Ordinary Differential Equations Notes

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§1 Week 1

§1.1 What are Differential Equations?

Mathematical modeling of physical phenomena often produces an equation which involves ordinary or partial derivatives of some unknown function. Such an equation is called a differential equation.

Definition — A differential equation involving only ordinary derivatives with respect to a single independent variable is called an ordinary differential equation. A differential equation involving partial derivatives with respect to more than one independent variable is called a partial differential equation.

Example — Newton's second law of motion applied to a free following body leads to an ordinary differential equation

$$m\frac{d^2h}{dt^2} = m\frac{dv}{dt} = -mg,$$

where m is the mass of the object, g is the gravitational acceleration, h is the height of the object, and $v = \frac{dh}{dt}$ is the velocity.

Example — Modeling of vibrating strings, under some ideal conditions, leads to a partial differential equation called the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0,$$

where t is the time, x is the location along the string, c is the wave speed, and $u = u(x,t)$ is the displacement of the string.

§1.2 Order and Normal Form

The order of a differential equation is the order of the highest derivative in the equation. For example,

$$e^x y'' - 3(y')^2 + 2xy = xe^x$$

is a second order ordinary differential equation and the wave equation in the previous section is a second order partial derivative equation.

The most general nth order ordinary differential equation may be written as

$$F(x, y, y', ..., y^{(n)}) = 0,$$

where F is a real-valued function of $n+2$ variables $x, y, y', ..., y^{(n)}$.

If we can solve the equation for the highest order derivative $y^{(n)}$ as

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)}),$$

then we call it as the normal form of the differential equation. For example, the normal form of the differential equation above is

$$y'' = 3e^{-x}(y')^2 - 2xe^{-x}y + x.$$

Sometimes we write a first order ordinary differential equation $y' = \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$ as

$$M(x,y)dx + N(x,y) = 0.$$

§1.3 Linear and Nonlinear Equations

An nth order ordinary differential equation is said to be linear if F is linear in $y, y', ..., y^{(n)}$, that is if the equation $F(x, y, y', ..., y^{(n)})$ can be written as

$$a_n(x)y^{(n)} + a_{n-1}x^{(n-1)} + ... + a_1(x)y' + a_0(x)y = g(x).$$

Here we notice two things. All of the coefficients $a_0(x), a_1(x), ..., a_n(x)$ depend only on the independent variable x. All the terms involving the dependent variable y or its derivative $y^{(k)}$ have degree 1.

An ordinary differential, which is not linear, is said to be nonlinear.

- $y'' - 2xy' + 2y = 0$ (Hermite's Equation) is a second order, linear differential equation
- $y(2-3x)dx + x(3y-1)dy = 0$ is a first order, nonlinear differential equation
- $y[1+(y')^2] = c$ (Brachistonchrone Problem) is a first order, nonlinear differential equation
- $\sqrt{1-y}y'' + 2xy' = 0$ (Kiddler's Equation) is a second order, nonlinear differential equation

§1.4 Solution

Definition — Any function $\phi(x)$ is a solution or explicit solution of a differential equation $F(x, y, y', ..., y^{(n)}) = 0$ on an interval I if $\phi(x)$ is defined and has nth derivative on I and satisfies the equation for all x in I when $\phi(x)$ is substituted for y in the equation. In other words,

$$F(x, \phi(x), \phi'(x), ..., \phi^{(n)}(x)) = 0$$

on the interval I. In this case, we say that $\phi(x)$ satisfies the differential equation on I.

Example — $\phi(x) = x^2 - \frac{1}{x}$ is a solution of the differential equation

$$y'' - 2x^{-2}y = 0$$

on the interval $(-\infty, 0) \cup (0, \infty)$.

Example — $\phi(x) = 2x^3$ is a solution of the differential equation

$$xy' = 3y$$

on the interval (∞, ∞).

§1.5 Implicit Solution

Definition — A relation $\phi(x, y) = 0$ is an implicit solution of the differential equation $F(x, y, y', ..., y^{(n)}) = 0$ on an interval I if there is at least one function $\phi(x)$, which satisfies both $\phi(x, \phi(x)) = 0$ and $F(x, \phi(x), \phi'(x), ..., \phi^{(n)}(x)) = 0$ on the interval I.

Example — $x^2 + y^2 = 4$ is an implicit solution of the differential equation

$$xdx + ydy = 0$$

on the interval $(-2, 2)$.

Example — $4x^2 - y^2 = c$, where c is an arbitrary constant, is one parameter of a family of implicit solutions of the differential equation

$$yy' - 4x = 0.$$

§1.6 Types of Solutions

A solution of an ordinary differential equation is also called an integral or integral curve of the equation since integration is an inverse operation of differentiation. By solving an nth order differential equation $F(x, y, y', ..., y^{(n)}) = 0$, we usually expect to have an n-parameter family of solutions, $\phi(x, c_1, ..., c_n$, where $\{c_i\}_{i=1}^{n}$ are arbitrary constants. Such a solution if it exists, is called a general solution of the differential equation $F(x, y, y', ..., y^{(n)}) = 0$. A solution which is free of arbitrary parameters is called a particular solution. A solution is called a singular solution if it cannot be obtained by specializing the parameters in the general solution.

Example — Find the different solutions of the differential equation

$$y' - x\sqrt{y} = 0.$$

We have that $y = (\frac{x^2}{4} + c)^2$ is a general solution, $y = \frac{x^4}{16}$ is a particular solution, and $y = 0$ is a singular and trivial solution to this differential equation.

If the constant function $\phi(x) = 0$ on I is a solution of the differential equation $F(x, y, y', ..., y^{(n)}) = 0$, we call it a trivial solution. For example the differential equation $y(2 - 3x)dx + x(3y - 1)dy = 0$ has a trivial solution but the differential equation $xdx + ydy = 0$ doesn't have a trivial solution. Not every differential equation has a solution. For example the differential equation $(y')^2 + 1 = 0$ has no real-valued solution on any interval.

§1.7 Initial Value Problem

Often, interesting mathematical modeling leads to not only a differential equation but also some side conditions on the unknown function and its derivatives. We now introduce one such problem.

Find a solution to an nth order ordinary differential equation

$$F(x, y, y', ..., y^{(n)}) = 0$$

satisfying the n initial conditions

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(n-1)}(x_0) = y_{n-1}$$

on some interval I containing x_0. Here $x_0, y_0, y_1, ..., y_{n-1}$ are given constants. Such a problem is called an initial value problem.

Example — Show that $y = \frac{1}{x^2+c}$ is a general solution of the first order differential equation $y' + 2xy^2 = 0$. Find a solution of the initial value problem for this differential equation satisfying the initial condition $y(0) = \frac{1}{2}$.

We have that $y = (x^2 + c)^{-1}$ which implies that $y' = -2x(x^2 + c)^{-2}$. Plugging this into the differential equation and using the initial condition $y(0) = \frac{1}{2}$ gives us that $c = 2$ so the solution to this initial value problem is that $y = \frac{1}{x^2+2}$.

Example — Show that $y = \sin(x) - \cos(x)$ is a solution of the initial value problem $y'' + y = 0; y(0) = -1, y'(0) = 1$.

We have that $y = \sin(x) - \cos(x)$ which implies that $y' = \cos(x) + \sin(x)$ and also $y'' = -\sin(x) + \cos(x)$. Plugging this into the differential equation and using the initial conditions $y(0) = -1$ and $y'(0) = 1$ gives us that $y = \sin(x) - \cos(x)$ is the solution to this initial value problem.

§1.8 Picard's Theorem

Example — Show that both functions $y = 0$ and $y = \frac{x^4}{16}$ are solutions of the initial value problem

$$y' = x\sqrt{y}; y(0) = 0.$$

This example shows that some initial value problems can have more than one solution.

Therefore two questions naturally arise. Does every initial value problem have a solution and if it has a solution, then is that solution unique.

Theorem (Picard's Theorem on the Unique Existence of a Solution to the Initial Value Problem)
Consider the first order initial value problem

$$y' = f(x, y); y(x_0) = y_0.$$

If there is a rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

containing (x_0, y_0) such that both $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous in $\mathbb{R}$, then the above initial value problem has a unique solution $\phi(x)$ in some interval $(x_0 - h, x_0 + h)$ with $h > 0$.

Picard's theorem gives only sufficient conditions for a first order initial value problem to have a local unique solution near x_0. If any condition required in the theorem does not hold, then anything may happen. The initial value problem may still have a unique solution or it may have several solutions or it may have no solution at all.

Picard's theorem does not tell us anything on the size of the interval $(x_0 - h, x_0 + h)$ on which the unique solution lies.

A typical known proof of Picard's theorem is nonconstructive, that is, its proof tells us nothing on how to construct the unique solution.

Example — Use Picard's theorem to show that the initial value problem

$$y' = x\sqrt{y}; y(x_0) = y_0$$

doesn't have a unique solution.

Now since $f(x, y) = x\sqrt{y}$ and $\frac{\partial f}{\partial y}(x, y) = \frac{1}{2}x\frac{1}{\sqrt{y}}$ are continuous on $R = \{(x, y) | -\infty < x < \infty, y > 0\}$. Hence if $y_0 > 0$, then the initial value problem has a unique solution in some neighborhood of x_0. However for $(x_0, y_0) = (0, 0)$, $\frac{\partial f}{\partial y}(x, y)$ is not continuous at $(0, 0)$ so the initial value problem has at least 2 solutions which are $y = 0$ and $y = \frac{x^4}{16}$.

§2 Week 2

§2.1 Separable Equations

Here we introdue a subclass of first order differential equations which allow analytic methods to obtain their general solutions.

Definition — A first order differential equation is called separable if it is of the form

$$y' = g(x)h(y).$$

For example, $y' = \frac{x-2xy}{y^2-3}$ is separable but $y' = 2 - x^2y$ is not separable.

With $q(y) = \frac{1}{h(y)}$, we can rewrite the differential equation as

$$q(y)dy = g(x)dx.$$

Then integrating both sides leads to

$$Q(y) = G(x) + c,$$

where $Q(y)$ and $G(x)$ are any antiderivatives of $q(y)$ and $g(x)$ respectively and c is an arbitrary constant. This is an implicit form of a general solution to the separable differential equation.

Example — Solve the differential equation $y' = \frac{x+1}{y^2}$.

Rewriting it as $y^2dy = (x + 1)dx$ and then integrating gives us

$$\frac{y^3}{2} = \frac{x^2}{2} + x + c_1 \longrightarrow y^3 = \frac{3x^2}{2} + 2x + 3c_1$$
$$y = \left(\frac{3x^2}{3} + 3x + 3c_1\right)^{\frac{1}{3}} = \left(\frac{3x^2}{2} + 3x + c\right)^{\frac{1}{3}}.$$

Example — Solve the initial value problem

$$y' = \frac{y-1}{2-x}; y(-1) = 2.$$

Rewriting it as $\frac{1}{y-1}dy = \frac{1}{2-x}dx$ and then integrating gives us

$$\ln|y-1| = -\ln|x-2| + c_1$$

$$|y-1| = e^{c_1}e^{-\ln|x-2|} = \frac{e^{c_1}}{|x-2|} = \frac{c_2}{|x-2|}\ (c_2 = e^{c_1} > 0).$$

Hence we have

$$y-1 = \pm c_2\frac{1}{x-2} = \frac{c}{x-2}\ (c = \pm c_2 \neq 0).$$

When $c = 0$, $y = 1$ also satisfies the given differential equation. hence the one parameter family with arbitrary constant c is a general solution. Finally $y(-1) = 2$ gives us that $c = -3$ so the solution to the initial value problem is

$$y = -\frac{3}{x-2} + 1.$$

Example — Solve the differential equation

$$(3x^2 + 2x - 1)dx = (\sin(y) + e^{-y})dy.$$

Integrating gives us

$$x^3 + x^2 - x + c = -\cos(y) - e^{-y},$$

which is an implicit form of a general solution.

