Short Rate Modelling

Emiliano Papa

Deutsche Bank, London

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Reading List

- John Hull, Options, Futures and Other Derivatives, Prentice Hall 2006.
- Steve Shreve, Stochastic Calculus for Finance II, Springer 2004.
- Martin Baxter and Andrew Rennie, Financial Calculus: An Introduction to Derivative Pricing, Chapter 5, (CUP 1996).
- Mark Joshi, C++ Design Patterns and Derivative Pricing, (Cambridge University Press, 2004).

Interest Rates and IR Instruments

- 1. Zero-Coupon Bonds or Discount-Bond
- 2. Coupon-Bonds
- 3. Forward Rate Agreements (FRA)
- 4. Swaps
- 5. Associated Rates - Libor Rates, Spot Rates, Forward Rates, (Continuous, Simple)

Interest Rate Derivatives

- 1. Bond Options
  - European, Bermudan
  - Callable Bonds
- 2. Swaptions
  - European, Bermudan
  - Caps, Floors
- 3. Exotic Swap Contracts
  - IR Callable Range Accruals
  - Cancelable Swaps

Zero-Coupon Bonds

- Zero-coupon bond is a contract that guarantees the holder GBP at T.

  Its price at time t is denoted by $B(t, T)$. By definition

$$B(T, T) = 1 \quad , \tag{1}$$

  and

$$0 < B(t, T) < 1 \quad . \tag{2}$$

  The dependence of $B(t, T)$ on T is known as **Term-Structure** of discount factors, or **Zero-Coupon Curve** at time t. The curve is a decreasing function of maturity.

Simply Compounded Interest Rates

The **simply compounded spot rate** at time t for maturity T_1, defined as annualized rate of return from holding the bond from holding the bond from time t to time T_1.

- $F(t, T_1)$

$$B(t, T_1) = \frac{1}{1 + (T_1 - t)F(t, T_1)} \quad , \tag{3}$$

resulting in

$$F(t, T_1) = \frac{1}{(T_1 - t)} \left(\frac{1}{B(t, T_1)} - 1 \right) \quad . \tag{4}$$

Interest from t to T_2, $F(t, T_2)$, is usually different from $F(t, T_1)$ reflecting the fact that interest earned on the two sub-periods $[t, T_1]$, $[T_1, T_2]$, are different, e.g. if interest rates are expected to rise or fall in the future.

Simply Compounded Interest Rates

If one decomposes time into small interest rate periods $[T_i, T_{i+1}]$

$$
\begin{aligned}
1 + F(t, T_k)\,(T_k - t) \quad &= \quad \Big(1 + L(t; T_0, T_1)(T_1 - T_0)\Big) \cdot \\
&\qquad \dots \\
&\quad \cdot \Big(1 + L(t; T_{k-1}, T_k)(T_k - T_{k-1})\Big) \quad , (5)
\end{aligned}
$$

here $t = T_0$. If we know all $F(t, T_i)$, $i = 1, \dots k$, then we can calculate $L(t; T_{i-1}, T_i)$

$$
L(t, T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{1 + F(t, T_i)(T_i - t)}{1 + F(t, T_{i-1})(T_{i-1} - t)} - 1 \right) \quad . \qquad (6)
$$

$L(t; T_i, T_{i+1})$ is the interest earned over the future time period $[T_i, T_{i+1}]$ as given by a contract fixed at t. It is rational to represent each such rate by its own stochastic process.

Usually $F(t, T_i)$, $i = 1, \dots k$, are known from the yield curve at t.

- Example of simply compounded rate is the **London Interbank Offered Rate (LIBOR)**.

Continuously compounded rates

Rates quoted in the market are always simply compounded. But it can be mathematically more convenient to work with continuously compounded rates. Denoted by $R(t, T)$ and expressed in terms of T-Bond as

$$B(t, T) = e^{-R(t,T)(T-t)} \quad , \quad \text{(time is annualized)} \quad . \tag{7}$$

$$R(t, T) = -\frac{\ln B(t, T)}{(T-t)} \quad . \tag{8}$$

The continuously compounded spot rate can be thought of as a measure of the implied interest rate offered by the bond and is referred to as the **yield to maturity**.
The graph of $R(t, T)$ versus maturity T is known as the **yield curve**.

Forward Interest Rates

- What would cost to enter into an agreement to buy a T-bond at time S, where $t < S < T$?
  At time S we are going to pay A to buy the T-Bond. The time t value of the cashflows will be

$$-AB(t, S) + B(t, T) = 0 \quad . \tag{9}$$

  This will be the value at time t of the forward discount bond

$$A = B(t; S, T) = \frac{B(t, T)}{B(t, S)} \quad . \tag{10}$$

  The time t value of the simply compounded annualized forward rate from time S to time T,

$$B(t; S, T) = \frac{B(t, T)}{B(t, S)} = \frac{1}{1 + (T - S)F(t; S, T)} \tag{11}$$

Forward Bond Replication

The forward bond can be replicated at time t, by taking a short position on a T_1-bond, to generate just enough funding to buy one T_2-bond $B(t, T_2)$. Therefore we need to take a short position at the cost of $B(t, T_2)$ on $B(t, T_1)$ bonds, meaning we sell $B(t, T_2)/B(t, T_1)$ amount of $B(t, T_1)$ bonds. The overall cost is zero, as shown below

$$B(t, T_2) + B(t, T_1)\left(\frac{B(t, T_2)}{B(t, T_1}\right) = 0 \quad . \tag{12}$$

At time T_1 we have to pay the investment on the T_1 bond which equals $B(t, T_2)/B(t, T_1)$. And at T_2 we will receive one unit of currency from the long position in the T_2-bond.
Therefore at t we have locked the payment at T_1 to receive \$1 at T_2. The amount paid at T_1 will be $B(t, T_2)/B(t, T_1)$.

Continuously Compounded Forward Rate

In addition to simply compounded forward rates $F(t; S, T)$ we can define also the continuously compounded forward rates

$$B(t; S, T) = e^{-R(t;S,T)(T-S)} \quad , \tag{13}$$

which can be expressed as

$$R(t; S, T) = -\frac{\ln B(t; S, T)}{(T-S)} = -\frac{\ln B(t, T) - \ln B(t, S)}{(T-S)} \quad . \tag{14}$$

Using Eq. (8)

$$R(t; S, T) = \frac{R(t, T)(T-t) - R(t, S)(S-t)}{(T-S)} \quad . \tag{15}$$

Instantaneous rates

- Instantaneous forward rate $f(t,T)$.
  At time t for maturity T, the instantaneous fwd rate $f(t,T)$, can be thought of as rate of return over an infinitesimally small time interval $[T, T+\delta T]$

$$f(t,T) = \lim_{\delta T \to 0} R(t;T,T+\delta T) = -\frac{\ln B(t,T+\delta T) - \ln B(t,T)}{\delta T}$$

$$= -\frac{\partial \ln B(t,T)}{\partial T} \qquad (16)$$

  The dependence of $f(t,T)$ on T is called term structure of forward rates at time t.
- Short Rate $r(t)$ - the instantaneous short rate or the **risk free rate** is the rate of return over an infinitesimally small time interval $[t, t+\delta t]$

$$r(t) = f(t,t) \quad . \qquad (17)$$

Relationship between yield and inst fwd rates

What did the forward bond agreement tell us? It told us that \$1 at time T, will have at time S a price different from \$1. in fact the market now thinks that the fair price for that contract will be $A = B(t,T)/B(t,S)$. When we reach the time S the price for \$1 at T will most likely change, but A discussed above today is, the fair price.
By comparing the price for \$1 received at T with its market value at t for payment at S, we can infer what the todays value for discounting between the time points S and T is, namely $F(t; S, T)$ or $L(t; S, T)$

$$L(t; S, T) = \frac{1}{(T-S)}\left(\frac{B(t,S)}{B(t,T)} - 1\right) = \frac{1}{(T-S)}\left(\frac{1}{B(t;S,T)} - 1\right) . (18)$$

Taking the small distance limit of $T \to S$, or expressed differently $F(t, T, T+\delta T) \equiv f(t, T)$

$$f(t; T) = \frac{1}{\delta T}\left(\frac{B(t,T)}{B(t,T+\delta T)} - 1\right) \equiv B(t; T, T+\delta T) = e^{-f(t,T)\delta T} \quad . (19)$$

This is the consensus now for the rate of discounting between two future time intervals.

Relationship between yield and inst fwd rates

Putting together time intervals between t and T we have

$$\begin{aligned}
B(t,T) &= B(t;t,t+\delta t)B(t;t+\delta t,t+2\delta t)\cdots B(t;T-\delta t,T) \quad , \\
&= \frac{B(t,t+\delta t)}{B(t,t)}\cdot\frac{B(t,t+2\delta t)}{B(t,t+\delta t)}\cdots\frac{B(t,T)}{B(t,T-\delta t)} \quad , \qquad (20)\\
&= e^{-f(t,t)\delta t}\cdot e^{-f(t,t+\delta t)\delta t}\cdots e^{-f(t,T)\delta t} \quad , \qquad (21)\\
&= e^{-\int_t^T f(t,u)\mathrm{d}u} \quad . \qquad (22)
\end{aligned}$$

If we were to enter into a contract to receive Libor at T for a fixed payment K at T the logic would go similarly. For the forward bond the market sets today the fair strike to be A, which in terms of Libor rate that sets at S would be [\$1 at S, will get \$ $(1+(T-S)F(t;S,T))$ or A at S \$1 at T.] For the Libor rate the fair strike would be $K = F(t;S,T) = B(t;S,T/(T-S)$. This the the FRA, thus equivalent to the forward bond discussion.

Relationship between yield and inst fwd rates

Going back to Eq. (16)

$$f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T} = \frac{\partial\Big(R(t,T)(T-t)\Big)}{\partial T} \quad , \qquad (23)$$

$$= R(t,T) + R'(t,T)(T-t) \quad . \qquad (24)$$

The forward rate is higher than the yield curve when the yield curve is normal, and below when it is inverted.

Integrating Eq. (23),

$$\int_{T_i}^{T_{i+1}} f(t,T)\mathrm{d}T = \int_{T_i}^{T_{i+1}} \mathrm{d}\Big(R(t,T)(T-t)\Big) \quad , \qquad (25)$$

$$R(t,T_{i+1})(T_{i+1}-t) - R(t,T_i)(T_i-t) = \int_{T_i}^{T_{i+1}} f(t,T)\mathrm{d}T \quad . \qquad (26)$$

Notice that due to curvature of the R-curve the surfaces under $R_i T_i$ and $R_{i-1} T_{i-1}$ don't cancel exactly below T_{i-1}, therefore to adjust the F curve has to be a bit displaced upward/downward depending on the sign of R'_{i-1}, unless the yield curve is flat, in which case the yeild curve and f-curve coincide.

Relationship between yield and fwd rates (cont.)

Dividing both sides of Eq. (26) we get

$$\frac{R(t,T_{i+1})(T_{i+1}-t)-R(t,T_i)(T_i-t)}{T_{i+1}-T_i}=\frac{1}{T_{i+1}-T_i}\int_{T_i}^{T_{i+1}} f(t,T)\mathrm{d}T\ . \quad (27)$$

From Eqs. (14)-(15) [here we note $R_i = R(t,T_i)$]

$$R(t;T_{i-1},T_i)=\frac{R_i(T_i-t)-R_{i-1}(T_{i-1}-t)}{T_i-T_{i-1}}=\frac{1}{T_i-T_{i-1}}\int_{T_{i-1}}^{T_i} f(t,u)\mathrm{d}u\ , \quad (28)$$

The average of the instantaneous fwd rate is equal to the discrete fwd rate of the interval $[T_{i-1},T_i]$

Another useful relationship between the rates

$$R(t,T)(T-t)=R(t,T_{i-1})(T_{i-1}-t)+\int_{T_{i-1}}^{T} f(t,u)\mathrm{d}u\ ,\ \ T\in[T_{i-1},T_i]\ . \quad (29)$$

Money Market Account (Continuous)

The money market account is security where interest accrues continuously at the riske free rate $r(t)$. Money market account at t is denoted by $B(t)$, and is defined by the differential equation

$$\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t \quad , \quad B(0) = 1 \quad . \tag{30}$$

Solving (30) we find

$$B(t) = \exp\left(\int_0^t r(u)\mathrm{d}u\right) \quad . \tag{31}$$

Money invested at t earns $r(t)\delta t$ over the time period $[t, t + \delta t]$ and it is re-invested continuously.
With the money market account we usually build the bank-account numeraire.

Money Market Account (Discrete)

If 1 unit of currency is invested at $T_0 = 0$, at T_1 we receive $1 + L(T_0; T_0, T_1)(T_1 - T_0)$. If we reinvest this over the next period and so on, over $[T_j, T_{j+1}]$, for $j = 1, 2, \ldots$ then at T_N the one unit of ccy invested at T_0 has become

$$B(T_N) = \prod_{j=0}^{N-1} \Big(1 + L(T_j; T_j, T_{j+1})(T_{j+1} - T_j)\Big) \quad . \tag{32}$$

In terms of forward rates

$$B(T_N) = \prod_{j=0}^{N-1} \exp\left(\int_{T_j}^{T_{j+1}} f(T_j, u)\mathrm{d}u\right) \quad . \tag{33}$$

If consider small intervals $[T_j, T_j + \delta]$, the above becomes

$$B(T_N) = \prod_{j=0}^{N-1} \exp\left(\int_{T_j}^{T_{j+1}} f(T_j, T_j)\mathrm{d}T_j\right) = \exp\left(\int_{T_0}^{T_N} r(u)\mathrm{d}u\right) \quad , \tag{34}$$

same as in previous page.

Coupon-bearing Bonds

A **fixed-coupon bond** pays the holder deterministic amounts $c_1, c_2, \ldots c_n$ which are called coupon payments at times $T_1, T_2, \ldots T_n$ where $T_0 < T_1 < \ldots T_n$. At maturity time T_n the holder receives the coupon c_n but also the fulla notional or face value N. The value of the coupon-bond at t is

$$B_{\text{fixed}}(t) = \sum_{i=1}^{n} c_i B(t, T_i) + NB(t, T_n) \quad . \tag{35}$$

A **floating-coupon bond** pays at T_i the floating Libor rate $L(T_{i-1}, T_i)$ multiplied by $\tau_i = T_{i-1} - T_i$ and N-notional

$$c_i = \tau_i NL(T_{i-1}, T_i) = N\left(\frac{B(T_{i-1}, T_{i-1})}{B(T_{i-1}, T_i)} - 1\right) \quad , \tag{36}$$

Floating Bonds

Let us look at the coupon payments one by one. At T_i, we receive

$$N\left(\frac{B(T_{i-1},T_{i-1})-B(T_{i-1},T_i)}{B(T_{i-1},T_i)}\right) \quad . \tag{37}$$

If we discount this payment to T_{i-1}, the value of the floating coupon at T_{i-1}

$$N\left(B(T_{i-1},T_{i-1})-B(T_{i-1},T_i)\right) \quad , \tag{38}$$

which is a portfolio of bonds. The time t value of these portfolio of bonds is

$$N\Big(B(t,T_{i-1})-B(t,T_i)\Big) \quad . \tag{39}$$

The above is just the today's value of floating coupon received at T_i. Putting them all together, we have

$$\begin{aligned} B_{\text{floating}} &= N\sum_{i=1}^{n}\left(B(t,T_{i-1}-B(t,T_i)\right)+NB(t,T_n) \quad , & (40)\\ &= NB(t,T_0) \quad . & (41) \end{aligned}$$

Swaps

A **payer swap** is a swap that receives a floating payment and pays a fixed rate

$$
\begin{aligned}
PS(t) &= \sum_{i=1}^{n}(B(t, T_{i-1}) - B(t, T_i)) - K \sum_{i=1}^{n} \tau_i B(t, T_i) , && (42) \\
&= B(t, T_0) - B(t, T_n) - K \sum_{i=1}^{n} \tau_i B(t, T_i) \quad , && (43) \\
&= \sum_{i=0}^{n} c_i B(t, T_i) \quad , && (44)
\end{aligned}
$$

where

$$
c_0 = 1 \,, \; c_i = -\tau_i K \,, \; c_n = -(1 + \tau_n K) \quad . \tag{45}
$$

Swap rate, the rates that makes the swap's value equal to zero

$$
S_{0,n}(t) = \frac{B(t, T_0) - B(t, T_n)}{\sum_{i=1}^{n} \tau_i B(t, T_i)} \quad . \tag{46}
$$

Swaps as weighted sums of forward rates

The Swap Rate can be written as an weighted sum of Libor rates

$$S_{0,n}(t) = \sum_{j=0}^{n-1} w_j L_j \ , \quad \text{with } w_j \geq 0 \ , \quad \sum_{j=0}^{n-1} w_j = 1 \ . \tag{47}$$

The weights are given by

$$w_j(t) = \frac{B(t, T_{j+1}) \, (T_{j+1} - T_j)}{\sum_{i=1}^{n} \tau_i B(t, T_i)} \quad , \tag{48}$$

since

$$\begin{aligned} w_j L_j &= \frac{B(t, T_{j+1}) \, (T_{j+1} - T_j)}{\sum_{i=1}^{n} \tau_i B(t, T_i)} \frac{1}{(T_{j+1} - T_j)} \left(\frac{B(t, T_j) - B(t, T_{j+1})}{B(t, T_{j+1})} \right) \quad , \\ &= \frac{B(t, T_j) - B(t, T_{j+1})}{\sum_{i=1}^{n} \tau_i B(t, T_i)} \quad . \end{aligned} \tag{49}$$

Obviously, the sum over all $j = 0, \cdots n-1$, gives for the denominator $B(t, T_0) - B(t, T_n)$. Notice that weights $w_j(t)$ are stochastic and correlated with L_j's.

