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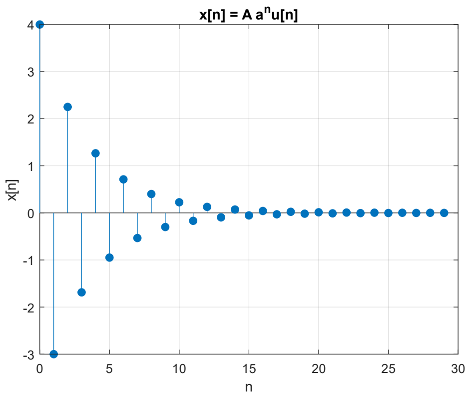
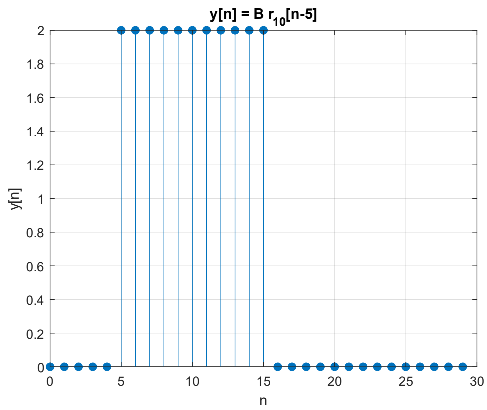
Signal analysis and processing
Lab Experience Session 2

Exercise 1

Analysis of the discrete-time convolution between $x[n] = A\,a^n\,u[n]$ (with $A = 4$ and $a = -3/4$) and $y[n] = B\,r_{10}[n-5]$ (with $B = 2$).





· Applying the definition formula

The discrete convolution in defined as:

$$z[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

· Using the matrix computation

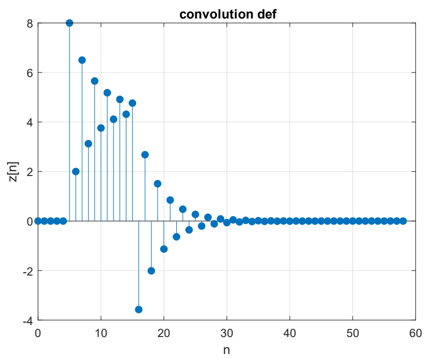
Given $x[n]$ (with extension $N_x = 30$) and $y[n]$ (with extension $N_y = 30$), we can compute the convolution vector z as:

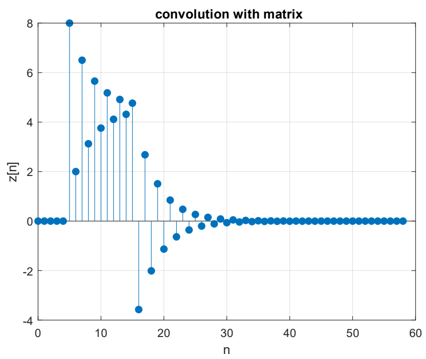
$$z = \begin{pmatrix} z[0] \\ z[1] \\ \vdots \\ z[N_z-1] \end{pmatrix} = M_y x = \begin{pmatrix} y[0] & 0 & \cdots & 0 \\ y[1] & y[0] & \cdots & 0 \\ \vdots & y[1] & \ddots & 0 \\ y[N_y-1] & \vdots & \ddots & y[0] \\ 0 & y[N_y-1] & \ddots & y[1] \\ 0 & 0 & \cdots & \vdots \\ 0 & 0 & \dots & y[N_y-1] \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N_x-1] \end{pmatrix}$$

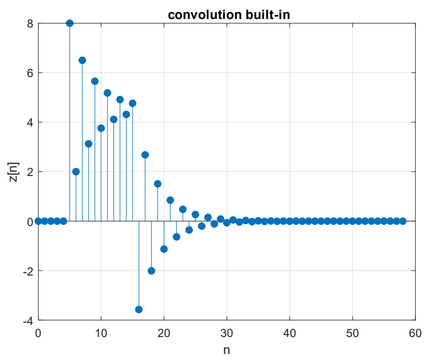
The matrix M_y is rectangular with N_x columns, each of which contains $y[n]$ with a downward shift of one position as the number of columns increases by one. If $x[n]$ has N_x samples and $y[n]$ has N_y samples, then the matrix has N_x columns and $N_x + N_y - 1$ rows. So, the extension of the convolution between $x[n]$ and $y[n]$ is given by $N_z = N_x + N_y - 1 = 59$.

