Estimation and control of discrete fractional order states-space systems

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ABSTRACT

This paper presents the estimation and the state feedback control of the discrete fractional order states-space systems. First of all the state-space model of a fractional order dynamic system is introduced. For this model the stability condition is derived and discussed. Then the parameters identification problem is presented and least squares solution to it is given. Having established the parameters of the model, in order to produce a state feedback control the state vector is needed. In most cases, however it is not available from measurements. Therefore, the state estimation problem arises. In the paper a modification of Kalman filtering technique for this purpose is proposed. At the end the state feedback pole-placement control is applied to a fractional order system. All the techniques used are illustrated with simulational examples.

Keywords: fractional calculus, fractional system estimation, Fractional Kalman Filter, state feedback control

1. INTRODUCTION

The fractional calculus (generalization of a traditional integer order integral and differential calculus) idea has been mentioned in 1695 by Leibniz and L'Hospital. In the end of 19th century Liouville and Riemann introduced first definition of fractional derivative. However, only just in late 60-ties of the 20th century this idea started to be interesting for engineers. Especially, when it was observed that the description of some systems are better and more accurate when fractional derivative is used. For example modeling of behavior of some materials like polymers and rubber especially macroscopic properties of materials with very complicated microscopic structure[1] . In article[2] the frequency dependence of a dynamic of the rubber isolator is modeled with success by a fractional calculus element. In papers[3,4] the relaxation phenomena of organic dielectric materials such as the semi-crystalline polymers are successfully modeled by mechanical and dielectric fractional models. The relaxation processes in organic dielectric materials are associated to molecular motions to a new structural equilibrium of less energy. The Lagrangian and Hamiltonian mechanics can be formulated to include fractional order derivatives, what lead directly to equations of motion with nonconservative forces such as friction.[5]

In article[6] the electrochemical processes and flexible robot arm are modeled by a fractional order models. Even for modeling traffic in information network fractional calculus is found as a useful tool.[7]

More examples and areas of using fractional calculus (eg. fractal modeling, Brownian motion, rheology, viscoelasticity, thermodynamics and others) are to be found in books.[1,8]

In articles[9,10] some geometrical and physical interpretation of fractional calculus are presented.

There are plenty of papers dealing with the continious fractional order state space systems. This paper presents the discrete fractional order state space system approach. In section 2 the discrete fractional order state space model is defined and some general properties of the model are discussed. Section 3 introduces the difference equation corresponding to the model defined earlier. The difference equation is used to parameters identification of the model. Section 4 presents the Fractional Kalman Filter and section 5 describes the state feedback control.

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2. STATE-SPACE MODEL

2.1. Fractional difference definition

Let us start our exposition with some basics of the fractional calculus used throughout the paper. In this paper, as a definition of fractional difference, a following Grünwald-Letnikov definition[11–13] will be used.

$$\Delta^n x_k = \frac{1}{h^n}\sum_{j=0}^{k}(-1)^j \binom{n}{j} x_{k-j} \tag{1}$$

where, $n \in \mathbb{Q}$, is a fractional degree, $\mathbb{Q}$, is a set of rational numbers, and h is a sampling time (this later will be equal to 1), $k \in \mathbb{N}$ is a number of sample for which the difference is calculated.

For $n > 0$ one gets the discrete equivalent of the derivative, for $n < 0$ the discrete equivalent of the integral and for $n = 0$ the function itself is obtained.

The difference operator the has following important properties that are used in this paper:

- linearity

$$\Delta^n(ax_k + by_k) = a\Delta^n x_k + b\Delta^n y_k \tag{2}$$

- exponential law

$$\Delta^{n_1}(\Delta^{n_2})x_k = \Delta^{n_1+n_2}x_k \tag{3}$$

The others properties can be found e.g. in the following papers.[14, 15]

2.2. State-space model

Let us assume a traditional (integer order) discrete space system,

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k & (4)\\ y_k &= Cx_k & (5) \end{aligned}$$

