4. Stochastic Integral

4.1. Continuous Time Processes

Throughout, $(\Omega, \mathcal{F}, P)$ denotes a probability space. From now on however we no longer assume that $\mathbf{T} = \mathbb{N}_0$ but[41]

$$\mathbf{T} = [0, \infty[\quad \text{and} \quad T = \sup \mathbf{T} = \infty$$

Remark 4.1. If not otherwise specified, elements of $\mathbf{T}$ are designed by the letter $s, t, u, \ldots$. Countable and dense subset of $\mathbf{T}$ is denoted by $\mathbf{Q}$ and their elements are designed by the letter q, r. Often $\mathbf{Q} = \mathbb{Q}$.♦

As random variable may be identified in the P-almost sure sense, for processes we have several possibilities at hand.

Definition 4.2. Let X and Y be two processes. We say that

- X is a *modification* of Y if $X_t = Y_t$ P-almost surely for every $t \in \mathbf{T}$, that is

$$P[X_t = Y_t] = 1, \quad \text{for every } t \in \mathbf{T}.$$

- X is indistinguishable from Y if $X_t = Y_t$ for all $t \in \mathbf{T}$ P-almost surely, that is

$$P[X_t = Y_t;\ \text{for every } t \in \mathbf{T}] = 1.$$

In the case where $\mathbf{T}$ is countable, these two notions are equivalent. However, if $\mathbf{T}$ is no longer countable, modification and indistinguishability are different in that we may have to take into account uncountably many null sets as the following example shows.

Example 4.3. Set $(\Omega, \mathcal{F}, P) := ([0,1], \mathcal{B}([0,1]), dt)$, where dt is the Lebesgue measure and $\mathcal{B}([0,1])$ is the Borel σ-algebra, and $\mathbf{T} = [0,1]$. Define the processes X and Y by

$$X_t = 0 \quad \text{and} \quad Y_t = \begin{cases} Y_t(\omega) = 1 & \text{if } \omega = t \\ Y_t(\omega) = 0 & \text{otherwise} \end{cases}$$

for every $t \in [0,1]$. It follows that $P[X_t = Y_t] = P[\{\omega \in [0,1] : \omega \neq t\}] = 1$, whereas $P[X_t = Y_t : \text{for every } t] = P[\{\omega \in [0,1] : \omega \neq t \text{ for every } t\}] = P[\emptyset] = 0$. ♢

We see that uncountably many sets of measure zero can add up to something that may no longer have measure zero. However, if the time set is countable, or if we can infer from the structure of the trajectories that it is sufficient to consider countably many times, then these two conditions will coincide. We say that a process X

- has "*des limites à gauche*" – left limits – or "*des limites à droite*" – right limits – if

$$P\left[\liminf_{s \nearrow t} X_s = \limsup_{s \nearrow t} X_s :\ \text{for every } t > 0, t \in \mathbf{T}\right] = 1,$$

  or

$$P\left[\liminf_{s \searrow t} X_s = \limsup_{s \searrow t} X_s :\ \text{for every } t < T, t \in \mathbf{T}\right] = 1,$$

  respectively.

[41] Eventually, $\mathbf{T}$ may be equal to $[0, T]$ for some $T < \infty$.

- is "*continue à gauche*" – left-continuous – or "*continue à droite*" – right-continuous – if

$$P\left[\lim_{s\nearrow t} X_s = X_t : \text{ for every } t > 0, t \in \mathbf{T}\right] = 1,$$

or

$$P\left[\lim_{s\searrow t} X_s = X_t : \text{ for every } t < T, t \in \mathbf{T}\right] = 1,$$

respectively.[42]

A process X is said to be càdlàg, càglàd, or làdlàg, if it is "continue à droite avec des limites à gauche", "continue à gauche avec des limites à droite" or "limité à gauche comme à droite", respectively. In general, these concepts are meaningful if $\mathbf{T}$ is a dense subset of an interval in $[0, \infty]$.