· Using MATLAB built-in function

Convolution was computed in MATLAB as: `z_builtin = conv(x, y);`







As expected, the results obtained with the different implementation methods are the same.

Exercise 2

For the given block diagram, the following equation holds:

$$Y(Z) = z^{-1}X(Z) + a\,z^{-2}X(Z) + c\,z^{-2}Y(Z) + bz^{-1}Y(Z)$$

Evaluating the inverse z-transform, we obtain the difference equation associated to the LTI system:

$$y[n] = x[n-1] + ax[n-2] + cy[n-2] + by[n-1]$$

Its transfer function in the Z domain is:

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{z^{-1} + az^{-2}}{1 - bz^{-1} - cz^{-2}} = \frac{z + a}{z^2 - bz - c}$$

Given $a = 2$, $b = 3/5$ and $c = 1/9$, our transfer function becomes: $H(Z) = \frac{z+2}{z^2 - \frac{3}{9}z - \frac{1}{9}}$

The zeros of the LTI systems are the roots of the numerator of $H(z)$, so just $z = -2$, while the poles are the roots of the denominator (calling them p_1 and p_2), $p_{1,2} = \frac{3}{10} \pm \frac{\sqrt{181}}{30}$

The region of convergence (ROC) for the described LTI system is obtained as:

$$|z| > max\{p_1, p_2\} \quad \rightarrow \quad |z| > \frac{3}{10} + \frac{\sqrt{181}}{30}$$

Impulse response

We can determine the impulse response as inverse z-transform of $H(z)$. The degree of the numerator is less than that of the denominator, so we can rewrite the transfer function as:

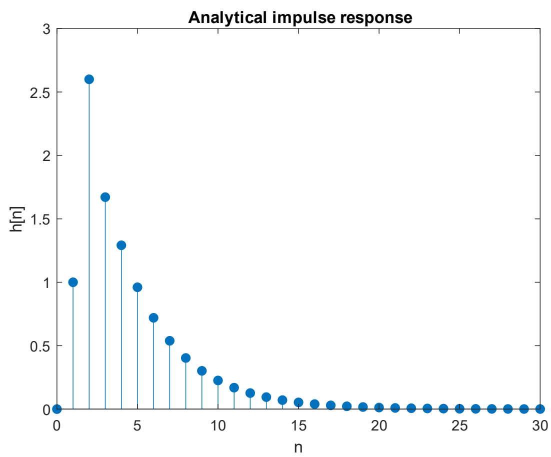
$$H(z) = \sum_{i=1}^{N_p} \frac{R_i}{z - p_i} = \frac{R_1}{z - \left(\frac{3}{10} + \frac{\sqrt{181}}{30}\right)} + \frac{R_2}{z - \left(\frac{3}{10} - \frac{\sqrt{181}}{30}\right)}$$

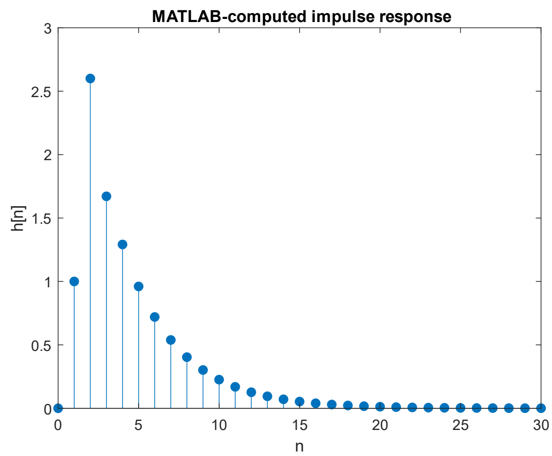
where $R_i = \lim_{z \to p_i} [(z - p_i)H(z)]$. We obtain: $R_1 = \frac{\sqrt{181} - 69}{2\sqrt{181}} \cong 3.06$ and $R_2 = \frac{\sqrt{181} + 69}{2\sqrt{181}} \cong -2.06$