Swaps as weighted sums of forward rates

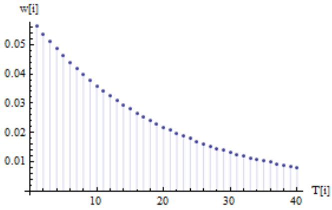


Figure: Swap weights w_i of Eq. (48) for annual 40y Swap. The rate here is taken to be flat at $r = 5\%$. The biggest contribution in the Swap Rate comes from the short end Libors on the curve. Here $w_1/w_{40} \approx 10$.

Yield Curve Construction

Yield Curve Construction

There are several ways of building a yield curve from quoted swap rates in the market. We are given a number of spot start swap rates of various maturities T_{n_i}, usually $T_{n_i} = 1, \dots 5, 7, 10, 12, 15, 20, 25, 30$. For quarterly paying swaps, with 4 Libors per year, $n_i = 4T_i = 4, 8, \dots 120$.
The most usual way to build the yield curve is by assuming that the instantaneous forward rates are piecewise flat between the two swap maturity dates T_{n_i} and $T_{n_{i+1}}$, where $n_1 < n_2 \cdots$

$$f(t, T) = \begin{cases} f_1\ , & \text{for } t \leq T \leq T_{n_1}\ , \\ f_{i+1}\ , & \text{for } T_{n_i} \leq T \leq T_{n_{i+1}}\ . \end{cases} \tag{50}$$

For $i = 1$ and $j = 0, \dots, n_1$

$$B(t, T_j) = \exp\left(-\int_t^{T_j} f(t, u)\mathrm{d}u\right) = \exp\left(-f_1(T_j - t)\right) \quad . \tag{51}$$

Yield Curve Construction (cont.)

Bond $B(t, T_{n_1})$:

$$\exp(-f_1(T_0 - t)) - \exp(-f_1(T_{n_1} - t)) = S_1 \sum_{j=1}^{n_1} \tau_j \exp(-f_1(T_j - t)) \,. \quad (52)$$

Solve this for f_1 by Newton-Raphson method.

For the ith forward rate proceed iteratively. We know all the forward rates up to maturity T_{n_i}. We assume the next forward rate is constant f_{i+1} out to maturity $T_{n_{i+1}}$. We now solve the following for f_{i+1}

$$\begin{aligned} B(t, T_0) - B(t, T_{n_{i+1}}) &= S_{i+1} \sum_{j=1}^{n_{i+1}} \tau_j B(t, T_j) && (53) \\ &= S_{i+1} \left(\sum_{j=1}^{n_i} \tau_j B(t, T_j) + \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(t, T_j) \right) && (54) \end{aligned}$$

$$B(t, T_{n_{i+1}}) + S_{i+1} \sum_{j=n_i+1}^{n_{i+1}} \tau_j B(t, T_j) = B(t, T_0) - S_{i+1} \sum_{j=1}^{n_i} \tau_j B(t, T_j) \,. \quad (55)$$

Yield Curve Construction (cont.)

For $j = n_i + 1, \ldots, n_{i+1}$ we have

$$\begin{aligned} B(t, T_j) &= \exp\left(-\int_t^{T_j} f(t,u)\mathrm{d}u\right) = B(t, T_{n_i}) \exp\left(-\int_{T_{n_i}}^{T_j} f(t,u)\mathrm{d}u\right) \, , \\ &= B(t, T_{n_i}) \exp\left(-f_{i+1}(T_j - T_{n_i})\right) \quad . \end{aligned} \tag{56}$$

We now solve (55) for f_{i+1}

$$\begin{aligned} \exp\left(-f_{i+1}(T_{n_{i+1}} - T_{n_i})\right) &+ S_{i+1} \sum_{j=n_i+1}^{n_{i+1}} \tau_j \exp\left(-f_{i+1}(T_j - T_{n_i})\right) \qquad (57) \\ &= B^{-1}(t, T_{n_i}) \left(B(t, T_0) - S_{i+1} \sum_{j=1}^{n_i} \tau_j B(t, T_j) \right) \, . \end{aligned}$$

Yield Curve Construction (cont.)

Think in terms of filling missing values of the swaps at empty spaces in the tenor structure. Filling out the missing info is usually done through an interpolation. There are several kinds. The one described above is linear in the $R(t,T)T$. But there are others, like linear interpolation in the yield $R(t,T)$, etc.

To make use of the interpolation schemes given below we can think as follows. Write Eq. (46)

$$B(t,T_0) - B(t,T_n) = S_n \sum_{j=1}^{n} \tau_j B(t,T_j) \quad , \tag{58}$$

$$B(t,T_0) - B(t,T_n) = S_n \sum_{j=1}^{n-1} \tau_j B(t,T_j) + S_n \tau_n B(t,T_n) \quad ,$$

$$B(t,T_0) - S_n \sum_{j=1}^{n-1} \tau_j B(t,T_j) = B(t,T_n) + S_n \tau_n B(t,T_n) \quad . \tag{59}$$

$$B(t,T_n)(1 + S_n \tau_n) = B(t,T_0) - S_n \sum_{i=1}^{n-1} \tau_j B(t,T_j) \tag{60}$$

Yield Curve Construction (cont.)

We can write

$$B(t, T_n) = \frac{B(t, T_0) - S_n \sum_{j=1}^{n-1} \tau_j B(t, T_j)}{1 + S_n \tau_n} \quad . \tag{61}$$

From here we write

$$R(t, T_n) = -\frac{1}{\tau_n} \ln \left[\frac{B(t, T_0) - S_n \sum_{j=1}^{n-1} \tau_j B(t, T_j)}{1 + S_n \tau_n} \right] \quad . \tag{62}$$

Some of the discount factors inside the sum on the right of Eq. (62) are unknown and we need to interpolate with one of the schemes below. Those interpolated points will depend on the unknown value of R_n and the known values from the previous tenors, like $R_{n_i - 1}$ (using previous notation of T_{n_i}, $i = 1, 2, \ldots, n$).

Yield Curve Construction (cont.)

For the interpolation above, use one of the following schemes:

- **(1).** Linear in $R(t, T)(T - t)$ - the one explained above,
  To simplify notation, set $t = 0$, and $R(0, T_i) = R_i$, then (28)

$$f(0, T) = \frac{R_i T_i - R_{i-1} T_{i-1}}{T_i - T_{i-1}} \;,\; T \in [T_{i-1}, T_i] \quad , \tag{63}$$

and Eq. (29)

$$\begin{aligned} R(0, T)T &= R_{i-1} T_{i-1} + (T - T_{i-1}) \frac{R_i T_i - R_{i-1} T_{i-1}}{T_i - T_{i-1}} \;, & (64) \\ &= \frac{T - T_{i-1}}{T_i - T_{i-1}} R_i T_i + \frac{T_i - T}{T_i - T_{i-1}} R_{i-1} T_{i-1} \;, & (65) \end{aligned}$$

thus the “linear R(T) T” naming.

Yield Curve Construction (cont.)

- **(2).** Linear in R(0,T)

$$R(0,T) = \frac{T - T_{i-1}}{T_i - T_{i-1}} R_i + \frac{T_i - T}{T_i - T_{i-1}} R_{i-1} \quad , \tag{66}$$

resulting in

$$f(0,T) \quad = \quad \frac{2T - T_{i-1}}{T_i - T_{i-1}} R_i + \frac{T_i - 2T}{T_i - T_{i-1}} R_{i-1} \quad , \tag{67}$$

Yield Curve Construction (cont.)

- **(3).** Linear on log of rates

$$\ln R(0,T) = \frac{T - T_{i-1}}{T_i - T_{i-1}} \ln R_i + \frac{T_i - T}{T_i - T_{i-1}} \ln R_{i-1} \quad , \tag{68}$$

resulting in (does not allow for negative rates)

$$R(0,T) = R_i^{\frac{T - T_{i-1}}{T_i - T_{i-1}}} R_{i-1}^{\frac{T_i - T}{T_i - T_{i-1}}} \quad . \tag{69}$$

Yield Curve Construction (cont.)

- **(4).** Linear on discount factors

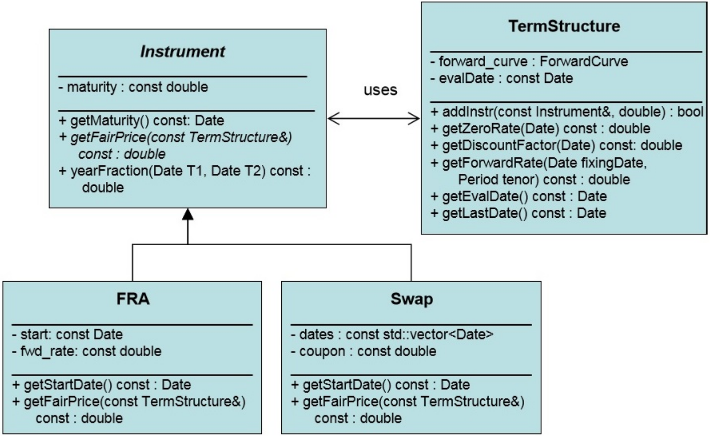
  For $T_{i-1} < T < T_i$,

  $$B(0,T) = \frac{T - T_{i-1}}{T_i - T_{i-1}} B(0,T_i) + \frac{T_i - T}{T_i - T_{i-1}} B(0,T_{i-1}) \quad , \qquad (70)$$

  which for the rates results in

  $$R(0,T) = -\frac{1}{T} \ln \left[\frac{T - T_{i-1}}{T_i - T_{i-1}} e^{-R_i T_i} + \frac{T_i - T}{T_i - T_{i-1}} e^{-R_{i-1} T_{i-1}} \right] \quad . (71)$$

Pricing Framework



TimeFunction Base Class

Define a TimeFunction interface to allow for function evaluation and interpolation.

```
class TimeFunction {

public:

        TimeFunction( const Date & start) : m_start( start ) {}
        virtual double operator() ( Date T ) const = 0 ;
        virtual double integrate( Date T1, Date T2 ) = 0 ;
        Date getStart( ) { return m_start ; }
        virtual Date getEnd( ) = 0;
private:

        const Date m_start;
}
```

Class ForwardCurve

```
class ForwardCurve : public TimeFunction {

public:

        ForwardCurve( Date start, const std::vector <Date>& dates,
                  const std::vector <double>forward_rates );
        ForwardCurve(const ForwardCurve & rhs);
        virtual double operator() (Date T) const ; // linear interp
        virtual double integrate(Date T1, Date T2) ; // trapezoidal
        virtual Date getEnd() ;
        void removeDate (Date T ) ;
        void setForwardRate (Date T, double rate ) ;
private:

        std::map<Date, double >m_forwardRates;
}
```

Instrument Base Class

```
class TermStructure;
typedef double Date;
class Instrument {

public:

    Instrument(Date maturity) : m_maturity(maturity) { }
    Instrument(const Instrument& rhs):
        m_maturity(rhs.m_maturity(rhs.m_maturity) {}
    Date getMaturity() const { return m_maturity ; }
    virtual double getFairPrice(const TermStructure& ts) const=0;
    // calculate #years from T1 to T2
    static double yearFraction(Date T1, Date T2);
private:

    const Date m_maturity;
}
```

Swap Rates

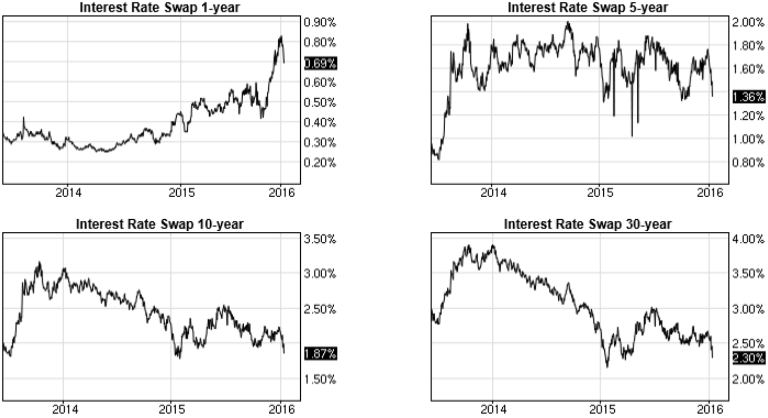
Swaps FlipCharts

Name	Current	1M ago	3M ago	6M ago	1Y ago	Date	Links
Interest Rate Swap 1 Year	0.69%	0.82%	0.55%	0.48%	0.35%	01/15/16	
Interest Rate Swap 2 Years	0.94%	1.13%	0.81%	0.91%	0.64%	01/19/16	
Interest Rate Swap 3 Years	1.11%	1.35%	1.08%	1.25%	0.92%	01/19/16	
Interest Rate Swap 4 Years	1.30%	1.62%	1.32%	1.54%	1.26%	01/14/16	
Interest Rate Swap 5 Years	1.40%	1.72%	1.49%	1.78%	1.36%	01/19/16	
Interest Rate Swap 7 Years	1.63%	1.96%	1.79%	2.13%	1.58%	01/19/16	
Interest Rate Swap 10 Years	1.89%	2.19%	2.03%	2.43%	1.89%	01/19/16	
Interest Rate Swap 15 Years	2.05%	2.35%	2.20%	2.62%	2.05%	01/14/16	
Interest Rate Swap 30 Years	2.32%	2.65%	2.52%	2.88%	2.35%	01/19/16	

In an interest rate swap agreement, one party undertakes payments linked to a floating interest rate index and receives a stream of fixed interest payments. The second party undertakes the reverse arrangement. The interest rate swap rate represents the fixed rate paid on a rate swap to receive payments based on a floating rate. Our Dollar Interest Rate Swaps page shows 1-, 5-, 10-, and 30-year rate swap charts, as well as historical rate swap data tables.

http://www.interestrateswapstoday.com/swap-rates.html
http://www.barchart.com/economy/swaps.php
http://www.thefinancials.com/free/EX_Interest_Swaps.html

Swap Rates (cont. 2)



Swap Rates (cont. 3)

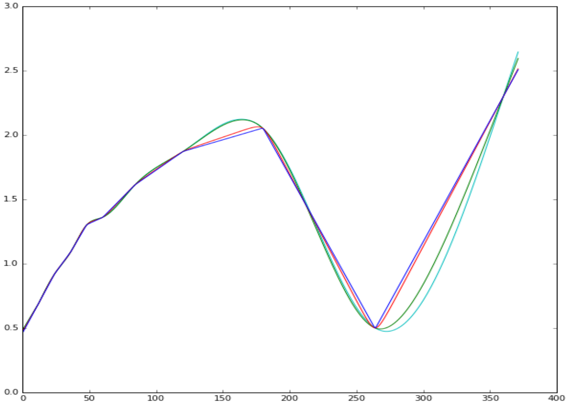


Figure: Swap Rates interpolated with tension Splines of increasing σ. $\sigma = 0.001$, $\sigma = 0.01$, $\sigma = 0.1$. The limit $\sigma \to 0$ approaches the cubic spline. Strong σ approaches the linear interpolation.

Swap Rates (cont. 4)

The cubic spline $f(x)$ interpolating a set of given points (x_i, f_i), $i = 1, \dots, N$. Such spline is piecewise linear in its second derivative

$$f''(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} f_i'' + \frac{x - x_i}{x_{i+1} - x_i} f_{i+1}'' \quad , \tag{72}$$

where the second derivative is continuous at knot points, $\lim_{x \to x_i^-} f''(x) = \lim_{x \to x_i^+} f''(x)$. We supply these equations with $f''(x_1) = f''(x_N) = 0$. This type of interpolation is called *natural cubic spline*.

An improvement of the cubic spline is by applying tensile force at the end points, measured by σ. Formally this is accomplished by replacing Eq. (72) by

$$f''(x) - \sigma^2 f(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i}(f_i'' - \sigma^2 f_i) + \frac{x - x_i}{x_{i+1} - x_i}(f_{i+1}'' - -\sigma^2 f_{i+1}) \quad . \tag{73}$$

See figure of previous strengths of σ. At strong σ the interpolation approaches a linear Swap Rate interpolation. Small σ limit approaches the natural cubic spline.

