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Numerical methods for multi-term fractional (arbitrary) orders differential equations

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Abstract

Our main concern here is to give a numerical scheme to solve a nonlinear multi-term fractional (arbitrary) orders differential equation.

Keywords: Multi-term fractional differential equation; Caputo derivative; Euler's method; Product trapezoidal quadrature formula; Product rectangle rule

1. Introduction

The use of fractional orders differential and integral operators in mathematical models has become increasingly widespread in recent years (see [12,23,27]). Several forms of fractional differential equations have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solution (see [12,16,23,27]). However, much of the work published to date has been concerned with linear single term equations and, of these, equations of order less than unity have been most often investigated (see [2,4–7]).

Let $\alpha \in (n, n+1]$, $\alpha_k \in (k-1, k]$, $k = 1, 2, \ldots, n$ and $\alpha_0 = 0$.

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Consider the initial value problem

$$D^{\alpha}x(t) = f(t, x(t), D^{\alpha_1}x(t-r), D^{\alpha_2}x(t-2r), \dots, D^{\alpha_n}x(t-nr)), \quad t \in I,$$

$$D^{j}x(t) = 0 \quad \text{for } t \leqslant 0, \ \ j = 0, 1, 2, \dots, n.$$

The existence of at least one solution of this initial value problem has been proved (see [15]) where the function $f(t, U) = f(t, u_1(t), \dots, u_n(t))$ satisfies Caratheodory conditions, i.e., $t \to f(t, U)$ is measurable for every $U \in R^{n+1}$ and $U \to f(t, U)$ is continuous for every $t \in I$. $f(t, U)$ is nondecreasing for all variables, and there exist a function $a(t) \in L_1$ and constants $b_k \geqslant 0$, such that

$$|f(t, U)| \leqslant a(t) + \sum_{k=0}^{n} b_k |u_k(t)| \quad \text{for all } (t, U) \in I \times R^{n+1}.$$

In this paper we focus on providing a numerical solution to the nonlinear multi-term fractional (arbitrary) orders differential equation.

$$D^{n}x(t) = f(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)), \quad t > 0 \tag{1.1}$$

subject to the initial values

$$D^{j}x(0) = a_j, \quad j = 0, 1, 2, \dots, n-1, \tag{1.2}$$

where α_i are real numbers $(i = 1, 2, \dots, m)$, such that

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < n$$

and n is any positive integer number.

Applications for such equations arise, e.g., in various areas of mechanics [27], the Bagley–Torvik equation [12] and the Basset equation [23].

Now we give the definition and some properties of the fractional order differential and integral operators.

Let $L_1 = L_1[a, b]$ be the class of Lebesgue integrable functions on $[a, b], a < b < \infty$.

Definition 1.1. Let $f(t) \in L_1$, $\beta \in R^{+}$. The fractional (arbitrary) order integral of the function $f(t)$ of order β is defined by (see [19,22,25,28])

$$I_a^{\beta} f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, \mathrm{d}s,$$

when $a = 0$ we can write $I^{\beta} f(t) = I_0^{\beta} f(t) = f(t) * \phi_{\beta}(t)$, where $\phi_{\beta}(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $t > 0$, $\phi_{\beta}(t) = 0$ for $t \leqslant 0$ and $\phi_{\beta} \to \delta(t)$ (the delta function) as $\beta \to 0$ (see [18]).

Definition 1.2. The fractional derivative D^{α} of order $\alpha \in (0, 1]$ of the absolutely continuous function $g(t)$ is defined as (see [3,15,19,25,28])

$$D_a^{\alpha} g(t) = I_a^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} g(t), \quad t \in [a,b].$$

2. Formulation of the problem

Eq. (1.1) can be written in the form

$$D^n x(t) = f(t, y_0(t), y_1(t), \ldots, y_m(t)), \quad t > 0, \tag{2.1}$$

where

$$y_i(t) = D^{\alpha_i} x(t), \quad i = 0, 1, 2, \ldots, m, \;\; \alpha_0 = 0. \tag{2.2}$$

Let

$$x_1(t) = x(t) \tag{2.3}$$

and

$$x_{i+1}(t) = \frac{\mathrm{d}}{\mathrm{d}t} x_i(t), \quad i = 1, 2, \ldots, n-1. \tag{2.4}$$

