Analytic calculation of option price kin Binomial Model

March 19, 2021

Contents

1 Class 1

1.1 Digital Call and Put

Extendability through Function Pointers:
1. Add two new payoffs, Digital Calls and Digital Puts and find the price of these options?
Digital Call $= 1$ if $z > K$, and zero otherwise,
Digital Put $= 1$ if $z < K$, and zero otherwise.
2. Write an Interchange() function of two variables, that interchanges the contents of two double variables, that are to be passed to the function by reference.
3. Write a code for ordering an array of double variables in increasing order.

1.2 Analytic calculation of option price in Binomial model

As we were discussing last time, one can find analytically the number of paths in each node of the binomial tree. Consider the node $(n, i) = (n = 4, i = 3)$, 3 up jumps and one down. The 3 up jumps can happen in any of the following ways, (u, u, u, d) - 3 consecutive up jumps and the last one down, or (u, u, d, u), (u, d, u, u),(d, u, u, u).

The number of paths to reach node (n, i) is

$$C_n^i = \frac{n!}{i!(n-i)!} \tag{1}$$

The probability for an up jump, in risk-neutral measure, is q, for a down jump is $(1-q)$. Therefore for i-up jumps and $(n-i)$-down jumps, the probability is $q^i(1-q)^{(n-i)}$.

So the option price will be

$$H(0) = \frac{1}{(1+R)^N} \sum_{i=0}^{N} \frac{N!}{i!(N-i)!} q^i (1-q)^{(N-i)} h(S(N,i)) \tag{2}$$

This is similar to

$$H(0) = \int_i \varphi(i) h(i) \mathrm{d}i \sim \int_x \varphi(x) h(x) \mathrm{d}x \quad , \tag{3}$$

where i is the position of the stock price along the vertical nodes line, at expiry.

Question 1: Can you compare the analytic and numerical prices for N=5, then N=10, N=30? The answers need to agree to all digits.

Question 2: Can you plot the probability density function for the stock to be at position i, $\varphi(i)$, at expiry?

$$\varphi(i) = \frac{N!}{i!(N-i)!} q^i (1-q)^{(N-i)} \tag{4}$$

1.3 Transition to Black-Scholes

To get to the Black-Scholes equation, divide the time interval $[0, T]$ into N steps of length $h = T/N$, and set the binomial parameters, as follows

$$U = e^{(r+\sigma^2/2)h + \sigma\sqrt{h}} - 1 \quad , \tag{5}$$

$$D = e^{(r+\sigma^2/2)h - \sigma\sqrt{h}} - 1 \quad , \tag{6}$$

and

$$R = e^{rh} - 1 \quad . \tag{7}$$

Here σ is the BS volatility, and r is the risk-free rate continuously compounded. Try to calculate the BS option price in the binomial model and compare it against the analytic BC price. It is easy to get for instance $T = 10y$, $N = 10$, and then increase N to a large number.

2 Class 2

2.1 Definite Integrals

Compute a definite integral using the trapezoidal approximation

$$\int_a^b f(x) \mathrm{d}x \approx \frac{h}{2} \left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{N-1}) + f(x_N) \right) \tag{8}$$

where

$$h = \frac{b-a}{N} \quad , \quad x_n = a + nh \quad n = 0, 1, \ldots, N \quad . \tag{9}$$

Write a class called *DefiniteInt* to compute the trapezoidal approximation of $\int_a^b f(x)\mathrm{d}x$ for a given function f.

The class should contain the following:

(1) Private members to hold the values of the integration limits a, b and a pointer to function f.

(2) A constructor function such that the integration limits a, b, and the pointer to the function f can be initiated at the time of creating an object of the class like

DefiniteInt MyInt(a,b,f)

(3) A public function ByTrapeziod(), taking N as an argument and returning the trapezoidal approximation to the integral when called by

MyInt.Trapezoidal(N)

(4) You can include also another public function *BySimpson(N)* to compute the integral by using the Simpson approximation.

2.2 Virtual Functions

As an exercise, inherit a DoubleDigital Option from the EurOption class, and extend the code to price, using the virtual function provided in Option06.h

$$h^{\text{doubDigit}}(z) = \begin{cases} 1, & if\ K_1 < z < K_2 \\ 0, & otherwise \quad . \end{cases} \tag{10}$$

Similarly, price **Bull Spreads**

$$h^{\text{bull}}(z) = \begin{cases} 0, & if\ z \le K_1 \\ z - K_1, & if\ K_1 < z < K_2 \\ K_2 - K_1, & if\ K_2 \le z \quad , \end{cases} \tag{11}$$

Bear Spreads

$$h^{\text{bear}}(z) = \begin{cases} K_2 - K_1, & if\ z \le K_1 \\ K_2 - z, & if\ K_1 < z < K_2 \\ 0, & if\ K_2 \le z \quad , \end{cases} \tag{12}$$

