2. Martingales

2.1. Stochastic Processes; Filtrations; Stopping Times

This lecture is about stochastic processes, that is we are interested in "time" dependent random outcome. We denote the set of different times t by $\mathbf{T}$.

Definition 2.1. A *stochastic process* – or simply process – is a family $X = (X_t)_{t\in\mathbf{T}}$ of random variables $X_t : \Omega \to \mathbb{R}$ indexed by $\mathbf{T}$.

Intending to model the time, $\mathbf{T}$ should have a *"direction"*. Therefore, throughout this lecture, we always assume that $\mathbf{T}$ is a subset of the positive extended real line $[0,\infty]$.[33] We will assume that $0 \in \mathbf{T}$ and denote $T := \sup \mathbf{T}$ which might be ∞. If not otherwise specified, elements of $\mathbf{T}$ are designed by the letter $s, t, u, \ldots$.

For the first part of the lecture $\mathbf{T}$ will be discrete, that is $\mathbf{T} = \{0, 1, \ldots\}$. Later, as we construct the stochastic integral, we consider more general times set such as $\mathbf{T} = [0, T]$ where $T > 0$ is a fixed time horizon or $\mathbf{T} = \{2^k T/2^n : 0 \leq k \leq 2^n, n \in \mathbb{N}\}$ the dyadic times points between 0 and T.

The mappings $t \mapsto X_t(\omega)$ for $\omega \in \Omega$ are called the paths – or sample paths, trajectories – of the process. A stochastic process $X = (X_t)_{t=0,\ldots,T}$ may also be viewed as

- a single random variable

$$\begin{aligned} X : \Omega \times \{0,\ldots,T\} &\longrightarrow \mathbb{R} \\ (\omega, t) &\longmapsto X_t(\omega) \end{aligned}$$

  where the σ-algebra on $\Omega \times \{0,\ldots,T\}$ is given by the product σ-algebra $\mathcal{F} \otimes 2^{\{0,\ldots,T\}}$.

- a measurable function with values in the sample space

$$\begin{aligned} X : \Omega &\longrightarrow \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R}^{T+1} \\ \omega &\longmapsto (X_0(\omega), \ldots, X_T(\omega)) \end{aligned}$$

  where the σ-algebra on the sample space is the product Borel σ-algebra on $\mathbb{R}^{T+1}$.

Exercice 2.2. Show that the three definition of a stochastic process in finite discrete time are equivalent.$\Diamond$

Example 2.3. Consider now our example of coin tossing but infinitely many times. As seen, the state space is defined as follows

$$\Omega = \prod_{t\in\mathbb{N}} \{-1, 1\} = \{-1, 1\}^{\mathbb{N}} = \{\omega = (\omega_t) \colon \omega_t = \pm 1 \text{ for every } t\}$$

On each $\Omega_t = \{-1, 1\}$ we consider the σ-algebra $\mathcal{F}_t = \{\emptyset, \{-1\}, \{1\}, \{-1, 1\}\}$ and on Ω the product σ-algebra $\mathcal{F} = \otimes \mathcal{F}_t$. We saw that it is generated by the finite product cylinders:

$$C = \{\omega = (\omega_t) \in \Omega \colon \omega_{t_k} = e_k, k = 1, \ldots n\} \tag{2.1}$$

for a given set of values $e_k \in \{-1, 1\}$, and times $t_k \in \mathbb{N}$, $k = 1, \ldots, n$. Suppose that the probability of getting head, that is 1, for the coin toss is equal to $p \in [0, 1]$, we can define on the collection of these finite product cylinder $\mathcal{R}$, which is a semi-ring, a content $P : \mathcal{R} \to [0, 1]$ given by

$$P[C] = p^l (1-p)^{n-l}$$

[33] More generally, though, any directed set can be considered with or without an origin. For instance in statistical mechanics, indexing a process by subsets of a countable set ordered by inclusion.

for every $C \in \mathcal{R}$ of the form (2.1) where l is equal to the number of those $k = 1, \ldots, n$ where $e_k = 1$. We will show later that this content fulfills the sub-additivity property required in Caratheodory's theorem and therefore extends to a probability measure on Ω.
Now that we have a probability space at hand, we can define the stochastic processes $X = (X_t)$ and $S = (S_t)$ by

$$X_0(\omega) = 0 \quad \text{and} \quad X_t(\omega) = \begin{cases} 1 & \text{if } \omega_t = 1 \\ -1 & \text{if } \omega_t = -1 \end{cases} = \omega_t, \quad t = 1, \ldots, \quad \omega \in \Omega$$

and

$$S_t = x_0 + \sum_{s=0}^{t} X_s, \quad t = 0, 1, \ldots$$

where $x_0 \in \mathbb{R}$ is the start value, or start price of S. The stochastic process S is called the *random walk* and the process X tells us what is the result of the coin toss at time t.
As an exercise in Ipython, make a plot of 5 sample paths of the random walk for