Say $0 < \alpha_1 \leqslant 1$ then

$$y_1(t) = I^{1-\alpha_1} x_2(t), \tag{2.5}$$

$1 < \alpha_2 \leqslant 2$ then

$$y_2(t) = I^{2-\alpha_2} x_3(t), \ldots, \tag{2.6}$$

$n-2 < \alpha_{m-1} \leqslant n-1$ then

$$y_{m-1}(t) = I^{n-1-\alpha_{m-1}} x_n(t) \tag{2.7}$$

and $n-1 < \alpha_m < n$ then

$$y_m(t) = I^{n-\alpha_m} f(t, y_0(t), y_1(t), \ldots, y_m(t)), \tag{2.8}$$

or

$$y_m(t) = I^{n-\alpha_m} \frac{\mathrm{d}}{\mathrm{d}t} x_n(t). \tag{2.9}$$

Lemma 2.1. *The fractional differential equation* (2.1) *can be transformed to the system*

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = AX(t) + B(t), \tag{2.10}$$

where

$$X(t) = (x_1(t), x_2(t), \ldots, x_n(t))', \tag{2.11}$$

$$B(t) = (0, 0, \dots, f(t, y_0(t), y_1(t), \dots, y_m(t))' \tag{2.12}$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}_{n\times n}, \tag{2.13}$$

where (′) *denotes the transpose of the matrix.*

Proof. Using Eqs. (2.1) and (2.4), we can easily obtain the result. □

Definition 2.1. By a solution of the system (2.10), we mean a column vector $X(t)$ with $X(0) = (a_0, a_1, \dots, a_{n-1})'$.

Set $I = [0, T]$, say, T is a suitable positive number and $\check{D} = I \times C^*(t)$ where $C^*(t)$ is the class of all continuous column vectors $(y_0(t), y_1(t), \dots, y_m(t))'$.

Assuming that the function $f(t, y_0(t), y_1(t), \dots, y_m(t))$ satisfies the Lipschitz condition

$$|f(t, y_0, y_1, \dots, y_m) - f(t, x_0, x_1, \dots, x_m)| \leqslant k \sum_{i=0}^{m} |y_i(t) - x_i(t)| \tag{2.14}$$

for $(t, y_0, y_1, \dots, y_m)$ and $(t, x_0, x_1, \dots, x_m) \in \check{D}$, $k > 0$.

The following theorem can be proved as in [1,14].

Theorem 2.1. *Let* $f(t, y_0, y_1, \dots, y_m) \in C(\check{D})$ *and satisfies the Lipschitz condition* (2.14), *then the system of first-order differential equations* (2.10) *has a unique solution.*

Now, we consider the initial value problem

$$D^{\alpha}x(t) = F(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)), \quad t > 0 \tag{2.15}$$

subject to the initial values

$$D^j x(0) = a_j, \quad j = 0, 1, 2, \dots, n-1 \tag{2.16}$$

and

$$D^n x(0) = 0, \tag{2.17}$$

where α_i are real numbers ($i = 1, 2, \dots, m$), such that

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha < n + 1.$$

Lemma 2.2. *Let $f = I^{\alpha-n}F$. If either F is monotonic in $t, t \in I$ or F satisfies Lipschitz condition in $t, t \in I$ then the initial value problems* (1.1), (1.2) *and* (2.15)–(2.17) *are equivalent.*

Proof. From Eq. (2.15), we get

$$D^{\alpha}x(t) = F(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)),$$
$$I^{n+1-\alpha}D^{n+1}x(t) = F(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)),$$
$$I^{\alpha-n}I^{n+1-\alpha}D^{n+1}x(t) = I^{\alpha-n}F(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)),$$
$$D^{n}x(t) = f(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)).$$

From Eq. (1.1), we get

$$D^{n}x(t) = f(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)),$$
$$D^{n}x(t) = I^{\alpha-n}F(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)),$$
$$D^{n+1}x(t) = DI^{\alpha-n}F(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)),$$
$$I^{n+1-\alpha}D^{n+1}x(t) = I^{n+1-\alpha}DI^{\alpha-n}F(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)),$$
$$D^{\alpha}x(t) = F(t, x(t), D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_m}x(t)).$$

Then the initial value problems (1.1), (1.2) and (2.15)–(2.17) are equivalent. □

3. Numerical methods and results

Numerical methods for the solution of linear fractional differential equations involving only one fractional derivative are well established (see for example [2,4,5,17]).

