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Signal analysis and processing
Lab Experience Session 1

Exercise 1

Analysis of the discrete-time signal $x[n] = a^{|n|}$ (with n integer from $-\infty$ to $+\infty$) considering three different cases of the absolute value of the parameter a. The signal was analyzed in MATLAB over the finite extension interval $[-25, 25]$.

· **Case (a):** $|a| < 1$ $\qquad x[n] = \frac{1}{2}^{|n|}$

By calculating the theoretically energy using a geometric series, we obtain:

$$E\{x[n]\} = \sum_{-\infty}^{+\infty}|x[n]|^2 = \sum_{-\infty}^{+\infty} a^{|2n|} = \sum_{-\infty}^{+\infty}\left(\frac{1}{4}\right)^{|n|} =$$

$$= \sum_{-\infty}^{-1}\left(\frac{1}{4}\right)^{|n|} + \sum_{0}^{+\infty}\left(\frac{1}{4}\right)^{n} = \sum_{1}^{+\infty}\left(\frac{1}{4}\right)^{n} +$$

$$+ \sum_{0}^{+\infty}\left(\frac{1}{4}\right)^{n} = \sum_{0}^{+\infty}\left(\frac{1}{4}\right)^{n} - 1 + \sum_{0}^{+\infty}\left(\frac{1}{4}\right)^{n} = \frac{5}{3}$$

The theoretical average power is clearly expected to approach zero as $N \to \infty$:

$$P\{x[n]\} = \lim_{N\to\infty} \frac{1}{2N+1} E\{x[n]\} = 0$$

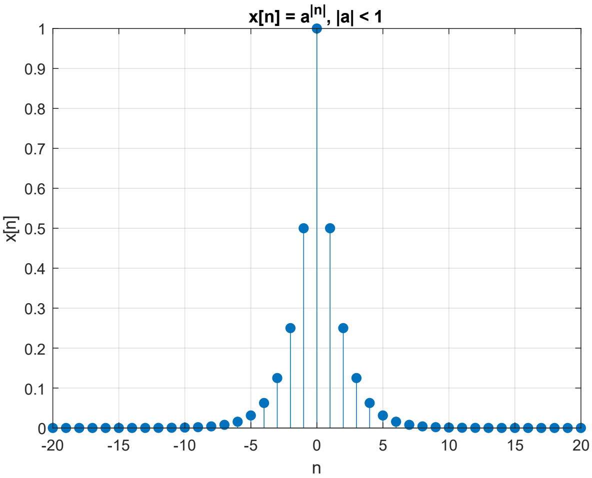

Figure 1: plot of x[n] with a=0.5

Numeric results
MATLAB-computed approximate energy and average power:

```
Approximate energy of the signal for |a| < 1 (a = 0.50): 1.6666666667
Approximate average power of the signal for |a| < 1 (a = 0.50): 0.0327
```

The MATLAB results closely match the theoretical values, confirming the value of energy, while average power trends towards zero as expected.

· **Case (b):** $|a| > 1$ $\qquad x[n] = \frac{3}{2}^{|n|}$

Theoretically, for this case energy results infinite because the summation $\sum_{-\infty}^{+\infty}|x[n]|^2$ diverges. Similarly, the average power also diverges since the energy across any finite interval grows unbounded as the interval size increases.

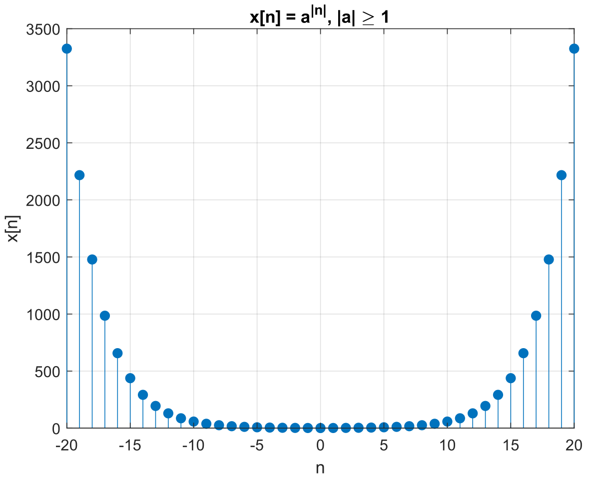

Figure 2: plot of x[n] with a=1.5

Numeric results

```
Approximate energy of the signal for |a|
> 1 (a = 1.50): 2295437398.1706
Approximate average of the signal for
|a| > 1 (a = 1.50): 45008576.4347
```

These high values are consistent with theoretical expectations.

