# Numerical methods for Extreme Mass Ratio Inspirals of Black Hole Binaries

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# Toward LISA EMRI templates

- Generating LISA EMRI templates requires evolving $10^6$ orbits with precision on the order of $\delta P/P \sim 10^{-6}$ *Danzmann, Karsten. LISA: A proposal in response to the ESA call for L3 mission concepts (2017)*
- For EMRI's, the self-force approximation is used in the limit where the mass ratio is large ($10^4$ to $10^6$)
- EMRI's have an orbital evolution timescale that scales as $M/\mu$, and a period that scales as $M$. These two widely different timescales necessitate a different numerical approach than numerical relativity.
- Self-force prescribes a perturbative approximation in the mass ratio.

# Self-force in general relativity

- In general relativity, test particles move along geodesics
- A compact object is not a test particle
- Motion $\rightarrow$ radiation $\rightarrow$ energy and angular momentum loss $\rightarrow$ inspiral
- Applies to scalar, electromagnetic, and tensor fields on a gravitational background
- Perturbative expansion in powers of the mass ratio

# Goals for the self-force community and this project

The long term goal for the field is to generate extremely precise EMRI gravitational wave templates for LISA.

- Scalar rather than tensor waves ($\Psi$ rather than $h_{\mu\nu}$)
- Non-rotating black holes: Schwarzschild spacetime

The state of the art is

- Kerr eccentric inclined geodesics for scalar fields
- Kerr geodesics for gravitational fields with nearly extremal spin
- Adiabatic evolution in Schwarzschild or Kerr spacetime in the orbital plane
- Scalar unbound orbits in Schwarzschild spacetime
- Scalar Schwarzschild self-consistent evolution to low accuracay

# Intermediate goals

1. To port a sample self-force simulation provided by Peter Diener from Fortran to C++ and parallelize it in HPX for the purpose of helping to develop the HPX parallelization system and numerical methods in the self-force field.
2. To assess the accuracy of Peter Diener's existing Discontinuous Galerkin eccentric orbit simulation.

# A generic wave equation solver

Wave equation:

$$\Box\Psi = RHS(r, t) \tag{1}$$

Can be written in state-vector form:

$$\frac{\partial u}{\partial t} = A\frac{\partial u}{\partial r} + Bu = RHS(u, t) \tag{2}$$

Use Discontinuous Galerkin method: truncation error scales as $h^{N+1}$

# Flat spacetime evolution

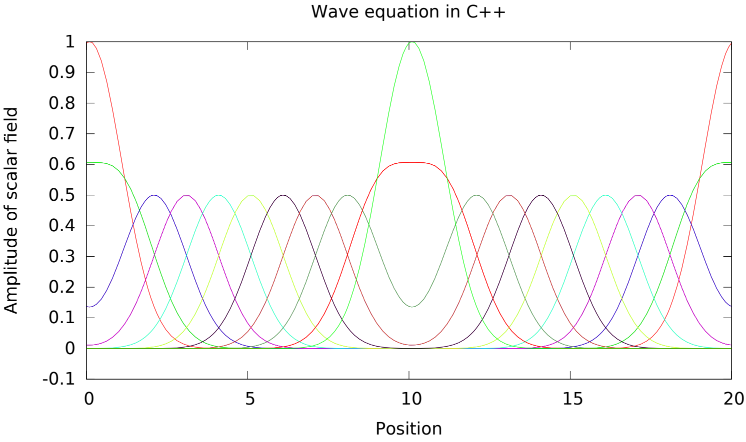



Figure: Gaussian initial conditions, flat spacetime

# Flat spacetime error convergence

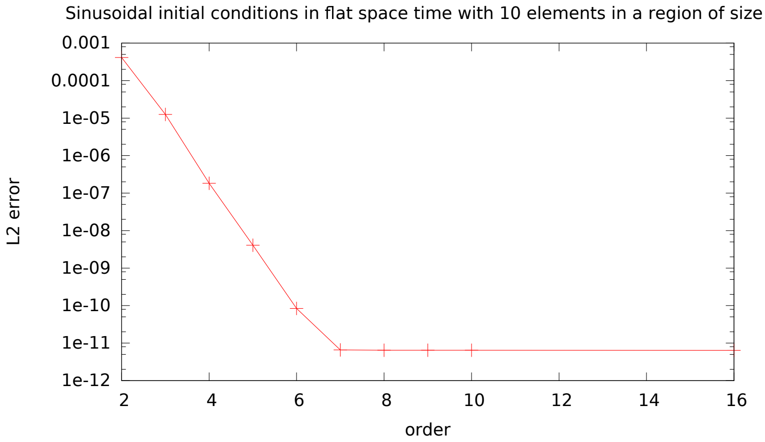



Figure: $L_2$ error converges exponentially until it hits roundoff noise with DG order for sinusiodal initial conditions with ten elements of size $h = 0.01$.

