"STOCHASTIC PROCESSES" – HOMEWORK SHEET 3

Exercise 3.1. *(10 points)*

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Given two random variables X and Y in L^2, the covariance of X and Y is given as

$$Cov(X,Y) = E\left[(X - E[X])(Y - E[Y])\right]$$

and the variance of X as

$$Var(X) = Cov(X,X) = E\left[(X - E[X])^2\right].$$

Let (X_n) be a sequence of pairwise uncorrelated random variables in L^2 such that $\sup Var(X_n) < \infty$. Defining

$$S_n = \frac{1}{n}\sum_{k\leq n}(X_k - E[X_k]),$$

show that $S_n \to 0$ in probability.[1]

Exercise 3.2. *(10 Points)*

We consider a very simple financial market with two stocks S^1 and S^2 which values tomorrow depends on three states, that is $\Omega := \{\omega_1, \omega_2, \omega_3\}$. The values are given as follows

$$S^1(\omega) := \begin{cases} 90 & \text{if } \omega = \omega_1 \text{ or } \omega = \omega_2 \\ 110 & \text{if } \omega = \omega_3 \end{cases} \quad \text{and} \quad S^1(\omega) := \begin{cases} 90 & \text{if } \omega = \omega_1 \\ 100 & \text{if } \omega = \omega_2 \\ 110 & \text{if } \omega = \omega_3 \end{cases}$$

We set $\mathcal{F} = 2^\Omega$ and consider that each state comes with the same probability, that is, we consider the uniform probability measure P on $\mathcal{F}$ given by $P[\{\omega_1\}] = P[\{\omega_2\}] = P[\{\omega_3\}] = 1/3$.

Suppose that you are an insider that have the knowledge about the outcome of X tomorrow. Compute the conditional expected value of Y with respect to this information, that is $E[Y|X]$.[2]

Exercise 3.3. *(18 points)*

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. Show that

(i) $E[|E[X|\mathcal{G}]|] \leq E[|X|]$ *for every $X \in L^1(\mathcal{F})$.*

(ii) *the mapping $X \mapsto E[X|\mathcal{G}]$ from $L^1(\mathcal{F})$ to $L^1(\mathcal{G})$ is linear and Lipschitz continuous;*

(iii) $E[X|\mathcal{G}] \geq 0$ *P-almost surely whenever $0 \leq X$ P-almost surely and $X \in L^1(\mathcal{F})$;*

(iv) $E[X_n|\mathcal{G}] \nearrow E[X|\mathcal{G}]$ *P-almost surely for every sequence (X_n) of elements in $L^1(\mathcal{F})$ such that $0 \leq X_n \nearrow X \in L^1(\mathcal{F})$;*

(v) $E[\liminf X_n|\mathcal{G}] \leq \liminf E[X_n|\mathcal{G}]$ *for every sequence (X_n) of elements in $L^1(\mathcal{F})$ such that $X_n \geq Y \in L^1(\mathcal{F})$;*

[1] What about Markov inequality?...

[2] Under this terminology, we understand conditional expectation of Y with respect to the σ-algebra generated by X, that is $\sigma(X)$.

(vi) $E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$ *P-almost surely and in* $L^1(\mathcal{F})$ *for every sequence* (X_n) *of elements in* $L^1(\mathcal{F})$ *such that* $|X_n| \leq Y \in L^1(\mathcal{F})$ *for every* n*;*

(vii) $E[YX|\mathcal{G}] = YE[X|\mathcal{G}]$ *whenever* Y *is* $\mathcal{G}$*-measurable and* XY *is integrable;*

(viii) $E[XE[Y|\mathcal{G}]] = E[E[X|\mathcal{G}]Y] = E[E[X|\mathcal{G}]E[Y|\mathcal{G}]]$ *whenever* X *and* Y *are in* L^2*;*

(ix) $E[E[X|\mathcal{G}_2]|\mathcal{G}_1]$ *whenever the* σ*-algebras are such that* $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$*.*

Exercise 3.4. *(Bonus 10 points)*
Let $(\Omega, \mathcal{F}, P)$ *be a probability space where* $\Omega = [0,1]$*,* $\mathcal{F}$ *is the Borel-*σ*-algebra of* $[0,1]$ *and* P *is the Lebesgue measure on* $[0,1]$*. On the vector space* L^0*, we consider the topology generated by the distance*[3]

$$d(X,Y) = E\left[\frac{|X-Y|}{1+|X-Y|}\right].$$

Let $C \subseteq L^0$ *be a convex set with non empty interior*[4]*. Show that* $C = L^0$*.*[5] *Deduce that the only continuous linear function* $I : L^0 \to \mathbb{R}$ *with this topology is the constant* 0*, that is,* $I(X) = 0$ *for every* $X \in L^0$*.*[6]

Due date: Upload before Monday 2015.10.19 14:00.

[3] Which according to the last homework sheet corresponds to the topology of convergence in probability.
[4] That is, there exists a open ball $B_\varepsilon(Y) = \{X : d(X,Y) < \varepsilon\}$ with $Y \in C$ and $\varepsilon > 0$ such that $B_\varepsilon(Y) \subseteq C$.
[5] Show that without loss of generality you can assume that $B_\varepsilon(0) \subseteq C$ for some $\varepsilon > 0$ and approximate any $X \in L^0$ by smart convex combinations – with more than two elements – of elements in $B_\varepsilon(0)$.
[6] If I is linear and continuous, it follows that $I^{-1}(]-\varepsilon,\varepsilon[)$ is convex and open.