· **Case (c):** $|a| = 1$ $\qquad x[n] = 1^{|n|}$

In this final case, $x[n]$ remains constantly equals to 1 along the entire extension. Theoretically, this implies an infinite energy, since the summation extends from $-\infty$ to $+\infty$, while the average power converges to 1:

$$P\{x[n]\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-\infty}^{+\infty} 1^{2|n|} = 1$$

Numeric Results
In MATLAB we can only deal with finite intervals:

```
Approximate energy of the signal for |a| > 1 (a = 1.00): 51.0000
```
(corresponds to the interval size)
```
Approximate average of the signal for |a| > 1 (a = 1.00): 1.0000
```

Energy and average power trends
About numerical results in MATLAB, in case (a) ($|a| < 1$), as the interval size increases, the energy remains constant, while the power gets closer and closer to zero. For case (b) ($|a| > 1$), both energy and power grow as the interval size increases. For case (c) ($|a| = 1$), the energy grows linearly with the interval size, while the average power remains constantly at 1.

Minimum interval size
To determine the minimum interval size required to achieve a relative error below 0.001% for $|a| < 1$, values of N were tested from 5 to 500. Results indicate that for $a = 0.5$, an interval of $N = 10$ is sufficient for this accuracy level. In fact, the tendency is that the closer the parameter a gets to 1, the higher the minimum dimension, the larger the error. As a gets closer to 1, the rate at which the series converges decreases, which implies that more terms are needed to achieve acceptable accuracy in the energy estimate. Consequently, the relative error tends to increase, making it more difficult to obtain a good estimate of the total energy of the signal.

Exercise 2

a) Analysis of the discrete-time signal:

$$x[n] = e^{j4\pi\frac{n}{N}} = e^{j\frac{\pi}{2}n}$$

for n in $[0, \mathrm{N}-1]$ with $\mathrm{N} = 8$

Energy of $x[n]$: $E\{x[n]\} = \sum_{n=0}^{8} \left|e^{j\frac{\pi}{2}n}\right|^2 = 8$

Power of $x[n]$: $P\{x[n]\} = \frac{E\{x[n]\}}{N} = 1$

The numeric results obtained in MATLAB are consistent with theoretical ones.

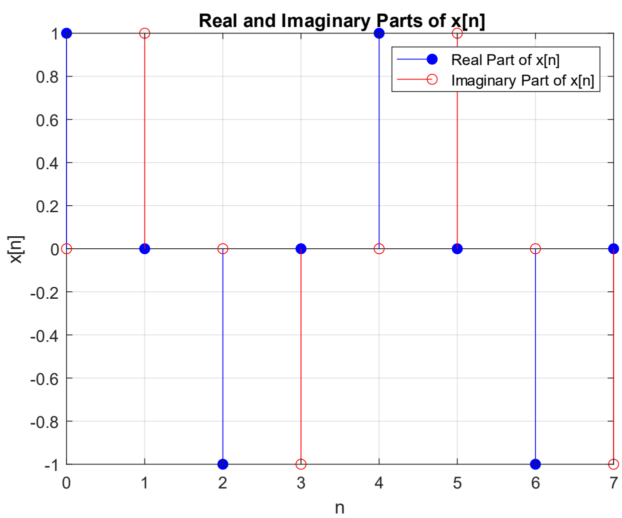

Figure 3: plot of x[n]

b) Analysis of the discrete-time signal

$$y[n] = e^{j8\pi\frac{n}{N}} = e^{j\pi n}$$

for n in $[0, \mathrm{N}-1]$ with $\mathrm{N} = 8$

Energy of $y[n]$: $E\{y[n]\} = \sum_{n=0}^{8} \left|e^{j\pi n}\right|^2 = 8$

Power of $y[n]$: $P\{y[n]\} = \frac{E\{y[n]\}}{N} = 1$

Also there, the numeric results obtained in MATLAB are consistent with theoretical ones.

