Complex Analysis Notes

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§1 History of Complex Numbers (Week 1)

Consider a quadratic equation

$$x^2 = mx + b.$$

Solutions to this equation are

$$x = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} + b}$$

and represent the intersection of $y = x^2$ and $y = mx + b$. But now what if $\frac{m^2}{4} + b < 0$? In particular, $x^2 = -1$ has no real solutions. It is often argued that this led to $i = \sqrt{-1}$. But historically, no interest in non-real solutions since the graphs of $y = x^2$ simply don't intersect in that case.

Cubic equations were the real reason. Consider the following equation $x^3 = px + q$. This represents the intersection of $y = x^3$ and $y = px + q$. There must always be a solution to this type of equation.

Del Ferro $(1465 - 1526)$ and Tartaglia $(1499 - 1577)$, followed by Cardano $(1501 - 1576)$, showed that the equation

$$x^3 = px + q$$

has a solution given by

$$x = \sqrt[3]{\sqrt{\frac{q^2}{4} - \frac{p^3}{27}} + \frac{q}{2}} - \sqrt[3]{\sqrt{\frac{q^2}{4} - \frac{p^3}{27}} - \frac{q}{2}}.$$

About 30 years after the discovery of this formula, Bombelli $(1526 - 1572)$ considered the equation

$$x^3 = 15x + 4.$$

Plugging in $p = 15$ and $q = 4$ into the formula yields

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Bombelli discovered that

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}, \sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1}.$$

Bombelli's discovery is considered the "Birth of Complex Analysis". It showed that perfectly real problems require complex arithmetic for their solution.

§2 Algebra and Geometry in the Complex Plane

Complex numbers are expressions of the form $z = x + iy$ where x is called the real part of z and denoted by $x = \text{Re}(z)$ and y is called the imaginary part of z and denoted by $y = \text{Im}(z)$.

The set of complex numbers can be denoted by $\mathbb{C}$. The set of real numbers $(\mathbb{R})$ is a subset of the complex numbers. The complex plane can be identified with $\mathbb{R}^{\not\models}$.

Definition — $\underbrace{(x + iy)}_{\text{z}} + \underbrace{(u + iv)}_{\text{w}} = \underbrace{(x + u)}_{\text{Re(z+w)}} + \underbrace{i(y + v)}_{\text{Im(z+w)}}$

Thus

$$\text{Re}(z + w) = \text{Re}(z) + \text{Re}(w), \text{Im}(z + w) = \text{Im}(z) + \text{Im}(w).$$

Graphically, this corresponds to vector addition.

Definition — The *modulus* of the complex number $z = x + iy$ is the length of the vector z :

$$|z| = \sqrt{x^2 + y^2}.$$

Motiation: $(x + iy) \cdot (u + iv) = xu + ixv + iyu + i^2yv$. This motivates the following definition.

Definition — $(x + iy) \cdot (u + iv) = (xu - yv) + i(xv + yu) \in \mathbb{C}$

Example — Evaluate $(3 + 4i)(-1 + 7i)$.

$$(3 + 4i)(-1 + 7i) = (-3 - 28) + i(21 - 4) = -31 + 17i.$$

The following properties also hold:

$$(z_1z_2)z_3 = z_1(z_2z_3)$$
$$z_1z_2 = z_2z_1$$
$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

Suppose that $z = x + iy$ and $w = u + iv$. What is $\frac{z}{w}$ assuming that $w \neq 0$?

$$\frac{z}{w} = \frac{x + iy}{u + iv}$$
$$= \frac{(x + iy)(u - iv)}{(u + iv)(u - iv)}$$
$$= \frac{(xu + yv) + i(-xv + yu)}{u^2 + v^2 + i(-uv + vu)}$$
$$\frac{xu + yv}{u^2 + v^2} + i\frac{yu - xv}{u^2 + v^2}$$

In particular:

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}, \text{as long as} z \neq 0.$$

Note the importance of the quality $x - iy$ in the previous calculation.

Definition — If $z = x + iy$ then $\overline{z} = x - iy$ is defined as the complex conjugate of z.

Some properties of the conjugate:
• $\overline{\overline{z}} = z$
• $\overline{z + w} = \overline{z} + \overline{w}$
• $|z| = |\overline{z}|$
• $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$
• $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}$
• $z + \overline{z} = (x + iy)(x - iy) = 2x, so$

$$\text{Re}(z) = \frac{z + \overline{z}}{2}, \text{ similarly, } \text{Im}(z) = \frac{z - \overline{z}}{2}$$

• $|z \cdot w| = |z| \cdot |w|$
• $\overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}}$, for $w \neq 0$
• $|z| = 0$ if and only if $z = 0$
• $-|z| \leq \text{Re}(z) \leq |z|$
• $-|z| \leq \text{Im}(z) \leq |z|$
• $|z + w| \leq |z| + |w|$ (triangle inequality)
• $|z - w| \geq |z| - |w|$ (reverse triangle inequality)

Theorem

If $a_0, a_1, ..., a_n$ are complex numbers with $a_n \neq 0$, then the polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

has n roots $z_1, z_2, ..., z_n \in \mathbb{C}$. It can be factored as

$$p(z) = a_n (z - z_1)(z - z_2)...(z - z_n).$$

We will be able to prove this theorem in later chapters. Consider the polynomial $p(x) = x^2 + 1$ in $\mathbb{R}$. It has no real roots. But in $\mathbb{C}$ it can be factored as such: $z^2 + 1 = (z + i)(z - 1)$.

§3 Polar Representation of Complex Numbers

Consider $z = x + iy \in \mathbb{C}$, such that $z \neq 0$. Note that z can also be described by the distance r from the origin ($r = |z|$) and the angle θ between the positive x axis and the line segment from 0 to z. We denote (r, θ) as the polar coordinates of z. The relation between Cartesian and polar coordinates are that $x = r\cos\theta$ and $y = r\sin\theta$. This implies that $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$. This is called the polar representation of z.

Definition — The principal argument of z, called $\text{Arg}(z)$, is the value of θ for which $-\pi < \theta \leq \pi$.

We define $\arg(z) = \{\text{Arg}(z) + 2\pi k : k = 0, \pm 1, \pm 2, ...\}$, for $z \neq 0$.

Some examples are that

$$\text{Arg}(i) = \frac{\pi}{2}$$
$$\text{Arg}(1) = 0$$
$$\text{Arg}{-1} = \pi$$
$$\text{Arg}(1 - i) = -\frac{\pi}{4}$$

Some convenient notation is $e^{i\theta} = \cos\theta + i\sin\theta$ which implies that $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Note that

$$e^{i(\theta+2\pi)} = e^{i(\theta+4\pi)} = e^{i(\theta+6\pi)} = ... = e^{i(\theta+2k\pi)}, \text{ for any } k \in \mathbb{Z}.$$

Some properties of exponential notation and the argument function are:

• $|e^{i\theta}| = 1$
• $|\overline{e^{i\theta}} = e^{-i\theta}$
• $\frac{1}{e^{i\theta}} = e^{-i\theta}$
• $e^{i(\theta+\phi)} = e^{i\theta} \cdot e^{i\phi}$
• $\arg(\overline{z}) = -\arg(z)$
• $\arg(\frac{1}{z}) = -\arg(z)$
• $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Suppose that $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$. Then the polar form of $z_1 z_2$ is

$$z_1 z_2 = r_1 e^{i\phi_1} r_2 e^{i\phi_2} = (r_1 r_2) e^{i(\phi_1+\phi_2)}.$$

Theorem
De Moivre's formula states that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

De Moivre's formula can be used to derive equations for terms like $\sin 3\theta$ and $\cos 3\theta$. For example,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

§4 Roots of Complex Numbers

Definition — Let w be a complex number. An nth root of w is a complex number z that satisfies the equation $z^n = w$.

Consider the following equation

$$z^n = w.$$

We will see that there are exactly n distinct nth roots that satisfy this equation. To solve this equation we will use the polar form of w and z. Let $w = pe^{i\phi}$ and $z = re^{i\theta}$. Then the equation $z^n = w$ becomes

$$r^n e^{in\theta} = pe^{i\phi}, \text{ so } r^n = p \text{ and } e^{in\theta} = e^{i\phi}.$$

Thus $r = \sqrt[n]{p}$ and $n\theta = \phi + 2k\phi, k \in \mathbb{Z}$ which implies that $\theta = \frac{\phi}{n} + \frac{2k\pi}{n}, k = 0, 1, ..., n-1$. Therefore we have

$$z = w^{\frac{1}{n}} = \sqrt[n]{p}e^{i(\frac{\phi}{n} + \frac{2k\pi}{n})}, k = 0, 1, ..., n-1.$$

Example — Find the square roots of the complex number $z = 4i$.

$$4i = 4e^{i\frac{\pi}{2}}, \text{ so } p = 4, \phi = \frac{\pi}{2}, \text{ and } n = 2.$$

$$\begin{aligned}(4i)^{\frac{1}{2}} &= \sqrt{4}\cdot e^{i(\frac{\pi}{4} + \frac{2k\pi}{2})}, k = 0, 1\\ &= 2\cdot e^{\frac{pi}{4}}, \text{ if } k = 0\\ &= 2\cdot e^{i(\frac{pi}{4} + \pi)} \text{ if } k = 1\\ &= \pm(\sqrt{2} + i\sqrt{2})\end{aligned}$$

Definition — The nth roots of 1 or the solutions to the equation $z^n = 1$ are called the *nth roots of unity*.

Using the formula that we found above and plugging in $w = 1$, we find that

$$1^{\frac{1}{n}} = \sqrt[n]{1}\cdot e^{i(\frac{0}{n} + \frac{2k\pi}{n})}, k = 0, 1, ..., n-1$$

$$= e^{i\frac{2k\pi}{n}}, k = 0, 1, ..., n-1.$$

§5 Topology in the Plane

We will talk about circles and disks in the complex plane. They are centered at $z_0 = x_0 + iy_0$ and have a radius of r.

- A disk of radius r centered at z_0 is denoted as $B_r(z_0) = \{z \in \mathbb{Z} : z$ has distance less than r from $z_0\}$
- A circle of radius r centered at z_0 is denoted as $K_r(z_0) = \{z \in \mathbb{Z} : z$ has distance r from $z_0\}$

Now how do we measure the distance between the center and any point on the circle. We will use the distance formula

$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2} = |(x - x_0)|i(y - y_0)| = |z - z_0|.$$

Therefore we can write $B_r(z_0)$ and $K_r(z_0)$ as

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

$$K_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

Definition — Let $E \subset \mathbb{C}$. A point z_0 is an *interior point of* E if there is some $r > 0$ such that $B_r(z_0) \subset E$.

Definition — Let $E \subset \mathbb{C}$. A point b is a *boundary point of* E if every disk around b contains a point in E and a point not in E. The *boundary* of the set $E \subset \mathbb{C}, \partial E$, is the set of all boundary points of E.

Definition — A set $U \subset \mathbb{C}$ is *open* if every one of its points is an interior point. A set $A \subset \mathbb{C}$ is *closed* if it contains all of its boundary points.

Examples:

- $\{z \in \mathbb{C} : |z - z_0| < r\}$ and $\{z \in \mathbb{C} : |z - z_0| > r\}$ are open
- $\mathbb{C}$ and $\emptyset$ are open
- $\{z \in \mathbb{C} : |z - z_0| \leq r\}$ and $\{z \in \mathbb{C} : |z - z_0| = r\}$ are closed
- $\mathbb{C}$ and $\emptyset$ are closed
- $\{z \in \mathbb{C} : |z - z_0| < r\} \cup \{z \in \mathbb{C} : |z - z_0| = r$ and $\mathrm{Im}(z - z_0) > 0\}$ is neither open nor closed

Definition — Let E be a set in $\mathbb{C}$. The *closure* of E is the set E together with all of its boundary points: $\overline{E} = E \cup \partial E$. The *interior* of E, $\dot{E}$ is the set of all interior points of E.

Examples:

- $\overline{B_r(z_0)} = B_r(z_0) \cup K_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$
- $\overline{K_r(z_0)} = K_r(z_0)$
- $\overline{B_r(z_0) \setminus K_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq r\}$
- With $E = \{z \in \mathbb{C} : |z - z_0| \leq r\}, \dot{E} = B_r(z_0)$
- With $E = K_r(z_0), \dot{E} = \emptyset$

Intuitively, a set is connected if it is "one piece". We will now make this definition rigorous.

Definition — Two sets X, Y in $\mathbb{C}$ are *separated* if there are disjoint open set U, V so that $X \subset U$ and $Y \subset V$. A set W in $\mathbb{C}$ is *connected* if it is impossible to find two separated non-empty sets whose union equals W.

Example — Prove that
$$X = [0, 1) \text{ and } Y = (1, 2]$$
are separated.

Choose $U = B_1(0)$ and $V = B_1(2)$. Thus
$$X \cup Y = [0, 2] \setminus \{1\}$$
is not connected. It is hard to check whether a set is connected.

For open sets, there is a much easier criterion to heck whether or not a set is connected.

Definition — Let G be an *open set* in $\mathbb{C}$. Then G is *connected if and only if* any two points in G can be joined in G by successive line segments.

Definition — A set A in $\mathbb{C}$ is *bounded* if there exists a number $R > 0$ such that $A \subset B_R(0)$. If no such R exists then A is called *unbounded.*

In $\mathbb{R}$, there are two directions that give rise to $\pm\infty$.
$$1, 2, 3, 4, 5, ... \to \infty; \;\; -1, -2, -3, -4, -5, ... \to \infty.$$
However in $\mathbb{C}$, there is only one ∞ which can be attained in many directions.
$$1, 2, 3, 4, ... \to \infty$$
$$-1, -2, -3, -4, ... \to \infty$$
$$i, 2i, 3i, 4i, ... \to \infty$$
$$1, 2i, -3, -4i, 5, 6i, -7, ... \to \infty$$
and many more.

§6 Complex Functions (Week 2)

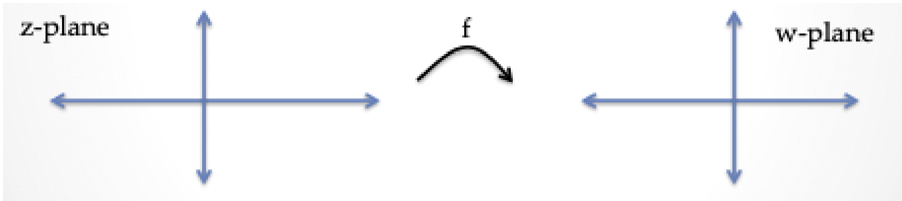
This week we will be focusing on Julia sets for quadratic polynomials and the Mandelbrot set. We will also need to study quadratic polynomials of the form $f(z) = z^2 + c$.

Recall that a function $f : A \to B$ is a rule that assigns each element of A to exactly one element of B. The graph of a function helps us understand the function.

Now what about complex functions. Consider the function $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2 + 1$. How would we graph this function? In order to graph this normally, we would need to graph it in 4 dimensions which is impossible. So we write $z = x + iy$ which gives us
$$w = f(z) = (x + iy)^2 + 1 = (x^2 - y^2 + 1) + i \cdot 2xy = u(x, y) + iv(x, y)$$

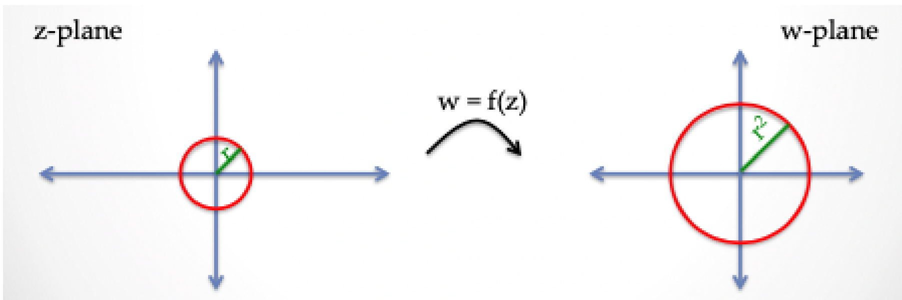
where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

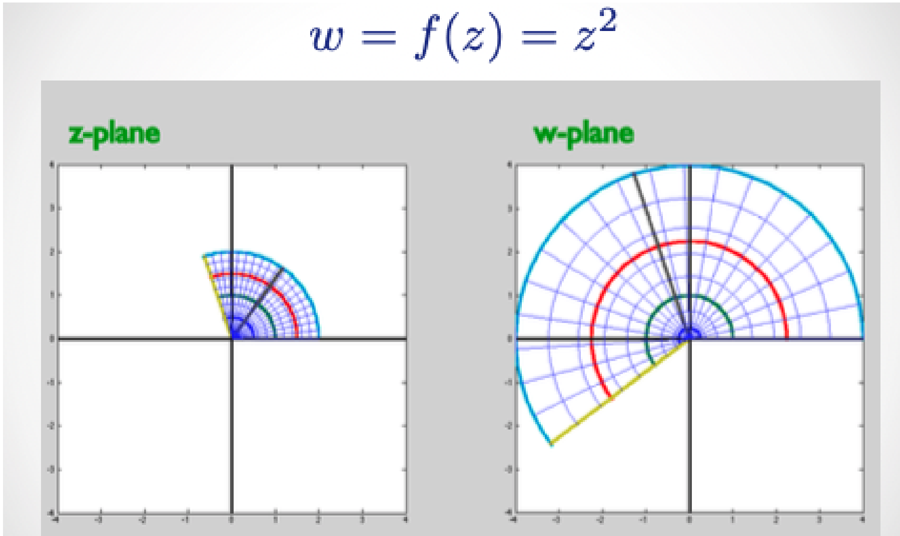
The idea behind graphing complex functions is that we will use two complex planes. One is for the domain and the other is for the range. The graph of the domain lies in the z plane while the graph of the range lies in the w plane. We will now take an example. Consider the function $f(z) = z^2$. Therefore $w = (x+iy)^2 = (x^2 - y^2) + 2ixy$



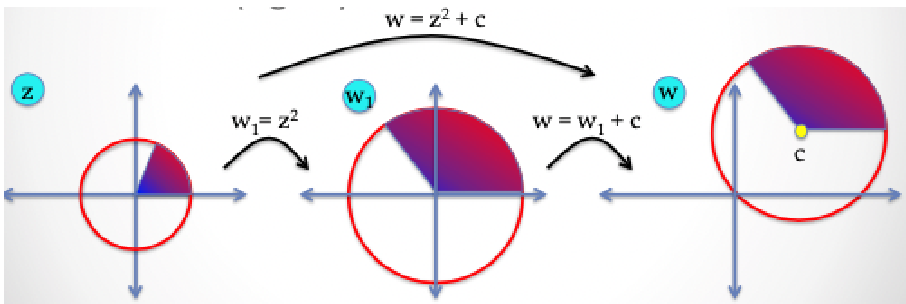
which isn't that helpful. We will instead use polar coordinates.

Therefore since $z = re^{i\theta}$, we have $w = r^2e^{2i\theta}$ so $|w| = |z|^2$ and $\arg(w) = 2\arg(z)$. Therefore as z moves around a circle of radius r once, w moves around the circle of radius r^2 at twice the speed.





Now what if we said that $f(z) = z^2 + c$ for some $c \in \mathbb{C}$. We will use the same method. Now we will look at



iterations of functions. Suppose that $f(z) = z + 1$. Then

$$f^2(z) = f(f(z)) = f(z+1) = z+2$$
$$f^3(z) = f(f(f(z))) = f(f(z+1)) = f(z+2) = z+3$$
$$f^n(z) = \underbrace{f(f(f(f(...(n)))))}_{\text{n times}}.$$

We also call f^n to be the nth iterate of f.

To study the Julia set of the polynomial $f(z) = z^2+c$, we will study the behavior of the iterates, $f, f^2, f^3, f^4, ..., f^n, ...$ of this function. The Julia set of f is the set of points z in the complex plane at which this sequence of iterates behaves "chaotically".

§7 Sequences and Limits of Complex Numbers

Consider the following sequences of complex numbers and see how they progress as more and more terms are added to the sequence.

- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ..., \frac{1}{n}, ... \to ?$
- $i, \frac{i}{2}, \frac{i}{3}, \frac{i}{4}, \frac{i}{5}, \frac{i}{6}, ..., \frac{i}{n}, ... \to ?$
- $i, \frac{-1}{2}, \frac{-i}{3}, \frac{1}{4}, \frac{i}{5}, \frac{-1}{6}, ..., \frac{i^n}{n}, ... \to ?$

Informally, a sequence $\{s_n\}$ converges to a limit s if the sequence eventually lies in any (every so small) disk centered at s. This gives rise to the following definition.

Definition — A sequence $\{s_n\}$ of complex numbers *converges* to $s \in \mathbb{C}$ if for every $\epsilon > 0$ there exists an index $N \geq 1$ such that

$$|s_n - s| < \epsilon \text{ for all } n \geq N.$$

In this case we write

$$\lim_{n\to\infty} s_n = s.$$

Examples:

- $\lim_{n\to\infty} \frac{1}{n} = 0$
- $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any $0 < p < \infty$
- $\lim_{n\to\infty} \frac{c}{n^p} = 0$ for any $c \in \mathbb{C}, 0 < p < \infty$
- $\lim_{n\to\infty} q^n = 0$ for $0 < q < 1$
- $\lim_{n\to\infty} z^n = 0$ for $|z| < 1$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Convergent sequences are always bounded. If $\{s_n\}$ converges to s and $\{t_n\}$ converges to t, then we know that $s_n + t_n \to s + t, s_n \cdot t_n \to s \cdot t$ (in particular, $a \cdot s_n \to a \cdot s$ for any $a \in \mathbb{C}$), $\frac{s_n}{t_n} \to \frac{s}{t}$ provided that $t \neq 0$.

Examples:

- $\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$ as $n \to \infty$
- $\frac{3n^2+5}{in^2+2in-1} = \frac{3+\frac{5}{n^2}}{i+\frac{2i}{n}-\frac{1}{n^2}} \to \frac{3}{i} = -3i$ as $n \to \infty$
- $\frac{n^2}{n+1} = \frac{n}{1+\frac{1}{n}}$ which is not bounded
- $\frac{3n+5}{in^2+2in-1} = \frac{\frac{3}{n}+\frac{5}{n^2}}{i+\frac{2i}{n}-\frac{1}{n^2}} \to \frac{0}{i} = 0$ as $n \to \infty$

Consider the sequence

$$\left\{\frac{i^n}{n}\right\} = i, \frac{-1}{2}, \frac{-i}{3}, \frac{1}{4}, \frac{i}{5}, \frac{-1}{6}, \ldots$$

This sequence appears to converge to 0 but we cannot show this with the previous rules since they don't seem to apply. We will first state a few facts: (1) A sequence of complex numbers, $\{s_n\}$, converges to 0 if and only if the sequence $\{|s_n|\}$ of absolute values converges to 0, (2) A sequence of complex numbers, $\{s_n\}$, with $s_n = x_n + iy_n$ converges to $s = x + iy$ if and only if $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Theorem (Squeeze Theorem)

Suppose that $\{r_n\}, \{s_n\}$, and $\{t_n\}$ are sequences of real numbers such that $r_n \leq s_n \leq t_n$ for all n. If both sequences $\{r_n\}$ and $\{t_n\}$ converge to the same limit, L, then the sequence $\{s_n\}$ has no choice but to converge to the limit L as well.

Theorem

A bounded, monotone sequence of real numbers converges.

Now we can apply these facts to the sequence $\left\{\frac{i^n}{n}\right\}$.

•$|\frac{i^n}{n}| = \frac{|i|^n}{n} = \frac{1}{n} \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} \frac{i^n}{n} = 0$
•$\frac{i^n}{n} = x_n + iy_n, x_n = 0$ if n is odd, 1 if $n = 4k$, -1 if $n = 4k+2$ and $y_n = 0$ if n is even, 1 if $n = 4k+1$, -1 if $n = 4k+3$

Since $= \frac{1}{n} \leq x_n \leq \frac{1}{n}$ and $-\frac{1}{n} \leq y_n \leq \frac{1}{n}$ for all n, the Squeeze Theorem implies that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ which implies that $\lim_{n\to\infty} \frac{i^n}{n} = 0$.

Definition — The complex-valued function $f(z)$ has limit L as $z \to z_0$ if the values of $f(z)$ are near L as $z \to z_0$.

