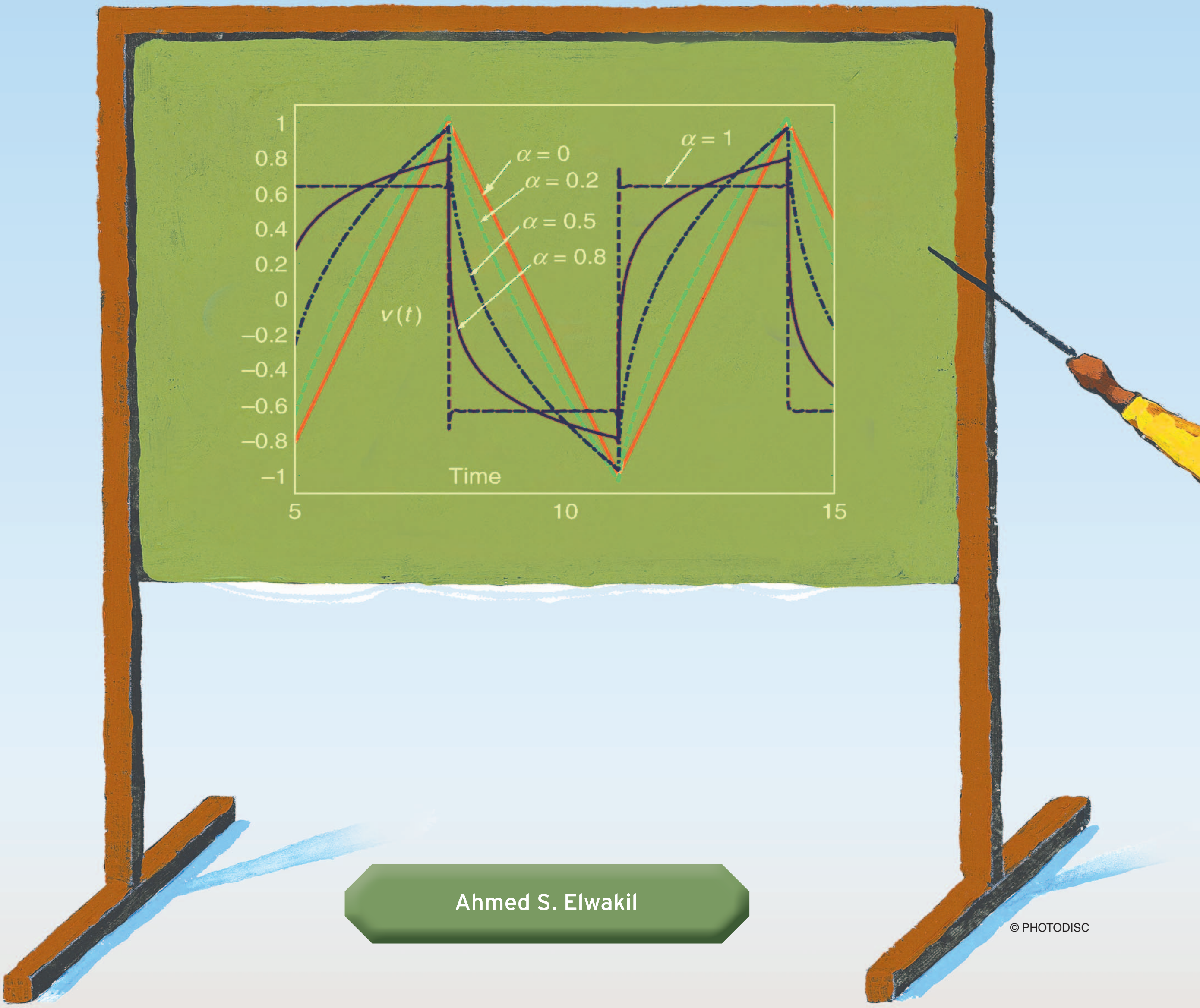

Ahmed S. Elwakil

Abstract

A recent article published in this magazine [1] has labeled fractional-order continuous-time systems as the "21st century systems". Indeed, this emerging research area is slowly gaining momentum among electrical engineers while its deeply rooted mathematical concepts also slowly migrate to various engineering disciplines. A very important aspect of research in fractional-order circuits and systems is that it is an interdisciplinary subject. Specifically, it is an area where biochemistry, medicine and electrical engineering over-lap giving rise to many new potential applications.

This article aims to provide an overview of the current status of research in this area, highlighting specific problems which need to be addressed particularly by electrical engineers.

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Fractional-Order Circuits and Systems: An Emerging Interdisciplinary Research Area

I. Mathematics

Fractional Calculus has been around for a long time. Although the mathematical basics were laid over 200 years ago, it remained a topic for the elite few of mathematicians. The basic idea behind this type of calculus is that it considers derivatives and integrals of arbitrary order. The classical first-order, second-order or third-order derivatives and integrals taught to students in their basic courses of mathematics are special cases of the more general situation where a derivative or integral can be for example of order 1/2 or order π. It may initially be difficult to grasp the physical meaning of such derivatives or integrals, but physical examples make it easier. Remember that it is easy to understand how a number x can be multiplied by itself n times to equal x^n if n is an integer. It is not however as easy to explain how a number can be multiplied by itself π times; i.e. x^π.

