HW1

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1. Use mathematical induction to prove the following statements

(a)
$\forall_n \geq 1$, $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$

lemma: $\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2}$ where $n \in N$
Base Case: $\sum_{i=1}^{1} i = 1 = \frac{1^2}{2} + \frac{1}{2}$
Inductive Hypothesis: $\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2}$
Inductive Steps:
Using induction on n
$\frac{(n+1)^2}{2} + \frac{n+1}{2} = \frac{n^2}{2} + \frac{3n}{2} + 1 = \frac{n^2}{2} + \frac{n}{2} + (n+1)$

So therefore the lemma is true for all natural numbers n greater than 1.

With the lemma we can solve the proof easier

Base Case: For n = 1 would be $\sum_{i=1}^{1} i^3 = 1^3 = 1$. $(\sum_{i=1}^{1} i)^2 = 1$.

Inductive Hypothesis: From the base n is true for $\forall_n \geq 1$, $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$

Inductive Step: Using lemma we can re-write this to $(\frac{n^2}{2} + \frac{n}{2})^2 = \sum_{i=1}^{n} i^3$
We will first add n+1, $(\frac{(n+1)^2}{2} + \frac{(n+1)}{2})^2 = \frac{n^4}{4} + \frac{3n^3}{2} + \frac{13n^2}{4} + 3\text{n} + 1$. Now we will do $(\frac{n^2}{2} + \frac{n}{2})^2 + (n+1)^3 = \frac{n^4}{4} + \frac{3n^3}{2} + \frac{13n^2}{4} + 3\text{n} + 1$. So therfore $\forall_n \geq 1$, $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$ is true for all natural numbers for n greater than or equal to 1.

(b) $\forall_n \geq 4$, $2^n < n!$

Basis: for n = 4 2^4 = 16 and 4! = 24

Inductive Hypothesis: $\forall_n \geq 4$, $2^n < n!$ is true for n.

Inductive Step: $2^{n+1} = 2^n * 2$. (n + 1)! = n! * (n + 1) Since we know from the inductive hypothesis that $\forall_n \geq 4$, $2^n < n!$ so for this to remain true the right side and the left would need to be multiplied by the same value or the right would need to be multiplied by a greater value. The left is multiplied by 1 and the right is multiplied by (n + 1). We know that n has to be 4 or greater so we know (n + 1) ¿ 2 so therfore $\forall_n \geq 4$, $2^n < n!$ for all natural numbers n.

2. Refer to the definition of Full Binary Tree from the notes. For a Full Binary Tree T, we use n(T), h(T), i(T) and ‘(T) to refer to number of nodes, height, number of internal nodes (non-leaf nodes) and number of leaves respectively. Note that the height of a tree with single node is 1 (not zero). Using structural induction, prove the following:

(a) For every Full Binary Tree T, n(T) ≥ h(T).

Basis: For a full binary tree T with a root node and two child leaf nodes $n(T_*) = 3$, and $h(T_*) = 2$ so $n(T_*) \geq h(T_*)$ is true.

Inductive Hypothesis: Suppose that for complete binary trees T, n(T) ≥ h(T) is true.

Recursive: Suppose we have two sub trees T_1, T_2, these threes are identical full binary trees and when we combine them together with an addictional node $T_1 + T_2 N_1 =$ T, $n(T_{1,2}) \geq h(T_{1,2})$. First we will compare $n(T_1)+n(T_2)+1 \geq h(T_{1,2})+1$. $n(T_1)+n(T_2)+1 =$ n(T) and h(T) = $h(T_{1,2})$ + 1. We can reduce it down to $2n(T_1) + 1 \geq h(T_1) + 1$ .... $2n(T_1) \geq h(T_1)$. By the IH we know that $n(T_1) \geq h(T_1)$ is true so $2n(T_1) \geq h(T_1)$ must also be true so therfore n(T) ≥ h(T) is true for all full binary trees T.

(b) For every Full Binary Tree T, i(T) ≥ h(T) - 1

Basis: For a full binary tree T_* with a single root node and two child leaf nodes. $i(T_*) = 1$ and $h(T_*) - 2 = 1$ so therfore $i(T_*) \geq h(T_*) - 1$

Inductive Hypothesis: Assume that i(T) $\geq$ h(T) - 1 is true.

Recursive: We will take two idential full binary trees $T_{1,2}$. If we combined them with a single node to connect them into a new full binary tree T then to find all the internal nodes of T we would do $i(T_1)+i(T_2)+1 \geq h(T)-1$. The height of T is one more than T_1 so $i(T_1)+i(T_2)+1 \geq h(T_1)$. This can also be reduced down to $i(2T_1)+1 \geq h(T_1)-1$. By the IH $i(T_1)+1 \geq h(T_1)-1$ so $i(2T_1)+1 \geq h(T_1)-1$ is also true and therfore i(T) $\geq$ h(T) - 1 is true for all complete binary trees T.

(c) For every Full binary Tree T, $l(T) = (n(T)+1)/2$

Basis: for a full binary tree T^2 with a single root node and two children has 2 leaf nodes and 3 alltogether. so (3 + 1)/2 = 2, so therefore $l(T^3) = (n(T^3)+1)/2$ is true for the basis.

Inductive Hypothesis: We assume $l(T) = (n(T)+1)/2$ is true.

Recursive Step: We will take two idential full binary trees $T_{1,2}$. If we combined them with a single node to connect them into a new full binary tree T. By the IH $l(T^3_{1,2}) = (n(T^3_{1,2})+1)/2$. To find the number of leaves for T we can add these together so $\frac{n(T_1)+1}{2} + \frac{n(T_2)+1}{2}$. We can simpllify this to because $T_1 = T_2$, $n(T_1)+1$ = l(T). Now me must prove this is correct. We know that $l(T) = (n(T)+1)/2$ so we will expand n(T) in terms of T_1 so $l(T) = (2n(T_1)+1+1)/2$ ... l(T) = $n(T_1)+1$. So because we were able to prove for T with two sub trees therefore $l(T) = (n(T)+1)/2$ is true for all full binary trees T.

3. Let a be an array of size n indexed by 0, 1, ... n 1. Consider the following code (inner loop of Bubble Sort)

```
for i in the range [0, n-2]
if (a[i] > a[i+1])
swap (a[i], a[i+1]); //swap the values of a[i] and a[i+1]
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Show the following using induction : At the start of ith iteration of the loop, a[i] is the maximum among a[0], a[1], ... a[i].

Basis: for a[0] there are no values index in a below 0 so therfore a[i] where i = 0 the start of the iteration of i is the maximum.

inductive Hypothesis: Assume that a[i] is the greatest max value during the iteration and that m(a[i]) is either true or false if a[i] is greatest during the ith iteration.

Inductive Step: By the IH assume that m(a[i]) is true so. If during the ith interation a[i] > a[i+1] then a[i+1] = a[i] in that case for m(a[i+1]) a[i+1] > a[i] and m(a[i]) is true by the inductive hypothesis so therefore for that that m(a[i+1]) is true. The other case is that a[i+1] > a[i] during the ith iteration. In that case its the same where m(a[i]) is already true so starting at the i+1 iteration we already know a[i] is greater than anything before it and the value after a[i], is greater a[i+1] > a[i] so therefore m(a[i]) or that a[i] is the greatest max for a[0],a[1],...a[i] for all natural numbers i where $0 \leq i \leq n-1$.