(More formally: $\lim_{z\to z_0} f(z) = L$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.) Also note that $f(z)$ needs to be defined near z_0 for this definition to make sense (but doesn't necessarily have to be defined at z_0).

Properties of limits:

• If f has a limit at z_0 then f is bounded near z_0
• If $f(z) \to L$ and $g(z) \to M$ as $z \to z_0$ then $f(z) + g(z) \to L + M, f(z) \cdot g(z) \to L \cdot M, \frac{f(z)}{g(z)} \to \frac{L}{M}$ as $z \to z_0$ provided that $M \neq 0$ for the last one.

Definition — The function f is continuous at z_0 if $f(z) \to f(z_0)$ as $z \to z_0$.

This definition implicitly states that f is defined at z_0, f has a limit as $z \to z_0$, and the limit equals $f(z_0)$.

§8 Iteration of Quadratic Polynomials, Julia Sets

We will be looking at polynomials of the form $f(z) = z^2 + c$, where $c \in \mathbb{C}$ is a constant. We will also study how the behavior of the iterates of f depend on c.

Another question is that what about other quadratic polynomials in the form $p(z) = az^2 + bz + d$, for constants $a, b, d \in \mathbb{C}$? It turns out that for each triple of constants $a, b, d \in \mathbb{C}$, there is exactly once constant c such that $p(z) = az^2 + bz + d$ and $f(z) = z^2 + c$ "behave the same under iteration". This is because given a, b, d, we can define $c = ad + \frac{b}{2} - (\frac{b}{2})^2$. Then letting $\pi(z) = az + \frac{b}{2}$, one can check that $p(z) = \phi^{-1}(f(\phi(z)))$ for all z. We write this as $p = \phi^{-1} \circ f \circ \phi$. Here is the miracle that happens under iteration:

$$p \circ p = (\phi^{-1} \circ f \circ \phi) \circ (\phi^{-1} \circ f \circ \phi) = \phi^{-1} \circ f \circ f \circ \phi, \text{ so}$$

$$p^2 = \phi^{-1} \circ f^2 \circ \phi$$

$$p^3 = \phi^{-1} \circ f^3 \circ \phi$$

$$p^n = \phi^{-1} \circ f^n \circ \phi$$

Thus it suffices to study the iteration of quadratic polynomials of the form $f(z) = z^2 + c$.

The *Julia set* (named after the French mathematician Gaston Julia, $1893 - 1978$) of $f(z) = z^2 + c$ is the set of all $z \in \mathbb{C}$ for which the behavior of the iterates is "chaotic" in a neighborhood. The *Fatou set* (named after the French mathematician Pierre Fatou, $1878 - 1929$) is the set of all $z \in \mathbb{C}$ for which the iterates behave "normally" in a neighborhood. The iterates of f behave normally near z if nearby points remain nearby under iteration. The iterates of f behave chaotically at z if in any small neighborhood of z, the behavior of the iterates depends sensitively on the initial point.

Let's look at $c = 0$, that is $f(z) = z^2$. Then $f^n(z) = z^{(2^n)}$. Writing $z = re^{i\theta}$, we see that $f^n(z) = r^{(2^n)} \cdot e^{i \cdot 2^n \theta}$. Thus:

- If $|z| < 1$, then $|f^n(z)| = |z|^{(2^n)} \to 0$ as $n \to \infty$, so $f^n(z) \to 0$ as $n \to \infty$
- If $|z| > 1$, then $|f^n(z)| \to \infty$ as $n \to \infty$, so we say that $f^n(z) \to \infty$ as $n \to \infty$
- If $|z| = 1$ then $z = e^{i\theta}$, so $f^n(z) = e^{i \cdot 2^n \theta}$, thus $|f^n(z)| = 1$ for all n

We notice that in any little disk around a point z with $|z| = 1$, there are points w with $|w| > 1$ (and for which thus $f^n(w) \to \infty$), and other points w with $|w| < 1$ (and for which thus $f^n(w) \to 0$).

The unit circle $\{z : |z| = 1\}$ is thus the locus of chaotic behavior, whereas $\underbrace{\{z : |z| > 1\}}_{\text{iterates attracted to } \infty}$ and $\underbrace{\{z : |z| < 1\}}_{\text{iterates attracted to } 0}$ form the locus of normal behavior.

We write $J(f) = \{z : |z| = 1|\}$ (Julia set) and $F(f) = \{z : |z| > 1\} \cup \{z : |z| < 1\}$ (Fatou set).

More generally, let's look at $f(z) = z^2 + c$. Let

$$A(\infty) = \{z : f^n(z) \to \infty\} \text{ "basin of attraction to } \infty\text{"}.$$

Theorem

The set $A(\infty)$ is open, connected, and unbounded. It is contained in the Fatou set of f. The Julia set of f coincides with the boundary of $A(\infty)$, which is a closed and bounded subset of $\mathbb{C}$.

Lets look at another example: $f(z) = z^2 - 2$. For this example, it is hard to calculate and understand the iterates $f^n(z)$. We will use a special trick though. Conjugate f with

$$\phi(w) = w + \frac{1}{w}, \phi : \{w : |w| > 1\} \to \mathbb{C} \setminus [-2, 2].$$

Then f maps $[-2, 2]$ to $[-2, 2]$ and $\mathbb{C} \setminus [-2, 2]$ to $\mathbb{C} \setminus [-2, 2]$. We can thus look at $\phi^{-1} \circ f \circ \phi$.

Recall that $f(z) = z^2 - 2, \phi(w) = w + \frac{1}{w}$. Then what is $\phi^{-1}(f(\phi(w)))$?

$$f(\phi(w)) = (\phi(w))^2 - 2 = (w + \frac{1}{w})^2 - 2 = w^2 + \frac{1}{w^2} + 2w\frac{1}{w} - 2 = w^2 + \frac{1}{w^2} = \phi(w^2)$$

so we have

$$\phi^{-1}(f(\phi(w))) = w^2.$$

Recall that $f(z) = z^2 - 2, \phi(w) = w + \frac{1}{w}$.

$$\phi^{-1}(f(\phi(w))) = w^2, \text{ or } f(z) = \phi(g(\phi^{-1}(z))), \text{ where } g(w) = w^2.$$

Thus, on $\mathbb{C} \setminus [-2, 2]$, the function $f(z) = z^2 - 2$ behaves like $g(w) = w^2$ behaves on the exterior of the closed unit disk. Since the iterates $g^n(w)$ tend to ∞ for $|w| > 1$, we conclude that $f^n(z) \to \infty$ as $n \to \infty$ for all $z \in \mathbb{C}[-2, 2]$. Thus $A(\infty) = \mathbb{C} \setminus [-2, 2]$ which implies that $J(f) = [-2, 2]$.

We have looked at two examples so far and found their Julia sets:

• $f(z) = z^2$, we found that $J(f) = \{z : |z| = 1\}$, the unit circle

•$f(z) = z^2 - 2$, we found that $J(f) = [-2, 2]$, the closed interval from -2 to 2 on the real axis

These two examples are exceptional in that their Julia sets are "smooth". In fact, they are the only examples amongst all $f(z) = z^2 + c$ with sooth Julia sets.

§9 Finding Julia Sets

Recall that for $f(z) = z^2 + c$, where $c \in \mathbb{C}$, the Julia set of f is the boundary of $A(\infty)$, where $A(\infty)$ is the "basin of attraction to infinity", i.e. $A(\infty) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$.

Let's start by finding all those $z \in \mathbb{C}$ for which $\{f^n(z)\}$ stays bounded:

$$K(f) = \{z \in \mathbb{C} : \{f^n(z)\} \text{ is bounded}\}$$

is the *filled-in Julia set* of f.

Examples:

•$f(z) = z^2$, then $K(f) = \{z : |z| \leq 1\}$ and $J(f) = \{z : |z| = 1\}$
•$f(z) = z^2 - 2$, then $K(f) = [-2, 2]$ and $J(f) = [-2, 2]$
•$f(z) = z^2 - 1$ then $K(f) =?$

We will now try and find the Julia set of $f(z) = z^2 - 1$.

Let $f(z) = z^2 - 1$ and we want to find $K(f)$. We will first check some orbits. Suppose that $z = 0$. Then $f(0) = -1, f(f(0)) = f(-1) = 0, f^3(0) = f(0) = -1, f^4(0) = 0, ...$ so the orbit is $0, -1, 0, -1, 0, -1, ...$ Since this is a *periodic orbit*, 0 is a *periodic point* of *period* 2. Clearly $0 \in K(f)$.

Suppose that $z = 1$. Then $f(1) = 0, f(f(1)) = f(0) = -1, f^3(0) = f(1) = 0, f^4(0) = -1, ...$ so the orbit is $0, -1, 0, -1, 0, -1, ...$ Since the orbit doesn't return back to 1, 1 isn't a periodic point, but since it runs into a periodic orbit, we call it a *pre-periodic* point. Again, $1 \in K(f)$.

Suppose that $z = \frac{1+\sqrt{5}}{2}$. Then $f(z) = \frac{(1+\sqrt{5})}{2}^2 - 1 = \frac{1+\sqrt{5}}{2} = z$. Therefore the point $z = \frac{1+\sqrt{5}}{2}$ is a *fixed point* of f and thus belongs to $K(f)$ as well.

So far everything we've tried belongs to $K(f)$. Let us find some more orbits.

•$z = -2, -2, 3, 8, 63, ... \to \infty$, so $-2 \in A(\infty)$
•$z = i, i, -2, 3, 8, 63, ... \to \infty$, so $i \in A(\infty)$

One can show that if x lies on the real axis and is smaller than $-\frac{1+\sqrt{5}}{2}$ or larger than $\frac{1+\sqrt{5}}{2}$, then $x \in A(\infty)$. More generally, if $z \in \mathbb{C}$ with $|z| > \frac{1+\sqrt{5}}{2}$, then $z \in A(\infty)$. Even more generally, if $z \in \mathbb{C}$ such that $|f^n(z)| > \frac{1+\sqrt{5}}{2}$, for some n, then $z \in A(\infty)$. A similar condition holds for general quadratic polynomials in the form $f(z) = z^2 + c$.

Theorem

Let $f(z) = z^2 + c$, and let $R = \frac{1+\sqrt{1+4|c|}}{2}$. Let $z_0 \in \mathbb{C}$. If for some $n > 0$, we have that $|f^n(z_0)| > R$, then $f^n(z_0) \to \infty$ as $n \to \infty$, i.e. $z_0 \in A(\infty)$, so $z_0 \notin K(f)$.

This is a condition that a computer could check. Given c, let $R = \frac{1+\sqrt{1+4|c|}}{2}$. Then we can choose a window $W = \{x + iy : a \leq x \leq b, c \leq y \leq d\}$ to display. If you want to see the entire Julia set, you'd want $B_R(0) \subset W$, but at some point it also may be interesting to zoom into the Julia set. The computer can't keep checking all iterates $f^n(z)$, at some point it will have to stop. So pick the largest number, *maxiter*, up to which it will check. The larger this number, the more accurate your picture will get, but the slower the calculation will be. For each pixel on your screen, choose a point z in your window W, corresponding to that pixel. Then calculate the iterates of z: $f(z), f(f(z)), f(f(f(z))), \ldots$ If one of these iterates satisfies that $|f^n(z)| > R$, color the initial pixel white. If you reach the maximum number of iterations, *maxiter*, without having left $B_R(0)$, there's a good chance that z belongs to $K(f)$. Color this pixel black.

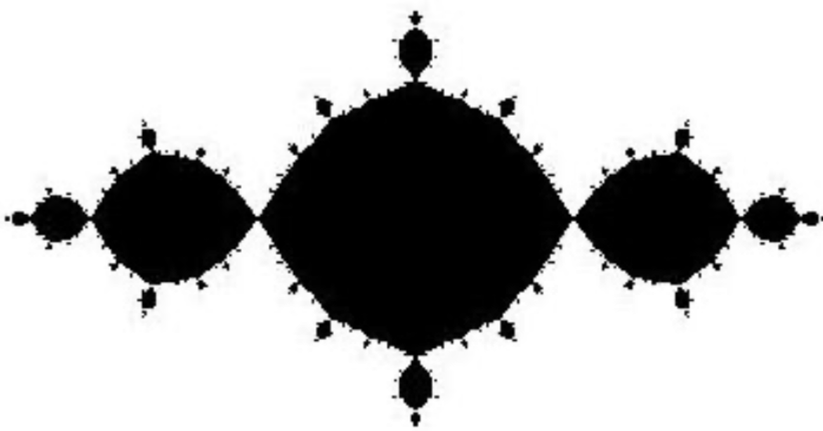


Figure 1: Julia set of $f(z) = z^2 - 1$

Figure 2: Julia set of $f(z) = z^2 + \frac{1}{4}$

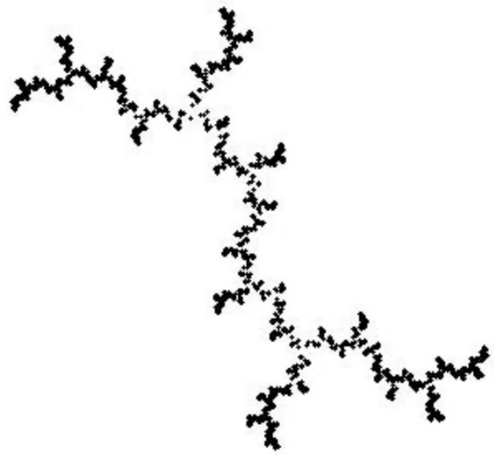
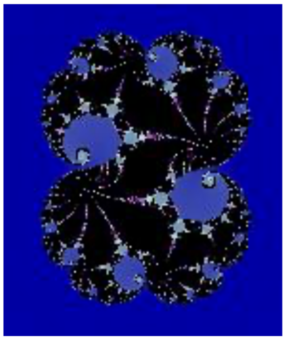


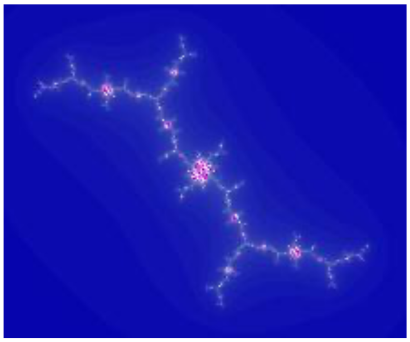
Figure 3: Julia set of $f(z) = z^2 + i$

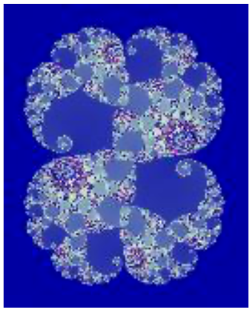
Figure 4: Julia set of $f(z) = z^2 + 1$

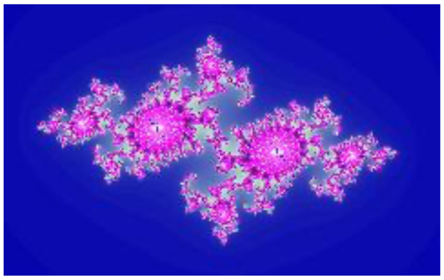
The pictures of the filled-in Julia sets we have produced so far are only black and white. For colorful pictures, choose colors, $c_0, c_1, c_2, ..., c_{maxiter}$ for $A(\infty)$ as well as a color for the filled-in Julia set. If $|z| > R$, color the corresponding pixel with color c_0. If $|f(f(z))| > R$, color the corresponding pixel with color c_1. And so on. Otherwise if $|f^n(z)| \leq R$ for all $n \leq maxiter$, color the corresponding pixel in your chosen color for the filled-in Julia set. The larger your constant *maxiter* is, the more detail you can see, but the longer the calculation will take. For small windows (i.e. zooms into the Julia set), a higher precision is required to avoid round-off errors.

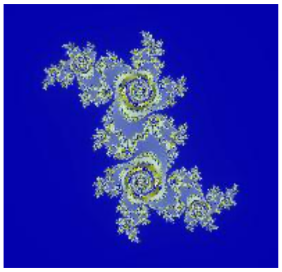
Let's look at some examples of colorful Julia sets.











We noticed the following in all of the pictures we looked at, the Julia set of $f(z) = z^2 + c$ is either "in one piece" or "totally dusty". This motivates the following definition.

> **Definition —** The *Mandelbrot set* M is the set of all parameters $c \in \mathbb{C}$ for which the Julia set $J(f)$ of $f(z) = z^2 + c$ is connected.

Note that the Mandelbrot set is a subset of the *parameter space* (the space of all possible c-values), whereas Julia sets are sets of z-values.

In the next section we will see how to determine whether for a given $c \in \mathbb{C}$, the Julia set $J(z^2 + c)$ is connected and if a computer can perform this task.

§10 The Mandelbrot Set

Recall that the Mandelbrot set is

$$M = \{c \in \mathbb{C} : J(z^2 + c) \text{ is connected}\}.$$

Now how would a computer check such a condition?

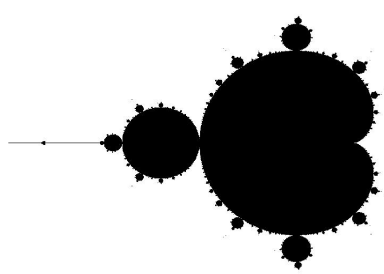
> **Theorem**
> Let $f(z) = z^2 + c$. Then $J(f)$ is *connected if and only if* 0 does not belong to $A(\infty)$, that is if and only if the orbit $\{f^n(0)\}$ remains bounded under iteration.

It is also possible to show the following result.

> **Theorem**
> A complex number c belongs to M if and only if $|f^n(0)| \leq 2$ for all $n \geq 1$ where $f(z) = z^2 + c$.

First we will choose a window $W = \{c_x + ic_y : c_{xmin} \leq c_x \leq c_{xmax}, c_{ymin} \leq c_y \leq c_{ymax}\}$ to display. If you want to see the entire Mandelbrot set, you'd want something like $W = \{c : -2 \leq c_z \leq 0.75, -1.5 \leq c_y \leq 1.5\}$. As before, pick a largest number of iterations, *maxiter*. The larger this number, the more accurate your picture will get, but the slower the calculation will be. For each pixel on your screen, choose a parameter c in your window W, corresponding to that pixel. We will look at the corresponding polynomial $f(z) = z^+c$. Calculate the iterates of 0 under this polynomial, $f(0) = c, f(f(0)) = c^2 + c, f(f(f(0))) = (c^2 + c)^2 + c, ...$ If one of these iterates satisfies $|f^n(0)| > 2$, color the initial pixel white. If you reach the maximum number of iterations, *maxiter*, without having left $\overline{B_2(0)}$, there's a good chance that the parameter c belongs to the Mandelbrot set M. Color this pixel black.

A first look at M: Again, you can use different colors for those parameters $c \in \mathbb{C}$ for which) escapes to



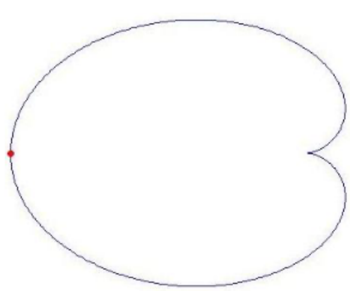
infinity under iteration, depending on how quickly the escape happens. If $|f(0| = |c| > 2$, color the corresponding pixel in color zero. If $|f(f(0))| = |c^2 + c| > 2$, color the corresponding pixel in color one. If $|f(f(f(0)))| = |(c^2 + c)^2 + c| > 2$, color the corresponding pixel to color two. And so on. If $|f^n(0)| \leq 2$ for all $n \leq maxiter$, color the pixel corresponding to c, black or whatever other color you choose for your M.

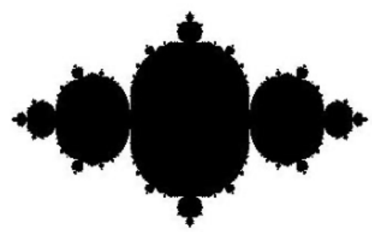
Zooming into the Mandelbrot set and coloring parameters by escape time yields beautiful pictures.

Properties of the Mandelbrot set:

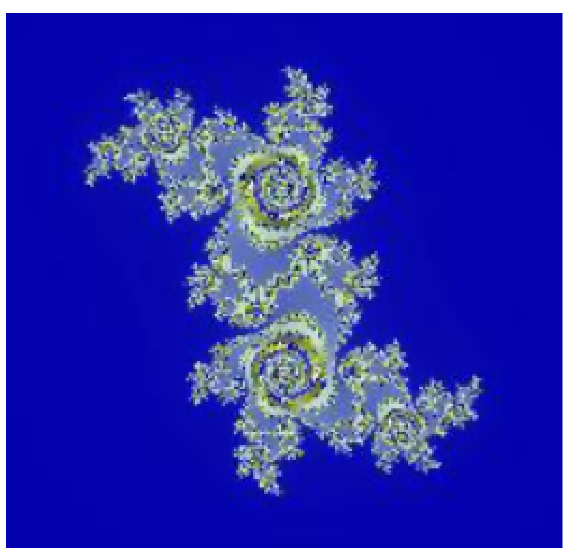
• M is a connected set (Douady, Hubbard, 1982)
• M is contained in the disk of radius 2, centered at 0
• The boundary of M is very intricate, this is where you will find the most beautiful zooms
• Moreover, for c-values near the boundary of M, their Julia sets have many different patterns

Let $c = \frac{1}{2}e^{2\pi i\alpha} - \frac{1}{4}e^{4\pi i\alpha}$, where $0 \leq \alpha < 1$ is a rational number. Then α is of the form $\frac{p}{q}$ where p and q are integers. The parameter c is an attachment point of another "bud" to the Mandelbrot set, and the Julia set for $f(z) = z^2 + c$ looks similar to the Julia sets for parameter values within the bud. An example is when $\alpha = \frac{1}{2}$. Then $c = \frac{1}{2}e^{\pi i} - \frac{1}{4}e^{2\pi i} = -0.75$. Here is a picture for $J(z^2 - 0.75)$: Now let $c = \frac{1}{2}e^{2\pi i\alpha} - \frac{1}{4}e^{4\pi i\alpha}$,





where $0 \leq \alpha < 1$ is an irrational number. Thus there are no values p and q such that $\alpha = \frac{p}{q}$. Julia sets for such values look more intricate and come in several "flavors". Here is an example: let $\alpha = \frac{1+\sqrt{5}}{2}$. Then $c = -0.390540870218401... - 0.586787907346969...i$. For $f(z) = z^2 + c$, the interior of $K(f)$ has a so-called "Siegel disk", in which iteration looks like a rotation by angle α.



Many points in the boundary of M are so-called *Misiurewicz points*:

• A point $c \in \mathbb{C}$ is called a Misiurewicz point if the orbit of 0 under $f(z) = z^2 + c$ is pre-periodic, but not periodic
• Example: $c = i$, then $f(z) = z^2 + i$ and the orbit of 0 under f is $0, i, -1+i, -i, -1+i, -i, ...$
• Clearly, Misiurewicz points c belong to M since the orbit of 0 under $f(z) = z^2 + c$ is bounded

Properties of Misiurewicz points:

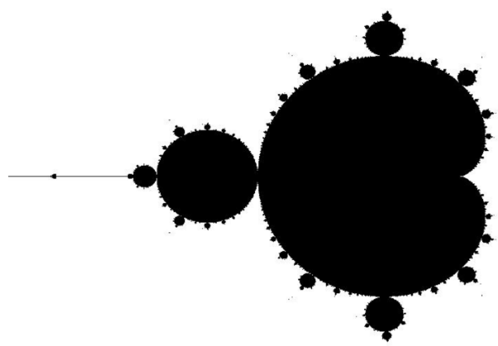
• Let c be a Misiurewicz point, then $J(f) = K(f)$, i.e. $K(f)$ has no interior
• Misiurewicz points are dense in ∂M
• The Mandelbrot set is self-similar under magnification near Misiurewicz points

Note that the Mandelbrot set is "quasi-self-similar" everywhere, small, slightly different versions of itself can be found at arbitrary small scales.

Here is one of the big outstanding conjectures in the field of complex dynamics:

Conjecture

The Mandelbrot set is locally connected, that is, for every $c \in M$ and every open set V with $c \in V$, there exists an open set U such that $c \in U \subset V$ and $U \cap M$ is connected.