Lemma 4.4. *Suppose that X and Y are modifications of each other and both are either right-continuous or left-continuous. Then X and Y are indistinguishable.*

Proof. Let X and Y be right-continuous and modifications of each other. Since $\mathcal{F}$ is a σ-algebra, it holds

$$P\left[X_q \neq Y_q : \text{for some } q \in \mathbb{Q}\right] = P\left[\cup_{q\in\mathbb{Q}}\{X_q \neq Y_q\}\right] \leq \sum P[X_q \neq Y_q] = 0$$

Hence $P[X_q = Y_q\colon \text{ for every } q] = P\left[\{X_q \neq Y_q : \text{for some } q \in \mathbb{Q}\}^c\right] = 1$. Now since X and Y are right-continuous, and $\mathbb{Q}$ is countable order dense, it follows that

$$\begin{aligned} P\left[X_t = Y_t : \text{for every } t \in \mathbf{T}\right] &= P\left[\lim_{q\searrow t} X_q = \lim_{q\searrow t} Y_q : \text{ for every } t \in \mathbf{T}\right] \\ &= P\left[X_q = Y_q : \text{for every } q \in \mathbb{Q}\right] = 1 \end{aligned}$$

□

Exercice 4.5. The assumption of left- or right-continuity is central. Find an example of two làdlàg processes X, Y, modifications of each other, which are not indistinguishable. ♢

As such, a process is nothing else than an arbitrary family of random variables indexed by the time. It can also be seen as a mapping $X : \Omega \times \mathbf{T} \to S$.

Definition 4.6. We say that a process X is *measurable* if it is measurable with respect to the product σ-algebra $\mathcal{F} \otimes \mathcal{B}(\mathbf{T})$.

The definition of a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ does not change as an increasing family of σ-algebra indexed by time. However, we can also define the right and left filtration $\mathbb{F}^+ = (\mathcal{F}_t^+)_{t\in\mathbf{T}}$ and $\mathbb{F}^- = (\mathcal{F}_t^-)_{t\in\mathbf{T}}$ as follows

$$\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s \quad \text{and} \quad \mathcal{F}_t^- = \bigvee_{s<t} \mathcal{F}_s := \sigma\left(\mathcal{F}_s : s < t\right)$$

for $t \in \mathbf{T}$. Clearly these definitions make sense if t is neither minimum or maximum of $\mathbf{T}$. In these extreme cases, we set $\mathcal{F}_0^- = \mathcal{F}_0$ and $\mathcal{F}_T^+ = \mathcal{F}_T$ if $T \in \mathbf{T}$. We say that the filtration is left- or right-continuous if $\mathbb{F} = \mathbb{F}^-$ or $\mathbb{F} = \mathbb{F}^+$, and continuous if it is both.

Remark 4.7. From the definition, $\mathbb{F}^\pm$ are themselves filtrations and it holds $\mathcal{F}_t^- \subseteq \mathcal{F}_t \subseteq \mathcal{F}_t^+$ as well as $\mathcal{F}_s^+ \subseteq \mathcal{F}_t^-$ whenever $s < t$. Hence $(\mathbb{F}^-)^+ = (\mathbb{F})^+ = (\mathbb{F}^+)^+$ as well as $(\mathbb{F}^+)^- = (\mathbb{F})^- = (\mathbb{F}^-)^-$. Clearly, if $\mathbf{T}$ is discrete, for instance $\mathbf{T} = \mathbb{N}_0$, left- and right-continuous filtrations make little sense. ♦

[42]Naturally, when we write $\lim$ we implicitly mean that it exists.

Definition 4.8. We say that a stochastic process X is

- $\mathbb{F}$-*adapted* – or simply adapted – if X_t is $\mathcal{F}_t$-measurable for every $t \in \mathbf{T}$;
- $\mathbb{F}$-*progressively measurable* – or simply progressively measurable – if $(\omega, s) \mapsto X_s(\omega)$, $s \le t$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{T} \cap [0,t])$-measurable for every $t \in \mathbf{T}$.

Proposition 4.9. *Let X be right-continuous or left-continuous $\mathbb{F}$-adapted process. Then X is progressively measurable.*

Proof. Suppose that X is right-continuous and fix $t \in \mathbf{T}$. Define

$$X_s^n = X_{\frac{k+1}{2^n}t}, \quad \text{for } \frac{k}{2^n}t < s \le \frac{k+1}{2^n}t$$

Since X is right-continuous, it follows that $\lim X^n = X$ on $\Omega \times [0,t]$ up to the null set of those ω on which X does not have right-continuous paths. Furthermore, since X is adapted, it follows that the piecewise constant process X^n is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{T} \cap [0,t])$-measurable. Hence X is progressively measurable. □

The previous result makes use of the regularity of paths to derive progressive measurability from adaptiveness. The following result goes a step further by showing that measurability together with adaptiveness yields progressive measurability, up to a modification though.