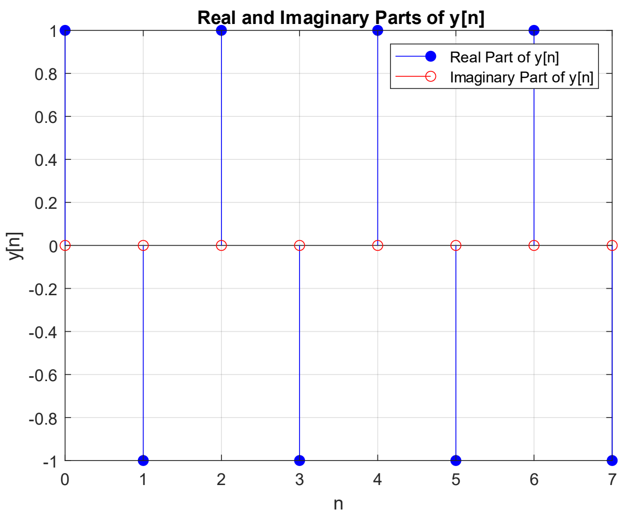

Figure 4: plot of y[n]

Orthogonality of the two signals
To check the orthogonality in $[0, N-1]$, let's calculate the inner product between $x[n]$ and $y[n]$:

$$(x[n], y[n]) = \sum_{n=0}^{N-1} x_n y_n^* = \sum_{n=0}^{N-1} e^{j\frac{\pi}{2}n}\, e^{-j\pi n} = \sum_{n=0}^{7} e^{-j\frac{\pi}{2}n} = 1 - j - 1 + j + 1 - j - 1 + j = 0$$

So, they're orthogonal.

Exercise 3

Given the discrete-time signal:

$$x[n] = 3\delta[n] + 2\delta[n-1] - \delta[n-2] + 4\delta[n-3]$$

$x[n]$ was created in MATLAB using the indicator functions $(n == k)$ that well simulate the delta function.

Since a set of orthonormal vectors $B = \{u_0, u_1, \dots, u_{N-1}\}$ forming a matrix U allows us to exactly represent any vector x as $x = c_o u_o + c_1 u_1 + + \cdots + c_{N-1} u_{N-1}$ with $c_k = x \cdot u_k$, the coefficients obtained by projecting the signal on three different basis were computed.

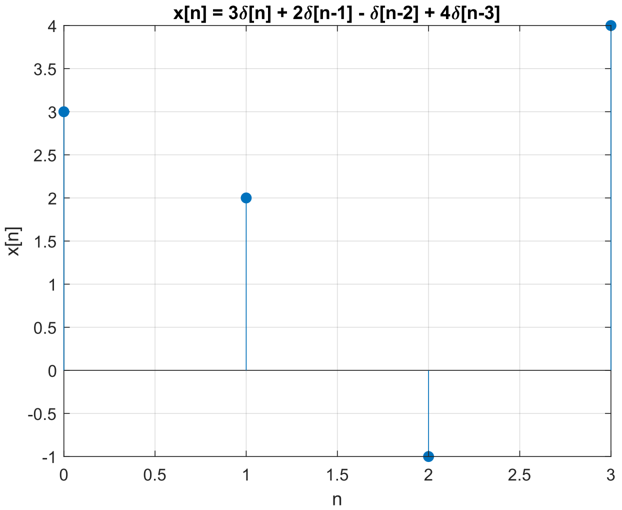

Figure 5: plot of x[n]

1. Canonical basis

The matrix representing the canonical basis is the identity matrix:

$$I = U_\delta = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0\,.,\dots & 0 & \cdots & 1 \end{pmatrix}$$

So, the coefficients c_k are obtained as:

$$c_k =< x[n], \delta[n-k] > = x[k]$$

In MATLAB, dealing with matrix, we compute the vector $c = U_\delta^T x$.

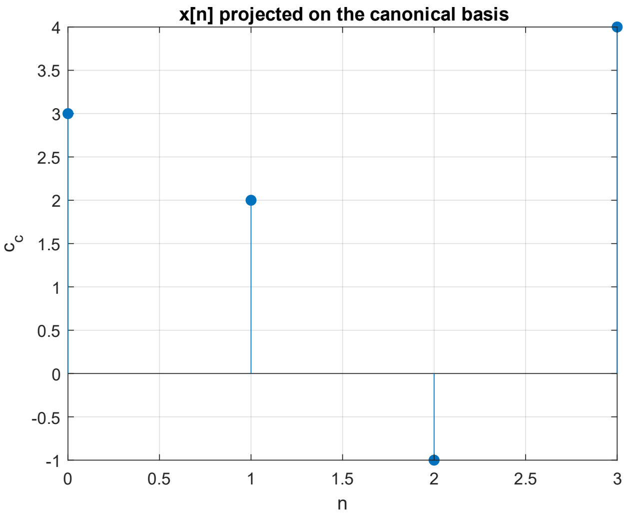

Figure 6: x[n] projected on the canonical basis

As we can see in Figure 6, the projection on the canonical basis corresponds to the original signal.