In the end:

$$h[n] = Z^{-1}\{H(z)\} = \sum_{i=1}^{N_p} \mathrm{R_i}\,\mathrm{p_i}^{n-1} u[n-1] = R_1\,p_1^{n-1}\,u[n-1] + R_2\,p_2^{n-1}\,u[n-1]$$

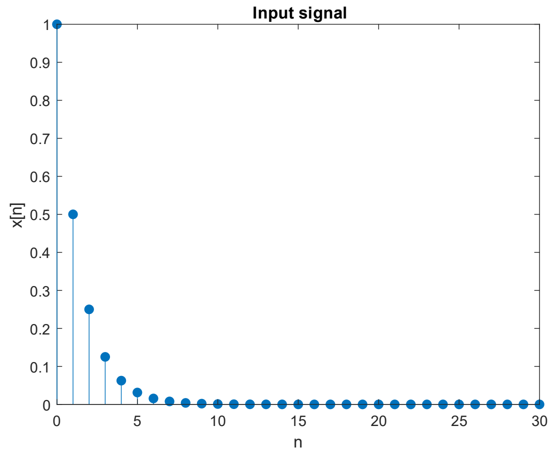
Correctness of the obtained analytical result was verified using the built-in MATLAB function `filter`, useful to determinate the response of a system, taking the delta function as input.





Output $y[n]$

Considering the input signal $x[n] = \left(\frac{1}{2}\right)^n u[n]$, whose z-transform is:



$$X(Z) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u[n]\, z^{-n} = \frac{z}{z - \frac{1}{2}}$$

The z-transform of the output is:

$$Y(z) = X(z)H(z) = \frac{z(z+2)}{\left(z - \frac{1}{2}\right)(z - p_1)(z - p_2)}$$

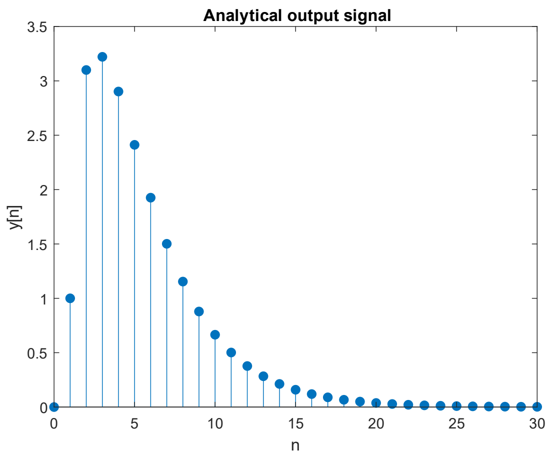
The analytical expression of the output $y[n]$ is obtained as the inverse z-transform of $Y(z)$, using the residue method as done previously. We can write:

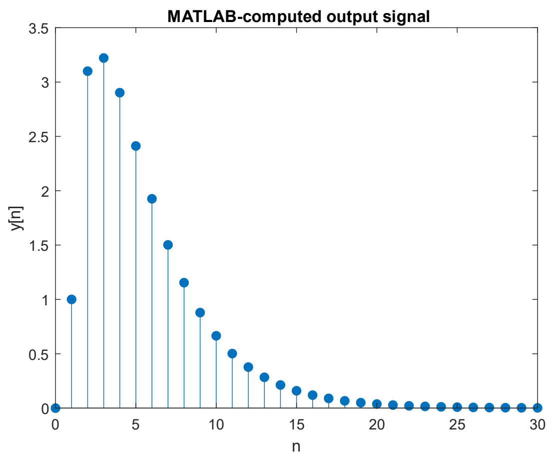
$$Y(z) = \frac{S_1}{z - \frac{1}{2}} + \frac{S_2}{z - p_1} + \frac{S_3}{z - p_2}$$

