Class Projects

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Contents

Abstract

Focusing on transitioning to the continuous case of the Binomial tree. There is freedom of choice here due to the fact that there are two equations for three unknowns. The one that is most useful that lend itself to further expansion into trinomial trees and PDEs is the CRR method of $ud = 1$.

0.1 Transition to continuous tree

If σ is given, there is only one equation for two unknowns u and d. One can chose them such that $ud = e^{2v\Delta t}$ for a generic v. One can select for v, one of the three popular choices, $v = 0$ (case of Cox-Ross-Rubinstein), $v = r - \sigma^2/2$ (case of Jarrow-Rudd), and $v = r + \sigma^2/2$ (general case) . To get to the Black-Scholes equation, divide the time interval $[0, T]$ into N steps of length $h = T/N$, and set the binomial parameters, in any of the three ways, as follows

$$1. \quad 1 + U = e^{\sigma\sqrt{\Delta t}} \quad , \quad 2. \quad 1 + U = e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t} e^{\sigma\sqrt{\Delta t}} \quad , \quad 3. \quad 1 + U = e^{\left(r + \frac{1}{2}\sigma^2\right)\Delta t} e^{\sigma\sqrt{\Delta t}} \quad , \tag{1}$$

$$1. \quad 1 + D = e^{-\sigma\sqrt{\Delta t}} \quad , \quad 2. \quad 1 + D = e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t} e^{-\sigma\sqrt{\Delta t}} \quad , \quad 3. \quad 1 + D = e^{\left(r + \frac{1}{2}\sigma^2\right)\Delta t} e^{-\sigma\sqrt{\Delta t}} \quad , \tag{2}$$

and

$$R = e^{r\Delta t} - 1 \quad . \tag{3}$$

Here σ is the BS volatility, and r is the risk-free rate continuously compounded. Try to calculate the BS option price in the binomial model and compare it against the analytic BS price.

Check the relationship between the U, D and the Black-Scholes volatilities, probabilities for up and down movements etc. We find the following

$$u = e^{v\Delta t + \sigma\sqrt{\Delta t}} \quad , \quad d = e^{v\Delta t - \sigma\sqrt{\Delta t}} \quad , \tag{4}$$

and the probability for the up down movement

$$q = \frac{e^{r\Delta t} - d}{u - d} \quad , \tag{5}$$

results in, (see below)

$$q = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu - v}{\sigma}\right)\sqrt{\Delta t} \quad , \quad 1 - q = \frac{1}{2} - \frac{1}{2}\left(\frac{\mu - v}{\sigma}\right)\sqrt{\Delta t} \quad . \tag{6}$$

There can be different choices for the value of v, as listed above. For each of those cases we have

1. CRR - $v = 0$

$$q = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{\Delta t} \quad , \quad 1 - q = \frac{1}{2} - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{\Delta t} \quad . \tag{7}$$

2. Jarrow-Rudd - $v = \mu$

$$q = \frac{1}{2} \quad , \quad 1 - q = \frac{1}{2} \quad . \tag{8}$$

3. Third case, the General case of any v for which the model converges, we select here the value of $v = \mu + \sigma^2$

$$q = \frac{1}{2} - \frac{1}{2}\sigma\sqrt{\Delta t} \quad , \quad 1 - q = \frac{1}{2} + \frac{1}{2}\sigma\sqrt{\Delta t} \quad . \tag{9}$$

4. A fourth case is to take v such that the tree is centered at the strike value, K. For this we take $v = \frac{1}{T}\ln\frac{S_0}{K}$, see further below.

Table 1: American and European Put Option Prices on Binomial Model. $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $r = 5\%$, $T = 1/12$. U and D calculated based on σ as input. (Give U and D for $\sigma = 0.2$).

Number of time steps (Jarrow - Rudd)	American Put (Bin. Tree)	European Put (Bin. Tree)	European Put (Analytic)
4	2.03904	2.00270	2.09627
10	2.10067	2.06804	2.09627
20	2.11946	2.08749	2.09627
80	2.12904	2.09802	2.09627
1,000	2.12740	2.09668	2.09627
5,000	2.12703	2.09638	2.09627
10,000	2.12693	2.09627	2.09627

Table 2: American and European Put Option Prices on Binomial Model. $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $r = 5\%$, $T = 1/12$, using Cox-Ross-Rubinstein of $v = 0$.