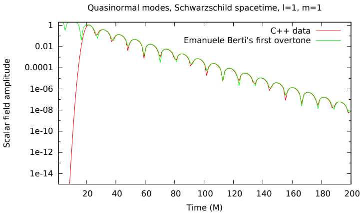
# Schwarzschild spacetime without a source

- Wave equation: $\Box\Psi = \frac{1}{\sqrt{-g}}\partial_\mu \left(g^{\mu\nu}(\partial_\nu\Psi)\sqrt{-g}\right) = 0$
- Multipole moment decomposition to account for angular dependence

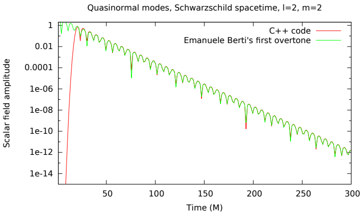
# Quasinormal mode (QNM) ringing

- Similar to black hole ringdown phase after a merger
- Due to interactions of the scalar field with the background at the peak of the potential
- Higher frequencies and faster decay for higher $l$

# Quasinormal modes




(a) $l = 1$



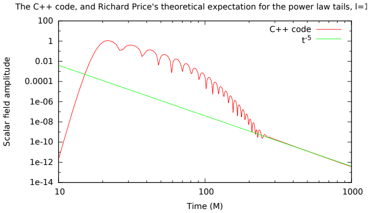

(b) $l = 2$

Figure: $l = 2$ has a higher frequency and a faster decay rate than $l = 1$

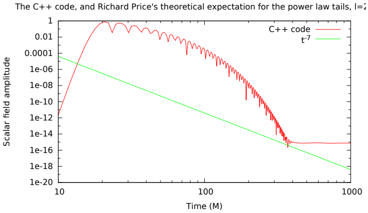
# Power law tails

- Due to scattering off spacetime far from the central black hole
- Follow the QNM
- Go as $t^{-(2l+3)}$

# Power law tails




(a) $l = 1$



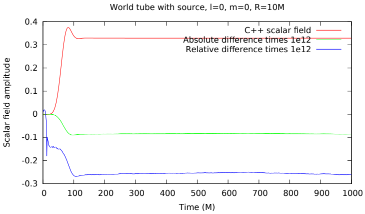

(b) $l = 2$

Figure: $l = 1$ decays as $t^{-}-5$ as expected; however, $l = 2$ has no power law tail due to truncation error

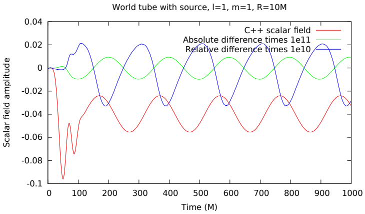
# A source on a geodesic

- $\Box\Psi^{ret} = -4\pi q \int \delta_4(x, z(\tau'))d\tau'$
- Detweiler-Whiting singular field: *Steven Detweiler, Bernard F. Whiting (2002). Phys. Rev. D 67, 024025*
- Source is siingular at location of particle
- Regularize field: $\Psi^R = \Psi^{ret} - \Psi^S$
- Only know approximation to DW singular field, $\tilde{\Psi}^S$
- Limit the effect of the approximation using a world tube window function
- $\Box\Psi^R = S_{eff} = \Box\Psi^{ret} - \Box(W\tilde{\Psi}^S)$
- HOW effective source: *Anna Heffernan, Adrian Ottewill, Barry Wardell (2013). Phys. Rev. D 82 104023*

# Circular orbit roundoff error comparison between languages




(a) l=0




(b) l=1

Figure: Relative and absolute errors are at the roundoff level– $10^{-10}$ to $10^{-12}$. Oscillations do not appear in the $l = 0$ mode but appear with the orbital period in the $l = 1$ mode.