Twentieth century mathematicians have indeed revived Fractional Calculus and have re-fined it to a greater extent. Books like [2], [3], [4], [5] and [6] contain most of the needed mathematical background. We will not attempt to review any of this mathematics here but for clarity we give below the widely accepted definition of a fractional derivative of order $0 < \alpha < 1$; the Riemann-Liouville definition [7]

$$\frac{d^\alpha}{dt^\alpha} f(t) \equiv D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau)\, d\tau, \quad (1)$$

where $\Gamma(\cdot)$ is the gamma function. A discrete version; the Grünwald-Letinkov approximation is given by [8]

$$D^\alpha f(t) \triangleq (\Delta t)^{-\alpha} \sum_{j=0}^{m} (-1)^j n_j^\alpha f((m-j)\Delta t), \quad (2)$$

where Δt is the integration step and $n_j^\alpha = (-1)^j (\Gamma(j-\alpha))/(\Gamma(-\alpha)\Gamma(j+1))$.

It is very fortunate that applying the Laplace transform to fractional derivatives is also valid. This is the basic driving force for further applications in biology, chemistry and engineering. Assuming zero initial conditions and applying the Laplace transform yields

$$L_0 d_t^\alpha f(t) = s^\alpha F(s). \quad (3)$$

Three specific issues are of importance to engineers:

Table 1.
Example approximations of $s^{-\alpha}$.

$\frac{1}{s^{0.5}}$	$\approx$	$\frac{15.97(s+0.127)(s+1.77)(s+35.23)}{(s+0.032)(s+0.501)(s+7.95)(s+125.8)}$
$\frac{1}{s^{0.7}}$	$\approx$	$\frac{4.85(s+0.006)(s+0.0058)(s+5.18)(s+46.41)}{(s+0.014)(s+0.125)(s+1.12)(s+10)(s+89.61)}$
$\frac{1}{s^{0.9}}$	$\approx$	$\frac{1.77(s+0.127)(s+21.79)}{(s+0.0013)(s+0.216)(s+35.93)}$

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A. Numerical Simulations

Simulating fractional-order systems is crucial to any application. In the time domain, the Grünwald-Letinkov approximation given above is the one widely used. However, there are still issues pertaining to numerical simulations of fractional-order systems [9], [10]. Nevertheless, two Matlab toolboxes have been developed: the Crone toolbox [11] and the freely available Ninteger toolbox [12]. It is notable that these toolboxes were developed by control systems engineers who were the earliest among engineers to adopt and utilize the concepts of fractional calculus.

In the frequency domain, it is necessary to simulate the Laplacian operator $s^{\alpha} = (j\omega)^{\alpha}$. Several methods have been developed to approximate this operator via integer-order transfer functions [13]–[17] to make use of existing simulation tools, which are based around integer-order differential equations. Oustaloup's recursive-approximation given by

$$s^{\alpha} = \prod_{n=1}^{N} \frac{1 + s/\omega_{z,n}}{1 + s/\omega_{p,n}} \tag{4}$$

is among the most widely used [18]. The approximation is only valid within a frequency range and the number of poles and zeros N has to be decided beforehand. The accuracy depends on the choice of N. Table 1 lists some possible approximations of $s^{-\alpha}$. Although these higher-order transfer functions approximate the magnitude and phase of the fractional $s^{-\alpha}$, a question which remains unresolved is the validity of approximating a system whose state-space dimension is less than one with that whose dimension is bigger than one. With reference to Table 1, one can see for example that $s^{-0.5}$ is being approximated by a 4th-order transfer function which is puzzling in the sense that a sub 1-D system is now being approximated by a 4-D one.

Table 2.
Stability conditions, pole frequency ω_o and pole quality factor Q for (8).

Relations	Stability condition and roots	ω_o, Q
$a^2 \geq b$	$\alpha < 2$	$\omega_{o_{1,2}} = g_{1,2}^{1/\alpha}$
$a, b > 0$	$r_{1,2} = -a \pm \sqrt{a^2 - b} = g_{1,2} e^{j\pi}$	$Q = \dfrac{-1}{2\cos\pi/\alpha}$
$a^2 < b$	$\alpha < \dfrac{2\delta}{\pi}$, $\delta = \cos^{-1}\dfrac{-a}{\sqrt{b}} > \dfrac{\pi}{2}$	$\omega_o = \sqrt{b}^{1/\alpha}$
$a, b > 0$	$r_{1,2} = \sqrt{b} e^{\pm j\delta}$	$Q = \dfrac{-1}{2\cos\delta/\alpha}$
$a^2 < b$	$\alpha < \dfrac{2\delta}{\pi}$, $\delta = \cos^{-1}\dfrac{-a}{\sqrt{b}} < \dfrac{\pi}{2}$	$\omega_o = \sqrt{b}^{1/\alpha}$
$a < 0, b > 0$	$r_{1,2} = \sqrt{b} e^{\pm j\delta}$	$Q = \dfrac{-1}{2\cos\delta/\alpha}$
$a^2 \geq b$, $a < 0$	Always unstable	–
or $b < 0$		–

