Strategies for computing the scalar self-force on a Schwarzschild background

Steven Dorsher

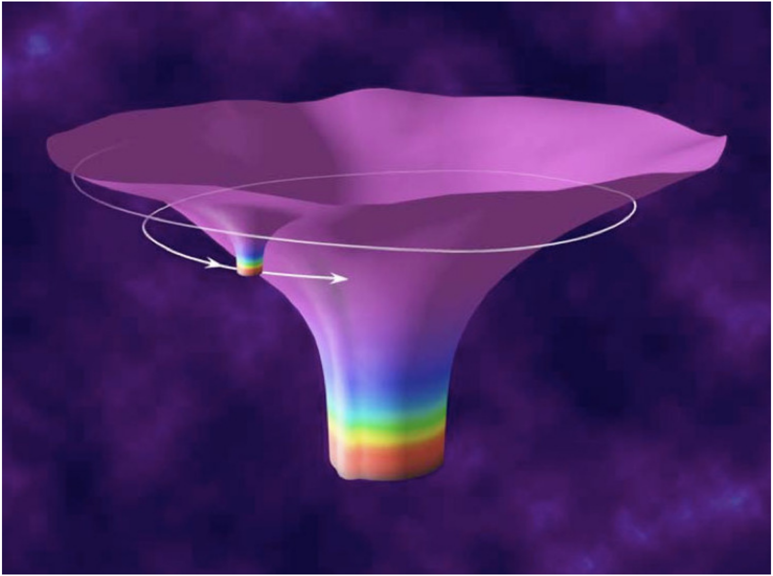
Louisiana State University

September 26, 2017

Overview

- Gravitational waves and Extreme Mass Ratio Inspirals
- The wave equation in flat spacetime
- Scalar waves on a Schwarzschild background without a source
- A scalar source on a Schwarzshild background on a circular orbit
- A scalar source on a Schwarzschild background on an eccentric orbit
- First order Richardson extrapolation
- The Discontinuous Galerkin method
- Fit to extend the mode sum to $\ell = \infty$
- Future work: a comparison of the self-consistent evolution and the geodesic evolution

Extreme Mass Ratio Inspirals



Laser Interferometer Space Antenna

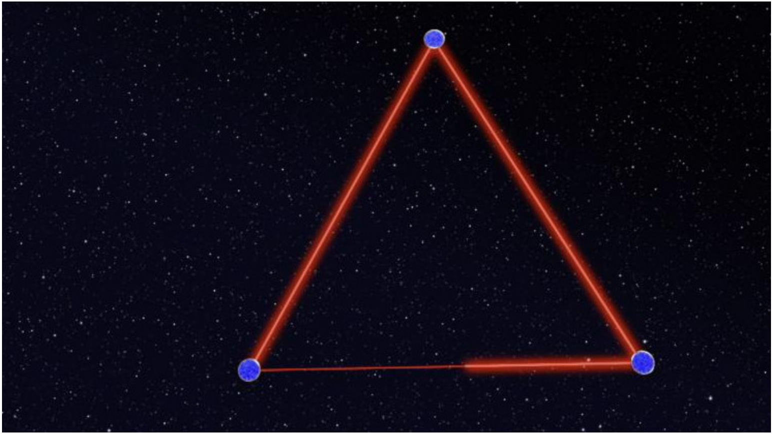


Figure: Laser Interferometer Space Antenna, which will operate around launches in early 2030's, ESA-NASA partnership, will detect EMRI's

Gravitational Waves

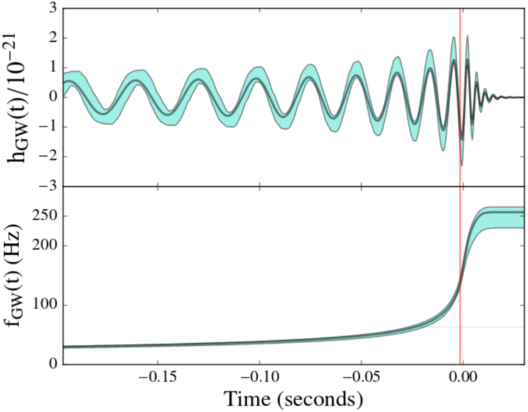


Figure: LIGO detection, September 14, 2015. General relativity was tested by comparing inpsiral with merger/ringdown phases.

