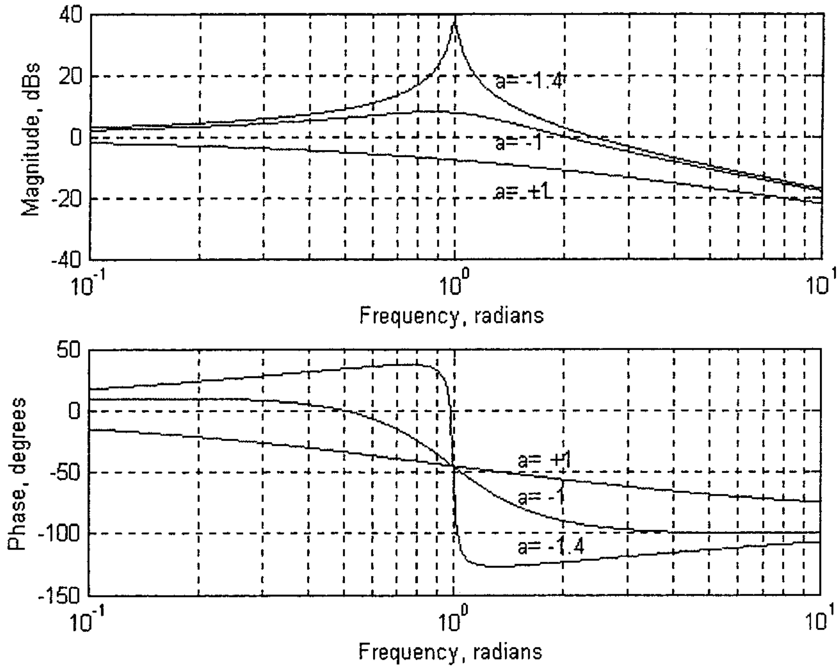

Figure 7. Bode plot for the system given in Equation (79).

cases in Figure 7). Thus, even though the high frequency asymptotes (as well as the Laplace transfer function) indicate that this system is only of first order, it can still go into resonance. Clearly, adding more fractional-order terms with a smaller value of q would allow even more resonances. *Consequently, it appears that the highest power of the Laplace variable in a transfer function is no longer an indicator of the effective order of a fractional-order system, or of the number of resonances to expect in its frequency response.* We will call resonances of the type considered here *fractional resonances*. More specifically, a transfer function with a factor of the form

$$\frac{1}{s^{q_1} + as^{q_2} + 1}, \quad q_1 > q_2 > 0$$

has a fractional resonance whenever the parameter a is such that its w-plane poles are in the underdamped region of the appropriate w-plane, or whenever the phase decrease near resonance is greater than $90q_1$ degrees, yet overall has a net decrease in phase of $90q_1$ degrees.

3.8. Nyquist Plane Design

Frequency response techniques have proven very useful for many control design problems. In control design, frequency response information is often presented in one of three equivalent forms; the Bode plot, the Nyquist plot, or the Nichols plot. Bode plots are widely used by engineers, and consequently, are the most widely understood presentation of the systems frequency response data. These plot the magnitude and phase of the system frequency response against a logarithmically increasing frequency. The Nyquist plot is an equivalent representation, however, it is a plot of the real *versus* the imaginary part of the system frequency response. This plot contains much useful information for feedback systems, as is shown in