B. Stability

Stability of a fractional-order system is of great importance if it is to be realizable. The study of stability has been mainly confined to s-domain techniques where the classical s-plane transforms into a cone whose face angle is $\theta = \alpha\pi/2$. For $\alpha = 1$, the cone disintegrates into the conventional s-plane [19]–[23]. In [24], the stability problem was studied more comprehensively and the following procedure was proposed:

For a general linear fractional-order system characteristic equation of the form

$$\sum_{i=0}^{j} a_i s^{\alpha_i} = 0 \tag{5}$$

and when α_i are rational, this equation may be rewritten as

$$\sum_{i=0}^{n} a_i s^{\frac{i}{m}} = 0, \tag{6}$$

where m is integer. Translating this equation into the W-plane $(W = s^{1/m})$ yields

$$\sum_{i=0}^{n} a_i W^i = 0, \tag{7}$$

which is a polynomial of order n. The general steps for stability analysis are then

1) for given a_i calculate the roots of equation (7).
2) find the absolute minimum phase of all roots $|\theta_{W\min}|$.
3) the condition for stability is $|\theta_{W\min}| > \pi/2m$, while the condition for oscillation is $|\theta_{W\min}| = \pi/2m$ otherwise the system is unstable.
4) roots in the W-plane which have corresponding physical roots in the s-plane can be obtained by finding all roots which lie in the region $|\theta_W| < \pi/m$ then applying the inverse transformation $s = W^m$. The time response of the system can be easily related to these roots.

As an example, the stability of a system whose transfer function is

$$T(s) = \frac{cs^{2\alpha} + ds^{\alpha} + h}{s^{2\alpha} + 2as^{\alpha} + b}, \tag{8}$$

where a, b, c, d and h are constants depends on both a and b, as summarized in Table 2. The pole frequency ω_o and the pole quality factor are also given in Table 2 for each case. Interesting enough, the quality factor Q is still defined for systems of order $\alpha < 1$ albeit Q is negative in this case.

C. Control System Theory

Control theorists, were early to realize the importance of incorporating fractional calculus techniques in control system design [13], [25]. Of particular importance was the introduction of the general $PI^{\lambda} D^{\mu}$ controllers instead of traditional PID controllers [26]. Such controllers use an integrator of order λ and a differentiator of order μ; $0 < \lambda, \mu < 1$. Various modifications and applications of such controllers have since been proposed [27]–[31]. A fractional-order controller for power converters was proposed in [32] while chaos controllers were designed in [33] and [34] following the design of several fractional-order chaotic systems [35], [36]. Despite the advanced level of control theory concepts that have already been generalized to the fractional-order domain, more work is still being done on the theoretical front [37], [38] and on the applications front [39], [40] particularly in robotics. A recent contribution in model reduction showed that higher-order system models can be well-presented by fractional-order reduced parameters systems [41]. This can be appreciated from Table 1.

II. Biochemistry and Medicine

Biochemists were also early to adopt and utilize fractional-order dynamics. The Impedance Spectroscopy measurement technique [42] has been used for a long time to characterize the electrical properties of materials and biological tissues [43]. The technique is centered around measuring the impedance of the material under investigation and plotting its magnitude and phase versus frequency. Biochemists have realized long ago that the impedances they were measuring always scaled with frequency as $Z(s) = 1/s^{\alpha}$; α being non-integer. This means that the phase angle between the applied current stimulus and the measured voltage is constant and equals $\alpha\pi/2$. Hence, the term Constant-Phase-Element (CPE) was adopted and is widely used by biochemists to describe this impedance [44]. The Cole-Cole model [45] is a famous model which has been found to fit many tissues. According to this model, the impedance of a tissue can be given as

$$Z = R_{\infty} + \frac{R_0 - R_{\infty}}{1 + (\tau s)^{\alpha}}, \tag{9}$$

where R_0 is the resistance at very low frequency, R_{∞} is the resistance at very high frequency, τ is a characteristic time constant and α is the so-called dispersion coefficient. This simple model thus contains two resistors and one CPE. Applied to model the human skull in [46], it was found that the skull tissue has a characteristic frequency $f_c = 1/2\pi\tau$ in the range 0.955–1.395 kHz and a characteristic $\alpha \approx 0.6$.

Another notation widely used is the Warburg Impedance [47], [48] which is a special case of the CPE when $\alpha = 0.5$. Measurements of tissue properties of protein fibres, fruits and vegetables have shown that their impedance behavior can only be modelled by using Warburg impedances in conjunction with ideal RC circuits or more generally using RC networks that incorporate CPE elements [49]–[51]. Extending these modeling techniques to human tissue characterization has shown to be very important for improving medical diagnostics [52]–[56]. Of particular importance is the diagnosis of cancer [57] and lung diseases [58], [59] as well as improved magnetic resonance measurements [60]. Figure 1 shows the model proposed in [61] for the human intestine tissue. The model contains two CPEs of different orders.