Self-force

- The self-force is a particle's interaction with its own field
- Applies to scalar, electromagnetic, and tensor fields on a gravitational background
- Motion $\rightarrow$ radiation $\rightarrow$ energy and angular momentum loss $\rightarrow$ inspiral
- Electromagnetic or gravitational field: source from perturbation theory
- Scalar: delta function source

Approximations and Goals

The long term goal for the field is to generate extremely precise EMRI gravitational wave templates for LISA.
Approximations:

- Scalar rather than tensor waves (Ψ rather than $h_{\mu\nu}$)
- Non-rotating black holes: Schwarzschild spacetime
- 4D spacetime: time + radius + spherical harmonics
- Discontinuous Galerkin spatial grid
- Self-force causes a particle to inspiral as it emits radiation
- We use the Detweiler-Whiting effective source as implemented by Barry Wardell
- Niels Warburton assumes: the particle has been on the same geodesic for all time when he calculates the self-force

Our goal is to implement a highly accurate self-consistent evolution and do a comparison study with Niels Warburton.

The Discontinuous Galerkin (DG) method

- Method for solving ODEs dependent on space and time.
- Break space into evenly spaced elements.
- Within each element there are N unevenly spaced nodes connected by $N+1$ interpolating polynomials.
- Allows discontinuities at element boundaries– numerical fluxes
- The DG method returns a derivative matrix that takes a derivative across an element and a lift matrix that accounts for numerical fluxes,
- Beneficial because it has truncation error that decreases exponentially with DG order N and because it naturally handles discontinuities in ϕ and ρ.

A simple test of the DG method: a wave equation in flat spacetime

For no source, the D'Alembertial equals zero:

$$\Box\Psi = 0 \tag{1}$$

In 1-dimension:

$$\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial x^2} = 0 \tag{2}$$

Rewrite as ODE

$$\begin{aligned} \partial_t\psi &= \rho \\ \partial_t\rho &= \partial_r\phi \\ \partial_t\phi &= \partial_r\rho \end{aligned} \tag{3}$$

Refer to $u = (\psi, \rho, \phi)$ as the state vector.

Flat space evolution

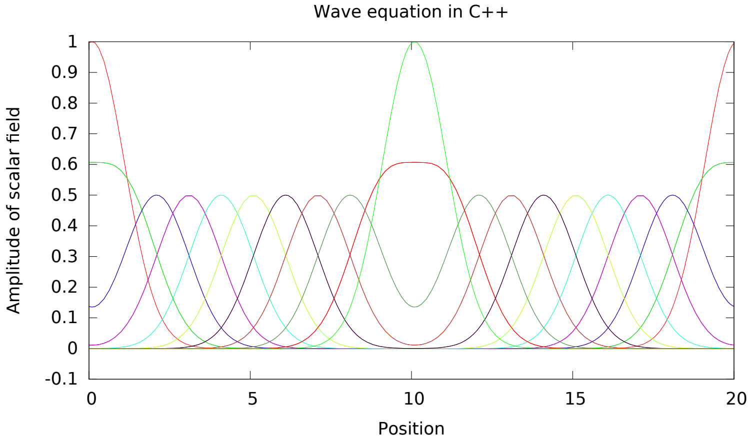


Figure: Gaussian initial conditions, flat spacetime

Flat spacetime error convergence

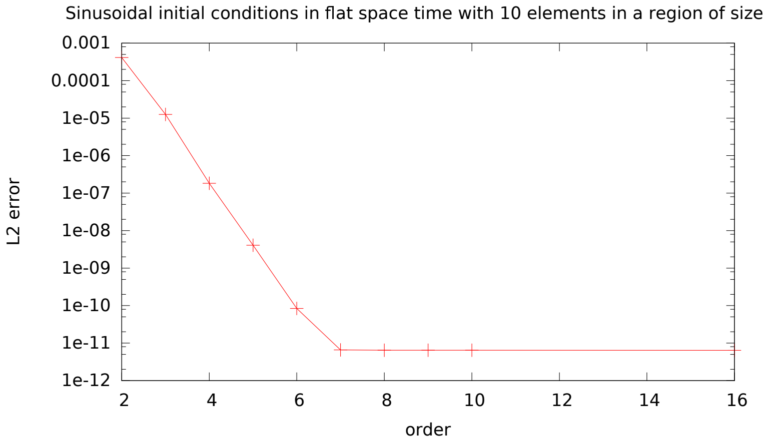
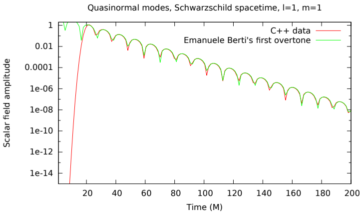


Figure: L_2 error converges exponentially until it hits roundoff noise with DG order for sinusiodal initial conditions with ten elements of size $h = 0.01$.

