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Numerical Solution of a Family of Fractional Differential Equations by Use of RBF Method

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Abstract: Radial basis function (RBF) interpolation methods are theoretically spectrally accurate. In applications this accuracy is seldom realized due to the necessity of solving a very poorly conditioned linear system to evaluate the methods. Some numerical methods such as Adomian decomposition method and Homotopy perturbation method works for a family of fractional differential equations. In this work, we approximate the exact solution by use of Radial Basis Functions method (RBF). It is important to note that the RBF method finally converges to a linear system and we can solve that system very easily by some Math Microsoft such as MAPLE, MATLAB or MATHEMATICA. Our results show the accurate of this method for these kinds of differential equations.

Keywords: Fractional differential equations; RBF interpolation.

I. INTRODUCTION

In the past century notable contributions have been made to the theory of the fractional calculus [1-3]. In recent decades, the fractional calculus provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. Furthermore, the fractional order models of real systems are regularly more adequate than usually used integer order models. Consequently, the field of the fractional differential equations has attracted interest of researchers in several areas including physics, chemistry, engineering and even finance and social sciences [4-6].

During the last decades, several methods have been used to solve fractional differential equations. Diethelm et al. [7] and Ford et al. [8] have reviewed some of the existing methods and explained their respective strengths and weaknesses. There are some further methods, such as operational method [9], homotopy analysis method [10], differential transform method [11-13] and other methods [6, 14-18]. Over the last 25 years, RBF methods have become an important tool for the interpolation of scattered data and for solving partial differential equations [19]. RBF methods that use infinitely differentiable basis functions that contain a free parameter are theoretically spectrally accurate. The implementation of RBF methods involves solving a linear system that is extremely ill-conditioned when the parameters of the method are such that the best accuracy is theoretically realized. Thus, in applications, RBF methods are not able to produce as accurate of results as they are theoretically capable of.

For appropriately chosen interpolation sites in 1d, the non-polynomial RBF methods are known to be equivalent to polynomial methods in the limit as the shape parameter goes to zero [20]. RBF methods with small values of the shape parameter have been evaluated with bypass algorithms [21] that evaluate the method without solving the associated ill-conditioned linear systems. The bypass algorithms are applicable only for use with a small number of interpolations sites and thus are not well suited for applications. However, the bypass algorithms have been used to show that RBF methods are often more accurate than polynomial based methods when a small, nonzero value of the shape parameter is used.

Unlike polynomial based methods, with RBF methods, it is not possible to rearrange the basic functions into an equivalent cardinal basis which reduces the interpolation matrix in the new basis to the identity matrix. It has been recently shown [22] that if the Gaussian RBF interpolation method is restricted to a uniform grid, that an approximate cardinal basis can be used to efficiently implement the method without any loss of accuracy. With the approximate cardinal approach, the Gaussian RBF method can be accurately implemented with very small values of the shape parameter where it is most accurate. In this work, we use the approximate cardinal approach to approximate derivatives and to numerically solve nonlinear time-dependent PDEs. As a particular application, we use Gaussian collocation method to numerically simulate a family of fractional differential equations.

We note that there also exists a linear system-free Gaussian method for use with equally spaced centers on bounded domains [23]. The method in [23] is based on the connection of the RBF method to polynomial methods and on potential theory rather than on the approximate cardinal function approach of [22]. In the present article, we apply the RBF method for solving fractional differential equations.
The paper is organized as follows. In Section 2, we show our main initial problem. In Section 3, we explain a summery about the RBF method. In section 4, we bring some test examples.

II. PROBMEL STATEMENT

This paper concerns the numerical solution of three-term fractional differential equations which have the general form,

(2.1) $$a\mathfrak{D}_*^{\alpha}y(t) + b\mathfrak{D}_*^{\beta}y(t) + cy(t) = g(t), \qquad 0 < \beta < \alpha \leqslant 2,$$

such that $t \in [0, T]$,

(2.2) $$y(0) = c_0, y'(0) = c_1,$$

where the second initial condition is for $\alpha > 1$ only.
Here, $\mathfrak{D}_*^{\alpha}y(t)$ is the Caputo type fractional derivative of order $q > 0$, defined by,

(2.3) $$\mathfrak{D}_*^{\alpha}y(t) = \frac{1}{\Gamma(m-q)}\int_0^t (t-\tau)^{m-q-1}y^{(m)}(\tau)\,d\tau, \quad t > 0,$$