Example — Solve the differential equation

$$y' = y^2 - 1.$$

Separating the variables gives us that

$$dx = \frac{dy}{y^2-1} = \frac{1}{2}\Big(\frac{1}{y-1} - \frac{1}{y+1}\Big)dy.$$

Integrating gives us that

$$2x + c_! = \ln\Big|\frac{y-1}{y+1}\Big|$$

$$\frac{y-1}{y+1} = \pm e^{c_1}e^{2x} = ce^{2x}\ (c = \pm e^{c_1} \neq 0)$$

$$y = \frac{1+ce^{2x}}{1-ce^{2x}}.$$

Note that when $c = 0$, $y = 1$ is also a solution so c can be an arbitrary constant. Simple inspection shows that $y = -1$ is a singular solution since it can't be obtained for any choice of c.

§2.2 Integrating Factor

Definition — A first order differential equation is linear if it is of the form

$$a_1(x)y' + a_0(x)y = f(x).$$

It is homogeneous if $g(x) = 0$ and nonhomogeneous otherwise.

With $p(x) = \frac{a_0(x)}{a_1(x)}$ and $q(x) = \frac{g(x)}{a_1(x)}$, we call

$$y' + p(x)y = q(x)$$

the standard form of the previous equation. If $p(x) = 0$ then the equation becomes $y' = q(x)$ which has a general solution of

$$y(x) = \int q(x)dx + c,$$

where c is an arbitrary constant.

In general, we seek a function $\mu(x)$ such that

$$\mu(x)y' + \mu(x)p(x)y = (\mu(x)y)'$$

or equivalently

$$\mu' = p(x)\mu \longrightarrow \frac{1}{\mu}d\mu = p(x)dx$$

which is a separable differential equation. Integrating gives us that

$$\mu(x) = e^{\int p(x)dx}.$$

If we multiply $\mu(x)$ to the differential equation $y' + p(x)y = q(x)$ then we will have

$$(\mu(x)y)' = \mu(x)q(x),$$

which has a general solution

$$y(x) = \frac{1}{\mu(x)}\Big(\int \mu(x)q(x)dx + c\Big).$$

We call $\mu(x)$ an integrating factor for the linear differential equation $y' + p(x)y = q(x)$.

§2.3 Does the Initial Value Problem Have a Unique Solution?

Example — Solve the initial value problem

$$y' + \frac{3y}{x} = \frac{1}{x^2}; y(1) = \frac{1}{2}.$$

We start by multiplying the differential equation by its integrating factor which is $\mu(x) = e^{\int \frac{3}{x}dx} = x^3$. Therefore we have

$$x = x^3(y' + \frac{3y}{x}) = (x^3y)'$$

$$y = \frac{1}{2x} + \frac{c}{x^3}.$$

Now since the initial conditions are $y(1) = \frac{1}{2}$, we have that $c = 0$ so the solution to this initial value problem is

$$y = \frac{1}{2x}.$$

Theorem

Assume that $p(x)$ and $q(x)$ are continuous on an interval (a, b). Then for any x_0 in (a, b) and any y_0, the initial value problem

$$y' + p(x)y = q(x); y(x_0) = y_0$$

has a unique solution $\phi(x)$ on (a, b).

Example — Solve the differential equation

$$xy' + (1 + x)y = e^{-x} \cos(3x).$$

We start by dividing the equation by x and then multiply it by its integrating factor which is $\mu(x) = e^{\int \frac{1+x}{x} dx} = xe^x$. Therefore we have

$$(xe^x y)' = \cos(3x)$$

which implies that

$$y = \frac{1}{xe^x}(\frac{\sin(3x)}{3} + c).$$

§2.4 Exact Equations

For a function $F(x, y)$ of two variables with continuous first partial derivatives, we call $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$ the total differential of F. Then along any level curve $F(x, y) = c$ for some constant c, we have

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0 \longrightarrow \frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}},$$

which is a first order differential equation.

Conversely for a differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

if its left hand side is the total differential of $F(x, y)$, then the equation can be written as $dF(x, y) = 0$ so that $F(x, y) = c$ is a general solution of the differential equation.

Definition — A first order differential equation $M(x, y)dx + N(x, y)dy = 0$ is exact in a plane region R if there is a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \text{ and } \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R. When R is the whole plane $\mathbb{R}^2$, we simply say that $M(x, y)dx + N(x, y)dy = 0$ is exact.

Theorem

Assume that $M(x,y)$ and $N(x,y)$ have continuous first partial derivatives in a rectangle

$$R = \{(x,y) \in \mathbb{R}^2 : a < x < b, c < y < d\}.$$

Then the differential equation $Mdx + Ndy = 0$ is exact in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ in } R.$$

Proof of the necessity. Assume that $Mdx + Ndy = 0$ is exact in R so that

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \text{ and } \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

holds for some function $F(x,y)$. Then we have that

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial x}\Big) = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial y}\Big) = \frac{\partial N}{\partial x}.$$

Proof of sufficiency Now assume that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

holds. Then we need to find a function $F(x,y)$ which satisfies

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \text{ and } \frac{\partial F}{\partial y}(x,y) = N(x,y).$$

The first equation of the condition above suggests that

$$F(x,y) = \int M(x,y)dx + g(y)$$

where arbitrary differentiable function $g(y)$ is the integration constant.

We then need to choose $g(y)$ wisely so that $F(x,y)$ also satisfies the second equation of the condition

$$N(x,y) = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y),$$

which gives us that

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y_d x.$$

Note that $g'(y)$ is independent of x since

$$\frac{\partial}{\partial x}\Big(N - \frac{\partial}{\partial y}\int Mdx\Big) = \frac{\partial N}{\partial x} = \frac{\partial}{\partial y}\Big(\frac{\partial}{\partial x}\int Mdx\Big) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Hence $g'(y)$ is solvable for $g(y)$ and $F(x,y)$ satisfies the conditions

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \text{ and } \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

so that $Mdx + Ndy = 0$ is exact.

Example — Solve the differential equation

$$y' = \frac{e^{-y} - y\sin(xy)}{xe^{-y} + x\sin(xy) + 2y}.$$

Write the differential equation in the form

$$(e^{-y} - y\sin(xy))dx - (xe^{-y} + x\sin(xy) + 2y)dy = 0.$$

It is exact since

$$\frac{\partial M}{\partial y} = -e^{-y} - \sin(xy) - xy\cos(xy) = \frac{\partial N}{\partial x}.$$

Therefore we have that

$$\frac{\partial F}{\partial x} = e^{-y} - y\sin(xy) \text{ and } \frac{\partial F}{\partial y} = -(xe^{-y} + x\sin(xy) + 2y)$$

$$F = \int (e^{-y} - y\sin(xy))dx + g(y) = e^{-y}x + \cos(xy) + g(y)$$

$$\frac{\partial F}{\partial y} = -xe^{-y} - x\sin(xy) + g'(y) = -xe^{-y} - x\sin(xy) - 2y$$

which implies that

$$g'(y) = -2y \longrightarrow = -y^2.$$

Therefore we have that

$$F(x, y) = xe^{-y} + \cos(xy) - y^2 = c.$$

Example — Solve the initial value problem

$$(e^{-x}y - xe^{-x}y + 3)dx + (xe^{-x} - 2y)dy = 0; y(1) = 0.$$

Note that this differential equation is exact since

$$\frac{\partial M}{\partial y} = e^{-x} - xe^{-x} = \frac{\partial N}{\partial x}.$$

Now we have

$$\frac{\partial F}{\partial y} = xe^{-x} - 2y$$

$$F(x, y) = \int (xe^{-x} - 2y)dy + g(x) = xe^{-x} - y^2 + g(x)$$

$$\frac{\partial F}{\partial x} = e^{-x}y - xe^{-x}y + g'(x) = e^{-x}y - xe^{-x}y + 3$$

which gives us that $g'(x) = 3$ and $g(x) = 3x$. Thus

$$F(x, y) = xe^{-x}y - y^2 + 3x = c.$$

Finally using the fact that $y(1) = 0$ we get that $c = 3$ so the solution to this initial value problem is

$$xe^{-x}y - y^2 + 3x = 3.$$

§3 Week 3

§3.1 Integrating Factor Part 1

Sometimes, a nonexact differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

can be made exact by multiplying the equation by a function

$$\mu M dx + \mu N dy = 0$$

is exact. In this case, we call $\mu(x,y)$ an *integrating factor* of $Mdx + Ndy = 0$.

Example — Show that $\mu(x,y) = xy$ is an integrating factor of

$$(-xy\sin(x) + 2y\cos(x))dx + 2x\cos(x)dy = 0$$

and then solve the equation.

It is easy to see that the differential equation shown above is not exact since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Now we multiply the above equation by an integrating factor which the problem said was $\mu(x,y) = xy$. Therefore we have

$$(-x^2y^2\sin(x) + 2xy^2\cos(x))dx + 2x^2y\cos(x)dy = 0,$$

which is exact since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Therefore xy is an integrating factor of the differential equation shown above.

Now we have found its integrating factor but we still have to find the solutions to this differential equation.

We know that this new differential equation

$$(-x^2y^2\sin(x) + 2xy^2\cos(x))dx + 2x^2y\cos(x)dy = 0$$

is exact which means that there exists a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = -x^2y^2\sin(x) + 2xy^2\cos(x)$ and $\frac{\partial F}{\partial y} = 2x^2y\cos(x)$.

Now consider the fact that $\frac{\partial F}{\partial y} = 2x^2y\cos(x)$. Then we have

$$F(x,y) = \int 2x^2y\cos(x)dy + g(x) = x^2y^2\cos(x) + g(x)$$

$$\frac{\partial F}{\partial x} = 2xy^2\cos(x) - x^2y^2\sin(x) + g'(x) = -x^2y^2\sin(x) + 2xy^2\cos(x)$$

which implies that $g'(x) = 0$ and $g(x) = 0$. Thus

$$x^2y^2\cos(x) = c$$

is a general solution of the differential equation in the problem.

§3.2 Integrating Factor Part 2

Now we have the obvious question which is how to find an integrating factor? Suppose that $\mu(x, y)$ is an integrating factor of a nonexact differential equation $M(x, y)dx + N(x, y)dy = 0$. Then we have that

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x},$$

that is,

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mu,$$

which is a partial differential equation for μ. Usually it is harder to solve the partial differential equation shown above rather than solving the original equation $Mdx + Ndy = 0$.

However, if μ depends on only one variable x or y, then the partial differential equation becomes much simpler. We have that

$$\frac{d\mu}{dx} = \frac{1}{N}\Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big)\mu \text{ when } \mu(x, y) = \mu(x)$$
$$\frac{d\mu}{dx} = \frac{1}{M}\Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mu \text{ when } \mu(x, y) = \mu(y).$$

§3.3 Integrating Factor Part 3

Now we will look at some examples.

Example — Solve the differential equation

$$(x + y)dx + x \ln x dy = 0.$$

First we will see if the differential equation is exact or not. If the equation is exact then we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

We can find that $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = \ln x + 1$ so this differential equation isn't exact since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore we will attempt to find one of this differential equation's integrating factors to make this an exact equation.

Notice that

$$\frac{1}{N}\Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big) = -\frac{1}{x}$$

is a function of x only so we can let $\mu(x) = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$ which is an integrating factor. Therefore the differential equation

$$\frac{(x + y)}{x}dx + \ln x dy = 0$$

is an exact equation.

Now since this new equation is exact, we know that there exists some function $F(x, y)$ such that $\frac{\partial F}{\partial x} = \frac{(x+y)}{x}$ and $\frac{\partial F}{\partial y} = \ln x$. Therefore using the fact that $\frac{\partial F}{\partial x} = \frac{(x+y)}{x}$, we can figure out that $F(x, y) = x + y \ln x + g(y)$. Therefore $\frac{\partial F}{\partial y} = \ln x + g'(y) = \ln x$ which gives us that $g'(y) = g(y) = 0$. Thus the general solution to this differential equation is

$$x + y \ln x = c.$$

Example — Solve the initial value problem

$$6xydx + (4y + 9x^2)dy = 0; y(0) = 1.$$

First we will see if the differential equation is exact or not. If the equation is exact then we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

We can find that $\frac{\partial M}{\partial y} = 6x$ and $\frac{\partial N}{\partial x} = 18x^2$ so this differential equation isn't exact. Therefore we will attempt to find one of this differential equation's integrating factors to make this an exact equation.