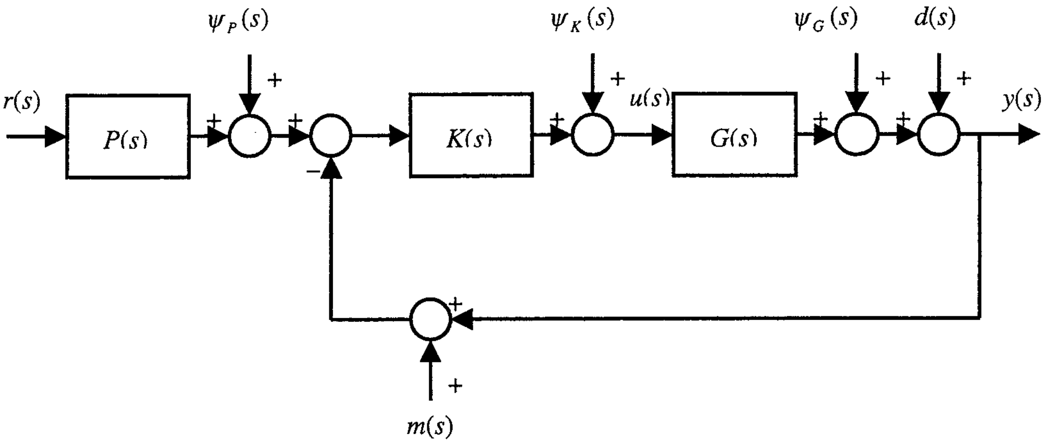

Figure 8. Closed-loop control configuration.

the next section. The Nichols plot is a conformal mapping of the Nyquist plot, and is probably in less use due to the distortion of the information. The remainder of this section contains a brief review of the utility of the Nyquist plot, and its use in a fractional-order system setting.

Referring to the single-input-single-output system of Figure 8, the signal $r(s)$ is a desired reference input, $d(s)$ is some generally unknown disturbance acting on the plant $G(s)$ and can include the plant uncertainty, $m(s)$ is the error incurred in measuring the plant output $y(s)$, and $u(s)$ is the control signal applied to the plant from the controller $K(s)$, and $P(s)$ is a prefilter used to shape the system inputs. The functions $\psi_P(s)$, $\psi_K(s)$, and $\psi_G(s)$ are the composite initialization functions associated with the corresponding transfer functions in the block diagram. The closed-loop plant response can then be determined to be

$$\begin{aligned} y(s) &= \frac{G(s)K(s)}{1+G(s)K(s)}[P(s)r(s)] + \frac{1}{1+G(s)K(s)}[d(s)+\psi_G(s)+G(s)\psi_K(s)] \\ &\quad - \frac{G(s)K(s)}{1+G(s)K(s)}[m(s)-\psi_P(s)]. \end{aligned} \tag{80}$$

This equation is organized so that the effects of the time varying initializations can be directly understood. The prefilter initialization due to any and all fractional-order components, $\psi_P(s)$, is combined with the measurement noise term, and has the same effect as measurement noise. The plant initialization due to any and all fractional-order components is $\psi_G(s)$, and the controller initialization due to any and all fractional-order components in the controller is $\psi_K(s)$. These initializations are combined with the disturbance term, and have the same effect as plant disturbances or uncertainty.

If the plant initialization is well known, the compensator initialization can be designed to eliminate the disturbance due to the plant initialization. For example, a regulator problem would usually have $y(s) = 0$ and $r(s) = 0$ in Equation (80). Then, ignoring $d(s)$ and $m(s)$ as persistent noises, we would have

$$G(s)\psi_K(s) + G(s)K(s)\psi_P(s) = -\psi_G(s) \tag{81}$$

to be solved for the controller initialization function $\psi_K(s)$, and the prefilter initialization function, $\psi_P(s)$, being careful to properly deal with any unstable w-plane poles and zeros in the inversion of $G(s)$.

Normally the following definitions are made; the loop gain

$$L(s) = G(s)K(s) \tag{82}$$

is the product of the plant and controller transfer functions. The sensitivity function

$$S(s) \equiv \frac{1}{1 + G(s)K(s)} \tag{83}$$

represents the sensitivity of the output to plant disturbances, which now includes the plant and controller initializations. The complementary sensitivity function

$$T(s) \equiv \frac{G(s)K(s)}{1 + G(s)K(s)} \tag{84}$$

is the sensitivity of the output to the reference input as well as measurement errors and now includes prefilter disturbances. This is normally referred to as the closed-loop transfer function.