Number of time steps (CRR)	American Put (Bin. Tree)	European Put (Bin. Tree)	European Put (Analytic)
4	2.03305	1.95799	2.09627
10	2.08962	2.03947	2.09627
20	2.10801	2.06766	2.09627
80	2.12224	2.08908	2.09627
1,000	2.12654	2.09569	2.09627
5,000	2.12684	2.09615	2.09627
10,000	2.12687	2.09621	2.09627

Table 3: American and European Put Option Prices on Binomial Model. $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $r = 5\%$, $T = 1/12$, using the general case of $v = r + \frac{1}{2}\sigma^2$.

Number of time steps (General)	American Put (Bin. Tree)	European Put (Bin. Tree)	European Put (Analytic)
4	2.07463	2.05602	2.09627
10	2.12712	2.09960	2.09627
20	2.13734	2.10732	2.09627
80	2.13414	2.10325	2.09627
1,000	2.12748	2.09679	2.09627
5,000	2.12703	2.09638	2.09627
10,000	2.12689	2.09623	2.09627

3. b). Remaining still in the general tree, with $ud = e^{2v\Delta t}$, where v is a scalar number and can be chosen

$$u = e^{v\Delta t + \sigma\sqrt{\Delta t}} \quad , \quad d = e^{v\Delta t - \sigma\sqrt{\Delta t}} \quad , \tag{10}$$

and the probability for the up down movement

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{(r-v)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} = \frac{1 + (r-v)\Delta t - \left(1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t\right)}{2\sigma\sqrt{\Delta t}} = \frac{\sigma\sqrt{\Delta t} + \left(r - \frac{1}{2}\sigma^2 - v\right)\Delta t}{2\sigma\sqrt{\Delta t}} \quad . \tag{11}$$

As shown earlier

$$q = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu - v}{\sigma}\right)\sqrt{\Delta t} \quad . \tag{12}$$

Table 4: American and European Put Option Prices on PDE. $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $r = 5\%$, $T = 1/12$, in the General case (3).

Number of time steps	American Put(PDE)	European (PDE)	Analytic European
i=3000, j=1000	2.12666	2.09602	2.09627

Then the interesting choice, selected here for the value of v, is $v = -\frac{1}{N\Delta t}\ln\frac{S_0}{K}$, use also $N\Delta t = T$ to write (v looks like small value in this case but check the numerical value - it should be easy to implement, in particular, when S_0 is close to the strike value K)

$$v = -\frac{1}{N\Delta t}\Big[\ln S_0 - \ln K\Big] = -\frac{1}{T}\Big[\ln S_0 - \ln K\Big] \quad . \tag{13}$$

This choice centers the tree around the value of the strike K, and it is like a strike-calibration of the tree. For instance take the position of the up node and down node and the ones in the middle $S_0 u^0 d^N = Ke^{-\sigma N\sqrt{\Delta t}}$, and $S_0 u^N d^0 = Ke^{\sigma N\sqrt{\Delta t}}$. The tree is centered right at the strike K. If we select the number of time steps as odd then the strike K is in the middle of two nodes, and if the number of time steps is even then the middle node will coincide with the strike K.

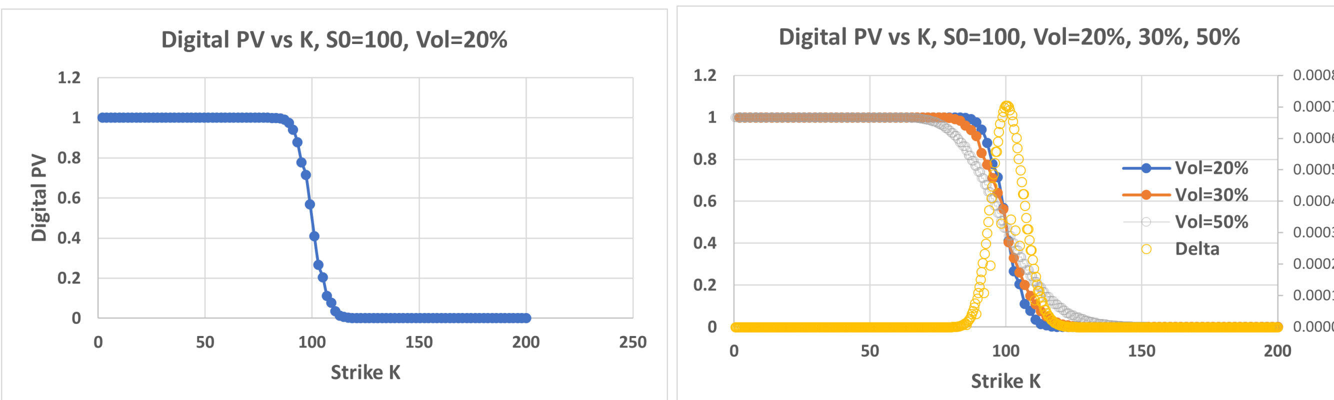

Figure 1: Digital Option PV versus strike K for different BS volatilities, 20%,30%,50%, respectively. These have been obtained with CRR. Joshi scheme does not converge well. The Bump in the middle, secondary axis, shows Delta obtained by using a bump size of $\epsilon = 0.01$.