Expression of the signal using the canonical basis:
$x[n] = c_0\delta[n] + c_1\delta[n-1] + \cdots + c_{N-1}\delta[n-(N+1)] = \sum_{k=0}^{N-1} x[k]\delta[n-k]$

The reconstruction of the signal in MATLAB is obtained as $x = U_c\, c_c$;

2. Fourier basis

The Fourier matrix is defined as:

$$U_F = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)(N-1)} \end{pmatrix}$$

where $w = e^{j2\pi\frac{n}{N}}$ and each row corresponds to a vector u_k.

Expression of the signal using the Fourier basis: $x[n] = \sum_{k=0}^{N-1} c_k u_k[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c_k e^{j2\pi k \frac{n}{N}}$

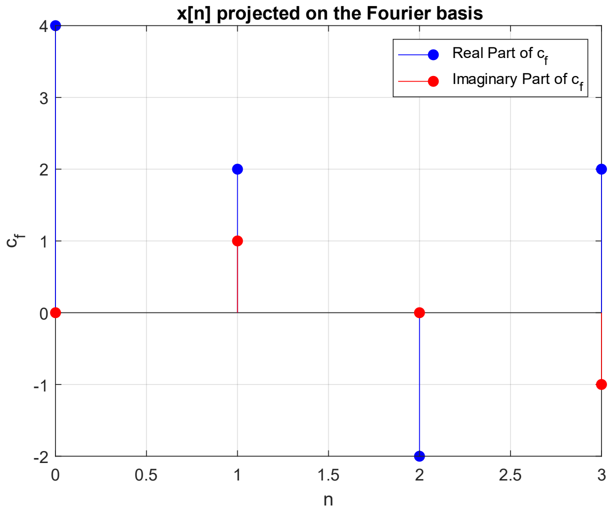

Figure 7: x[n] projected on the Fourier basis

3. Generic orthonormal basis

For the last point, a generic orthonormal basis U_r was created from a random real matrix A by using the Gram-Shmidt algorithm.

$$A = \begin{pmatrix} 1.0933 & -1.2141 & -0.7697 & -1.0891 \\ 1.1093 & -1.1135 & 0.3714 & 0.0326 \\ -0.8637 & -0.0068 & -0.2256 & 0.5525 \\ 0.0774 & 1.5326 & 1.1174 & 1.1006 \end{pmatrix} \rightarrow U_r = \begin{pmatrix} 0.6133 & -0.2101 & -0.5993 & 0.4697 \\ 0.6223 & -0.1469 & 0.7630 & 0.0953 \\ -0.4845 & -0.3747 & 0.2285 & 0.7567 \\ 0.0434 & 0.8910 & 0.0806 & 0.4446 \end{pmatrix}$$

As before, the vector containing the coefficients c_k was computed via MATLAB and the signal reconstruction appears correct.

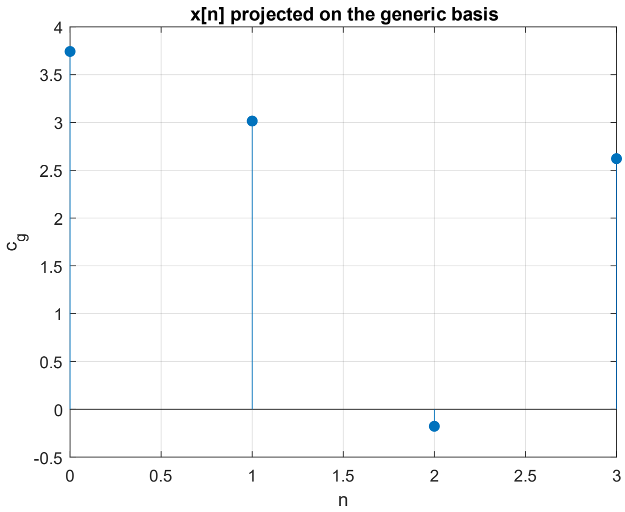

Figure 8: x[n] projected on the generic basis

Exercise 4

Analysis of the discrete-time signal $x[n] = r_{88}[n-5]$ defined in $[0, N-1]$ with $N = 128$.