# Eccentric orbits using Peter Diener's simulation

- $\chi$: radial evolution angle, $\phi$: angular evolution angle $\rightarrow$ precession because they are out of phase
- The orbit is held fixed on a geodesic to counteract the self-force generating the scalar waves
- $r_{periastron} = \frac{pM}{1+e}$, $r_{apastron} = \frac{pM}{1-e}$
- Radial self-force: $F_r = q\partial_r \Psi^R$

# Precession of the eccentric orbit

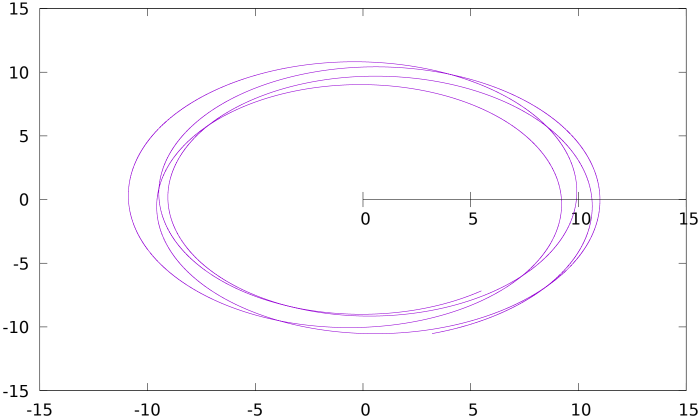



Figure: Precession of the eccentric orbit. $p = 9.9$, $e = 0.0$.

# Evolution of the radial self-force for different initial conditions

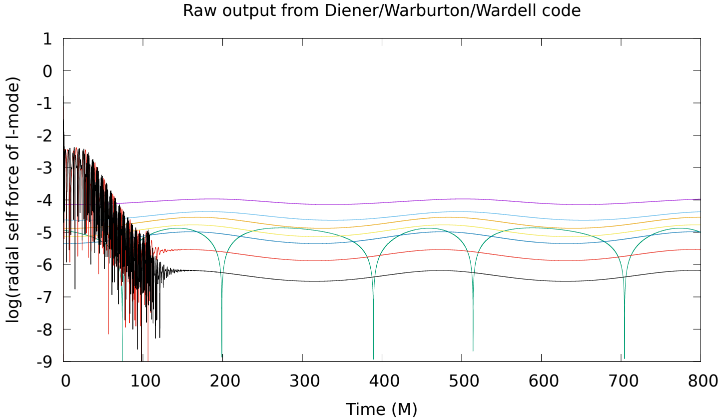



Figure: Low $l$ behavior of the self-force for a given l-mode summed over $m$

# Self-force spherical harmonic components depend on time

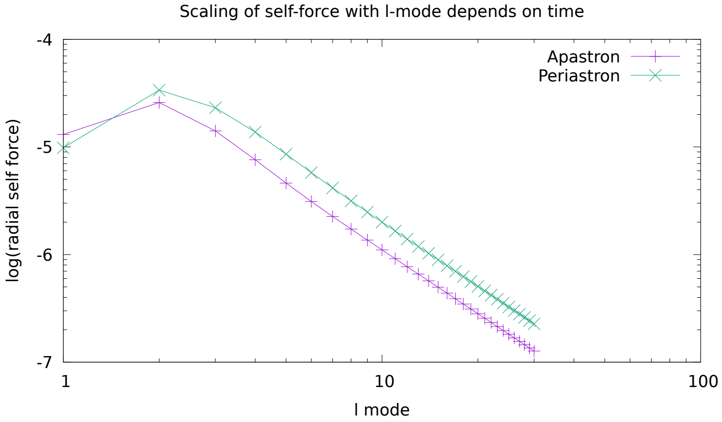



Figure: Periastron has a higher self-force amplitude for nearly every spherical harmonic than apastron.