- an horizon of $T = 10, 100, 1.000, 100.000$;
- for $p = 1/3, 1/2, 2/3$. $\Diamond$

As such, a process is nothing else than an arbitrary family of random variables indexed by the time. However, our intuitive understanding of a process rather corresponds to X_s "having less, or knowing less" than X_t whenever $s \leq t$. To model this intuition we use an increasing set of information.

Definition 2.4. A *filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbf{T}}$ is a family of σ-algebras on Ω indexed by $\mathbf{T}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ whenever $s \leq t$ with $s, t \in \mathbf{T}$. A measurable space together with a filtration is called a *filtered* space. A stochastic process X is $\mathbb{F}$-*adapted* – or simply adapted – if X_t is $\mathcal{F}_t$-measurable for every $t \in \mathbf{T}$;

The σ-algebras in a filtration becomes finer and finer due the inclusion. It means that the considered events at time t provide more information than the ones at previous times. Filtration can be given, but also generated by stochastic processes.

Definition 2.5. Let X be a stochastic process. The family of σ-algebra

$$\mathcal{F}_t^X = \sigma(X_s \colon s \leq t) := \sigma\left(\left\{X_s^{-1}(A) \colon A \in \mathcal{B}(\mathbb{R}), s \leq t\right\}\right), \quad t \in \mathbf{T}$$

is a filtration called the filtration generated by X which we denote by $\mathbb{F}^X$.

The fact that the filtration generated by a stochastic process is indeed a filtration is easy to verify.

Example 2.6. In our random walk example, we did not specify a filtration, but we can consider the following sequences of σ-algebras for $t \in \mathbb{N}_0$

- $\mathcal{F}_t^X$;
- $\mathcal{F}_t^S$;
- $\mathcal{G}_t := \sigma(S_t)$;
- $\mathcal{H}_t := \sigma(X_t)$;

As an exercise, try to figure out which sequence of $sigma$-algebra is a filtration give an expression for their generators in the case where $x_0 = 0$. $\Diamond$

From now on, until we mention otherwise,

$$\mathbf{T} = \{0, 1, \dots\}!!!!! \quad \text{and} \quad (\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}) \text{ is a filtrated space.}$$

A further important notion in the theory of stochastic processes, are the so called stopping time. As an illustration of which, consider the following game. You pay 10 Kuai and a coin will be tossed every minute. If it is head you win one Kuai, if it is tail you loose one Kuai. So the evolution of your wealth as times goes by follows

$$S_t := 10 + \sum_{k=1}^{t} X_t$$

However you would like to leave the game before loosing too much money, that is, you stop the first time you reach let's say 3 kuai.

$$\tau = \inf\{t\colon S_t \leq 3\}$$

This time however is no longer known but random since it depends on the random outcomes $S_t(\omega)$. This is the same on financial markets, where investors wants to know the time until which a company might be bankrupt for instance, or the time until they reach a certain level of wealth in their strategic investment.

Exercice 2.7. In the case when the coin toss is fair, what is the probability that you exit the game before 100 minutes? $\Diamond$

Intuitively, a *random time* gives information about when a random event occurs.

Definition 2.8. On a measurable space, a *random time* is a measurable mapping $\tau : \Omega \to \mathbf{T} \cup T$. Given a filtration, a random time is a *stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \in \mathbf{T}$.

For a process X and a subset B of $\mathbb{R}$, we define the *hitting time* of X in B as

$$\tau_B(\omega) = \inf\{t \in \mathbf{T} : X_t(\omega) \in B\}.$$

This function is not necessarily random even if X is adapted, however we have the following.

Proposition 2.9. *If X is an adapted process and B is Borel, then τ_B is a stopping time.*

Proof. Let $t \in \mathbf{T}$. Since $\mathbf{T}$ is discrete, the infimum is in fact a minimum. Hence, it follows that

$$\{\tau_B \leq t\} = \bigcup_{s=0,\dots,t} \{X_s \in B\}$$

Since X is adapted, it follows that $A_s = \{X_s \in B\} \in \mathcal{F}_s$ for every s. Furthermore, $\mathbf{F}$ being a filtration, it holds $\mathcal{F}_s \subseteq \mathcal{F}_t$. Hence, $A_s \in \mathcal{F}_s$ for every $s \leq t$. Finally, $\mathcal{F}_t$ being a σ-algebra, the finite union of A_s for $s \leq t$ is also in $\mathcal{F}_t$ showing that $\{\tau_B \leq t\}$ is a stopping time. $\square$

Let us collect some standard properties of stopping times.