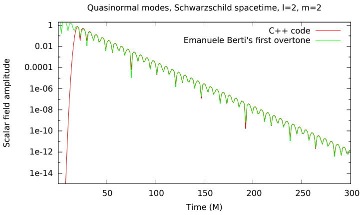
Schwarzschild spacetime without a source

- Wave equation: $\Box\Psi = \partial_\mu \left(\frac{1}{\sqrt{-g}} g^{\mu\nu} \partial_\nu \Psi \right) = 0$
- Tortoise coordinates (Eddington-Fingkelstein coordinates) in interior
- Hyperboloidal coordinates near horizon and at infinity to bring lightlike infinity, $\mathcal{I}^+$, to a finite coordinate
- Multipole moment decomposition to account for angular dependence
- Expect quasinormal mode ringing with higher frequencies for higher l followed by power law tails that go as $t^{-(2l+3)}$
- QNM ringing: interactions near peak of potential
- Tails: scattering off spacetime far from peak of potential

Quasinormal modes



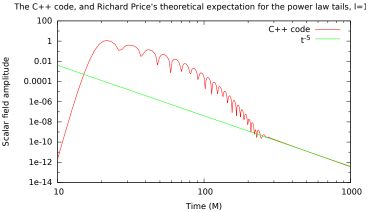
(a) $l = 1$



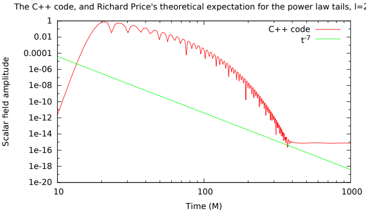
(b) $l = 2$

Figure: $l = 2$ has a higher frequency and a faster decay rate than $l = 1$

Power law tails



(a) $l = 1$



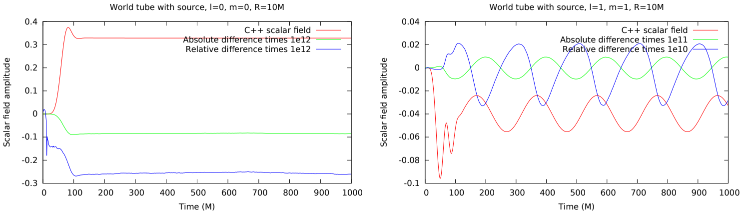
(b) $l = 2$

Figure: $l = 1$ decays as $t^{-}-5$ as expected; however, $l = 2$ has no power law tail due to truncation error

Scalar charge on a circular orbit with an effective source

- Self-force causes the particle to emit radiation and inspiral. An artifical force holds it on a circular orbit.
- Wave equation for retarded field has delta function source proportional to charge
- Delta function source is singular
- Regularize field: $\Psi^R = \Psi^{ret} - \Psi^S$
- Perturbative expansion to Detweiler-Whiting singular field in terms of powers of radius around singular field
- Detweiler-Whiting singular field: *Steven Detweiler, Bernard F. Whiting (2002). Phys. Rev. D 67, 024025*
- Perturbative expansion: *Anna Heffernan, Adrian Ottewill, Barry Wardell (2012). Phys. Rev. D 86, 104023*
- World tube window function
- I have ported this from Fortran to C++ with partially redesigned structure. I have succeeded in reproducing the results to roundoff error precision.

Circular orbit roundoff error comparison between languages



(a) l=0 (b) l=1

Figure: Relative and absolute errors are at the roundoff level– 10^{-10} to 10^{-12}. Oscillations do not appear in the $l = 0$ mode but appear with the orbital period in the $l = 1$ mode.