# The first order Richardson extrapolation

- Discontinuous Galerkin: ODE solver with truncation errors that scale as $h^{n+1}$
- Assume $F_r(n,l) = F_{inf}(l) + c(l)\exp(-\alpha n)$
- Obtain extrapolation by solving system of equations for $F_r(n_i, l)$, $i = 1, 2, 3$

# Well-converging data

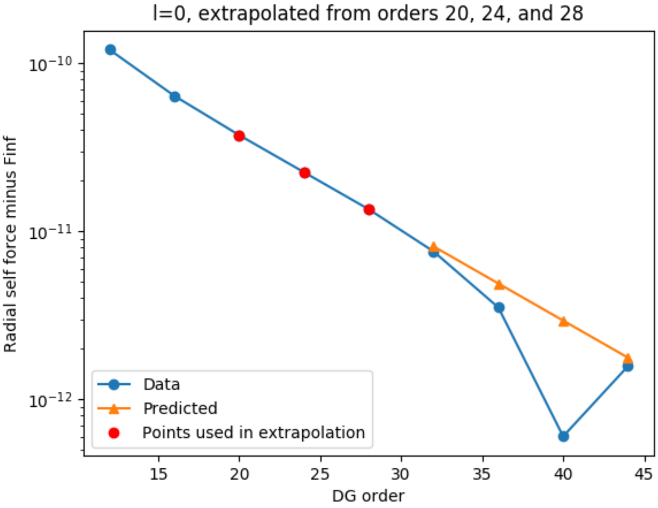



Figure: l=0 at t=370. This data converges very cleanly until it hits roundoff noise at high DG orders.

# Error due to neglecting the first order Richardson extrapolation

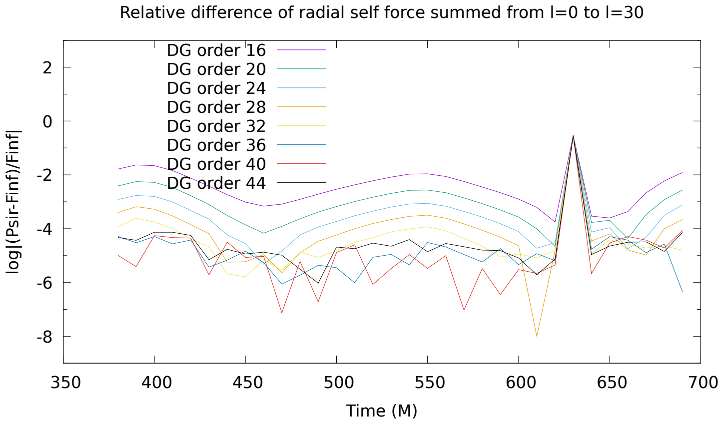



Figure: Relative error between DG starting orders and $F_{inf}$ vs time. For DG order 36 where roundoff error sets in, the error is about $10^{-4}$.

# The l-mode sum and fit

- Use spherical harmonic decomposition of $\Psi$ to encode angular information
- must sum over $l$ and $m$ at end to obtain self force
- Use fit to extend sum to $l = \infty$

$$F_r(l,t) = \frac{A(t)}{(2l-1)(21+3)} + \frac{B(t)}{(2l-3)(2l-1)(2l+3)(2l+5)} + \frac{C(t)}{(2l-5)(2l-3)(2l-1)(2l+3)(2l+5)(2l+7)} + \ldots \quad (3)$$

*Anna Heffernan, Adrian Ottewill, Barry Wardell (2012). Phys. Rev. D 86, 104023*

- Fit from $l_{min}$ to $l_{max}$.
- Sum numerically from zero to $l_{max}$ then use fit coefficients to analytically sum to $l = \infty$

# l-mode fit

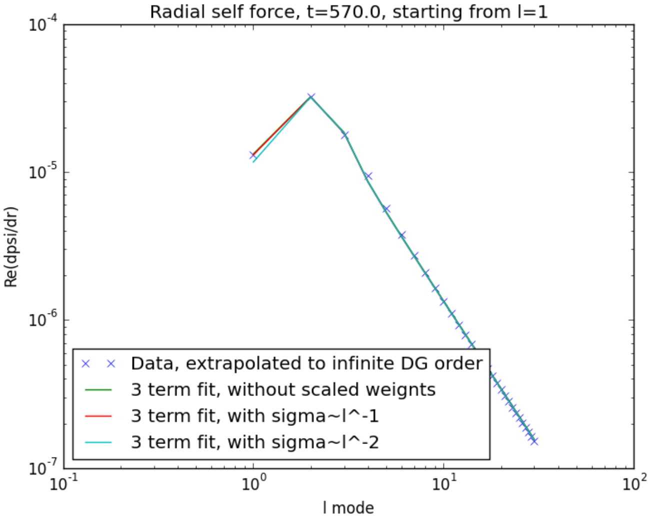
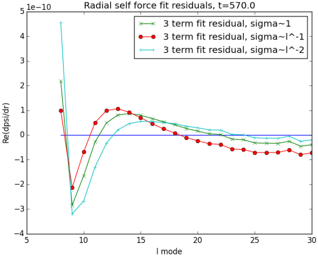
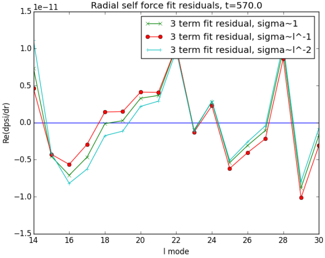