To further demonstrate how natural tissues behave, Fig. 2 shows the measured current-voltage phase difference in a simple voltage-driven RC circuit once with a normal 12 nF capacitor and again with an apricot fruit whose measured capacitance was 12 nF. Note the constant phase of approximately 50° for the apricot fruit. More results for other different types of fruits have been reported in [51] and are being considered for noninvasive testing of fruits via impedance measurements.

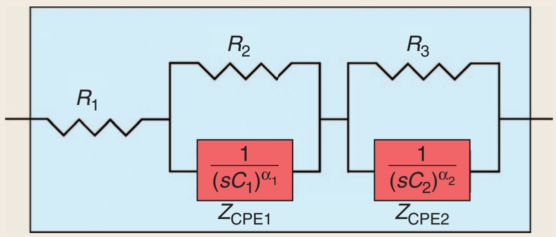

Figure 1. Model proposed in [61] for the intestine tissue.

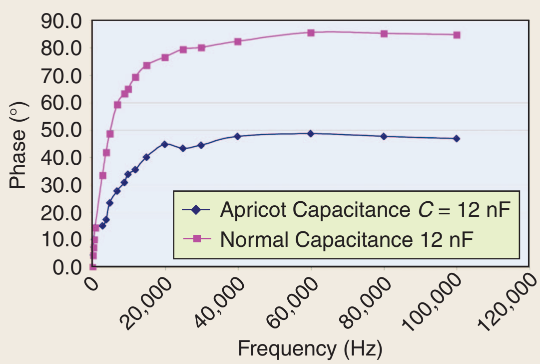

Figure 2. Measuring the I-V phase shift in a normal 12 nF capacitor and an apricot fruit whose capacitance is also 12 nF.

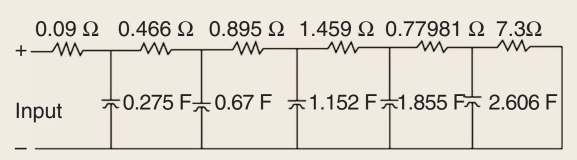

Figure 3. Possible RC approximation of $1/\sqrt{s}$.

III. Circuit Theory and Design

Although circuit theorists did approach the area of fractional-order systems some time ago, the concept never really found its way into real circuit design. The reasons will be highlighted below.

A. The Fractional Capacitor, Fractance Device or Fractal Immittance

In 1964, a paper was published proposing a method to approximate a "fractional capacitor" whose impedance is $(1/s)^{-n}$ [62]. The notation "fractional capacitor" was thus born to represent the same element readily known to biochemists as the "constant phase element". Also, in 1964, an RC approximation of the special case $s^{-1/2}$ was given in [63]. This special case is the same as the "Warburg impedance" readily known by biochemists. In 1967, another paper was published showing how the "constant-argument immittance" can be approximated [64]. Note that the reason behind trying to approximate a fractional-impedance via RC networks is simply the lack of off-shelf dielectric-material-based capacitors whose impedance is fractional. It is really unfortunate that while Westerlund has pointed out that dielectric capacitors can only be accurately modeled using fractional derivatives [65], [66], no effort was put into fabricating and commercializing fractional-order capacitors.

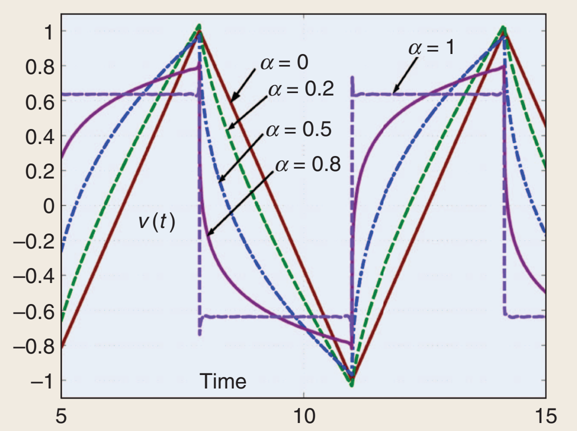

Figure 4. $V(t)$ across a fractional-order capacitor for different α if $I(t)$ is a triangle waveform.

However, this can also be attributed to the lack of a general circuit theory that incorporates fractional impedances. It is thus obvious that this feedback loop where circuit designers did not (and still do not) consider fractional-order circuits seriously because a fractional-order capacitor does not exist and on the other hand fabricating fractional-order capacitors was not (and still is not) taken seriously because there are no motivating circuit applications, needs to be broken. The proper way to break this loop is to intensify the research work in circuit theory in order to move it out of its current integer-order cage.

The exercise of proposing RC approximations of the fractional-impedance was re-initiated again in [67]–[69] calling such an impedance a "fractance device" and a "fractal immittance". These approximations have been appealing only to those seeking to experiment with fractional-order circuits to prove a concept but have never been appealing to real-world circuit designers. The reason is obviously economic since several capacitors are needed just to approximate one capacitor with a non-integer-order. Figure 3 shows an example RC network approximating $s^{-0.5}$ [70].