Proposition 2.10. *The following assertions hold*

(a) $\tau + \sigma$, $\tau \vee \sigma$ and $\tau \wedge \sigma$ are stopping times as soon as τ, σ are stopping times.

(b) $\lim \tau^n$ is a stopping time as soon as (τ^n) is an increasing sequence of stopping times.

(c) If τ is a stopping time, then the collection $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$ is a σ-algebra and τ is $\mathcal{F}_\tau$-measurable.

(d) *For any two stopping times, it holds $\mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma\wedge\tau}$. In particular, $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$, if $\sigma \leq \tau$. For every integrable random variable X with respect to some probability on $\mathcal{F}$, it holds $E[E[X \mid \mathcal{F}_\sigma] \mid \mathcal{F}_\tau] = E[X \mid \mathcal{F}_{\sigma\wedge\tau}]$.*

Proof. *(a)* follows from

$$\{\tau + \sigma \leq t\} = \bigcup_{q\leq t} \{\sigma \leq t - q\} \cap \{\tau \leq q\} \in \mathcal{F}_t$$

$$\{\tau \vee \sigma \leq t\} = \{\tau \leq t\} \cup \{\sigma \leq t\} \in \mathcal{F}_t \quad \text{and} \quad \{\tau \wedge \sigma \leq t\} = \{\tau \leq t\} \cap \{\sigma \leq t\} \in \mathcal{F}_t$$

(b) follows from $\{\lim \tau^n \leq t\} = \{\tau^n \leq t : \text{ for all } n\} = \cap_n \{\tau^n \leq t\} \in \mathcal{F}_t$.

(c) Clearly $\emptyset, \Omega \in \mathcal{F}_\tau$. For $A \in \mathcal{F}_\tau$ it holds $A^c \cap \{\tau \leq t\} = (A \cup \{\tau > t\})^c = [(A \cap \{\tau \leq t\}) \cup \{\tau \leq t\}^c]^c \in \mathcal{F}_t$. Finally, for $(A_n) \subseteq \mathcal{F}_\tau$ it holds $(\cup A_n) \cap \{\tau \leq t\} = \cup(A_n \cap \{\tau \leq t\}) \in \mathcal{F}_t$.

(d) Follows from $\{\tau \wedge \sigma \leq t\} = \{\tau \leq t\} \cap \{\sigma \leq t\}$. □

Proposition 2.11. *Let X be an adapted process and τ a stopping time. Then $X_\tau(\omega) := X_{\tau(\omega)}(\omega)$ is an $\mathcal{F}_\tau$-measurable random variable. Furthermore, $X^\tau := (X_{\cdot\wedge\tau})$ is an adapted process.*

Proof. Exercise. □

For a stopping time τ, we denote by $[\tau] = \{(\omega, t) \in \Omega \times \mathbf{T} : \tau(\omega) = t\}$ its graph.

2.2. Martingales

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}, P)$ be a filtrated probability space.

Definition 2.12. A process X is called a *martingale* if

a) X is adapted;

b) X_t is integrable for every $t \in \mathbf{T}$;

c) $E[X_t \mid \mathcal{F}_s] = X_s$ whenever $s \leq t$, $s, t \in \mathbf{T}$.

A process X is called a *super-martingale* if instead of c) we require

c') $E[X_t \mid \mathcal{F}_s] \leq X_s$ whenever $s \leq t$, $s, t \in \mathbf{T}$.

A process X is called a *sub-martingale* if instead of c) we require

c”) $E[X_t \mid \mathcal{F}_s] \geq X_s$ whenever $s \leq t$, $s, t \in \mathbf{T}$.

We say that a martingale, super-martingale or sub-martingale X is closed on the right if there exists $\xi \in L^1$ such that $E[\xi \mid \mathcal{F}_t] = X_t$, $E[\xi|\mathcal{F}_t] \leq X_t$ or $E[\xi|\mathcal{F}_t] \geq X_t$, respectively, for every $t \in \mathbf{T}$.

Remark 2.13. Note that a martingale is in particular a super- and a sub-martingale at the same time. Furthermore, given $\xi \in L^1$, the process given by $X_t = E[\xi|\mathcal{F}_t]$ for $t \in \mathbf{T}$ defines a martingale.[34] ♦

[34] Why?