After generating the Walsh basis using the algorithm in the attached MATLAB scripts (function in `generateWalshBasis.m`), the coefficient vectors were computed considering the required three different cases, in the same way of Exercise 3.

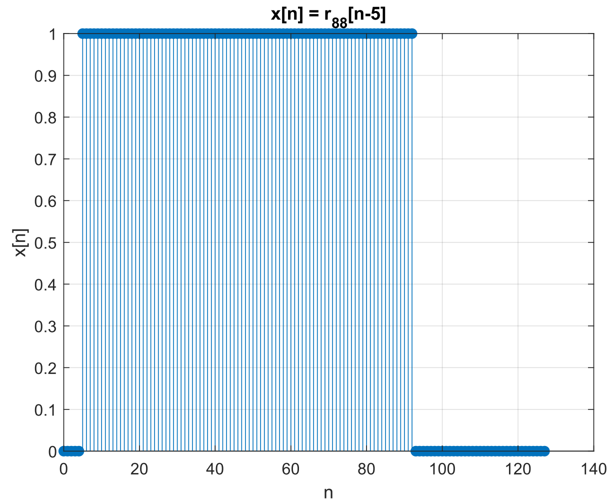

Figure 9: plot of x[n]

1. Canonical basis

As already showed in Exercise 3, the projection on the canonical basis corresponds to the original signal. It can be verified by executing the attached MATLAB script that displays all the plots.

2. Fourier basis

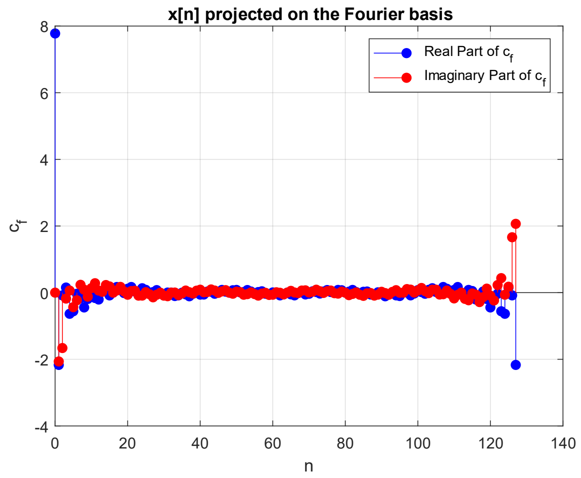

Figure 11: x[n] projected on the Fourier basis

3. Walsh basis

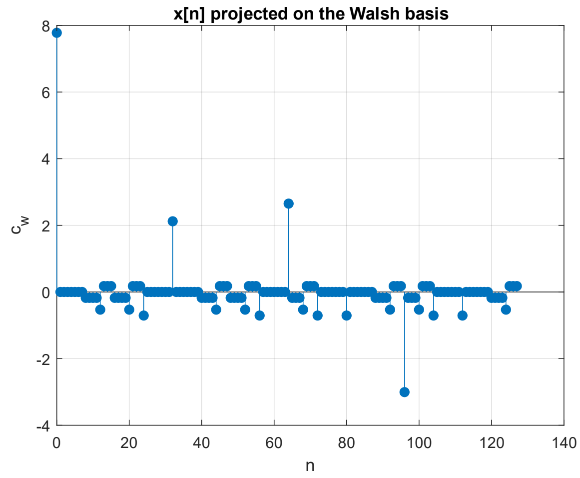

Figure 10: x[n] projected on the Walsh basis

Verification of the Parseval equality

The following equivalences must be satisfied for all three vectors of coefficients:

$$E\{x[n]\} = \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

From MATLAB output we can confirm Parseval equality is verified:

```
E_x  = sum(abs(x_n).^2) = 88

E_c = sum(abs(c_c).^2) = 88
E_f = sum(abs(c_f).^2) = 88.0000
E_w = sum(abs(c_w).^2) = 88.0000
```