§ Vasicek model of Bond Price (as discount) / Stock Price (as growth)

OU process is the rate of price of security a.k.a return

- Dynamics: short rate being OU process

$$dr_t = -\lambda r_t\,dt + \sigma\,dW_t$$

or

$$= -\lambda r_t\,dt + \sigma\,\xi_t(\cdot)\,dt$$

where $\xi_t(\cdot) \sim \mathcal{N}(0,1^2)$ is random generator i.i.d on t

← ignore asymptotic mean $\theta \to 0$ here. From full process $dr_t = (\theta - k r_t)dt + \sigma dW_t$ or $dr_t = \lambda(r_\infty - r_t)dt + \sigma dW_t$

Ansatz becomes $f(r_t, t) = r_t e^{\lambda t}$ where $r_t(t=0) := r_0$ is $f(r_0, 0)$

where $\frac{d f(r_t,t)}{dt} = \frac{dr_t}{dt}\cdot e^{\lambda t} + \lambda r_t e^{\lambda t}$

$$= (-\lambda r_t + \sigma \xi_t(\cdot))e^{\lambda t} + \lambda r_t e^{\lambda t} = \sigma \xi_t(\cdot)\, e^{\lambda t} \quad ①$$

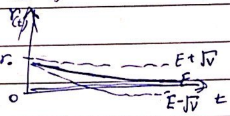
then $\int_0^t \frac{df(r_s,s)}{ds}\,ds := f(r_t,t) - f(r_0,0) = r_t e^{\lambda t} - r_0$

is equal to $\int^t \sigma \xi_s(\cdot) e^{\lambda s} ds$ due to ①

↳ $r_t = r_0 e^{-\lambda t} + \sigma e^{-\lambda t}\int_0^t \xi_s(\cdot) e^{\lambda s} ds$ (or $r_0 e^{-\lambda t} + r_\infty(1-e^{-\lambda t}) + \sigma e^{-\lambda t}\int_0^t \xi_s(\cdot) e^{\lambda s} ds$)

⇒ (Optional) key properties of the short rate becomes

$$\begin{cases} E[r_t | \mathcal{F}_t] = r_0 e^{-\lambda t} \\ V[r_t | \mathcal{F}_t] = (\sigma/\lambda)^2 \cdot (1 - e^{-2\lambda t}) \end{cases}$$



- Zero Coupon Bond price : $P_T = \exp\left[-\frac{r_0}{\lambda}\rho + \frac{1}{2}\left(\frac{\sigma}{\lambda}\right)^2 \cdot \left\{T - \frac{1}{2\lambda}(\rho^2 + 2\rho)\right\}\right]$

$$P_T := \cancel{P_0}^{\,1} \cdot E\left[e^{-\int_0^T r_t dt} \,\middle|\, \mathcal{F}_{t\in[0,T]}\right]$$

(for ZCB. For Stock Forward flip sign s.t. r_t grows price)

$\Leftrightarrow\ S_T = S_0 E[e^{(+)\int_0^T r_t dt} | \mathcal{F}_t]$

$$= E\left[e^{-\int_0^T r_0 e^{-\lambda t} dt} \cdot e^{-\int_0^T e^{-\lambda t} dt \int_0^t \sigma \xi_s(\cdot) e^{\lambda s} ds} \,\middle|\, \cdot\right]$$

$$= e^{-\frac{r_0}{\lambda}\rho} \cdot E\left[e^{-\sigma\int_0^T dt \int_0^t ds\, e^{-\lambda t} e^{\lambda s} \xi_s(\cdot)} \,\middle|\, \cdot\right] \quad \text{where } \rho = 1 - e^{-\lambda T}$$

(Applying Fubini's theorem)

$$= e^{-\frac{r_0}{\lambda}\rho} \cdot E\left[e^{-\frac{\sigma}{\lambda}\int_0^T ds\, \xi(s)(1 - e^{\lambda(s-T)})}\right] \quad \text{then by } \int_0^T f(t)\xi_t(\cdot)dt \sim \mathcal{N}(0; \textstyle\int f^2)$$

$$= e^{-\frac{r_0}{\lambda}\rho} \cdot E\left[e^{-x} \,\middle|\, x \sim \mathcal{N}\left(0; \mathrm{Var} = \left(\tfrac{\sigma}{\lambda}\right)^2 \left(T - \tfrac{1}{2\lambda}(\rho^2 + 2\rho)\right)\right)\right] \quad \text{then } E[e^{-x} | \mathcal{N}(0; \mathrm{Var})] = \tfrac{1}{2}\cdot \mathrm{Var}$$