In [13] Luchko and Diethelm discussed a new algorithm for the numerical solution of initial value problems for general linear multi-term differential equations of fractional order with constant coefficients and fractional derivatives in the Caputo sense. Which is obtained by applying the convolution quadrature and discretized operational calculus to the analytical solution of the problem given in terms of the Mittag–Leffer type function (see [20]). That may require a large amount of computational effort to calculate its weights.

In [16] Ford et al. showed how the numerical approximation of the solution of a linear multi-term fractional differential equation can be calculated by reduction of the problem to a system of ordinary and fractional differential equations each of order at most unity (see [4]).

In [21] Leszczynski and Ciesielski proposed an algorithm for the numerical solution of arbitrary differential equations of fractional order. Which is obtained by using the following decomposition of the differential equation into a system of differential equation of integer order connected with inverse forms of

Abel-integral equations (see [24,26]). The algorithm is used for solution of the linear and nonlinear equations.

In [12] Diethelm and Ford considered the reformulation of the Bagley–Torvik equation as a system of fractional differential equations of order 1/2.

If the coefficient of D^{α_m} is zero then we have two methods:

Method 1 (ET): Apply the Euler's method (see [1]) in Eqs. (2.1) and (2.4), then replace the integrals by using the product trapezoidal quadrature formula (see [10]).

Method 2 (ER): Apply the Euler's method in Eqs. (2.1) and (2.4), then replace the integrals by using the product rectangle rule (see [10]).

If the coefficient of D^{α_m} is nonzero then we have two methods:

Method 1 (2E): Apply the Euler's method in Eqs. (2.1) and (2.4), then replace the integrals by using the product rectangle rule.

Method 2 (3E): Apply the Euler's method in Eqs. (2.1), (2.4) and (2.9), then replace the integrals by using the product rectangle rule.

4. Numerical examples

As an example that arises in application, we solve the Bagley–Torvik equation which arises, for instance, in modelling the motion of a rigid plate immersed in a Newtonian fluid.

Example 1. The equation

$$D^2x(t) + D^{0.5}x(t) + x(t) = 0 \tag{4.1}$$

subject to

$$x(0) = 1, \quad x'(0) = 0.$$

This problem was solved in [16]. We show the approximate solutions in Figs. 1 and 2 for step size $h = 0.01$.

Example 2. Consider the nonlinear equation

$$aD^2x(t) + bD^{\alpha_2}x(t) + cD^{\alpha_1}x(t) + e(x(t))^3 = f(t), \quad 0 < \alpha_1 < \alpha_2 \leqslant 1 \tag{4.2}$$

and

$$f(t) = 2at + \frac{2b}{\Gamma(4-\alpha_2)}t^{3-\alpha_2} + \frac{2c}{\Gamma(4-\alpha_1)}t^{3-\alpha_1} + e\left(\frac{1}{3}t^3\right)^3$$

subject to

$$x(0) = x'(0) = 0.$$

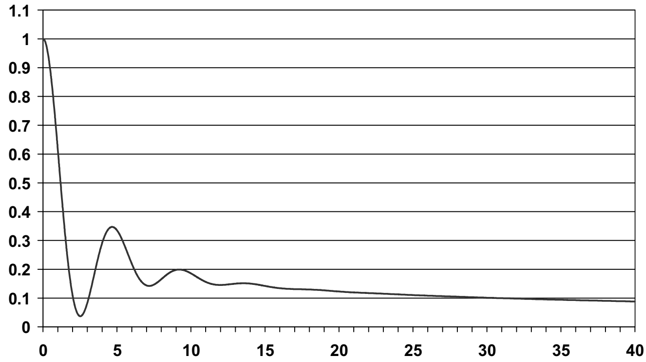

Fig. 1. By ET.

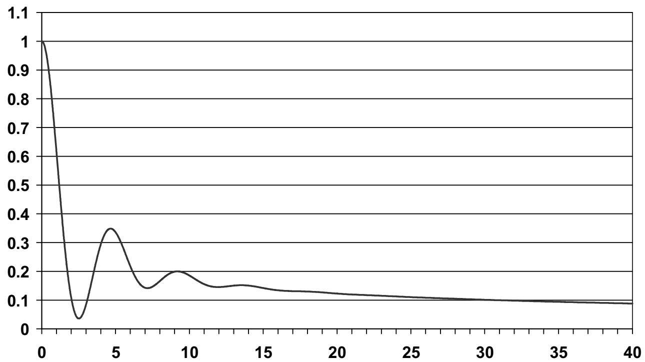

Fig. 2. By ER.