If a sinusoidal current $I(t) = I_o \sin(\omega t)$ is used to excite a fractional capacitor, the voltage developed can be shown to be given by [71]

$$V(t) = L^{-1}[s^{\alpha} I(s)] = I_o \omega^{\alpha}\left[\sin_{\alpha}(\omega t)\cos\left(\frac{\alpha\pi}{2}\right) + \cos_{\alpha}(\omega t)\sin\left(\frac{\alpha\pi}{2}\right)\right], \quad (10)$$

where $\sin_{\alpha}(\omega t)$ and $\cos_{\alpha}(\omega t)$ are the generalized trigonometric functions defined respectively as [71]

$$\sin_{\alpha}(t) = \sum_{k=0}^{\infty} e^{t}_{k-\alpha} \sin(k-\alpha)\frac{\pi}{2} \quad (11a)$$

$$\cos_{\alpha}(t) = \sum_{k=0}^{\infty} e^{t}_{k-\alpha} \cos(k-\alpha)\frac{\pi}{2}. \quad (11b)$$

The steady-state voltage is then given by

$$V_{ss}(t) = \lim_{t\to\infty} V(t) = I_o \omega^{\alpha} \sin\left(\omega t + \frac{\alpha\pi}{2}\right). \quad (12)$$

If $I(t)$ is a periodic triangle waveform with fundamental frequency $\omega_o = 2\pi/T$, it can be shown that $V(t)$ in this case is given by

$$V(t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^{2-\alpha}}\left[\sin_{\alpha}((2n-1)t)\cos\left(\frac{\alpha\pi}{2}\right) + \cos_{\alpha}((2n-1)t)\sin\left(\frac{\alpha\pi}{2}\right)\right]. \quad (13)$$

Figure 4 shows the waveform of $V(t)$ for different values of α. Note that at $\alpha = 0$, $V(t)$ is also a triangle waveform following $I(t)$ while at $\alpha = 1$, $V(t)$ is a square-wave indicating that $I(t)$ is order-one differentiated. The $V(t)$ waveform at half differentiation $\alpha = 0.5$ is seen to be somewhere in between.

The authors of [72] have recently proposed a capacitive probe fabricated in such a way that it displays a metal-insulator-liquid interface when placed into a liquid. The analysis in [72] reveals that this capacitive probe then has a fractional-order impedance $(1/s)^{\alpha}$ and α can be varied with the amount of liquid immersion. The same authors have used this probe to realize a fractional-order differentiator circuit in [73]. Although this capacitive probe is bulky and cannot be used in real circuit settings, it shows that further research in material properties is needed in order to fabricate a commercial device.

More interesting are the silicon-based fractional capacitors, fabricated and tested in [74]–[76]. The idea was to use fractal geometry to lithographically implement a capacitor using a standard silicon process. The experimental measurements of the fabricated device

Figure 5. Silicon-based fractional-order capacitors fabricated in [74]–[76].