Notice that

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{2}{y}$$

is a function of y only so we can let $\mu(y) = e^{\int \frac{2}{y} dy} = y^2$ which is an integrating factor. Therefore

$$6xy^3dx + (4y^3 + 9x^2y^2)dy = 0$$

is an exact equation.

Therefore there exists some function $F(x, y)$ such that $\frac{\partial F}{\partial x} = 6xy^3$ and $\frac{\partial F}{\partial y} = 4y^3 + 9x^2y^2$. Therefore using the fact that $\frac{\partial F}{\partial x} = 6xy^3$, we can figure out that $F(x, y) = 2x^2y^3 + g(y)$. Differentiating this gives us that $\frac{\partial F}{\partial y} = 9x^2y^2 + g'(y) = 4y^3 + 9x^2y^2$ which gives us that $g'(y) = 4y^3$ so $g(y) = y^4$. Thus the general solution to this equation is

$$3x^2y^3 + y^4 = c.$$

However since $y(0) = 1$, we have that $c = 1$. Therefore the solution of the initial value problem is

$$3x^2y^3 + y^4 = 1.$$

§3.4 Substitutions

A first order differential equation, which is neither linear, separable, nor exact, may often be transformed into another form by a suitable substitution, which is easy to solve.

Example — Solve the differential equation

$$xe^yy' + e^y = \cos(x).$$

We can make the substitution $u = e^y$. Then we have that $u' = e^yy'$, so the equation becomes

$$xu' + y = (xu)' = \cos(x).$$

Now we can integrate both sides to get that

$$xu = xe^y = \sin(x) + c$$

which is the general solution to this differential equation.

§3.5 Homogeneous Equations

We call a function $f(x, y)$ to be *homogeneous* of degree α, where $\alpha \in \mathbb{R}$, if $f(tx, ty) = t^\alpha f(x, y)$. For example, $x^2 - 2xy + 3y^2$ is homogeneous of degree 2 and $e^{\frac{y}{x}}$ is homogeneous of degree 0. A first order differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be homogeneous if both M and N are homogeneous of the same degree.

Now suppose that both M and N in the equation, $M(x, y)dx + N(x, y)dy = 0$, are both homogeneous of degree α. Now suppose we set $u = \frac{y}{x}$. Then $y = xu$ which gives us that $dy = udx + xdu$ and the equation becomes

$$M(x, xu)dx + N(x, xu)(udx + xdu) = x^\alpha[M(1, u)dx + N(1, u)(udx + xdu)] = 0$$
$$M(1, u)dx + N(1, u)(udx + xdu) = 0$$
$$[M(1, u) + uN(1, u)]dx + xN(1, u)du = 0$$
$$\frac{1}{x}dx + \frac{N(1, u)}{M(1, u) + uN(1, u)}du = 0$$

which is separable. Likewise if we use the substitution $v = \frac{x}{y}$, this leads to a separable equation also.

Note that if the original equation,

$$M(x, y)dx + N(x, y)dy = 0,$$

is homogeneous, then it can be written as

$$\frac{dy}{dx} = \frac{-M(x, y)}{N(x, y)} = \frac{-M(1, \frac{y}{x})}{N(1, \frac{y}{x})} = f(\frac{y}{x}),$$

of which the right hand side is a function of the ratio $\frac{y}{x}$ alone and vice versa.

§3.6 More Substitution Examples

Example — Solve the equation

$$\frac{dy}{dx} = \frac{y^2 - x^2}{xy}.$$

Note that the equation is homogeneous. We make the substitution $u = \frac{y}{x}$. Then the equation becomes

$$u + xu' = u - \frac{1}{u}$$

$$udu = -\frac{1}{x}dx$$

so we have that

$$u^2 = (\frac{y}{x})^2 = -\ln x^2 + c$$

so the general solution to this differential equation is

$$y^2 = x^2(c - \ln x^2).$$

Now consider a differential equation in the form $y' = f(ax + by)$ where $b \neq 0$. Now consider the substitution $u = ax + by$. Then we have $u' = a + by'$. Note that $y' = f(u)$ which gives us that $u' = bf(u) + a$. This gives us that $\frac{bf(u)+a}{d}u = dx$ which is a separable differential equation that we can now solve.

Example — Solve the initial value problem

$$y' = \sqrt{x+y} - 1; y(0) = 1.$$

We will make the substitution $u = x + y$. Then $u' = 1 + y'$, so the equation becomes $u' = \sqrt{u}$ or $u^{-\frac{1}{2}}du = dx$. Therefore $2\sqrt{u} = x + c$ which gives us the general solution of $2\sqrt{x+y} = x + c$. Since $y(0) = 1$, we get that $c = 2$. Therefore $y = \frac{x^2}{4} + 1$ is the solution to the initial value problem.

§3.7 Bernoulli Equation

A first order differential equation of the form

$$y' + p(x)y = q(x)y^n \text{ (n is a constant)}$$

is called a Bernoulli equation. We assume that $n \neq 0, 1$ and that $q(x) \neq 0$ otherwise the differential equation becomes linear. Note that for $n > 0, y = 0$ is the trivial solution. We can make the substitution $u = y^{1-n}$ which will transform the equation into a linear differential equation

$$u' + (1-n)p(x)u = (1-n)q(x).$$

Example — Solve the differential equation

$$y' + \frac{2y}{x} = 2x\sqrt{y}.$$

Consider the substitution $u = \sqrt{y}$. Then we have that $u' = \frac{1}{2}y^{-\frac{1}{2}}y'$. Then the equation becomes

$$u' + \frac{u}{x} = x \longrightarrow xu' + u = (xu)' = x^2.$$

Therefore we can integrate both sides to get that

$$xu = x\sqrt{y} = \frac{x^3}{3} + c \longrightarrow y = \left(\frac{x^2}{3} + \frac{c}{x}\right)^2$$

which is the general solution with $y = 0$ being a singular solution.

§3.8 Equations with Linear Coefficients

Consider a first order differential equation of the form

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0.$$

Note that if $c_1 = c_2 = 0$, then the differential equation is homogeneous. Now set $x = u - h$ and $y = v - k$, where h and k are constants to be determined later. Then the equation becomes

$$(a_1u + b_1v - a_1h - b_1k + c_1)du + (a_2u + b_2v - a_2h - b_2k + c_2)dv = 0.$$

Now if we can choose h and k so that

$$a_1h + b_1k = c_1$$
$$a_2h + b_2k = c_2,$$

then the equation becomes homogeneous. If $a_1b_2 - a_2b_1 = 0$, then the equation can be put into the form $y' = f(ax+by)$ so we assume that $a_1b_2 - a_2b_1 \neq 0$. Then the equations

$$a_1h + b_1k = c_1$$

$$a_2h + b_2k = c_2$$

have a unique solution for h and k. In fact by the Cramer rule,

$$h = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, k = \frac{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

With h and k being the values above, the equation becomes a homogeneous equation

$$(a_1u + b_1v)du + (a_2u + b_2v)dv = 0.$$

Example — Solve the initial value problem

$$(x + 2y - 1)dx + (2x + y + 1)dy = 0; y(-1) = 2.$$

We can do some calculation to find the values of h and k and we get that $h = -1$ and $k = 1$. So we can set $x = u + 1$ and $y = v - 1$. Then the equation becomes a homogeneous equation

$$(u + 2v)du + (2u + v)dv = 0.$$

Now make the substitution $w = \frac{v}{u}$. Then we get

$$(2w + 1)du + (w + 2)(udw + wdu) = 0$$

$$(w^2 + 4w + 1)du + u(w + 2)dw = 0$$

$$\frac{1}{u}du + \frac{w + 2}{w^2 + 4w + 1}dw = 0$$

so that

$$\ln|u| + \frac{1}{2}\ln|w^2 + 4w + 1| = c_1$$

$$u^2(w^2 + 4w + 1) = c \text{ such that } c \neq 0$$

$$v^2 + 4uv + u^2 = c$$

$$(y - 1)^2 + 4(x + 1)(y - 1) + (x + 1)^2 = c.$$

Finally we have that $y(-1) = 2$ which gives us that $c = 1$ so the solution to this initial value problem is

$$(y - 1)^2 + 4(x + 1)(y - 1) + (x + 1)^2 = 1.$$

§3.9 Ricatti's Equation

Ricatti's equation is a nonlinear first order equation of the form

$$y' = p(x) + q(x)y + r(x)y^2.$$

Now assume that y_1 is a solution of that differential equation. Then $y = y_1 + u$ is also a solution of that differential equation if and only if

$$u' - (q + 2ry_1)u = ru^2,$$

which is a Bernoulli equation.

> **Example —** Solve the differential equation
>
> $$y' = 2x^2 + \frac{y}{x} - 2y^2, \text{ knowing that } y_1 = kx \text{ is a solution for some constant } k.$$

We start by substituting $y_1 = kx$ back into the differential equation. We get that $k = 2x^2 + k - 2k^2x^2$ so we have that $k = \pm 1$. Now take $k = 1$ and set $y = x + u$. Then we have

$$u' + (4x - \frac{1}{x})u = -2u^2.$$

Now we make the substitution $w = \frac{1}{u}$. Then $w' + (\frac{1}{x} - 4x)w = 2$, which has an integrating factor of xe^{-2x^2}. Hence we have

$$\left(xe^{-2x^2}w\right)' = 2xe^{-2x^2}$$

and so we have

$$w = \frac{1}{x}\left(ce^{2x^2} - \frac{1}{2}\right).$$

Hence we have that $u = \frac{x}{ce^{2x^2} - \frac{1}{2}}$ and so we have that $y = x + \frac{2x}{ce^{2x^2} - 1}$ (after replacing $2c$ with c).

§3.10 Clairaut's Equation

Consider a first order differential equation of the form where

$$y = xy' + f(y')$$

where $f(t)$ is twice differentiable and $f''(t) \neq 0$. If we differentiate the equation, we get that

$$y''(x + f'(y')) = 0$$

so we have that either $y'' = 0$ or $x + f'(y') = 0$. Taking $y'' = 0$ gives the general solution $y = cx + f(c)$ of the equation, which is a family of straight lines.

Moreover, the equation has a singular solution in parametric form

$$x = -f'(t), y = f(t) - tf'(t),$$

which is the envelope of the family $\{y = cx + f(c) : c \text{ arbitrary }\}$, that is, the curve whose tangent lines are given by the family. It is a solution since

$$x = -f'(t), y = f(t) - tf'(t)$$

implies that

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} = (-tf''(t))(-f''(t))^{-1}t$$

so $xy' + f(y') = -f'(t)t + f(t) = y$.

Example — Solve the differential equation

$$y = (x+4)y' + (y')^2.$$

Note that $y = xy' + f(y')$, where $f(t) = t^2 + 4t$ so the general solution is $y = cx + c^2 + 4c$ and a singular solution is $x = -2t - 4, y = -t^2$, i.e., $y = -\frac{(x+4)^2}{4}$.

§4 Week 4

§4.1 Radioactive Decay

Mathematical modeling is a process of simulating reality by using mathematical language. In formulating real world problems in mathematical terms, in most cases, we need some idealization and simplification in order for the resulting model to be mathematically tractable. When a problem involves a rate of change (i.e. derivative), its modeling naturally leads to a differential equation. Modeling sometimes involves conceptual replacing of a discrete process by a continuous one as one can see in the study of population dynamics.

Radioactive decay is the process by which an unstable atomic nucleus loses energy by radiation. Radioactive material decays at a rate proportional to its present amount. That is, with $Q(t)$ being the amount of radioactive material at time t, we have

$$Q'(t) = -rQ(t)(r > 0 \text{ decay rate})$$

so that

$$Q(t) = Q_0 e^{-rt}, Q_0 = Q(0).$$

Time period during which the mass of a radioactive material is reduced to $\frac{1}{2}$ of its original mass is called the half-life of the material.