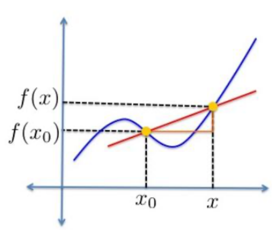


§11 The Complex Derivative (Week 3)

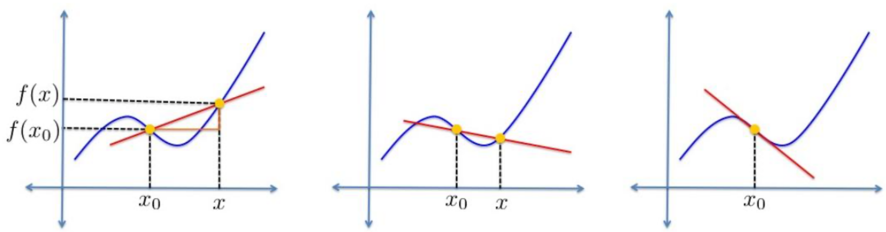
Let $f : (a, b) \to \mathbb{R}$ be a real valued function of a real variable, and let $x_0 \in (a, b)$. The function f is *differentiable* at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

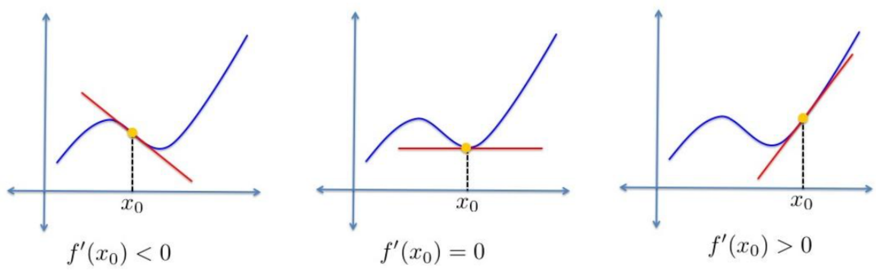
exists. If so, we call this limit the *derivative* of f at x_0 and denote it by $f'(x_0)$.



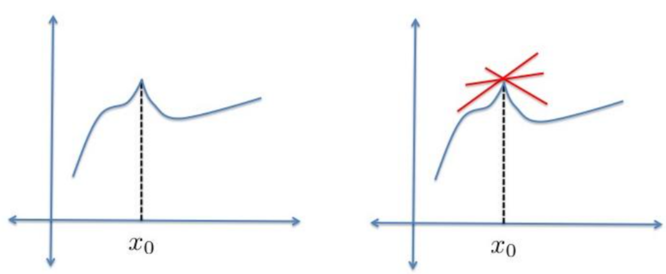
Note that $\frac{f(x)-f(x_0)}{x-x_0}$ is the slope of the secand line through the points $(x_0, f(x_0))$ and $(x, f(x))$. As x approaches x_0, the slope of the secant line changes.



In the limit, the slopes approach the slope of the tangent line to the graph of f at x_0.



The derivative does not always exist. Here is an example.



Definition — A complex-valued function f of a complex variable is *(complex) differentiable* at $z_0 \in$ domain(f) if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

If this limit exists, it is denoted as $f'(z_0)$, $\frac{df}{dz}(z_0)$, or $\frac{d}{dz} f(z)|_{z=z_0}$.

Instead of

$$\frac{(f(z) - f(z_0)}{z - z_0}$$

we often write $z = z_0 + h$ (where $h \in \mathbb{C}$), and the difference quotient becomes

$$\frac{f(z_0 + h) - f(z_0)}{h} \text{ or simply } \frac{f(z+h) - f(z)}{h},$$

where we'll take the limit as $h \to 0$.

Theorem

Suppose f and g are differentiable at z and h is differentiable at $f(z)$. Let $c \in \mathbb{C}$. Then

- $(cf)'(z) = cf'(z)$
- $(f+g)'(z) = f'(z) + g'(z)$
- $(f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z)$
- $(\frac{f}{g})'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}$, for $g(z) \neq 0$
- $(h \circ f)'(z) = h'(f(z))f'(z)$

Now let $f(z) = \text{Re}(z)$. Write $z = x + iy$ and $h = h_x + ih_y$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{(x + h_x) - x}{h} = \frac{h_x}{h} = \frac{\text{Re}(h)}{h}.$$

Does this have a limit as $h \to 0$?

- $h \to 0$ along real axis, then $h = h_x + i \cdot 0$, so $\text{Re}(h) = h$ and thus the quotient evaluates to 1 and the limit equals 1
- $h \to 0$ along imaginary axis, then $h = 0 + i \cdot h_y$, so $\text{Re}(h) = 0$ and thus the quotient evaluates to 0 and the limit equals 0
- $h_n = \frac{i^n}{n}$ then $\frac{\text{Re}(h_n)}{h_n} = \frac{\text{Re}(i^n)}{i^n}$ which equals 1 if n is even and 0 when n is odd which has no limit as $n \to \infty$

Therefore $f = \text{Re}(z)$ is not differentiable anywhere in $\mathbb{C}$.

Let $f(z) = \overline{z}$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z} + \overline{h} - \overline{z}}{h} = \frac{\overline{h}}{h}.$$

- If $h \in \mathbb{R}$ then $\frac{\overline{h}}{h} = 1 \to 1$ as $h \to 0$
- If $h = i\mathbb{R}$ then $\frac{\overline{h}}{h} = -1 \to -1$ as $h \to 0$

Thus $\frac{\overline{h}}{h}$ does not have a limit as $h \to 0$, and f is not differentiable anywhere in $\mathbb{C}$.

It is a well known fact that if f is differentiable at z_0, then f is continuous at z_0.

Definition — A function f is *analytic* in an open set $U \subset \mathbb{C}$ if f is (complex) differentiable at each point $z \in U$. A function which is analytic in all of $\mathbb{C}$ is called an *entire* function.

For example, polynomials are analytic in $\mathbb{C}$ and are therefore entire functions. Rational functions in the form $\frac{p(z)}{q(z)}$ are analytic wherever $q(z) \neq 0$ and $f(z) = \overline{z}$ is not analytic.

Let $f(z) = |z|^2$. Then

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{|z+h|^2 - |z|^2}{h} \\
&= \frac{(z+h)(\overline{z} + \overline{h}) - |z|^2}{h} \\
&= \frac{|z|^2 + z\overline{h} + h\overline{z} + h\overline{h} - |z|^2}{h} \\
&= \overline{z} + \overline{h} + z \cdot \frac{\overline{h}}{h}
\end{aligned}$$

Thus, if $z \neq 0$ then the limit as $h \to 0$ does not exist, if $z = 0$ then the limit equals 0, thus f is differentiable at 0 with $f'(0) = 0$, f is not analytic anywhere, and f is continuous in $\mathbb{C}$.

§12 The Cauchy-Riemann Equations

Recall that a complex function f can be written as

$$f(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$ and u, v are real-valued functions that depend on the two real variables x and y.

An example is $f(z) = z^2$. Then $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i \cdot 2xy$ which implies that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. We already know that f is differentiable everywhere in $\mathbb{C}$ and $f'(z) = 2z$ for all $z \in \mathbb{C}$.

Now lets look at the function $u(x, y)$ a little more carefully. If we fix the variable y at a certain value, then u only depends on x. For example, if we let $y = 3$ then $u(x, y) = u(x, 3) = x^2 - 9$. We can now differentiate this function with respect to x according to the rules of calculus and find that the derivative is $2x$. We write $\frac{\partial u}{\partial x}(x, y) = u_x(x, y) = 2x$, and, more generally, for arbitrary (fixed) y,

$$\frac{\partial u}{\partial x}(x, y) = u_x(x, y) = 2x.$$

This is called the *partial derivative of* u *with respect to* x.

Similarly, we can do the same thing by fixing x and differentiating with respect to x. Keeping $f(z) = z^2$, for an arbitrary (fixed) y, we have

$$\frac{\partial v}{\partial x}(x, y) = v_x(x, y) = 2y.$$

This is called the *partial derivative of* v *with respect to* x. Now we can do the same thing by fixing y and differentiating with x and differentiating with y. These results are

$$\frac{\partial u}{\partial y}(x, y) = u_y(x, y) = -2y \text{ and}$$

$$\frac{\partial v}{\partial y}(x, y) = v_y(x, y) = 2x.$$

These are called the *partial derivative of* u *with respect to* y and *partial derivative of* v *with respect to* y respectively.

Comparing these derivatives with each other, notice that $u_x(x, y) = v_y(x, y), u_y(x, y) = -v_x(x, y), f' = \underbrace{u_x(x, y) + iv_x(x, y)}_{=f_x} = \underbrace{-i(u_y(x, y) + iv_y(x, y))}_{=-if_y}$. This motivates the Cauchy Riemann equations.

Theorem (Cauchy Riemann Equations)
Suppose that $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point z_0. Then the partial derivatives u_x, u_y, v_x, v_y exist at z_0 and satisfy the equations:

$$u_x = v_y \text{ and } u_y = -v_x.$$

These equations are called the Cauchy-Riemann equations. Also

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = f_x(z_0)$$

$$= -i(u_y(x_0, y_0) + iv_y(x_0, y_0)) = -if_y(z_0).$$

To prove this theorem, you'd look at the difference quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

whose limit as $h \to 0$ must exist if f is differentiable at z_0.

• You'd let h approach 0 along the real axis only, then along the imaginary axis only, both times the limit must exist and both limits must be the same
• Equating these two limits with each other and recognizing the partial derivatives in the expressions yields the Cauchy-Riemann equations

We will look at an example. Let $f(z) = \overline{z} = x - iy$. Then $u(x, y) = x$ and $v(x, y) = -y$, so

$$u_x(x, y) = 1, v_x(x, y) = 0$$

$$u_y(x, y) = 0, v_y(x, y) = -1$$

Since $u_x \neq v_y$ (while $u_y = -v_x$) for all z, the function is not differentiable anywhere.

Now we stumble upon an interesting question. Are the Cauchy-Riemann equations a guarantee for differentiability? In other words, if f satisfies the Cauchy-Riemann equations at a point z_0 then does this imply that f is differentiable at z_0? The answer is almost.

Theorem
Let $f = u + iv$ be defined on a domain $D \subset \mathbb{C}$. Then f is analytic in D if and only if $u(x, y)$ and $v(x, y)$ have continuous first partial derivatives on D that satisfy the Cauchy-Riemann equations.

Example: $f(z) = e^x \cos y + ie^x \sin y$. Then

$$u_x(x, y) = e^x \cos y, v_x(x, y) = e^x \sin y$$

$$u_y(x, y) = -e^x \sin y, v_y(x, y) = e^x \cos y$$

Thus the Cauchy-Riemann equations are satisfied, and in addition, the functions u_x, u_y, v_x, v_y are continuous in $\mathbb{C}$. There, the function f is analytic in $\mathbb{C}$ and is also entire.

In the next section, we will be studying the function

$$f(z) = e^x \cos y + ie^x \sin y$$

which is the complex exponential function.

§13 The Complex Exponential Function

Recall from the last section that the function

$$f(z) = e^x \cos y + ie^x \sin y$$

(where again, $z = x + iy$) is an entire function. Let's look at some of its properties.

• If $y = 0$, then $f(z) = f(x + i \cdot y) = f(x) = e^x$, so f agrees with the "regular" exponential function on $\mathbb{R}$
• $f(z) = e^x(\cos y + i \sin y) = e^x e^{iy}$.

Definition — The complex exponential function, e^z, sometimes also denoted $\exp(z)$, is defined by

$$e^z = e^x \cdot e^{iy}, \text{ where } z = x + iy.$$

This definition also motivates a few other properties.

- $|e^z| = |e^x||e^{iy}| = e^x$
- $\arg(e^z) = \arg(e^x e^{iy}) = y$
- $e^{z+2\pi i} = e^x ei(y + 2\pi) = e^x e^{iy} = e^z$
- $e^{z+w} = e^{(x+u)+i(y+v)} = e^{x+u} e^{i(y+v)} = e^x e^u e^{iy} e^{iv} = (e^x e^{iy})(e^u e^{iv}) = e^z e^w$
- $\frac{1}{e^z} = e^{-z}$

Now what is the derivative of the complex exponential function. Recall that

$$u(x, y) = e^x \cos y, v(x, y) = e^x \sin y$$

and

$$u_x(x, y) = e^x \cos y, v_x(x, y) = e^x \sin y$$

$$u_y(x, y) = -e^x \sin y, v_y(x, y) = e^x \cos y$$

Thus $f'(z) = u_x(x, y) + iv_x(x, y) = e^x \cos y + ie^x \sin y = e^z$ so $\frac{d}{dz} e^z = e^z$.

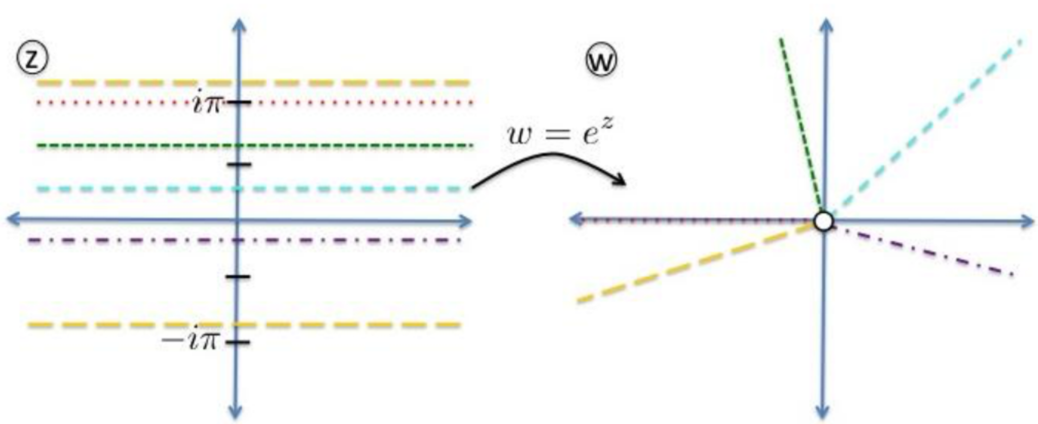
Even more properties of the complex exponential function.

- $\frac{d}{dz} e^{az} = a \cdot e^{az} (a \in \mathbb{C}$ by the chain rule
- $e^{\overline{z}} = e^{x-iy} = e^x e^{-iy} = e^x e^{\overline{iy}} = \overline{e^x e^{iy}} = \overline{e^z}$
- $e^z = 1$ if and only if $e^x e^{iy} = 1$, the complex number in polar form, $e^x e^{iy}$, equals 1 when its length equals 1 and its argument equals 0, i.e. when $e^x = 1$ and $y = 2k\pi$, thus

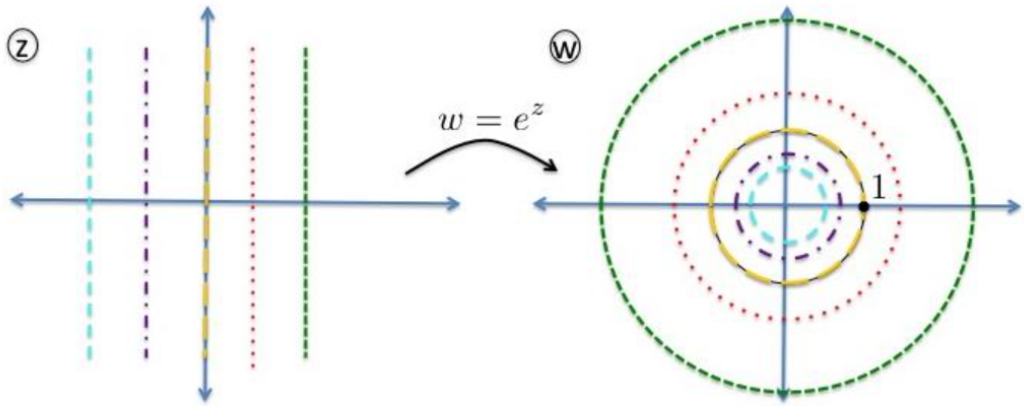
$$e^z = 1 \longrightarrow z = 2\pi ik, k \in \mathbb{Z}$$

- $e^z = e^w \longrightarrow e^{z-w} = 1 \longrightarrow z - w = 2\pi ik \longrightarrow z = w + 2\pi ik$ where $k \in \mathbb{Z}$

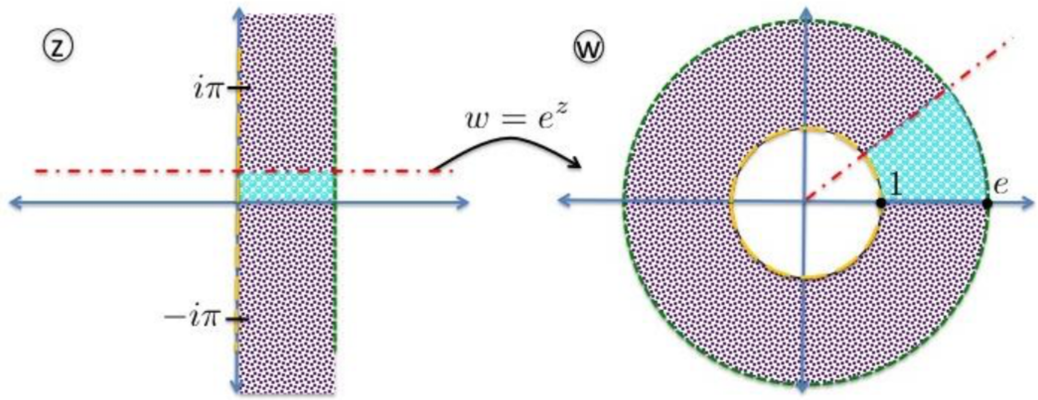
The function $w = e^z$ is a mapping from $\underbrace{\mathbb{C}}_{\text{z plane}}$ to $\underbrace{\mathbb{C}}_{\text{w plane}}$. What are the images of horizontal lines? $L = \{x + iy_0 | x \in \mathbb{R}\}$ for fixed $y_0 \in \mathbb{R}$. Then $e^{x+iy_0} = e^x e^{iy_0}$.



What are the images of vertical lines? $L = \{x_0 + iy | y \in \mathbb{R}\}$ for fixed $x_0 \in \mathbb{R}$. Then $e^{x_0+iy} = e^{x_0}e^{iy}$.



What is the image of a vertical strip? $S = \{z : 0 < \text{ Re}(z) < 1\}$.



For a given $z \in \mathbb{C}\{0\}$, is there a $w \in \mathbb{C}$ such that $e^w = z$? Writing $z = |z|e^{i\theta}$ and $w = u + iv$ this is equivalent to

$$e^w = z \longrightarrow e^u e^{iv} = |z|e^{i\theta}$$

$$e^u = |z| \text{ and } e^{iv} = e^{i\theta}$$

$$u = \ln|z| \text{ and } v = \theta + 2k\pi$$

$$w = \ln|z| + i\arg(z)$$

This is the complex logarithm.

§14 Complex Trigonometric Functions

Now that we have extended the exponential function to the complex plane, can we do the same thing for trigonometric functions?

Recall that $e^{i\theta} = \cos\theta + i\sin\theta$. Therefore, $e^{-i\theta} = \cos -\theta + i\sin -\theta = \cos\theta - i\sin\theta$. Hence we have $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$. Thus $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

Definition — The complex cosine and since functions are defined via

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Note that $\sin z$ and $\cos z$ are analytic functions and entire functions. For real-valued z, the complex sine and cosine functions agree with the real-valued sine and cosine functions. We also have that $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$. We also have $\cos(z+w) = \cos z \cos w - \sin z \sin w$ and $\sin(z+w) = \sin z \cos w + \cos z \sin w$. Also note that $\cos(z + 2\pi) = \cos z$ and $\sin(z+2\pi) = \sin z$. We also have that $\sin^2 z + \cos^2 z = 1$ and that $\sin(z + \frac{\pi}{2}) = \cos z$.

When is $\sin z = 0$? Remember that $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$.

$$\sin z = 0 \longrightarrow e^{iz} = e^{-iz}$$

$$(iz) - (iz) = 2k\pi i, k \in \mathbb{Z}$$

$$2iz = 2k\pi i, k \in \mathbb{Z}$$

$$z = k\pi, k \in \mathbb{Z}$$

Similarly $\cos z = 0$ when $z = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$.

What are the derivatives of the sine and the cosine functions?

$$\frac{d}{dz} \sin z = \cos z \text{ and } \frac{d}{dz} \cos z = -\sin z$$

It is possible to express the complex sine and cosine functions solely in terms of the real sine and cosine functions as well as the real hyperbolic sine and cosine functions

$$\sin z = \sin(x + iy)$$

$$= \sin x \cos(iy) + \cos x \sin(iy)$$

$$= \sin x \frac{e^{i(iy)} + e^{-i(iy)}}{2} + \cos x \frac{e^{i(iy)} - e^{-i(-y)}}{2i}$$

$$= \sin x \frac{e^{y} + e^{-y}}{2} + \cos x \frac{e^{y} - e^{-y}}{2}$$

$$= \sin x \cosh y + i \cos x \sinh y$$

Similarly

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

§15 First Properties of Analytic Functions

Theorem

If f is analytic on a domain D, and if $f'(z) = 0$ for all $z \in D$, then f is constant in D.

Recall the 1−dimensional analog of this, if $f : (a, b) \to \mathbb{R}$ is differentiable and satisfies $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

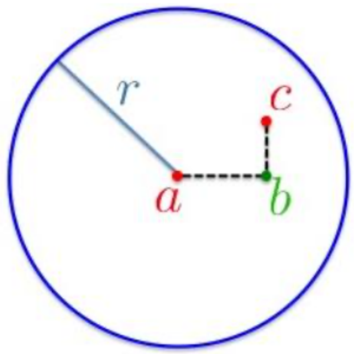
Sketch of proof:

(1) First show that f in any disk $B_r(a)$, contained in D, using the 1-dimensional fact.

(2) Second, use the fact that D is connected and the first fact to show that f is constant in D.

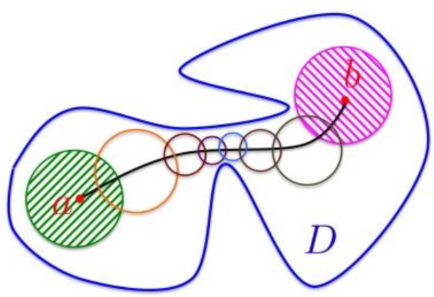
Here are the main steps in these proofs:

(1) Let $B_r(a)$ be a disk contained in D, and let $c \in B_r(a)$. Pick the point $b \in B_r(a)$ as in the picture.



Since $f'(z) = 0$ in D, we have $u_x = u_y = v_x = v_y = 0$ in D. In particular, look at u on the horizontal line segment from a to b. It depends only on one parameter (namely x) there, and $u_x = 0$. By the $1-$dimensional fact, we find that u is constant on the line segment, in particular, $u(a) = u(b)$. Similarly, $u(b) = u(c)$ thus $u(a) = u(c)$. Since c was an arbitrary point in $B_r(a)$, u is thus constant in $B_r(a)$. Similarly, v is constant in $B_r(a)$ and thus f is constant in $B_r(a)$.

(2) Let a and b be two arbitrary points in D. Since D is connected, there exists a nice curve in D, joining a and b.



By the previous step, f is constant in the disk around the point a. Furthermore, f is also constant in the neighboring disk. Since these two disks overlap, the two constants must agree. Continue on in this matter until you reach b. Therefore $f(a) = f(b)$. Thus f is constant in D.

The previous theorem, together with the Cauchy-Riemann equations, has strong consequences.

• Suppose that $f = u + iv$ is analytic in a domain D. Suppose furthermore, that u is constant in D. Then f must be constant in D. Proof: u constant in D implies that $u_x = u_y = 0$ in D. Since f is analytic, the Cauchy-Riemann equations now imply that $v_x = v_y = 0$ as well. Thus $f' = u_x + iv_x = 0$ in D. By our theorem, f is constant in D.
• Similarly, if $f = u + iv$ is analytic in a domain D with v being constant, then f must be constant in D.
• Suppose next that $f = u + iv$ is analytic in a domain D with $|f|$ being constant in D. This too implies that f

itself must be constant.