It is noted that

$$T(s) + S(s) = 1. \tag{85}$$

This equation indicates that there is a trade off between reducing the effects of plant disturbances, which now includes the plant and controller initializations, and reducing the effects of measurement noise, which now includes prefilter disturbances. The magnitude of $T(s)$ is also an indication of the control effort required, $u(t)$. This trade-off is usually managed by making $T(j\omega)$ small at one set of frequencies and then making $S(j\omega)$ small at another set of frequencies. Most often, the measurement noise $m(t)$ will have a large bandwidth, and thus $T(j\omega)$ will be made small at high frequencies. Correspondingly, the disturbances, including the plant and controller initializations, usually have a low frequency nature, and thus $S(j\omega)$ is made small at low frequencies. The trade-off is usually accomplished by forcing the loop gain, $L(s) = G(s)K(s)$, to be large at low frequencies and small at high frequencies. A consequence of not allowing $T(j\omega)$ to be equal to one at all frequencies is that the system cannot track an input perfectly, and equivalently, the control actuators will not be required to be effective at unrealistically high frequencies.

One of the most useful benefits of using frequency response techniques in linear control system design is that closed-loop information can be directly obtained from open-loop information. Referring to the Nyquist approach, not only can stability be determined by looking at encirclements of the minus-one point, but also the frequency response plots of $T(s)$ and $S(s)$ can be determined directly from the frequency response plot of $L(s)$. To see these, lines are added to the Nyquist plane representing contours of constant magnitude of both $T(s)$ and $S(s)$. A detailed discussion of these contours can be found in [40] or [41], and a brief discussion follows. For $T(j\omega)$, its magnitude is zero at the origin of the Nyquist plot, the magnitude is infinity at the minus-one point of the Nyquist plane, and the unity magnitude contour is the vertical line at $\mathrm{Real}(L(j\omega)) = -0.5$. For $S(j\omega)$, its magnitude is zero at a radius of infinity on the Nyquist plane, its magnitude is infinity at the minus-one point, and the unity magnitude contour is a circle of radius one centered at the minus-one point (see Figure 9 as an example). The consequences of this are that the sensitivity to disturbances at low frequencies can be made zero by the addition of an integrator to the controller. The addition of the integrator will also allow tracking of inputs at low frequencies as well. If

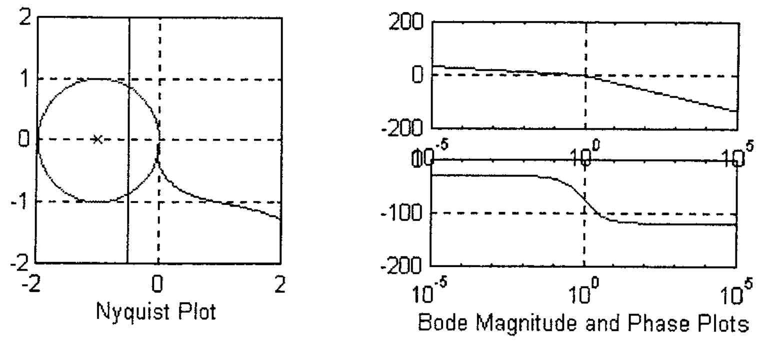

Figure 9. Frequency response analysis of the combination of the system of Equation (88) used to compensate the system of Equation (87).

the pole excess is greater than one, the Nyquist plot of $L(s)$ will enter the unity sensitivity circle at high frequencies, indicating that the cost of lowering the sensitivity to low frequency disturbances is an increase in sensitivity to high frequency disturbances. With the availability of fractional-order integrators, we now have the ability to customize this integral behavior in the Nyquist plane. A fractional integrator will rotate the Nyquist plot counter-clockwise relative to an integer integrator, keeping the Nyquist plot at infinity at zero frequency, but reducing the amount inside the unity sensitivity circle at high frequency. Thus, by using a fractional integrator rather than an integer integrator, we can still maintain, or improve, the low frequency tracking behavior, while reducing the high frequency noise sensitivity.