Where m is the smallest integer greater than q. These equations are also refereed to multi-term fractional differential equations. There are a number of instances where such problems arise; the earliest example seems to be the Bagley-Torvik equation with $\alpha = 2$ and $\beta = 1.5$ that describes the motion of a rigid plate immersed into a Newtonian viscous fluid [2, 24, 25]. Another special case of (2.1) is Basset's problem with $\alpha = 1$ and $\beta = 0.5$ that was first interpreted by Mainardi in terms of a fractional derivative [26, 27]. Also, we can refer to Koeller equation [5] with $\alpha = 2\beta$. As some numerical solutions of problem (2.1)-(2.2), we can mention the works of Edwards et al. [28] and Ford and Connolly [29], where they transferred the problem to a system of fractional differential equations, each of order at most unity. The Bagley-Torvik equation is numerically considered by Podlubny [2] and Diethelm and Ford [30].

III. RBF INTERPOLATION

The RBF interpolation method uses linear combinations of translates of one function $\phi(r)$ of a single real variable. Given a set of centers $x_1^c, x_2^c, \ldots, x_N^c$ in $\mathbb{R}^d$, the RBF interpolant takes the form,

(3.1) $$y(x) = \sum_{j=1}^{N}\lambda_j\phi\left(\left\|x - x_j^c\right\|_2\right).$$

Many different basis functions $\phi(r)$ have been used such as,

Multi-quadratic (MQ) $$\phi(r_j) = \sqrt{r_j^2 + c^2},$$

Inverse multi-quadric (IMQ) $$\phi(r_j) = \frac{1}{\sqrt{r_j^2+c^2}},$$

Inverse quadric (IQ) $$\phi(r_j) = \frac{1}{r_j^2+c^2},$$

Thin plate spline (TPS) $$\phi(r_j) = r_j^2\log(r_j),$$

Gaussian (G) $$\phi(r_j) = e^{(-c^2 r_j)},$$

where c is the shape parameter of the radial basis functions. Optimal shape parameter values are found experimentally and these values are written for exact text problems. But in this paper we concentrate on the Gaussian RBF,

(3.2) $$\phi(r) = e^{(-c^2 r)}.$$

The coefficients, λ, are chosen by enforcing the interpolation condition,

(3.3) $$y(x_j) = f(x_j),$$

at a set of nodes that typically coincide with the centers. Enforcing the interpolation condition at N centers results in a $N \times N$ linear system,

(3.4) $$B\lambda = f,$$

to be solved for the RBF expansion coefficients λ. The matrix B with entries,

(3.5) $$b_{ij} = \phi\left(\|x_i^c - x_j^c\|_2\right), \qquad i, j = 1, \dots, N$$

is called the interpolation matrix or the system matrix. For distinct center locations, the system matrix for the GA RBF is known to be nonsingular [31] if a constant shape parameter is used. To evaluate the interpolant at M points x_i using (3.1), the $M \times N$ evaluation matrix H is formed with entries,

(3.6) $$h_{ij} = \phi\left(\|x_i - x_j^c\|_2\right), \qquad i = 1, \dots, M \;\; and \;\; j = 1, \dots, N.$$

Then the interpolant is evaluated at the M points by the matrix multiplication,

(3.7) $$f_\alpha = H\lambda.$$

Theoretically, RBF methods are most accurate when the shape parameter is small. However, the use of small shape parameters results in system matrices that are very poorly conditioned. The by now very established fact that in RBF methods is that we cannot have both good accuracy and good conditioning at the same is known as the uncertainty principle [32]. Recent books [19, 33-35] on RBF methods can be consulted for more information. Now we apply (3.1) and (2.3) for (2.1), then we have, and for the last array of Matrix coefficient by initial conditions we have,

(3.8)

$$\sum_{i=0}^{N} \lambda_i \left(\frac{a}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \phi_j^{(m)}(\|t_j - t_i\|)\, d\tau + \frac{b}{\Gamma(m-\beta)} \int_0^t (t-\tau)^{m-\beta-1} \phi_j^{(m)}(\|t_j - t_i\|)\, d\tau + \right.$$
$$\left. + c\phi_j(\|t_j - t_i\|)\right) = g(t), \qquad j = 0, \dots, N-1,$$

and for the last array of Matrix coefficient by initial conditions we have,

(3.9) $$y(0) + y'(0) = \sum_{i=0}^{N} \lambda_i \left(\phi(\|t_i\|) + \phi'(\|t_i\|)\right) = c_0 + c_1.$$

In the next section, we apply RBF method for some examples.