Example — If Einsteinium 237 loses $\frac{1}{3}$ of its mass in 12 days, find its half-life.

We know that $Q(12) = Q_0 e^{-12r} = \frac{2}{3}Q_0$ so we have that $r = \frac{\ln\frac{3}{2}}{12}$. Now let α be the half-life of Einsteinium 237. Then we have

$$\frac{1}{2}Q_0 = Q(\alpha) = Q_0 e^{-r\alpha}$$

which gives us that

$$\alpha = \frac{\ln 2}{r} = \frac{12\ln 2}{\ln\frac{3}{2}}.$$

Example — The half-life of a radioactive isotope C$-$14 is 5600 years. Determine the age of a wooden fossil if it contains only 5% of the original amount of C$-$14.

We know that $\frac{1}{2}Q_0 = Q_0 e^{-5600r}$ which gives us that $r = \frac{\ln 2}{5600}$. Now we have the equation

$$\frac{1}{20}Q_0 = Q_0 e^{-rt}$$

which gives us that

$$t = \frac{\ln 20}{r} = \frac{5600\ln 20}{\ln 2}.$$

§4.2 Population Dynamics

In many applications like medicine, economics, ecology, etc, it is important to predict the change of population of a certain species. Let $p(t)$ be the population of a species at time t. Even though $p(t)$ is always an integer, we may assume that $p(t)$ is a continuous or even differentiable function when $p(t)$ is large enough.

In its simplest case, we assume that population grows at a rate proportional to its present population as in the case of bacteria growth under enough space and food supply. Then

$$p'(t) = rp(t) \ (r > 0 \text{ growth rate})$$

so that $p(t) = p_0 e^{rt}, p(0) = p_0$. We call the equation above the exponential (or Malthusian) model of population growth.

However in a more realistic solution, it is reasonable to assume that the individual growth rate $\frac{p'}{p}$ depends on the population and decreases as p increases. As a simple model, we take $\frac{p'}{p}$ to be linear and decreasing in p, for example

$$p' = (r - ap)p \ (r > 0, a > 0),$$

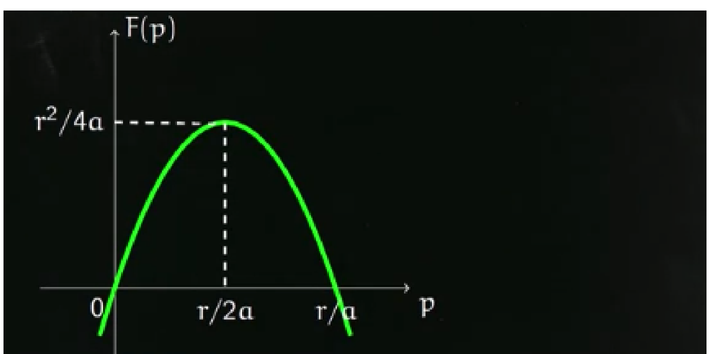
which we call the logistic equation with birth rate r and death rate ap.

We first try to investigate the behavior of its integral curves, which are called the logistic curves, without solving it explicitly by inspecting the function $F(p) = (r - ap)p$.

First note that $F(p) = 0$ when $p = 0$ and $p = \frac{r}{a}$, both of which are constant solutions of the equation and are called equilibrium solutions.

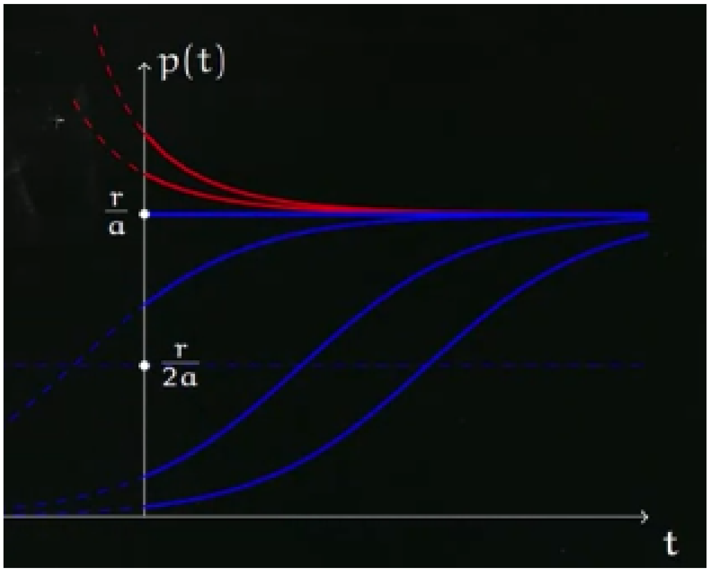
An integral curve is increasing if $p'(t) = p(r - ap) > 0$ and is decreasing if $p'(t) = p(r - ap) < 0$. An integral curve is concave up if $p''(t) = p'(r - 2ap) > 0$ and is concave down if $p''(t) = p'(r - 2ap) < 0$.

Combining these two facts and the graph of $F(p)$, we have



p	p′	p″	p(t)	
$0 < p < \frac{r}{2a}$	+	+	inc. & concave up	
$\frac{r}{2a} < p < \frac{r}{a}$	+	-	inc. & concave down	
$p > \frac{r}{a}$	-	+	dec. & concave up	

Any logistic curve starting from $p_0 = p(0) > 0$ approaches the equilibrium solution $p = \frac{r}{a}$ but never reaches $\frac{r}{a}$ at a finite time by Picard's theorem. For $0 < p_0 < \frac{r}{a}, p(t) < \frac{r}{a}$ for all t and $\lim_{t\to\infty} p(t) = \frac{r}{a}$ so that we call $\frac{r}{a}$ the carrying capacity or the saturation level of the environment.



It is quite simple to solve the differential equation

$$p'(t) = (r - ap)p.$$

Separating the variables in the equation gives us

$$\frac{1}{(r-ap)p}dp = \left(\frac{\frac{1}{r}}{p} + \frac{\frac{a}{r}}{r-ap}\right)dp = dt$$

$$\frac{1}{r}\ln\left|\frac{p}{r-ap}\right| = t + c_1$$

so that the equation has a general solution

$$p(t) = \frac{rc}{ac + e^{-rt}} \text{ where } c \text{ is an arbitrary constant.}$$

With $p(0) = p_0 \geq 0$, we have that $c = \frac{p_0}{r - ap_0}$ which gives us the equation

$$p(t) = \frac{rp_0}{ap_0 + (r - ap_0)e^{-rt}} \text{ (logistic function).}$$

> **Example** — The number of people in a village of size 10000 people, who are exposed to a certain rumor is governed by the logistic equation with birth rate 3. Initially only 10 people hear the rumor. How many days are needed for one half of the whole villagers to hear the rumor? Here time is measured in days.

Let $p(t)$ be the number of people hearing the rumor after t days. Then we have $r = 3, p(0) = p_0 = 10$, and $a = 3 \cdot 10^{-4}$ since its carrying capacity is $\frac{r}{a}$ which is $10000 = 10^4$. Therefore by the logistic function we get that

$$p(t) = \frac{3 \cdot 10}{3 \cdot 10^{-3} + (3 - 3 \cdot 10^{-3})e^{-3t}} = \frac{10^4}{1 + (10^3 - 1)e^{-3t}}.$$

Finally we have that

$$5 \cdot 10^3 = \frac{10^4}{1 + (10^3 - 1)e^{-3t}}$$

which gives us that $t = \frac{\ln(10^3 - 1)}{3}$.

Example — Consider a large tank holding 100 liters of brine solution in which 10 kilograms of salt is dissolved. A brine solution with salt concentration 0.2 kilograms per liter flows into the tank at a rate of 5 liters per minute, and then the well stirred solution flows out of the tank at a rate of 3 liters per minute. Determine the mass of salt inside the tank at any time t.

Let $m(t)$ be the mass of the salt in the tank at time t. Then

$$m'(t) = \text{ input rate } - \text{ output rate}, m(0) = 10.$$

Now, input rate of salt is

$$(5 \text{ L/min})(0.2 \text{ kg/L}) = 1 \text{ kg/min}$$

and output rate of salt is

$$(3 \text{ L/min})(\frac{m}{100 + (5-3)t} \text{ kg/L}) = \frac{3m}{100 + 2t} \text{ kg/min}.$$

hence $m(t)$ must satisfy the initial value problem

$$m' = 1 - \frac{3m}{100 + 2t}; m(0) = 10$$

which gives us that

$$m(t) = \frac{1}{5}(2t + 100) - 10^4(2t + 100)^{-\frac{3}{2}}.$$

§5 Week 5

§5.1 Linear Differential Equations

Consider a linear nth order differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = b(x),$$

where $a_n(x), a_{n-1}(x), ..., a_0(x)$, and $b(x)$ are continuous functions on an interval I. When $a_n, a_{n-1}, ..., a_0$ are constants, we say that the equation has constant coefficients, otherwise it has variable coefficients. When $b(x) = 0$ on I, we say that the equation is homogeneous, otherwise we say it is nonhomogeneous.

In Week 4, we always assume that $a_n(x) \neq 0$ on I and then we may rewrite the differential equation in its standard form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = g(x),$$

where $p_k(x) = \frac{a_k(x)}{a_n(x)}$ for $k = 0, 1, ..., n-1$ and $g(x) = \frac{b(x)}{a_n(x)}$.

With the notation $D^k y = y^{(k)}$ for $k = 0, 1, 2, ...$, we may express the differential equation as

$$L[y] = [a_n(x)D^n + a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x)](y) = b(x),$$

where

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + ... + a_1(x)D + a_0(x)$$

is called a (linear) differential operator.

The differential operator is linear in the sense that

$$L[\alpha f(x) + \beta g(x)] = \alpha L[f(x)] + \beta L[g(x)],$$

where α and β are constants. Because of the linearity, we call the operator L, a linear differential operator.

§5.2 Superposition Principle

Theorem (Superposition Principle for Homogeneous Equations)
If $\{y_i\}_{i=1}^k$ are solutions of the homogeneous equation

$$L[y] = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = 0,$$

then their linear combination

$$\sum_{i=1}^k c_i y_i$$

is also a solution of the differential equation where the c_i are constants.

Proof. By linearity and $L[y_i] = 0$ for $i = 1, 2, ..., k$, we have

$$L[\sum_{i=1}^k c_i y_i] = \sum_{i=1}^k c_i L[y_i] = 0.$$

Example — Find a solution to the initial value problem

$$x^2y'' - xy' + y = 0; y(1) = 1, y'(1) = -1$$

after confirming $y_1 = x$ and $y_2 = x \ln x$ are solutions of the differential equation above on the interval $(0, \infty)$.

It is obvious to see that $y_1 = x$ and $y_2 = x \ln x$ are solutions. Then by the superposition principle, $y = c_1x+c_2x \ln x$ is also a solution for arbitrary constants c_1 and c_2. Applying the initial conditions gives us $1 = y(1) = c_1$ and $-1 = y'(-1) = c_1 + c_2$ so we have that $c_1 = 1$ and $c_2 = -2$ giving us the solution of $y = x - 2x \ln x$ to the initial value problem. As the next theorem will show, this is the only solution to the initial value problem.

Theorem (Unique Existence of a Solution to IVP)
Let $\{a_i(x)\}_{i=0}^n$ and $b(x)$ be continuous on an interval I and assume $a_n(x) \neq 0$ on I. Then for any x_0 in I and any constants $\{y_i\}_{i=0}^{n-1}$, the initial value problem

$$L[y] = b(x); y^{(i)}(x_0) = y_i \text{ for } i = 0, 1, ..., n - 1$$

has a unique solution $\phi(x)$ on I.

