Reference

1. Fabrice Douglas Rouah, Euler and Milstein Discretization
2. Quant Next, The Merton Jump Diffusion Model. 2024.
3. Rama Cont et al., Financial Modelling with Jump Processes. 2004.
4. Heston, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. 1993.
5. Forde et al, The small-time smile and term structure. 2012.
6. Patrick Labadie, Pricing American Options in Python. 2023.

Tasks

Problems

Using Monte-Carlo sampling methods, implement below stochastic processes and answer to below questions regarding the option price and greeks.

General Parameters:

- $S_0 = 80$
- $r = 5.5\%$
- $\sigma = 35\%$
- Time to maturity = 3 months

Stochastic Volatility Modeler For the Heston model, you can use the following parameters:

- $v_0 = 3.2\%$
- $\kappa_v = 1.85$
- $\theta_v = 0.045$
- Q5. Using the Heston Model and Monte-Carlo simulation, price an ATM European call and an ATM European put, using a correlation value of -0.30.
- Q6. Using the Heston Model, price an ATM European call and an ATM European put, using a correlation value of -0.70.
- Q7. Calculate delta and gamma for each of the options in Questions 5 and 6.
Hint: You can numerically approximate this by forcing a change in the variable of interest—i.e., underlying stock price and delta change—and recalculating the option price.
- Q13. Repeat Questions 5 and 7 for the case of an American call option (no need to price the put). Comment on the differences you observe from original Questions 5 and 7.
- Q14. Using Heston model data from Question 6, price a European up-and-in call option (CUI) with a barrier level of 95 and a strike price of 95 as well. This CUI option becomes alive only if the stock price reaches (at some point before maturity) the barrier level (even if it ends below it). Compare the price obtained to the one from the simple European call.

Jump Modeler For the Merton model, you can use the following parameters:

- $\mu = -0.5$
- $\delta = 0.22$

- Q8. Using the Merton Model, price an ATM European call and an ATM European put with jump intensity parameter equal to 0.75.
- Q9. Using the Merton Model, price an ATM European call and an ATM European put with jump intensity parameter equal to 0.25.
- Q10. Calculate delta and gamma for each of the options in Questions 8 and 9.
  Hint: You can use the same trick as in Question 7.
- Q15. Using Merton model data from Question 8, price a European down-and-in put option (PDI) with a barrier level of 65 and a strike price of 65 as well. This PDI option becomes alive only if the stock price reaches (at some point before maturity) the barrier level (even if it ends above it). Compare the price obtained to the one from the simple European put.

Assume no stock dividend throughout this report unless specified otherwise per question.

Heston Modeler

The Heston model is a stochastic volatility process where the variance develops with a mean-reverting process:

$$dS_t = \mu S dt + \sqrt{\nu_t} S dZ_t^{(1)}$$
$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dZ_t^{(2)}$$

The functional form of the Heston process is [1]

$$S_t = S_{t-1} e^{(r-\nu_t/2)dt + \sigma\sqrt{\nu_t} dZ_t^{(1)}}$$

, where $\nu_t = \nu_{t-1} + \kappa(\theta - \nu_{t-1})dt + \sigma\sqrt{\nu_{t-1}} dZ_t^{(2)}$ is the variance process and $< Z^{(1)}, Z^{(2)} >_t = \rho$ are correlated bivariate guassian distribution.

For Q5, Q6, Q17, Q13, and Q14, we base on following configuration by default and change variables as demanded per question.

`HestonParameters`

- 'S0': 80.0, 'r': 0.055, 'T': 0.25, 'v0': 0.032, 'kappa': 1.85, 'sigma': 0.35, 'theta': 0.045, 'rho': -0.3,
- `SamplingParameters`:{'M': 500, 'I': 10000, 'random_seed': 0}

Q5. At-the-money option price of `EUR CALL: 3.398` and `EUR PUT: 2.276` with $\rho = -0.30$

Q6. At-the-money option price of `EUR CALL: 3.356` and `EUR PUT: 2.278` with $\rho = -0.70$

Q7. Like Q10, we perturb 1% for demo.

side	rho	delta
call	-0.3	0.5
call	-0.7	0.49
put	-0.3	-0.5
put	-0.7	-0.51

Delta barely changes.

side	rho	gamma
call	-0.3	0.053
call	-0.7	0.053
put	-0.3	0.053
put	-0.7	0.053

Change of Gamma is not observed.

Without putting much detail (beyond the level of curriculum of this course), Heston stated [4] that the ρ only contributes to the skewness of the underlying asset price return but not variance. However, the Black-Scholes formula - the European option of our interest abides to - shows that the delta only depends on the variance not skewness nor kurtosis. Hence the delta does not change by the ρ and hence its derivative Gamma as well. The sligh change of numbers in Delta is small enough such that we can regard it as the numerical precision error at the sample random paths.

More concretely, this means the model sensitivity to price apart from the Black-Scholes model contribution is determined by $\rho \ln(K/S)$ in short-term [5], which is 0 when the moneyness is 1 (at-the-money). the correlation has no impact on the call price sensitivity to the underlying price. Hence the first order sensitivity (Delta) and the second order sensitivty (Gamma) are not related to the correlation in that context.

Also, as in for the Jump model - see Q10 in more details -, the Call-Put parity holds hence the gamma is symmetry on the option side.