IV. NUMERICAL EXAMPLES AND COMPARISONS

4.1 Test problem 1. Let us consider the following fractional differential equation,

$$\mathfrak{D}_*^\alpha y(t) + y(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + t^2 - t, \qquad 1 < \alpha < 2.$$

The initial values were chosen as $y(0) = 0$ and as $y'(0) = -1$. The exact solution is $y(t) = t^2 - t$.

This problem is considered in [8, 38].

4.2 Test problem 2. Consider the following fractional differential equation,

$$\mathfrak{D}_*^{\frac{1}{2}} y(t) + y'(t) - 2y(t) = 0\,, \qquad t > 0,$$

which arises, for instance, in the study of the generalized Basset force occurring when a sphere sinks in a (relatively less dense) viscous fluid [8, 38]. The analytical solution, obtained with the help of Laplace transformation of Caputo fractional derivatives, under the initial condition $y(0) = 1$, is given by,

$$y(t) = \frac{2}{3\sqrt{t}} E_{\frac{1}{2},\frac{1}{2}}(\sqrt{t}) - \frac{1}{6\sqrt{t}} E_{\frac{1}{2},\frac{1}{2}}(-2\sqrt{t}) - \frac{2}{2\sqrt{\pi t}},$$

where $E_{\lambda,\mu}$ is Mittag-Leffler function [2,6] with parameters $\lambda, \mu > 0$,

$$E_{\lambda,\mu}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\lambda k + \mu)}.$$

4.3 Test problem 3. As the third example, we consider the following fractional differential equation,

$$\mathfrak{D}_*^\alpha y(t) + y(t) = 1, \qquad 1 < \alpha < 2,$$

The analytical solution, obtained with the help of Laplace transformation of Caputo fractional derivatives, under the initial conditions $y(0) = 1$ and $y'(0) = -1$, is given by the expression [14, 38],

$$y(t) = E_{\alpha,1}(-t^{\alpha}) - tE_{\alpha,2}(-t^{\alpha}) + t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha}).$$

4.4 Test problem 4. Next we look at $\mathcal{D}_*^{\frac{1}{2}}y(t) - y(t) = 0$ with the initial condition chosen as $y(0) = 1$ such that the exact solution is [8, 38]

$$y(t) = e^{t}\,\mathrm{erf}\left(\sqrt{t} + 1\right).$$

The approximation and exact solutions are showed in Figures 1-8. Also some Approximation and Exact values are brought in tables 1 to 4.

V. CONCLUSIONS

In this paper, we presented a numerical scheme for solving a family of fractional differential equations. We have approximated y(t) by RBF method. Numerical results obtained, show high accuracy of the method, as compared with exact solution. In some cases, the exact solution is very complex or to be very hard but approximation solution is easy to take. For this matter we use RBF method for Convenience. The existence of exact solution of fractional differential equation was discussed in Reference [36] and also the convergence of using method, I mean RBF method, was discussed in reference [37].

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APPENDIXES

Table 1: Ex. 1 for $N = 100 \; and \; h = \frac{1}{100}$

t	0.2	0.4	0.6	0.8
$Approximation\ value$	-0.159999999999999	-0.239999999999999	-0.239999999999999	-0.160000000000000
$Exact\ value$	-0.16	-0.24	-0.24	-0.16

Table 2: Ex. 2 for $N = 50 \; and \; h = \frac{1}{50}$

t	0.2	0.4	0.6	0.8
$Approximation\ value$	1.35206	1.74187	2.20187	2.75505
$Exact\ value$	1.3520935037228606	1.74160634085918	2.2020475880652728	2.7552125985451705

Table 3: Ex. 3 for $N = 50 \; and \; h = \frac{1}{50}$

t	0.2	0.4	0.6	0.8
$Approximation\ value$	0.798360	0.615081	0.452395	0.315972
$Exact\ value$	0.8026910516388106	0.6178883585921764	0.4536053724544029	0.3156113463130239

Table 4: Ex. 4 for $N = 50 \; and \; h = \frac{1}{50}$

t	0.2	0.4	0.6	0.8
$Approximation\ value$	1.790	2.421	3.135	3.976
$Exact\ value$	1.7990172441881772	2.4300431414976621	3.1462130322103350	3.9928358341927076