It is easily verified that the exact solution of this problem is

$$x(t) = \frac{1}{3}t^3.$$

For $a = 1$, $b = 2$, $c = 0.5$, $e = 1$, $\alpha_1 = 0.00196$ and $\alpha_2 = 0.07621$ we found the following results.

Step size	Maximal error by ET	Maximal error by ER
0.1	0.086705100000	0.080849080000
0.01	0.008907288000	0.007917225000
0.001	0.000892400700	0.000789135700

For $a=1$, $b=0.1$, $c=0.2$, $e=0.3$, $\alpha_1=\frac{\sqrt{5}}{5}$ and $\alpha_2=\frac{\sqrt{2}}{2}$ we found the following results.

Step size	Maximal error by ET	Maximal error by ER
0.1	0.091559170000	0.089693500000
0.01	0.009669006000	0.009382159000
0.001	0.000971794100	0.000943839600

Example 3. Consider the equation

$$aD^2x(t)+b(t)Dx(t)+c(t)D^{\alpha_2}x(t)+e(t)D^{\alpha_1}x(t)+k(t)x(t) \\ =f(t),\quad 0<\alpha_1<\alpha_2<1 \tag{4.3}$$

and

$$f(t)=-a-b(t)t-\frac{c(t)}{\Gamma(3-\alpha_2)}t^{2-\alpha_2}-\frac{e(t)}{\Gamma(3-\alpha_1)}t^{2-\alpha_1}+k(t)\left(2-\frac{1}{2}t^2\right)$$

subject to

$$x(0)=2,\quad x'(0)=0.$$

It is easily verified that the exact solution of this problem is

$$x(t)=2-\frac{1}{2}t^2.$$

For $a=0.1$, $b(t)=t$, $c(t)=t+1$, $e(t)=t^2$, $k(t)=(t+1)^2$, $\alpha_1=0.781$ and $\alpha_2=0.891$ we found the following results.

Step size	Maximal error by ET	Maximal error by ER
0.1	0.032531260000	0.043517230000
0.01	0.003163218000	0.000342846400
0.001	0.000315427800	0.000027898520

For $a=5$, $b(t)=\sqrt{t}$, $c(t)=t^2-t$, $e(t)=3t$, $k(t)=t^3-t$, $\alpha_1=\frac{\sqrt{7}}{70}$ and $\alpha_2=\frac{\sqrt{13}}{13}$ we found the following results.

Step size	Maximal error by ET	Maximal error by ER
0.1	0.050356980000	0.049063440000
0.01	0.005046725000	0.004850388000
0.001	0.000503540000	0.000484466600

Example 4. The Bagley–Torvik equation

$$aD^2x(t) + bD^{3/2}x(t) + cx(t) = f(t), \quad 0 \leqslant t \leqslant 5, \tag{4.4}$$

where

$$f(t) = c(t+1)$$

subject to

$$x(0) = x'(0) = 1.$$

It is easily verified that the exact solution of this problem is

$$x(t) = t + 1.$$

This problem was solved in [12,13]. We evaluate the maximal error (m.e.) and the results are compared to results obtained by the methods of [12,13].

For $a = 1$, $b = 0.5$ and $c = 0.5$ we found the following results.

Step size	M.e. by our methods	E. at $t = 5$ by Ref. [12]
0.5	0	−0.15131473519232
0.25	0	−0.04684102179946
0.125	0	−0.01602947553912
0.0625	0	−0.00562770408881

For $a = 1$, $b = 1$ and $c = 1$ we found the following results.