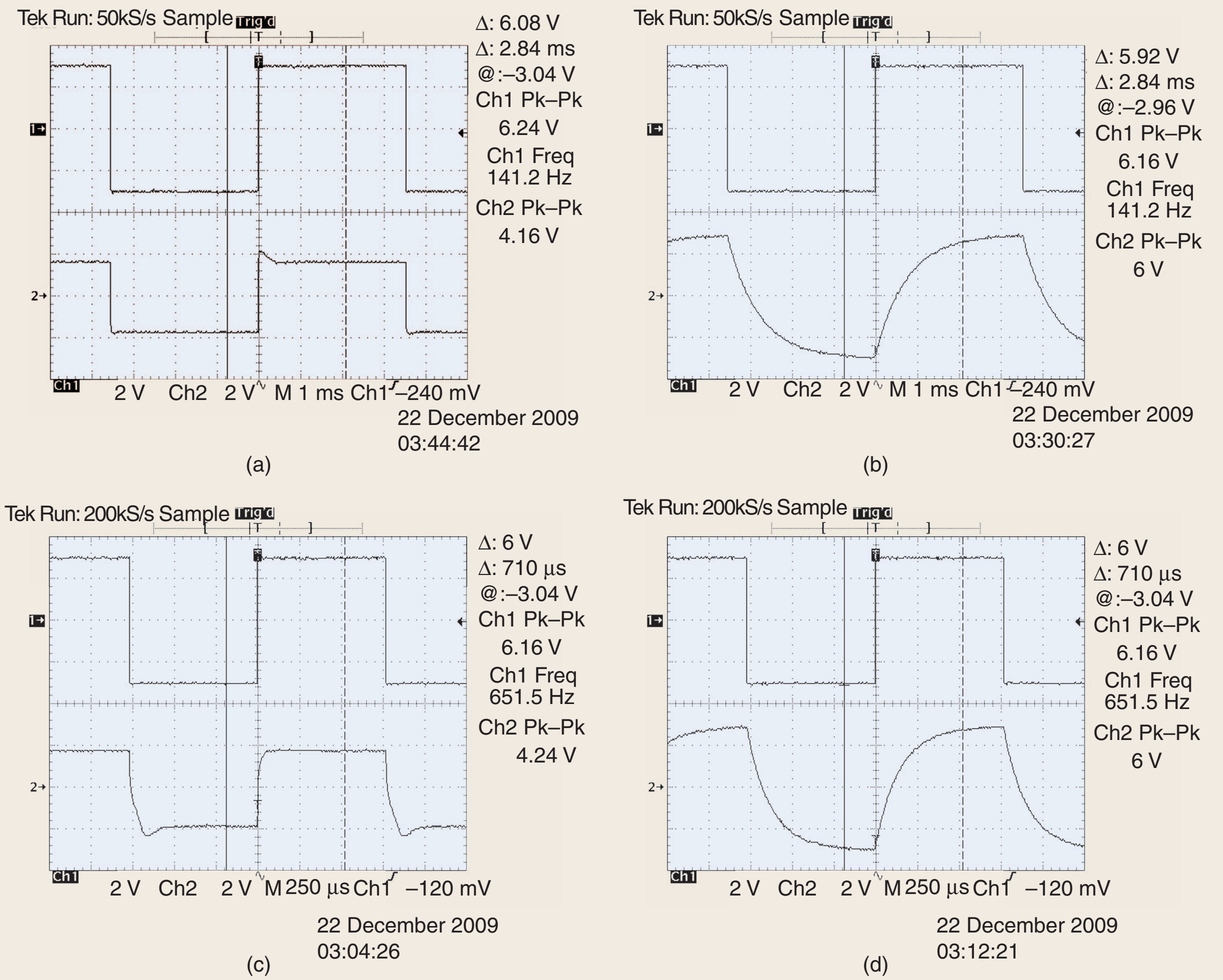

Figure 6. Step response of the fractional capacitors [(a) and (c)] compared to normal capacitors of the same value [(b) and (d)].

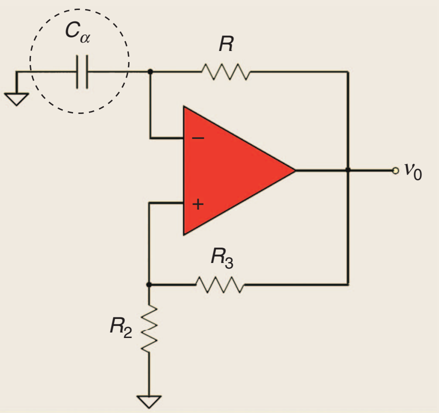

Figure 7. Simple multivibrator with a fractional capacitor C_α.

indeed indicate that there is a wide frequency band in which the device operates as a fractional capacitor. The fractional-order of the capacitor α is related to the geometry of the fractal pattern used. This work is very promising indeed. The authors of [74]–[76] have kindly supplied the author of this article with the two fractional-order capacitors shown in Fig. 5. The Hilbert-type capacitor has a capacitance of approximately $C = 7.5$ nF and order $\alpha \approx 0.5$ (i.e. $|Z_c| \approx 11547/\sqrt{\omega}$; $\measuredangle \approx \pi/4$) while the Arbre-type has $C = 1.5$ nF and $\alpha \approx 0.5$ A comparison between the step response of these two fractional capacitors versus normal capacitors of the same value is shown in Fig. 6. We shall present circuit design results using these integrated capacitors below.

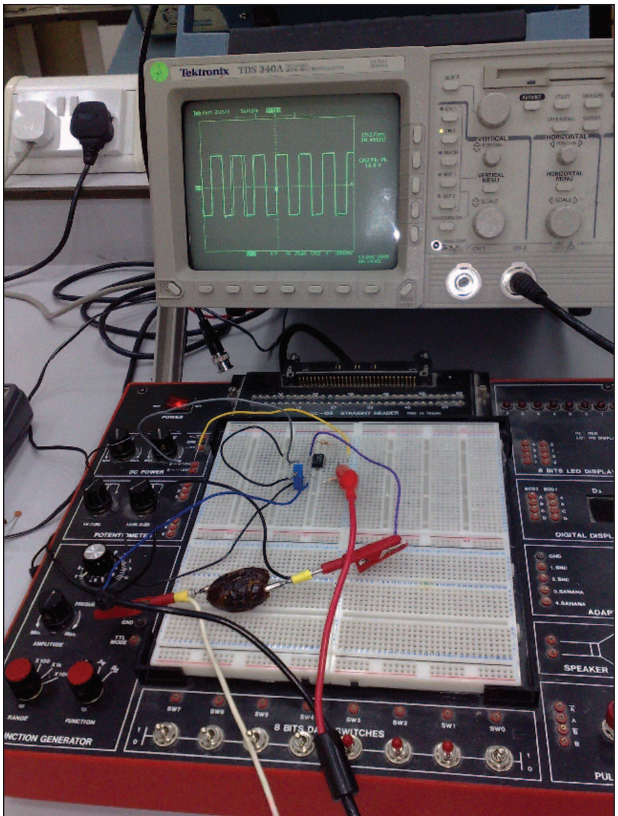

Figure 8. Multivibrator implemented with a date fruit.