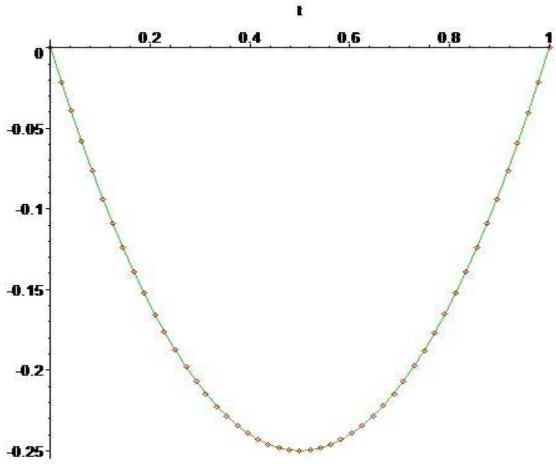

Figure 1: Example 1. Exact solution (Line plot) and Approximation solution (Dot plot)

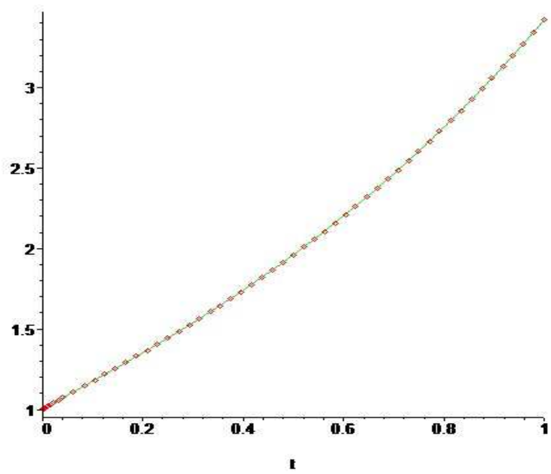

Figure 2: Example 2. Exact solution (Line plot) and Approximation solution (Dot plot)

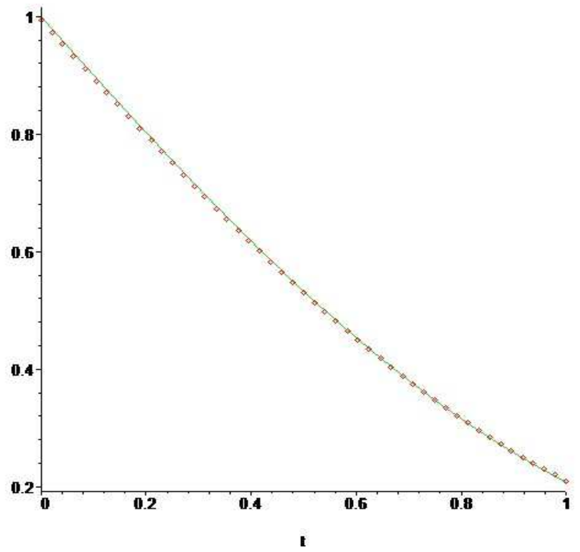

Figure 3: Example 3. Exact solution (Line plot) and Approximation solution (Dot plot)

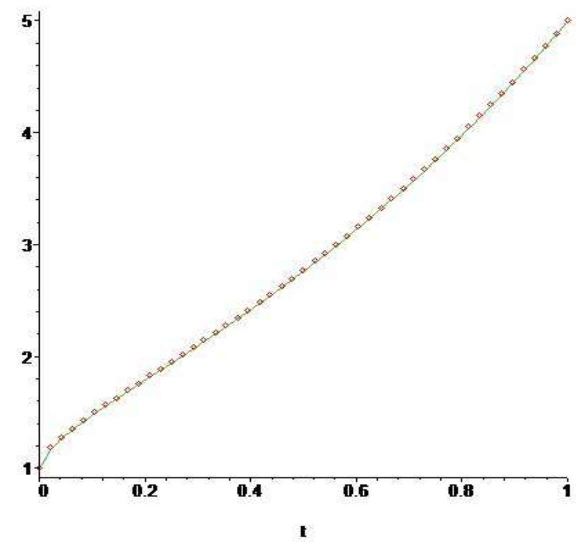

Figure 4: Example 4. Exact solution (Line plot) and Approximation solution (Dot plot)