Step size	M.e. by our methods	E. at $t = 5$ by Ref. [13]	E. at $t = 5$ by Ref. [13]
0.5	0	0.3831	0.00741
0.25	0	0.0904	0.00630
0.125	0	0.0265	0.00196
0.0625	0	0.0084	0.00056
0.01325	0	0.0028	0.00016

Example 5. Consider the equation

$$\begin{aligned} &aD^2x(t) + b(t)D^{\alpha_2}x(t) + c(t)Dx(t) + e(t)D^{\alpha_1}x(t) + k(t)x(t) \\ &\quad = f(t), \quad 0 < \alpha_1 < 1, \ 1 < \alpha_2 < 2 \end{aligned} \tag{4.5}$$

and

$$f(t) = -a - \frac{b(t)}{\Gamma(3-\alpha_2)} t^{2-\alpha_2} - c(t)t - \frac{e(t)}{\Gamma(3-\alpha_1)} t^{2-\alpha_1} + k(t)\left(2 - \frac{1}{2}t^2\right)$$

subject to

$$x(0) = 2, \quad x'(0) = 0.$$

It is easily verified that the exact solution of this problem is

$$x(t) = 2 - \frac{1}{2}t^2.$$

For $a = 1$, $b(t) = t^{1/2}$, $c(t) = t^{1/3}$, $e(t) = t^{1/4}$, $k(t) = t^{1/5}$, $\alpha_1 = 0.333$ and $\alpha_2 = 1.234$ we found the following results.

Step size	Maximal error by 2E	Maximal error by 3E
0.1	0.040959840000	0.040959840000
0.01	0.003975630000	0.003975630000
0.001	0.000396132500	0.000396490100

For $a = 3$, $b(t) = t$, $c(t) = t + 1$, $e(t) = t^2$, $k(t) = (t + 1)^2$, $\alpha_1 = \frac{\sqrt{3}}{30}$ and $\alpha_2 = \sqrt{3}$ we found the following results.

Step size	Maximal error by 2E	Maximal error by 3E
0.1	0.045955060000	0.045955060000
0.01	0.004457474000	0.004457474000
0.001	0.000444412200	0.000444293000

Example 6. Consider a nonlinear form of the fractional differential equation

$$D^2x(t) + 0.5D^{3/2}x(t) + 0.5x^3(t) = f(t), \quad t > 0, \tag{4.6}$$

where

$$f(t) = \left\{ \begin{array}{ll} 8 & \text{for } 0 \leqslant t \leqslant 1, \\ 0 & \text{for } t > 1 \end{array} \right\}$$

subject to

$$x(0) = x'(0) = 0.$$

This is a Bagley–Torvik equation where nonlinear term $x^3(t)$ is introduced. This problem was solved in [21]. Figs. 3 and 4 show a behaviour of the numerical solution for step size $h = 0.01$.

Example 7. Consider the nonlinear equation

$$D^{1.455}x(t) = -t^{0.1}\frac{E_{1.545}(-t)}{E_{1.445}(-t)}\mathrm{e}^t x(t)D^{0.555}x(t) + \mathrm{e}^{-2t} - (Dx(t))^2, \tag{4.7}$$

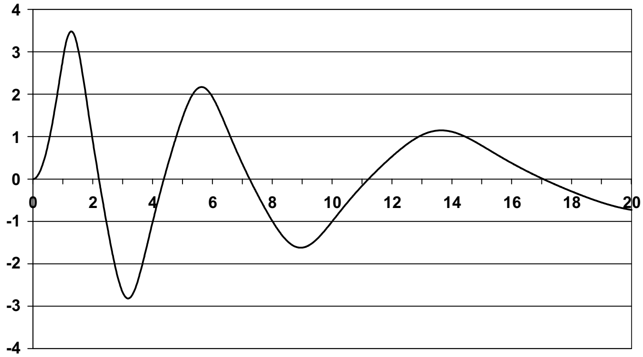

Fig. 3. By 2E.

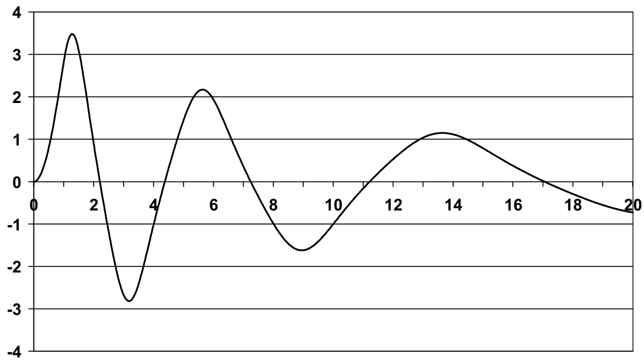

Fig. 4. By 3E.