Theorem 4.10. *Any measurable and adapted process admits a progressive modification.*

The proof of this theorem is not trivial and somewhat lengthy in the standard literature, see for instance [1]. Since this result will be needed for the construction of the stochastic integral we provide a recent simpler version of the proof from Ondrejat and Seidler [3] in Section 4.3.
The notion of stopping times also has to be slightly modified in the continuous time

Definition 4.11. On a probability space, a *random time* is a measurable mapping $\tau : \Omega \to \mathbf{T} \cup T$. Given a filtration, a random time is a(n)

- *optional time* if $\{\tau < t\} \in \mathcal{F}_t$ for every $t \in \mathbf{T}$.
- *stopping time* if $\{\tau \le t\} \in \mathcal{F}_t$ for every $t \in \mathbf{T}$.

Proposition 4.12. *Every stopping time is is an optional time, and every optional time is a stopping time for the right-filtration. In particular, the two notions coincide if the filtration is right-continuous.*

Proof. The first assertion is trivial. As for the second, let τ be an optional time, and $t \in \mathbf{T}$. It follows that $\{\tau \le t\} = \cap_n \{\tau < t + 1/n\} \in \mathcal{F}_t^+$. □

For a process X and a subset V of the state space we define the *hitting time* of X in V as

$$\tau_V(\omega) = \inf\{t \in \mathbf{T} : X_t(\omega) \in V\}.$$

This function is not necessarily random even if X is adapted, however we have the following.

Proposition 4.13. *If X is an adapted right-continuous process and V is open, then τ_V is an optional time. If X is a continuous adapted process and V is closed, then τ_V is a stopping time.*

Proof. It holds $\{\tau_V < t\} = \{\omega \in \Omega : X_s(\omega) \in V, s < t\}$. Since X is right-continuous and V is open, $X_s(\omega) \in V$ implies the existence of a rational $q < s$ such that $X_q(\omega) \in V$. Hence $\{\tau_V < t\} = \{X_q \in V : q < t\} = \cup_{q<t}\{X_q \in V\} \in \mathcal{F}_t$. For the case of X being continuous and V closed, define the open sets $V_n = \{x : d(x, V) < 1/n\} \supseteq V$. Then by continuity of X we obtain

$$\{\tau_V \leq t\} = \{X_t \in V\} \bigcup \left(\bigcap_n \bigcup_{q<t} \{X_q \in V_n\} \right) \in \mathcal{F}_t. \qquad \square$$

Let us collect some standard properties of optional and stopping times.

Proposition 4.14. *The following assertions hold*

(a) *Every constant t is a stopping time.*

(b) *$\tau + \sigma$, $\tau \vee \sigma$ and $\tau \wedge \sigma$ are stopping/optional times as soon as τ, σ are stopping/optional times.*

(c) *$\lim \tau^n$ is a stopping time as soon as (τ^n) is an increasing sequence of stopping times.*

(d) *$\lim \tau^n$ is an optional time as soon as (τ^n) is a decreasing sequence of optional times. It is a stopping time if (τ^n) are stationary stopping times, that is, $\tau^m(\omega) = \tau^n(\omega)$ for all m greater than a given n, for P-almost all $\omega \in \Omega$.*[43]

(e) *If τ is a stopping time, then the collection $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$ is a σ-algebra and τ is $\mathcal{F}_\tau$-measurable.*

(f) *For any two stopping times, it holds $\mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma \wedge \tau}$. In particular, $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$, if $\sigma \leq \tau$. For every random variable X, it holds $E[E[X \,|\, \mathcal{F}_\sigma] \,|\, \mathcal{F}_\tau] = E[X \,|\, \mathcal{F}_{\sigma \wedge \tau}]$.*

Proof. The proof follows the same argumentation as in the discrete time by noting that $\mathbb{Q}$ is a countable dense subset of $\mathbf{T}$. Only the following point needs a certain care.