All of the frequency response tools from integer-order system theory carry over to fractional-order systems directly, as long as the frequency response plots are taken to be on the primary Riemann sheet. Consequently, the availability of fractional-order components for control allows much new design freedom in reshaping the Nyquist plane response. The next section will discuss this further.

3.9. Controllers in a Fractional-Order Universe

Let us consider a fractional-order plant with components that are multiples of 1/2-order. To control such a system, we have at our disposal all the standard compensators from integer-order control systems, with the addition of 1/2-order terms such as semi-integrals, semi-derivatives, semi-leads, semi-lags, semi-pole placement, etc. This extra design freedom allows improved control designs. Usually an improved control design will increase the stability, the tracking performance, the speed of response, the insensitivity to disturbances, or some similar measure of goodness. With more general values of q, we have q-integrals, q-derivatives, q-order leads, q-order lags, etc., which gives us even more design freedom (see [11, 23, 24, 42, 43] for more information).

Many generalizations of integer control design are possible with the freedom allowed by fractional-order systems. Some of the many generalizations are listed below:

(a) Integral control:
Fractional integrals, $H(s) = ks^{-q}$, are now available for compensators. The interesting feature of fractional integrals is that they still allow closed-loop tracking of step reference signals, while allowing the freedom to tune the low and high frequency behavior by tuning the value of q, although the tracking will be slower.

(b) Derivative control:
Although pure derivative control is seldom used, derivatives of any fractional-order, $H(s) = ks^q$, are now available, and these will have less noise amplification at high frequencies than integer-order derivatives.

(c) PI, PD, and PID control:
Fractional elements allow the use of any values of q for the integral and the derivative in these controllers. If fractional PID control is implemented, the fraction in the derivative need not be the same as the fraction in the integral, $H(s) = k_p + k_i s^{-q_1} + k_d s^{+q_2}$. Podlubny [11, 23] discusses controllers of the form PI$^\mu$D$^\nu$ more thoroughly.

(d) Leads and lags:
Lead compensators are often used to help stabilize marginally unstable systems. Lag compensators are often used to help reduce the magnitude of the high frequency loop gain of the system. Using fractional-order components, it is now possible to design fractional leads and fractional lags,

$$H(s) = \frac{k(s^{q_1} + a)}{s^{q_2} + b}.$$

The benefit to these is that it is easier to shape the open-loop and closed-loop frequency responses using them than using exclusively integer-order elements, due to the extra freedom offered by the continuum of values of q.

Another design criterion unique to fractional-order systems deals with time domain singularities occurring at time zero. If a plant transfer function, $G(s)$, does not contain a term in the denominator with an exponent of at least 1, that is the leading term in the denominator is s^q, then by the initial value theorem

$$g(0) = \lim_{s\to\infty} sG(s), \tag{86}$$

the impulse response, $g(t) = L^{-1}\{G(s)\}$, will have a singularity at time zero. This may or may not be desirable, however, using the appropriate fractional compensator in the forward path (say $H(s) = ks^{q-1}$) we can eliminate the singularity in the output of the plant.