This equation (4) could be rewritten as follows,

$$\Delta^1 x_{k+1} = A_d x_k + Bu_k \tag{6}$$

where $\Delta^1 x_k$ is a first order difference for x_k in the following form

$$\Delta^1 x_{k+1} = x_{k+1} - x_k.$$

and $A_d = A - I$, I being an identity matrix. Taking into account the definition of a first order difference the value of the space vector for time instance $k+1$ could be obtained from the following relation,

$$x_{k+1} = \Delta^1 x_{k+1} + x_k.$$

Following similar considerations to those presented in integer case one can come to the following in general case when n is a rational number (not necessarily an integer)

$$\begin{aligned} \Delta^n x_{k+1} &= A_d x_k + Bu_k & (7)\\ x_{k+1} &= \Delta^n x_{k+1} \\ &\quad - \sum_{j=1}^{k+1}(-1)^j \binom{n}{j} x_{k-j+1} & (8)\\ y_k &= Cx_k & (9) \end{aligned}$$

The values of fractional order differences are obtained according to (7), from this values the next state vector is calculated using relation (8). Output equation is given by (9).

2.3. General state-space model

When system equations do not have the same degree, then the discrete state-space model has a somewhat different form. It can be presented as follows:

$$\begin{bmatrix} \Delta^{n_1} x_{1,k+1} \\ \vdots \\ \Delta^{n_N} x_{N,k+1} \end{bmatrix} = A_d \begin{bmatrix} x_{1,k} \\ \vdots \\ x_{N,k} \end{bmatrix} + Bu_k \tag{10}$$

In this case the next state is evaluated, according to (8) as follows

$$x_{k+1} = \Delta^{\Upsilon} x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_j x_{k-j+1}. \tag{11}$$

where:

$$\Upsilon_k = \operatorname{diag} \begin{bmatrix} \begin{pmatrix} n_1 \\ k \end{pmatrix} \\ \vdots \\ \begin{pmatrix} n_N \\ k \end{pmatrix} \end{bmatrix}, \quad \Delta^{\Upsilon} x_{k+1} = \begin{bmatrix} \Delta^{n_1} x_{1,k+1} \\ \vdots \\ \Delta^{n_N} x_{N,k+1} \end{bmatrix}$$

and $n_1, \ldots, n_N$ are orders of system equations.

The output equation is the same as in the previous case, i.e., (9)

In practical realization the number of elements in the sum in equation (11) has to be limited to predefined value L. The equation (11) in this case has the following form:

$$x_{k+1} = \Delta^{\Upsilon} x_{k+1} - \sum_{j=1}^{L} (-1)^j \Upsilon_j x_{k-j+1}. \qquad L < k \tag{12}$$

This simplification speeds up calculations and in real applications makes them possible at all. However it has an effect on the accuracy of the model realization.[16, 17]

2.4. Stability Conditions

Let us assume fractional discrete state-space system in the form

$$\Delta^{\Upsilon} x_{k+1} = A_d x_k \tag{13}$$

this equation can by rewritten as

$$x_{k+1} = (A_d + \Upsilon_1) x_k - \sum_{j=2}^{k+1} (-1)^j \Upsilon_j x_{k-j+1} \tag{14}$$

This system can be treated as an infinite order state delayed system and rewritten as

$$\begin{bmatrix} x_{k+1} \\ x_k \\ x_{k-1} \\ \vdots \end{bmatrix} = \mathbb{A} \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \end{bmatrix} \tag{15}$$

where

$$\mathbb{A} = \begin{bmatrix} (A_d + \Upsilon_1) & -(-1)^2\Upsilon_2 & -(-1)^3\Upsilon_3 & \dots \\ I & 0 & 0 & \dots \\ 0 & I & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{16}$$

The stability conditions for such a redefined system are as follows.

THEOREM 2.1. *The system defined by equation (13) is asymptotically stable iff*

$$||\mathbb{A}|| < 1 \tag{17}$$

where $||.||$ *denotes matrix norm defined as a* $\max||\lambda_i||$ *where* λ_i *is an* $i-th$ *eigenvalue of the matrix* $\mathbb{A}$.

The number of factors Υ_j in matrix $\mathbb{A}$ has to be reduced in the same way as it is presented in equation 12. This reduction may cause the decrease of accuracy of the stability determination. Especially when the system is close to the stability margin.