Let $f = u + iv$ be analytic in a domain D, and suppose that $|f|$ is constant in D. Then $|f|^2$ is also constant, i.e. there exists $c \in \mathbb{C}$ such that

$$|f(z)|^2 = u^2(z) + v^2(z) = c \text{ for all } z \in D.$$

• If $c = 0$ then u and v must be equal to zero everywhere, and so f is equal to zero in D.
• If $c \neq 0$ then in fact $c > 0$. Taking the partial derivative with respect to x (and similarly with respect to y) of the above equation yields:

$$2uu_x + 2vv_x = 0 \text{ and } 2uu_y + 2vv_y = 0.$$

Substituting $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second equation gives

$$uu_x - vu_y = 0 \text{ and } uu_y + v_u x = 0.$$

Multiplying the first equation by u and the second equation by v, we find

$$u^2u_x - uvu_y = 0 \text{ and } uvu_y + v^2u_x = 0.$$

We now add these two equations to get

$$u^2u_x + v^2u_x = 0.$$

Since $u^2 + v^2 = c$, this last equation becomes

$$cu_x = 0.$$

But since $c > 0$, so it must be the case that $u_x = 0$ in D. We can similarly find that $u_y = 0$ in D and using the Cauchy-Riemann equations, we also obtain that $v_x = v_y = 0$ in D. Hence $f'(z) = 0$ in D, and our theorem yields that f is constant in D.

Let's look at a strange example: Let

$$f : \mathbb{C} \to \mathbb{C}, f(z) = e^{-\frac{1}{z^4}} \text{ for } z \neq 0, 0 \text{ for } z = 0$$

• One can find u, v, u_x, u_y, v_x, v_y that actually satisfy the Cauchy-Riemann equations in $\mathbb{C}$
• Clearly, f is analytic in $\mathbb{C} \setminus \{0\}$
• At the origin, one can show that $u_x(0) = u_y(0) = v_x(0) = v_y(0) = 0$
• However, f is not differentiable at the origin, the reason is because f isn't even continuous at the origin

Why isn't f continuous at the origin?

• Consider z approaching the origin along the real axis, i.e. $z = x + i \cdot 0 \to 0$. Then

$$f(z) = f(x) = e^{-\frac{1}{x^4}} \to 0.$$

• Next, consider z approaching the origin along the imaginary axis, i.e. $z = 0 + iy \to 0$. Then

$$f(z) = f(iy) = e^{-\frac{1}{i^4y^4}} \to 0.$$

• However, consider $z = re^{\frac{\pi}{4}} \to 0$. Then $z^4 = r^4e^{i\frac{\pi}{4}4} = -r^4$, so

$$f(z) = e^{-\frac{1}{-r^4}} = e^{\frac{1}{r^4}} \to \infty \neq f(0).$$

What happened? The functions u and v satisfy the Cauchy-Riemann equations, yet, f is not differentiable at the origin?

Theorem

Let $f = u + iv$ be defined on a domain $D \subset \mathbb{C}$. Then f is analytic in D if and only if $u(x, y)$ and $v(x, y)$ have continuous first partial derivatives on D that satisfy the Cauchy-Riemann equations.

In our strange example, even though the Cauchy-Riemann equations were satisfied, the partial derivatives were not continuous at 0 so the assumptions of the theorem are not satisfied.

§16 Inverse Functions of Analytic Functions (Week 4)

Motivation: Given $z \in \mathbb{C} \setminus \{0\}$, find $w \in \mathbb{C}$ such that $e^w = z$. To solve this problem, we will write $z = |z|e^{i\theta}$, then $e^w = |z|ei\theta$. Next, we write $w = u + iv$. Then $e^u e^{iv} = |z|e^{i\theta}$. Thus $e^u = |z|$ and $e^{iv} = e^{i\theta}$, so $u = \ln |z|$ and $v = \theta + 2k\pi = \arg(z)$.

Definition — For $z \neq 0$ we define

$$\text{Log}z = \ln |z| + i\text{Arg}(z), \text{ the principal branch of logarithm,}$$

and

$$\begin{aligned}\log z &= \ln |z| + i\arg(z), \text{a multi-valued function}\\ &= \text{Log}z + 2k\pi i, k \in \mathbb{Z}.\end{aligned}$$

$$f(z) = \text{Log}z = ln|z| + i\text{Arg}z$$

Some properties of this function are

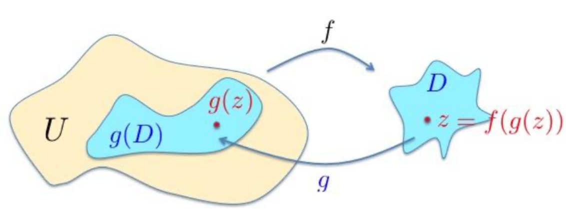
• $z \longrightarrow |z|$ is continuous in $\mathbb{C}$
• $z \longrightarrow \ln |z|$ is continuous in $\mathbb{C} \setminus \{0\}$
• $z \longrightarrow$ Arg$z$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$
• Thus, Log$z$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$
• However as $z \to -x \in (-\infty, 0)$ from above, Log$z \to \ln x + i\pi$ and as $z \to -x$ from below, Log$z \to \ln x - i\pi$, so Log$z$ is not continuous on $(-\infty, 0)$ and is not defined at 0

The principal branch of logarithm, Log$z$, is analytic in $\mathbb{C} \setminus (-\infty, 0]$. Also note that $\frac{d}{dz}\text{Log}z = \frac{1}{z}$.

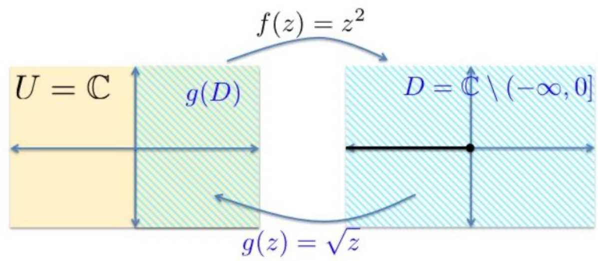
Theorem

Suppose that $f : U \to \mathbb{C}$ is an analytic function and there exists a continuous function $g : D \to U$ from some domain $D \subset \mathbb{C}$ into U such that $f(g(z)) = z$ for all $z \in D$. Then g is analytic in D, and

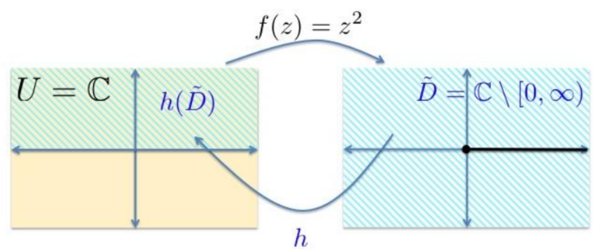
$$g'(z) = \frac{1}{f'g(z))} \text{ for } z \in D.$$



Let $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2$. Then $f'(z) = 2z$. Let $g : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}, g(z) = \sqrt{z}$ be the principal branch of the square root. Then $f(g(z)) = z$ for all $z \in D = \mathbb{C} \setminus (-\infty, 0]$, g is continuous in D which implies that g is analytic in D, and $g'(z) = \frac{1}{f'(g(z))} = \frac{1}{2g(z)} = \frac{1}{2\sqrt{z}}$. Again let $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2$. Then $f'(z) = 2z$. This



time let $h : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}, h(z) = \sqrt{z}$ when $\text{Im}(z) \geq 0$ and $-\sqrt{z}$ when $\text{Im}(z) < 0$. Then $f(h(z)) = z$ for all $z \in \tilde{D} = \mathbb{C} \setminus [0, \infty), h$ is continuous in $\tilde{D}$ which implies that h is analytic in $\tilde{D}$, and $h'(z) = \frac{1}{f'(h(z))} = \frac{1}{2h(z)}$. Let's



finish up by recalling some terminology. Let $f : U \to V$ be a function. Then

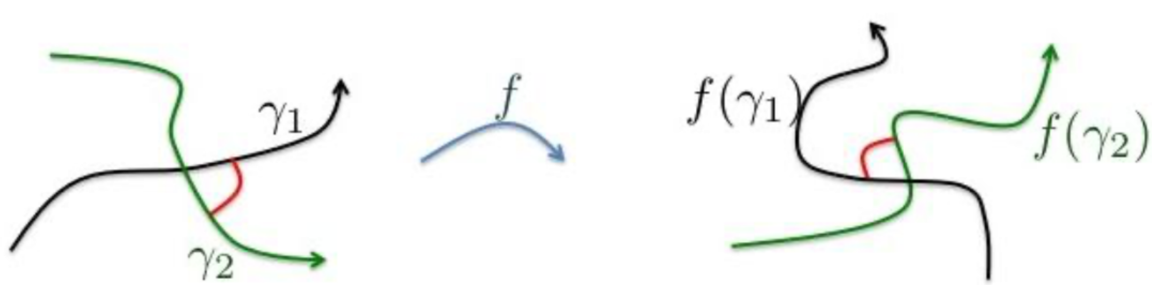
- f is *injective* (also called $1-1$) provided that $f(a) \neq f(b)$ whenever $a, b \in U$ with $a \neq b$
- f is *surjective* (also called onto) provided that for every $y \in V$ there exists an $x \in U$ such that $f(x) = y$
- f is a *bijection* (also called $1-1$ and onto) if f is both injective and surjective

Examples:

- $f : \{z \in \mathbb{C} |\ \text{Re}(z) > 0\} \to \mathbb{C} \setminus (-\infty, 0], f(z) = z^2$ is a bijection
- $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2$ is not injective but is surjective
- $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}, f(z) = \sqrt{z}$ is injective but not surjective

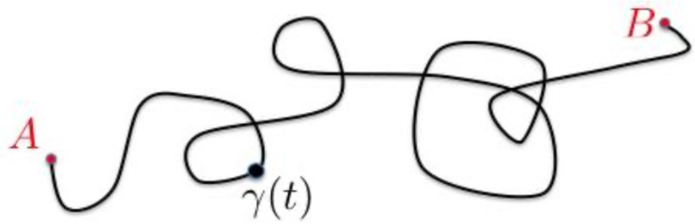
§17 Conformal Mappings

Intuitively, a conformal mapping is a "mapping that preserves angles between curves".



To make this precise, we need to define curves as well as angle between curves.

Definition — A *path* in the copmlex plane from a point A to a point B is a continuous function $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) = A$ and $\gamma(b) = B$.



Definition — A path $\gamma : [a, b] \to \mathbb{C}$ is *smooth* if the functions $x(t)$ and $y(t)$ in the representation $\gamma(t) = x(t) + iy(t)$ are smooth, that is, have as many derivatives as desired.

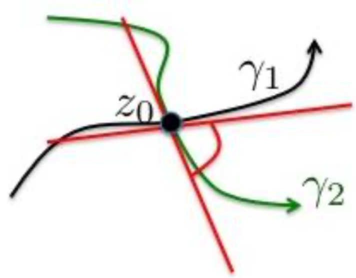
The term *curve* is typically used for a smooth or piecewise smooth path.

If $\gamma = x + iy : [a, b] \to \mathbb{C}$ is a smooth curve and $t_0 \in (a, b)$, then

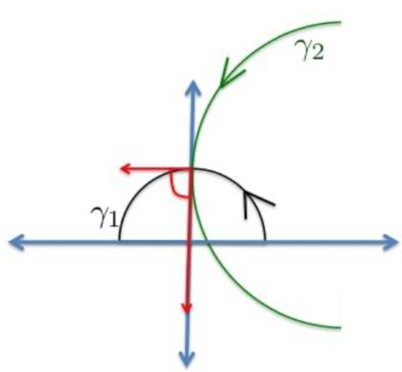
$$\gamma'(t_0) = x'(t_0) + iy'(t_0)$$

is a tangent vector to γ at $z_0 = \gamma(t_0)$.

Definition — Let γ_1 and γ_2 be two smooth curves, intersection at a point z_0. The *angle* between the two curves at z_0 is defined as the angle between the two tangent vectors at z_0.



Let $\gamma_1 : [0, \pi] \to \mathbb{C}, \gamma_1(t) = e^{it}$ and $\gamma_2 = [\frac{\pi}{2}, \frac{3\pi}{2}] \to \mathbb{C}, \gamma_2(t) = 2 + i + e^{it}$. Then $\gamma_1(\frac{\pi}{2}) = \gamma_2(\pi) = i$. Furthermore, $\gamma_1'(t) = ie^{it}, \gamma_1'(\frac{\pi}{2}) = -1$ and $\gamma_2'(t) = 2ie^{it}, \gamma_2'(\pi) = -2i$.



The angle between these curves at i is thus $\frac{\pi}{2}$.

Definition — A function is conformal if it preserves angles between curves. More precisely, a smooth complex-valued function g is *conformal at* z_0 if whenever γ_1 and γ_2 are two curves that intersect at z_0 with non-zero tangents, then $g \circ \gamma_1$ and $g \circ \gamma_2$ have non-zero tangents at $g(z_0)$ that intersect at the same angle.

A *conformal mapping* of a domain D onto V is a continuously differentiable mapping that is conformal at each point in D and maps D one-to-one onto V.

Theorem

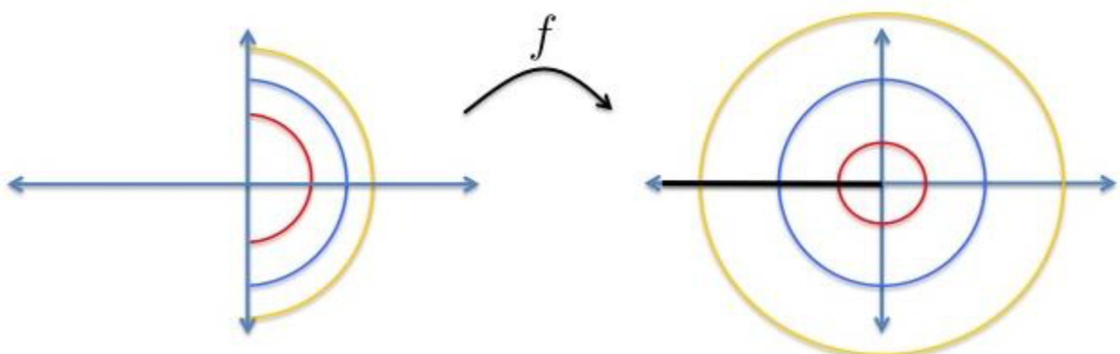
If $f : U \to \mathbb{C}$ is analytic and if $z_0 \in U$ such that $f'(z_0) \neq 0$, then f is conformal at z_0.

Reason: If $\gamma : [a, b] \to U$ is a curve in U with $\gamma(t_0) = z_0$ for some $t_0 \in (a, b)$, then

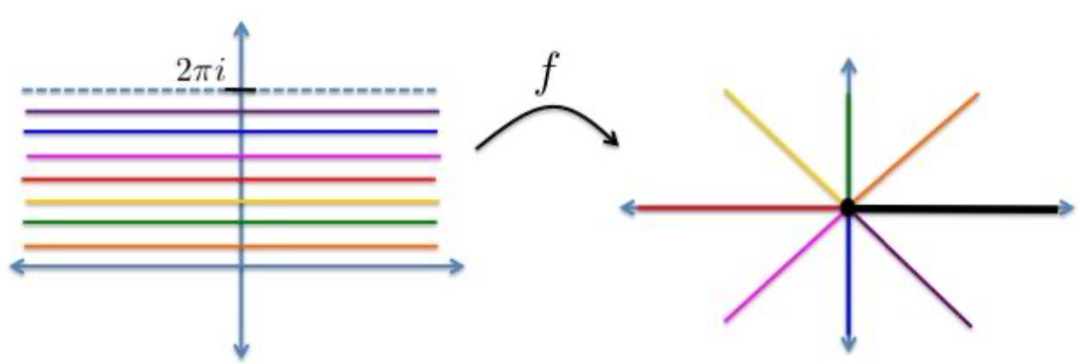
$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0)) \cdot \gamma'(t_0) = \underbrace{f'(z_0)}_{\in \mathbb{C} \setminus \{0\}} \cdot \gamma'(t_0).$$

Thus $(f \circ \gamma)'(t_0)$ is obtained from $\gamma'(t_0)$ via multiplication by $f'(z_0)$ (rotation sketching). If γ_1 and γ_2 are two curves in U through z_0 with tangent vectors $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$, respectively, via multiplication by $f'(z_0)$. The angle between them is thus preserved.

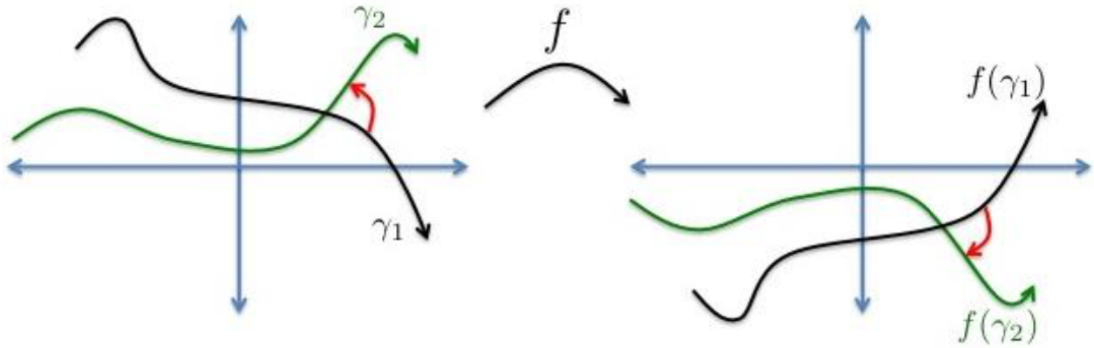
$f(z) = z^2$ maps $U = \{z \in \mathbb{C} | \operatorname{Re}(z) > 0\}$ conformally onto $\mathbb{C} \setminus (-\infty, 0]$.



$f(z) = e^z$ is conformal at each point in $\mathbb{C}$(f is analytic in $\mathbb{C}$ and $f'(z) \neq 0$ in $\mathbb{C}$). Since f is not one-to-one in $\mathbb{C}$, it is not a conformal mapping from $\mathbb{C}$ onto $\mathbb{C} \setminus \{0\}$. However, if you choose $D = \{z | 0 < \operatorname{Im}(z) < 2\pi\}$, then f maps D conformally onto $f(D) = \mathbb{C} \setminus [0, \infty)$.



$f(z) = \overline{z}$ is one-to-one and onto from $\mathbb{C}$ to $\mathbb{C}$, however, angles between curves are reversed in orientation.



Thus f is not conformal anywhere.

§18 Möbius Transformations, Part 1

Definition — A Möbius transformation (also called fractional linear transformation) is a function of the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

• As $z \to \infty$, $f(z) \to \frac{a}{c}$ if $c \neq 0$ and $f(z) \to \infty$ if $c = 0$. We thus allow $z = \infty$ and define $f(\infty) = \frac{a}{c}$ if $c \neq 0$ and $f(\infty) = \infty$ if $c = 0$.
• Similarly, $f(-\frac{d}{c}) = \infty$, if $c \neq 0$.
• We thus regard f as a mapping from the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ to the extended complex plane $\hat{\mathbb{C}}$.
• $f'(z) = \frac{(cz+d)a-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$. The condition $ad - bc \neq 0$ simply guarantees that f is non-constant.
• If we multiply each parameter a, b, c, d by a constant $k \neq 0$, we obtain the same mapping, so a given mapping does not uniquely determine a, b, c, d.
• A Möbius transformation is one-to-one and onto from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. To prove this, pick $w \in \hat{\mathbb{C}}$ and observe $f(z) = w$. Then $z = \frac{wd-b}{-wc+a}$. So for each $w \in \hat{\mathbb{C}}$, there is one and only one $z \in \hat{\mathbb{C}}$ such that $f(z) = w$.

Möbius transformations are thus conformal mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. In fact, Möbius transformations are the only conformal mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

Suppose $c = 0, d = 1$, then $f(z) = az + b (a \neq 0)$. These are also called affine transformations. They map ∞ to ∞ and therefore map $\mathbb{C}$ to $\mathbb{C}$. They are hence conformal mappings from $\mathbb{C}$ to $\mathbb{C}$ (and in fact, these are the only conformal mappings from $\mathbb{C}$ to $\mathbb{C}$.

• $f(z) = az$ (i.e. $b = 0$) is a rotation dilation
• $f(z) = z + b$ (i.e. $a = 1$) is a translation
• $a = 0, b = 1, c = 1, d = 0$, then $f(z) = \frac{1}{z}$ and is an inversion. If $z = re^{i\theta}$ then $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$. f interchanges outside and inside of the unit circle. A circle, centered at 0, is clearly mapped to a circle, centered at 0, of reciprocal radius. What about other circles?

What are the images of circles under f where $f(z) = \frac{1}{z}$?

Let $K = \{z : |z-3| = 1\}$ be the circle of radius 1, centered at 3. What is its image $f(K)$?

$$w \in f(K) \iff \frac{1}{w} \in K \iff |\frac{1}{w} - 3| = 1$$

$$\iff |1-3w|^2 = |w|^2$$

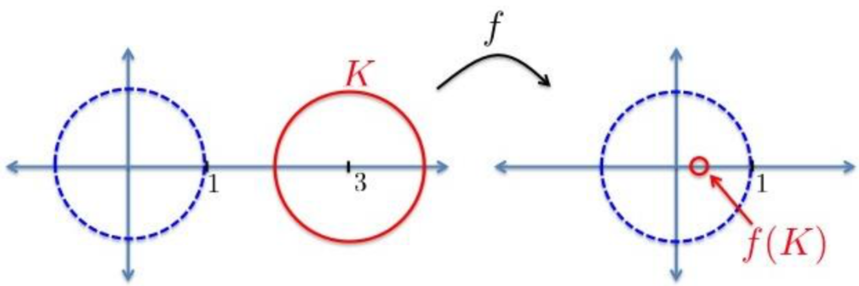
$$\iff 1 - 3w - 3\overline{w} + 9|w|^2 = |w|^2$$

$$\iff 8|w|^2 - 3w - 3\overline{w} = -1$$

$$\iff (w - \frac{3}{8})(\overline{w} - \frac{3}{8}) = \frac{9}{64} - \frac{1}{8}$$

$$\iff |w - \frac{3}{8}| = \frac{1}{8}$$

Thus the image of the circle $K = \{z : |z-3| = 1\}$ under $f(z) = \frac{1}{z}$ is another circle, namely the circle of radius $\frac{1}{8}$, centered at $\frac{3}{8}$.



Let now $K = \{z : |z-1| = 1\}$ be the circle of radius 1, centered at 1. What is $f(K)$?

$$w \in f(K) \longleftrightarrow |\frac{1}{w} - 1| = 1$$

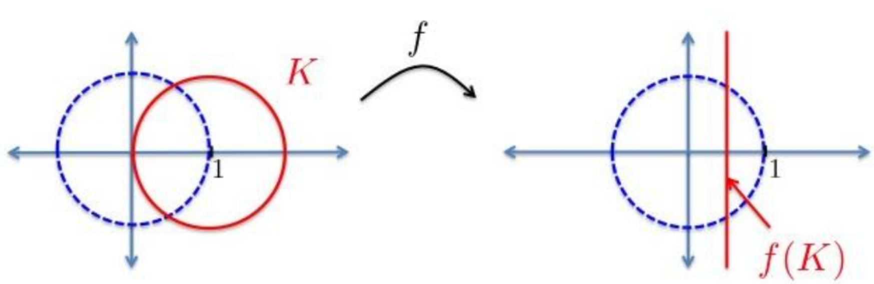
$$\longleftrightarrow |1-w|^2 = |w|^2$$

$$\longleftrightarrow 1 - w - \overline{w} + |w|^2 = |w|^2$$

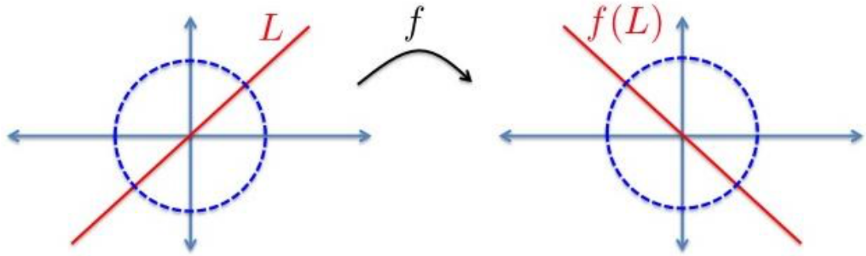
$$\longleftrightarrow w + \overline{w} = 1$$

$$\longleftrightarrow \ \mathrm{Re}(w) = \frac{1}{2}$$

Thus the image of the circle $K = \{z : |z-1| = 1\}$ is the vertical line $f(K) = \{w : \ \mathrm{Re}(w) = \frac{1}{2}\}$.