Figure: l-mode versus $F_{inf}$.

# Residuals to the l-mode fit



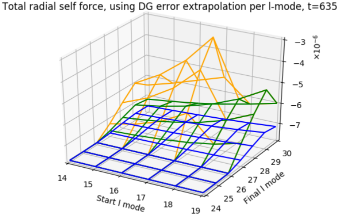

(a) $l_{min} = 8$



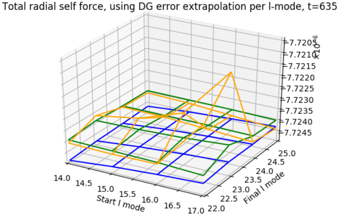

(b) $l_{min} = 14$

Figure: $l_{min} = 14$ is a better fit than $l_{min} = 8$ both because it is less systematically biased and because it has an amplitude an order of magnitude smaller. Both fits end at $l_{max} = 30$.

# Roundoff noise in $F_{inf}$ at high $l_{max}$



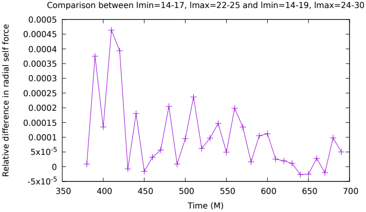

(a) Large range



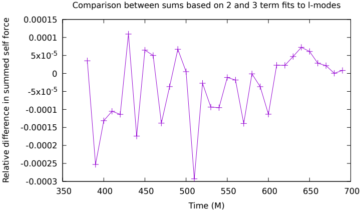

(b) Small range

Figure: $l_{min} = 14$ and $l_{max} = 25$ appear to be good start and stop values. Roundoff noise is evident at higher l.

# Error due to l-mode selection, number of terms




(a) large versus small range




(b) 2 versus 3 terms

Figure: Relative errors in both of these effects appear to be at the $10^{-4}$ level.

# Smooth evolution of total radial self-force

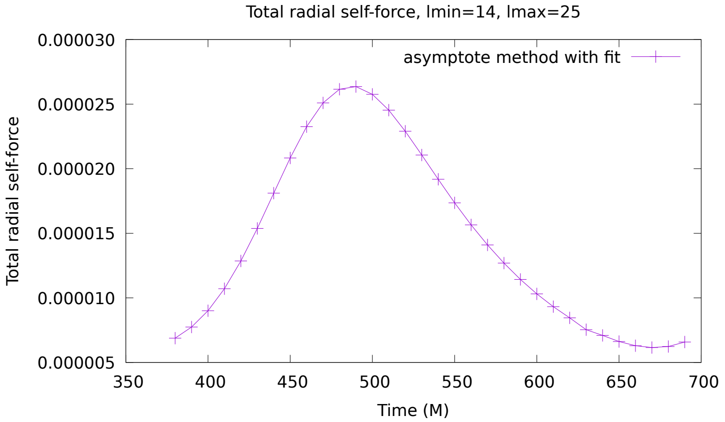



Figure: Total radial self-force including sum to $l = \infty$ over time

# Error analysis conclusions

- The relative error due to neglecting the first order Richardson extrapolation is $10^{-4}$. The best order at which to run is DG order 36. Limit dominated by roundoff error.
- The best $l_{min} = 14$ and $l_{max}$=25. The relative error in these choice of values is $10^{-4}$.
- The relative error due to the number of terms used in the fit is $10^{-4}$.
- The error due to the use of weights in fitting is insignificant.
- These results are preliminary and need further investigation.