3. DIFFERENCE EQUATION

In order to identify the system parameters the difference equation describing input-output dynamic relation is more convenient than the state-space representation. It can be introduced by using the shift operator q. Let the system equations (9) and (10) be rewritten as follows

$$\begin{aligned} q\Delta^{\Upsilon}x_k &= A_d x_k + Bu_k \\ y_k &= Cx_k \end{aligned}$$

This immediately gives the relation

$$\frac{Y(q)}{U(q)} = C(I(q\Delta^{\Upsilon}) - A)^{-1}B$$

where Δ^{Υ} is also a polynomial of q.

THEOREM 3.1. *The Fractional Difference Equation corresponding to the discrete fractional order state space system (10) is defined by the following relation:*

$$\begin{aligned} \Delta^{n_N^*}y_k + a_{N-1}\Delta^{n_{N-1}^*}y_{k-1} + \dots + a_0 y_{k-N} = \\ b_{N-1}\Delta^{n_{N-1}^*}u_{k-1} + \dots + b_0 u_{k-N} \end{aligned} \tag{18}$$

where

$$n_N^* = n_1 + n_2 + \dots + n_N \tag{19}$$

and the initial conditions are $x_j = 0$ *for* $j \leq 0$.

The coefficients a_i and b_j, for example, are entries of following matrices :

$$A_d = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{N-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} b_0 & b_1 & \dots & b_{N-1} \end{bmatrix} \tag{20}$$

It is worth to mention that each state variable introduce one time shift in the difference equation.

3.1. Parametric Identification

The equation (18) describes dynamic dependence of the system output in relation to the system input. In order to estimate parameters of the equation, let us introduce vectors φ_k, Y_k, θ. The vector φ contains shifted differences of output and input of order from n_{N-1} to n_0 for the k-th sample and has the form:

$$\varphi_k^T = [\ -\Delta^{n^*_{N-1}} y_{k-1} \quad \dots \quad -y_{k-N} \tag{21}$$

$$\Delta^{n^*_{N-1}} u_{k-1} \quad \dots \quad u_{k-N}\ ] \tag{22}$$

The vector Y_k contains the output difference of the order n_N and has the following form:

$$Y_k = [\ \Delta^{n^*_N} y_k s\ ] \tag{23}$$

The vector θ contains parameters of the difference equation and is defined as follows:

$$\theta^T = \begin{bmatrix} a_{N-1} \\ \vdots \\ a_0 \\ b_{N-1} \\ \vdots \\ b_0 \end{bmatrix} \tag{24}$$

The parameters may be obtained by solving the following equation.

$$\begin{bmatrix} Y_k \\ Y_{k-1} \\ \dots \end{bmatrix} = \begin{bmatrix} \varphi_k^T \\ \varphi_{k-1}^T \\ \dots \end{bmatrix} \theta \tag{25}$$

For the higher number of samples than N the overdetermined set of equation is obtained. In that case the pseudo inversion of the matrix φ has to be used to solve it.

3.2. Parametric Identification Example

Let us assume a system given by the following matrices

$$A_d = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [\ b_0 \quad b_1\ ], N = [\ n_1 \quad n_2\ ]^T \tag{26}$$

where

$a_0 = 0.1, a_1 = 0.4, b_0 = 0.6, b_1 = 0.3, n_1 = 1.2, n_2 = 0.7$

The identified parameters for the system without noise are:

$a_0 = 0.1000, a_1 = 0.4000, b_0 = 0.6000, b_1 = 0.3000$

For output with additive gaussian noise with $\mathrm{E}[\nu_k \nu_k^T] = 0.1$ (see equations (27), (28) and (29)) the results are

$a_0 = 0.1088, a_1 = 0.4700, b_0 = 0.6557, b_1 = 0.2591.$

The accuracy of identification results strongly depends on the data noise. The difference computation process is very sensitive to noise. In Fig. 1 the output of the identified system is compared to the original one for system with noise.