It can occur by design, or otherwise, that a fractional-order system with a given base order, say 1/2, must be combined with a fractional-order system that has a different base order. That different order can be rationally related, say 1 or 1/3, or it can be non-rationally related, say $1/\pi$. In the first situation, all fractional components are related to one another by rational fractions. The plant may contain only integer-order devices, whereas the controller could be of half-order. The second situation deals with systems whose components are not rationally related fractions. It should be noted that although any of the methods discussed above can very closely approximate systems with irrationally related components by choosing a nearby rational fraction, they cannot solve it exactly as is discussed below. An example will illustrate the situation. Consider the plant

$$G(s) = \frac{1}{s+1} \tag{87}$$

to be compensated by the specific fractional-order integral

$$H(s) = \frac{1}{s^{1/\pi}} = \frac{1}{s^{0.318309886\ldots}}. \tag{88}$$

In this situation, one would normally choose the base fraction for constructing the appropriate w-plane. This value of q is usually the greatest common denominator of all the fractions appearing in the system. Unfortunately, for the above system in closed-loop, the greatest common denominator goes to zero, or some very small number, because pi is irrational. This would imply that the corresponding w-plane would be almost entirely a hyperdamped region, with a small sliver representing the usual s-plane. This would imply that the corresponding branch points in the s-plane are of infinite order, which means that the number of Riemann sheets below the primary sheet goes to infinity. Finally, a fractional vector space for such a system would require an infinite number of fractional vectors. This is not a very practical situation.

The most straightforward approach to analyzing such systems is to use Nyquist design from the beginning. The fact that there are non-rationally related terms in the system is of no consequence when using Nyquist analysis, as the frequency response uses the response on the primary Riemann sheet. Thus the frequency response and the Nyquist analysis tools can be used as usual, without any confusion or modification. This is shown for the given system in Figure 9, with the inclusion of the unity sensitivity circle (center at $s = -1$, radius 1) and the unity complementary sensitivity line (at $s = -0.5$) on the Nyquist plot. Consideration of the Nyquist plot of the plant-controller combination with respect to the unity sensitivity circle and complementary sensitivity line shows that the closed-loop system will track steps, however sluggishly, and that there will be just a little bit of overshoot for large enough gains.

3.10. Generalized PI-Control and PID-Control

This section first presents a generalization of the proportional-plus-integral-(PI)-controller. This is followed by a generalization of proportional-plus-integral-plus-derivative-(PID)-control. These generalizations are possible only because of the availability of fractional-order elements. In this section and the next two, initialization effects due to the fractional-order elements are not considered. However the discussion of Section 3.8 concerning the initialization effects in compensator implementations still applies. The standard integer-order PI-controller has a transfer function of the form

$$H(s) = k_p + \frac{k_i}{s}. \tag{89}$$

This can be written as

$$H(s) = k_p + k_i s^{-1}. \tag{90}$$

This controller can now be generalized using fractional-order integrals. Assuming a base fraction $q = 1/N$, for now, a generalized PI-controller can be written as

$$H(s) = k_0 + k_1 s^{-q} + k_2 s^{-2q} + \cdots + k_{N-1} s^{-(N-1)q} + k_N s^{-1}, \quad 0 < q < 1, \tag{91}$$

or

$$H(s) = \sum_{n=0}^{N} k_n s^{-nq}, \quad Nq = 1. \tag{92}$$

Again performing some algebra, this can be written as

$$H(s) = \frac{k_0 s + k_1 s^{(N-1)q} + \cdots + k_{N-1} s^q + k_N}{s}, \quad Nq = 1 \tag{93}$$

or

$$H(s) = \frac{\sum_{n=0}^{N} k_n s^{(N-n)q}}{s}, \quad Nq = 1. \tag{94}$$

Clearly, this controller allows a much greater variety of possible compensation results. Inserting this into the standard closed-loop control configuration with plant $G(s)$, gives the closed-loop transfer function, which is $T(s) = H(s)G(s)/(1 + H(s)G(s))$, to be

$$T(s) = \frac{\left[\sum_{n=0}^{N} k_n s^{(N-n)q}\right] G(s)}{s + \left[\sum_{n=0}^{N} k_n s^{(N-n)q}\right] G(s)}, \quad Nq = 1. \tag{95}$$

With $G(s) = N(s)/D(s)$ this becomes

$$T(s) = \frac{\left[\sum_{n=0}^{N} k_n s^{(N-n)q}\right] N(s)}{sD(s) + \left[\sum_{n=0}^{N} k_n s^{(N-n)q}\right] N(s)}, \quad Nq = 1. \tag{96}$$

This major generalization of PI-control possesses considerably more design capability and freedom, as both closed-loop poles and closed-loop zeros can be placed by proper selection of the gains.