where E_μ denotes the Mittage–Leffler function of order μ, defined by

$$E_\mu(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(j\mu + 1)}.$$

We combine Eq. (4.7) with the initial conditions

$$x(0) = 1, \quad x'(0) = -1.$$

It is easily verified that the exact solution of this problem is

$$x(t) = \mathrm{e}^{-t}.$$

This problem was solved in (see [8,9,11]). Set

$$y(t) = x(t) + t,$$

Table 1

Step size	Maximal error by 2E	Run time (s)
1/100	0.018165301723	6.97
1/200	0.006423074841	28.17
1/400	0.001267305832	195.5
1/500	0.000786614454	294.17

Table 2

Step size	Maximal error by 3E	Run time (s)
1/100	0.013091171613	3.62
1/200	0.003916009823	14.39
1/400	0.000945018748	97.45
1/430	0.000824219209	106.29

then we have the following equation

$$\begin{aligned} D^{1.455}y(t) = -t^{0.1}\frac{E_{1.545}(-t)}{E_{1.445}(-t)}\mathrm{e}^{t}(y(t)-t)\bigg(D^{0.555}y(t) \\ -\frac{1}{\Gamma(1.445)}t^{0.445}\bigg) + \mathrm{e}^{-2t} - (Dy(t)-1)^{2} \end{aligned} \tag{4.8}$$

subject to

$$y(0) = 1, \quad y'(0) = 0.$$

Now, we can apply our method in Eq. (4.8). We found the following results (Tables 1 and 2).

Example 8. Consider the equation

$$D^{2.2}x(t) + 1.3D^{1.5}x(t) + 2.6x(t) = \sin(2t) \tag{4.9}$$

subject to

$$x(0) = x'(0) = x''(0) = 0.$$

We show the approximate solutions in Figs. 5 and 6 for step size $h = 0.01$.

Example 9. Consider the equation

$$\begin{aligned} &aD^{\alpha}x(t) + bD^{\alpha_2}x(t) + cD^{\alpha_1}x(t) + ex(t) = f(t), \quad 0 < \alpha_1 \leqslant 1, \\ &\quad 1 < \alpha_2 \leqslant 2, \;\; 3 < \alpha < 4 \end{aligned} \tag{4.10}$$

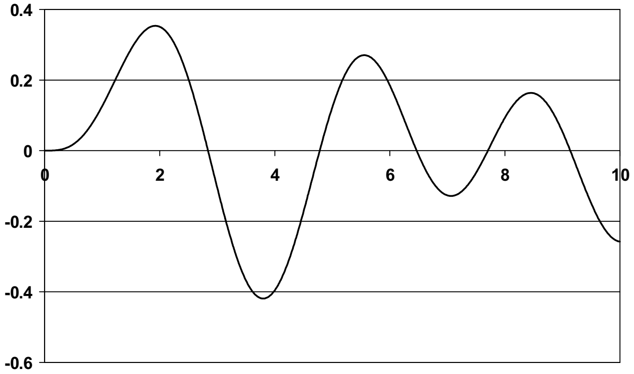

Fig. 5. By 2E.

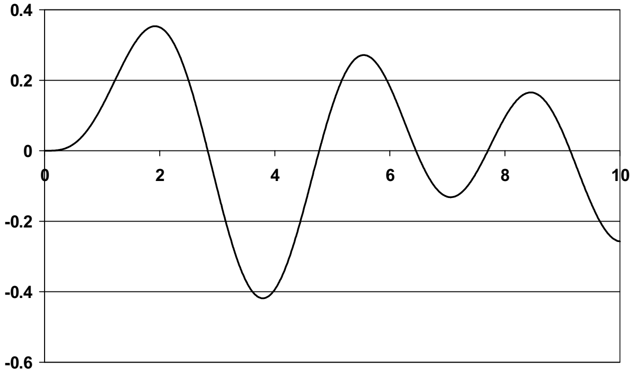

Fig. 6. By 3E.

and

$$f(t) = \frac{2b}{\Gamma(3-\alpha_2)} t^{2-\alpha_2} + \frac{2c}{\Gamma(3-\alpha_1)} t^{2-\alpha_1} - \frac{c}{\Gamma(2-\alpha_1)} t^{1-\alpha_1} + e(t^2 - t)$$

subject to

$$x(0) = 0, \quad x'(0) = -1, x''(0) = 2 \quad \text{and} \quad x'''(0) = 0.$$

It is easily verified that the exact solution of this problem is

$$x(t) = t^2 - t.$$

For $a = 1$, $b = 1$, $c = 1$, $e = 1$, $\alpha_1 = 0.77$, $\alpha_2 = 1.44$ and $\alpha = 3.91$ we found the following results.

