HW1

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1. Use mathematical induction to prove the following statements

(a)
$\forall_n \geq 1$, $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$

lemma: $\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2}$ where $n \in N$
Base Case: $\sum_{i=1}^{1} i = 1 = \frac{1^2}{2} + \frac{1}{2}$
Inductive Hypothesis: $\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2}$
Inductive Steps:
Using induction on n
$\frac{(n+1)^2}{2} + \frac{n+1}{2} = \frac{n^2}{2} + \frac{3n}{2} + 1 = \frac{n^2}{2} + \frac{n}{2} + (n+1)$

So therefore the lemma is true for all natural numbers n greater than 1.

With the lemma we can solve the proof easier

Base Case: For n = 1 would be $\sum_{i=1}^{1} i^3 = 1^3 = 1$. $(\sum_{i=1}^{1} i)^2 = 1$.

Inductive Hypothesis: From the base n is true for $\forall_n \geq 1$, $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$

Inductive Step: Using lemma we can re-write this to $(\frac{n^2}{2} + \frac{n}{2})^2 = \sum_{i=1}^{n} i^3$ We will first add n+1, $(\frac{(n+1)^2}{2} + \frac{(n+1)}{2})^2 = \frac{n^4}{4} + \frac{3n^3}{2} + \frac{13n^2}{4} + 3$n + 1 . Now we will do $(\frac{n^2}{2} + \frac{n}{2})^2 + (n+1)^3 = \frac{n^4}{4} + \frac{3n^3}{2} + \frac{13n^2}{4} + 3$n + 1 . So therfore $\forall_n \geq 1$, $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$ is true for all natural numbers for n greater than or equal to 1.

(b) $\forall_n \geq 4$, $2^n < n!$

Basis: for n = 4 2^4 = 16 and 4! = 24

Inductive Hypothesis: $\forall_n \geq 4$, $2^n < n!$ is true for n.

Inductive Step: $2^{n+1} = 2^n * 2$. (n + 1)! = n! * (n + 1) Since we know from the inductive hypothesis that $\forall_n \geq 4$, $2^n < n!$ so for this to remain true the right side and the left would need to be multiplied by the same value or the right would need to be multiplied by a greater value. The left is multiplied by 1 and the right is multiplied by (n + 1). We know that n has to be 4 or greater so we know (n + 1) ¿ 2 so therfore $\forall_n \geq 4$, $2^n < n!$ for all natural numbers n.

2. Refer to the definition of Full Binary Tree from the notes. For a Full Binary Tree T, we use n(T), h(T), i(T) and '(T) to refer to number of nodes, height, number of internal nodes (non-leaf nodes) and number of leaves respectively. Note that the height of a tree with single node is 1 (not zero). Using structural induction, prove the following:

(a) For every Full Binary Tree T, n(T) ≥ h(T).

Basis: For a full binary tree T with a root node and two child leaf nodes $n(T_*) = 3$, and $h(T_*) = 2$ so $n(T_*) \geq h(T_*)$ is true.

Inductive Hypothesis: Suppose that for complete binary trees T, n(T) ≥ h(T) is true and n(T) = $2^{h(T)} - 1$.

Recursive: Suppose we have two sub trees T_1, T_2 that when combined with a node to connect them $n_0 + T_1 + T_2$ = T. So to find the total number of nodes for n(T) = $n(T_1) + n(T_2) + 1$ which because $T_1 = T_2$ n(T) = $n(T_1) + n(T_2)$ can be reduced to n(T) = $2n(T_1) + 1$ or by using the IH $2^{h(T_1)+1}$. We know that the height of T will increase by 1 so $h(T) = h(T_1) + 1$. So $2^{h(T_1)+1} \geq h(T)$ or $2^{h(T)} \geq h(T)$

(b) For every Full Binary Tree T, i(T) ≥ h(T) - 1

(c) For every Full binary Tree T, $l(T) = (n(T) + 1)/2$

Basis: for a full binary tree T^2 with a single root node and two children has 2 leaf nodes and 3 alltogether. so $(3 + 1)/2 = 2$, so therefore $l(T^3) = (n(T^3) + 1)/2$ is true for the basis.

Inductive Hypothesis: We assume $l(T) = (n(T) + 1)/2$ is true and that the n(T) $= 2^{h(T)} - 1$.

Recursive Step: Suppose we have T_1 and T_2 full binary trees where $l(T_{1,2}) = (n(T_{1,2}) + 1)/2$. If we combined T_1 and T_2 into a full tree T then the leaves of the new tree would be $\frac{(n(T^1)+1)}{2} + \frac{(n(T^2)+1)}{2}$. Since T_1 and T_2 are equivilant we can say $n(T_1) + 1$. With our IH $n(T_1) + 1 = 2^{h(T_1)-1}$. also from the IH we can say that IH of $l(T) = 2^{h(T)-1}$. Because $h(T) = h(T_1) + 1$ $l(T) = 2^{h(T)-1} = 2^{h(T_1)}$. So therefore $l(T^3) = (n(T^3) + 1)/2$ is true for all full binary trees T.