

“STOCHASTIC PROCESSES” – HOMEWORK SHEET 4

Throughout, $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space where $\mathbf{T} = \mathbb{N}_0$.

Exercise 4.1. (10 points)

- (a) Let X and Y be two super-martingales. Show that $X \wedge Y$ is a super-martingale;
- (b) Let X be a predictable martingale, show that $X_t = X_0$ for every t .
- (c) Let $\xi \in L^1$, show that $X_t := E[\xi | \mathcal{F}_t]$ defines a martingale X ;
- (d) Show, or find a counter example for the following assertion: Let X be an adapted process such that X_t is integrable and $E[X_0] = E[X_t]$ for every t . Then X is a martingale.

Exercise 4.2. (5 points) Let $I \subseteq \mathbf{T}$ be such that $\sup I = \infty$. Show that every super-martingale X such that $E[X_0] \leq E[X_t]$ for every $t \in I$ is a martingale.

Exercise 4.3. (10 points) Let X be an adapted process such that $E[\sup_t |X_t|] < \infty$. Denote by \mathcal{T} the set of all stopping times and let $T \in \mathbb{N}_0$ be an arbitrary time horizon. We define recursively

$$S_T = X_T \quad \text{and} \quad S_t = \max \{E[S_{t+1} | \mathcal{F}_t]; X_t\}, \quad t \leq T-1.$$

Define further $\tau^t = \inf\{s : s \geq t \text{ and } S_s = X_s\}$ for every $t = 0, \dots, T$.

- (a) Show that $S = (S_t)_{t=0, \dots, T}$ is a super-martingale such that $S_t \geq X_t$ for every $t = 0, \dots, T$;
- (b) Let $U = (U_t)_{t=0, \dots, T}$ be a super-martingale such that $U_t \geq X_t$ for every $t = 0, \dots, T$. Show that $U_t \geq X_t$ for every $t = 0, \dots, T$;
- (c) Show that $E[X_{\tau^t} | \mathcal{F}_s] = E[S_t | \mathcal{F}_s]$ for every $s \leq t \leq T$ and conclude that

$$E[X_{\tau^t}] = E[S_t] = \max_{\{\tau \in \mathcal{T} : t \leq \tau \leq T\}} E[X_\tau]$$

- (d) We denote by $S^T = (S_t^T)_{t=0, \dots, T}$ the process defined in (a) whereby, we stress the dependence on the time horizon due to its recursive definition. Clearly, $S_t = \lim_{T \rightarrow \infty} S_t^T$ defines a process S . Show that S is a super-martingale for which holds $S \geq X$. Show further that for every other super-martingale U such that $U \geq X$ it follows $U \geq S$.

Exercise 4.4. (Bonus 10 points) Consider now our example of coin tossing but infinitely many times. As seen, the state space is defined as follows

$$\Omega = \prod_{t \in \mathbb{N}} \{-1, 1\} = \{-1, 1\}^{\mathbb{N}} = \{\omega = (\omega_t) : \omega_t = \pm 1 \text{ for every } t\}$$

On each $\Omega_t = \{-1, 1\}$ we consider the σ -algebra $\mathcal{F}_t = \{\emptyset, \{-1\}, \{1\}, \{-1, 1\}\}$ and on Ω the product σ -algebra $\mathcal{F} = \otimes \mathcal{F}_t$.

(a) Show that the collection \mathcal{R} of finite product cylinders

$$A = \{\omega = (\omega_t) \in \Omega : \omega_{t_k} = e_k, k = 1, \dots, n\} \quad (4.1)$$

for a given set of values $e_k \in \{-1, 1\}$, and times $t_k \in \mathbb{N}$, $k = 1, \dots, n$, together with the empty-set is a semi-ring.

(b) For $p \in [0, 1]$, we define $P : \mathcal{R} \rightarrow [0, 1]$ as follows

$$P[\emptyset] = 0 \quad \text{and} \quad P[A] = p^l (1 - p)^{n-l}$$

for every $A \in \mathcal{R}$ of the form (4.1) where l is equal to the number of those $k = 1, \dots, n$ where $e_k = 1$. Show that P defines a content.

(c) Show that for every finite family $(A_k)_{k \leq n}$ of elements in \mathcal{R} and $A \in \mathcal{R}$ such that $A \subseteq \cup_{k \leq n} A_k$, it holds¹

$$P[A] \leq \sum_{k \leq n} P[A_k].$$

(d) Show that for every countable family (A_n) of elements in \mathcal{R} and $A \in \mathcal{R}$ such that $A \subseteq \cup A_n$, it holds²

$$P[A] \leq \sum P[A_n],$$

and deduce using Caratheodory's theorem that P extends uniquely to a probability measure P on \mathcal{F} .

(e) Defining the process X by $X_t(\omega) = \omega_t$, describe the filtration generated by X .

Due date: Upload before Monday 2015.10.26 14:00.

¹Consider first the case where $n = 2$ and A_1, A_2 are one dimensional product cylinder.

²Show that there exists a finite n_0 such that $A \subseteq \cup_{k \leq n_0} A_k$.