Step size	Maximal error by ET	Maximal error by ER
0.1	0.099579330000	0.098006640000
0.01	0.009911000000	0.009503874000
0.001	0.000998064200	0.000953856500

For $a = 1$, $b = 1$, $c = 0.5$, $e = 0.5$, $\alpha_1 = \frac{\sqrt{2}}{20}$, $\alpha_2 = \sqrt{2}$ and $\alpha = \sqrt{11}$ we found the following results.

Step size	Maximal error by ET	Maximal error by ER
0.1	0.099395790000	0.099119690000
0.01	0.009884602000	0.009760696000
0.001	0.000995854100	0.000980524900

Example 10. Consider the nonlinear equation

$$aD^{\alpha}x(t) + bD^{\alpha_2}x(t) + cD^{\alpha_1}x(t) + e(x(t))^3 = f(t), \quad 0 < \alpha_1 \leqslant 1, \quad 1 < \alpha_2 \leqslant 2, \; 2 < \alpha < 3 \tag{4.11}$$

and

$$f(t) = \frac{2a}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{2b}{\Gamma(4-\alpha_2)}t^{3-\alpha_2} + \frac{2c}{\Gamma(4-\alpha_1)}t^{3-\alpha_1} + e\left(\frac{1}{3}t^3\right)^3$$

subject to

$$x(0) = x'(0) = x''(0) = 0.$$

It is easily verified that the exact solution of this problem is

$$x(t) = \frac{1}{3}t^3.$$

For $a = 1$, $b = 2$, $c = 0.5$, $e = 1$, $\alpha_1 = 0.00196$, $\alpha_2 = 1.07621$ and $\alpha = 2.55$ we found the following results.

Step size	Maximal error by 2E	Maximal error by 3E
0.1	0.135641400000	0.127290200000
0.01	0.013735230000	0.012267770000
0.001	0.001347393000	0.001200527000

For $a = 1$, $b = 0.1$, $c = 0.2$, $e = 0.3$, $\alpha_1 = \frac{\sqrt{7}}{7}$, $\alpha_2 = \frac{\sqrt{7}}{2}$ and $\alpha = \sqrt{7}$ we found the following results.

Step size	Maximal error by 2E	Maximal error by 3E
0.1	0.146481700000	0.145938400000
0.01	0.015419600000	0.015312880000
0.001	0.001508594000	0.001497775000

5. Conclusion

Method ER produces better results than ET method on the other hand methods 2E and 3E produce almost the same results.

In our method $\alpha_1, \alpha_2, \ldots, \alpha_m$ take arbitrary values such that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < n$. On the other hand a specific conditions must be satisfied on αs (see [8,9,11]). Example 7 has been solved (see [8,9,11]) by transforming the given initial value problem into a system of equations, all of which must have the same order and the dimension of the system is $d = \alpha/q$ $(q = \gcd(1, \alpha_1, \alpha_2, \ldots, \alpha_m, \alpha))$ which in most cases is very large because the order of the system q is small and it is obvious that this leads to much larger requirements concerning computer memory and run time.

In Example 7 the order of the system $q = 0.005$ and the dimension of the system $d = 400$ and after approximations $\alpha_1 = 0.55$ and $\alpha = 1.45$ the order of the system becomes 0.05 and the dimension of the system becomes 40 further after approximations $\alpha_1 = 0.5$ and $\alpha = 1.5$ the order of the system becomes 0.5 and the dimension of the system becomes 4.

This produce two sources of errors one due to the approximations of αs to reduce the dimension of the system and the other in the numerical solution of the resulting system. From Tables 1 and 2 it is obvious that our method produces less maximal error with higher step size and very small run time. For Example 7 maximal error by 2E = 0.000786614454 at step size 1/500 and takes 294.17 s run time and maximal error by 3E = 0.000824219209 at step size 1/430 and takes 106.29 s run time on the other hand [9] for step size 1/1600 maximal error = 0.0600 and run time = 5017.4 s.

In Example 4 we obtain that the maximal error is exactly zero.

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