(d) Suppose that τ^n is a decreasing sequence of optional times. It follows from $\{\lim \tau^n < t\} = \{\tau^n < t : \text{ for some } n\} = \cup_n\{\tau^n < t\} \in \mathcal{F}_t$ that $\lim \tau^n$ is an optional time. If τ^n are stopping times, it only holds $\{\lim \tau^n \leq t\} = \cap_{q>0}\{\tau^n \leq t + q : \text{for some } n\} \in \mathcal{F}_t^+$ and therefore $\lim \tau^n$ is optional. However, defining $A_n = \{\tau^n = \tau^m : \text{for all } m \geq n\}$, it follows from stationarity that A_n is increasing to Ω. Furthermore, $A_n \in \mathcal{F}_{\tau^n}$ and hence $\{\lim \tau^n \leq t\} = \cup_n\{\tau^n \leq t\} \cap A_n \in \mathcal{F}_t$. $\square$

Proposition 4.15. *Let X be a progressively measurable process and τ a stopping time. Then $X_\tau(\omega) := X_{\tau(\omega)}(\omega)$ is an $\mathcal{F}_\tau$-measurable random variable. Furthermore, $X^\tau := (X_{\cdot \wedge \tau})$ is a progressive process.*

Proof. First, τ being a stopping time implies that $(\omega, s) \mapsto h(\omega, s) := (\omega, \tau(\omega) \wedge s)$ from $\Omega \times \mathbf{T} \cap [0, t]$ onto itself is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{T} \cap [0, t])$-measurable for every t. Since X is progressive and $X_s^\tau(\omega) = X \circ h(\omega, s)$ for every $s \leq t$, it follows that $(s, \omega) \mapsto X_s^\tau(\omega)$ is also $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{T} \cap [0, t])$-measurable. Thus X^τ is progressive and, in particular, X_τ is $\mathcal{F}_\tau$ measurable. $\square$

The null sets on a probability space plays a central role. They allow to identify random variables in the almost sure sense. With regard to a filtration indexed by an uncountable time set, this may yield some tricky problems – this is mainly due to the problem of right continuous version of processes not further discussed here, see [1, Theorem III-44 p. 64, Theorems IV-32-33 pp. 102–103]. In order to get rid of these problems and the identification between optional and stopping times we will work with the following assumption.

[43] Note that n may depend on ω.

Definition 4.16. A filtration $\mathbb{F}$ is said to

- be *complete* if $\mathcal{F}_0$ contains all the P-negligible sets of $\mathcal{F}$;
- satisfy the *usual conditions* if it is complete and right-continuous, that is $\mathbb{F}^+ = \mathbb{F}$.

From now on:

$$\mathbb{F} = \mathbb{F}^+ \quad \text{and} \quad \mathcal{F}_0 \text{ contains all the } P \text{ null sets of } \mathcal{F}$$

For a stopping/optional time τ, we denote by $[\tau] = \{(\omega, t) \in \Omega \times \mathbf{T} : \tau(\omega) = t\}$ its graph.

Proposition 4.17. *Let X be a càdlàg, adapted process on a filtration satisfying the usual conditions. Then there exists a sequence of stopping times (τ^n) which exhausts the jumps*[44] $\Delta X = X - X_-$ *of X, that is*

$$\{\Delta X \neq 0\} \subseteq \bigcup_n [\tau^n].$$

The definition of (Super-/Sub-)Martingales do not change in the continuous time. The martingale properties and theorems extend to the continuous time. However some restrictions have to be made in terms of path regularity. Indeed, as mentioned earlier, the infinite amount of null sets that may add up has to be countably controlled.

Definition 4.18. We denote by $\mathcal{S}$ be the set of *simple predictable* processes H of the form

$$H_t = H_0 1_{\{0\}} + \sum_{k=1}^{n} H_k 1_{]\tau_{k-1}, \tau_k]}(t)$$

for a finite sequence $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_n$ of stopping times where τ_n is bounded, and $H_k \in L^0_{\tau_{k-1}}$ for $k = 0, \ldots, n$.