The compensator can be further generalized by considering the powers of q to be unrelated. In this case the compensator is

$$H(s) = k_0 + k_1 s^{-q_1} + k_2 s^{-q_2} + \cdots + k_{N-1} s^{-q_{N-1}} + k_N s^{-1}, \tag{97}$$

with q_N assumed to be unity and q_0 assumed to be zero, and $0 \le q_i \le 1$, $0 \le i \le N$. This can also be rewritten as

$$H(s) = \frac{\sum_{n=0}^{N} k_n s^{q_{N-n}}}{s}. \tag{98}$$

This result effectively generalizes the work of [44] to the design of system controllers.

The above approach can also be applied to PID-controllers, where fractional-order derivatives are allowed. A similar approach has been presented in [11]. Like most discussions of PID-control we will not consider the causality problem associated with the derivative. The PI-controller of Equations (91) and (92) can be generalized to include fractional derivatives as follows:

$$\begin{aligned} H(s) \;=\; & k'_N s^{+1} + k'_{N-1} s^{+(N-1)q} + \cdots + k'_2 s^{+2q} + k'_1 s^{+q} + k_0 \\ & + k_1 s^{-q} + k_2 s^{-2q} + \cdots + k_{N-1} s^{-(N-1)q} + k_N s^{-1}, \end{aligned} \tag{99}$$

or equivalently, with $q = 1/N$,

$$H(s) = \sum_{n=-N}^{N} h_n s^{nq}, \quad Nq = 1. \tag{100}$$

3.11. Generalized Dynamic Compensator

Based on the results of the previous section, it should be clear that a compensator, such as Equation (94), can be further generalized by expanding not only the numerator into a series of fractional operators, but also the denominator (initialization effects are ignored here). Hence, a generalized dynamic compensator has the form

$$H(s) = \frac{\sum_{n=0}^{N} b_n s^{n/N}}{\sum_{n=0}^{N} a_n s^{n/N}} \tag{101}$$

if the powers of s are related and are bounded by one. Alternatively, allowing arbitrary powers of s yields

$$H(s) = \frac{\sum_{n=0}^{N} b_n s^{p_n}}{\sum_{n=0}^{N} a_n s^{q_n}}, \tag{102}$$

where here the numerator and denominator can have different powers of s. In a closed-loop feedback configuration with plant $G(s) = N(s)/D(s)$, the closed-loop transfer function would be

$$T(s) = \frac{\left[\sum_{n=0}^{N} b_n s^{p_n}\right] N(s)}{\left[\sum_{n=0}^{N} a_n s^{q_n}\right] D(s) + \left[\sum_{n=0}^{N} b_n s^{p_n}\right] N(s)}. \tag{103}$$

To determine the effectiveness of such a controller, the analysis must be performed in the Nyquist plane. It would seem that any desired Nyquist locus may be approximated by using a generalized compensator. The quality of the approximation is limited only by the acceptable number of terms.

3.12. Continuum Feedback: Order-Distributions

Proceeding from the generalized compensator of the last section, the idea of taking the sum to the limit occurs. In this case, the summations would be replaced by integrals over the variable q, which is the power of s. A continuum-of-q compensator generalizing Equation (102) would then have the form

$$H(s) = \frac{\int_a^b K_N(q) s^q \, dq}{\int_a^b K_D(q) s^q \, dq}, \tag{104}$$

where the functions $K(q)$ must be chosen so that $H(s)$ remains causal and so that the integrals are convergent. Integrals of the form shown in Equation (104) are called 'order-distributions' in [25]. Other forms of order-distributions are considered in [26].