Strangle

$$h^{\text{strangle}}(z) = \begin{cases} K_1 - z, & if\ z \le K_1 \\ 0, & if\ K_1 < z \le K_2 \\ z - K_2, & if\ K_2 \le z \quad , \end{cases} \tag{13}$$

Butterfly

$$h^{\text{butterfly}}(z) = \begin{cases} z - K_1, & if\ K_1 < z \le \frac{K_1+K_2}{2}\,, \\ K_2 - z, & if\ \frac{K_1+K_2}{2} < z \le K_2\,, \\ 0, & \text{otherwise} \quad . \end{cases} \tag{14}$$

3 Class Monte Carlo Path-Dependent Option Pricing: Variance reduction

3.1 Exercise: Continuous Arithmetic Asian

Price an Arithmetic Asian Call

$$H(T) = \left(\frac{1}{T} \int_0^T S(t) \mathrm{d}t - K \right)^+ \quad , \tag{15}$$

using

$$G(T) = \left(\exp \left\{ \frac{1}{T} \int_0^T \ln S(t) \mathrm{d}t \right\} - K \right)^+ \quad , \tag{16}$$

as control variate.

Hint: Notice that $G(T)$ is easy to compute analytically

$$G(T) = \left(c_1 \exp \left\{ \left(r - \frac{1}{2}\sigma^2 \right) T + c_2 \sqrt{T} N(0,1) \right\} - K \right)^+ \quad . \tag{17}$$

The way to think about it is the following:

$$S(t) = S(0) \exp \left\{ \left(r - \frac{1}{2}\sigma^2 \right) t + \sigma W(t) \right\} \tag{18}$$

and its logarithm integrated needs calculation of straightforward integrals, apart perhaps from the one we have discussed in the class

$$\sigma \int_0^T W(t) \mathrm{d}t = \sigma \int_0^T (T-s) \mathrm{d}W(s) \quad , \tag{19}$$

whose variance is

$$\sigma^2 \int_0^T (T-s)^2 \mathrm{d}s = \frac{\sigma^2}{3} T^3 \quad . \tag{20}$$

Therefore

$$\frac{1}{T} \int_0^T \ln S(t) \mathrm{d}t \quad = \quad \frac{1}{T} \left[T \ln S(0) + \left(r - \frac{1}{2}\sigma^2 \right) \frac{T^2}{2} + \sigma \sqrt{\frac{T^3}{3}} N(0,1) \right] \tag{21}$$

$$= \quad \ln S(0) + \left(r - \frac{1}{2}\sigma^2 \right) \frac{T}{2} + \sigma \sqrt{\frac{T}{3}} N(0,1) \tag{22}$$

Call $c_2 = \sigma/\sqrt{3}$ and re-write in the standard Black-Scholes form

$$\ln S(0) + \frac{1}{2} \left(-r + c_2^2 - \frac{1}{2}\sigma^2 \right) T + \left(r - \frac{1}{2} c_2^2 \right) T + c_2 N(0,1) \tag{23}$$

$$\ln S(0) - \frac{1}{2} \left(r + \frac{1}{6}\sigma^2 \right) T + \left(r - \frac{1}{2} c_2^2 \right) T + c_2 N(0,1) \tag{24}$$

which now can be written as a new stock price with dynamics

$$\left[S(0)\exp\left\{-\frac{1}{2}\left(r+\frac{1}{6}\sigma^2\right)T\right\}\right]\exp\left\{\left(r-\frac{1}{2}c_2^2\right)T+c_2Z\right\} \tag{25}$$

$$=c_1\exp\left\{\left(r-\frac{1}{2}c_2^2\right)T+c_2Z\right\} \quad . \tag{26}$$

4 Class Monte Carlo Path-Dependent Option Pricing: Basket Options

4.1 Exercise: Option on sum of stocks

Write a class for pricing of a European basket option with payoff

$$H(T)=\left(\sum_{j=1}^{d}S_j(T)-K\right)^{+} \quad . \tag{27}$$

Use as control variate to reduce the MC error the following basket of european calls

$$G(T)=\sum_{j=1}^{d}\left(S_j(T)-K_j\right)^{+} \quad , \quad K_j=K\frac{S_j(0)}{\sum_{j=1}^{d}S_j(0)} \quad . \tag{28}$$

The above is similar to pricing options on sums of Libors, a very well known type of trade on the fixed income derivatives

$$H(T)=\left(\sum_{j=1}^{d}\tau_j L_j(T_j)-K\right)^{+} \quad , \tag{29}$$

where τ_j are accrual periods for each Libor L_j.

Notice also that Swaps are given as follows:

$$Swap(T)=B(T,T_0)-B(T,T_n)-K\sum_{j=1}^{n}\tau_j B_J(T,T_j)=\sum_{j=0}^{n}c_j B_j(T,T_j) \quad , \tag{30}$$

where you can extract the value of c_j.

Swaptions are simply options on the $Swap(T)$, with payoff at time T equal to

$$Swaption(T)=(Swap(T))^{+}=\left(\sum_{j=0}^{n}c_j B_j(T,T_j)\right)^{+} \tag{31}$$