Some active implementations of the fractional derivative have also been proposed using for example switched capacitor circuits [77], [78]. It is important to note here that the development of a fractional capacitor is also very much related to the generalization of electromagnetic concepts to the fractional-order domain; something that has only been done recently [79], [80]. In conclusion, more work still needs to be done to develop robust fractance devices. Using a silicon-base as in [74]–[76] is appealing but alternative materials, particularly natural tissue properties is also very interesting and should not be overlooked.

B. Circuit Design

Circuit designers have not yet fully understood the need to start working and developing fractional-order circuits. It is true that no commercial fractional-order capacitor yet exists, but more important than waiting for it to exist is to make sure that when it exists we know how to make use of it. In particular, the development of clear applications may actually speed-up the process of standardizing the use of fractional-order circuits.

1. Fractional-order oscillators

In 2001, a letter was published on the use of fractional capacitors to implement a classical Wien-bridge sinusoidal oscillator [81]. In that letter, a Wien-bridge oscillator was assumed to have two identical fractional-order capacitors of equal capacitance C and equal order α. The oscillation start-up condition was then found to be $K = 3 + \cos(\alpha\pi/2)$; where K is the amplifier gain. evidently, if normal capacitors are used, i.e. $\alpha = 1$, the start-up condition reduces to $K = 3$ as well-known. Interestingly, the oscillation frequency would be given by $\omega_0 = (1/RC)^{1/\alpha}$. What this means is that we can actually obtain very high oscillation frequencies independent of the capacitance C if α is made less than one. For a Warburg impedance case ($\alpha = 0.5$), the oscillation frequency would simply be $(1/RC)^2$. In [81], only numerical simulations were shown based on the Grünwald-Letinkov approximation of (2). In 2002, a book was published [82] on nonlinear noninteger-order circuits and systems. The circuit examples given there were of fractional-order chaotic oscillators, a topic which later found many other publications (see [83] and the references therein). Still, the motivation for circuit designers was not really clear.

In [84] and [85], the authors re-visited again fractional-order sinusoidal oscillators; this time in a more generalized way where two or three fractional-order capacitors (or inductors) of orders α, β and γ are assumed. Many verified examples using Spice and also experimentally using the capacitive probe described in [73] were given. The findings of these two articles confirmed the findings of [81]. In particular, [85] reported an experimental example of an oscillator whose oscillation frequency was measured as 181 kHz while the $1/2\pi RC$ frequency was only 265 Hz! This increase in frequency with a factor of over 650 times is due to a small fractional order α and not to a reduction in the capacitance value C. Moreover, in [86] a multivibrator which employs a single fractional capacitor, as shown in Fig. 7, was studied and an experiment was setup to show that a square wave of frequency 1000 times higher can be obtained for the same capacitance C if its order is $\alpha = 0.5$ instead of a normal capacitor whose order is $\alpha = 1$. Spice simulations and experimental results were given. However, the experimental results were performed with a composite circuit (like that shown in Fig. 3) rather than a real fractional capacitor. A closed-form expression relating the period T of the generated square-wave to the time constant $\tau = RC$ was derived and is given by

$$\sum_{n=0}^{\infty} \frac{\tau^{-n}\left(\frac{T}{2}\right)^{\leq n}}{\Gamma(\alpha n + 1)} = \frac{K+1}{K-1}, \tag{14}$$

where $K = 1 + R_3/R_2$ and $\Gamma(\cdot)$ is the gamma function. This implies a nonlinear relationship between T and τ which is not the case when a normal capacitor is used since in this case we obtain the well-known relationship

$$T = 2\tau \ln\left(\frac{K+1}{K-1}\right). \tag{15}$$

Table 3.
Measured oscillation frequencies in kHz for the multivibrator of Fig. 7 using a date fruit.

Date fruit	Normal capacitor
254	1.49
167	1.04
125.6	0.908
110	0.836
70.7	0.813

Table 4.
Measured oscillation frequencies in kHz for the multivibrator of Fig. 7 using the Hilbert-type fractional capacitor ($C = 7.5$ nF, $\alpha \approx 0.5$).

Hilbert capacitor	Normal capacitor
47.4	0.666
90	1.009
222.5	2.181
440	4.631

Here, we will repeat the same multivibrator example described in [86], fixing $K = 3.3$ but using two different fractional-capacitors. Firstly, we use a date fruit as a fractional capacitor, as shown in Fig. 8. The date fruit, similar to most other fruits and vegetables, behaves as a fractional capacitor [51]. Table 3 shows the measured oscillation frequency when the date fruit is used compared to that when a normal capacitor of the same capacitance value is used at different oscillation frequencies adjusted through varying R. The large difference is clear.