Eccentric orbits using Peter Diener's simulation

- Radial oscillations and angular oscillations have different periods
- The orbit precesses
- The orbit is artifically held on a geodesic to counteract the self-force generating the scalar waves
- Oribital parameters can be entirely expressed in terms of p, e, ϕ, χ, and t.
- $r_{periastron} = \frac{pM}{1+e}$, $r_{apastron} = \frac{pM}{1-e}$
- A time dependent transformation keeps the particle's coordinate fixed at a Discontinuous Galerkin element boundary
- Radial self-force: $F_r = q\partial_r \Psi^R$
- Use Niels Warburton's frequency domain initial conditions for $l = 0$ through $l = 6$. No transients expected in these modes.
- Warburton initial conditions: particle has been on the same geodesic for all time
- Diener initial conditions: field starts at zero with particle at aphelion

Precession of the eccentric orbit

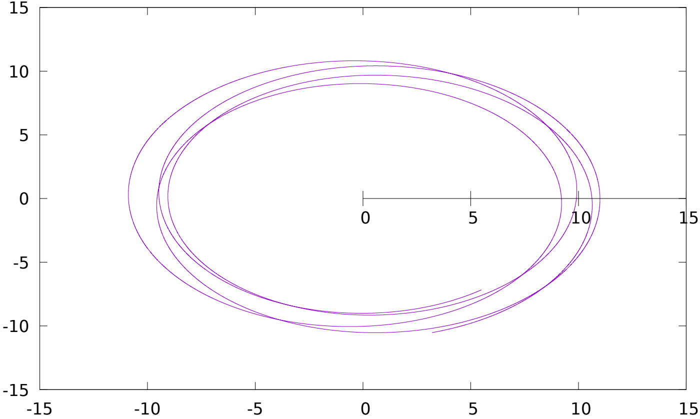


Figure: Precession of the eccentric orbit. $p = 9.9$, $e = 0.0$.

Evolution of the radial self-force for different initial conditions

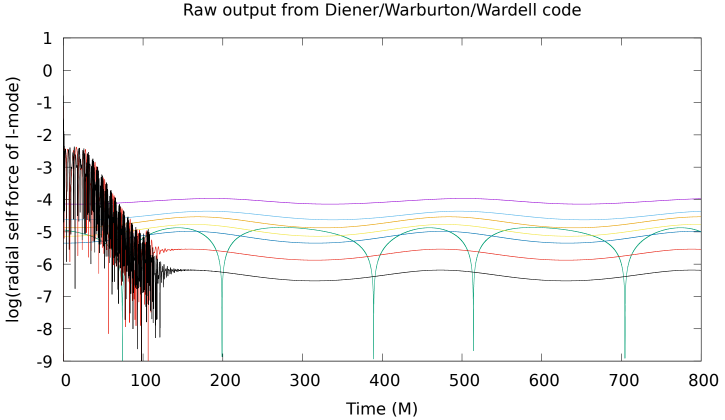


Figure: Modes zero through seven are shown. Zero through five have smooth starts. Six and seven have transients.

Addressing instabilities and error

- To address instabilities in accelerated evolutions, I ran eccentric orbits with accelerations of zero.
- Analyzed error due to:
  - Neglecting the first order Richardson extrapolation
  - Choice of start and end modes in mode sum fit
  - Choice of number of terms in mode sum fit
  - Selection of weights associated with mode sum fit
- Concluded the first three errors were comparable and at the 10^{-4} level

The first order Richardson extrapolation

- Truncation error is skewed entirely to one side when measured using L_0 or L_2 error
- Convergent code approaches an asymptote
- Assume $F_r(n, l) = F_{inf}(l) + c(l)\exp(-\alpha n)$
- Use three different DG orders $n = n_1$, $n_1 + 4$, and $n_1 + 8$ to obtain an extrapolation to F_{inf} from starting order n_1
- Not all starting orders have solutions because some fail to follow the expected exponential behavior due to roundoff noise– get NaNs

Well-converging data

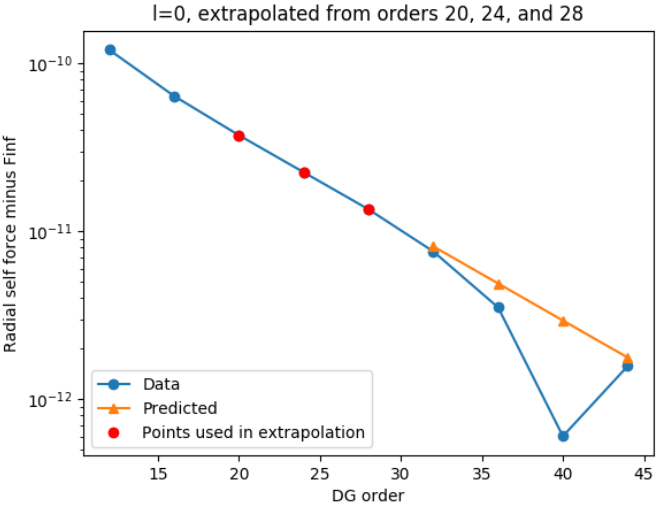


Figure: l=0 at t=370. This data converges very cleanly until it hits roundoff noise at high DG orders.