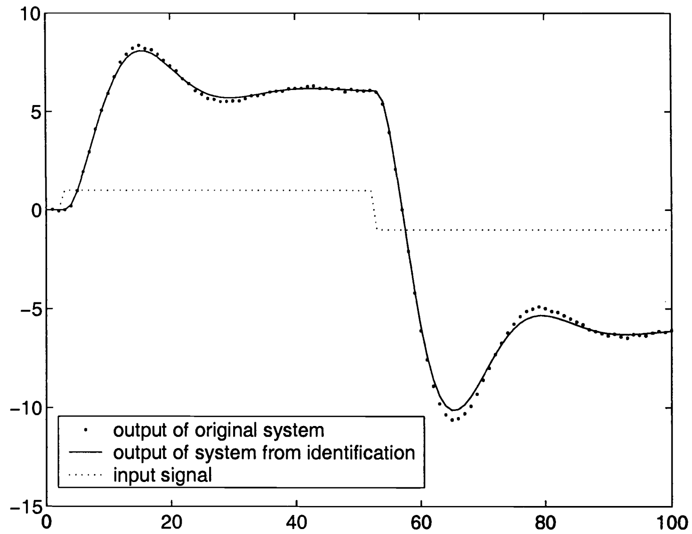

Figure 1. Plant and identified plant output

4. FRACTIONAL KALMAN FILTER

The Kalman Filter is an optimal state vector estimator using the knowledge about the system model, input and output signals. The stochastic discrete fractional order state space model is defined as follows:

$$\Delta^{\Upsilon} x_{k+1} = A_d x_k + B u_k + \omega_k \tag{27}$$

$$x_{k+1} = \Delta^{\Upsilon} x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_j x_{k-j+1} \tag{28}$$

$$y_k = C x_k + \nu_k \tag{29}$$

Results of estimation are obtained by minimizing in each step the following cost function[18]:

$$\begin{aligned}\hat{x}_k &= \arg\min_x [(\tilde{x}_k - x)\tilde{P}_k^{-1}(\tilde{x}_k - x)^T + \\ &+ (y - Cx)R_k^{-1}(y - Cx)^T]\end{aligned}$$

where $\tilde{x}$ is a state vector prediction and $\hat{x}_k$ is a state vector estimation. $\tilde{P}_k$ is a prediction of estimation error symmetric covariance matrix, Q_k is symmetric covariance matrix of system noise ω_k in (27) and R_k is symmetric covariance matrix of output noise ν_k in (28)

For such a system defined in state-space equations form we may introduce a Kalman Filter by the following equations (see article[17] for details):

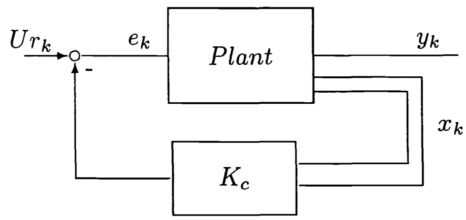

Figure 2. State feedback controller scheme

$$
\begin{aligned}
\Delta^{\Upsilon}\tilde{x}_{k+1} &= A_d\hat{x}_k + Bu_k \\
\tilde{x}_{k+1} &= \Delta^{\Upsilon}\tilde{x}_{k+1} - \sum_{j=1}^{k+1}(-1)^j\Upsilon_j\hat{x}_{k-j+1} \\
\tilde{P}_k &= (A_d + \Upsilon_1)\,P_{k-1}\,(A_d + \Upsilon_1)^T + \\
&\quad +Q_{k-1} + \sum_{j=2}^{k}\Upsilon_j P_{k-j}\Upsilon_j^T \\
K_k &= \tilde{P}_k C^T(C\tilde{P}_k C^T + R_k)^{-1} \\
\hat{x}_k &= \tilde{x}_k + K_k(y_k - C\tilde{x}_k) \\
P_k &= (I - K_k C)\tilde{P}_k
\end{aligned}
$$

with initial conditions

$$x_0, P_0 = \mathrm{E}[(\hat{x}_0 - x_0)(\hat{x}_0 - x_0)^T]$$

where: ν_k is an output noise and ω_k is a system noise . Both of the noise signals are assumed to be independent noises with zero expected value.