Per definition, any $H \in \mathcal{S}$ is a predictable progressive process. For a progressive process X and $H \in \mathcal{S}$, we denote by $H \bullet X$ the process

$$H \bullet X := H_0 X_0 + \sum_{k=1}^{n} H_k \left(X^{\tau_k} - X^{\tau_{k-1}} \right).$$

This process may be seen as the simple integral of H with respect to X, also denoted by $\int H dX$.

Theorem 4.19. *Let X be a process either on $\mathbf{T} = \mathbb{Q}$ countable, or right-continuous on $\mathbf{T} = [0, \infty[$, and $H \in \mathcal{S}$. The following assertions hold true.*

(a) If X is a martingale and $H \bullet X_t$ is integrable for every t, then $H \bullet X$ is a martingale. If X is a super/submartingale, $H \bullet X_t$ is integrable for every t and H is positive, then $H \bullet X$ is a super/submartingale. In particular if τ is a bounded stopping time then X^τ is a martingale or super/submartingale, respectively.

(b) Let X be a submartingale, $t \in \mathbf{T}$ and $\lambda > 0$. Then it holds

$$\lambda P\left[\bar{X}_t \geq \lambda\right] \leq E\left[1_{\{\bar{X}_t < \lambda\}} X_t\right] \leq E\left[X_t^+\right] \tag{4.1}$$

$$\lambda P\left[\underline{X}_t \leq -\lambda\right] \leq E\left[1_{\{\underline{X}_t > -\lambda\}} X_t\right] - E\left[X_0\right] \leq E\left[X_t^+\right] - E\left[X_0\right] \tag{4.2}$$

[44]For X càdlàg, the jump process ΔX is the difference of X with the càglàd version X_- of X

(c) Let X be a positive submartingale and $p > 1$, it holds

$$\left\| \sup_{s \le t} X_s \right\|_p \le q \left\| X_t \right\|_p$$

where $q = p/(p-1)$ is the conjugate of p.

(d) Let X be a supermartingale, then for every two reals $x < y$, the numbers of up-crossing of $[x, y]$ by X up to time t, $U_{[0,t]}(x, y, X)$ is a random variable and it holds

$$E\left[U_{[0,t]}\left(x, y, X\right)\right] \le \frac{E\left[\left(x - X_t\right)^+\right]}{y - x} \le \frac{|x| + E\left[|X_t|\right]}{y - x}, \quad t \in \mathbf{T}.$$

In particular, if X is a martingale, and $p > 1$, then $|X|^p$ is a positive submartingale – Jensen inequality – and so

$$\left\| X_t^* \right\|_p \le \left(\frac{p}{p-1} \right) \left\| X_t \right\|_p$$

for every $p > 1$.

Proof. The inequalities hold true if the process X is sampled on any finite discretization of $[0, t]$ containing 0 and t. Hence, passing to the limit, these inequalities hold for $([0, t] \cap \mathbb{Q}) \cup \{0, t\}$, showing the case $\mathbf{T} = \mathbb{Q}$. In case where $\mathbf{T}$ is continuous and the paths of X are right-continuous, the inequalities also follow as seen before. The single thing to check is whether $U_{[0,t]}(x, y, X)$ is a well defined random variable. However, for any finite $F \subseteq [0, t]$, since X is right-continuous, the τ^k and σ^k in the construction of the $U_F(x, y, X)$ are stopping times according to Proposition 4.13, therefore $U_F(x, y, X)$ is a random variable. It follows that $U_{([0,t] \cap \mathbb{Q}) \cup \{0,t\}}\left(x, y, X\right)$ is a random variable. Since X is right-continuous, this set takes into account all the up-crossing on $[0, t]$. $\square$

Theorem 4.20. *Any right-continuous submartingale is càdlàg and every sample path is almost surely bounded on any compact interval. Furthermore, X is a submartingale with respect to $\mathbb{F}^+$ as well as with respect to the augmentation of $\mathbb{F}$.*

Proof. Let X be a right-continuous submartingale. The boundedness of the sample paths on any compact interval almost surely follows from (4.1) and (4.2). As for the làg property, for $x < y$ two reals, define

$$A = \bigcup_{n \in \mathbb{N}} \bigcup_{p, q \in \mathbb{Q}, p < q} \left\{ \omega \in \Omega : U_{[0,n]}(p, q, X(\omega)) = \infty \right\}.$$