Short Rate Modeling

Short Rate Modeling

In a **one-factor short-rate model** we assume that $r(t)$ satisfies the following

$$\mathrm{d}r(t) = \mu(t, r(t))\mathrm{d}t + \sigma(t, r(t))\mathrm{d}W(t) \tag{74}$$

Table: Short Rate Models

Model	$\mu(t, r(t))$	$\sigma(t, r(t))$
Merton	θ	σ
Vasicek	$\theta - \kappa r(t)$	σ
Cox–Ingersoll–Ross	$\kappa(\beta - r(t))$	$\sigma\sqrt{r(t)}$
Dothan	$\kappa r(t)$	$\sigma r(t)$
Black–Derman–Toy	$\theta(t) r(t)$	$\sigma r(t)$
Ho–Lee	$\theta(t)$	$\sigma(t)$
Hull–White (extended Vasicek)	$\theta(t) - \kappa r(t)$	$\sigma(t)$
Black–Karasinski	$r(t)(\theta(t) - \kappa \ln r(t))$	$\sigma r(t)$

Bond Price

Starting point, $\frac{B(t,T)}{B(t)}$ is a martingale under the Q risk-neutral measure. $B(t) = \exp\left(\int_0^t r(s)\mathrm{d}s\right)$ and

$$\frac{B(t,T)}{B(t)} = E^Q\left[\frac{B(T,T)}{B(T)}\middle|\mathcal{F}_t\right] = E^Q\left[\frac{1}{\exp\left(\int_0^T r(s)\mathrm{d}s\right)}\middle|\mathcal{F}_t\right]. \quad (75)$$

$$B(t,T) = B(t)E^Q\left[e^{-\int_0^T r(s)\mathrm{d}s}\middle|\mathcal{F}_t\right], \quad (76)$$

$$= E^Q\left[e^{-\int_t^T r(s)\mathrm{d}s}\middle|\mathcal{F}_t\right]. \quad (77)$$

Compare this with the expected values for the stocks case

$$E^Q\left[(S(T)-K)^+\right], \quad \text{and} \quad E^Q\left[\left(\exp\left(\int_0^T S(u)\mathrm{d}u\right)-K\right)^+\right]. \quad (78)$$

The equation to the right demonstrates the complexity of interest rate modeling compared with the equity modeling.

Merton Model

This is the simplest model from the one-factor short rate models

$$\mathrm{d}r(t) = \theta \mathrm{d}t + \sigma \mathrm{d}W(t) \quad . \tag{79}$$

Integrating this gives

$$r(s) = r(t) + \theta(s-t) + \sigma(W(s) - W(t)) \quad . \tag{80}$$

In Eq. (77) we need the surface under the path of the short rate:

$$\int_t^T r(s)\mathrm{d}s = r(t)(T-t) + \frac{1}{2}\theta(T-t)^2 + \sigma\int_t^T W(s)\mathrm{d}s - \sigma W(t)(T-t)\ . \tag{81}$$

We need to calculate the integral

$$\int_t^T W(s)\mathrm{d}s \quad , \tag{82}$$

and there are two ways of doing it, either a) by parts, or b) using Fubini's theorem. I will show both below.

Merton Model

$$\int_t^T W(s)\mathrm{d}s = W(s)s\Big|_t^T - \int_t^T s\mathrm{d}W(s) \quad , \tag{83}$$

$$= W(T)T - W(t)t - \int_t^T s\mathrm{d}W(s) \quad , \tag{84}$$

$$= (W(T)T - W(t)T) + (W(t)T - W(t)t) - \int_t^T s\mathrm{d}W(s) \, ,$$

$$= (T-t)W(t) + \int_t^T (T-s)\mathrm{d}W(s) \quad .$$

The calculation above can analogously be thought as follows:

$$\mathrm{d}\left((T-s)W(s)\right) = -W(s)\mathrm{d}s + (T-s)\mathrm{d}W(s) \quad , \tag{85}$$

we get

$$X = \int_t^T r(s)\mathrm{d}s = r(t)(T-t) + \frac{1}{2}\theta(T-t)^2 + \sigma\int_t^T (T-s)\mathrm{d}W(s) \, , \tag{86}$$

Merton Model

Notice that putting together the last two terms of Eq. (81), we get:

$$\sigma \int_t^T W(s)\mathrm{d}s - \sigma W(t)(T-t) = \sigma \int_t^T (T-s)\mathrm{d}W(s) \quad . \tag{87}$$

We can now write

$$\begin{aligned} X = \int_t^T r(s)\mathrm{d}s &= r(t)(T-t) + \frac{1}{2}\theta(T-t)^2 + \sigma \int_t^T (T-s)\mathrm{d}W(s) \ , \\ &= \bar{X} + \sigma \int_t^T (T-s)\mathrm{d}W(s) \ , \\ &= \bar{X} + \left(\sigma^2 \int_t^T (T-s)^2 \mathrm{d}s\right)^{1/2} y \ , \\ &= \bar{X} + (\mathrm{Var}(X))^{1/2} y \ , \end{aligned} \tag{88}$$

where y is the standard Gaussian distributed variable

$$\mathrm{PDF}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \quad . \tag{89}$$

Merton Model (cont.)

Let us use another method of calculating the following integral, namely use Fubini's method

$$\int_t^T W(s)\mathrm{d}s = \int_0^{\tilde{T}} \mathrm{d}s \int_0^s \mathrm{d}W(u) \quad , \tag{90}$$

$$= \int_0^{\tilde{T}} \mathrm{d}W(u) \int_u^{\tilde{T}} \mathrm{d}s \quad , \tag{91}$$

$$= \int_0^{\tilde{T}} (\tilde{T} - u)\mathrm{d}W(u) \quad , \tag{92}$$

whose variance is given by

$$\mathrm{Var}(X) = \int_0^{\tilde{T}} (\tilde{T} - u)^2 \mathrm{d}u = \int_0^{\tilde{T}} u^2 \mathrm{d}u = \frac{1}{3}(T - t)^3 \quad , \tag{93}$$

same as what we found with the integration by parts.

Merton Model (cont.)

Since the variance of the integral is

$$\mathrm{var}(X) = \sigma^2 \int_t^T (T-s)^2 \mathrm{d}s = \frac{1}{3}\sigma^2 (T-t)^3 \quad , \tag{94}$$

and expected value of $e^{-X} = e^{-\bar{X}+\frac{1}{2}\mathrm{var}(\mathrm{X})}$, the zero-coupon bond price in the Merton model becomes

$$B(t,T) = \exp\left(-r(t)(T-t) - \frac{1}{2}\theta(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3\right) \quad . \tag{95}$$

If there would be no drift term then the surface under the process would be just $r(t)(T-t)$. The drift adds to this the additional area of $(1/2)[\theta \cdot (T-t)](T-t)$. But there is an additional term that adds to the area, $\sigma^2(T-t)^3/6$, and this is the convexity that we have mentioned earlier due to the fact that the stochastic process is in the exponential.

Merton Model (cont.)

What did this calculation buy us? Why did we need to calculate the bond prices which we can exctract from the yield curve?

The answer found in these two points:

1. We can calibrate this to the existing Bonds to extract θ and σ
2. We know the stochastic behavior of the bond prices, namely

$$
\begin{aligned}
B(t,T) &= \exp\left(-r(t)(T-t) - \frac{1}{2}\theta(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3\right) \quad (96) \\
&= \exp\left(-[r(0)+\theta t + \sigma W(t)]\tilde{T} - \frac{1}{2}\theta\tilde{T}^2 + \frac{1}{6}\sigma^2\tilde{T}^3\right), \quad (97)
\end{aligned}
$$

were $\tilde{T} = T - t$.

Therefore now we know the stochastic process for all bonds and we will be able to calculate options prices.

Expectation Calculations

$$
\begin{aligned}
E\left[e^{-X}\right] &= E\left[e^{-\bar{X}-(X-\bar{X})}\right] = e^{-\bar{X}} E\left[e^{-\sqrt{\mathrm{Var}(X)}y}\right] = e^{-\bar{X}} E\left[e^{-\sigma y}\right] \;, \\
&= e^{-\bar{X}} \int_{-\infty}^{\infty} \left[e^{-\sigma y}\right] e^{-\frac{1}{2}y^2} \frac{\mathrm{d}y}{\sqrt{2\pi}} \;, && (98) \\
&= e^{-\bar{X}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(y^2+2\sigma y+\sigma^2-\sigma^2\right)} \frac{\mathrm{d}y}{\sqrt{2\pi}} \;, \\
&= e^{-\bar{X}} e^{\frac{1}{2}\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(y^2+2\sigma y+\sigma^2\right)} \frac{\mathrm{d}y}{\sqrt{2\pi}} \;, \\
&= e^{-\bar{X}} e^{\frac{1}{2}\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y+\sigma)^2} \frac{\mathrm{d}y}{\sqrt{2\pi}} \;, && (99) \\
&= e^{-\bar{X}} e^{\frac{1}{2}\sigma^2} \;, \\
&= e^{-\bar{X}} e^{\frac{1}{2}\mathrm{Var}(X)} \;. && (100)
\end{aligned}
$$

Shortcomings of Merton Model

Note: A model to be used to price exotics should at least recover the discount factors $B(0, T)$ for all T. Other instruments it should recover are the the trades "natural hedging instruments".
But

1. Merton model has just two parameters, see Eq. (95), the parameters θ and σ.
2. Merton model doesn't recover an important empirical feature: When rates are high, they tend to revert to their historical values over time. Analogously, when the rates are low they will tend to reverse drift and go higher towards their historical values. Merton doesn't have mean reversion and the process leads to strong "convexity" as it will be seen in next few pages.

The Vasicek Model for Bond Price

The Vasicek Model for Bond Price

The Vasicek model in the risk-neutral measure $\mathcal{Q}$,

$$\mathrm{d}r(t) = (\theta - \kappa r(t))\mathrm{d}t + \sigma \mathrm{d}W(t) \quad , \tag{101}$$

or equivalently in terms of the ratio $\theta/\kappa = \vartheta$

$$\mathrm{d}r(t) = \kappa(\vartheta - r(t))\mathrm{d}t + \sigma \mathrm{d}W(t) \quad . \tag{102}$$

Vasicek remediates the mean-reversion shortcoming of Merton model, however, it still has a limited number of parameters to fit the intial discount curve.

The drift is positive if $r(t) < \theta/\kappa$ and negative if $r(t) > \theta/\kappa$.

The Vasicek Model for Bond Price (cont. 2)

Re-write the model as follows:

$$\mathrm{d}r(t) + \kappa r(t)\mathrm{d}t = \theta \mathrm{d}t + \sigma \mathrm{d}W(t) \quad , \tag{103}$$

Multiply both sides with the factor $e^{\kappa t}$, resulting in

$$\mathrm{d}\left(e^{\kappa t} r(t)\right) = e^{\kappa t}\theta \mathrm{d}t + e^{\kappa t}\sigma \mathrm{d}W(t) \quad . \tag{104}$$

Now this equation can equation can be integrated on both sides from t to s

$$e^{\kappa s} r(s) - e^{\kappa t} r(t) = \theta \int_t^s e^{\kappa u}\mathrm{d}u + \sigma \int_t^s e^{\kappa u}\mathrm{d}W(u) \quad , \tag{105}$$

$$r(s) = e^{\kappa(t-s)} r(t) + \theta \int_t^s e^{\kappa(u-s)}\mathrm{d}u + \sigma \int_t^s e^{\kappa(u-s)}\mathrm{d}W(u) \quad . \tag{106}$$

The Vasicek Model for Bond Price (cont. 3)

Notice the long term values of the short rate and its variance ($t = 0$ to simplify)

$$E^{\mathcal{Q}}\left[r(s)\right] = r(0)e^{-\kappa s} + \theta e^{-\kappa s} \int_0^s e^{\kappa u} \mathrm{d}u \quad , \tag{107}$$

$$= r(0)e^{-\kappa s} + \theta e^{-\kappa s}\left(e^{\kappa s} - 1\right) \quad , \tag{108}$$

$$= r(0)e^{-\kappa s} + \theta \frac{\left(1 - e^{-\kappa s}\right)}{\kappa} \quad . \tag{109}$$

In the long term, when $s \to \infty$, the short rate will tend to the historical level of $r(s) \to \theta/\kappa$.
The variance of the short rate as function of time is going to change

$$\mathrm{Var}(r(s)) = \sigma^2 \int_0^s e^{-2\kappa(s-u)} \mathrm{d}u = \sigma^2 \frac{1 - e^{-2\kappa s}}{2\kappa} \quad . \tag{110}$$

The forward variance is given by (see Fig. in next page; Forward vols are higher for stronger κ)

$$\mathrm{Var}(r(t \to T)) = \sigma^2 \int_t^T e^{-2\kappa(T-u)} \mathrm{d}u = \sigma^2 \frac{1 - e^{-2\kappa(T-t)}}{2\kappa} \quad . \tag{111}$$

The Vasicek Model for Bond Price (cont. 4)

Question 1 : What is the asymptotic value of the variance at long times, $s \to \infty$?

Question 2 : What is the asymptotic value of the variance at high values of the mean reversion, $\kappa \to \infty$?

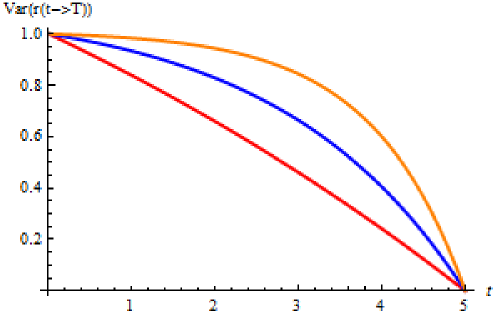


Figure: Three different Forward Variance term-structures $\mathrm{Var}(t, T)$ obtained with $\kappa = 0.9$ (top orange line), $\kappa = 0.45$, (blue middle line), and $\kappa = 0.1$ (red bottom line).

The Vasicek Model for Bond Price (cont. 5)

To find the price of the bond under the Vasicek model, we need the distribution of the integral

$$X = \int_t^T r(s)\mathrm{d}s \tag{112}$$

Its mean value is

$$E^Q\left[\int_t^T r(s)\mathrm{d}s\right] = \int_t^T e^{-\kappa(s-t)} r(t)\mathrm{d}s + \theta \int_t^T \left(\int_t^s e^{-\kappa(s-u)}\mathrm{d}u\right)\mathrm{d}s \tag{113}$$

The Vasicek Model for Bond Price (cont. 5)

We will repeatedly need these integrals

$$\int_t^s e^{-\kappa(s-u)} \mathrm{d}u = \int_t^s e^{-\kappa(u-t)} \mathrm{d}u = \frac{1 - e^{-\kappa(s-t)}}{\kappa} \quad . \tag{114}$$

We denote by

$$G(t, T) = \int_t^T e^{-\kappa(s-t)} \mathrm{d}s = \frac{1 - e^{-\kappa(T-t)}}{\kappa} \quad . \tag{115}$$

Therefore the mean value of the integral over $r(t)$,

$$E^Q \left[\int_t^T r(s) \mathrm{d}s \right] = \int_t^T e^{-\kappa(s-t)} r(t) \mathrm{d}s + \theta \int_t^T \left(\int_t^s e^{-\kappa(s-u)} \mathrm{d}u \right) \mathrm{d}s$$

$$\bar{X} = r(t) G(t, T) + \theta \int_t^T G(t, s) \mathrm{d}s \tag{116}$$

$$\bar{X} = r(t) G(t, T) + \theta \int_t^T G(u, T) \mathrm{d}u \tag{117}$$

The Vasicek Model for Bond Price (cont. 7)

The variance can be inferred from the variance of the following normally distributed variable

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{var}\left(\sigma \int_t^T \mathrm{d}s \left(\int_t^s e^{\kappa(u-s)} \mathrm{d}W(u)\right)\right) && (118)\\
&= \mathrm{var}\left(\sigma \int_t^T G(u,T) \mathrm{d}W(u)\right) && (119)\\
&= \sigma^2 \int_t^T G^2(u,T)\mathrm{d}u \quad . && (120)
\end{aligned}$$

Here we make use of

$$\mathrm{d}\left(\int_t^s e^{-\kappa(s-u)} \mathrm{d}u\right) = \mathrm{d}s - \kappa \left(\int_t^s e^{-\kappa(s-u)} \mathrm{d}u\right) \mathrm{d}s \qquad (121)$$

The Vasicek Model for Bond Price (cont. 8)

From Eq.(121) we get

$$\left(\int_t^s e^{-\kappa(s-u)}\mathrm{d}u\right)\mathrm{d}s = \mathrm{d}\left(\int_t^s \frac{1-e^{-\kappa(s-u)}}{\kappa}\mathrm{d}u\right) \quad , \qquad (122)$$

$$= \mathrm{d}\left(\int_t^s G(u,s)\mathrm{d}u\right) \quad . \qquad (123)$$

Similarly

$$\left(\int_t^s e^{-\kappa(s-u)}\mathrm{d}W(u)\right)\mathrm{d}s = \mathrm{d}\left(\int_t^s \frac{1-e^{-\kappa(s-u)}}{\kappa}\mathrm{d}W(u)\right) , \qquad (124)$$

$$= \mathrm{d}\left(\int_t^s G(u,s)\mathrm{d}W(u)\right) \quad . \qquad (125)$$

Therefore the mean value and the variance will be given by

$$\bar{X} = r(t)G(t,T) + \theta\int_t^T G(u,T)\mathrm{d}u \quad , \quad \mathrm{var(X)} = \sigma^2\int_t^T G^2(u,T)\mathrm{d}u \qquad (126)$$

The Vasicek Model for Bond Price (cont. 9)

We use again the equality $E[e^{-X}] = e^{-\bar{X}+\frac{1}{2}\mathrm{var}(X)}$.