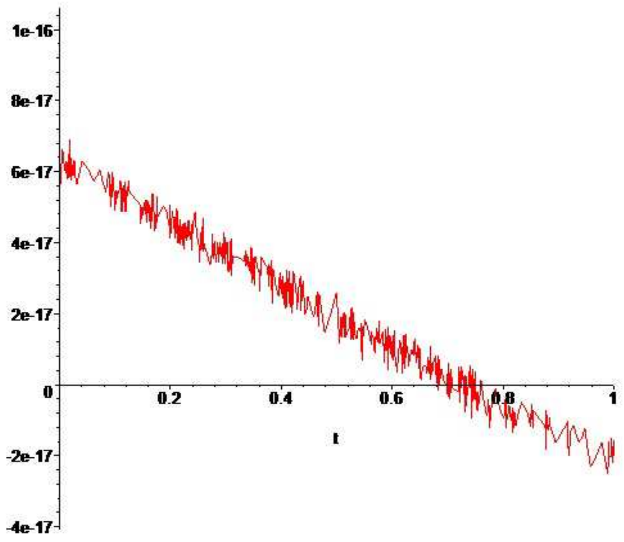

Figure 5: Error plot in Example 1

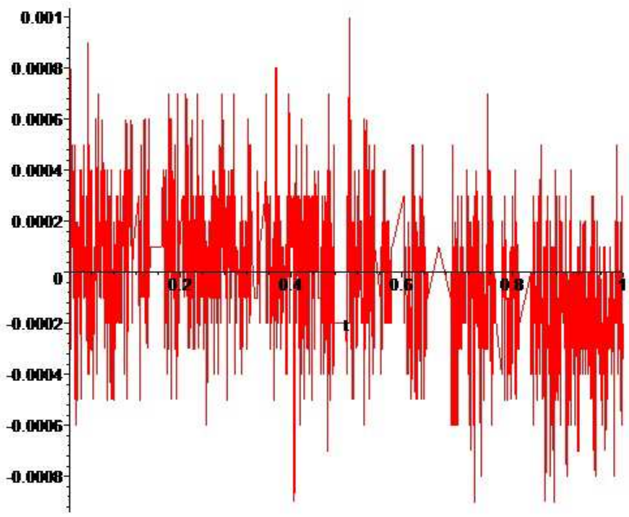

Figure 6: Error plot in Example 2

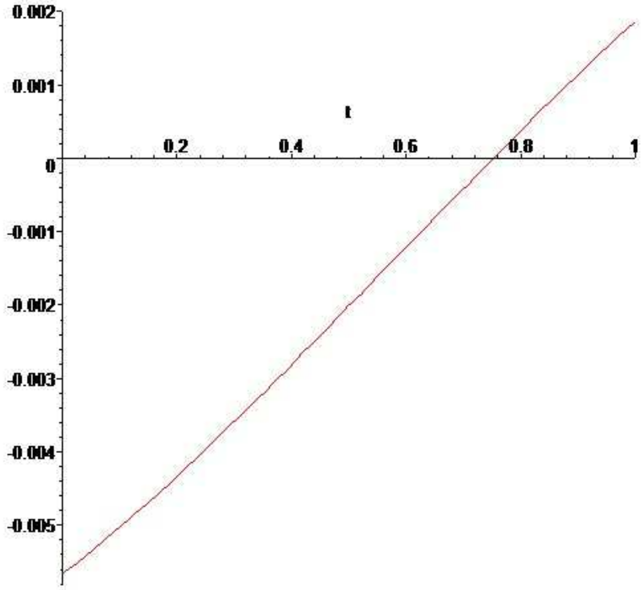

Figure 7: Error plot in Example 3

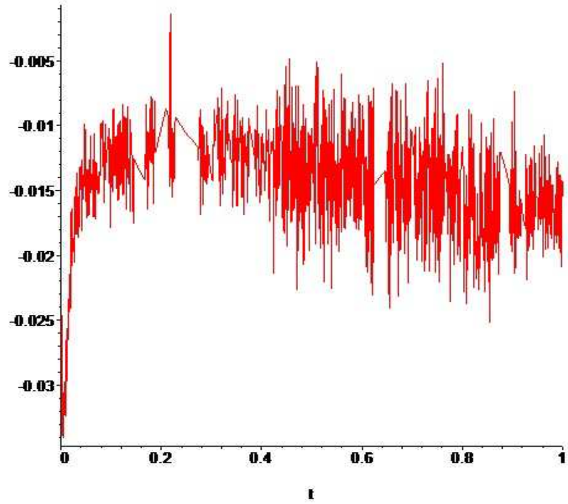

Figure 8: Error plot in Example 4