Since $f(f(z)) = f(\frac{1}{z}) = z$, we also find that f maps the line $\{z : \ \mathrm{Re}(Z) = \frac{1}{2}\}$ to the circle $\{z : |z-1| = 1\}$ and f maps the circle $\{z : |z - \frac{3}{8}| = \frac{1}{8}\}$ to the circle $\{z : |z-3| = 1\}$.



Let now L be the line $\{z : z = t + it, -\infty < t < \infty\}$. If $z = t + it$, then

$$f(z) = \frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{t - it}{2t^2} = \frac{1}{2t} - i\frac{1}{2t} = s - is.$$

Thus the image of the line $\{t + it : t \in \mathbb{R}\}$ is the line $\{s - is : s \in \mathbb{R}\}$.
Images of lines and circles seem to be lines or circles.

Fact: Every Möbius transformation maps circles and lines to circles or lines, a line could be viewed as a "circle through infinity".

Fact: Given three distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, there exists a unique Möbius transformation f such that $f(z_1) = 0, f(z_2) = 1$, and $f(z_3) = \infty$. You can actually write this Möbius transformation down:

$$f(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}.$$

§19 Möbius Transformations, Part 2

Recall that a Möbius transformation is a function $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Also f maps $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to $\mathbb{C}$, f is a conformal map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, if $c = 0, d = 1$: $f(z) = az + b$ is a conformal map from $\mathbb{C}$ to $\mathbb{C}$, and Möbius transformations map circles and lines to circles or lines. For distinct z_1, z_2, z_3, the Möbius transformation $f(z) = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$ maps z_1, z_2, z_3 to $0, 1, \infty$, respectively. The composition of two Möbius transformations is a Möbius transformation, and so is the inverse.

Given three distinct points z_1, z_2, z_3 and three distinct points w_1, w_2, w_3, there exists a unique Möbius transformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that maps z_j to w_j where $j = 1, 2, 3$.

Proof: Let f_1 be the Möbius transformation that maps z_1, z_2, z_3 to $0, 1, \infty$. Let f_2 be the Möbius transformation that maps w_1, w_2, w_3 to $0, 1, \infty$. Then $f_2^{-1} \circ f_1$ maps z_1, z_2, z_3 to w_1, w_2, w_3, respectively.

Find the Möbius transformation f that maps 0 to -1, i to 0, and ∞ to 1.

$$f_1(z) = \frac{z - 0}{\text{"}z - \infty\text{"}} \cdot \frac{\text{"}i - \infty}{i - 0} = \frac{z}{i}$$

maps $0, i, \infty$ to $0, 1, \infty$.

$$f_2(z) = \frac{z+1}{z-1} \cdot \frac{0-1}{0+1} = \frac{z+1}{-z+1}$$

maps $-1, 0, 1$ to $0, 1, \infty$. To find f_2^{-1}, solve for z:

$$w = \frac{z+1}{-z+1} \longleftrightarrow w(-z+1) = z + 1$$

$$\longleftrightarrow -wz + w = z + 1$$

$$\longleftrightarrow w - 1 = z(1 + w)$$

$$\longleftrightarrow z = \frac{w-1}{w+1}$$

So far $f_1(z) = \frac{z}{i}$ maps $0, i, \infty$ to $0, 1, \infty$ and $f_2^{-1}(w) = \frac{w-1}{w+1}$ maps $0, 1, \infty$ to $-1, 0, 1$. Thus $f = f_2^{-1} \circ f_1$ is the desired map,

$$(f_2^{-1} \circ f_1)(z) = \frac{f_1(z) - 1}{f_1(z) + 1} = \frac{\frac{z}{i} - 1}{\frac{z}{i} + 1} = \frac{z - i}{z + i}.$$

Lets look at another approach. f is of the form $f(z) = \frac{az+b}{cz+d}$. Since $f(i) = 0$, we have $a \neq 0$, we can thus assume that $a = 1$. Thus $f(z) = \frac{z+b}{cz+d}$. Since $f(i) = 0$, we have that $b = -i$. Thus $f(z) = \frac{z-i}{cz+d}$. Since $f(\infty) = 1$, we have $c = 1$, and since $f(0) = -1$, we have that $d = i$. Thus $f(z) = \frac{z-i}{z+i}$.

Fact: Every Möbius transformation is the composition of maps of the type

$$z \to az \text{ (rotation and dilation)}$$

$$z \to z + b \text{ (translation)}$$

$$z \to \frac{1}{z} \text{ (inversion)}$$

Proof: Let f be a Möbius transformation.

(1) Suppose first that $f(\infty) = \infty$. Then $f(z) = az + b$. This corresponds to the following composition:

$$z \underbrace{\to}_{\text{rot and dil}} az \underbrace{\to}_{\text{translation}} az + b.$$

(2) Suppose next that $f(\infty) \neq \infty$. Then

$$f(z) = \frac{az+b}{cz+d} \text{ with } c \neq 0$$

$$= \frac{\frac{a}{c}z + \frac{b}{c}}{z + \frac{d}{c}},$$

we can thus assume that $c = 1$. So

$$f(z) = \frac{az+b}{z+d} = \frac{a(z+d) + b - ad}{z+d} = a + \frac{b - ad}{z+d}.$$

This corresponds to the following composition:

$$z \underbrace{\to}_{\text{trans}} z + d \underbrace{\to}_{\text{inv}} \frac{1}{z+d} \underbrace{\to}_{\text{dil and rot}} \frac{b-ad}{z+d} \underbrace{\to}_{\text{trans}} a + \frac{b-ad}{z+d}.$$

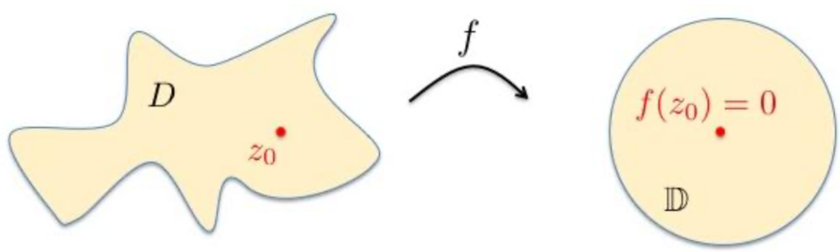
We mentioned earlier the fact that Möbius transformations map circles and lines to circles and lines. How would one prove this? By the previous composition result it suffices to prove this fact for the three standard types (translation, rotation and dilation, inversion). Clearly, dilations, rotations, and translations preserve circles and lines as circles and lines, so all that is left to show is that $f(z) = \frac{1}{z}$ maps circles and lines to circles and lines. We saw the main ideas on how to do so during the last section.

§20 The Riemann Mapping Theorem

So far we have seen that the conformal mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ are of the form $z \to \frac{az+b}{cz+d}$ and the conformal mappings from $\mathbb{C}$ to $\mathbb{C}$ are of the form to $z \to az+b$. It is also true that there is no conformal mapping $f : \mathbb{C} \to D$, where $D \subset \mathbb{C}, D \neq \mathbb{C}$ and there is no conformal mapping $f : \hat{\mathbb{C}} \to D$, where $D \subset \mathbb{C}$.

Question: What conformal mappings are there of the form $f : \mathbb{D} \to D$, where $\mathbb{D} = B_1(0)$ is the unit disk and $D \subset \mathbb{C}$?

Theorem (Riemann Mapping Theorem)
If D is a simply connected domain (open, connected, no holes) in the complex plane but not the entire complex plane, then there is a confmal map (analytic, one-to-one, onto) of D onto the open unit disk $\mathbb{D}$.

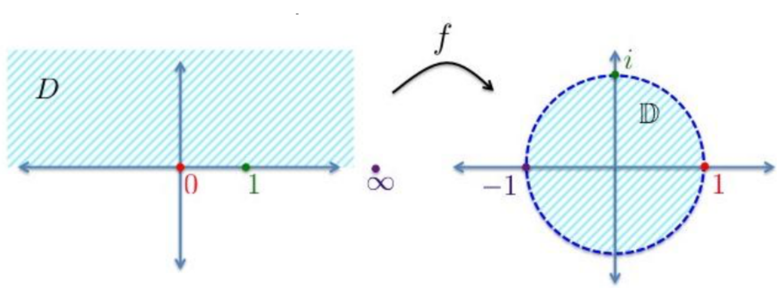


We say that "D is conformally equivalent to $\mathbb{D}$."

Let D be a simply connected domain. In order to find a *unique* conformal mapping f from D onto $\mathbb{D}$, we need to specify 3 real parameters. For example, specify a point $z_0 \in D$ that is to be mapped to 0 under f (2 real parameters x_0, y_0 and the argument of $f'(z_0)$ (1 real parameter), for example by requiring that $f'(z_0) > 0$.

We will now look at some examples.

Let D be the upper half plane, i.e. $D = \{z : \text{Im}(z) > 0\}$. Then D can be mapped to $\mathbb{D}$ via (the restriction of) a Möbius transformation. Let f be the Möbius transformation that maps $0, 1, \infty$ to $1, i, -1$,



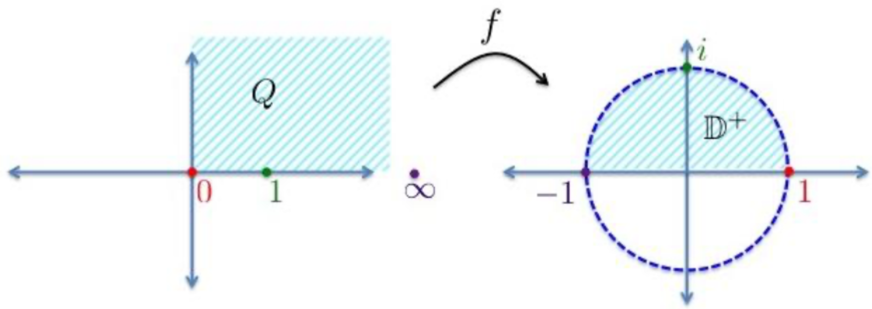
Then the line through $0, 1, \infty$ (the real axis) must be mapped to the circle through $1, i, -1$ (the unit circle). Furthermore, the domain to the left of the real axis (D) is then mapped to the domain to the left of the unit circle ($\mathbb{D}$), oriented by the ordering of the given points.

The restriction of the Möbius transformation f to the upper half plane D thus maps D onto $\mathbb{D}$. Can we find a formula for f, given that f maps $0, 1, \infty$ to $1, i, -1$?

f is of the form $f(z) = \frac{az+b}{cz+d}$. Since $f(\infty) \neq \infty$, we have $c \neq 0$ and can thus assume that $c = 1$. Thus

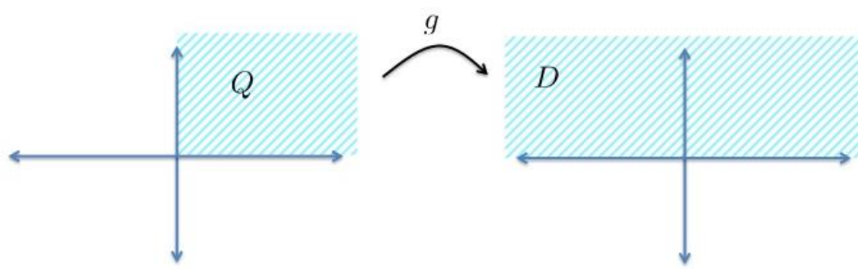
$f(z) = \frac{az+b}{z+d}$. Since $f(\infty) = -1$, we conclude that $a = -1$. Then $f(z) = \frac{-z+b}{z+d}$. Since $f(0) = 1$, we have $\frac{b}{d} = 1$, so $b = d$. So $f(z) = \frac{-z+b}{z+b}$. Since $f(1) = i$, we have $\frac{-1+b}{1+b} = i$, so $b = i$. Thus $f(z) = \frac{-z+i}{z+i}$ maps the upper half plane D conformally onto the unit disk $\mathbb{D}$.

Let Q be the first quadrant, i.e. the domain in the complex plane bounded by the positive real axis and the positive imaginary axis. Since the map f from the previous example maps 0 to 1, i to 0, and ∞ to -1, it maps the line through $0, i, \infty$ (the imaginary axis) to the line through $1, 0, -1$ (the real axis).



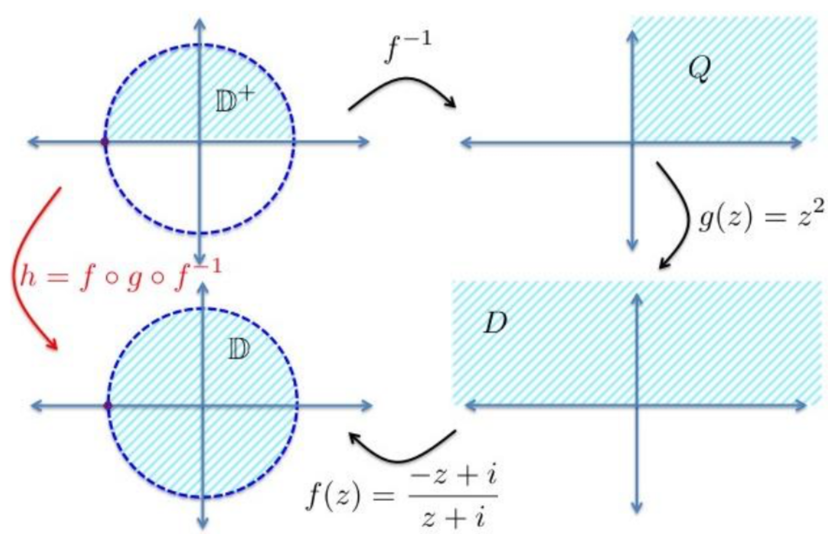
Hence the restriction of f to Q maps Q conformally onto the upper half of the unit disk, $\mathbb{D}^+$.

The map $g(z) = z^2$ is injective and analytic in the first quadrant Q.

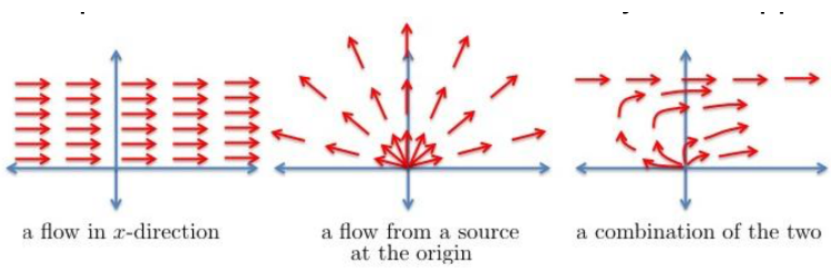


g maps Q conformally onto its image, namely the upper half plane D.

The previous three examples help us construct the Riemann map from $\mathbb{D}^+$ to $\mathbb{D}$



Many problems are easier to solve in the unit disk (or some other "nice" standard region) than in the region they are formulated in. Solutions can be found in the standard region, then transported back to the original region via a Riemann map. For example, fluid flow can be modeled nicely in the upper half plane:



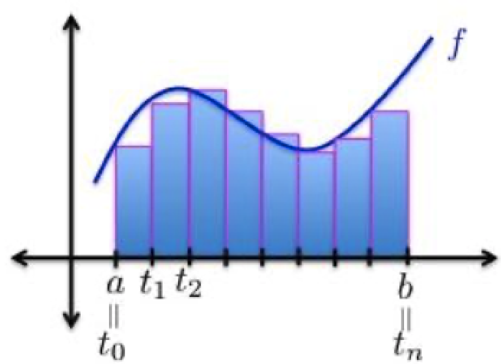
To understand a similar fluid flow in another region, map this flow from the upper half plane to the desired region using a Riemann map. Other examples include electrostatics, head conduction, aerodynamics, etc.

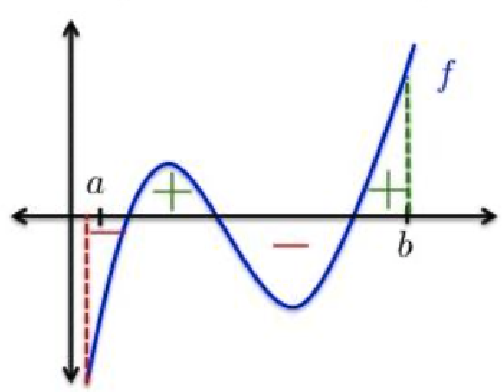
§21 Complex Integration (Week 5)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then

$$\int_a^b f(t)dt = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(t_j)(t_{j+1} - t_j),$$

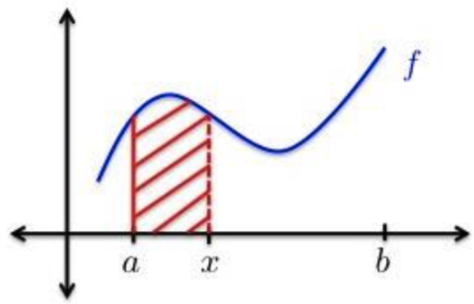
where $a = t_0 < t_1 < ... < t_n = b$. If $f \geq 0$ on $[a, b]$ then $\int_a^b f(t)dt$ is the "area under the curve". Otherwise $\int_a^b f(t)dt$ is the sum of the areas above the x axis minus the sum of the areas below the x axis.





Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous and define $F(x) = \int_a^x f(t)dt$. Then F is differentiable and $F'(x) = f(x)$ for $x \in [a, b]$.



Let $f : [a, b] \to \mathbb{R}$ as above. A function $F : [a, b] \to \mathbb{R}$ that satisfies $F'(x) = f(x)$ for all $x \in [a, b]$ is called an antiderivative of f. If F and G are both antideriavtives of the same function f, then

$$(G - F)'(x) = G'(x) - F'(x) = f(x) - f(x) = 0 \text{ for all } x \in [a, b],$$

so $G - F$ is constant.

Let G be any antiderivative of f. Then

$$\int_a^b f(t)dt = G(b) - G(a).$$

Instead of integrating over an interval $[a, b] \subset \mathbb{R}$, we are now in $\mathbb{C}$. So we will integrate over curves.

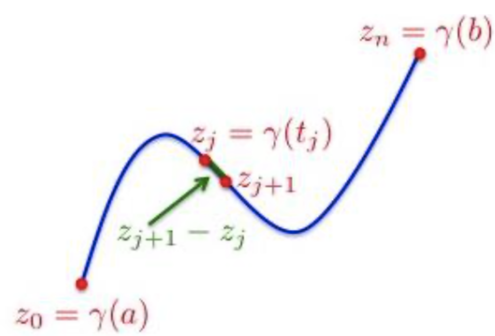
Recall that a curve is a smooth or piecewise smooth function

$$\gamma : [a, b] \to \mathbb{C}, \gamma(t) = x(t) + iy(t).$$

If f is complex-valued on γ, we define

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j),$$

where $z_j = \gamma(t_j)$ and $a = t_0 < t_1 < ... < t_n = b$.



One can show that if $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve and f is continuous on γ, then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Proof idea:

$$\sum_{j=0}^{n-1} f(z_j)(z_{j+1}-z_j) = \sum_{j=0}^{n-1} f(\gamma(t_j))\frac{\gamma(t_{j+1})-\gamma(t_j)}{t_{j+1}-t_j}(t_{j+1}-t_j) \to \int_a^b f(\gamma(t))\gamma'(t)dt \text{ as } n\to\infty.$$

Examples:

Evaluate $\int_{|z|=1}\frac{1}{z}dz$.

Let $\gamma(t)=e^{it}, 0\le t\le 2\pi$. Then $\gamma'(t)=ie^{it}$. Therefore we have

$$\int_{|z|=1}\frac{1}{z}dz = \int_0^{2\pi}\frac{1}{\gamma(t)}\gamma'(t)dt = \int_0^{2\pi}\frac{1}{e^{it}}ie^{it}dt = i\int_0^{2\pi}dt = 2\pi i.$$

Evaluate $\int_{|z|=1} zdz$.

Let $\gamma(t)=e^{it}, 0\le t\le 2\pi$. Then $\gamma'(t)=ie^{it}$. Therefore we have

$$\int_{|z|=1} zdz = \int_0^{2\pi}\gamma(t)\gamma'(t) = \int_0^{2\pi} e^{it}ie^{it}dt = i\int_0^{2\pi} e^{2it}dt = 0.$$

Evaluate $\int_{|z|=1}\frac{1}{z^2}dz$.

Let $\gamma(t)=e^{it}, 0\le t\le 2\pi, \gamma'(t)=ie^{it}$. Therefore we have

$$\int_{|z|=1}\frac{1}{z^2}dz = \int_0^{2\pi}\frac{1}{\gamma^2(t)}\gamma'(t)dt = \int_0^{2\pi}\frac{ie^{it}}{e^{2it}}dt = \int_0^{2\pi} ie^{-it}dt = 0.$$

In general we have

$$\int_{|z|=1} z^m dz = 2\pi i \text{ if } m=-1 \text{ and } 0 \text{ otherwise.}$$

§22 Complex Integration - Examples and First Facts

Recall that if $\gamma:[a,b]\to\mathbb{C}$ be a smooth curve, and let f be complex-valued and continuous on γ, then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Let $\gamma(t)=1-t(1-i), 0\le t\le 1$ and let $f(z)=\text{Re}(z)$. Then

$$\int_\gamma f(z)dz = \int_0^1 \text{Re}(1-t(1-i))(-1)(1-i)dt = (i-1)\int_0^1(1-t)dt = \frac{i-1}{2}.$$

Let $\gamma(t)=re^{it}, 0\le t\le 2\pi$. Then $\gamma'(t)=rie^{it}$. Let $f(z)=\overline{z}$. Then

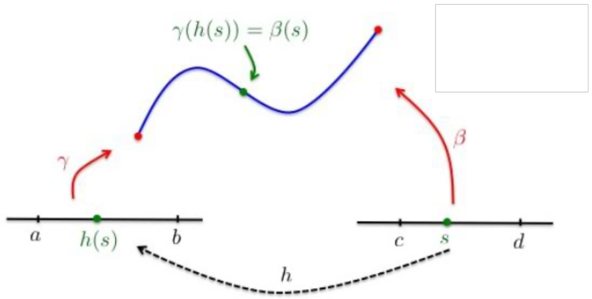
$$\int_\gamma f(z)dz = \int_\gamma \overline{z}dz = \int_0^{2\pi}\overline{\gamma(t)}\gamma'(t)dt = \int_0^{2\pi} re^{-it}rie^{i}tdt = r^2 i\int_0^{2\pi}dt = 2\pi i r^2 = (2i)\cdot \text{ area}(B_r(0)).$$

Let $[a,b]$ and $[c,d]$ be intervals in $\mathbb{R}$ and let $h:[c,d]\to[a,b]$ be smooth. Suppose that $f:[a,b]\to\mathbb{R}$ is a continuous function. Then

$$\int_{h(c)}^{h(d)} f(t)dt = \int_c^d f(h(s))h'(s)ds.$$

Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve and let $\beta : [c, d] \to \mathbb{C}$ be another smooth parametrization of the same curve given by $\beta(s) = \gamma(h(s))$ where $h : [c, d] \to [a, b]$ is a smooth bijection. Let f be a complex-valued function defined on γ. Then

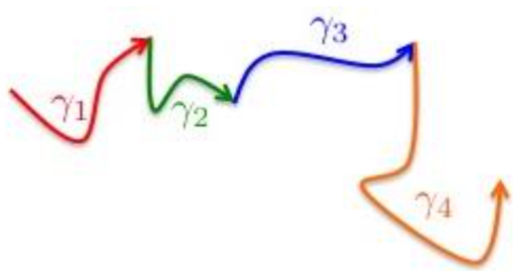
$$\int_\beta f(z)dz = \int_c^d f(\beta(s))\beta'(s)ds = \int_c^d f(\gamma(h(s)))\gamma'(h(s))h'(s)ds = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_\gamma f(z)dz.$$



Therefore, the complex path integral is independent of the parametrization.