0.2 Additive Binomial model for Black-Scholes dynamics

Introduction: The binomial and trinomial models contain two important information. The first is related with the value of $x_{i,j}$ at every node (i,j). The second is related with the probabilities p_u, p_d, (and/or p_m). Below we have obtained those probabilities for the Black-Scholes model.

The way that has proven to have advantages in the modeling of derivatives is to use instead of the price the logarithm $x = \ln S$. While it looks like it is a trivial substitution it has advantages to using directly the price to building a grid. The main benefit of it is that it can be scaled with ease to larger dimensions, and removes the complications that come with the factors u and d on the stock prices of the binomial trees that were historically modeled, compare for instance $S_{i,j} = S_0 u^j d^{i-j}$ versus $S_{i,j} = S_0 \exp\{j\Delta x\}$ for $j = -i, \ldots, i$. x is present also in fixed income derivatives modeling as part of the Bond reconstruction equations and the ideas can be extended there easily as well.

Applying Ito's lemma, we get the following equation for x

$$\mathrm{d}x = \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W(t) \quad , \tag{14}$$

with starting point of $x_0 = \ln S_0$ as the center node, solution to the above $x = x_0 + \mu t + \sigma W(t)$. Then the variable x in a small interval Δt can go either up to the level $x_0 + \Delta x_u$ or down to $x_0 + \Delta x_d$, with probabilities p_u and $p_d = 1 - p_u$, respectively.

Table 5: American and European Put Option Prices on PDE. $S_0 = 100$, $K = 100$, $B = 102$,$\sigma = 0.2$, $r = 5\%$, $T = 1/12$, in the General case (3).

Number of time steps	European Up-and-Out Put(PDE)
5,000	1.33021

We equate the mean and variance of the binomial process for x with the mean and variance of the continuous time process over the time interval Δt. The mean of the continuous process in Eq. (14)

$$E[\Delta x] = p_u \Delta x_u + p_d \Delta x_d = \mu \Delta t \quad , \tag{15}$$

where $\mu = r - \sigma^2/2$, and

$$E[\Delta x^2] = p_u \Delta x_u^2 + p_d \Delta x_d^2 = \sigma^2 \Delta t + \mu^2 \Delta t^2 \quad . \tag{16}$$

The term $\mu^2 \Delta t^2$ was introduced by Trigeorgis and can improve the numerical accuracy in not so small time steps trees. From these two equations, combined with $p_u + p_d = 1$, we have three equations and four unknowns. Therefore we have a free choice for one of the parameters. Two are the obvious choices of solutions, the CRR, and the JR.

The JR case, the equal up and down probabilities (get the same for the HW and the Cheyette model)

$$\frac{1}{2}\Delta x_u + \frac{1}{2}\Delta x_d = \mu \Delta t \quad , \tag{17}$$

$$\frac{1}{2}\Delta x_u^2 + \frac{1}{2}\Delta x_d^2 = \sigma^2 \Delta t + \mu^2 \Delta t^2 \quad . \tag{18}$$

This gives:

$$\Delta x_u = \frac{1}{2}\mu \Delta t + \frac{1}{2}\sqrt{4\sigma^2 \Delta t - 3\mu^2 \Delta t^2} \quad , \tag{19}$$

$$\Delta x_d = \frac{3}{2}\mu \Delta t - \frac{1}{2}\sqrt{4\sigma^2 \Delta t - 3\mu^2 \Delta t^2} \quad . \tag{20}$$

The CRR case, of Equal Jump size case:

$$p_u \Delta x + p_d(-\Delta x) = \mu \Delta t \quad , \tag{21}$$

$$p_u \Delta x^2 + p_d \Delta x^2 = \sigma^2 \Delta t + \mu^2 \Delta t^2 \quad , \tag{22}$$

gives

$$\Delta x = \sqrt{\sigma^2 \Delta t + \mu^2 \Delta t^2} \quad . \tag{23}$$

Then from Eq. (21)

$$p_u \Delta x - (1 - p_u)\Delta x = \mu \Delta t \quad , \quad 2 p_u \Delta x - \Delta x = \mu \Delta t \quad , \tag{24}$$

$$p_u = \frac{1}{2} + \frac{1}{2}\frac{\mu \Delta t}{\Delta x} \quad . \tag{25}$$