5. STATE FEEDBACK CONTROL

State feedback configuration under consideration is depicted in Fig. 2.

where the plant is given in the form (10) and (9) For the control signal given by the following expression

$$e_k = Ur_k - K_c x_k \tag{30}$$

where $K_c = [k_0, k_1, \ldots, k_N]$ is a state feedback controller matrix, the system matrix is given by the equation

$$A_d^* = A_d - BK_c \tag{31}$$

For system matrices defined in (20) the control defined by (30) leads to the direct link between plant, controller and closed loop system parameters

$$
\begin{aligned}
a_0^* &= a_0 + k_0 \\
a_1^* &= a_1 + k_1 \\
\vdots\ & \quad \vdots \\
a_{N-1}^* &= a_{N-1} + k_{N-1}
\end{aligned}
$$

This way it is possible to obtain the state feedback controller matrix parameters. The procedure is the same as in standard case.

In many real application the state variables could not be measured directly from the fractional order plant. In that case the Fractional Kalman Filter has to be used to estimate state variables.

5.1. State Feedback Control Example

Let us assume the stochastic model as

$$A_d = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{32}$$

$$C = \begin{bmatrix} 0.6 & 0.3 \end{bmatrix}, N = \begin{bmatrix} 1.2 & 0.7 \end{bmatrix}^T, \mathrm{E}[\nu_k \nu_k] = [0.01], \mathrm{E}[\omega_k \omega_k] = [0] \tag{33}$$

Reference matrix A_d

$$A_d = \begin{bmatrix} 0 & 1 \\ -0.2 & -1.2 \end{bmatrix} \tag{34}$$

regulator matrix $K_c = [0.1, 0.8]$

In the example the state variables are estimated through the Fractional Kalman Filter is needed. The FKF parameters are:

$$P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, R = [0.01], x_0 = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{35}$$

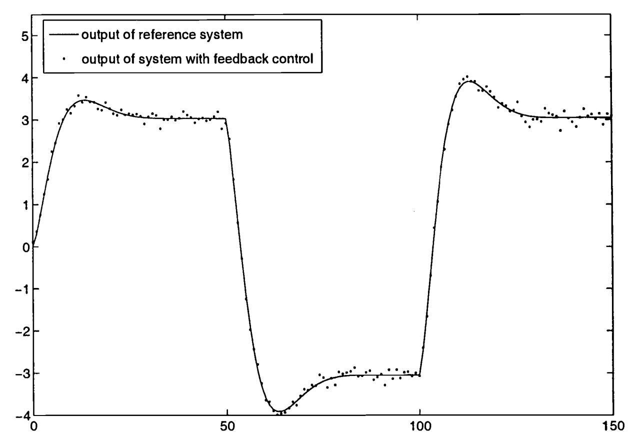

Figure 3. Reference plant and plant with controller output.

In Fig. 3 the output of the the plant with reference matrix A without noise is compare with the output of the original plant with controller.

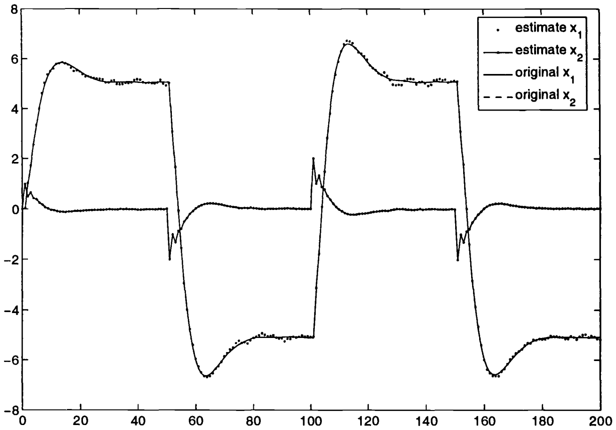

Figure 4. State variables estimated and original

6. CONCLUSIONS

The state feedback control of the discrete state space systems was presented and tested. In order to design such a controller the parameters and the state variables of the plant are needed. To get the plant parameters the parametric identification of the discrete fractional order system was introduced and its results were tested on simulation examples. The state variables estimator—fractional Kalman Filter—was proposed in order to obtain the states of the plant. This approach was confirmed in simulations, even with a substantial degree of noise present in the system. Finally, the control law was designed and applied to a fractional order system. This linked together parameters identification, state estimation and pole-placement control and it gave the desired dynamic properties of the closed-loop fractional order system.

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