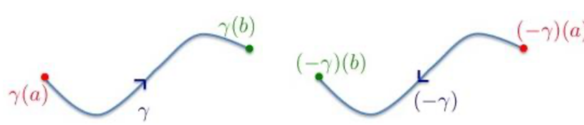
Let $\gamma = \gamma_1 + \gamma_2 + ... + \gamma_n$ be a piecewise smooth curve (i.e. γ_{j+1} starts where γ_j ends). Then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + ... + \int_{\gamma_n} f(z)dz.$$



If $\gamma : [a, b] \to \mathbb{C}$ is a curve, then a curve $(-\gamma) : [a, b] \to \mathbb{C}$ is defined by

$$(-\gamma)(t) = \gamma(a + b - t).$$



Note that $(-\gamma)'(t) = \gamma'(a + b - t)(-1)$. If f is continuous and complex-valued on γ, then

$$\int_{(-\gamma)} f(z)dz = \int_a^b f((-\gamma)(s))(-\gamma)'(s)ds = -\int_a^b f(\gamma(a + b - s))\gamma'(a + b - s)ds = \int_b^a f(\gamma(t))\gamma'(t)dt$$

$$= -\int_a^b f(\gamma(t))\gamma'(t)dt = -\int_\gamma f(z)dz.$$

Theorem

If γ is a curve, c a complex constant, and f, g are continuous and complex-valued on γ, then

$$\int_\gamma (f(z) + g(z))dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz$$

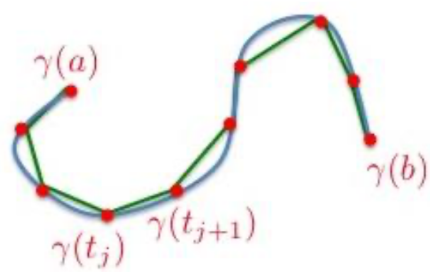
$$\int_\gamma c(f(z))dz = c\int_\gamma f(z)dz$$

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$$

Given a curve $\gamma : [a, b] \to \mathbb{C}$, how can we find its length?

Let $a = t_0 < t_1 < ... < t_n = b$. Then

$$\text{length}(\gamma) \approx \sum_{j=0}^{n} |\gamma(t_{j+1}) - \gamma(t_j)|.$$



If the limit exists as $n \to \infty$, then this is the length of γ.

$$\sum_{j=0}^{n} |\gamma(t_{j+1}) - \gamma(t_j)| = \sum_{j=0}^{n} \frac{|\gamma(t_{j+1}) - \gamma(t_j)|}{t_{j+1} - t_j}(t_{j+1} - t_j) \to \int_a^b |\gamma'(t)|dt.$$

Thus we have

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)|dt.$$

Example — Let $\gamma(t) = Re^{it}, 0 \le t \le 2\pi$, for some $R > 0$. Find length(γ).

So $\gamma'(t) = Rie^{it}$, and we have

$$\text{length}(\gamma) = \int_0^{2\pi} |Rie^{it}|dt = \int_0^{2\pi} Rdt = 2\pi R.$$

Example — Let $\gamma(t) = t + it, 0 \le t \le 1.$. Find length$(\gamma)$.

So $\gamma'(t) = 1 + i$, and we have

$$\text{length}(\gamma) = \int_0^1 |1 + i|dt = \int_0^1 \sqrt{2}dt = \sqrt{2}.$$

Definition — Let γ be a smooth curve and let f be a complex-valued and continuous function on γ. Then

$$\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|dt$$

is the integral of f over γ with respect to arc length.

Note that piecewise smooth curves are allowed as well. We just break up the integral into a sum over smooth pieces.

Theorem (ML Estimate)
If γ is a curve and f is continuous on γ, then

$$|\int_\gamma f(z)dz| \leq \int_\gamma |f(z)||dz|.$$

In particular, if $|f(z)| \leq M$ on γ, then

$$|\int_\gamma f(z)dz| \leq M \cdot \text{ length}(\gamma).$$

Let $\gamma(t) = t + it, 0 \leq t \leq 1$. We'd like to find an upper bound for $\int_\gamma z^2 dz$. We'll first use the second part of the theorem which states that

$$|\int_\gamma f(z)dz| \leq M \cdot \text{ length}(\gamma).$$

For this example, $f(z) = z^2$ and we have that $|f(z)| = |z|^2 \leq (\sqrt{2}^2 = 2$ on γ, so $M = 2$. Also recall that $\text{length}(\gamma) = \sqrt{2}$. Thus

$$|\int_\gamma z^2 dz| \leq 2\sqrt{2}.$$

We'd like to find a better estimate for $\int_\gamma z^2 dz$ using the first part of the theorem which states that

$$|\int_\gamma f(z)dz| \leq \int_\gamma |f(z)||dz|.$$

Thus

$$|\int_\gamma z^2 dz| \leq \int_\gamma |z|^2|dz| = \int_0^1 |\gamma(t)^2||\gamma'(t)|dt = \int_0^1 |t+it|^2\sqrt{2}dt = \int_0^1 2t^2\sqrt{2}dt = \frac{2\sqrt{2}}{3}.$$

§23 The Fundamental Theorem of Calculus for Analytic Functions

Definition — If $f : [a, b] \to \mathbb{R}$ is continuous and has an antiderivative $F : [a, b] \to \mathbb{R}$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

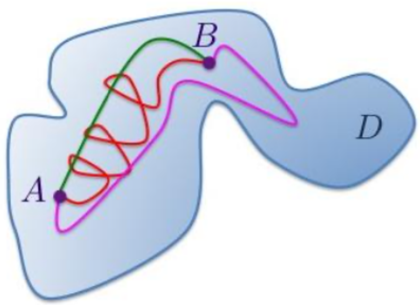
Is there a complex equivalent to this?

Definition — Let $D \subset \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be a continuous function. A primitive of f on D is an analytic function $F : D \to \mathbb{C}$ such that $F' = f$ on D.

Theorem

If f is continuous on a domain D and if f has a primitive F in D, then for any curve $\gamma : [a, b] \to D$, we have that

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(b)).$$



The integral only depends on the initial point and the terminal point of γ. A big "hidden" assumption is that f needs to have a primitive in D. Under what assumptions does f have a primitive?

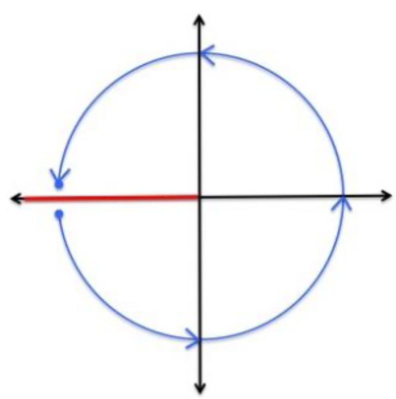
Let $\gamma : [a, b] \to \mathbb{C}$ be the line segment from 0 to $1 + i$. What is $\int_\gamma z^2 dz$?

The function $f(z) = z^2$ has a primitive in $\mathbb{C}$, namely $F(z) = \frac{z^3}{3}$. Therefore

$$\int_0^{1+i} z^2 dz = \int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) = F(1 + i) - F(0) = \frac{2}{3}(-1 + i).$$

Can we evaluate $\int_{|z|=1} \frac{1}{z} dz$ using a primitive?

The function $F(z) = \text{Log}z$ satisfies that $F'(z) = \frac{1}{z}$ but not in all of $\mathbb{C}$.



We know that $F = \text{Log}z$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$. Let $\tilde{\gamma} = [a, b] \to \mathbb{C}$ be the part of the unit circle started just below the negative x axis and to just above the negative x axis. Then

$$\int_\gamma \frac{1}{z} dz \approx \int_{\tilde{\gamma}} \frac{1}{z} dz = \text{ Log}(\tilde{\gamma}(b)) - \text{ Log}(\tilde{\gamma}(b)) \approx \pi i - (-\pi i) = 2\pi i.$$

Let γ be any curve in $\mathbb{C}$ from i to $\frac{i}{2}$. Then

$$\int_\gamma e^{\pi z} dz = \frac{1}{\pi} e^{\pi z}|_i^{\frac{i}{2}} = \frac{1}{\pi}(i+1).$$

Let γ be any path in $\mathbb{C}$ from $-\pi i$ to πi. Then

$$\int_\gamma \cos z dz = \sin z|_{-\pi i}^{\pi i} = \sin(\pi i) - \sin(-\pi i) = i(e^\pi - e^{-\pi}).$$

Theorem (Goursat)

Let D be a simply connected domain in $\mathbb{C}$ and let f be analytic in D. Then f has a primitive in D. Moreover, a primitive is given *explicitly* by picking $z_0 \in D$ and letting

$$F(z) = \int_{z_0}^{z} f(w) dw,$$

where the integral is taken over an arbitrary curve in D from z_0 to z.

One way to prove this theorem is as follows:

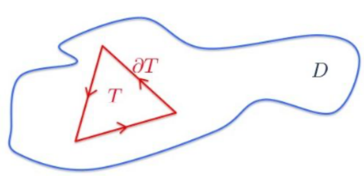
(1) First show Morera's theorem which states that if f is continuous on a simply connected domain D and if $\int_\gamma f(z)dz = 0$ for any triangular curve γ in D, then f has a primitive in D.
(2) Next, show the Cauchy theorem for triangles which states that for any triangle T that fits into D (including its boundary), then $\int_{\partial T} f(z)dz = 0$.

Theorem (Cauchy for Triangles)

Let D be an open set in $\mathbb{C}$ and let f be analytic in D. Let T be a triangle that fits into D (including its boundary), and let ∂T be its boundary, oriented positively. Then

$$\int_{\partial T} f(z) dz = 0.$$

Proof idea:



Subdivide the triangle into four equal-sized triangles. The integral of f over ∂T is the same as the sum of the four integrals over the boundaries of the smaller triangles. Use the $ML-$estimate and delicate balancing of boundary length of triangles and the fact that

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \epsilon(z - z_0)$$

for z near a point z_0 inside T.

Theorem (Morera)
If f is continuous on a simply connected domain D and if $\int_\gamma f(z)dz = 0$ for any triangular curve in D, then f has a primitive in D.

Proof idea:

(1) First, show Morera's theorem in a disk (the proof is not hard and resembles the proof of the real-valued fundamental theorem of calculus).
(2) Extending the result to the arbitrary simply connected domains is not that easy. This part of the proof requires the use of Cauchy's Theorem for simply connected domains. This theorem will be discussed in the next section.

§24 Cauchy's Theorem and Integral Formula

Theorem (Cauchy's Theorem for Simply Connected Domains)
Let D be a simply connected domain in $\mathbb{C}$ and let f be analytic in D. Let $\gamma : [a, b] \to D$ be a piecewise smooth, closed curve in D (i.e. $\gamma(b) = \gamma(a)$). Then

$$\int_\gamma f(z)dz = 0.$$

Example: $f(z) = e^{(z^3)}$ is analytic in $\mathbb{C}$ and $\mathbb{C}$ is simply connected. Therefore

$$\int_\gamma e^{(z^3)}dz = 0$$

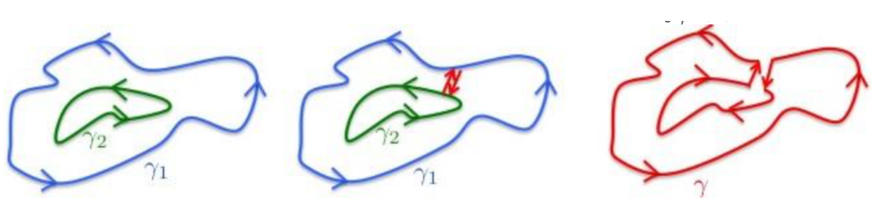
for any closed, piecewise smooth curve in $\mathbb{C}$.

Proof idea: Since D has no holes, γ can be deformed continuously to a point in D. Show that the integral does not change along the way by using the Cauchy Theorem in a disk.

Corollary
Let γ_1 and γ_2 be two simple closed curves (i.e. neither of the curves intersects itself), oriented counterclockwise, where γ_2 is inside γ_1. If f is analytic in a domain D that contains both curves as well as the region between them, then

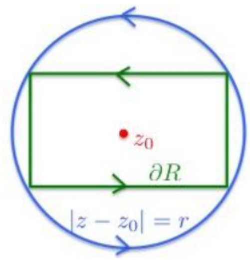
$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof idea: Here is a neat trick. Form a "joint curve" γ as in the picture below. As f is analytic in a simply connected region containing γ, we have $\int_\gamma f(z)dz = 0$.



Let R be a rectangle centered at z_0. Prove that $\int_{\partial R} \frac{1}{z - z_0} dz = 2\pi i$.

Proof: We calculate that $\int_{|z-z_0|=r} \frac{1}{z-z_0} dz = 2\pi i$. Since f is analytic "between" the two curves, the integrals must agree.



Example — Evaluate $\int_{|z|=1} \frac{1}{z^2+2z} dz$.

$$\frac{1}{z^2+2z} = \frac{1}{z(z+2)} = \frac{1}{2}\frac{(z+2)-z}{z(z+2)} = \frac{1}{2}(\frac{1}{z} - \frac{1}{z+2}).$$

Thus

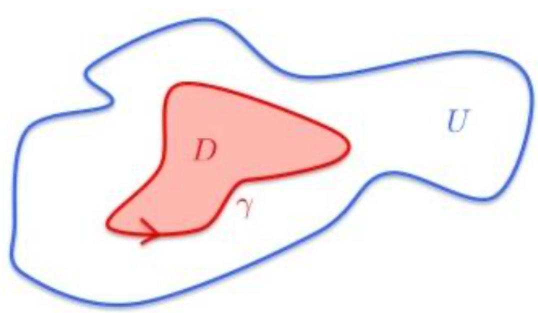
$$\int_{|z|=1} \frac{1}{z^2+2z} dz = \frac{1}{2}\left(\int_{|z|=1} \frac{1}{z} dz - \int_{|z|=1} \frac{1}{z+2} dz\right) = \frac{1}{2}(2\pi i - 0) = \pi i$$

since the function $z \to \frac{1}{z+2}$ is analytic in the simply connected domain $B_{1.5}(0)$, which in turn contains the curve we're integrating over.

Theorem (Cauchy Integral Formula)

Let D be a simply connected domain bounded by a piecewise smooth curve γ and let f be analytic in a set U that contains the closure of D (i.e. D and γ). Then

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz \text{ for all } w \in D.$$



The proof of the Cauchy Integral Formula goes as follows:

- Let $w \in D$ and pick $\epsilon > 0$ such that $\overline{B_\epsilon(w)} \subset D$.
- Using Cauchy's theorem, we see that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz = \frac{1}{2\pi i} \int_{\partial B_\epsilon(w)} \frac{f(z)}{z-w} dz,$$

since the integrand is analytic in a region containing these two curves and the area between them.
• It is easily seen that

$$\frac{1}{2\pi i}\int_{\partial B_\epsilon(w)}\frac{f(z)}{z-w}dz=\frac{1}{2\pi}\int_0^{2\pi}f(w+\epsilon e^{it})dt.$$

• This is true for any (small) $\epsilon>0$ and as $\epsilon\to 0$, the right hand side approaches $f(w)$.

Example — Evaluate $\int_{|z|=2}\frac{z^2}{z-1}$.

Here we have $f(z)=z^2$ and $w=1$, so

$$\frac{1}{2\pi i}\int_{|z|=2}\frac{f(z)}{z-1}dz=f(1)=1.$$

Hence

$$\int_{|z|=2}\frac{z^2}{z-1}dz=2\pi i.$$

Example — Evaluate $\int_{|z|=1}\frac{z^2}{z-2}dz$.

Our first thought is to let $f(z)=z^2$ and $w=2$, however $w=2$ is not inside the curve. Since the function $z\to\frac{z^2}{z-2}$ is analytic in $B_{1.5}(0)$, this implies (using Cauchy's theorem) that

$$\int_{|z|=1}\frac{z^2}{z-2}dz=0.$$

Example — Evaluate $\int_{|z|=1}\frac{\text{Log}(z+e)}{z}dz$.

Here we have $f(z)=\text{Log}(z+e)$ and $w=0$. The function f is analytic in $\{\text{Re}(z)>0\}$ which contains the curve we're integrating over along with its inside. Also w is inside the curve. Thus

$$\int_{|z|=1}\frac{\text{Log}(z+e)}{z}dz=2\pi i\text{Log}(0+e)=2\pi i.$$

Here is an amazing consequence of the Cauchy Integral Formula.

Theorem

If f is analytic in an open set U, then f' is also analytic in U.

Proof idea:

• We first use the Cauchy Integral Formula to show that for any $w\in U$, the derivative $f'(w)$ can be found via

$$'(w)=\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{(z-w)^2}dz,$$

where γ is the boundary of a small disk centered at w; small enough so that it fits into U.
• We next show that the right hand side defines an analytic function in w and therefore f' must be analytic.

Repeated application of the previous theorem shows that an analytic function has infinitely many derivatives. Continuing along the same lines as the previous proof yields the following extension of the Cauchy Integral Formula.

Theorem (Cauchy Integral Formula for Derivatives)
Let D be a simply connected domain bounded by a piecewise smooth curve γ and let f be analytic in a set U that contains the closure of D (i.e. D and γ). Then

$$f^{(k)}(w) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{k+1}}dz \text{ for all } w \in D, k \geq 0.$$

Here $f^{(k)}$ denotes the kth derivative of f.

Example — Evaluate $\int_{|z|=2\pi} \frac{z^2 \sin z}{(z-\pi)^3} dz$.

Here we have $f(z) = z^2 \sin z, w = \pi$, and $k = 2$. We thus need to find $f''(\pi)$. So

$$f'(z) = 2z \sin z + z^2 \cos z$$

$$f''(z) = 2\sin z + 2z\cos z + 2z\cos z - z^2 \sin z, \text{ so } f''(\pi) = -4\pi$$

Thus

$$\int_{|z|=2\pi} \frac{z^2 \sin z}{(z-\pi)^3} dz = \frac{2\pi i}{2!} f^{(2)}(\pi) = -4\pi^2 i.$$

Example — Evaluate $\int_{|z|=2} \frac{e^z}{(z+1)^2} dz$.

Here we have $f(z) = e^z, w = -1$, and $k = 1$. We thus need to find $f'(-1)$. So we have

$$f'(z) = e^z, \text{ so } f'(-1) = \frac{1}{e}.$$

Thus

$$\int_{|z|=2} \frac{e^z}{(z+1)^2} dz = \frac{2\pi i}{1!} f'(-1) = \frac{2\pi i}{e}.$$

§25 Consequences of Cauchy's Theorem and Integral Formula

From the last section, we had learned two theorems.

Theorem (Cauchy's Theorem for Simply Connected Domains)
Let D be a simply connected domain in $\mathbb{C}$ and let f be analytic in D. Let $\gamma : [a, b] \to D$ be a piecewise smooth, closed curve in D (i.e. $\gamma(b) = \gamma(a)$). Then

$$\int_\gamma f(z)dz = 0.$$

Theorem (Cauchy Integral Formula for Derivatives)
Let D be a simply connected domain bounded by a piecewise smooth curve γ and let f be analytic in a set U that contains the closure of D (i.e. D and γ). Then

$$f^{(k)}(w) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{k+1}}dz \text{ for all } w \in D, k \geq 0.$$

Here $f^{(k)}$ denotes the kth derivative of f.

In this section, we will use these theorems to prove other theorems.

Theorem (Cauchy's Estimate)
Suppose that f is analytic in an open set that contains $\overline{B_r(z_0)}$ and that $|f(z)| \leq m$ holds on $\partial B_r(z_0)$ for some constant m. Then for all $k \geq 0$ we have

$$|f^{(k)}(z_0)| \leq \frac{k!m}{r^k}.$$

Proof: By the Cauchy Integral Formula we have that

$$|f^{(k)}(z_0)| = \frac{k!}{2\pi}\left|\int_{|z-z_0|=r}\frac{f(z)}{(z-z_0)^{k+1}}dz\right| \leq \frac{k!}{2\pi}\int_{|z-z_0|=r}\frac{|f(z)|}{|z-z_0|^{k+1}}|dz| \leq \frac{k!m}{2\pi r^{k+1}}\cdot 2\pi r = \frac{k!m}{r^k}.$$

Theorem (Liouville)
Let f be analytic in the complex plane (thus f is an entire function). If f is bounded then f must be constant.

Proof: Suppose that $|f(z)| \leq m$ for all $z \in \mathbb{C}$. Pick $z_0 \in \mathbb{C}$. Since $\mathbb{C}$ contains $\overline{B_r(z_0)}$ for any $r > 0$, we obtain from Cauchy's estimate

$$|f'(z_0)| \leq \frac{m}{r}$$

for any $r > 0$. Letting $r \to \infty$, we find that $f'(z_0) = 0$. Since z_0 was arbitrary, $f'(z) = 0$ for all z implying that f is constant.

Example — Suppose that f is an entire function, $f = u + iv$, and suppose that $u(z) \leq 0$ for all $z \in \mathbb{C}$. Then f must be constant.

Proof: Consider the function $g(z) = e^{f(z)}$. Then g is an entire function as well. Furthermore

$$|g(z)| = e^{\text{Re}(f(z))} = e^{u(z)} \leq e^0 = 1.$$

Thus g is an entire and bounded function, so by Liouville's theorem, it implies that g is constant which implies that f is constant.

Theorem (Fundamental Theorem of Algebra)
Any polynomial $p(z) = a_0 + a_1 z + ... + a_n z^n$ (with $a_0, ..., a_n \in \mathbb{C}, n \geq 1$ and $a_n \neq 0$) has a zero in $\mathbb{C}$, i.e. there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof: Suppose to the contrary that there exists a polynomial p as in the theorem that has no zeroes. Then $f(z) = \frac{1}{p(z)}$ is an entire function. We wish to apply Liouville's theorem to f.

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right), \text{ so}$$

$$|p(z)| \geq |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_0|}{|z|^n} \right) \underbrace{\rightarrow}_{|z| \rightarrow \infty} \infty.$$

One consequence of the Fundamental Theorem of Algebra is that polynomials can be factored in $\mathbb{C}$ as

$$p(z) = a_n(z - z_1)(z - z_2) \ldots (z - z_n),$$

where $z_1, z_2, \ldots, z_n \in \mathbb{C}$ are the zeroes of p and are not necessarily distinct.

Another consequence of the Cauchy Integral Formula is the following powerful result.

Theorem (Maximum Principle)
Let f be analytic in a domain D and suppose that there exists a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Then f is constant in D.

Corollary (Consequence of Maximum Principle)
If $D \subset \mathbb{C}$ is a bounded domain and if $f : \overline{D} \rightarrow \mathbb{C}$ is continuous in $\overline{D}$ and analytic in D, then $|f|$ reaches its maximum on ∂D.

Example — Let $f(z) = z^2 - 2z$. What is $\max(|f(z)|)$ on the square $Q = \{z = x + iy : 0 \leq x, y \leq 1\}$?

Since f is analytic inside Q and continuous on Q, we know that the maximum of $|f|$ occurs on ∂Q.

• On $\gamma_1 : 0 \leq x \leq 1, y = 0$, so

$$|f(z)| = |f(x)| = |x^2 - 2x| = |x(x - 2)|.$$

The maximum on γ_1 occurs at $x = 1$, so $|f(z)| \leq |f(1)| = 1$ on γ_1.

• On $\gamma_2 : 0 \leq y \leq 1, x = 1$, so

$$|f(z)| = |f(1 + iy)| = |1 - y^2 + 2iy - 2 - 2iy| = |-1 - y^2| = y^2 + 1.$$

The maximum on γ_2 occurs at $y = 1$, so $|f(z)| \leq |f(1 + i)| = 2$ on γ_2.

• On γ_3, γ_4, one can similarly see that $|f(z)| \leq |f(i)| = |-1 - 2i| = \sqrt{5}$ on γ_3 and $\gamma 4$.

Thus $|f(z)| \leq |f(i)| = \sqrt{5}$ on Q.

§26 Infinite Series of Complex Numbers (Week 6)

Definition — An infinite series

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + ... + a_n + a_{n+1} + ...$$

(with $a_k \in \mathbb{C}$ converges to S if the sequence of partial sums $\{S_n\}$, given by

$$S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + ... + a_n$$

converges to S.

Example — Find when the series

$$\sum_{k=0}^{\infty} z^k, \text{ for some } z \in \mathbb{C}$$

converges and what is its closed form.