Then the nodes (i, j) for $i = 0, \ldots, N$ and $j = 0, \ldots, i$, on the tree will be given by

$$S_{i,j} = \exp\Big(x_0 + j\Delta x_u + (i - j)\Delta x_d\Big) \quad . \tag{26}$$

In the above $e^{x_0} = S_0$, meaning that in the above we have $S_{i,j} = S_0 \exp\Big(j\Delta x_u + (i-j)\Delta x_d\Big)$.

However, there is a better implementation in the case of equal up and down jumps. We change the structure of the tree, and now use an index j to account for the level of the asset price rather then to account for the number of the up jumps. At the last time slice, N, the level of the index takes values $j = -N, -N+2, \ldots, N-2, N$, and the position in the tree will be given by

$$S_{i,j} = \exp\left(x_0 + j\Delta x\right) \quad . \tag{27}$$

0.3 Additive Trinomial model for Black-Scholes dynamics

CRR for Trinomial tree: Similar ideas can apply to trinomial trees. We apply again the equal jumps size case

$$p_u \Delta x + p_m(0) + p_d(-\Delta x) = \mu \Delta t \quad , \tag{28}$$

$$p_u \Delta x^2 + p_m(0) + p_d \Delta x^2 = \sigma^2 \Delta t + \mu^2 \Delta t^2 \quad , \tag{29}$$

$$p_u + p_m + p_d = 1 \quad . \tag{30}$$

The first two equations give

$$p_u - p_d = \frac{\mu \Delta t}{\Delta x} \quad , \tag{31}$$

$$p_u + p_d = \frac{\sigma^2 \Delta t + \mu^2 \Delta t^2}{\Delta x^2} \quad , \tag{32}$$

which gives

$$p_u = \frac{1}{2} \left(\frac{\sigma^2 \Delta t + \mu^2 \Delta t^2}{\Delta x^2} + \frac{\mu \Delta t}{\Delta x} \right) \quad , \tag{33}$$

and subtracting the first from the second equation we obtain

$$p_d = \frac{1}{2} \left(\frac{\sigma^2 \Delta t + \mu^2 \Delta t^2}{\Delta x^2} - \frac{\mu \Delta t}{\Delta x} \right) \quad . \tag{34}$$

The third probability, p_m, comes from the other two, $p_m = 1 - (p_u + p_d)$,

$$p_m = 1 - \frac{\sigma^2 \Delta t + \mu^2 \Delta t^2}{\Delta x^2} \quad . \tag{35}$$

The space step Δx can not be chosen independent of the time step, and we will take it to be

$$\Delta x = \sigma \sqrt{3 \Delta t} \quad . \tag{36}$$

As opposed to the case of the binomial model, we cannot find a firm value for the space step in terms of the time step, as there not enough equations to solve. However one can put restrictions based on the probabilities remaining positive.

Table 6: American and European Put Option Prices on Binomial Model. $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $r = 5\%$, $T = 1/12$, using Cox-Ross-Rubinstein of $v = 0$. Trigeorgis (1992) correction is used here.

Number of time steps (CRR)	E Put PDE	European Put (Trin. Tree)	European Put (Bin. Tree)	European Put (Analytic)
4	1.93951	1.9389	1.95799	2.09627
10	2.03753	2.03732	2.03947	2.09627
20	2.06734	2.06724	2.06766	2.09627
80	2.08911	2.08908	2.08908	2.09627
1,000	2.09570	2.09569	2.09569	2.09627
5,000	2.09615	2.09615	2.09615	2.09627
10,000	2.09621	2.09621	2.09621	2.09627

0.4 Trinomial and partial differential equations for Black-Scholes

Finally, the time step takes values $t = i\Delta t$, where $i = 0, 1, \ldots, N$, and in general $j = -i, \ldots, i$. At the last node we have $j = -N, \ldots, N$. The asset price at the node (i, j) is given by

$$S_{i,j} = S_0 \exp\left(j \Delta x\right) \quad , \quad j = -i, \ldots, i \quad , \quad i = 0, \ldots, N \quad . \tag{37}$$

The values of j are constrained by the values of i, in the symmetric form $j = -i, \ldots, 0, \ldots, i$.