By means of the up-crossing inequality, it follows that this countable union is of measure 0. However, A contains the set

$$\left\{ \omega \in \Omega : \liminf_{s \nearrow t} X_s(\omega) < \limsup_{s \nearrow t} X_s(\omega), t \in \mathbf{T} \right\}.$$

Hence X is càdlàg. The fact that X is a supermartingale with respect to $\mathbb{F}^+$ is immediate. As for the augmentation, observe that null sets do not modify the supermartingale inequalities. $\square$

As noticed, the up-crossing inequality shows that supermartingales have some nice regularity of paths. However, we assumed from the beginning that these supermartingale were right-continuous, central to derive Doob's maximal inequalities. Let us show that up to modification, any supermartingale has nice properties, however in the right-continuous filtration or in a filtration satisfying the usual conditions. From now on, $\mathbf{T}$ is a continuous time interval and $\mathbb{Q}$ is a countable order dense subset of it.

Theorem 4.21. *Let X be a submartingale. Then the following holds true.*

(a) Almost surely, the limits

$$X_{t+} = \lim_{q \searrow t} X_q \quad \text{and} \quad X_{t-} = \lim_{q \nearrow t} X_q$$

exist for every $t \in \mathbf{T}$ and thereby define two processes X_+ and X_-,respectively.

(b) The process X_+ is a $\mathbb{F}^+$ submartingale and is a martingale if X is. Analogously, the process X_- is a $\mathbb{F}^-$ submartingale and is a martingale if X is. Furthermore

$$X_t \geq E\left[X_{t+} \,|\, \mathcal{F}_t\right] \tag{4.3}$$

$$X_{t-} \geq E\left[X_t \,|\, \mathcal{F}_{t-}\right] \tag{4.4}$$

with equality in (4.3) if $t \mapsto E[X_t]$ is right-continuous and equality in (4.4) if $t \mapsto E[X_t]$ is left-continuous. In particular, equality holds in (4.3) and (4.4) if X is a martingale.

Proof. (a) Unlike in the previous proof we can only estimate the up-crossing of X over a countable bounded interval. Define

$$A = \bigcup_{n \in \mathbb{N}} \bigcup_{p<q, p,q \in \mathbb{Q}} \left\{\omega \in \Omega : U_{[0,n]\cap\mathbb{Q}}(p,q,X(\omega)) = \infty\right\}.$$

This set is of measure 0. Hence with the same argumentation as in the previous proof, it follows that

$$P\left[\liminf_{q \nearrow t, q\in\mathbb{Q}} X_q < \limsup_{q \nearrow t, q \in \mathbb{Q}} X_q : \text{for some } t \in \mathbf{T}\right] = 0,$$

$$P\left[\liminf_{q \searrow t, q\in\mathbb{Q}} X_q < \limsup_{q \searrow t, q \in \mathbb{Q}} X_q : \text{for some } t \in \mathbf{T}\right] = 0.$$

We can then define the processes X_- and X_+ by

$$X_{t+} = \lim_{q \searrow t} X_q, \text{ for } t < T \quad \text{and} \quad X_{t-} = \lim_{q \nearrow t} X_q, \text{ for } t > 0,$$

with the conventions that $X_{0-} = X_0$ and $X_{T+} = X_T$ if $T \in \mathbf{T}$.

(b) Clearly X_+ and X_- are $\mathbb{F}^+$- and $\mathbb{F}^-$-adapted processes, respectively. Let $(q_n) \subseteq \mathbb{Q}$ be a sequence decreasing to $t \in \mathbf{T}$. From the previous step, X_{q_n} converges P-almost surely to X_{t+}. Further, $E[X_t] \leq E[X_{q_n}] \leq E[X_{q_0}]$ for every n, so (X_{q_n}) is uniformly bounded in L^1, and $E[X_{q_n}]$ is a decreasing sequence converging to $\lim E[X_{q_n}] > E[X_t] > -\infty$. Hence, for $\lambda > 0$, and $\varepsilon > 0$, let n_0 be such that $E[X_{q_n}] \geq E[X_{q_{n_0}}] - \varepsilon$ for every $n \geq n_0$. As X is a submartingale, it follows that