Asymptote method for finding the best starting order

- Might expect second order truncation error to also follow a convergent form on theoretical grounds but data is sparsely populated due to NaN modes
- When there are four or five data points, concave. Use first and second derivative test.
- When there are fewer, average, then veto one sigma outliers. Average again. Take point closest to second average as best choice.

Error due to neglecting the first order Richardson Extrapolation

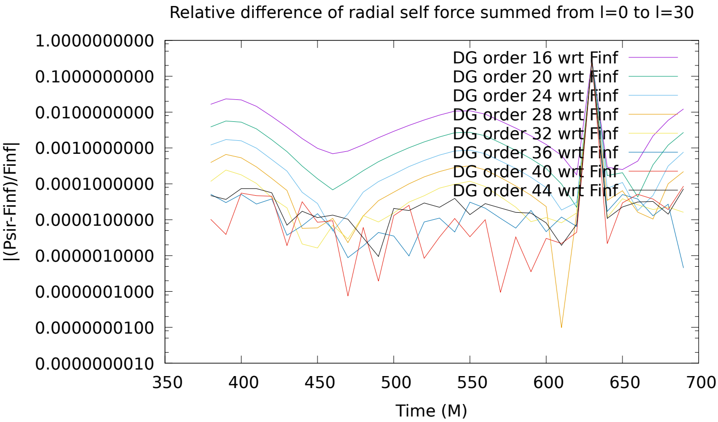


Figure: Relative error between DG starting orders and F_{inf} vs time. For DG order 36 where roundoff error sets in, the error is about 10^{-4}.

The l-mode sum and fit

$$F_r(l,t) = \frac{A(t)}{(2l-1)(21+3)} + \frac{B(t)}{(2l-3)(2l-1)(2l+3)(2l+5)} + \frac{C(t)}{(2l-5)(2l-3)(2l-1)(2l+3)(2l+5)(2l+7)} + \ldots \quad (4)$$

Barry Wardell, Ian Vega, Jonathan Thornburg, Peter Diener (2012). Phys. Rev D. 85, 104044

- Fit from l_{min} to l_{max}.
- Sum numerically from zero to l_{max} then use fit coefficients to analytically sum to $l = \infty$
- Least squares fit: minimize χ^2

$$\chi^2 = \sum_i \frac{(y_i - f(x_i))^2}{\sigma_i^2} \quad (5)$$

l-mode fit with three weight models

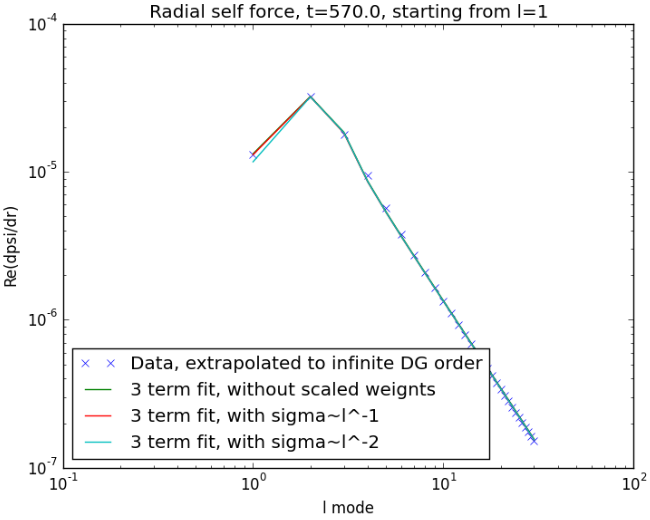
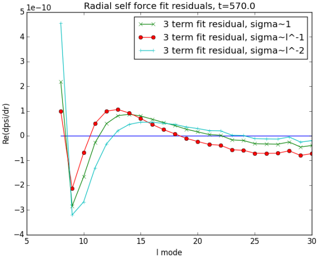
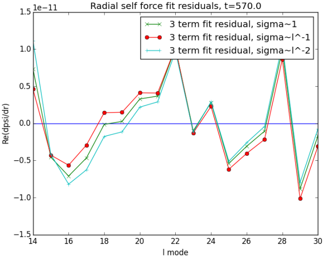


Figure: l-mode versus F_{inf}.