Note that

$$S_n = 1 + z + z^2 + ... + z^n$$
$$z \cdot S_n = z + z^2 + ... + z^n + z^{n+1}$$

Now we can subtract the 2nd equation from the first equation to get that

$$S_n - z \cdot S_n = 1 - z^{n+1}.$$

Hence $S_n = \frac{1-z^{n+1}}{1-z}$ for $z \neq 1$, and since $z^{n+1} \to 0$ as $n \to \infty$ as long as $|z| < 1$, we have that

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \text{ for } |z| < 1.$$

Theorem

If a series $\sum_{k=0}^{\infty} a_k$ converges then $a_k \to 0$ as $k \to \infty$.

Let's now analyze the real and imaginary parts of the equation

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

for $|z| < 1$.

Writing $z = re^{i\theta}$, we have that $z^k = r^k e^{ik\theta} = r^k \cos(k\theta) + ir^k \sin(k\theta)$. Thus

$$\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} r^k \cos(k\theta) + \sum_{k=0}^{\infty} r^k \sin(k\theta).$$

Furthermore

$$\frac{1}{1-z} = \frac{1}{1-re^{i\theta}} = \frac{1-re^{-i\theta}}{(1-re^{i\theta})(1-re^{-i\theta})} = \frac{1-r\cos\theta + ir\sin\theta}{1-re^{i\theta}-re^{-i\theta}+r^2} = \frac{1-r\cos\theta+ir\sin\theta}{1-2r\cos\theta+r^2}.$$

Thus we have

$$\sum_{k=0}^{\infty} r^k \cos(k\theta) = \frac{1-r\cos\theta}{1-2r\cos theta + r^2}$$

and

$$\sum_{k=0}^{\infty} r^k \sin(k\theta) = \frac{r\sin\theta}{1-2r\cos\theta+r^2}.$$

Consider the series

$$\sum_{k=1}^{\infty} \frac{i^k}{k}.$$

Does this series converge?

We note that

$$\sum_{k=1}^{\infty} \left|\frac{i^k}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$$

is the harmonic series, which is known to diverge. But does the series itself (without the absolute values) converge? Let's split it up into real and imaginary parts. We have

$$\sum_{k=1}^{\infty} \frac{i^k}{k} = \sum_{n=1} \frac{i^{2n}}{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{2n+1}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} + i\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

But since

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4}...$$

is the alternating harmonic series which converges, we know that the original series must also converge.

Definition — A series $\sum_{k=0}^{\infty} a_k$ converges absolutely if the series $\sum_{k=0}^{\infty} |a_k|$ converges.

$\sum_{k=0}^{\infty} z^k$ converges and converges absolutely for $|z| < 1$. $\sum_{k=1}^{\infty} \frac{i^k}{k}$ converges, but not absolutely.

Theorem

If $\sum_{k=0}^{\infty} a_k$ converges absolutely, then it also converges and

$$\left|\sum_{k=0}^{\infty} a_k\right| \leq \sum_{k=0}^{\infty} |a_k|.$$

If $|z| < 1$, then the series

$$\sum_{k=0}^{\infty} z^k$$

converges absolutely s,

$$\left|\sum_{k=0}^{\infty} z^k\right| \leq \sum_{k=0}^{\infty} |z|^k.$$

But the left hand side equals $\left|\frac{1}{1-z}\right|$ and the right hand side equals $\frac{1}{1-|z|}$ so we have

$$\left|\frac{1}{1-z}\right| \leq \frac{1}{1-|z|}.$$

§27 Power Series

Definition — A power series (also called Taylor series), centered at $z_0 \in \mathbb{C}$, is a series of the form

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k.$$

Example — $\sum\limits_{k=0}^{\infty} z^k$ is a power series with $a_k = 1, z_0 = 0$. It converges for $|z| < 1$.

Example — $\sum\limits_{k=0}^{\infty} \frac{(-1)^l}{2^k} z^{2k} = 1 - \frac{z^2}{4} + \frac{z^4}{4} - \frac{z^6}{8} \ldots = \sum\limits_{k=0}^{\infty} \left(\frac{-z^2}{2}\right)^k = \sum_{k=0}^{\infty} w^k$, where $w = \frac{-z^2}{2}$. This series converges when $|w| < 1$ and diverges when $|w| \geq 1$. Therefore, the original series converges when $|z| < \sqrt{2}$ and diverges when $|z| \geq \sqrt{2}$.

For what values of z does a power series converge?

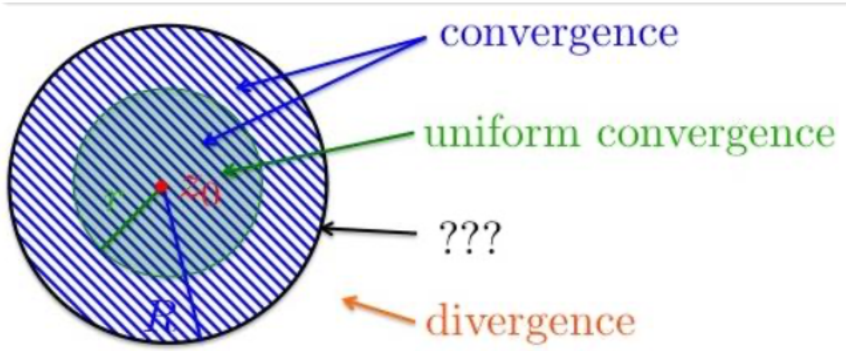
Theorem

Let $\sum\limits_{k=0}^{\infty} a_k(z-z_0)^k$ be a power series. Then there exists a number R with $0 \leq R \leq \infty$ such that the series converges absolutely in $\{|z-z_0| < R\}$ and diverges in $\{|z-z_0| > R\}$. Furthermore, the convergence is uniform in $\{|z-z_0| \leq r\}$ for each $r < R$.

We call R the *radius of convergence* of the power series.

Example — Find the radius of convergence of $\sum\limits_{k=0}^{\infty} k^k z^k$.

Pick any arbitrary $z \in \mathbb{C} \setminus \{0\}$. Observe that $|k^k z^k| = (k|z|)^k \geq 2^k$ as soon as $k \geq \frac{2}{|z|}$ thus the series does not converge for any $z \neq 0$. The radius of convergence of this power series is 0.



Example — Find the radius of converge of $\sum_{k=0}^{\infty} \frac{z^k}{k^k}$.

Pick any arbitrary $z \in \mathbb{C}$. Observe that $\left|\frac{z^k}{k^k}\right| = \left(\frac{|z|}{k}\right)^k \leq \left(\frac{1}{2}\right)^k$ as soon as $k \geq 2|z|$. Thus the series converges absolutely for all $z \in \mathbb{C}$ so our radius of converge is ∞.

Theorem

Suppose that $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ is a power series with radius of convergence $R > 0$. Then

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k \text{ is analytic in } \{|z-z_0| < R\}.$$

Furthermore, the series can be differentiated term by term, i.e.

$$f'(z) = \sum_{k=1}^{\infty} a_k \cdot k(z-z_0)^{k-1}, f''(z) = \sum_{k=2}^{\infty} a_k \cdot k(k-1)(z-z_0)^{k-2}, \ldots$$

In particular, $f^{(k)}(z_0) = a_k \cdot k!$, i.e. $a_k \frac{f^{(k)}(z_0)}{k!}$ for $k \geq 0$.

Recall that $\sum_{k=0}^{\infty} z^k$ has radius of convergence 1 and so by the theorem,

$$f(z) = \sum_{k=0}^{\infty} z^k \text{ is analytic in } \{|z| < 1\}.$$

Taking the derivative and differentiating term by term (as in the theorem), we find

$$f'(z) = \sum_{k=1}^{\infty} kz^{k-1} = \sum_{k=0}^{\infty} (k+1)z^k.$$

But we also know that $f(z) = \frac{1}{1-z}$ and so $f'(z) = \frac{1}{(1-z)^2}$. Thus

$$\sum_{k=0}^{\infty} (k+1)z^k = \frac{1}{(1-z)^2}.$$

Similarly power series can be integrated term by term.

Theorem

If $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ has radius of convergence R, then for any w with $|w-z_0|<R$, we have that

$$\int_{z_0}^{w} \sum_{k=0}^{\infty} a_k(z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \int_{z_0}^{w} (z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \frac{1}{k+1}(w-z_0)^{k+1}.$$

Here the integral is taken over any curve in the disk $\{|z-z_0|<R\}$ from z_0 to w.

Lets look at the power series $\sum_{k=0}^{\infty} z^k$ which has $R=1$. Then for any w with $|w|<1$, we thus have

$$\int_0^w \sum_{k=0}^{\infty} z^k dz = \sum_{k=0}^{\infty} \int_0^w z^k dz = \sum_{k=0}^{\infty} \frac{1}{k+1} w^{k+1} = \sum_{k=1}^{\infty} \frac{w^k}{k}.$$

We also know that $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ so

$$\int_0^w \sum_{k=0}^{\infty} z^k dz = \int_0^w \frac{1}{1-z} dz = -\text{Log}(1-z)|_0^w = -\text{Log}(1-w).$$

Here we used that $\text{Log}(z)$ is analytic in $\mathbb{C}\setminus(-\infty,0]$ so -$\text{Log}(1-z)$ is analytic in $\mathbb{C}\setminus[1,\infty)$, in particular in $\{|z|<1\}$, where it is a primitive of $\frac{1}{1-z}$.

We have shown that

$$\int_0^w \sum_{k=0}^{\infty} z^k dz = \sum_{k=1}^{\infty} \frac{w^k}{k} \text{ and } \int_0^w \sum_{k=0}^{\infty} z^k dz = -\text{Log}(1-w)$$

which implies that

$$\sum_{k=1}^{\infty} \frac{w^k}{k} = -\text{Log}(1-w) \text{ for } |w|<1.$$

Letting $z=1-w$, this gives us

$$\text{Log}(z) = -\sum_{k=1}^{\infty} \frac{(1-z)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(z-1)^k \text{ for } |z-1|<1.$$

Now how do we find the radius of convergence of a power series? We will discuss this in the next section.

§28 The Radius of Convergence of a Power Series

Recall that given a power series $\sum_{k=0}^{\infty}$, there exists a number R with $0 \le R \le \infty$ such that the series converges (absolutely) in $\{|z-z_0|<R\}$ and diverges in $\{|z-z_0|>R\}$. So how do we find R?

Theorem (Ratio Test)

If the sequence $\left\{ \left| \frac{a_k}{a_{k+1}} \right| \right\}$ has a limit as $k\to\infty$ then this limit is the radius of convergence, R, of the power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$.

Theorem (Root Test)

If the sequence $\left\{\sqrt[k]{|a_k|}\right\}$ has a limit as $k \to \infty$, then $R = \frac{1}{\lim_{k\to\infty}\left\{\sqrt[k]{|a_k|}\right\}}$.

For some series like $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^{2k}$, neither the ratio test nor the root test give us a limit. For that series we can let $w = z^2$ and then use the root test which gives us that the series converges for $|w| < 2$ or for $|z| < \sqrt{2}$. Is there a formula that finds this?

Theorem (Cauchy-Hadamard)

The radius of convergence of the power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ equals

$$R = \frac{1}{\lim_{k\to\infty}\ \sup\ \sqrt[k]{|a_k|}}.$$

Theorem

Let $f : U \to \mathbb{C}$ and let $\{|z - z_0| < r\} \subset U$. Then in this disk, f has a power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k, |z-z_0| < r,\ \text{where } a_k = \frac{f^{(k)}(z_0)}{k!}, k \geq 0.$$

The radius of convergence of this power series is $R \geq r$.

Example — Let $f(z) = e^z$. Find the power series for $f(z)$ about $z = 0$.

Since $f(z) = e^z$, we have $f^{(k)}(z) = e^z$. Letting $z_0 = 0$, we have $f^{(k)}(z_0) = e^0 = 1$ for all k. Thus $a_k = \frac{1}{k!}$ so we have

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \text{ for all } z \in \mathbb{C}.$$

Example — Let $f(z) = e^z$. Find the power series for $f(z)$ about $z = 1$.

Since $f(z) = e^z$, we have $f^{(k)}(z) = e^z$. Letting $z_0 = 1$, we have $f^{(k)}(z_0) = e^1 = e$ for all k. Thus $a_k = \frac{e}{k!}$ for all k so we have

$$e^z = \sum_{k=0}^{\infty} \frac{e}{k!}(z-1)^k \text{ for all } z \in \mathbb{C}.$$

Power series for trigonometric functions (about $z = 0$):

$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$

$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$ Note that the theorem implies that an analytic function is entirely determined in a disk by all of its derivatives $f^{(k)}(z_0)$ at the center z_0 of the disk.

Corollary

If f and g are analytic in $\{|z - z_0| < r\}$ and if $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all k, then $f(z) = g(z)$ for all z in $\{|z - z_0| < r\}$.

§29 The Riemann Zeta Function and the Riemann Hypothesis

The zeta function was first introduced by Leonhard Euler (1707 – 1783) who used it in the study of prime numbers. In particular, he used its properties to show that $\sum_{p \text{ prime}} \frac{1}{p}$ diverges. This shows that there are infinitely many primes but also shows some information about their distribution. Bernhard Riemann (1826 – 1866) used this function to obtain results on the asymptotic distribution of prime numbers. In this section, we will study the Riemann zeta function and in the next section, we will study its relation to prime numbers.

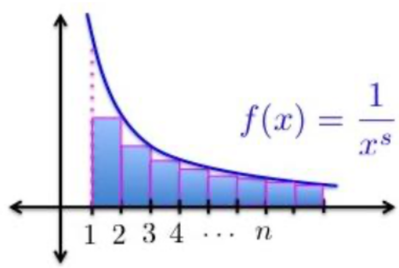
Recall that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

but the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \text{ converges for all } s > 1.$$

We can show this fairly easily.



Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \leq 1 + \int_1^{\infty} \frac{1}{x^s} dx = 1 + \frac{1}{1-s} \frac{1}{x^{s-1}} |_1^{\infty} = 1 - \frac{1}{1-s} = \frac{s}{s-1} (s > 1).$$

We will now consider $s \in \mathbb{C}$ instead of $s \in \mathbb{R}$.

Definition — For $z \in \mathbb{C}$ with $\text{Re}(s) > 1$, the zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is traditional to call the complex variable s instead of z. But what is n^s for $s \in \mathbb{C}$? Note that for real s, we have that $n^s = e^{\ln(n^s)} = e^{s \ln(n)}$, so we define

$$n^s = e^{s \ln(n)} \text{ for } s \in \mathbb{C}.$$

Does the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converge for $\text{Re}(s) > 1$?

Since $n^s = e^{s\ln(n)}$, we have that $|n^s| = |e^{s\ln(n)}| = e^{\mathrm{Re}(s)\ln(n)} = n^{\mathrm{Re}(s)}$. Thus

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{\mathrm{Re}(s)}},$$

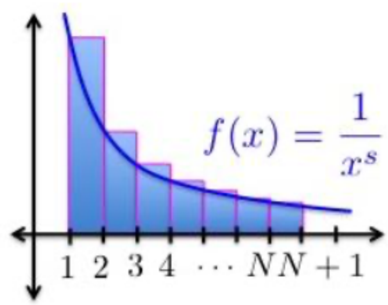
and since $\mathrm{Re}(s) > 1$, the series on the right converges. Thus $\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges absolutely in $\{\mathrm{Re}(s) > 1\}$.

In fact, the convergence is uniform in $\{\mathrm{Re}(s) \geq r\}$ for any $r > 1$ and this can be used to show that $\zeta(s)$ is analytic in $\{\mathrm{Re}(s) > 1\}$.

Theorem

The zeta function has analytic continuation into $\mathbb{C} \setminus \{1\}$ and this continuation satisfies that $\zeta(s) \to \infty$ as $s \to 1$.

It is slightly easier to construct an extension to the right half plane $\{\mathrm{Re}(s) > 0\}$ excluding the point 1. We outline the construction here. Motivation in $\mathbb{R}$:



$$\sum_{n=1}^{N}\frac{1}{n^s} = \int_1^{N+1}\frac{1}{x^s}dx + \sum_{n=1}^{N}\delta_n(s) \text{ where } \delta_n(s) = \frac{1}{n^s} - \int_n^{n+1}\frac{1}{x^s}dx$$

$$\underbrace{\sum_{n=1}^{N}\frac{1}{n^s}}_{\to\,\zeta(s)} = \underbrace{\int_1^{N+1}\frac{1}{x^s}dx}_{\to\,\frac{1}{s-1}} + \sum_{n=1}^{N}\delta_n(s) \text{ where } \delta_n(s) = \frac{1}{n^s} - \int_n^{n+1}\frac{1}{x^s}dx$$

Observe that $\sum_{n=1}^{N}\delta_n(s)$ is analytic in $\{\mathrm{Re}(s) > 0\}$. One can show that $\sum_{n=1}^{N}\delta_n(s)$ converges as $N \to \infty$ to an analytic function $H(s)$ in $\{\mathrm{Re}(s) > 0\}$. Thus

$$\zeta(s) = \frac{1}{s-1} + H(s) \text{ holds for } \mathrm{Re}(s) > 1 \text{ (*)}$$

where $H(s)$ is analytic in $\{\mathrm{Re}(s) > 0\}$.

We can therefore use (*) to define the zeta function in all of $\{\mathrm{Re}(s) > 0\} \setminus \{1\}$. This definition agrees with the original definition in $\{\mathrm{Re}(s) > 1\}$. Riemann was actually able to extend the zeta function to an analytic function in all of $\mathbb{C} \setminus \{1\}$.

Of much interest are the zeroes of the zeta function, i.e. those $s \in \mathbb{C}$ for which $\zeta(s) = 0$.

Theorem

The only zeroes of the zeta function outside of the strip $\{0 \le \text{ Re}(s) \le 1$ are at the negative even integers, i.e. $-2, -4, -6, ...$

The zeroes $-2, -4, -6, ...$ are often called the "trivial zeroes" and the region to be studied remains the strip $\{0 \le \text{ Re}(s) \le 1\}$. A key result is that the zeta function has no zeroes on the line $\{\text{Re}(s) = 1\}$. This is an essential fact in the proof of the prime number theorem. From the fact that the zeta function has no zeroes on $\{\text{Re}(s) = 0\}$, it can easily be deduced that it has no zeroes on $\{\text{Re}(s) = 0\}$ via a functional equation.

In his seminal paper in which he proved the analytic continuation of the zeta function to $\mathbb{C} \setminus \{1\}$, Riemann initiated important insights into the distribution of prime numbers.

Conjecture (Riemann Hypothesis)

In the strip $\{0 < \text{ Re}(s) < 1\}$, all zeroes of ζ are on the line $\{\text{Re}(s) = \frac{1}{2}\}$.

Much research has been done in attempts to prove this conjecture:

- $\zeta(s)$ has infinitely many zeroes in $\{0 < \text{ Re}(s) < 1\}$
- The asymptotic distribution of the zeroes of ζ in $\{0 < \text{ Re}(s) < 1\}$ is known
- At least one third of the zeroes in $\{0 < \text{ Re}(s) < 1\}$ lie on the critical line $\{\text{Re}(s) = \frac{1}{2}\}$.
- Trillions of zeroes of the zeta function have been calculated; so far all of them lie on the critical line
- The Riemann Hypothesis has strong implications on the distribution of prime numbers and on the growth of many other important arithmetic functions. It would greatly sharpen many number-theoretic results.

§30 The Prime Number Theorem

Let $\pi(x)$ = number of primes less than or equal to x. This function is called the prime counting function.

It seems impossible to find an explicit formula for $\pi(x)$. One thus studies the asymptotic behavior of $\pi(x)$ as x becomes very large.

Theorem (Prime Number Theorem)

$$\pi(x) \sim \frac{x}{\ln(x)} \text{ as } x \to \infty.$$

The symbol "$\sim$" means that the quotient of the two quantities approaches 1 as $x \to \infty$, i.e.

$$\frac{\pi(x)}{\frac{x}{\ln(x)}} \to 1 \text{ as } x \to \infty.$$

Euler discovered the connection between the zeta function $\zeta(s)$ (for real values of s) and the distribution of prime numbers. 60 years later, Legendre and Gauss conjectured the prime number theorem, after numerical calculations. Another 60 years later, Tchebychev showed that there are constants A, B (with $0 < A < B$) such that $A\frac{x}{\ln(x)} \le \pi(x) \le B\frac{x}{\ln(x)}$. In 1859, Riemann published his seminal paper "On the Number of Primes Less Than a Given Magnitude." In this paper, he constructed the analytic continuation of the zeta function and introduced revolutionary ideas, connection to its zeroes of the distribution of prime numbers. Hadamard and de la Vallée Poussin used these ideas and independently proved the Prime Number Theorem in 1896. The main step in their proofs is to establish that $\zeta(s)$ has no zeroes on the line $\{\mathrm{Re}(s) = 1\}$.

Euler discovered that

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}},$$

where the product is over all primes.

$$\begin{aligned}\zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + ... \\ &= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + ...\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + ...\right)\left(1 + \frac{1}{5^s} + \frac{1}{25^s} + ...\right)... \\ &= \prod_p \sum_{k=0}^{\infty} \frac{1}{p^{ks}} \\ &= \prod_p \frac{1}{1-p^{-s}}\end{aligned}$$

- This product formula shows that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$
- The key step in the proof of the prime number theorem is that ζ has no zeroes on $\{\mathrm{Re}(s) = 1\}$.
- The prime number theorem says that $\pi(x) \sim \frac{x}{\ln(x)}$ but it doesn't have any information about the difference $\pi(x) - \frac{x}{\ln(x)}$.
- The prime number theorem can also be written as $\pi(x) \sim \mathrm{Li}(x)$, where $\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)} dt$ is the (offset) logarithmic integral function.
- The proofs of the prime number theorem by Hadamard and de la Vallée Poussin actually show that $\pi(x) = \mathrm{Li}(x) +$ error term, where the error term grows to infinity at a controlled rate.
- Von Koch (in 1901) was able to give best possible bounds on the error term, assuming the Riemann Hypothesis is true. Schoenfeld (in 1976) made this precise and proved that the Riemann Hypothesis is equivalent to

$$|\pi(x) - \mathrm{li}(x)| < \frac{\sqrt{x}\ln(x)}{8\pi},$$

where $\mathrm{li}(x) = \int_0^x \frac{1}{\ln(t)} dt$ is the (un-offset) logarithmic integral function related to $\mathrm{Li}(x)$ via $\mathrm{Li}(x) = \mathrm{li}(x) - \mathrm{li}(2)$.

The veracity of the Riemann Hypothesis would therefore imply further results about the distribution of prime numbers, in particular, they'd be distributed beautifully regularly about there "expected" locations.

§31 Laurent Series (Week 7)

Recall that if $f : U \to \mathbb{C}$ is analytic and $\{|z - z_0| < R \subset U$ then f has a power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \text{ where } a_k = \frac{f^{(k)}(z_0)}{k!}, k \geq 0.$$

What happens if f is not differentiable at some point?

Theorem (Laurent Series Expansion)
If $f : U \to \mathbb{C}$ is analytic and $\{r < |z - z_0| < R\} \subset U$ then f has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

that converges at each point of the annulus and converges absolutely and uniformly in each sub annulus $\{s \leq |z - z_0| \leq t\}$, where $r < s < t < R$.

Note that the coefficients a_k are uniquely determined by f but how do we find them?

Example — Find the Laurent series of $f(z) = \frac{1}{(z-1)(z-2)}$ in the annulus $\{1 < |z| < 2\}$.

Note that the function $f(z) = \frac{1}{(z-1)(z-2)}$ is analytic in $\mathbb{C} \setminus \{1, 2\}$.

$$\begin{aligned}\frac{1}{(z-1)(z-2} &= \frac{(z-1)-(z-2)}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \\ &= \frac{-1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z(1-\frac{1}{z})} \\ &= \frac{-1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k - \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k}\end{aligned}$$

$$\sum_{k=0}^{\infty}\frac{-1}{2^{k+1}}z^k + \sum_{k=1}^{\infty}\frac{-1}{z^k} = \sum_{k=0}^{\infty}\frac{-1}{2^{k+1}}z^k + \sum_{k=-\infty}^{-1}(-1)z^k$$

Therefore we have

$$\frac{1}{(z-1)(z-2)} = \sum_{k=0}^{\infty}\frac{-1}{2^{k+1}}z^k + \sum_{k=-\infty}^{-1}(-1)z^k \text{ in } \{1 < |z| < 2\}.$$

What if we choose a different annulus?

Example — Find the Laurent series of $f(z) = \frac{1}{(z-1)(z-2)}$ in the annulus $\{2 < |z| < \infty\}$.