§5.3 Linear Independence

Definition — A finite set of functions $\{f_i(x)\}_{i=1}^n$ is said to be linearly dependent on an interval I if there are constants $\{c_i\}_{i=1}^n$, not all zero such that

$$\sum_{i=1}^n c_i f_i(x) = 0 \text{ on } I.$$

Otherwise we say that $\{f_i(x)\}_{i=1}^n$ are linearly independent on I. In other words, $\{f_i(x)\}_{i=1}^n$ are linearly independent on I if

$$\sum_{i=1}^{n} c_i f_i(x) = 0$$

on I implies that $c_i = 0$ for all $i = 1, 2, ..., n$.

In particular, two functions $f(x)$ and $g(x)$ are linearly dependent on I if and only if one is a constant multiple of the other. For example, $\sin(2x)$ and $\sin(x)\cos(x)$ are linearly dependent on $(-\infty, \infty)$ since $\sin(2x) = 2\sin(x)\cos(x)$. Linear dependence or independence of functions not only depends on functions but also on the interval on which we test linear dependence or independence.

Example — Show that $f(x) = x$ and $g(x) = |x|$ are linearly dependent on $(0, \infty)$ but linearly independent on $(-\infty, \infty)$.

In general for $n \geq 3$, it is not easy to use the definition itself to check linear independence or dependence of a family of functions. However when they are solutions of a linear homogeneous differential equation like $L[y] = 0$, there is a much easier test to apply.

For any family $\{f_i()\}_{i=1}^{n}$ of functions, which are $(n-1)$-times differentiable on I, we call

$$W(f_1, ..., f_n)(x) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ f_1^{(n-1)} & f_2^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix}$$

the Wronskian of $\{f_i()\}_{i=1}^{n}$.

§5.4 Wronskian Test

From last section we have, for any family $\{f_i()\}_{i=1}^{n}$ of functions, which are $(n-1)$-times differentiable on I, we call

$$W(f_1, ..., f_n)(x) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ f_1^{(n-1)} & f_2^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix}$$

the Wronskian of $\{f_i()\}_{i=1}^{n}$.

Theorem (Wronskian Test)

Let $\{y_i\}_{i=1}^n$ be n solutions of the linear nth order homogeneous differential equation

$$L[y] = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = 0$$

on an interval I. Then the following three statements are equivalent:

$$(a)\{y_i\}_{i=1}^n \text{ are linearly independent on } I;$$

$$(b)W(y_1, ..., y_n)(x) \neq 0 \text{ for all } x \text{ in } I;$$

$$(c)W(y_1, ..., y_n)(x_0) \neq 0 \text{ for some } x_0 \text{ in } I.$$

Proof. For simplicity, we assume that $n = 2$. Then we have

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$$

Now we have

$$y_1(a_2y_2'' + a_1y_2' + a_0y_2) - y_2(a_2y_1'' + a_1y_1' + a_0y_1) = a_2(y_1y_2'' - y_1''y_2) + a_1(y_1y_2' - y_1'y_2) = a_2w' + a_1w = 0.$$

This gives us that

$$w' + p(x)w = 0 \text{ with } p(x) = \frac{a_1(x)}{a_2(x)} \text{ and } w(x) = W(y_1, y_2)(x).$$

Hence $w(x) = ce^{-\int p(x)dx}$ by Abel's identity so that either $w(x) \neq 0$ for all x in I when $c \neq 0$ or $w(x) = 0$ on I when $c = 0$. Therefore the conditions (b) and (c) are equivalent. Now assume that (a) is true, i.e., y_1 and y_2 are linearly independent on I but $W(y_1, y_2)(x_0) = 0$ for some x_0 in I, i.e., the 2×2 matrix

$$\begin{bmatrix} y_1(x_0) & y_2(X_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is singular (determinant is nonzero). Then there exist two constants c_1 and c_2 such that at least one is nonzero that satisfy

$$\begin{bmatrix} y_1(x_0) & y_2(X_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

that is, $c_1y_1(x_0) + c_2y_2(x_0) = c_1y_1'(x_0) + c_2y_2'(x_0) = 0$.

Now consider the initial value problem

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0; y(x_0) = y'(x_0) = 0.$$

Both $\phi_1(x) = 0$ and $\phi_2(x) = c_1y_1 + c_2y_2$ are solutions to this initial value problem on I for some constants c_1 and c_2. Since there is a unique solution to the initial value problem, we must have $c_1y_1 + c_2y_2 = 0$ on I, which contradicts the fact that y_1 and y_2 are linearly independent on I since $c_1^2 + c_2^2 \neq 0$. Therefore (a) implies (c).

Conversely assume that (c) is true and $c_1y_1(x) + c_2y_2(x) = 0$ on I. Then $c_1y_1'(x) + c_2y_2'(x) = 0$ on I also. Hence we have

$$\begin{bmatrix} y_1(x_0) & y_2(X_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $c_1 = c_2 = 0$ since $W(y_1, y_2)(x_0) \neq 0$ by (c). Hence y_1 and y_2 are linearly independent on I which implies that (a) holds.

§6 Week 6

§6.1 Fundamental Set of Solutions

Definition — We call any set $\{y_i\}_{i=1}^n$ of n linearly independent solutions of a linear nth order homogeneous differential equation $L[y] = 0$ on an interval I, a fundamental set of solutions of the differential equation on I.

Theorem (Existence of a Fundamental Set)
A linear homogeneous differential equation $L[y] = 0$ always has a fundamental set of solutions on I.

Proof. Again, we assume that $n = 2$. By the uniqueness of solutions to the initial value problem, we know that for any x_0 in I, the following two IVP's

$$a_2y'' + a_1y' + a_0y = 0; y(x_0) = 1, y'(x_0) = 0$$

$$a_2y'' + a_1y' + a_0y = 0; y(x_0) = 0, y'(x_0) = 1$$

have unique solutions, say $y_1(x)$ and $y_2(x)$ respectively. Then

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(X_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0) & 1 \end{bmatrix} = 1 \neq 0$$

so that y_1 and y_2 are linearly independent on I by the Wronskian test. Hence $\{y_1, y_2\}$ is a fundamental set of solutions of the differential equation $L[y] = 0$ on I.

We had seen in a previous section that $y_1 = x$ and $y_2 = x \ln x$ are solutions of the differential equation

$$x^2y'' - xy' + y = 0 \text{ on } (0, \infty).$$

Moreover since $W(y_1, y_2)(x) = x \neq 0$ on $(0, \infty)$, $\{x, x \ln x\}$ is a fundamental set of solutions of the differential equation

$$x^2y'' - xy' + y = 0 \text{ on } (0, \infty).$$

§6.2 General Solutions

Theorem (General Solution of Linear Homogeneous Equations)
Let $\{y_i\}_{i=1}^n$ be a fundamental set of solutions of the differential equation $L[y] = 0$ on I. Then any solution of this differential equation on I must be of the form

$$y(x) = \sum_{i=1}^n c_i y_i(x),$$

where $\{c_i\}_{i=1}^n$ are arbitrary constants.

This theorem implies that the expression

$$y(x) = \sum_{i=1}^n c_i y_i(x)$$

with arbitrary constants $\{c_i\}_{i=1}^n$ is a general solution of the differential equation $L[y] = 0$ and that the set

$$\{\sum_{i=1}^{n} c_i y_i(x) : c_i's \text{ arbitrary constants}\}$$

is the set of all possible solutions of the differential equation $L[y] = 0$ on I. In particular, any linear homogeneous differential equation $L[y] = 0$ cannot have a singular solution.

Proof. Again, we assume that $n = 2$. Let $\{y_1, y_2\}$ be a fundamental set of solutions and $\phi(x)$ be a solution of the differential equation

$$a_2 y'' + a_1 y' + a_0 y = 0$$

on I.

now for any point x_0 in I, the simultaneous equation

$$\begin{bmatrix} y_1(x_0) & y_2(X_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \phi(x_0) \\ \phi'(x_0) \end{bmatrix}$$

has a unique set of solutions for c_1 and c_2 since $W(y_1, y_2)(x_0) \neq 0$, With this c_1 and c_2, both $\phi(x)$ and $c_1 y_1(x) + c_2 y_2(x)$ satisfy the initial value problem

$$a_2 y'' + a_1 y' + a_0 y = 0; y_(x_0) = \phi(x_0), y'(x_0) = \phi'(x_0).$$

Since every initial value problem has a unique solution, we have that $\phi(x) = c_1 y_1(x) + c_2 y_2(x)$ on I.

Example — Find the general solution of the differential equation

$$x^2 y'' - xy' - 3y = 0$$

on $(0, \infty)$ by finding particular solutions of the form x^r with constant r.

Setting $y = x^r$ in the equation gives us $[r(r-1)-r-3]x^r = (r^2-2r-3)x^r = 0$ which gives us that $r = -1$ or $r = 3$. So we have that $y_1 = \frac{1}{x}$ and $y_2 = x^3$ are solutions of the equation on $(0, \infty)$. Moreover, $W(y_1, y_2)(x) = 4x \neq 0$ on $(0, \infty)$ so we have that y_1 and y_2 are linearly independent on $(0, \infty)$. Hence $\frac{c_1}{x} + c_2 x^3$ with arbitrary constants c_1 and c_2 is the general solution to this differential equation.

§6.3 Nonhomogeneous Equations

We will now consider a general nonhomogeneous equation.

Theorem (Superposition Principal for Nonhomogeneous Equations)
If $L[\phi_i] = b_i(x)$ on an interval I for $i = 1, 2, ..., k$, where L is a linear differential operator, then

$$L[\sum_{i=1}^{k} c_i \phi_i] = \sum_{i=1}^{k} c_i b_i(x)$$

for arbitrary constants $\{c_i\}_{i=1}^k$.

Theorem (General Solution of Nonhomogeneous Equations)
Let y_p be any particular solution of the nonhomogeneous differential equation

$$L[y] = b(x)$$

on an interval I. Then its general solution is

$$y = y_c + y_p = \sum_{i=1}^{n} c_i y_i + y_p,$$

where $\{c_i\}_{i=1}^n$ are arbitrary constants and $\{y_i\}_{i=1}^n$ is any fundamental set of solutions of the corresponding homogeneous differential equation $L[y] = 0$ on I.

Proof. Let y be any solution of the differential equation $L[y] = b(x)$. Then $L[y - y_p] = b(x) - b(x) = 0$. Hence $y - y_p = \sum\limits_{i=1}^{n} c_y y_i$ for some constants $\{c_i\}_{i=1}^n$ so that we have $y = \sum\limits_{i=1}^{n} c_i y_i + y_p$. In fact, the family of solutions is the set of all solutions of the differential equation $L[y] = b(x)$. We call $y_c = \sum\limits_{i=1}^{n} c_i y_i$ the complementary solution of the nonhomogeneous differential equation $L[y] = b(x)$.

Example — Show that $\{e^x, e^{-2x}\}$ is a fundamental set of solutions of the equation $y'' + y' - 2y = 0$. Then solve the initial value problem

$$y'' + y' - 2y = 4; y(0) = 0, y'(0) = -1.$$

We can see by inspection that $y_p = -2$ is a particular solution of $y'' + y' - 2y = 4$. Hence its general solution is $y = c_1 e^x + c_2 e^{-2x} - 2$. Now the initial conditions gives us that $c_1 + c_2 - 2 = 0$ and $c_1 - 2c_2 = -1$ which implies that $c_1 = c_2 = 1$. Hence

$$e^x + e^{-2x} - 2$$

is the solution to the initial value problem.

§6.4 Reduction of Order

Consider a second order linear homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are continuous functions on an interval I. Then solving this differential equation is equivalent to finding its two linearly independent solutions.

Sometimes, one solution, say y_1, is easy to find and then we need to find the other solution y_2, which is linearly independent of y_1. Since y_1 and y_2 are linearly independent, their ratio $\frac{y_2}{y_1}$ cannot be constant on I. So we set

$$y_2(x) = u(x)y_1(x),$$

where a nonconstant function $u(x)$ will be determined later.