The zero-coupon Bond price in the Vasicek model becomes

$$B(t,T) = \exp\left(-r(t)G(t,T) - \theta\int_t^T G(u,T)\mathrm{d}u + \frac{1}{2}\sigma^2\int_t^T G^2(u,T)\mathrm{d}u\right) \quad (127)$$

Compare this with the Merton zero-coupon Bond price, Eq.(95)

$$B(t,T) = \exp\left(-r(t)(T-t) - \frac{1}{2}\theta(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3\right) \quad . \quad (128)$$

Notice that Eq.(127) contains one more variable, κ, than (128). The κ parameter in Eq.(127) is contained within the $G(t,T)$ function.

The Vasicek Model for Bond Price (cont. 10)

Look at the limit $\kappa \to 0$ of the Vasicek model below (sometimes also known as the PhiT limit of the PhiG model an acronym for Vasicek)

$$G(t,T) = \int_t^T e^{-\kappa(s-t)} \mathrm{d}s = \frac{1-e^{-\kappa(T-t)}}{\kappa} \quad . \tag{129}$$

In the limit

$$\kappa \to 0 \quad , \quad G(t,T) = \frac{1-e^{-\kappa(T-t)}}{\kappa} \to (T-t) \quad . \tag{130}$$

and Eq. (127) reduces to Eq. (128). $\kappa \to 0$ is the Merton limit of the Vasicek model.

In the equations in Vasicek $G(t,T)$ should be seen as a transformation of time caused by the presence of the mean reversion κ.

Calibration of the Vasicek Model

Project 2: Calibrate the Vasicek model to this zero-discount curve.

Table: Calibration of Vasicek model

Maturity	$B^{\mathrm{mkt}}(0, T_i)$	$B^{\mathrm{mdl}}(0, T_i)$
1	0.999181	?
2	0.988664	.
3	0.976716	.
4	0.963709	.
5	0.950122	.
6	0.936292	.
7	0.922252	.
8	0.908675	.
9	0.895657	.
10	0.883039	.

Use the least square method

$$\min \sum^{N} \left(B^{\mathrm{mkt}}(0, T_i) - B^{\mathrm{mdl}}(0, T_i; \theta, \kappa, \sigma) \right)^2 \tag{131}$$

Bonds on the Vasicek Model

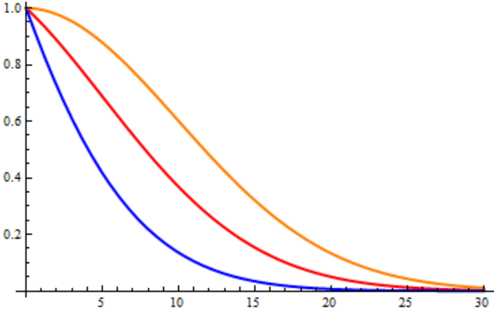


Figure: Three different bond term-structures $B(t, T)$ obtained with $r(t) = 1\%$ (top orange line), $r(t) = 5\%$, (red middle line), and $r(t) = 15\%$ (blue bottom line).

As $r(t)$ changes over time, so does the line $B(t, T)$, moving continuously from orange to the red and blue when the interest rates rise, and opposite direction when rates move to lower values.

Bonds on the Vasicek Model

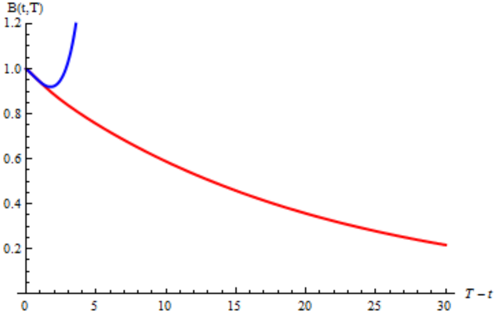


Figure: Graph of Merton and Vasicek models term-structure of the Bonds. The Merton bonds (blue line) show the erroneous behaviour due to presence of the term in $y(t, T) = -\sigma_r^2 (T-t)^2/2 + \ldots$.

Overlapping graphs of Merton and Vasicek Bonds...

Alternative Solution to the Vasicek Model

Project 3: Based on the fact that

$$B(t,T) = F(t,r(t);T) = e^{f(t,T)+r(t)g(t,T)} \tag{132}$$

and the equation for $\mathrm{d}r(t)$, find the values of $f(t,T)$ and $g(t,T)$. Use also the fact that $B(T,T) = 1$.

Alternative derivation of Vasicek

Project 4: Based on the fact that

$$B(t,T) = F(t, r(t); T) = e^{f(t,T)+r(t)g(t,T)} \quad , \tag{133}$$

and the equation for $\mathrm{d}r(t)$, find the values of $f(t,T)$ and $g(t,T)$.
Solution:
1. Make use of the fact that the value of the bond at maturity is equal to 1\$, $B(T,T) = 1$. What does this mean for the volatility of the Bond as we approach the maturity point $t \to T$? It means that the Bonds volatility tends to zero, $\Sigma(t,T) \to 0$ when $t \to T$, as there is no uncertainty on the payoff of the bond at T. Another useful information we can extract from $B(T,T) = 1$ is that $f(T,T) = 0$ and $g(T,T) = 0$,

$$B(T,T) = 1 \Leftrightarrow \begin{cases} \Sigma(T,T) = 0 & , \\ f(T,T) = 0 & , \\ g(T,T) = 0 & . \end{cases} \tag{134}$$

Alternative derivation of Vasicek (cont. 2)

2. To simplify intuition, denote the process $X(t)$ instead of $r(t)$

$$\mathrm{d}X(t) = \mu(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W_r(t) \quad . \tag{135}$$

In the risk-neutral measure $\mathcal{Q}$, the bond is a trading asset with risk-free rate drift

$$\mathrm{d}B(t,T) = r(t)B(t,T)\mathrm{d}t + \Sigma(t,T)B(t,T)\mathrm{d}W_B \quad , \tag{136}$$

with the lognormal choice due to the fact that the short rate in the exponent is normal.

$$\mathrm{d}B = \frac{\partial B}{\partial t}\mathrm{d}t + \frac{\partial B}{\partial X}\mathrm{d}X + \frac{1}{2}\frac{\partial^2 B}{\partial X^2}\mathrm{d}X^2 \quad , \tag{137}$$

$$\begin{aligned}
\mathrm{d}B &= \frac{\partial B}{\partial t}\mathrm{d}t + \frac{\partial B}{\partial X}\Big(\mu(t,X)\mathrm{d}t + \sigma(t,X)\mathrm{d}W\Big) + \frac{1}{2}\frac{\partial^2 B}{\partial X^2}\mathrm{d}X^2 \quad , \\
&= \left(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial X}\mu(t,X) + \frac{1}{2}\frac{\partial^2 B}{\partial X^2}\sigma^2\right)\mathrm{d}t + \frac{\partial B}{\partial X}\sigma(t,X)\mathrm{d}W_r \quad .
\end{aligned}$$

Alternative derivation of Vasicek (cont. 3)

The Bond volatility is

$$\Sigma(t,T) = \sigma(t,X)\frac{\partial B}{\partial X}\frac{1}{B} = \sigma_r \frac{\partial \ln B(t,T)}{\partial X} = \sigma_r g(t,T)\,. \qquad (138)$$

Equating the drift terms

$$\frac{\partial B}{\partial t} + \frac{\partial B}{\partial X}\mu(t,X) + \frac{1}{2}\frac{\partial^2 B}{\partial X^2}\sigma^2 = XB(t,T) \quad . \qquad (139)$$

Look at the derivatives one by one

$$1. \quad \frac{\partial B}{\partial t} = \frac{\partial}{\partial t} e^{f(t,T)+Xg(t,T)} = \left(\frac{\partial f}{\partial t} + X\frac{\partial g}{\partial t}\right) e^{f(t,T)+Xg(t,T)} \quad , \qquad (140)$$

$$2. \quad \frac{\partial B}{\partial X} = g(t,T)e^{f(t,T)+Xg(t,T)} \quad , \quad \mu(t,X) = \theta - \kappa X \quad , \qquad (141)$$

$$3. \quad \frac{\partial^2 B}{\partial X^2} = g^2(t,T)e^{f(t,T)+Xg(t,T)} \quad , \qquad (142)$$

Alternative derivation of Vasicek (cont. 3)

Putting all the derivatives back in Eq. (139), and canceling out the exponential on both sides, we obtain

$$\frac{\partial f}{\partial t}(t,T) + g(t,T)\theta + \frac{1}{2}\sigma^2 g^2(t,T) + x\left(\frac{\partial g}{\partial t}(t,T) - g(t,T)\kappa\right) = x \quad (143)$$

Equating terms proportional to x we get

$$\partial_t g(t,T) - g(t,T)\kappa = 1 \quad , \quad \mathrm{d}\left(e^{-\kappa t} g(t,T)\right) = e^{-\kappa t} \quad . \qquad (144)$$

For $g(t,T)$ we find

$$g(t,T) = -\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) = -G(t,T) \quad . \qquad (145)$$

The bond volatility is therefore equal to

$$\Sigma(t,T) = -\sigma_r G(t,T) \, . \qquad (146)$$

Alternative derivation of Vasicek (cont. 4)

Equating the terms independent of x

$$f_t(t,T) + \theta g(t,T) + \frac{1}{2}\sigma^2 g^2(t,T) = 0 \quad , \tag{147}$$

$$f(T,T) - f(t,T) = -\theta \int_t^T g(u,T)\mathrm{d}u - \frac{1}{2}\sigma^2 \int_t^T g^2(u,T)\mathrm{d}u\,, \tag{148}$$

$$f(t,T) = -\theta \int_t^T G(u,T)\mathrm{d}u + \frac{1}{2}\sigma^2 \int_t^T G^2(u,T)\mathrm{d}u\,. \tag{149}$$

(keep in mind $f(T,T) = 0$ from Eq. (134).)
Compare this with Eq. (127) and we conclude that we got an equivalent derivation based on very simple facts assumed in the exponential proposition (133).

Yield to Maturity of Merton and Vasicek Models

In the Merton model, the yield increases and then decreases to $y_\infty \to -\infty$, an unrealistic situation

$$y(t,T) = -\frac{\ln B(t,T)}{T-t} = r(t) + \frac{1}{2}\theta(T-t) - \frac{1}{6}\sigma_r^2(T-t)^2 \quad , \qquad (150)$$

the yield increases and then decreases reaching

$$y_\infty = \lim_{T\to\infty} y(t,T) = -\frac{1}{6}\sigma_r^2(T-t)^2 \to -\infty \quad . \qquad (151)$$

In the Vasicek model

$$y(t,T) = r(t)\frac{G(t,T)}{T-t} + \theta\int_t^T \frac{G(u,T)}{T-t}\mathrm{d}u - \frac{1}{2}\sigma_r^2\int_t^T \frac{G^2(u,T)}{T-t}\mathrm{d}u \qquad (152)$$

In the $T-t\to 0$ limit, which is equivalent to $\kappa \to 0$, the yield is as in Eq. (150). At $T-t\to\infty$

$$y(t,T) \to r(t) + \frac{\theta}{\kappa} - \frac{1}{2}\frac{\sigma_r^2}{\kappa^2} \quad , \qquad (153)$$

reaching the long term limit of $r_\infty = \theta/\kappa = \vartheta$ corrected by finite convexity $\sigma_r^2/(2\kappa^2)$. It does not have issue of Merton model (convexity).

Yield to Maturity of Merton and Vasicek Models (cont.)

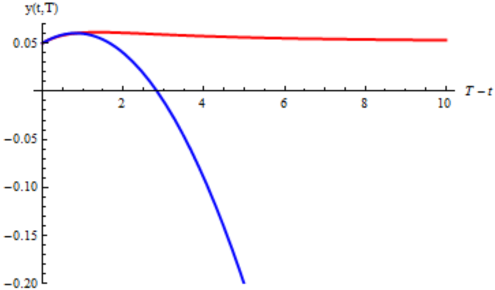


Figure: The yield-to-maturity $y(t, T)$ for the Merton model (blue line) and the Vasicek model (red line). obtained with $r(t) = 5\%$, $\theta = 5\%$, $\kappa = 1$, $\sigma_r = 10\%$ (double check the numbers).

The Merton model leads to a very strong convexity term $-\sigma_r^2(T - t)^2/2$ which eventually becomes negative.

The Hull-White Model

The Hull-White Model

The Hull-White model is an extension of the Vasicek model, by letting $\theta = \theta(t)$ be a function of t and $\sigma = \sigma(t)$, (κ remains a constant as before).
(How many variables do we have to fit the discount curve (?))

$$\mathrm{d}r(t) = (\theta(t) - \kappa r(t))\,\mathrm{d}t + \sigma(t)\mathrm{d}W(t) \quad . \tag{154}$$

Integrating the above equation we get a similar equation to Vasicek's model

$$r(s) = e^{-\kappa(s-t)}r(t) + \int_t^s \theta(u)e^{-\kappa(s-u)}\mathrm{d}u + \int_t^s \sigma(u)e^{-\kappa(s-u)}\mathrm{d}W(u) \tag{155}$$

The Hull-White Model (cont. 2)

The zero-coupon Bond price in the Hull-White model becomes

$$B(t,T) = \exp\Big(-r(t)G(t,T) - \int_t^T \theta(u)G(u,T)\mathrm{d}u + \frac{1}{2}\int_t^T \sigma^2(u)G^2(u,T)\mathrm{d}u\Big) \quad . \tag{156}$$

The only thing to be done here is extract $\theta(u)$ such that it fits the initial discount curve.

Compare this with zero-coupon Bond price in the Vasicek model

$$B(t,T) = \exp\left(-r(t)G(t,T) - \theta\int_t^T G(u,T)\mathrm{d}u + \frac{1}{2}\sigma^2\int_t^T G^2(u,T)\mathrm{d}u\right) \quad . \tag{157}$$

The Hull-White Model (cont. 3)

The only place where θ is present in Eq. (156) is inside the integral

$$\int_t^T \theta(u)G(u,T)\mathrm{d}u \quad . \tag{158}$$

We need to express it in terms of the current yield curve i.e. the current term-structure of the bonds. We will use Eq. (156), once for $B(0,t)$ and also $B(0,T)$.
Most importantly we will make use of the following "Taylor"-like expansion

$$G(0,T) = G(0,t) + G_t(0,t)G(t,T) \quad , \tag{159}$$

where $G(t,T)$ is given by Eq. (115). This function is used as $G(t,T) \sim (T-t)$. Notice that the equality Eq. (159) is exact and not an approximation

$$\int_0^T e^{-\kappa s}\mathrm{d}s = \int_0^t e^{-\kappa s}\mathrm{d}s + e^{-\kappa t}\int_t^T e^{-\kappa(s-t)}\mathrm{d}s \quad . \tag{160}$$

The Hull-White Model (cont. 4)

The "Taylor"-like equality holds for more complex form of $G(t,T)$ as well, like non-constant $\kappa(t)$-functions

$$G(t,T) = \int_t^T e^{-\int_t^u \kappa(s)\mathrm{d}s}\mathrm{d}u \quad , \tag{161}$$

with

$$G(0,T) = G(0,t) + e^{-\int_0^t \kappa(s)\mathrm{d}s} G(t,T) \quad . \tag{162}$$

From Eq. (156), we have

$$\ln B(0,T) = -r(0)G(0,T) - \int_0^T \theta(u)G(u,T)\mathrm{d}u + \frac{1}{2}\int_0^T \sigma^2(u)G^2(u,T)\mathrm{d}u$$

$$\ln B(0,t) = -r(0)G(0,t) - \int_0^t \theta(u)G(u,t)\mathrm{d}u + \frac{1}{2}\int_0^t \sigma^2(u)G^2(u,t)\mathrm{d}u \tag{163}$$

The Hull-White Model (cont. 5)

Notice that in the two integrands occur the functions $G(u, T)$ and $G(u, t)$ which do not cancel out. Therefore some extra terms need to be added and subtracted when the difference $\ln B(0, T) - \ln B(0, t)$ is taken

$$\int_0^T \theta(u) G(u, T) \mathrm{d}u - \int_0^t \theta(u) G(u, t) \mathrm{d}u \tag{164}$$

$$= \int_t^T \theta(u) G(u, T) \mathrm{d}u + \int_0^t \theta(u) \Big(G(u, T) - G(u, t) \Big) \mathrm{d}u \quad ,$$

$$= \int_t^T \theta(u) G(u, T) \mathrm{d}u + \int_0^t \theta(u) \Big(G_t(u, t) G(t, T) \Big) \mathrm{d}u \quad ,$$

$$= \int_t^T \theta(u) G(u, T) \mathrm{d}u + G(t, T) \int_0^t \theta(u) G_t(u, t) \mathrm{d}u \quad .$$

($G(t, T)$ comes out of the second integral.) However, we now got an extra integral containing θ, the second integral above.