Q13. At-the-money option price of `USA CALL: 3.398` and `USA PUT: 2.277` with $\rho = -0.30$. This is expected as the American call has optimal exercise at expiry and the put barely exploits the early exercise if the moneyness is 1 or less (ATM or OOM).

side	rho	delta
call	-0.3	0.17
call	-0.7	0.17
put	-0.3	-0.83
put	-0.7	-0.83

side	rho	gamma
call	-0.3	0.027
call	-0.7	0.028
put	-0.3	0.027
put	-0.7	0.028

About the Delta and Gamma, rho does not affect as we stated in Q7. What differs from the European option's property in Q7 is the magnitude of the Delta in which the PUT's Delta becomes much higher reflecting the fact that the self-exciting volatility increases the chance of early exercise opportunity raising the Delta - or the proxy of the exercisability - and the call's delta as the result decreases to compensate the sum of absolute to be 1. Reduced gamma indicates the early exercise property offers the smoother concavity of the option premium as even deeper OOMs can have more odds to get positive exercise with the American optionality.

- Disclaimer: Pricing the American Option in Monte-Carlo is not a direct aggregation of sample paths' values down to the valuation timestamp, as the optimal exercise depends on the expected value of the option at each time. This means the pricer suppose to estimate the distribution of future values based on the sampled paths. The Least-Square Monte-Carlo method seems to be the most viable way within our understanding of the course and we benchmarked the open-source implementation such as [6].

Q14. The Up-and-in's payoff is as Payoff = Payoff.European $\cdot$ $1_{max[S_{path}]>=barrier}$ for the call option, so it's a touch to trigger money.

type	side	S0	K	Barrier	option_price
european	call	80	95	95	0.401
upandin	call	80	95	95	0.401
european	put	80	95	95	14.1
upandin	put	80	95	95	0.289

type	side	S0	K	Barrier	option_price
european	call	65	95	95	0.0191
upandin	call	65	95	95	0.0191
european	put	65	95	95	28.7
upandin	put	65	95	95	0.00953

The result shows the up-and-in condition does not change the option value when the knock-in condition opens to the same direction of the positive pay-off of the option. This is obvious in that out-the-money call with barrier at strike means those you reach to the barrier can only has the non-zero payoff even for the european option case and vice-versa.

The deep ITM of PUT case is more interesting for the up-and-in case as the underlying price must hit deep out of strike price then back into in the money zone, which is a subset of non-zero paid-off paths of the European option case.

Jump Modeler

The functional form of the process is

$$S_t = S_{t-1}(e^{(r-r_j-\sigma^2/2)dt+\sigma\sqrt{dt}z_t^{(1)}} + (e^{\mu_j+\delta z_t^{(2)}} - 1)y_t)$$

, where $r_j := \lambda(e^{\mu_j+\delta^2/2} - 1)$, $y_t \sim Poisson(\lambda dt)$, and $z_t^{(1)}, z_t^{(2)} \sim \mathcal{N}(0, 1^2)$

For Q8, Q9, Q10, and Q15, we base on following configuration by default and change variables as demanded per question.

`JumpParameters`

- 'S0': 80.0, 'r': 0.055, 'T': 0.25, 'lambd': 0.75, 'mu': -0.5, 'delta': 0.22, 'sigma': 0.35,
- `SamplingParameters`:{'M': 500, 'I': 10000, 'random_seed': 0}

Q8. At-the-money option price of `EUR CALL: 3.69` and `EUR PUT: 11.1` with $\lambda = 0.75$

Q9. At-the-money option price of `EUR CALL: 5.18` and `EUR PUT: 6.97` with $\lambda = 0.25$

Q10. We perturb very big for demo - S0 by 5% for both up/down for delta calculation, and perturb another 5% for each for gamma calculation

side	lambda	delta
call	0.25	0.5
call	0.75	0.38
put	0.25	-0.46
put	0.75	-0.51

The delta goes down for both CALL and PUT when the λ increases. This is expected as more frequence jumps at $\mu_j < 0$ drops the chance of excersie for CALL and increases for PUT.

side	lambda	gamma
call	0.25	0.026
call	0.75	0.023
put	0.25	0.026
put	0.75	0.023

The gamma goes down for both CALL and PUT when the λ increases. This aligns with the property of the Jump model that the underlying price paths distribution has higher kurtosis than the usual GBM - the European option price based on the Merton Jump model dynamics can be understood as the that of the GBMs combined [3] - hence the squared sensitivity to the discounted payoff or the curvature of the option price becomes smoother by bigger lambda. Lastly, the symmetry of the gamma on CALL/PUT holds with the Jump process dynamics, assuring the arbitrage-free call-put parity property of the European option where

$$\Gamma_C - \Gamma_P := \frac{\partial^2(C-P)}{\partial S^2} = \frac{\partial^2(S - Ke^{-rT})}{\partial S^2} = 0$$

.

Q15. The payoff structure of down-and-in option is as Payoff = Payoff.European $\cdot\, 1_{min[S_{path}]<=barrier}$ for the call option, so it's a conditioned option with a low-bound to touch.

The knock-down-and-in is a tough condition for deep in-the-money option hence the option price of such Down-and-in is significantly lower than that of the European option's. In contrast, the exercis-ability of the at-the-money option with sufficient amount of time to expire is not compromised by the condition hence the option price between the two option types are almost equal.

type	side	S0	K	Barrier	option_price
european	call	80	65	65	11
downandin	call	80	65	65	0.58

type	side	S0	K	Barrier	option_price
european	call	65	65	65	2.999
downandin	call	65	65	65	2.999