Secondly, we use the Hilbert-type fractional capacitor shown in Fig. 5. The obtained results are given in Table 4 and Fig. 9 shows the waveform corresponding to

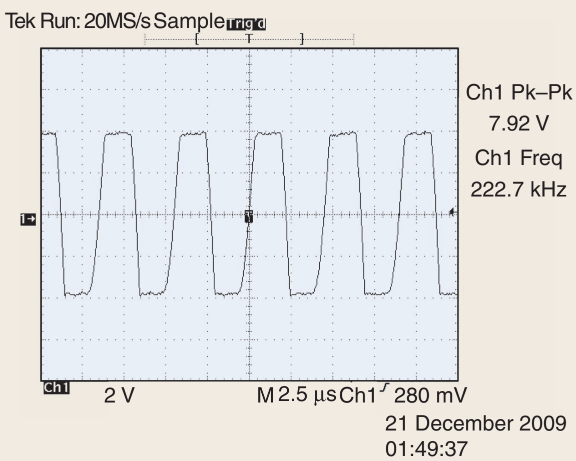

Figure 9. Measured waveform at the output of the multivibrator of Fig. 7 using the Hilbert-type fractional capacitor.

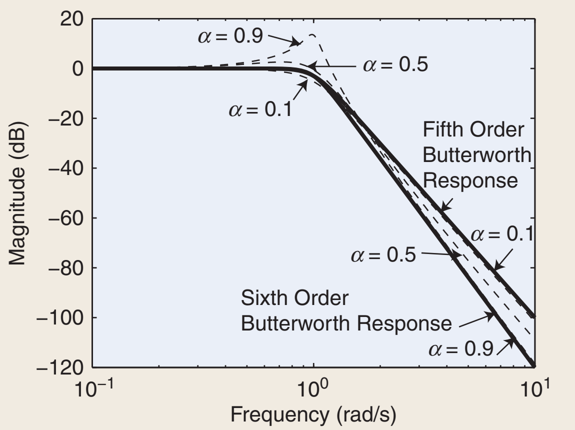

Figure 10. Response of 5 + α fractional-order lowpass filter.

There is no reason why we should remain modeling and designing systems and circuits in the integer-order sub-space.

the third measurement in the Table. The advantage of implementing oscillators using fractional-order capacitors is clearly enabling the boost of frequency independent of the capacitance value.

2. Fractional-order filters

On the other hand, analog filters are also important circuit building blocks that are classically classified as 1st-order, 2nd-order or *n*th-order filters. Although the specifications required by many applications result in a non-integer-order filter (e.g. 6.7 lowpass filter or 3.3 bandpass filter response), the current implementation practice is to round-up the required order to the nearest integer number in order for the filter it to be realizable. In [87] and [88], the authors have generalized 1st and 2nd-order filters to the fractional domain showing some clear advantages. In particular, instead of a classical stop-band attenuation of $-(20 * n)$dB/dec, the attenuation of a fractional-order filter is $-[(20 * n) * \alpha]$dB/dec which allows for stepping the attenuation very precisely. Further, the filter cutoff frequency (center frequency) and quality factor are functions of the fractional-order α, as is clear from Table 2. More work is ongoing in the topic [89] to realize for example higher-order fractional-step filters with maximally-flat response (Butterworth response) with some preliminary results already reported in [90] and [91]. Figure 10 shows a plot of the response of a fractional-order lowpass filter of order $5 + \alpha$ in steps of 0.1. Electronic fractional-order filters can also be used in biomedical tissue characterization.

IV. Conclusion

Fractional-order circuits and systems design is definitely an emerging area of interdisciplinary research. It is unfortunate to see that knowledge of fractional calculus and its various techniques is wide spread among biochemists, control theorists and signal processing researchers while circuit theory and circuit applications are lagging behind. Fractional-order circuits may revolutionize the way we educate future engineers and its applications in biomedicine are of particular importance. There is no reason why we should remain modeling and designing systems and circuits in the integer-order subspace. One other important aspect of the interdisciplinary nature of this topic is that it may pave the way for more environmentally friendly materials that can be used to implement electronic circuits. At the moment, one has to admit that the silicon industry is far from being environmentally friendly. The end products of this industry are a major concern when decommissioned. Interesting enough would be the merge of biochemical impedance properties with electronic circuits in a "biochemistronics" setup. Specific targets which should be set may be

1) to make off-shelf fractional capacitors available in the market, just like normal capacitors are, with reliable performance and with specified tolerances both in the capacitance value C and the capacitance order α. Although super capacitors, which are used for hybrid battery-powered systems [92] show fractional-order behavior [93], [94]; they have very large values not suitable for electronic circuit applications.
2) to intensify research work on constructing robust and well-characterized fractional capacitors on the integrated circuit level using the current silicon technology. The work in [74]–[76] and the earlier step in [95] are very important to build upon.
3) to capitalize on the fact that natural tissues and membranes already show fractional impedance behavior. Instead of using this only as a diagnostic tool, investigating the use of more environmentally friendly materials in building electronic circuits should receive significant attention.
4) to incorporate fractional impedances into circuit simulators like Spice.

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