The Hull-White Model (cont. 6)

Such an integral over θ appears in the Eq. (163) for $\ln B(0,t)$. Taking its derivative

$$\begin{aligned} f(0,t) &= -\frac{\partial \ln B(0,t)}{\partial t} \quad , \qquad (165) \\ &= r(0)G_t(0,t) + \int_0^t \theta(u)G_t(u,t)\mathrm{d}u - \int_0^t \sigma^2(u)G(u,t)G_t(u,t)\mathrm{d}u\,. \end{aligned}$$

The second the integral over θ therefore can be calculated from the above, after multiplying with with $G(t,T)$

$$\begin{aligned} f(0,t)G(t,T) = r(0)G(t,T)G_t(0,t) &+ G(t,T)\int_0^t \theta(u)G_t(u,t)\mathrm{d}u \qquad (166) \\ &- G(t,T)\int_0^t \sigma^2(u)G(u,t)G_t(u,t)\mathrm{d}u \end{aligned}$$

Also use the same formula for the difference of $G(0,T)-G(0,t)$ to get

$$r(0)(G(0,T)-G(0,t)) = r(0)G(t,T)G_t(0,t) \quad . \qquad (167)$$

The Hull-White Model (cont. 7)

The other terms can be arranged better as follows

$$\frac{1}{2}\int_0^T \sigma^2(u)G^2(u,T)\mathrm{d}u - \frac{1}{2}\int_0^t \sigma^2(u)G^2(u,t)\mathrm{d}u \tag{168}$$

$$= \frac{1}{2}\int_t^T \sigma^2(u)G^2(u,T)\mathrm{d}u + \frac{1}{2}\int_0^t \sigma^2(u)\Big(G^2(u,T) - G^2(u,t)\Big)\mathrm{d}u$$

and

$$\frac{1}{2}\int_0^t \sigma^2(u)\Big(G^2(u,T) - G^2(u,t)\Big)\mathrm{d}u \tag{169}$$

$$= \frac{1}{2}\int_0^t \sigma^2(u)G_t(u,t)G(t,T)\Big(G(u,T) + G(u,t)\Big)\mathrm{d}u$$

The Hull-White Model (cont. 8)

Again replace $G(u,T) = G(u,t) + G_t(u,t)G(t,T)$ in the brackets in the last equation to obtain

$$\frac{1}{2}\int_0^T \sigma^2(u)G^2(u,T)\mathrm{d}u - \frac{1}{2}\int_0^t \sigma^2(u)G^2(u,t)\mathrm{d}u \tag{170}$$

$$= \frac{1}{2}\int_t^T \sigma^2(u)G^2(u,T)\mathrm{d}u + G(t,T)\int_0^t \sigma^2(u)G(u,t)G_t(u,t)\mathrm{d}u$$

$$+\frac{1}{2}G^2(t,T)\int_0^t \sigma^2(u)G_t^2(u,t)\mathrm{d}u \quad . \tag{171}$$

The Hull-White Model (cont. 9)

Putting all the results together, the θ ntegral of Eq. (156) can be extracted from the following equation

$$\begin{aligned} \ln\frac{B(0,T)}{B(0,t)} &= -f(0,t)G(t,T) - \int_t^T \theta(u)G(u,T)\mathrm{d}u \\ &+ \frac{1}{2}\int_t^T \sigma^2(u)G^2(u,T)\mathrm{d}u + \frac{1}{2}G^2(t,T)\int_0^t \sigma^2(u)G_t(u,t)\mathrm{d}u\,. \end{aligned} \tag{172}$$

The Hull-White Model (cont. 10)

After some obvious rearranging, the bond process $B(t, T)$ can be written as

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left\{-(r(t) - f(0,t))G(t,T) - \frac{1}{2}G^2(t,T)\int_0^t \sigma^2(u)e^{-2\kappa(t-u)}\mathrm{d}u\right\} \quad , \tag{173}$$

$$= \frac{B(0,T)}{B(0,t)} \exp\left\{-(r(t) - f(0,t))G(t,T) - \frac{1}{2}G^2(t,T)\int_0^t \sigma^2(u)G_t^2(u,t)\mathrm{d}u\right\} \quad , \tag{174}$$

$$= \frac{B(0,T)}{B(0,t)} \exp\left\{-x(t)G(t,T) - \frac{1}{2}y(t)G^2(t,T)\right\} \tag{175}$$

The Hull-White Model (cont. 11)

Let us look a bit more closely at $x(t) = r(t) - f(0,t)$

$$r(t) = e^{-\kappa t} r(0) + \int_0^t \theta(u) e^{-\kappa(t-u)} \mathrm{d}u + \sigma \int_0^t e^{-\kappa(t-u)} \mathrm{d}W(u) \quad (176)$$

$$= r(0) G_t'(0,t) + \int_0^t \theta(u) G_t'(u,t) \mathrm{d}u + \sigma \int_0^t G_t'(u,t) \mathrm{d}W(u) \quad (177)$$

We get rid of the integral over θ by using Eq. (165)

$$r(t) = r(0) G_t'(0,t) + \left(f(0,t) - r(0) G_t'(0,t) + \sigma^2 \int_0^t G(u,t) G_t'(u,t) \mathrm{d}u \right) + \sigma \int_0^t G_t'(u,t) \mathrm{d}W(u) \quad (178)$$

$$x(t) = r(t) - f(0,t) = \sigma^2 \int_0^t G(u,t) G_t'(u,t) \mathrm{d}u + \sigma \int_0^t G_t'(u,t) \mathrm{d}W(u) \quad ,$$

$$= \sigma^2 \int_0^t G(u,t) D_2 G(u,t) \mathrm{d}u + \sigma \int_0^t D_2 G(u,t) \mathrm{d}W(u) \quad . \quad (179)$$

The Hull-White Model (cont. 12)

Therefore in the risk-neutral measure the variable $x(t)$ will be random normally distributed but will have a drift, or mean value different from zero.

If we change the measure from Q to $\mathcal{P}_t$, done by changing the Brownian motion drift

$$\begin{aligned} W^t(u) &= W(u) - \int_0^u \Sigma(v,t)\mathrm{d}v = W(u) + \sigma_r \int_0^u G(v,t)\mathrm{d}v \quad (180) \\ \mathrm{d}W(u) &= \mathrm{d}W^t(u) - \sigma_r G(u,t)\mathrm{d}u \quad . \quad (181) \end{aligned}$$

Substituting in Eq. (179) we see that $x(t)$ becomes driftless in the t-forward measure

$$x(t) = \sigma \int_0^t G_t'(u,t)\mathrm{d}W^t(u) \quad , \quad (182)$$

and $y(t)$ is its variance

$$y(t) = \sigma^2 \int_0^t G_t'^2(u,t)\mathrm{d}u \quad . \quad (183)$$

No T dependence in either $x(t)$ or $y(t)$ and this is why this formula is important.

The Hull-White Model (cont. 13)

Using the relationship

$$G'_t(u,t)G(t,T) = G(u,T) - G(u,t) \quad , \tag{184}$$

the Brownian part becomes

$$\sigma \int_0^t G'_t(u,t)G(t,T)\mathrm{d}W(u) = \sigma \int_0^t \Big(G(u,T) - G(u,t) \Big) \mathrm{d}W(u) \tag{185}$$

and the drift part together with the $y(t)$ part become

$$\begin{aligned} & \sigma^2 \int_0^t G(u,t)G'_t(u,t)G(t,T)\mathrm{d}u + \frac{1}{2}\sigma^2 G^2(t,T) \int_0^t G'^2_t(u,t)\mathrm{d}u \\ = \quad & \frac{1}{2}\sigma^2 \int_0^t \left[2G(u,t)(G(u,T) - G(u,t)) + (G(u,T) - G(u,t))^2 \right] \mathrm{d}u \\ = \quad & \frac{1}{2}\sigma^2 \int_0^t \left[G^2(u,T) - G^2(u,t) \right] \mathrm{d}u \end{aligned} \tag{186}$$

The Hull-White Model (cont. 14)

In the risk-neutral measure the bond reconstruction formula is given by

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left\{-x(t)G(t,T) - \frac{1}{2}y(t)G^2(t,T)\right\} \quad (187)$$

$$= \frac{B(0,T)}{B(0,t)} \exp\left\{-\sigma \int_0^t \Big(G(u,T) - G(u,t)\Big)\mathrm{d}W(u) - \frac{1}{2}\sigma^2 \int_0^t \left[G^2(u,T) - G^2(u,t)\right]\mathrm{d}u\right\} \quad , \quad (188)$$

In the t-forward measure the bond reconstruction formula is given by

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left\{-x(t)G(t,T) - \frac{1}{2}y(t)G^2(t,T)\right\} \quad (189)$$

$$= \frac{B(0,T)}{B(0,t)} \exp\left\{-\sigma \int_0^t \Big(G(u,T) - G(u,t)\Big)\mathrm{d}W^t(u) - \frac{1}{2}\sigma^2 \int_0^t \Big(G(u,T) - G(u,t)\Big)^2\mathrm{d}u\right\} \quad , \quad (190)$$

The Hull-White Model (cont. 15)

In the <u>t-forward</u> measure the bond reconstruction formula is given by

$$\begin{aligned} B(t,T) &= B(0;t,T)\exp\left\{-x(t)G(t,T) - \frac{1}{2}y(t)G^2(t,T)\right\} && (191) \\ &= B(0;t,T)\exp\left\{-\sigma\int_0^t \Big(G(u,T) - G(u,t)\Big)\mathrm{d}W^t(u) \right. \\ &\quad \left. -\frac{1}{2}\sigma^2\int_0^t \Big(G(u,T) - G(u,t)\Big)^2 \mathrm{d}u\right\} \quad . && (192) \end{aligned}$$

The discounting from t to T will be on average equal to what the forward bond discounting was at time $t = 0$, i.e. $B(0;t,T) = B(0,T)/B(0,t)$. On average only , however, the proper discounting is stochastically distributed around the forward bond.

The Hull-White Model (cont. 16)

In the T-forward measure we use again the measure change formula

$$\mathrm{d}W(u) \quad = \quad \mathrm{d}W^T(u) - \sigma_r G(u,T)\mathrm{d}u \quad . \tag{193}$$

In the bond reconstruction formula the sign of the variance in the exponent changes now to $(+)$

$$B(t,T) \quad = \quad B(0;t,T)\exp\left\{-x(t)G(t,T) + \frac{1}{2}y(t)G^2(t,T)\right\} \tag{194}$$

$$= \quad B(0;t,T)\exp\left\{-\sigma\int_0^t \Big(G(u,T) - G(u,t)\Big)\mathrm{d}W^t(u) + \frac{1}{2}\sigma^2\int_0^t \Big(G(u,T) - G(u,t)\Big)^2\mathrm{d}u\right\} \quad . \tag{195}$$

Notice now that the inverse of the bond $B(t,T)$ is a martingale in the T-forward measure. This is to be expected indeed.

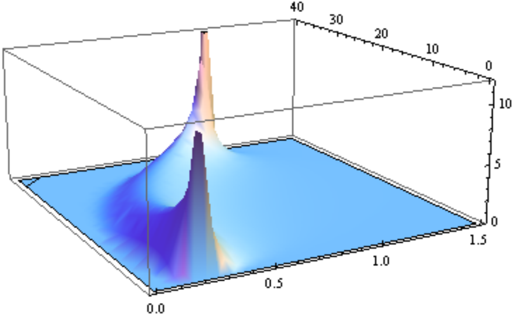


Figure: The distribution of the bond price as t varies in $(0, T)$. Notice the two delta functions at the beginning, $t \to 0$, and end $t \to T$. Expected value moves along $B(0; t, T)$ as t moves in $(0, T)$, $(T = 40\text{y})$.

The distribution of the Bond price $B(t, T)$ can be written in the two measures by using the formula

$$F(x) = \frac{1}{\sigma(t, T)x\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln\frac{x}{x_0} - \mu(t, T)}{\sigma(t, T)}\right)^2\right\} \quad . \qquad (196)$$

as function of the bond price $x = B(t, T)$, where $x_0 = B(0; t, T)$. While $\sigma(t, T)$ is the same in both measures, the drift is different.

The Hull-White Model (cont. 18)

Risk-neutral measure:

- $\sigma(t, T)$:

$$\sigma(t,T) = \sigma\left[\int_0^t \Big(G(u,T) - G(u,t)\Big)^2 \mathrm{d}u\right]^{\frac{1}{2}} \quad , \tag{197}$$

$$= \sigma\left[\frac{1}{2\kappa^3}\left(1 - e^{-\kappa(T-t)}\right)^2 \left(1 - e^{-2\kappa t}\right)\right]^{\frac{1}{2}} \quad , \tag{198}$$

$$\sim \sigma(T-t)\sqrt{t} \quad , \quad (\kappa \to 0) \quad . \tag{199}$$

- $\mu(t, T)$:

$$\mu(t,T) = -\frac{1}{2}\sigma^2 \int_0^t \left[G^2(u,T) - G^2(u,t)\right] \mathrm{d}u \quad . \tag{200}$$

The Hull-White Model (cont. 19)

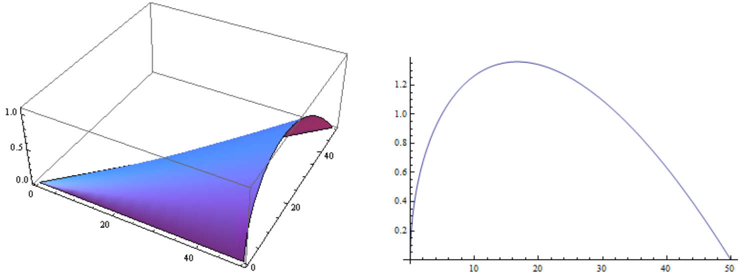


Figure: $\sigma(t, T)$ given by Eq. (198) as 3D plot (left) and as 2D plot (right). The variance tends to 0, on both limits, $t \to 0$ and $t \to T$, reaching the maximum somewhere at $t \approx T/2$. Here T is fixed to 50y.

The Hull-White Model (cont. 20)

t-forward measure:

- $\sigma(t, T)$:

$$\sigma(t, T) = \sigma \left[\int_0^t \Big(G(u, T) - G(u, t) \Big)^2 \mathrm{d}u \right]^{\frac{1}{2}} \quad . \tag{201}$$

- $\mu(t, T)$:

$$\mu(t, T) = -\frac{1}{2}\sigma^2 \int_0^t \left(G(u, T) - G(u, t) \right)^2 \mathrm{d}u \quad . \tag{202}$$

The Hull-White Model (cont. 21)

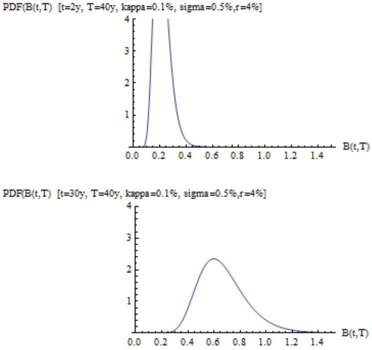


Figure: Plots of the PDF of $B(t,T)$ for $t = 2y, 20y, 30y, 35y$, as given by Eq. (196) are shown. Here $T = 40y$ and other parameters are shown on the graphs labels. The variance tends to 0, on both limits, $t \to 0$ and $t \to T$, reaching the maximum somewhere at $t \approx T/2$. The distribution takes into account the time dependence of the variance.

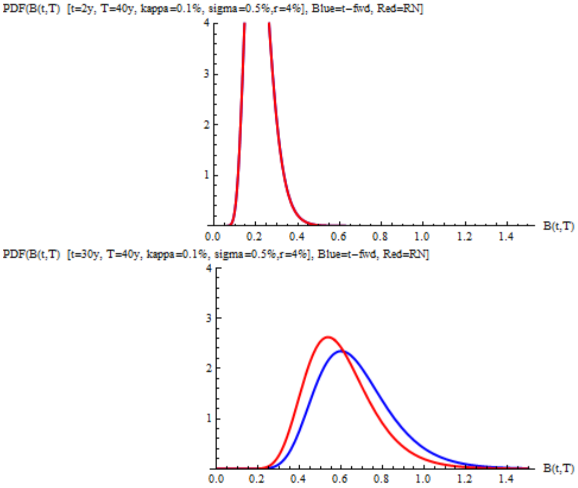


Figure: Plots of the PDF of $B(t, T)$ for $t = 2y$, and 30y, as given by Eq. (196) are shown. Here $T = 40y$ and other parameters are shown on the graphs labels. The two measure start to show discrepancies as $t \to T$.

The Hull-White Model (cont. 22)

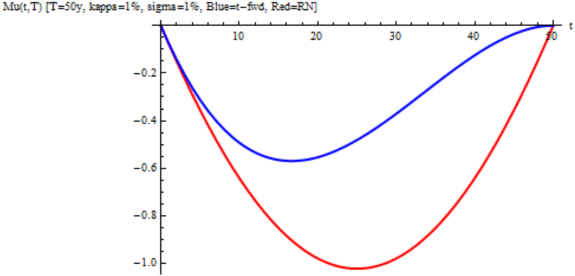


Figure: Plots of the drift of $B(t,T)$ for $t \in (0,T)$, as given by Eq. (196) are shown. Here $T = 40y$ and other parameters are shown on the graphs labels. The Blue line shows the t-forward measure drift, whereas the Red line shows the Risk-Neutral drift.