We an substitute this expression for y_2 back into the original equation which gives us

$$(u''y_1 + 2u'y_1' + uy_1'') + p(u'y_1 + y_1') + quy_1$$

$$= y_1 u'' + (2y_1' + py_1)u' + (y_1'' + py_1' + qy_1)u$$
$$= y_1 u'' + (2y')1 + py_1 u' = y_1 v' + (2y_1' + py_1)v = 0$$

where we have $v = u'$ which is a linear first order differential equation for v.

Solving this differential equation gives us

$$v = u' = \frac{c_1}{y_1^2} e^{-\int p(x)dx} \text{ and } u = c_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx + c_2.$$

We take $c_1 \neq 0$ and $c_2 = 0$. Then y_1 and $y_2 = y_1 u$ are linearly independent solutions of the original differential equation since $u(x) \neq$ constant. We call this process as the reduction of order.

Example — Knowing that e^x is a solution of $xy'' - (x+1)y' + y = 0$, solve the differential equation

$$xy'' - (x+1)y' + y = 2$$

on $(0, \infty)$.

First consider the corresponding homogeneous equation $xy'' - (x+1)y' + y = 0$, of which $y_1 = e^x$ is a solution. Set $y_2 = e^x u(x)$. Then we have

$$xy_2'' - (x+1)y_2' + y_2 = e^x[xu'' + (x-1)u'] = 0$$

so that

$$xu'' + (x-1)u' = xv' + (x-1)v = 0$$

where $v = u'$. Hence $v(x) = u'(x) = c_1 x e^{-x}$ and $u(x) = -c_1(x+1)e^{-x} + c_2$. Choosing $c_1 = -1$ and $c_2 = 0$ gives us $y_2 = e^x u(x) = x + 1$. Now the complementary solution is $y_c = c_1 e^x + c_2(x+1)$. Finally we know that the particular solution is $y_p = 2$ so the general solution is

$$y = c_1 e^x + c_2(x+1) + 2.$$

§6.5 Homogeneous Linear Equations with Constant Coefficients

Consider a second order homogeneous linear equation with constant coefficients

$$ay'' + b' + c = 0,$$

where a, b, c are real constants such that $a \neq 0$.

By inspection, we guess that this differential equation may have a solution of the form e^{rx}. Setting $y = e^{rx}$ in the differential equation gives us $(ar^2 + br + c)e^{rx} = 0$. Hence e^{rx} is a solution of the differential equation if and only if

$$ar^2 + br + c = 0,$$

which is called the characteristic or auxiliary equation of the differential equation.

The roots of this equation are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, of which their nature depends on the sign of the discriminant $b^2 - 4ac$. The characteristic equation has 2 distinct real roots $r_1 \neq r_2$ when $b^2 - 4ac > 0$, 1 double real root $r_1 = r_2$ when $b^2 - 4ac = 0$, and 2 complex conjugate roots $\alpha \pm i\beta$ $(\beta > 0)$ when $b^2 - 4ac < 0$.

When this equation has two distinct real roots, r_1 and r_2, $\{e^{r_1 x}, e^{r_2 x}\}$ form a fundamental set of solutions of the differential equation so its general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

When the equation has one double root, $r_1 = -\frac{b}{2a}$, $y_1 = e^{r_1 x}$ is a solution and we obtain, by reduction of order, a second solution $y_2 = e^{r_1 x} \int \frac{1}{e^{2r_1 x}} e^{-\int \frac{b}{a}} = xe^{r_1 x}$, which is linearly independent of y_1. Hence a general solution of the differential equation is

$$y = e^{r_1 x}(c_1 + c_2 x).$$

When the equation has complex conjugate roots, $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ where α and β are real with $\beta > 0$, $y_1 = e^{(\alpha+i\beta)x}$ and $y_1 = e^{(\alpha-i\beta)x}$ are linearly independent complex-valued solutions of the differential equation. Since we prefer to have real-valued solutions, consider the real and imaginary parts of y_1.

$$\mathrm{Re}(y_1) = \frac{1}{2}(y_1 + \overline{y_1}) = \frac{1}{2}(y_1 + y_2) = e^{\alpha x}\cos(\beta x)$$
$$\mathrm{Im}(y_1) = \frac{1}{2i}(y_1 - \overline{y_1}) = \frac{1}{2i}(y_1 - y_2) = e^{\alpha x}\sin(\beta x).$$

By the superposition principle, $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are also linearly independent real-valued solutions of the differential equation so its general solution is

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$

Example — Solve the differential equations

$$(a)\ y'' - 4y = 0;$$
$$(b)\ y'' + 4y' + 4y = 0;$$
$$(c)\ y'' - 2y' + 2y = 0.$$

(a) Note that $r^2 - 4 = (r+2)(r-2) = 0$ when $r = \pm 2$ so $y = c_1 e^{-2x} + c_2 e^{2x}$. Note that $\frac{1}{2}(e^{2x} + e^{-2x}) = \cosh(2x)$ and $\frac{1}{2}(e^{2x} - e^{-2x}) = \sinh(2x)$ are also two linearly independent solutions so $y = c_1 \cosh(x) + c_2 \sinh(x)$ is a general solution to the differential equation.

(b) Note that $r^2 + 4r + 4 = (r+2)^2 = 0$ when $r = -2$ so the general solution is $y = e^{-2x}(c_1 + c_2 x)$.

(c) Note that $r^2 - 2r + 2 = (r - (1+i))(r - (1-i)) = 0$ when $r = 1 + i$ or $r = 1 - i$ so the general solution is $y = e^x(c_1 \cos(x) + c_2 \sin(x))$.

§6.6 Higher Order Equations

We can extend previous arguments to any higher order equations with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

(a_i's are constants) of which the associated characteristic equation is

$$a_n r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0 = 0.$$

If r_1 is a real root of the characteristic equation with multiplicity $k \geq 1$ then $\{x^i e^{r_1 x}\}_{i=0}^{k-1}$ are k linearly independent solutions of the differential equation. If $\alpha + i\beta$ with α and β real with $\beta > 0$ is a complex root of the differential equation with multiplicity $k \geq 1$, then $\{x^i e^{\alpha x}\cos(\beta x), x^i e^{\alpha x}\sin(\beta x)\}_{i=0}^{k-1}$ are $2k$ linearly independent solutions of the differential equation. In this way, we can obtain n linearly independent solutions of the differential equation.

Example — Solve the differential equation

$$y^{(4)} - 2y^{(3)} + 2y^{(2)} - 2y' + y = 0.$$

The characteristic equation is $r^4 - 2r^3 + 2r^2 - 2r + 1 = (r-1)^2(r+i)(r-i) = 0$ when $r = 1, r = -i, r = i$. Therefore the general solution to this differential equation is

$$y = c_1 e^x + c_2 x e^x + c_3 \cos(x) + c_4 \sin(x).$$

§7 Week 7

§7.1 Differential Polynomials

Consider a nonhomogeneous constant coefficients linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = g(x)$$

(a_i's are constants with $a_n \neq 0$).

Then the general solution of this differential equation is

$$y = y_c + y_p,$$

where the complementary solution y_c is a general solution of the corresponding homogeneous differential equation

$$a_n y^{(y)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

and y_p is any particular solution of the differential equation

$$a_n y^{(y)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = g(x).$$

In this section, we see how to find y_c so we now consider the problem of finding y_p.

With the notation $D = \frac{d}{dx}$ and constants $\{a_i\}_{i=0}^n$, we call any expression of the form

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + ... + a_1 D + a_0$$

a differential polynomial. Then its easy to see that any two differential polynomials $P(D)$ and $Q(D)$ commute:

$$P(D)Q(D)y = Q(D)P(D)y.$$

Example — $(D-2)(D+1)y = (D+1)(D-2)y = (D^2 - D - 2)y = y'' - y' - 2y.$

§7.2 Annihilator

For a function $f(x)$ differentiable sufficiently many times, we say that a differential polynomial $P(D)$ annihilates $f(x)$ if $P(D)f = 0$. In this case, we call $P(D)$, where $P(D)$ is a differential polynomial, an annihilator of $f(x)$. For example, D^n for an integer n annihilates x^k for $k = 0, 1, ..., n-1$ but does not annihilate x^k for $k = n, n+1, n+2, ...$ In fact, D^n annihilates any polynomial of degree $< n$.

Example — Find the annihilators of the following expressions:

$$(a)\ 3e^{-x} - 2xe^{2x};$$

$$(b)\ -2e^{-x}\sin(2x);$$

$$(c)\ 3e^{-x}\sin(2x) - 2xe^{-x}\sin(2x) + x^2e^{-x}\cos(2x).$$

(a) $(D+1)e^{-x} = 0$ and $(D-2)^2xe^{2x} = 0$ so $(D+1)(D-2)^2(3e^{-x} - 2xe^{2x}) = 0$.

(b) We seek a quadratic polynomial in D with $-1 \pm 2i$ as its roots, which is $(D - (-1+2i))(D - (-1-2i)) = (D+1)^2 - (2i)^2 = D^2 + 2D + 5$. Then $(D^2 + 2D + 5)(-2e^{-x}\sin(2x)) = 0$.

(c) $(D^2 + 2D + 5)^3(3e^{-x}\sin(2x) - 2xe^{-x}\sin(2x) + x^2e^{-x}\cos(2x) = 0$.

Generalizing these above examples, we can see that for an integer $n \geq 1$ and real numbers α and $\beta \neq 0$, we have

- D^n annihilates $\{x^k\}_{k=0}^{n-1}$;
- $(D-\alpha)^n$ annihilates $\{x^ke^{\alpha x}\}_{k=0}^{n-1}$;
- $[(D - (\alpha + i\beta))(D - (\alpha - i\beta))]^n = (D^2 - 2\alpha D + \alpha^2 + \beta^2)^n$ annihilates $\{x^ke^{\alpha x}\cos(\beta x), x^ke^{\alpha x}\sin(\beta x)_{k=0}^{n-1}$.

§7.3 Method of Undetermined Coefficients

Now let's return to our original problem of finding a particular solution y_p of the differential equation $P(D)y = g$, where we assume that g has an annihilator $Q(D)$, a differential polynomial of degree m. Choose a fundamental set of solutions of a homogeneous differential equation

$$Q(D)P(D)y = 0$$

in the form $\{y_i\}_{i=1}^n \cup \{z_j\}_{j=1}^m$, where $\{y_i\}_{i=1}^n$ is a fundamental set of solutions of $P(D)y = 0$. Then

$$y = \sum_{i=1}^n c_iy_i + \sum_{j=1}^m d_jz_j$$

where the c_i's and d_j's are arbitrary constants, is a general solution of the differential equation

$$Q(D)P(D)y = 0.$$

Since

$$Q(D)P(D)(y_p) = Q(D)(g) = 0,$$

y_p must be of the form $\sum\limits_{i=1}^n c_iy_i + \sum\limits_{j=1}^m d_jz_j$ for suitable $\{c_i\}_{i=1}^n$ and $\{d_j\}_{j=1}^m$. In particular, we may take all of the $c_i = 0$ since $P(D)(y_i) = 0$ for $1 \leq i \leq n$. Hence, there must be a particular solution y_p of $P(D)(y) = g$ of the form

$$y_p = \sum_{j=1}^m d_jz_j$$

for suitable constants $\{d_j\}_{j=1}^m$, which are determined from $P(D)(y_p) = g$. We call this process the method of undetermined coefficients.