Now the pricing in the risk-neutral world is given by

$$C_{i,j} = e^{-r\Delta t} \left(p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j-1} \right) \tag{38}$$

One thing to mention, with importance for the PDE implementation, is the change in the PDE of the Black-Scholes equation to the following

$$-\frac{\partial C}{\partial t} = \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 C}{\partial S^2} + (r - \delta) S \frac{\partial C}{\partial S} - rC \quad , \tag{39}$$

substituting $x = \ln S$, the equation changes to the following;

$$-\frac{\partial C}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2} + (r - \delta - \frac{1}{2}\sigma^2)\frac{\partial C}{\partial x} - rC \quad , \quad \text{use } \mu = (r - \delta - \frac{1}{2}\sigma^2) \quad . \tag{40}$$

In the discrete version this can be approximated as follows

$$-\frac{C_{i+1,j} - C_{i,j}}{\Delta t} = \frac{1}{2}\sigma^2 \frac{C_{i+1,j+1} - 2C_{i+1,j} + C_{i+1,j-1}}{\Delta x^2} + \mu \frac{C_{i+1,j+1} - C_{i+1,j-1}}{2\Delta x} - rC_{i+1,j} \quad . \tag{41}$$

which can be written as

$$C_{i,j} = p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j-1} \tag{42}$$

$$p_u = \Delta t \left(\frac{\sigma^2}{2\Delta x^2} + \frac{\mu}{2\Delta x} \right) \quad , \tag{43}$$

$$p_m = 1 - \Delta t \frac{\sigma^2}{\Delta x^2} - r\Delta t \quad , \tag{44}$$

$$p_d = \Delta t \left(\frac{\sigma^2}{2\Delta x^2} - \frac{\mu}{2\Delta x} \right) \quad . \tag{45}$$

If we approximate the $rC_{i+1,j}$ term with $rC_{i,j}$, then we get the discounted expectation

$$C_{i,j} = \frac{1}{1 + r\Delta t} \Big(p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j-1} \Big) \tag{46}$$

where now p_m doesn't have the $-r\Delta t$ any more

$$p_u = \Delta t \left(\frac{\sigma^2}{2\Delta x^2} + \frac{\mu}{2\Delta x} \right) \quad , \tag{47}$$

$$p_m = 1 - \Delta t \frac{\sigma^2}{\Delta x^2} \quad , \tag{48}$$

$$p_d = \Delta t \left(\frac{\sigma^2}{2\Delta x^2} - \frac{\mu}{2\Delta x} \right) \quad . \tag{49}$$

This is about extending the Trinomial tree into a rectangular grid in the shape of a PDE. To transition from trinomial tree into a rectangular grid we only need to supply the boundary conditions. The upper boarder of the grid lies at the high asset prices, and the lower bound extends at the lower asset prices. Therefore, we set the boundary conditions based on the behavior of the option at the high asset prices and at the lower asset prices, respectively.

For European Calls, we have

$$\left.\frac{\partial C}{\partial S}\right|_{S_{i,N_j}} = 1 \quad \text{for large } S \tag{50}$$

$$\left.\frac{\partial C}{\partial S}\right|_{S_{i,-N_j}} = 0 \quad \text{for small } S \tag{51}$$

In terms of the grid, the above translate into the following:

$$\frac{C_{i,N_j} - C_{i,N_{j-1}}}{S_{i,N_j} - S_{i,N_{j-1}}} = 1 \quad \text{for large } S \tag{52}$$

$$\frac{C_{i,-N_j+1} - C_{i,-N_j}}{S_{i,-N_j+1} - S_{i,-N_j}} = 0 \quad \text{for small } S \tag{53}$$

Basically, the above says that the change of the option price in the vertical axis, in the asset price axis, is the same as the asset price change, as at high asset prices, options and underlying have close values. In the lower bound, the option price does not change as the asset price decreases.

To simplify, use always N, ie do not extend beyond the square size $N \times N$. For European Calls, we have

$$\left.\frac{\partial C}{\partial S}\right|_{S_{i,N}} = 1 \quad \text{for large } S \quad , \quad \left.\frac{\partial C}{\partial S}\right|_{S_{i,-N}} = 0 \quad \text{for small } S \tag{54}$$

In terms of the grid, the above translate into the following:

$$\frac{C_{i,N} - C_{i,N-1}}{S_{i,N} - S_{i,N-1}} = 1 \quad \text{for large } S \quad , \quad \frac{C_{i,-N+1} - C_{i,-N}}{S_{i,-N+1} - S_{i,-N}} = 0 \quad \text{for small } S \quad . \tag{55}$$

Build a table for European Call Option prices as function of N and see how good the convergence is.