Origins of Cheyette

Notice in Eq. (175) that the term containing $y(t)$

$$-\frac{1}{2}G^2(t,T)\int_0^t \sigma^2(u)G_t'^2(u,t)\mathrm{d}u \quad , \tag{203}$$

has a component that extends from $t \to T$ and another one from $0 \to t$,

$$y(t) = \int_0^t \sigma^2(u)G_t^2(u,t)\mathrm{d}u \quad . \tag{204}$$

The differential equation for $y(t)$ can be obtained by taking the differential (keep in mind $G_t(t,t) = 1$)

$$\mathrm{d}y(t) = \sigma^2(t)\mathrm{d}t + \int_0^t \sigma^2(u)\Big(2G_t(u,t)G_{tt}(u,t)\mathrm{d}t\Big)\mathrm{d}u \tag{205}$$

but notice that

$$G_t(u,t) = e^{-\kappa(t-u)} \quad , \quad G_{tt}(u,t) = -\kappa e^{-\kappa(t-u)} = -\kappa G_t(u,t) \quad . \tag{206}$$

In the Cheyette model σ can be stochastic due to its dependence on the variable $x(t)$, and we write in general

$$\sigma = \sigma(t;\omega) \quad . \tag{207}$$

Origins of Cheyette

$$\mathrm{d}y(t) = \Big(\sigma^2(t;\omega) - 2\kappa y(t)\Big)\mathrm{d}t \quad . \tag{208}$$

This together with the equation for $x(t)$

$$\mathrm{d}x(t) = \Big(y(t) - \kappa x(t)\Big)\mathrm{d}t + \sigma_r(t;\omega)\mathrm{d}W(t) \quad , \tag{209}$$

constitute the Cheyette model. Usually $\sigma(t;\omega) = \sigma(t;x(t),y(t))$.

Option Pricing: Bond Option, European Options

Option Pricing: Bond Option

We start with the Bond option, of strike K, expiry S, on a bond of maturity T. With Bond options there are always two relevant future time points, expiry S, and bond maturity T. Between these two points there is a Libor rate relevant for interest accrual, now known as $L(t; S, T)$ which changes over time eventually becoming $L(S; S, T)$ at reset S. We will see that the option on bonds are similar to options on this Libor rate (see Caplet/Floorlet sections).

We apply the general pricing theorem

$$\frac{V(t)}{B(t)} = E^{\mathcal{Q}}\left[\left. \frac{(B(S,T)-K)^{+}}{B(S)} \right| \mathcal{F}_t \right] \tag{210}$$

In the risk-neutral measure the bond satisfies

$$\mathrm{d}B(t,T) = r(t)B(t,T)\mathrm{d}t + \Sigma(t,T)B(t,T)\mathrm{d}W(t) \quad . \tag{211}$$

with a measure change the price can be written as

$$V(t) = B(t,S)E^{\mathcal{P}_S}\left[\left. (B(S,T)-K)^{+} \right| \mathcal{F}_t \right] \quad . \tag{212}$$

Option Pricing: Bond Option (cont. 2)

Now we use the following trick, writing pricing equation as follows:

$$V(t) = B(t,S)E^{\mathcal{P}_S}\left[\left(\frac{B(S,T)}{B(S,S)} - K\right)^+ \middle| \mathcal{F}_t\right] \quad , \tag{213}$$

since $B(S,S) = 1$. The ratio $\frac{B(t,T)}{B(t,S)}$ is a martingale in the S-forward measure. Then we find the distribution of the ratio of the two bonds (which is the forward bond Eq. (10)), as follows

$$B(t;S,T) = \frac{B(t,T)}{B(t,S)} \quad . \tag{214}$$

To get the distribution of the forward bond at the expiry time S, we use the following:

$$\mathrm{d}\left(\frac{X}{Y}\right) = \mathrm{d}X\frac{1}{Y} + X\mathrm{d}\left(\frac{1}{Y}\right) + \mathrm{d}X\mathrm{d}\left(\frac{1}{Y}\right) \quad . \tag{215}$$

Option Pricing: Bond Option (cont. 3)

Notice the Ito lemma calculus

$$\mathrm{d}\left(\frac{1}{Y}\right) = -\frac{\mathrm{d}Y}{Y^2} + \frac{1}{2}(-1)(-2)\frac{(dY)^2}{Y^3} \quad , \tag{216}$$

and

$$\mathrm{d}X\mathrm{d}\left(\frac{1}{Y}\right) = -\frac{\mathrm{d}X\mathrm{d}Y}{Y^2} + \cdots \quad . \tag{217}$$

The Differential equation for the forward Bond $B(t; S, T)$ becomes

$$\begin{aligned} \mathrm{d}\left(\frac{B(t,T)}{B(t,S)}\right) &= \frac{B(t,T)}{B(t,S)}\Big(r(t)\mathrm{d}t + \Sigma(t,T)\mathrm{d}W(t)\Big) \\ &- \frac{B(t,T)}{B(t,S)}\Big(r(t)\mathrm{d}t + \Sigma(t,S)\mathrm{d}W(t) - \Sigma^2(t,S)\mathrm{d}t\Big) \\ &- \frac{B(t,T)}{B(t,S)}\Sigma(t,T)\Sigma(t,S)\mathrm{d}t \quad . \end{aligned} \tag{218}$$

Option Pricing: Bond Option (cont. 4)

$$\mathrm{d}\left(\frac{B(t,T)}{B(t,S)}\right) = \frac{B(t,T)}{B(t,S)}\Big(\Sigma^2(t,S)\mathrm{d}t - \Sigma(t,S)\Sigma(t,T)\mathrm{d}t + (\Sigma(t,T) - \Sigma(t,S))\,\mathrm{d}W(t)\Big) \quad . \tag{219}$$

The final point is to change the measure from Q to $\mathcal{P}_S$. This is done by changing the Brownian motion drift

$$W^S = W - \int_0^t \Sigma(u,S)\mathrm{d}u \quad . \tag{220}$$

Therefore the SDE of the forward bond in the $\mathcal{P}_S$ measure

$$\mathrm{d}B(t;S,T) = \Big(\Sigma(t,T) - \Sigma(t,S)\Big)B(t;S,T)\mathrm{d}W^S(t) \quad . \tag{221}$$

Notice that there is no drift for the forward bond in this measure.

Option Pricing: Bond Option (cont. 5)

$$
\begin{aligned}
V(t) &= B(t,S)E^{\mathcal{P}_S}\left[\left(\frac{B(S,T)}{B(S,S)} - K\right)^+ \middle| \mathcal{F}_t\right] \quad , & (222) \\
&= B(t,S)\left[\frac{B(t,T)}{B(t,S)}N(d_+) - KN(d_-)\right] \quad . & (223)
\end{aligned}
$$

where

$$
\begin{aligned}
d_+ &= \frac{1}{\sqrt{\xi(t,S)}}\left(\ln\frac{B(t;S,T)}{K} + \frac{1}{2}\xi(t,S)\right) \quad , & (224) \\
d_- &= \frac{1}{\sqrt{\xi(t,S)}}\left(\ln\frac{B(t;S,T)}{K} - \frac{1}{2}\xi(t,S)\right) \quad . & (225)
\end{aligned}
$$

Option Pricing: Bond Option (cont. 6)

Summing up

$$V(t) = B(t,T)N(d_+) - KB(t,S)N(d_-) \quad . \tag{226}$$

The variance $\xi(t,S)$ be extracted by Eq. (155) by Eq. (138)

$$\xi(t,S) = \sigma^2 \int_t^S |G(u,S) - G(u,T)|^2 \, \mathrm{d}u \quad , \tag{227}$$

$$= \frac{\sigma^2}{2\kappa^3} \left(1 - e^{-\kappa(T-S)}\right)^2 \left(1 - e^{-2\kappa(S-t)}\right) . \tag{228}$$

For the Hull-White model the variance is the one that corresponds to the $\ln B(S,T)$ Eq. (175), $r(S)G(S,T)$, were $r(S)$ is given by Eq. (155)

$$\xi_{\mathrm{Hull-White}}(t,S) = G^2(S,T) \int_t^S \sigma^2(u) e^{-2\kappa(S-u)} \mathrm{d}u \tag{229}$$

$$= \sigma^2 G^2(\kappa; S,T) G(2\kappa; t,S) \quad . \tag{230}$$

Option Pricing: Bond Option (cont. 7)

The same result can be obtained much faster if we use the bond reconstruction formula Eq. (195) which expresses already $B(S,T)$ in the S-forward measure

$$
\begin{aligned}
V(0) &= B(0,S)E^{\mathcal{P}_S}\left[\left.(B(S,T)-K)^+\right|\mathcal{F}_0\right] \quad , \qquad (231)\\
&= B(0,S)E^{\mathcal{P}_S}\left[\left.\left(\frac{B(0,T)}{B(0,S)}e^{-\sigma(t,T)y-\frac{1}{2}\sigma^2(t,T)}-K\right)^+\right|\mathcal{F}_0\right] \quad ,\\
&= E^{\mathcal{P}_S}\left[\left.\left(B(0,T)e^{-\sigma(t,T)y-\frac{1}{2}\sigma^2(t,T)}-KB(0,S)\right)^+\right|\mathcal{F}_0\right] \quad .
\end{aligned}
$$

y in the above equations is a standard normal distributed variable. The Black-Scholes results of the previous pages can be obtained immediately without further elaboration.

Option Pricing: Caps and Floors

Option Pricing: Caps and Floors

A Caplet is given by the following

$$\text{Caplet}:\ \tau_i\left(L(T_{i-1},T_i)-K\right)^+ \quad , \quad \text{Floorlet}:\ \tau_i\left(K-L(T_{i-1},T_i)\right)^+ \tag{232}$$

The payment occurs at T_i. The Libor rate resets at T_{i-1}. If we discount both payment to T_{i-1}, we get

$$\tau_i B(T_{i-1},T_i)\left(L(T_{i-1},T_i)-K\right)^+ ,\ \tau_i B(T_{i-1},T_i)\left(K-L(T_{i-1},T_i)\right)^+ \tag{233}$$

Use

$$L(T_{i-1},T_i)=\frac{1}{\tau_i}\left(\frac{1}{B(T_{i-1},T_i)}-1\right) \quad , \tag{234}$$

to get for the payoff at T_{i-1}

$$\left[1-(1+\tau_i K)B(T_{i-1},T_i)\right]^+=(1+\tau_i K)\left[\frac{1}{1+\tau_i K}-B(T_{i-1},T_i)\right]^+ \tag{235}$$

Option Pricing: Caps and Floors (cont. 2)

The Caplet becomes a Put option on a Bond maturing at T_i expiring at T_{i-1} of $\mathrm{Notional} = (1 + \tau_i K)$, of strike $1/(1 + \tau_i K)$. Notice the strike in the Libor rate $L(T_{i-1}; T_{i-1}, T_i)$ is equivalent to a strike in the discounting from T_{i-1} to T_i, $B(T_{i-1}, T_i) = 1/(1 + \tau_i L(T_{i-1}; T_{i-1}, T_i))$ set at $\mathcal{K} = 1/(1 + \tau_i K)$.

Putting together results from Bond options, we have

$$\mathrm{Caplet} = (1 + \tau_i K) Put\left(t; T_{i-1}, T_i, \frac{1}{(1 + \tau_i K)}\right) , \tag{236}$$

$$\mathrm{Cap} = \sum_{i=1}^{n} (1 + \tau_i K) Put\left(t; T_{i-1}, T_i, \frac{1}{(1 + \tau_i K)}\right) , \tag{237}$$

$$\mathrm{Floor} = \sum_{i=1}^{n} (1 + \tau_i K) Call\left(t; T_{i-1}, T_i, \frac{1}{(1 + \tau_i K)}\right) . \tag{238}$$

Option Pricing: Caps and Floors (cont. 3)

We write the Caplet price explicitly as it will be useful to compare with the results obtained for the Caplet from the HJM model, by extracting directly the price of the volatility of the Libor

$$\mathrm{Caplet} = (1+\tau_i K)B(t,T_i)\left[\frac{1}{1+\tau_i K}N(-d_-) - \frac{B(t,T_{i+1})}{B(t,T_i)}N(-d_+)\right] \quad (239)$$

where

$$d_\pm = \frac{1}{\sqrt{\xi}}\left[\ln\frac{\frac{B(t,T_{i+1})}{B(t,T_i)}}{\frac{1}{1+\tau_i K}} \pm \frac{1}{2}\xi\right] \quad . \quad (240)$$

$$-d_- \quad = \quad \tilde{d}_+ = \frac{1}{\sqrt{\xi}}\left[\ln\frac{1}{1+\tau_i K}\frac{B(t,T_i)}{B(t,T_{i+1})} + \frac{1}{2}\xi\right] \quad , \quad (241)$$

$$-d_+ \quad = \quad \tilde{d}_- = \frac{1}{\sqrt{\xi}}\left[\ln\frac{1}{1+\tau_i K}\frac{B(t,T_i)}{B(t,T_{i+1})} - \frac{1}{2}\xi\right] \quad . \quad (242)$$

Option Pricing: Caps and Floors (cont. 4)

Here we give explicitly the Caplet PV

$$\mathrm{Caplet} = B(t, T_i) N(\tilde{d}_+) - B(t, T_{i+1})(1 + \tau_i K) N(\tilde{d}_-) \quad , \tag{243}$$

where the new effective $\tilde{d}_\pm$ are as follows

$$\tilde{d}_\pm = \frac{1}{\sqrt{\xi(t, T_i, T_{i+1})}} \left[\ln \frac{1}{1 + \tau_i K} \frac{B(t, T_i)}{B(t, T_{i+1})} \pm \frac{1}{2} \xi(t, T_i, T_{i+1}) \right] \tag{244}$$

Compare this with Eq. (336). Both ways give exactly same result.

Option Pricing: Volatility Bonds

Option Pricing: Vol-Bonds

A Vol-Bond is a straddle where at reset date T_i one compares the Libor rate which fixes at T_i with the Libor that fixes at T_{i-1}. At T_{i+1} the payoff of the Vol-Bond is

$$\theta \left| L(T_i; T_i, T_{i+1}) - L(T_{i-1}; T_{i-1}, T_i) \right| \quad . \tag{245}$$

Use the standard notation

$$L(T_a; T_a, T_b) = \frac{1}{\theta} \left(\frac{1}{B(T_a; T_a, T_b)} - 1 \right) \quad . \tag{246}$$

The VB can be written as follows

$$VB = \sum_{i=1}^{N} (\mathcal{C}_i + \mathcal{F}_i) \quad , \tag{247}$$

where the ith Caplet is

$$\mathcal{C}_i = \theta E \left[e^{-\int_0^{T_{i+1}} r(t) \mathrm{d}t} \Big(L(T_i; T_i, T_{i+1}) - L(T_{i-1}; T_{i-1}, T_i) \Big)^+ \right] \ . \tag{248}$$

For the Floorlet the difference is reverted $L_{i-1} - L_i$ in the above equation.

Option Pricing: Vol-Bonds (cont. 2)

$$
\begin{aligned}
(L_i - L_{i-1}) &= \frac{1}{\theta}\left(\frac{1}{B(T_i; T_i, T_{i+1})} - \frac{1}{B(T_{i-1}; T_{i-1}, T_i)}\right), && (249)\\
&= \frac{1}{\theta}\frac{1}{B(T_i; T_i, T_{i+1})}\left(1 - \frac{B(T_i; T_i, T_{i+1})}{B(T_{i-1}; T_{i-1}, T_i)}\right), && (250)
\end{aligned}
$$

The fwd bond in front cancels $\exp\{-\int_{T_i}^{T_{i+1}} r(t)\mathrm{d}t\}$ resulting in

$$
\begin{aligned}
\mathcal{C}_i &= E\left[e^{-\int_0^{T_i} r(t)\mathrm{d}t}\left(1 - \frac{B(T_i; T_i, T_{i+1})}{B(T_{i-1}; T_{i-1}, T_i)}\right)^+\right], && (251)\\
&= B(0, T_i)E^{T_i}\left[\left(1 - \frac{B(T_i; T_i, T_{i+1})}{B(T_{i-1}; T_{i-1}, T_i)}\right)^+\right]. && (252)
\end{aligned}
$$

$$
W^{(T_i)} = W - \int_0^{T_i} \Sigma_i(t)\mathrm{d}t \quad , \qquad (253)
$$

Option Pricing: Vol-Bonds (cont. 3)

$$\mathrm{d}B(t;T_i,T_{i+1}) = B(t;T_i,T_{i+1})\left[(\Sigma_{i+1}-\Sigma_i)\mathrm{d}W + (\Sigma_i^2 - \Sigma_i\Sigma_{i+1})\mathrm{d}t\right]$$

$$\mathrm{d}B(t;T_{i-1},T_i) = B(t;T_{i-1},T_i)\left[(\Sigma_i-\Sigma_{i-1})\mathrm{d}W + (\Sigma_{i-1}^2 - \Sigma_{i-1}\Sigma_i)\mathrm{d}t\right]$$

In the $W^{(T_i)}$ measure the above equations change to

$$\mathrm{d}B(t;T_i,T_{i+1}) = B(t;T_i,T_{i+1})\left[(\Sigma_{i+1}-\Sigma_i)\mathrm{d}W^{T_i}\right] \quad ,$$

$$\mathrm{d}B(t;T_{i-1},T_i) = B(t;T_{i-1},T_i)\left[(\Sigma_i-\Sigma_{i-1})\mathrm{d}W^{T_i} + (\Sigma_{i-1}^2 - \Sigma_{i-1}\Sigma_i)^2\mathrm{d}t\right]$$