Example — Solve the following differential equations:

$$(a)\ y'' + y' + y = -5\cos(2x) + \sin(2x);$$

$$(b)\ y'' - 4y' + 4y = -e^{2x};$$

$$(c)\ y'' + 4y = -4\sin^2(x).$$

(a) The characteristic equation of the corresponding homogeneous equation is $r^2 + r + 1 = 0$ so $r = \frac{-1\pm\sqrt{3}i}{2}$ and the complementary solution is $y_c = e^{-\frac{x}{2}}(c_1\cos(\frac{\sqrt{3}x}{2}) + c_2\sin(\frac{\sqrt{3}x}{2}))$. $(D^2+4)(-5\cos(2x) + \sin(2x)) = 0$ so applying (D^2+4) to the differential equation gives us

$$(D^2+4)(D^2+D+1)y = 0$$

of which a general solution is $y = y_c + d_1\cos(2x) + d_2\sin(2x)$. Hence there is a particular solution of the form

$$y_p = d_1\cos(2x) + d_2\sin(2x).$$

Substituting y_p back into the differential equation gives us

$$y_p'' + y_p' + y_p = (-3d_1 + 2d_2)\cos(2x) + (-2d_1 - 3d_2)\sin(2x) = -5\cos(2x) + \sin(2x).$$

Equating coefficients gives us

$$-3d_1 + 2d_2 = -5, -2d_1 - 3d_2 = 1$$

which implies that $d_1 = 1$ and $d_2 = -1$. Hence $y_p = \cos(2x) - \sin(2x)$ and a general solution is

$$y = y_c + y_p = e^{-\frac{x}{2}}(c_1\cos(\frac{\sqrt{3}x}{2}) + c_2\sin(\frac{\sqrt{3}x}{2})) + \cos(2x) - \sin(2x).$$

(b) The complementary solution is $y_c = e^{2x}(c_1 + c_2x)$. Since $(D-2)(-e^{2x}) = 0$, we have $(D-2)^3y = 0$, of which a general solution is $y = y_c + dx^2e^{2x}$. Hence, we try a particular solution $y_p = dx^2e^{2x}$ into the differential equation to obtain

$$y_p'' - 4y_p' + 4y_p = 2de^{2x} = -e^{2x}.$$

This implies that $d = -\frac{1}{2}$ and a general solution to this differential equation is

$$y = e^{2x}(c_1 + c_2x) - \frac{1}{2}x^2e^{2x}.$$

(c) The complementary solution is $y_c = c_1\cos(2x) + c_2\sin(2x)$. Since $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $D(D^2 + 4)(\sin^2(x)) = 0$ so that $D(D^2+2)^2y = 0$, of which a general solution is

$$y = y_c + d_1 + d_2x\cos(2x) + d_3x\sin(2x).$$

Hence, we try a particular solution

$$y_p = d_1 + d_2x\cos(2x) + d_3x\sin(2x)$$

into the differential equation to obtain

$$y_p'' + 4y_p = 4d_3\cos(2x) - 4d_2\sin(2x) + 4d_1 = -4\sin^2(x) = -4\cdot\frac{1}{2}(1 - \cos(2x)).$$

This implies that $d_1 = -\frac{1}{2}, d_2 = 0$, and $d_3 = \frac{1}{2}$ and a general solution to this differential equation is

$$y = c_1\cos(2x) + c_2\sin(2x) - \frac{1}{2} + \frac{1}{2}x\sin(2x).$$

The method of undetermined coefficients cannot be used for a linear differential equation $P(D)(y) = g$ unless g has a differential polynomial annihilator. For example $\frac{1}{x}$ and $\sec(x)$ have no differential polynomial annihilator. The class of functions $g(x)$ which allow the method of undetermined coefficients includes polynomials, $e^{\alpha x}, \cos(\beta x), \sin(\beta x)$, or finite linear combinations and products of these functions.

Example — Determine the form of a particular solution for the differential equation

$$y''' - 2y'' + y' = -3x^2 + 2x - xe^x + 2e^{2x}.$$

Since $D^3(-3x^2 + 2x) = 0, (D-1)^2 xe^x = 0$ and $(D-2)e^{2x} = 0$, we have

$$D^3(D-1)^2(D-2)(D^3 - 2D^2 + D)y = D^4(D-1)^4(D-2)y = 0,$$

which has a general solution

$$c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^x + c_6xe^x + c_7x^2e^x + c_8x^3e^x + c_9e^{2x}.$$

Since $y_c = c_1 + c_5e^x + c_6xe^x$ is a complementary solution, there is a particular solution of the form

$$y_p = d_1x + d_2x^2 + d_3x^3 + d_4x^2e^x + d_5x^3e^x + d_6e^{2x}.$$

§8 Week 8

§8.1 Variation of Parameters

Though the method of undetermined coefficients is simple, it has two strong constraints: for a linear nonhomogeneous differential equation $L[y] = g(x)$, differential operator $L[\cdot]$ must have constant coefficients and the input function $g(x)$ must have a differential polynomial annihilator. Here, we introduce the method of variation of parameters to obtain a particular solution of a linear nonhomogeneous differential equation $L[y] = g(x)$, where $L[\cdot]$ may have variable coefficients and $g(x)$ can be arbitrary.

Consider a nonhomogeneous linear second order differential equation with variable coefficients

$$y'' + p(x)y' + q(x)y = g(x).$$

Assume that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation $y'' + p(x)y' + q(x)y = 0$ and we seek a particular solution of this differential equation of the form

$$y_p = u_1y_1 + u_2y_2,$$

where $u_1(x)$ and $u_2(x)$ are determined later.

Since we have two unknowns u_1 and u_2, we need two conditions. One comes from

$$y_p'' + py_p' + qy_p = g.$$

We take another condition which makes the computations easy. Differentiating y_p gives us

$$y_p' = (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2).$$

We requires $u_1'y_1 + u_2'y_2 = 0$ so that $y_p' = u_1y_1' + u_2y_2'$ and $y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'$. Then we have

$$y_p'' + py_p' + qy_p = u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) + u_1'y_1' + u_2'y_2' = u_1'y_1' + u_2'y_2' = g.$$

Hence we obtain a system of equations for two unknowns u_1' and u_2':

$$y_1u_1' + y_2u_2' = 0$$

$$y_1'u_1' + y_2'u_2' = g.$$

Solving this system by Cramer's rule gives us

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{W} = -\frac{y_2 g}{W}, u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{W} = \frac{y_1 g}{W}$$

where $W = W(y_1, y_2) = y_1 y_2' - y_1' y_2 \neq 0$ is the Wronskian of y_1 and y_2. With u_1 and u_2 obtained from the equations above by integrating, y_p is a particular solution of the differential equation. We call this process of obtaining a particular solution of a nonhomogeneous differential equation, the variation of parameters.

Theorem (Variation of Parameters)

Assume that p, q, and g are continuous on (α, β) and that y_1 and y_2 are linearly independent solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$. Then we have

$$y_p = -y_1 \int \frac{y_2 g}{W} dx + y_2 \int \frac{y_1 g}{W} dx \text{ where } (W = W(y_1, y_2))$$

is a particular solution and $y = c_1 y_1 + c_2 y_2 + y_p$ is a general solution of the nonhomogeneous differential equation

$$y'' + p(x)y' + q(x)y = g(x).$$

Example — Solve the initial value problem

$$y'' + 2y' + y = \frac{e^{-x}}{x^2+1}; y(0) = \frac{1}{2}, y'(0) = -\frac{1}{2}.$$

The characteristic equation for this differential equation is $r^2 + 2r + 1 = (r+1)^2 = 0$ when $r = -1$. Therefore we have $y_1 = e^{-x}$ and $y_2 = xe^{-x}$. We also have

$$W = \begin{vmatrix} e^{-x} & e^{-x} - xe^{-x} \\ -e^{-x} & y_2' \end{vmatrix} = e^{-2x}.$$

We also have that $y_p = u_1 y_1 + u_2 y_2$ where

$$y_1 u_1' + y_2 u_2' = 0;$$
$$y_1' u_1' + y_2' u_2' = \frac{e^{-x}}{x^2+1}.$$

This gives us that

$$u_1' = -\frac{x}{x^2+1} \longrightarrow u_1 = -\frac{1}{2} \ln{(x^2+1)} \text{ and } u_2' = \frac{1}{x^2+1} \longrightarrow u_2 = \arctan{(x)}.$$

So we have $y_p = -\frac{1}{2} e^{-x} \ln{(x^2+1)} + xe^{-x} \arctan{(x)}$ and a general solution is

$$y = c_1 e^{-x} + c_2 x e^{-x} - \frac{1}{2} e^{-x} \ln{(x^2+1)} + xe^{-x} \arctan{(x)}.$$

From the initial conditions we have $y(0) = c_1 = \frac{1}{2}$ and $y'(0) = -c_1 + c_2 = -\frac{1}{2}$ which gives us that $c_2 = 0$. Therefore the solution to this initial value problem is

$$y = \frac{1}{2} e^{-x} - \frac{1}{2} e^{-x} \ln{(x^2+1)} + xe^{-x} \arctan{(x)}.$$

Example — Solve the differential equation

$$xy'' - (1+x)y' + y = x^2 e^{2x}.$$

(Hint: e^x satisfies the differential equation $xy'' - (1+x)y' + y = 0$.)

Set $y_1 = e^x$ and we wish to find a second solution of the differential equation

$$xy'' - (1+x)y' + y = 0$$

of the form $y_2 = e^x u(x)$. Then

$$xy_2'' - (1+x)y_2' + y_2 = e^x[xu'' + (x-1)u'] = 0$$

so we have

$$xu'' + (x-1)u' = xv' + (x-1)v = 0 \text{ where } v = u'.$$

Hence $v = u' = c_1 x e^{-x}$ and so $u = c_1(-xe^{-x} - e^{-x}) + c_2$. Take $c_1 = -1$ and $c_2 = 0$ so that $u = (x+1)e^{-x}$ and $y_2 = x+1$. Then y_1 and y_2 are linearly independent solutions of the differential equation

$$xy'' - (1+x)y' + y = 0 \text{ and } W = -xe^x.$$

Set $y_p = y_1 u_1 + y_2 u_2$ where we have

$$y_1 u_1' + y_2 u_2' = 0;$$
$$y_1' u_1' + y_2' u_2' = xe^{2x}.$$

Then we have

$$u_1' = \frac{-x(x+1)e^{2x}}{-xe^x} = (x+1)e^x \longrightarrow u_1 = xe^x$$

$$u_2' = \frac{xe^{3x}}{-xe^x} = -e^{2x} \longrightarrow u_2 = -\frac{1}{2}e^{2x}.$$

So we have $y_p = xe^x \cdot e^x - \frac{1}{2}e^{2x}(x+1) = \frac{1}{2}(x-1)e^{2x}$ and the general solution to this differential equation is

$$y = c_1 e^x + c_2(x+1) + \frac{1}{2}(x-1)e^{2x}.$$

§9 Week 9

§9.1 Spring-Mass System

Here, we provide two physical phenomena, of which their mathematical modelings involve linear second order differential equations with constant coefficients.

Attach a mass m to a spring of length l, which is suspended from a rigid support so that the spring is stretched with elongation δl and reaches its equilibrium state. Then there are two forces acting on the mass, the gravitational force ($w = mg$) and the spring's restoring force (F_s) acting upward. Here we adopt the convention that the downward direction is positive.

By Hooke's law, when δl is small compared to l, F_s is proportional to the elongation so that $F_s = -k\delta l$, where $k > 0$ is the spring constant. Since the mass is at its equilibrium state, we have $mg = k\delta l = 0$. Now let the mass be displaced further from its equilibrium state by an amount $x(t)$, measured positive downward and then released.

Then there are 4 forces acting on the system:

• The weight $w = mg$, due to gravitation, acting downward;
• Spring force $F_s = -k(\delta l + x)$, acting upward when spring is extended and and when spring is compressed, it acts downward
• Damping force F_d due to the viscosity of the fluid in which the mass is moving, which is proportional to the mass speed so that $F_d = -cx'(t)$, where $c > 0$ is the damping constant when the speed of the mass is small. The negative sign here means that the damping force acts in the direction opposite to the mass moving direction;
• Any external force $f(t)$ acting on the mass.

Then by Newton's second law of motion, we have

$$mx'' = mg + F_s + F_d + f = mg - k(\delta l + x) - cx + f \longrightarrow mx'' + cx' + kx = f$$

since $mg = k\delta l$, which is a second order linear differential equation with constant coefficients called the differential equation of forced damped motion (or free damped motion if $f = 0$).