$$\begin{aligned}
E\left[|X_{q_n}| : |X_{q_n}| > \lambda\right] &= E\left[X_{q_n} : X_{q_n} > \lambda\right] - E\left[X_{q_n} : X_{q_n} < -\lambda\right] \\
&= E\left[X_{q_n} : X_{q_n} > \lambda\right] - E\left[X_{q_n}\right] + E\left[X_{q_n} : X_{q_n} \geq -\lambda\right] \\
&\leq E\left[X_{q_{n_0}} : X_{q_n} > \lambda\right] + \varepsilon - E\left[X_{q_{n_0}}\right] + E\left[X_{q_{n_0}} : X_{q_n} \geq -\lambda\right] \\
&= E\left[X_{q_{n_0}} : X_{q_n} > \lambda\right] - E\left[X_{q_{n_0}} : X_{q_n} < -\lambda\right] + \varepsilon \\
&\leq E\left[\left|X_{q_{n_0}}\right| : |X_{q_n}| > \lambda\right] + \varepsilon.
\end{aligned}$$

By Markov's inequality, $P[|X_{q_n}| > \lambda] \leq \sup_n E[|X_{q_n}|]/\lambda = C/\lambda$ for $0 < C < \infty$, showing therefore that (X_{q_n}) is uniformly integrable. Together with the P-almost sure convergence, it follows that X_{q_n} converges in L^1 to X_{t+}. Thus X_{t+} is integrable and it holds

$$X_t \leq \lim E\left[X_{q_n} \,|\, \mathcal{F}_t\right] = E\left[X_{t+} \,\middle|\, \mathcal{F}_t\right].$$

Further, for $s < t$, and $q_n \searrow s$ with $q_n < t$, it holds

$$X_{q_n} \leq E\left[X_t \,|\, \mathcal{F}_{q_n}\right] \leq E\left[E\left[X_{t+} \,|\, \mathcal{F}_t\right] \,|\, \mathcal{F}_{q_n}\right] = E\left[X_{t+} \,|\, \mathcal{F}_{q_n}\right]$$

for every n. The same arguments as above show that $E\left[X_{t+} \,|\, \mathcal{F}_{q_n}\right]$ is uniformly integrable and converges P-almost surely and in L^1 and that the limit is $E[X_{t+} \,|\, \mathcal{F}_{s+}]$. Thus X_+ is a $\mathbb{F}^+$-submartingale. Finally, if $t \mapsto E[X_t]$ is right-continuous, it follows that $E[X_{t+}] = \lim E[X_{q_n}] = E[X_t]$. Hence, the positive random variable $X_t - E\left[X_{t+} \,|\, \mathcal{F}_t\right]$ has zero expectation and therefore is zero.

As for the case of X_-, the a similar argumentation holds using submartingale convergence theorem for the existence and integrability of X_{t-} and inequality (4.4). Furthermore, by $X_{s-} \leq E[X_s \,|\, \mathcal{F}_{s-}] \leq E[E[X_{t-} \,|\, \mathcal{F}_s] \,|\, \mathcal{F}_{s-}] = E[X_{t-} \,|\, \mathcal{F}_{s-}]$ it follows that X is a $\mathbb{F}^-$ submartingale. The equality in (4.4) if $t \mapsto E[X_t]$ is left-continuous follows by an analogous argumentation. $\square$

Theorem 4.22. *Let X be a supermartingale with respect to a filtration satisfying the usual assumptions. Suppose further that $t \mapsto E[X_t]$ is right-continuous. Then X has a càdlàg modification.*

Proof. According to the previous theorem, set $Y = X_+$ outside the negligible set A up to which X_+ and X_- are defined, and 0 on A. Since $A \in \mathcal{F}_0$, it follows that Y is càdlàg. Furthermore, from $t \mapsto E[X_t]$ right-continuous, by the previous theorem it holds $X_t = E[X_{t+} \,|\, \mathcal{F}_t] = E[Y_t \,|\, \mathcal{F}_t]$. However, since $\mathbb{F}$ is right-continuous, it follows that Y_t is $\mathcal{F}_t$-measurable and so $X_t = Y_t$ almost surely for every t. $\square$