We use here

$$\mathrm{d}\left(\frac{x}{y}\right) = \frac{\mathrm{d}x}{y} - \frac{x}{y^2}\left(\mathrm{d}y - \frac{1}{y}(\mathrm{d}y)^2\right) - \frac{\mathrm{d}x\mathrm{d}y}{y^2} \quad , \tag{254}$$

$$= \frac{B(t;T_i,T_{i+1})}{B(t;T_{i-1},T_i)}\left[(\Sigma_{i+1}-\Sigma_i)\mathrm{d}W^{T_i} - (\Sigma_i-\Sigma_{i-1})\mathrm{d}W^{T_i}\right.$$

$$\left. -(\Sigma_{i+1}-\Sigma_i)(\Sigma_i-\Sigma_{i-1})\mathrm{d}t\right] \quad . \tag{255}$$

Option Pricing: Vol-Bonds (cont. 4)

Think in terms

$$\mathrm{d}Z = Z[\Sigma_i(\mathrm{fwdBond}_i)\mathrm{d}W^{T_i} - \Sigma_{i-1}(\mathrm{fwdBond}_{i-1})\mathrm{d}W^{T_{i-1}}]$$

$$-\Sigma_i(\mathrm{fwdBond}_i)\Sigma_{i-1}(\mathrm{fwdBond}_{i-1})\mathrm{d}t \quad . \tag{256}$$

$$\mathrm{d}\ln Z = \eta_i \mathrm{d}W^{T_i} - \mu_i \mathrm{d}t - \frac{1}{2}\eta_i^2 \mathrm{d}t \quad , \tag{257}$$

$$Z = Z_0 e^{\int_0^{T_i} \mu \mathrm{d}t} e^{\int_0^{T_i} \nu_i(t)\mathrm{d}W^{T_i}(t) - \frac{1}{2}\int_0^{T_i} \eta_i^2(t)\mathrm{d}t} \quad , \tag{258}$$

$$Z = Z_0 \mathcal{K} e^{\int_0^{T_i} \nu_i(t)\mathrm{d}W^{T_i}(t) - \frac{1}{2}\int_0^{T_i} \eta_i^2(t)\mathrm{d}t} \quad , \tag{259}$$

$$\frac{B(T_i; T_i, T_{i+1})}{B(T_{i-1}; T_{i-1}, T_i)} = \frac{B(0; T_i, T_{i+1})}{B(0; T_{i-1}, T_i)} \mathcal{K} e^{\int_0^{T_i} \nu_i(t)\mathrm{d}W^{T_i}(t) - \frac{1}{2}\int_0^{T_i} \eta_i^2(t)\mathrm{d}t} \quad , \tag{260}$$

Option Pricing: Swaptions

Option Pricing: Swaptions (cont. 1)

At the first reset time T_0 the price of the Swaption can be written as follows:

$$Swap = \sum_{i=1}^{n} \tau_i B(T_0, T_i) \left(F(T_0; T_{i-1}, T_i) - K \right) \quad , \tag{261}$$

$$= 1 - B(T_0, T_n) - K \sum_{i=1}^{n} \tau_i B(T_0, T_i) \quad . \tag{262}$$

To price Swaptions we can use Jamshidian's trick

Option Pricing: Swaptions (cont. 2)

Define at T_0 a critical value r^* for $r(T_0)$ for which the swap at T_0 has zero value

$$B(T_0, T_n; r^*) + K \sum_{i=1}^{n} \tau_i B(T_0, T_i; r^*) = 1 \quad . \tag{263}$$

r^* can be found by a root search of the above equation and equations for the bond prices.

Keep in mind that the bond prices are decreasing functions of r. Overall there are n bonds taking part in the swap rate. The one that matures at T_1 has higher value and all the rest are below this in a decreasing order, T_2, ...T_n. The value of each bond at r^* will give an ATM strike for each of them

$$K_i = B(T_0, T_i; r^*) \quad , \quad i = 1, \ldots, \quad . \tag{264}$$

All strikes (multiplied with τ_i and swap strike K) sum up to 1

$$K_n + K \sum_{i=1}^{n} \tau_i K_i = 1 \quad . \tag{265}$$

Option Pricing: Swaptions (cont. 3)

The values of the bond decrease if $r(T_0) > r^*$, but increase if $r(T_0) < r^*$. The Swaption pay out at T_0 will be positive only in the region $r(T_0) > r^*$

$$
\begin{aligned}
V_{\mathrm{Swaption}}(T_0) &= \left(1 - B(T_0, T_n; r(T_0)) \right. && (266)\\
&\qquad \left. - K \sum_{i=1}^{n} \tau_i B(T_0, T_i; r(T_0)) \right) 1_{\{r(T_0) > r^*\}} \quad , \\
&= \left(K_n + K \sum_{i=1}^{n} \tau_i K_i - B(T_0, T_n; r(T_0)) \right. && (267)\\
&\qquad \left. - K \sum_{i=1}^{n} \tau_i B(T_0, T_i; r(T_0)) \right) 1_{\{r(T_0) > r^*\}} \quad .
\end{aligned}
$$

Option Pricing: Swaptions (cont. 4)

Grouping together the strikes with the respective bond, we obtain

$$
\begin{aligned}
V_{\text{Swaption}}(T_0) &= \Big(K_n - B(T_0, T_n; r(T_0)\Big) 1_{\{r(T_0) > r^*\}} \\
&\quad + K \sum_{i=1}^{n} \tau_i \Big(K_i - B\big(T_0, T_i; r(T_0)\big)\Big) 1_{\{r(T_0) > r^*\}} \quad , \qquad (268)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(K_n - B\big(T_0, T_n; r(T_0)\big)\Big)^+ \\
&\quad + K \sum_{i=1}^{n} \tau_i \Big(K_i - B\big(T_0, T_i; r(T_0)\big)\Big)^+ \quad . \qquad (269)
\end{aligned}
$$

The price of a European Swaption has been decomposed into n put options on zero-coupon bonds. These options can be valued with the floorlet prices we discussed in the previous sections.

Option Pricing: Swaptions (cont. 5)

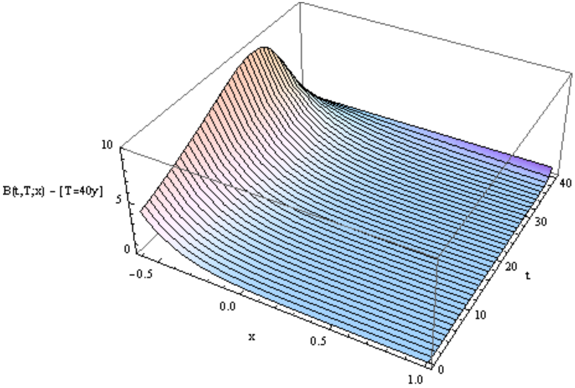


Figure: Graph of the dependence $B(t, T; x)$ as function of x for different t. There is a dumping term in front of the bond reconstruction formula $B(0; t, T) \sim \exp\{-r_0(T - t)\}$.

Option Pricing: Swaptions (cont. 6)

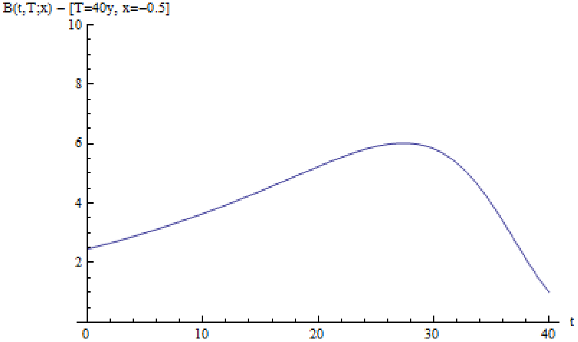


Figure: Graph of the dependence $B(t, T; x)$ as function of t for fixed x. There is a dumping term in front of the bond reconstruction formula $B(0; t, T) \sim \exp\{-r_0(T - t)\}$, otherwise the function would have been monotonically decreasing.

Two factor Hull-White

Two factor Hull-White

To handle exotic options that depend on the correlations between rates of different maturities we need a multi-factor model. One such model is the following

$$r(t) = \phi(t) + x(t) + y(t) \quad , \tag{270}$$

where $\phi(t)$ is a deterministic function and

$$\mathrm{d}x(t) = -\kappa x(t)\mathrm{d}t + \sigma \mathrm{d}W_1(t) \quad , \tag{271}$$

$$\mathrm{d}y(t) = -\beta y(t)\mathrm{d}t + \eta \mathrm{d}W_2(t) \quad . \tag{272}$$

Both process mean revert to zero. Also $x(0) = 0$ and $y(0) = 0$. The correlation between the Brownian motions is

$$\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \rho \mathrm{d}t \quad . \tag{273}$$

Two factor Hull-White (cont. 2)

Integrate equation for $\mathrm{d}r(t)$ from t to s

$$
\begin{aligned}
\int_t^s \mathrm{d}r(u) &= \int_t^s \mathrm{d}\phi(u) + \int_t^s \mathrm{d}x(u) + \int_t^s \mathrm{d}y(u) \quad , && (274) \\
r(s) &= \phi(s) + x(s) + y(s) \quad , && (275) \\
r(s) &= \phi(s) + x(t)e^{-\kappa(s-t)} + y(t)e^{-\beta(s-t)} && (276) \\
&+ \sigma \int_t^s e^{-\kappa(s-u)} \mathrm{d}W_1(u) + \eta \int_t^s e^{-\beta(s-u)} \mathrm{d}W_2(u) \quad .
\end{aligned}
$$

The bond price is given by

$$
B(t,T) = E^Q \left[e^{-\int_0^T r(s)\mathrm{d}s} \middle| \mathcal{F}_t \right] \quad . \tag{277}
$$

Two factor Hull-White (cont. 3)

We need to calculate the integral

$$E^{\mathcal{Q}}\left[\int_t^T r(s)\mathrm{d}s\right] \quad . \tag{278}$$

For the double integrals

$$\sigma \int_t^T \mathrm{d}s \left(\int_t^s e^{-\kappa(s-u)} \mathrm{d}W_1(u)\right) \quad , \tag{279}$$

we use the following:

$$\mathrm{d}\left(\int_t^s e^{-\kappa(s-u)} \mathrm{d}W_1(u)\right) = \mathrm{d}W_1(s) - \left(\kappa \int_t^s e^{-\kappa(s-u)} \mathrm{d}W_1(u)\right) \mathrm{d}s \, , \tag{280}$$

which results in

$$\mathrm{d}s \left(\int_t^s e^{-\kappa(s-u)} \mathrm{d}W_1(u)\right) = \mathrm{d}\left(\int_t^s \frac{1-e^{-\kappa(s-u)}}{\kappa} \mathrm{d}W_1(u)\right) \quad . \tag{281}$$

Two factor Hull-White (cont. 4)

$$\int_t^T \mathrm{d}s \left(\int_t^s e^{-\kappa(s-u)}\mathrm{d}W_1(u)\right) = \int_t^T \mathrm{d}\left(\int_t^s \frac{1-e^{-\kappa(s-u)}}{\kappa}\mathrm{d}W_1(u)\right) \quad ,$$

$$= \int_t^T \frac{1-e^{-\kappa(T-u)}}{\kappa}\mathrm{d}W_1(u) \quad , \qquad (282)$$

$$= \int_t^T G(u,T;\kappa)\mathrm{d}W_1(u) \quad . \qquad (283)$$

Putting it all together

$$\int_t^T r(s)\mathrm{d}s = \int_t^T \phi(s)\mathrm{d}s - x(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} - y(t)\frac{1-e^{-\beta(T-t)}}{\beta} \qquad (284)$$

$$+ \quad \sigma\int_t^T \frac{1-e^{-\kappa(T-u)}}{\kappa}\mathrm{d}W_1(u) + \eta\int_t^T \frac{1-e^{-\beta(T-u)}}{\beta}\mathrm{d}W_2(u)$$

Two factor Hull-White (cont. 5)

Writing in terms of G-functions:

$$\begin{aligned}\int_t^T r(s)\mathrm{d}s &= \int_t^T \phi(s)\mathrm{d}s - x(t)G(t,T;\kappa) - y(t)G(t,T;\beta) \\ &+ \sigma\int_t^T G(u,T;\kappa)\mathrm{d}W_1(u) + \eta\int_t^T G(u,T;\beta)\mathrm{d}W_2(u)\end{aligned} \quad (285)$$

To calculate the expected value

$$E^{\mathcal{Q}}\left[\int_t^T r(s)\mathrm{d}s\right] \quad . \quad (286)$$

we know the mean value of the integral. We need additionally also the variance

$$V(t,T) = \mathrm{var}(I_1^2) + \mathrm{var}(I_1^2) + 2\mathrm{var}(I_1 I_2) \quad , \quad (287)$$

Two factor Hull-White (cont. 6)

where

$$\mathrm{var}(I_1^2) = \sigma^2 \int_t^T G^2(u, T; \kappa)\mathrm{d}u = \frac{\sigma^2}{\kappa^2} \int_t^T \left(1 - e^{-\kappa(T-u)}\right)^2 \mathrm{d}u \quad , (288)$$

$$\mathrm{var}(I_2^2) = \eta^2 \int_t^T G^2(u, T; \beta)\mathrm{d}u = \frac{\eta^2}{\beta^2} \int_t^T \left(1 - e^{-\beta(T-u)}\right)^2 \mathrm{d}u \quad , (289)$$

and

$$\mathrm{var}(I_1 I_2) = \sigma\eta\rho \int_t^T G(u, T; \kappa) G(u, T; \beta)\mathrm{d}u \quad , \qquad (290)$$

$$= \frac{\sigma\eta\rho}{\kappa\beta} \int_t^T \left(1 - e^{-\kappa(T-u)}\right)\left(1 - e^{-\beta(T-u)}\right) \mathrm{d}u \quad , (291)$$

Two factor Hull-White (cont. 7)

Putting it all together, and using the equality $E[e^{-X}] = e^{-\bar{X}+\frac{1}{2}\mathrm{var}(X)}$, for the bond value we get:

$$\begin{aligned} B(t,T) &= \exp\left(-x(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} - y(t)\frac{1-e^{-\beta(T-t)}}{\beta}\right. \\ &- \left.\int_t^T \phi(s)\mathrm{d}s + \frac{1}{2}V(t,T)\right) \quad . \end{aligned} \tag{292}$$

We select $\phi(t)$ to fit the initial bond term-structure. The model fits the term-structure at time zero

$$B(0,T) = \exp\left(-\int_0^T \phi(s)\mathrm{d}s + \frac{1}{2}V(0,T)\right) \quad , \tag{293}$$

and

$$B(0,t) = \exp\left(-\int_0^t \phi(s)\mathrm{d}s + \frac{1}{2}V(0,t)\right) \quad . \tag{294}$$

Two factor Hull-White (cont. 8)

Taking the difference of the logarithms, we can find the integral over $\int_t^T \phi(s)\mathrm{d}s$

$$\int_t^T \phi(s)\mathrm{d}s = \ln \frac{B(0,t)}{B(0,T)} + \frac{1}{2}\left(V(0,t) + V(0,T)\right) \quad . \tag{295}$$

For the bond prices in the two-factor Hull-White model we find

$$\begin{aligned} B(t,T) \quad = \quad & \frac{B(0,T)}{B(0,t)} \exp\left(-x(t)\frac{1-e^{-\kappa(T-t)}}{\kappa} - y(t)\frac{1-e^{-\beta(T-t)}}{\beta}\right. \\ & \left. + \frac{1}{2}\left(V(0,t) - V(0,T) + V(t,T)\right)\right) \quad . \end{aligned} \tag{296}$$

Notice the term

$$V(t,T) - \Big(V(0,T) - V(0,t)\Big) \quad , \tag{297}$$

Two factor Hull-White (cont. 9)

is the difference of the variance from t to T, $V(t, T)$, with the terminal variance obtained by the simple difference between $V(0, T) - V(0, t)$. In the two-factor model the forward volatility is distributed such that this difference can be different from zero.

Bond options can be obtained with a similar formula

$$V(t) = B(0, T)N(d_+) - KB(0, S)N(d_-) \quad , \tag{298}$$

where

$$d_+ = \frac{\ln \frac{B(0,S)K}{B(0,T)} + \frac{1}{2}\xi(0, S)}{\sqrt{\xi(0, S)}} \quad , \tag{299}$$

$$d_- = \frac{\ln \frac{B(0,S)K}{B(0,T)} - \frac{1}{2}\xi(0, S)}{\sqrt{\xi(0, S)}} \quad , \tag{300}$$

where the variance can be obtained by taking the logarithm $\ln B(S, T)$. This should be same as $V(S, T)$ (Check this!)