Setting $\lambda = \frac{c}{2m}, \omega^2 = \frac{k}{m}$, and $g(t) = \frac{f(t)}{m}$, we can rewrite the differential equation as

$$x'' + 2\lambda x' + w^2 x = g(t).$$

§9.2 Free Undamped Motion

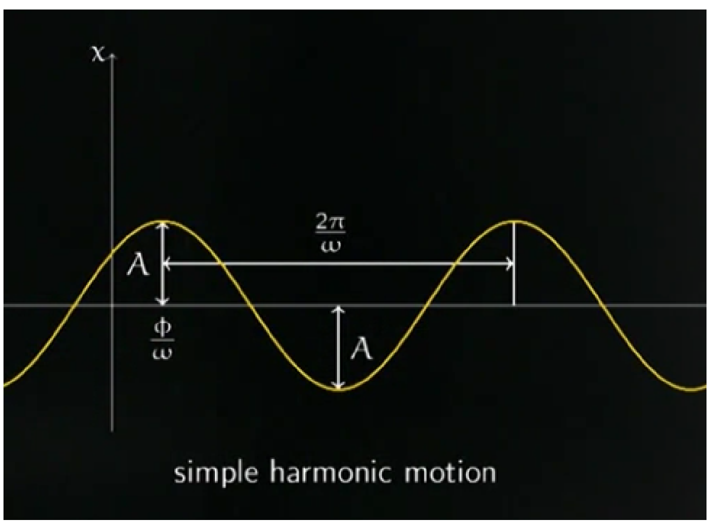
When there is no external force and no damping, i.e. $c = 0$ and $f = 0$, then the differential equation becomes

$$x'' + w^2 x = 0$$

of which the general solution is

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\cos(wt - \phi),$$

where $A = \sqrt{c_1^2 + c_2^2}, c_1 = A\cos(\phi)$, and $c_2 = A\sin(\phi)$. We call A the amplitude and ϕ the phase angle of the simple harmonic motion. Note that the simple harmonic motion has period $T = \frac{2\pi}{w}$ seconds and frequency $F = \frac{\omega}{2\pi}$ cycles per second.



Example — A 1 kg mass is attached to a spring with stiffness/spring constant of 4 kg/sec^2. At time $t = 0$, the mass is stretched downward $\frac{\sqrt{3}}{4}$ m from its equilibrium point and released with upward velocity $\frac{1}{2}$ m/sec. Assuming there is no damping and no external force, find its motion, amplitude, and phase angle in the interval $(-\pi, \pi]$. Find also the earliest time after release at which the mass passes through the equilibrium position.

Since the problem is a free undamped motion problem with $m = 1$ and $k = 4$, we have the following initial value problem:

$$x'' + 4x = 0; x(0) = \frac{\sqrt{3}}{4}, x'(0) = -\frac{1}{2}.$$

Solving this differential equation gives us that

$$x(t) = \frac{\sqrt{3}}{4}\cos(2t) - \frac{1}{4}\sin(2t) = \frac{1}{2}\cos(2t + \frac{\pi}{6})$$

so it has amplitude $\frac{1}{2}$ and phase angle $-\frac{\pi}{6}$. The mass passes through the equilibrium position when $x(t) = \frac{1}{2}\cos(2t + \frac{\pi}{6}) = 0$. Then $2t + \frac{\pi}{6} = \frac{\pi}{2} + k\pi$ so $t = \frac{\pi}{6} + \frac{k\pi}{2}$ for $k \in \mathbb{Z}$. Hence the first such time is when $t = \frac{\pi}{6}$ seconds.

§9.3 Free and Forced Damped Motion

When there is no external force, i.e. $f = 0$, the differential equation becomes

$$x'' + 2\lambda x' + \omega^2 x = 0.$$

Its characteristic equation is $r^2 + 2\lambda r + w^2 = 0$ so $r = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$. Hence there are three distinct cases depending on the sign of $\lambda^2 - \omega^2$. For each of the three cases, there is a solution to the differential equation.

Case 1: $\lambda^2 - \omega^2 > 0$ (overdamped)

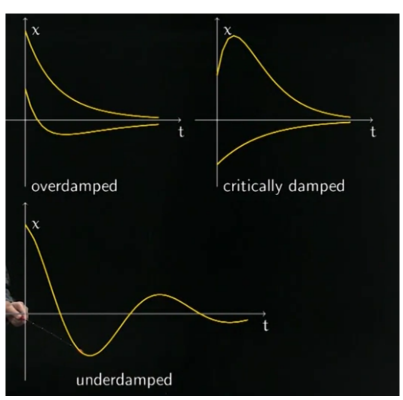
$$x(t) = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{\sqrt{-\lambda^2-\omega^2}t}.$$

Case 2: $\lambda^2 - \omega^2 = 0$ (critically damped)

$$x(t) = e^{-\lambda t}(c_1 + c_2 t).$$

Case 3: $\lambda^2 - \omega^2 < 0$ (underdamped)

$$x(t) = e^{-\lambda t}(c_1\cos(\sqrt{\omega^2 - \lambda^2}t) + c_2\sin(\sqrt{\omega^2 - \lambda^2}t).$$



Note that in cases 1 and 2, $x(t)$ is non-oscillatory but in case 3, $x(t)$ is oscillatory. In all 3 cases, we have

$$\lim_{t\to\infty} x(t) = 0.$$

Case 2 is said to be critically damped since any small decrease in the damping force results in oscillatory motion. In case 3, $x(t)$ can be written as

$$x(t) = Ae^{-\lambda t}\cos\left(\sqrt{\omega^2-\lambda^2}t-\phi\right)$$

where $A = \sqrt{c_1^2+c_2^2}$, $\cos(\phi) = \frac{c_1}{A}$, and $\sin(\phi) = \frac{c_2}{A}$. Here we call $Ae^{-\lambda t}$ the damped altitude, $\frac{2\pi}{\sqrt{\omega^2-\lambda^2}}$ the quasi period, and $\frac{\sqrt{\omega^2-\lambda^2}}{2\pi}$ the quasi frequency. The quasi period is the time interval between successive maxima of $x(t)$.

We will now move on to forced damped motion. We now consider the nonhomogeneous differential equation describing forced damped motion. Its general solution is

$$x(t) = x_c(t) + x_p(t)$$

where $x_c(t)$ is the general solution of the differential equation of free damped motion and $x_p(t)$ is a particular solution of the original differential equation. In particular, if $g(t) = a\cos(\gamma t) + b\sin(\gamma t)$ where $a^2+b^2 \neq 0$ is periodic, then there is a particular solution of the form $x_p(t) = A\cos(\gamma t) + B\sin(\gamma t)$. Since

$$\lim_{t\to\infty} x_c(t) = 0, x(t) \sim x_p(t) \text{ for } t \text{ large enough.}$$

In this sense, we call $x_c(t)$ a transient solution and $x_p(t)$ a steady state solution of the differential equation.

§9.4 Examples

Example — A 1 kg mass is attached to a 5 m long spring. At equilibrium the spring measures 6 m. We push the mass up and release it from rest at a point 50 cm above the equilibrium position. Assume that the gravitational acceleration is $g = 10$ m/sec^2 and that the damping constant of the medium surrounding the mass is 7 kg/sec. Find the displacement $x(t)$ of the mass from its equilibrium position. Can the mass pass through the equilibrium position?

Since $mg = 10 = k\delta l = k(6-5) = k$ we have that $k = 10$ kg/sec^2. Therefore we have that $x(t)$ satisfies the initial value problem

$$x'' + 7x' + 10x = 0; x(0) = -\frac{1}{2}; x'(0) = 0.$$

Then the characteristic equation of this differential equation is $r^2 + 7r + 10 = 0$ at $r = -5, -2$. Therefore we have that

$$x(t) = c_1e^{-5t} + c_2e^{-2t}.$$

Solving the initial value problem gives us that $c_1 = \frac{1}{3}$ and $c_2 = -\frac{5}{6}$. Therefore we have that

$$x(t) = \frac{1}{3}e^{-5t} - \frac{5}{6}e^{-2t}.$$

If the mass passes through the equilibrium position, then we must have that $x(t) = 0$. So we have that

$$x(t) = 0 \text{ at } t = -\frac{1}{3}\ln\frac{5}{2} < 0.$$

Therefore the mass cannot pass through the equilibrium position.

Example — In the spring-mass system as in the previous example, find its steady state solution when there is an external force $2\cos(2t)$ kg $\cdot$ m/sec^2 acting on the system.

The governing differential equation is

$$x'' + 7x' + 10x = 2\cos(2t)$$

and so there is a steady state solution (particular solution) of the form $x_p(t) = A\cos(2t) + B\sin(2t)$. Substituting $x_p(t)$ into the above differential equation gives us that

$$(6A + 14B)\cos(2t) + (6B - 14A)\sin(2t) = 2\cos(2t)$$

$$6A + 14B = 2, 6B - 14A = 0$$

$$A = \frac{3}{58}, B = \frac{7}{58}$$

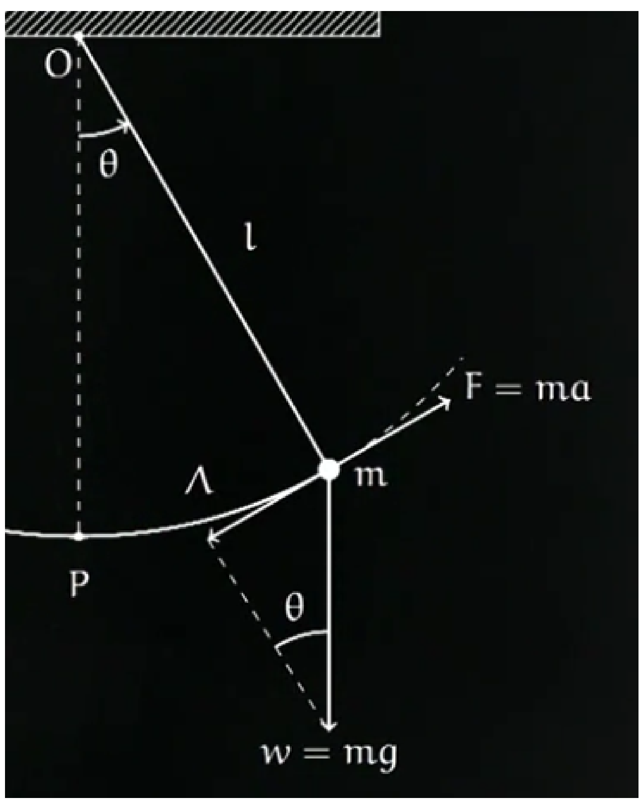
$$x_p(t) = \frac{1}{58}(3\cos(2t) + 7\sin(2t)).$$

§9.5 Pendulum

A mass m suspending from the end of a rod of length l is swinging in the vertical plane. Let $\theta(t)$ be the angle, measured in radians, from the vertical with the right side to be positive. Since the length of an arc with vertical angle θ is $\Lambda = l\theta$, the angular acceleration is $\alpha = \Lambda''(t) = l\theta''(t)$. Then Newton's second law of motion gives us

$$F = m\alpha = ml\theta'' = -mg\sin(\theta),$$

where the right hand side is the tangential component of the gravitational force $w = mg$ acting on the mass in the direction opposite to the motion after neglecting damping force and the mass of the rod.



Hence we obtain that

$$\theta'' + \frac{g}{l}\sin(\theta) = 0.$$

When $|\theta|$ is small, we have that

$$\sin(\theta) = \theta$$

so that we may approximate the equation above by

$$\theta'' + \frac{g}{l}\theta = \theta'' + \omega^2\theta = 0 \ (\omega^2 = \frac{g}{l}),$$

which is a linearized equation of the previous equation.

Note that this differential equation is the same as the differential equation for free undamped spring motion. The general solution of this differential equation is

$$\theta = c_1\cos(\omega t) + c_2\sin(\omega t) = A\cos(\omega t - \phi),$$

where $A = \sqrt{c_1^2 + c_2^2}$ is the amplitude, ϕ is the phase angle, $c_1 = A\cos(\phi)$, and $c_2 = A\sin(\phi)$.