Residuals to the l-mode fit



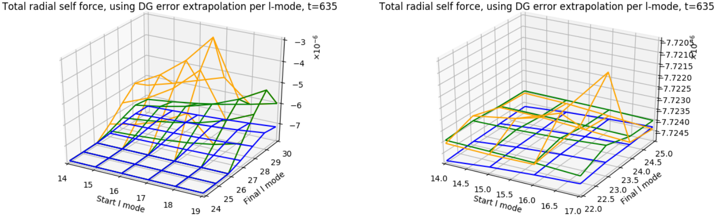
(a) $l_{min} = 8$



(b) $l_{min} = 14$

Figure: $l_{min} = 14$ is a better fit than $l_{min} = 8$ both because it is less systematically biased and because it has an amplitude an order of magnitude smaller. Both fits end at $l_{max} = 30$.

Roundoff noise in F_{inf} at high l_{max}



(a) Large range (b) Small range

Figure: $l_{min} = 14$ and $l_{max} = 25$ appear to be good start and stop values. Roundoff noise is evident at higher l.

Smooth evolution of total radial self-force

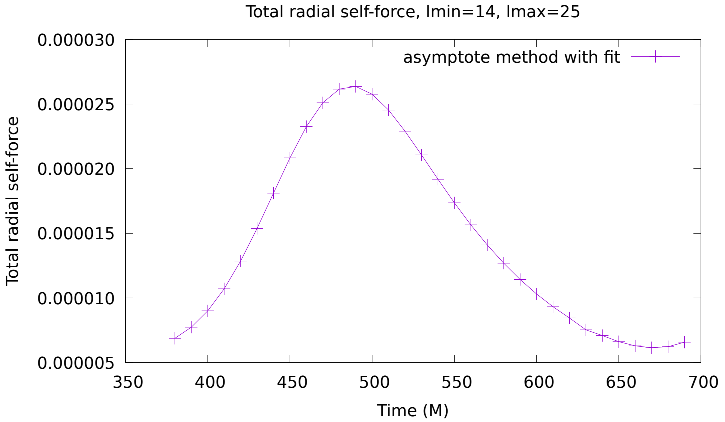
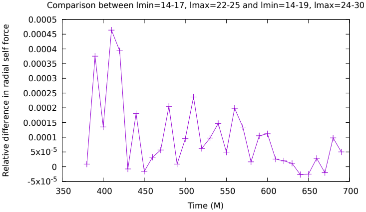
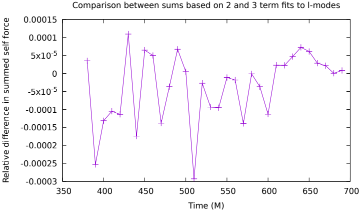


Figure: Total radial self-force including sum to $l = \infty$ over time

Error due to l-mode selection, number of terms



(a) large versus small range



(b) 2 versus 3 terms

Figure: Relative errors in both of these effects appear to be at the 10^{-4} level.

Comparison study of self-consistent evolution to geodesic evolution

- Self-consistent evolution accounts for the interaction of the particle with the field it has generated in the past naturally since it is evolved in the time domain
  - Uses the Detweiler-Whiting singular field
  - The particle position evolves according to the geodesic equation with an acceleration on the right hand side
  - The mass of the particle also evolves according to the work being done on it
- Geodesic evolution uses an self-force that assumes the particle has been evolving on the same geodesic for all time
  - Self-force can be efficiently calculated in the frequency domain due to periodicity
  - Frequency domain component is good for initial conditions without transients
  - Evolves in the time domain after generating the self-force in the frequency domain
  - Cannot handle effects where the timescale of the orbital evolution is short compared to the period

Long term goals

- Goal is to compare Diener's self-consistent code using Warburton's initial conditions and Wardell's effective source to Warburton's geodesic evolutions
- I will run code, analyze physics, and debug, as necessary.
- Timescale: 2-3 years