Two factor Hull-White - Volatility of Forward Bond

Solving Eqs. (271) and (272) we have the usual

$$x(s) = x(t)e^{-\kappa(s-t)} + \int_t^s \sigma(u)e^{-\kappa(s-u)}\mathrm{d}W_1(u) \quad , \tag{301}$$

$$y(s) = y(t)e^{-\kappa(s-t)} + \int_t^s \eta(u)e^{-\kappa(s-u)}\mathrm{d}W_2(u) \quad . \tag{302}$$

For the bond $B(S,T)$ in the two-factor Hull-White model we find

$$\begin{aligned} B(S,T) &= \frac{B(0,T)}{B(0,S)} \exp\left(-x(S)\frac{1-e^{-\kappa(T-S)}}{\kappa} - y(S)\frac{1-e^{-\beta(T-S)}}{\beta}\right. \\ &\quad \left. + \frac{1}{2}\left(V(0,S) - V(0,T) + V(S,T)\right)\right) \quad . \end{aligned} \tag{303}$$

Two factor Hull-White - Volatility of Forward Bond (cont. 2)

The variance of $\ln\left[B(S,T)\right]$ is given by

$$\xi(t,S)=\mathrm{var}\left(x(S)\frac{1-e^{-\kappa(T-S)}}{\kappa}+y(S)\frac{1-e^{-\beta(T-S)}}{\beta}\right)\quad, \tag{304}$$

where the stochastic part of $x(S)$ and $y(S)$ is as follows:

$$\int_t^s \sigma(u)e^{-\kappa(s-u)}\mathrm{d}W_1(u)\quad,\quad \int_t^s \eta(u)e^{-\kappa(s-u)}\mathrm{d}W_2(u)\quad. \tag{305}$$

The non-stochastic part is unnecessary in the variance calculation. Therefore

$$\begin{aligned}\xi(t,S) &= G^2(T,S;\kappa)\mathrm{var}\Big(x(S)\Big)+G^2(T,S;\beta)\mathrm{var}\Big(y(S)\Big)\\ &+ 2G(T,S;\kappa)G(T,S;\beta)\mathrm{var}\Big(x(S)y(s)\Big)\quad.\end{aligned} \tag{306}$$

Two factor Hull-White - Volatility of Forward Bond (cont. 2)

$$
\begin{aligned}
\xi(t,S) = G^2(S,T;\kappa)\int_t^S \sigma^2(u)e^{-2\kappa(S-u)}\mathrm{d}u + G^2(S,T;\beta)\int_t^S \eta^2(u)e^{-2\beta(S-u)}\mathrm{d}u \\
+G(S,T;\kappa)G(S,T;\beta)\int_t^S \sigma(u)\eta(u)\rho(u)e^{-\kappa(S-u)}e^{-\beta(S-u)}\mathrm{d}u \quad , \\
= \quad \frac{\sigma^2}{2\kappa^3}\left(1-e^{-\kappa(T-S)}\right)^2\left(1-e^{-2\kappa(S-t)}\right) \quad , \\
+\frac{\eta^2}{2\kappa^3}\left(1-e^{-\beta(T-S)}\right)^2\left(1-e^{-2\beta(S-t)}\right) \qquad (307) \\
+\frac{2\sigma\eta\rho}{\kappa\beta(\kappa+\beta)}\left(1-e^{-\kappa(T-S)}\right)\left(1-e^{-\beta(T-S)}\right)\left(1-e^{-(\kappa+\beta)(S-t)}\right) \quad .
\end{aligned}
$$

Forward Rate HJM-Models

Forward Rate Models

- HJM models stated in terms of the instantaneous forward rates $f(t,T)$

$$\mathrm{d}f(t,T) = \alpha(t,T)\mathrm{d}t + \sigma_f(t,T)\mathrm{d}W(t) \quad . \tag{308}$$

- the instantaneous forward rates are martingale in the T-forward measure in similarity to the Libor rates in Libor Market Model (LMM)

$$f(t,T) = E^T\Big[f(T,T)\Big] = E^T\Big[r(T)\Big] \quad . \tag{309}$$

- The volatility of the forward rates is defined by the volatility of the bonds

$$\sigma_f(t,T) = \frac{\partial \sigma_B(t,T)}{\partial T}\,, \quad \sigma_B(t,T) = \int_t^T \sigma_f(t,u)\mathrm{d}u \quad . \tag{310}$$

- The drift of the forward rates is function of the volatility $\alpha(t,T) = \sigma_f(t,T)\sigma_B(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,u)\mathrm{d}u$.

Forward Rate Models

The Libor rate that resets at T and pays at $T + \delta T$ can be expressed as a ratio of bonds. The one in denominator matures at $T + \delta T$. For this reason the Libor rate is a martingale under the $T + \delta T$ measure

$$\begin{aligned} L(t; T, T + \delta T) &= E^{T+\delta T}\left[L(T; T, T + \delta T) \middle| \mathcal{F}_t \right] \quad , \qquad (311) \\ &= E^{T+\delta T}\left[\frac{B(T, T) - B(T, T + \delta T)}{B(T, T + \delta T)} \middle| \mathcal{F}_t \right] \qquad (312) \end{aligned}$$

If we take the limit $\delta T \to 0$ then the equation becomes an equation for the instantaneous forward rate $f(t, T)$

$$f(t; T) = E^{T}\left[f(T, T) \middle| \mathcal{F}_t \right] = E^{T}\left[r(T) \middle| \mathcal{F}_t \right] \quad . \qquad (313)$$

Forward Rate Models

Let us take the usual tenor structure $T_0, T_1, \ldots, T_N$, and at time t consider the discount rates. If we think in terms of Libor rates

$$B(t, T_N) = \prod_{j=0}^{N-1} \frac{1}{\Big(1 + L(t; T_j, T_{j+1})(T_{j+1} - T_j)\Big)} \quad . \tag{314}$$

In the limit of $T_{j+1} - T_j \to 0$, we can write approximately

$$\Big(1 + L(t; T_j, T_{j+1})(T_{j+1} - T_j)\Big) \sim \exp\left(L(t; T_j, T_{j+1})(T_{j+1} - T_j)\right) \tag{315}$$

Since the short-tenor Libor rates are equivalent to forward rates, then (assumed here that $t = T_0$)

$$B(t, T_N) = \prod_{j=0}^{N-1} \exp\left(-\int_{T_j}^{T_{j+1}} f(t, u)\mathrm{d}u\right) \quad , \tag{316}$$

$$= exp\left(-\int_{t}^{T_N} f(t, u)\mathrm{d}u\right) \quad . \tag{317}$$

Forward Rate Models (cont. 2)

From

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\mathrm{d}u\right) \quad , \tag{318}$$

and the expression

$$1+\tau_i L(t;T_i,T_{i+1}) = \frac{B(t,T_i)}{B(t,T_{i+1})} = \frac{\exp\left(-\int_t^{T_i} f(t,u)\mathrm{d}u\right)}{\exp\left(-\int_t^{T_{i+1}} f(t,u)\mathrm{d}u\right)} \quad ,$$

$$= \exp\left(\int_{T_i}^{T_{i+1}} f(t,u)\mathrm{d}u\right) \tag{319}$$

For the usual values of Libor rates of the order of 3% and 3m or 6m Libors where $\tau_{3m} = 0.25$ and $\tau_{6m} = 0.5$, taking the logarithm of both sides Eq. (319)

$$\ln\Big(1+\tau_i L(t;T_i,T_{i+1})\Big) \approx \tau_i L(t;T_i,T_{i+1}) = \int_{T_i}^{T_{i+1}} f(t,u)\mathrm{d}u \quad . \tag{320}$$

Forward Rate Models (cont. 3)

From Eq. (320)

$$L(t; T_i, T_{i+1}) = \frac{1}{T_{i+1} - T_i} \int_{T_i}^{T_{i+1}} f(t,u)\mathrm{d}u \quad , \tag{321}$$

therefore the Libor rate is an average of the instantaneous rate that mature in the interval $[T_i, T_{i+1}]$.
In an exact way

$$\mathrm{d}(1 + \tau_i L_i(t)) = \mathrm{d}\exp\left(\int_{T_i}^{T_{i+1}} f(t,u)\mathrm{d}u\right) = \mathrm{d}\exp(X) \quad , \tag{322}$$

Using the usual

$$\mathrm{d}\exp(X) \quad = \quad \exp(X)(\mathrm{d}X + \frac{1}{2}\mathrm{d}X\mathrm{d}X) \quad , \tag{323}$$

$$= \exp(X)\left[\int_{T_i}^{T_{i+1}} (\mathrm{d}f(t,u))\mathrm{d}u + \frac{1}{2}\int_{T_i}^{T_{i+1}} (\mathrm{d}f(t,u))\mathrm{d}u \int_{T_i}^{T_{i+1}} (\mathrm{d}f(t,u))\mathrm{d}u\right] \quad .$$

Forward Rate Models (cont. 5)

$$\mathrm{d}\exp(X) = \exp(X)(\mathrm{d}X + \frac{1}{2}\mathrm{d}X\mathrm{d}X) \quad , \tag{324}$$

$$= \exp(X)\left[\int_{T_i}^{T_{i+1}}(\alpha(t,u)\mathrm{d}u)\mathrm{d}t + \frac{1}{2}\left(\int_{T_i}^{T_{i+1}}(\sigma(t,u))\mathrm{d}u\int_{T_i}^{T_{i+1}}(\sigma(t,u))\mathrm{d}u\right)\mathrm{d}t\right]$$

$$+\exp(X)\left(\int_{T_i}^{T_{i+1}}\sigma(t,u)\mathrm{d}u\right)\mathrm{d}W(t) \quad . \tag{325}$$

In the T-forward measure the drift is zero

$$\mathrm{d}\exp\left(\int_{T_i}^{T_{i+1}}f(t,u)\mathrm{d}u\right) = \exp\left(\int_{T_i}^{T_{i+1}}f(t,u)\mathrm{d}u\right)\left(\int_{T_i}^{T_{i+1}}\sigma(t,u)\mathrm{d}u\right)\mathrm{d}W(t) \tag{32}$$

Forward Rate Models (cont. 6)

Summarizing the above

$$\tau_i \mathrm{d}L_i(t) = (1 + \tau_i L_i(t)) \left(\int_{T_i}^{T_{i+1}} \sigma(t,u)\mathrm{d}u \right) \mathrm{d}W(t) \quad . \tag{327}$$

Let us think of finding the volatility of the Libor rate in terms of the volatility of the short rate or the vol of the instantaneous forward rates, with $\sigma(t, T) = \sigma e^{-(T-t)}$

$$\mathrm{d}L_i(t) = \frac{1 + \tau_i L_i(t)}{\tau_i} \left(\int_{T_i}^{T_{i+1}} \sigma(t,u)\mathrm{d}u \right) \mathrm{d}W(t) \quad , \tag{328}$$

$$= \frac{1 + \tau_i L_i(t)}{\tau_i} \sigma \Big(G(t, T_{i+1}) - G(t, T_i) \Big) \mathrm{d}W(t) \quad . \tag{329}$$

Forward Rate Models (cont. 7) - Libor Caplets

The equation for the Libor rate is a displaced lognormal model

$$\frac{\mathrm{d}(1+\tau_i L_i(t))}{1+\tau_i L_i(t)} = \Sigma(t, T_i, T_{i+1})\mathrm{d}W(t) \quad , \tag{330}$$

equivalently

$$\frac{\mathrm{d}\tilde{L}_i(t)}{\tilde{L}_i(t)} = \Sigma(t, T_i, T_{i+1})\mathrm{d}W(t) \quad , \tag{331}$$

in which we can now apply the Black-Scholes equation to find the price of a Caplet. Notice that a strike of K on $L_i(t)$ corresponds to a strike for $\tilde{L}$, $\tilde{K} \sim 1 + \tau_i K$. For $\tilde{L}(t)$ the Black-Scholes equation applies

$$\mathrm{Caplet} = B(t, T_{i+1})\tilde{L}_i N(d_+) - B(t, T_i)\tilde{K} N(d_-) \quad , \tag{332}$$

$$d_\pm = \frac{1}{\sqrt{\xi}}\left[\ln\frac{\tilde{L}}{\tilde{K}} \pm \frac{1}{2}\xi\right] = \frac{1}{\sqrt{\xi}}\left[\ln\frac{1+\tau_i L_i(t)}{1+\tau_i K} \pm \frac{1}{2}\xi\right] \quad . \tag{333}$$

Forward Rate Models (cont. 8) - Libor Caplets

Using

$$1 + \tau_i L_i(t) = \frac{B(t, T_i)}{B(t, T_{i+1})} \quad , \tag{334}$$

we substitute $B(t, T_{i+1})\tilde{L}_i(t) = B(t, T_i)$ in Eq. (332), and the above equation inside $d_\pm$ to get

$$\text{Caplet} = B(t, T_{i+1}) E^{T_{i+1}} \left[\Big(L(T_i, T_{i+1}) - K \Big)^+ \right] \quad , \tag{335}$$

$$= B(t, T_i) N(d_+) - B(t, t_{i+1})(1 + \tau_i K) N(d_-) \quad , \tag{336}$$

where

$$d_\pm = \frac{1}{\sqrt{\xi}} \left[\ln \left(\frac{1}{(1 + \tau_i K)} \frac{B(t, T_i)}{B(t, T_{i+1})} \right) \pm \frac{1}{2} \xi \right] \quad , \tag{337}$$

$$\xi(t, T_i, T_{i+1}) = \sigma^2 \int_t^{T_i} \Big| G(u, T_{i+1}) - G(u, T_i) \Big|^2 \mathrm{d}u \quad . \tag{338}$$

Forward Rate Models (cont. 9) - Libor Caplets

Compare this with Eq. (236) that we calculated through the short-rate models.

Forward Rate Models - Bond Volatility

The volatility of the bond $B(t, T)$ can be found from a similar equation to (326)

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\mathrm{d}u\right) \quad , \tag{339}$$

and in the T-forward measure (no drift)

$$\mathrm{d}B(t,T) = B(t,T)\left(-\int_t^T \sigma_f(t,u)\mathrm{d}u\right)\mathrm{d}W^T(t) \quad . \tag{340}$$

In the Q-risk neutral measure the drift will be $r(t)B(t,T)\mathrm{d}t$.
From

$$\mathrm{d}\ln B(t,T) = O(\mathrm{d}t) - \sigma_B(t,T)\mathrm{d}W^T(t)\,, \quad f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T} \tag{341}$$

we find

$$\mathrm{d}f(t,T) = \frac{\partial \sigma_B(t,T)}{\partial T}\mathrm{d}W^T(t) \quad , \tag{342}$$

Forward Rate Models - Bond Volatility

Using

$$\mathrm{d}W^T(t) = \mathrm{d}W(t) + \sigma_B(t,T)\mathrm{d}t \quad , \tag{343}$$

we find

$$\mathrm{d}f(t,T) = \sigma_f(t,T)\sigma_B(t,T)\mathrm{d}t + \sigma_f(t,T)\mathrm{d}W(t) \quad , \tag{344}$$

$$= \sigma_f(t,T)\left(\int_t^T \sigma_f(t,u)\mathrm{d}u\right)\mathrm{d}t + \sigma_f(t,T)\mathrm{d}W(t) \tag{345}$$

These features we mentioned at the beginning of the section, as one of the main features of HJM model. Notice the $(-)$ sign in the Bond equation Eq. (341). Due to this, $\sigma_B(t,T) = -\Sigma(t,T)$ and there is a sign change when compared with the measure change Eq. (220).

Short rate model from HJM

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)\mathrm{d}s + \int_0^t \sigma_f(s, T)\mathrm{d}W(s) \quad , \tag{346}$$

which results in

$$r(t) = f(t, t) = f(0, t) + \int_0^t \alpha(s, t)\mathrm{d}s + \int_0^t \sigma_f(s, t)\mathrm{d}W(s) \quad . \tag{347}$$

$$\mathrm{d}r(t) = \frac{\partial f(0, t)}{\partial t}\mathrm{d}t + \alpha(t, t)\mathrm{d}t + \left(\int_0^t \frac{\partial \alpha(s, t)}{\partial t}\mathrm{d}s\right)\mathrm{d}t \tag{348}$$

$$+ \sigma_f(t, t)\mathrm{d}W(t) + \left(\int_0^t \frac{\partial \sigma_f(s, t)}{\partial t}\mathrm{d}W(s)\right) \quad . \tag{349}$$

Take the case of $\sigma_f(t, T) = \sigma$, then the above equation will give for the short rate

Short rate model from HJM (cont. 2)

$$\alpha(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,u)\mathrm{d}u = \sigma^2\int_t^T \mathrm{d}u = \sigma^2(T-t) \quad . \tag{350}$$

$$\alpha(s,t) = \sigma^2(t-s) \quad , \quad \frac{\partial\alpha(s,t)}{\partial t} = \sigma^2 \quad , \quad \frac{\sigma_f(s,t)}{\partial t} = 0 \quad . \tag{351}$$

Putting them all together we get,

$$\mathrm{d}r(t) = \frac{\partial f(0,t)}{\partial t}\mathrm{d}t + \sigma^2 t\mathrm{d}t + \sigma\mathrm{d}W(t) \quad , \tag{352}$$

$$= \theta(t)\mathrm{d}t + \sigma\mathrm{d}W(t) \quad . \tag{353}$$

This is the the Ho-Lee model, with

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + \sigma^2 t \quad . \tag{354}$$

Short rate model from HJM (cont. 3)

If we chose $\sigma_f(t, T) = \sigma e^{-\kappa(T-t)}$, going through the calculations of the previous page we find

$$\alpha(t, T) = \sigma^2 \frac{1}{\kappa} \left(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)} \right) \quad , \tag{355}$$

and

$$\mathrm{d}r(t) = (\theta(t) - \kappa r(t))\mathrm{d}t + \sigma \mathrm{d}W(t) \quad , \tag{356}$$

where

$$\theta(t) = \frac{f(0, t)}{\partial t} + \kappa f(0, t) + \frac{\sigma^2}{2\kappa} \left(1 - e^{-2\kappa t} \right) \quad . \tag{357}$$

This is the Hull-White model. Its equivalent in the HJM framework is the model with $\sigma_f(t, T) = \sigma e^{-\kappa(T-t)}$.