Where: $S_1 = \lim_{z \to \frac{1}{2}} \left[\left(z - \frac{1}{2}\right) Y(z)\right] = \frac{-225}{29} \cong -7.7586$ and, in the same way, $S_2 \cong -0.4726$, $S_3 \cong 9.2312$. So:

$$y[n] = Z^{-1}\{Y(z)\} = S_1 \left(\frac{1}{2}\right)^{n-1} u[n-1] + S_2 p_1^{n-1} u[n-1] + S_3 p_2^{n-1} u[n-1]$$

Exactly as before, the analytical results just shown were compared with MATLAB-computed ones.



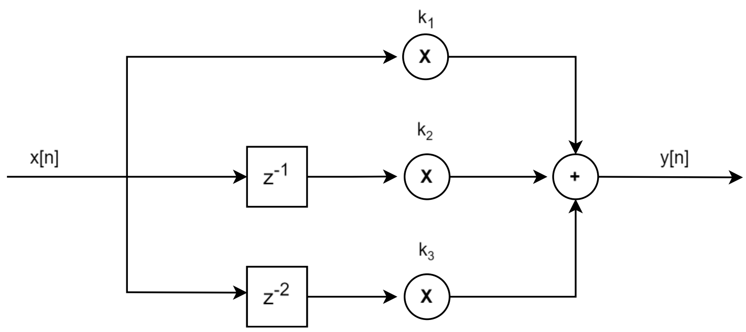


Exercise 3

Given the difference equation:

$$y[n] = k_1 x[n] + k_2 x[n-1] + k_3 x[n-2] \text{ where } k_1 = \frac{1}{4}, k_2 = \frac{3}{4} \text{ and } k_3 = -\frac{3}{4}$$

The block diagram of the LTI system described is:



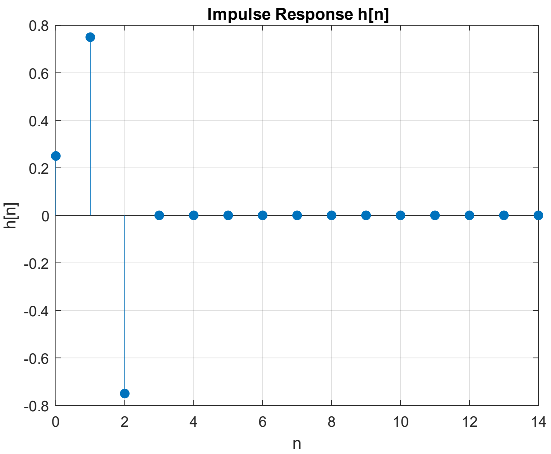
To get the transfer function in the Z domain:

$$Y(z) = \frac{1}{4}X + \frac{3}{4}Xz^{-1} - \frac{3}{4}Xz^{-2} \quad \rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4} + \frac{3}{4}z^{-1} - \frac{3}{4}z^{-2} = \frac{z^2 + 3z - 3}{z^2}$$

The system has a pole in $z = 0$ with multiplicity 2 and two zeros: $z_{1,2} = \frac{-3 \pm \sqrt{21}}{2}$

Since $H(z)$ it's just a sum of powers of z, the inverse z-transform is easy to obtain:

$$h[n] = \frac{1}{4}\delta[n] + \frac{3}{4}\delta[n-1] - \frac{3}{4}\delta[n-2]$$



Taking $x[n] = r_5[n-3]$ as input signal, the output was computed analytically, evaluating the convolution manually:

$$y[n] = \sum_{k=3}^{7} x[k]h[n-k]$$

n	0	1	2	3	4	5	6	7	8	9
$y[n]$	0	0	0	$\frac{1}{4}$	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{3}{4}$

Results were verified using a MATLAB script.