Row Echelon Form and Gaussian Elimination

Thm from last time: If M and N are row-equivalent, then $\mathcal{LS}(M)$ and $\mathcal{LS}(N)$ have same sol'ns

Goal: Given M find row equiv N where $\mathcal{LS}(N)$ is easy to solve

RREF: M where
1. all rows of 0's at bottom.
2. The left most nonzero entry is a 1 (leading 1)
3. A leading one is the only nonzero entry in its col
4. leading ones move down and to the right (if $i > s$, then $j > t$ for (i,j), (s,t))

$$\begin{pmatrix} \boxed{1} & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & \boxed{1} & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{pmatrix}$$

$(s,t) = (2,3)$

Ex:

Thm: Every Matrix M is row-equivalent to one in RREF

proof by algorithm

Gauss-Jordan Elimination: M $m \times n$ matrix

Denote rows R_i and cols C_j

0. Set $r = 0$ and $j = 0$
1. If $j \geq n$, stop and return current matrix, otherwise $j += 1$
2. If C_j is 0 below row r, go to 1
3. Set $r += 1$ and arrange that the j_{th} entry of R_r is nonzero by swapping with one of $R_{r+1}, \dots, R_m$ if needed. (ok because 2)
4. Scale R_r so jth entry is 1
5. For i in $1, \dots, m$ with $i \neq r$, replace R_i with $-cR_r + R_i$, where $c =$ jth entry of R_r
6. Go to 1

Ex:

$$\begin{pmatrix} 0 & 4 & 6 & 8 \\ 2 & 0 & -2 & 4 \\ -3 & 0 & 3 & 5 \end{pmatrix}$$

$r = 1$ $j = 1$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 2 & 0 & -2 & 4 \\ 0 & 4 & 6 & 8 \\ -3 & 0 & 3 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 4 & 6 & 8 \\ -3 & 0 & 3 & 5 \end{pmatrix}$$

c

$$\xrightarrow{3R_1 + R_3} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 4 & 6 & 8 \\ 0 & 0 & 0 & 11 \end{pmatrix}$$

$-c$

$r = 2$ $j = 2$

$$\xrightarrow{\frac{1}{4}R_2} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 11 \end{pmatrix}$$

$r = 3$

$j = 3$ increase j

$j = 4$ increase r

$$\xrightarrow{\frac{1}{11}R_3} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[-2R_3+R_2]{-2R_3+R_1} \begin{pmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix} \longleftarrow \text{output (In RREF)}$$

Proof of thm: (ie that the algorithm works)

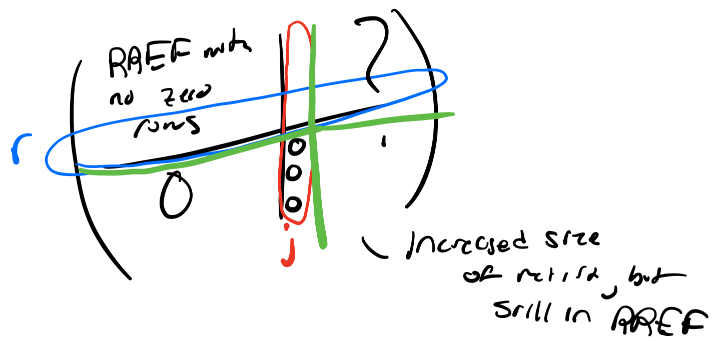
The algorithm terminates since reach ① at most n times

Will show output is RREF by induction

Claim: When we arrive at ① the matrix has the following form

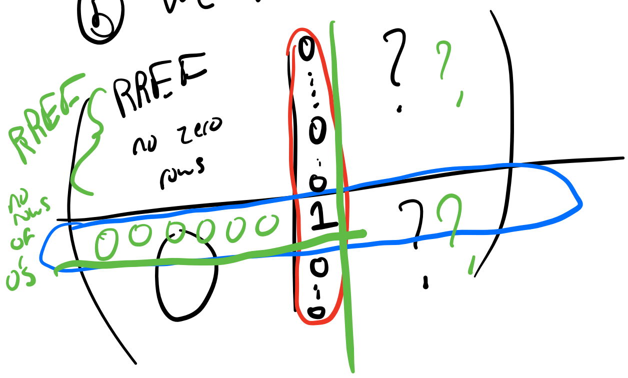
Base Case: Here $r=j=0$, so holds vacuously

Inductive Step: Assume claim holds at some visit to ①, prove it holds at the next visit



Two Cases: ⓐ We return via ②

ⓑ We return via ⑤



So by Induction, when we exit the algorithm with $n=j$, have

which is itself RREF,

So output is in RREF