The generalized transfer functions of the last section can be written as infinite series. Thus both a fractional power series

$$H(s) = \frac{\sum_{n=0}^{N} b_n s^{p_n}}{\sum_{n=0}^{N} a_n s^{q_n}} = \sum_{n=0}^{\infty} c_n s^{r_n} \tag{105}$$

and a fractional asymptotic series

$$H(s) = \frac{\sum_{n=0}^{N} b_n s^{p_n}}{\sum_{n=0}^{N} a_n s^{q_n}} = \sum_{n=0}^{\infty} c_n s^{-w_n} \tag{106}$$

representation can be used. Clearly these concepts may be generalized to the continuum. Thus a power integral representation for the continuum compensator would be

$$H(s) = \int_0^a K_{PS}(q) s^q \, dq, \tag{107}$$

while the asymptotic integral representation would be

$$H(s) = \int_0^\infty K_{AS}(q) s^{-q} \, dq, \tag{108}$$

or in a combined form

$$H(s) = \int_{-\infty}^a K(q) s^q \, dq, \tag{109}$$

where the $K(q)$ functions are chosen so that the integrals are convergent. For example, an order-distribution PID-controller generalizing Equation (100) is

$$H(s) = \int_{-1}^1 K(q) s^q \, dq. \tag{110}$$

Clearly, given the $K(q)$ functions, it is an easy task to perform analysis of a closed-loop system in the Nyquist plane. However, design of the $K(q)$ functions for a particular behavior is still an open question. Exact physical realization is also problematic, although accurate approximations are presently possible. Initialization effects in order-distributions are discussed in [26].

As another example of a possible situation, consider the plant to be controlled to be

$$G(s) = \frac{1}{s^2 + 1} \tag{111}$$

which represents an undamped oscillator, or an undamped spring-mass system. A possible compensator using order-distributions would be

$$H(s) = \int_0^2 K(q) s^q \, dq. \tag{112}$$

If this compensator were placed before the plant, the resulting closed-loop system would have the transfer function

$$T(s) = \frac{\int_0^2 K(q) s^q \, dq}{s^2 + \int_0^2 K(q) s^q \, dq + 1}. \tag{113}$$

This compensator allows an infinite number of frequencies in the closed-loop system, and thus allows considerable freedom to design the appropriate $K(q)$. The question arises as to how one chooses $K(q)$. A straightforward approach is to minimize some desired error for a given input, but much more research is required in this area.

4. Discussion

An overview of control system design techniques for fractional-order systems have been presented in this paper. Some of the important features of the discussion are listed below.

(a) Fractional-order behavior occurs in viscoelastic systems, electrochemical systems, heat transfer problems, and in other areas.
(b) Control of these systems is very important and becomes more so when it is recognized that fractional-order systems are a generalization that includes the normal integer-order systems as a special case.
(c) The issue of control is complicated by two factors. One is the presence of the time-varying initialization term (the fading memory), and the other is the presence of non-integer-order derivatives and integrals.
(d) Vector space control design is done in almost the same way as for integer-order systems, with the major differences being that the system equations are fractional-order differential equations, and hat the pole placement must be done in the w-plane corresponding to the basis value of q.
(e) Frequency response techniques are valid for the analysis of fractional-order systems, as long as the principal values of the fractional terms are used.
(f) With the validity of the frequency response, both Bode and Nyquist design techniques can be used with little modification.
(g) Root locus design methods can be applied directly to fractional-order systems where the root locus is performed in the w-plane.
(h) The order-distribution concept may be used to allow improved modeling ability and more flexible control design.

It is clear that the presence of fractional-order systems, and the availability of fractional-order compensators, allows a wide variety of new control approaches to be used. It will be interesting to see the effect of these fractional-order systems in future control applications. One should remember, however, that when using fractional-order components, it is important to properly include the effects of component initialization.

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