Note that the function $f(z) = \frac{1}{(z-1)(z-2)}$ is analytic in $\mathbb{C} \setminus \{1, 2\}$.

$$\begin{aligned}\frac{1}{(z-1)(z-2)} &= \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z(1-\frac{2}{z})} - \frac{1}{z(1-\frac{1}{z})} \\ &= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k - \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k\end{aligned}$$

$$\sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k} - \sum_{k=1}^{\infty}\frac{1}{z^k} = \sum_{k=-\infty}^{-1}(2^{-k-1} - 1)z^k$$

Therefore we have

$$\frac{1}{(z-1)(z-2)} = \sum_{k=-\infty}^{-1}(2^{-k-1} - 1)z^k \text{ in } \{2 < |z| < \infty\}.$$

Theorem

If f is analytic in $\{r < |z - z_0| < R\}$, then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

where

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)}{(z - z_0)^{k+1}} dz$$

for any s between r and R and all $k \in \mathbb{Z}$.

This does not seem all that useful for finding actual values of a_k, but it is useful to estimate a_k. We will use this when calculating integrals later.

§32 Isolated Singularities of Analytic Functions

Definition — A point z_0 is an *isolated singularity* of f if f is analytic in a punctured disk $\{0 < |z - z_0| < r\}$ centered at z_0.

- $f(z) = \frac{1}{z}$ has an isolated singularity at $z_0 = 0$
- $f(z) = \frac{1}{\sin(z)}$ has isolated singularities at $z_0 = 0, \pm\pi, \pm 2\pi, ...$
- $f(z) = \sqrt{z}$ and $f(z) = \mathrm{Log} z$ do not have isolated singularities at $z_0 = 0$ since these functions cannot be defined to be analytic in any punctured disk around 0
- $\frac{1}{z-2}$ has an isolated singularity at $z_0 = 2$

By Laurent's Theorem, if f has an isolated singularity at z_0 (so f is analytic in the annulus $\{0 < |z - z_0| < r\}$ for some $r > 0$) then f has a Laurent series expansion there

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$= \underbrace{... \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0}}_{\text{principal part}} + \underbrace{a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...}_{\text{analytic}}$$

Three fundamentally different things can happen that influence how f behaves near z_0.

Definition — Suppose z_0 is an isolated singularity of an analytic function f with Laurent series

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, 0 < |z - z_0| < r.$$

Then the singularity z_0 is

- *removable* if $a_k = 0$ for all $k < 0$.
- a *pole* if there exists $N > 0$ so that $a_{-N} \neq 0$ but $a_k = 0$ for all $k < -N$. The index N is the *order* of the pole.
- *essential* if $a_k \neq 0$ for infinitely many $k < 0$.

z_0 is a ...	Laurent series in $0 < \lvert z - z_0 \rvert < r$
Removable singularity	$a_0 + a_1(z - z_0) + \cdots$
Pole of order N	$\dfrac{a_{-N}}{(z-z_0)^N} + \cdots + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$
Simple pole	$\dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$
Essential singularity	$\cdots + \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$

Recall that z_0 is a removable singularity of f if its Laurent series, centered at z_0, satisfies that $a_k = 0$ for all $k < 0$.

Consider the series

$$f(z) = \frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} ..., 0 < |z| < \infty.$$

The Laurent series looks like a Taylor series. Note that Taylor series are analytic within their region of convergence. Thus if we define $f(z)$ to have the value 1 at $z_0 = 0$, then f becomes analytic in $\mathbb{C}$,

$$f(z) = \frac{\sin(z)}{z} \text{ if } z \neq 0, f(z) = 1 \text{ if } z = 0 \text{ is analytic in } \mathbb{C}.$$

We have removed the singularity.

Theorem (Riemann's Theorem)

Let z_0 be an isolated singularity of f. Then z_0 is a removable singularity if and only if f is bounded near z_0.

Theorem

Let z_0 be an isolated singularity of f. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Note that if $f(z)$ has a pole at z_0, then $\frac{1}{f(z)}$ has a removable singularity at z_0 and vice versa.

Recall that z_0 is an essential singularity of f if its Laurent series, centered at z_0, satisfies that $a_k \neq 0$ for infinitely many $k < 0$.

The series

$$f(z) = e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^k} = 1 + \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + ...$$

has an essential singularity at $z_0 = 0$. Note that if $z = x \in \mathbb{R}$, then $f(z) = e^{\frac{1}{x}}$ as $x \to 0$ from the right and $f(z) = e^{\frac{1}{x}} \to 0$ as $x \to 0$ from the left. Also if $z = ix \in i\mathbb{R}$, then $f(z) = e^{\frac{1}{ix}}$ lies on the unit circle for all x. It appears that f does not have a limit as $z \to z_0$.

Theorem (Casorati-Weierstraß)

Suppose that z_0 is an essential singularity of f. Then for every $w_0 \in \mathbb{C}$ there exists a sequence $\{z_n\}$ with $z_n \to z_0$ such that $f(z_n) \to w_0$ as $n \to \infty$.

Let $f(z) = e^{\frac{1}{z}}$. Then f has an essential singularity at 0. Let's pick a point $w_0 \in \mathbb{C}$, say, $w_0 = 1 + i\sqrt{3}$. By Casorati-Weierstraß, there must exist $z_n \in \mathbb{C} \setminus \{0\}$ such that $e^{\frac{1}{z_n}} \to 1 + i\sqrt{3}$ as $n \to \infty$. How do we find z_n? We can find z_n such that $e^{\frac{1}{z_n}} = 1 + i\sqrt{3}$, namely $\frac{1}{z_n} = \log(1 + i\sqrt{3})$. Recall that $\log(z) = \ln(|z|) + i\arg(z)$, so $\log(1 + i\sqrt{3}) = \ln(2) + i\frac{\pi}{3} + 2n\pi i$. Pick $z_n = \frac{1}{\ln(2) + i\frac{\pi}{3} + 2n\pi i}$. Then $z_n \to 0$ as $n \to \infty$. Furthermore

$$\begin{aligned} e^{\frac{1}{z_n}} &= e^{\ln(2) + i\frac{\pi}{3} + 2n\pi i} \\ &= 2e^{i\frac{\pi}{3}} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= 1 + i\sqrt{3} = w_0 \end{aligned}$$

for all n. We thus found z_n with $z_n \to 0$ such that $f(z_n) = w_0$ for all n.

We just observed a much stronger result that is true (but much harder to prove) for essential singularities.

Theorem (Picard)
Suppose that z_0 is an essential singularity of f. Then for every $w_0 \in \mathbb{C}$ with at most one exception, there exists a sequence $\{z_n\}$ with $z_n \to z_0$ such that $f(z_n) = w_0$.

Note that $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z_0 = 0$. Also we have that $f(z) \neq 0$ for all z so by Picard's theorem, for every $w_0 \neq 0$, there must exist infinitely many z_n with $z_n \to 0$ such that $f(z_n) = w_0$.

Pick $w_0 = 1$ for example. Then $f(z) = w_0$ if $e^{\frac{1}{z}} = 1$, that is $\frac{1}{z} = 2n\pi i$ for some $n \in \mathbb{Z}$. Then $z_n \to 0$ as $n \to \infty$ and $f(z_n) = 1$ for all n.

§33 The Residue Theorem

Recall that if f has an isolated singularity at z_0 if f is analytic in $\{0 < |z - z_0| < r\}$ for some $r > 0$. In that case, f has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, 0 < |z - z_0| < r.$$

Observe that if $0 < \rho < r$ then

$$\int_{|z-z_0|=\rho} f(z)dz = \sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=\rho} (z - z_0)^k dz.$$

But what is $\int_{|z-z_0|=\rho} (z - z_0)^k$?

• For $k \neq 1$, the function $h(z) = (z - z_0)^k$ has a primitive, namely $H(z) = \frac{1}{k+1}(z - z_0)^{k+1}$. Therefore, $\int_{|z-z_0|=\rho} (z - z_0)^k dz = 0$ for $k \neq 1$.
• For $k = -1$, the integral is $\int_{|z-z_0|=\rho} \frac{1}{z-z_0} dz$. We can use the Cauchy Integral Formula (or compute this directly) and find $\int_{|z-z_0|=\rho} (z - z_0)^k dz = 2\pi i$ for $k = -1$.

Hence

$$\int_{|z-z_0|=\rho} f(z)dz = \sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=\rho} (z - z_0)^k dz = 2\pi i a_{-1}.$$

Therefore, a_{-1} gets special attention.

Definition — If f has an isolated singularity at z_0 with Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k, 0 < |z-z_0| < r,$$

then the *residue* of f at z_0 is $\text{Res}(f, z_0) = a_{-1}$.

If we have

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \sum_{k=0}^{\infty}(-1)(z-1)^k \text{ in } 0 < |z-1| < 1,$$

then $\text{Res}(f, 1) = -1$.

If we have

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \sum_{n=0}^{\infty}(-1)^n(z-2)^n \text{ in } 0 < |z-2| < 1,$$

then $\text{Res}(f, 2) = 1$.

If we have

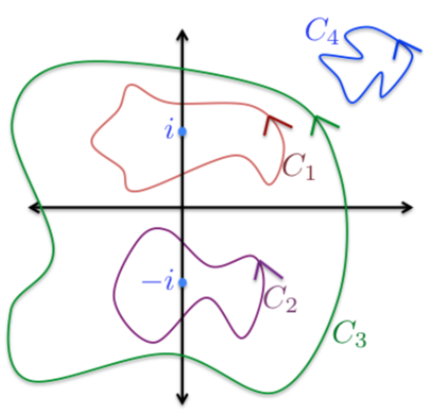
$$f(z) = \frac{\sin(z)}{z^4} = \frac{1}{z^3} - \frac{1}{3!}\cdot\frac{1}{z} + \frac{1}{5!}\cdot z - \frac{1}{7!}\cdot z^3 \ldots \text{ in } 0 < |z| < \infty,$$

then $\text{Res}(f, 0) = \frac{-1}{3!} = -\frac{1}{6}$.

Theorem (Residue Theorem)
Let D be a simply connected domain and let f be analytic in D, except for isolated singularities. Let C be a simple closed curve in D (oriented counterclockwise) and let $z_1, z_2, \ldots, z_n$ be those isolated singularities of f that lie inside of C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k).$$

Note that the function $f(z) = \frac{1}{z^2+1}$ is analytic in $D = \mathbb{C}$, except for isolated singularities at $z = \pm i$.



- $\int_{C_1} f(z)dz = 2\pi i \text{Res}(f, i) = 2\pi i(-\frac{1}{2}i) = \pi$
- $\int_{C_2} = 2\pi i \text{Res}(f, -i) = 2\pi i(\frac{1}{2}i) = -\pi$
- $\int_{C_3} f(z)dz = 2\pi i(\text{Res}(f, i) + \text{Res}(f, -i)) = 2\pi i(-\frac{1}{2}i + \frac{1}{2}i) = 0$
- $\int_{C_4} f(z)dz = 0$

§34 Finding Residues

Recall that f has an isolated singularity at z_0 if f is analytic in the punctured disk $\{0 < |z - z_0| < r\}$. In that case, f has a Laurent series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

in this punctured disk. The representation is unique. The residue of f at z_0 is $\operatorname{Res}(f, z_0) = a_{-1}$, the coefficient of the term $\frac{1}{z-z_0}$.

Recall that z_0 is a removable singularity if $a_k = 0$ for all $k < 0$. In particular $a_{-1} = 0$ in that case so we have $\operatorname{Res}(f, z_0) = 0$.

Recall that z_0 is a simple pole if $a_{-1} \neq 0$ and $a_k = 0$ for all $k \leq 1$. So

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

How do we isolate a_{-1}? Idea:

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + ...$$

so that

$$\operatorname{Res}(f, z_0) = a_{-1} = \lim_{z \to z_0} ((z - z_0)f(z)).$$

Example — Calculate $\operatorname{Res}(f(z), i)$ given that $f(z) = \frac{1}{z^2+1}$.

Note that $f(z) = \frac{1}{z^2+1}$ has a simple pole at $z_0 = i$ and another one at $-i$.

$$\begin{aligned}
\operatorname{Res}(\frac{1}{z^2+1}, i) &= \lim_{z \to i} \left((z - i) \frac{1}{z^2 + 1} \right) \\
&= \lim_{z \to i} \left(z - i) \frac{1}{(z - i)(z + i)} \right) \\
&= \lim_{z \to i} \frac{1}{z + i} = \frac{1}{2i} = -\frac{i}{2}.
\end{aligned}$$

Recall that z_0 is a double pole if $a_{-2} \neq 0$ and $a_k = 0$ for all $k \leq -3$. So

$$f(z) = \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 ....$$

How do we isolate a_{-1}? Idea:

$$(z - z_0)^2 f(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + ...,$$

so that

$$\frac{d}{dz} \left((z - z_0)^2 f(z) \right) = a_{-1} + 2a_0(z - z_0) + ...$$

Hence

$$\operatorname{Res}(f, z_0) = a_{-1} = \lim_{z \to z_0} \left((z - z_0)^2 f(z) \right).$$

Example — Calculate $\text{Res}(f(z), 1)$ given that $f(z) = \frac{1}{(z-1)^2(z-3)}$.

Note that $f(z) = \frac{1}{(z-1)^2(z-3)}$ has a double pole at $z_0 = 1$ and a simple pole at 3.

$$\begin{aligned}\text{Res}(\frac{1}{(z-1)^2(z-3)}, 1) &= \lim_{z\to 1} \frac{d}{dz}\left((z-1)^2 \frac{1}{(z-1)^2(z-3)}\right) \\ &= \lim_{z\to 1} \frac{d}{dz}\left(\frac{1}{(z-3)}\right) = \lim_{z\to 1} \frac{-1}{(z-3)^2} = -\frac{1}{4}.\end{aligned}$$

Recall that z_0 is a pole of order n if $a_{-n} \neq 0$ and $a_k = 0$ for all $k \leq -(n+1)$.

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + ... + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ...$$

Then

$$\text{Res}(f, z_0) = a_{-1} = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}\left((z-z_0)^n f(z)\right).$$

Corollary

If $f(z) = \frac{g(z)}{h(z)}$, where g and h are analytic near z_0, and h has a simple zero at z_0 then

$$\text{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}.$$

For example, let $f(z) = \frac{1}{(z-1)^2(z-3)}$ and choose $g(z) = \frac{1}{(z-1)^2}$ and $h(z) = (z-3)$. Then g and h are analytic near $z_0 = 3$ and h has a simple zero at $z_0 = 3$. Thus

$$\text{Res}(\frac{1}{(z-1)^2(z-3)}, 3) = \frac{g(3)}{h'(3)} = \frac{\frac{1}{(3-1)^2}}{1} = \frac{1}{4}.$$

§35 Evaluating Integrals via the Residue Theorem

Recall the residue theorem.

Theorem (Residue Theorem)

Let D be a simply connected domain and let f be analytic in D, except for isolated singularities. Let C be a simple closed curve in D (oriented counterclockwise) and let $z_1, z_2, ..., z_n$ be those isolated singularities of f that lie inside of C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k).$$

Example — Evaluate $\int_{|z|=1} e^{\frac{3}{z}} dz$.

Note that by the residue theorem we have

$$\int_{|z|=1} e^{\frac{3}{z}} dz = 2\pi i \text{Res}(f(z), 0), \text{ where } f(z) = e^{\frac{3}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{3}{z}\right)^k.$$

Thus since $\text{Res}(f(z), 0) = 3$, we have

$$\int_{|z|=1} e^{\frac{3}{z}} dz = 6\pi i.$$

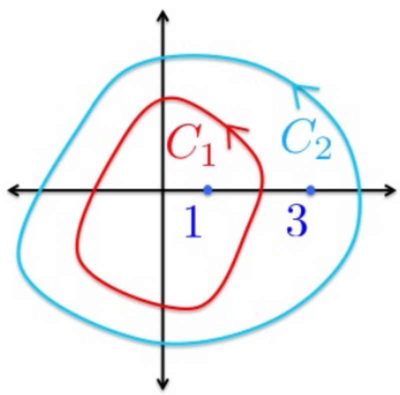
Example — Evaluate $\int_{|z|=2} \tan(z) dz$.

Note that by the residue theorem we have

$$\int_{|z|=2} \tan(z) dz = 2\pi i \left(\text{Res}(f(z), \frac{\pi}{2}) + \text{Res}(f(z), -\frac{\pi}{2})\right), \text{ where } f(z) = \tan(z) = \frac{\sin(z)}{\cos(z)}.$$

To find $\text{Res}(f(z), \frac{\pi}{2})$, note that $f(z) = \frac{g(z)}{h(z}$, where $g(z) = \sin(z)$ and $h(z) = \cos(z)$ are analytic near $\frac{\pi}{2}$ and $h(\frac{\pi}{2}) = 0$. Thus $\text{Res}(f(z), \frac{\pi}{2}) = \frac{g(\frac{\pi}{2})}{h'(\frac{pi}{2})} = \frac{\sin(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = -1$. Similarly $\text{Res}(f(z), -\frac{\pi}{2}) = \frac{g(-\frac{\pi}{2})}{h'(-\frac{pi}{2})} = \frac{\sin(-\frac{\pi}{2})}{-\sin(-\frac{\pi}{2})} = -1$. Thus we have

$$\int_{|z|=2} \tan(z) dz = 2\pi i(-1 - 1) = -4\pi i.$$



Example — Evaluate $\int_{C_1} \frac{1}{(z-1)^2(z-3)} dz$.

Note that by the residue theorem we have

$$\int_{C_1} \frac{1}{(z-1)^2(z-3)} dz = 2\pi i \text{Res}(f(z), 1), \text{ where } f(z) = \frac{1}{(z-1)^2(z-3)}.$$

Now we can calculate the residue

$$\text{Res}(f(z), 1) = \lim_{z\to 1} \frac{d}{dz}\left(\frac{(z-1)^2}{(z-1)^2(z-3)}\right)$$

$$= \lim_{z\to 1} \frac{d}{dz} \frac{1}{z-3} = \lim_{z\to 1} \frac{-1}{(z-3)^2} = -\frac{1}{4}.$$

Thus we have

$$\int_{C_1} \frac{1}{(z-1)^2(z-3)} dz = -\frac{1}{4} 2\pi i = -\frac{\pi i}{2}.$$

Example — Evaluate $\int_{C_2} \frac{1}{(z-1)^2(z-3)} dz$.

Note that by the residue theorem we have

$$\int_{C_2} \frac{1}{(z-1)^2(z-3)} dz = 2\pi i(\text{Res}(f(z), 1) + \text{ Res}(f(z), 3)), \text{ where } f(z) = \frac{1}{(z-1)^2(z-3)}.$$

Now we can calculate the residue

$$\text{Res}(f(z), 3) = \lim_{z \to 3} \left(\frac{z-3}{(z-1)^2(z-3)} \right) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{4}.$$

Therefore we have

$$\int_{C_2} \frac{1}{(z-1)^2(z-3)} dz = 2\pi i(\text{Res}(f(z), 1) + \text{ Res}(f(z), 3)) = 2\pi i(-\frac{1}{4} + \frac{1}{4}) = 0.$$

The Residue theorem can also be used to evaluate real integrals, for example the ones below in the following forms:

- $\int_0^{2\pi} R(\cos(t), \sin(t)) dt$, where $R(x, y)$ is a rational function of the real variables x and y.
- $\int_{-\infty}^{\infty} f(x) dx$, where f is a rational function of x.
- $\int_{-\infty}^{\infty} f(x) \cos(\alpha x) dx$, where f is a rational function of x.
- $\int_{-\infty}^{\infty} f(x) \sin(\alpha x) dx$, where f is a rational function of x.

In the next section, we will demonstrate an example of the last type.

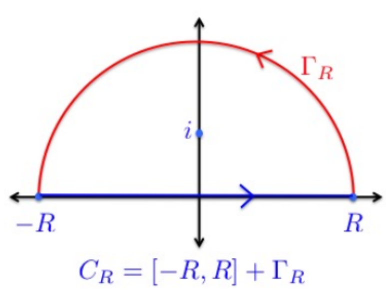
§36 Evaluating an Improper Integral via the Residue Theorem (Bonus)

Goal. Evaluate the following integral,

$$\int_0^{\infty} \frac{\cos(x)}{1+x^2} dx.$$

Note that $\int_0^{\infty} f(z) dz$ means $\lim_{R \to \infty} \int_0^R f(x) dz$, so we need to consider $\int_0^R \frac{\cos(x)}{1+x^2} dx$ and then let $R \to \infty$. Idea:

$$\int_0^R \frac{\cos(x)}{1+x^2} dx = \frac{1}{2} \int_{-R}^R \frac{\cos(x)}{1+x^2} dx = \frac{1}{2} \int_{-R}^R \frac{\cos(x) + i \sin(x)}{1+x^2} dx = \frac{1}{2} \int_{-R}^R \frac{e^{ix}}{1+x^2} dx.$$



Therefore we can do

$$\frac{1}{2} \int_{-R}^R \frac{e^{ix}}{1+x^2} dx = \frac{1}{2} \int_{[-R,R]} \frac{e^{iz}}{1+z^2} dz$$

$$\begin{aligned}
&= \frac{1}{2}\int_{C_R} \frac{e^{iz}}{1+z^2}dz - \frac{1}{2}\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz \\
&= \frac{1}{2}\cdot 2\pi i \text{Res}\left(\frac{e^{iz}}{1+z^2}, i\right) - \frac{1}{2}\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz.
\end{aligned}$$

We thus need to find the residue of $f(z) = \frac{e^{iz}}{1+z^2}$ at $z_0 = i$ and estimate the integral over Γ_R.

Finding the residue of $f(z) = \frac{e^{iz}}{1+z^2}$ at $z_0 = i$:

- f has a simple pole at $z = i$.
- Thus $\text{Res}(f(z), i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{(z-i)e^{iz}}{1+z^2} = \lim_{z\to i} \frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}$.
- Hence $\frac{1}{2}\int_{C_R} \frac{e^{iz}}{1+z^2}dz = \frac{1}{2}2\pi i \frac{1}{2ie} = \frac{\pi}{2e}$.

Estimating $\frac{1}{2}\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz$:

- We are only interested in what happens as $R \to \infty$.
- We want to show that $\frac{1}{2}\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz \to 0$ as $R \to \infty$.
- Therefore it suffices to show that $\left|\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz\right| \leq \text{ const}(R)$, where the constant goes to zero as $R \to \infty$.
- Recall that $\left|\int_{\Gamma_R} f(z)dz\right| \leq \text{ length}(\Gamma_R)\cdot \max_{z\in\Gamma_R}|f(z)|$.
- Thus $\left|\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz\right| \leq \text{ length}(\Gamma_R)\cdot \max_{z\in\Gamma_R}\left|\frac{e^{iz}}{1+z^2}\right|$.
- $\left|\frac{e^{iz}}{1+z^2}\right| = \frac{e^{\text{Re}(iz)}}{|1+z^2|} = \frac{e^{-y}}{|1+z^2|} \leq \frac{e^{-y}}{R^2-1} \leq \frac{1}{R^2-1}$ for $z \in \Gamma_R$, since $|z| = R$ and $y \geq 0$ on Γ_R.
- So $\left|\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz\right| \leq \pi R \cdot \frac{1}{R^2-1} \to 0$ as $R \to \infty$.
- Thus $\int_{\Gamma_R} \frac{e^{iz}}{1+z^2}dz \to 0$ as $R \to \infty$.

Here is what we have:

- $\int_0^\infty \frac{\cos(x)}{1+x^2}dx = \lim_{R\to\infty}\int_0^R \frac{\cos(x)}{1+x^2}dx$.
- $\int_0^R \frac{\cos(x)}{1+x^2}dx = \frac{1}{2}\int_{C_R}\frac{e^{iz}}{1+z^2}dz - \frac{1}{2}\int_{\Gamma_R}\frac{e^{iz}}{1+z^2}dz$.
- $\frac{1}{2}\int_{C_R}\frac{e^{iz}}{1+z^2}dz = \frac{1}{2}\cdot 2\pi i\text{Res}\left(\frac{e^{iz}}{1+z^2}, i\right) = \frac{\pi}{2e}$.
- $\int_{\Gamma_R}\frac{e^{iz}}{1+z^2}dz \to 0$ as $R \to \infty$.

Hence we have

$$\int_0^\infty \frac{\cos(x)}{1+x^2}dx = \frac{\pi}{2e}.$$