"STOCHASTIC PROCESSES" – HOMEWORK SHEET 2

Throughout, $(\Omega, \mathcal{F}, P)$ be a probability space.

Exercise 2.1. (10 Points) Given a sequence (A_n) of events, we define

$$\liminf A_n = \cap_n \cup_{k \geq n} A_k \quad \text{and} \quad \limsup A_n = \cup_n \cap_{n \geq k} A_k.$$

In other terms

$$\begin{aligned}\limsup A_n &= \{\omega \colon \omega \in A_n \text{ for all } n \geq n_0 \text{ for } n_0 \text{ large enough}\}\\ \limsup A_n &= \{\omega \colon \omega \in A_n \text{ for infinitely many } n\}\end{aligned}$$

Show that

(a) $P[\liminf A_n] \leq \liminf P[A_n] \leq \limsup P[A_n] \leq P[\limsup A_n]$ and give an example for which all inequalities are strict.[1]

(b) if $\sum P[A_n] < \infty$, then $P[\limsup A_n] = 0$.[2]

Exercise 2.2. (20 Points + 4 Bonus point question (f)) Recall that a sequence (X_n) of random variables converges to X in probability if $P[|X_n - X| \geq \varepsilon] \to 0$ for every $\varepsilon > 0$. Throughout the exercise (X_n) and (Y_n) denote sequences of random variables and X, Y two random variables.

(a) Show that

$$d(X, Y) = E\left[\frac{|X - Y|}{1 + |X - Y|}\right],$$

defines a metric on L^0 and that convergence in this metric is equivalent to convergence in probability.[3]

(b) Show that $X_n \to X$ P-almost surely implies that $X_n \to X$ in probability. Give and example that the reciprocal is not true.

(c) Suppose that $\sum P[|X_n - X| \geq \varepsilon] < \infty$ for every $\varepsilon > 0$. Show that $X_n \to X$ P-almost surely.

(d) Show that each converging sequence of random variables that converges in probability has a subsequence that converges P-almost surely.

(e) Suppose that any subsequence of (X_n) admits itself another subsequence that converges to X P-almost surely. Show that $X_n \to X$ in probability.

(f) (this one is Bonus) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function.[4] Show that if $X_n \to X$ and $Y_n \to Y$ both in probability, then it holds $f(X_n, Y_n) \to f(X, Y)$ in probability.

Exercise 2.3. (20 Points)

[1] To this end, show that $\liminf 1_{A_n} = 1_{\liminf A_n}$ and $\limsup 1_{A_n} = 1_{\limsup A_n}$.

[2] Recall that if $\sum a_n < \infty$ for $a_n > 0$, then it holds $\sum_{k \geq n} a_k \to 0$ as $n \to \infty$.

[3] That is $X_n \to X$ in probability is equivalent to $d(X_n, X) \to 0$. Make use of Markov's inequality, and the fact that $f(x) = x/(1 + x)$ on $\mathbb{R}_+$ is bounded by 1, and strictly increasing.

[4] Use the fact that f is uniformly continuous on compact

(a) Find a sequence of positive random variables (X_n) such that $E[X_n] \to 0$ but $P[\limsup X_n > \liminf X_n] = 1$, that is X_n converges P-almost nowhere.

(b) Find a sequence of positive random variables (X_n) such that $X_n \to X$ P-almost surely and in L^1, but $\sup_n X_n$ is not integrable.

(c) Show that if $X_n \to X$ in L^1, then $X_n \to X$ in probability. Find an example such that the reciprocal is not true.

(d) Show that the dominated convergence theorem holds when instead of requiring $X_n \to X$ P-almost surely, on suppose that $X_n \to X$ in probability.

(e) Let $\alpha \geq 1$ and X be an integrable positive random variable. Show that $\lim E[n \ln(1 + (X/n)^\alpha)]$ exists and compute its value.[5]

Exercise 2.4. (Bonus, 10 Points) Recall that the $\|\cdot\|_\infty$ operator is defined as[6]

$$\|X\|_\infty = \inf \{m \in \mathbb{R}_+ \colon P[|X| \geq m] = 0\}$$

for a random variable X.

Let now (X_n) be a sequence of random variables which converges P-almost surely to a random variable X. Show that for every $\varepsilon > 0$, there exists a measurable set A with $P[A^c] < \varepsilon$ such that

$$\lim \|(X_n - X)1_A\|_\infty = 0.$$

Hint: Define $A_{n,k} = \cup_{m \geq n}\{|X_m - X| \geq 1/k\}$ and show that its probability can be made arbitrarily small.

Due date: Upload before Monday 2015/10/12 14:00.

[5] Hint, show that $\ln(1 + x^\alpha) \leq \alpha x$ for $\alpha \geq 1$ and $x \geq 0$. Then use some Taylor expansion.

[6] With the convention that $\inf \emptyset = \infty$.