From 3D to 2D

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1 Introduction

If you know what you are going to read then you can confidently skip this introduction. So, in this note we are going to study about how the 3D world is perceived in our brain, at least mathematically. Basically we will read about two things mainly- rotations of object in space and mapping of 3D point in 2D space. We need some prerequisites, like Euler formula and basics of linear algebra, to understand what will be going on down there.

After understanding how to bring down the 3D space into 2D space you will be able to render things from 3D to 2D space by just knowing the formula of respective shapes which you want to render.

2 Rotation

What does it mean to rotate something? That means that the object is going in some kind of curvy path rather than going in straight line, but when specifically said rotation it means that the distance from the point of pivot is constant and only the angle is changing with time. So, understand that mathematically we have to keep the distance from origin constant and change the slope of the line connecting and we have a perfect solution to that using complex numbers.

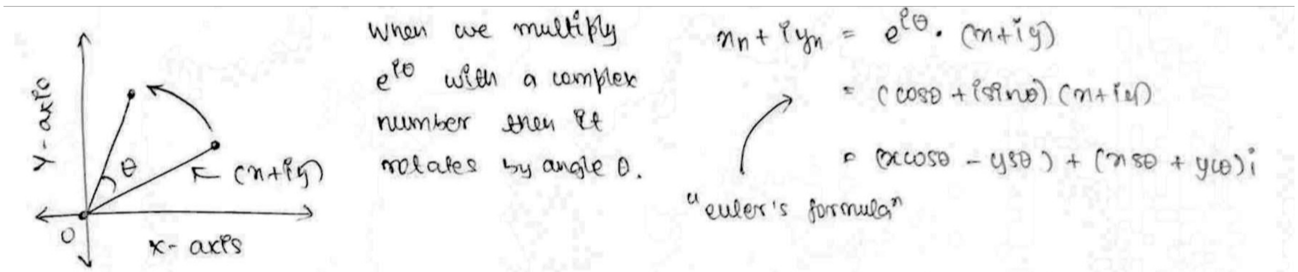

Figure 1: Rotation in complex plane

In the above [figure 1] I have written how we can use complex numbers concept to rotate a point about origin. When this process is done collectively to all the points in an object then the whole object is rotated with the specified degree which is θ here.

Now the same thing can be done with X-Z plane as well. For writing down the formulae in short we can make use of matrix as follows.

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

Now we have to write down this matrix with all the three axes, then we will the the matrix as following:

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}$$

As mentioned before we do the same thing with other plane and multiple both the matrices to get the combined rotation.

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} cos\phi & 0 & -sin\phi \\ 0 & 1 & 0 \\ sin\phi & 0 & cos\phi \end{bmatrix} \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}$$

So, this is the matrix form of rotating a point in 3D space, which can be applied to all the points of a shape to make the entire shape to be rotated in some orientation.

3 To capture the shape in 2D

Mapping from 3D to 2D is actually pretty easy. We just have to assume there is a screen in between you and the object and the 2D mapped point of the 3D point is just the intersection of the line joining your vision and the point itself. Refer to [Figure 2] for visual understanding.

Speaking in terms of maths we have to take some parameters and get a good formula on top of which we can work on. Let us assume that the distance of screen from the origin is $(d_s, 0, 0)$ and our eye's distance is at $(d_e, 0, 0)$, also assume that the screen is parallel to X-Y plane which will make our calculations much simpler. Now suppose we have to map the 2D value of a point say (x, y, z), then we have to join a line between the point (x, y, z) and $(d_e, 0, 0)$ and find the point of intersection. From the above data we get the equation of line and point of intersection as following:

$$\frac{X - x}{d_e - x} = \frac{Y - y}{-y} = \frac{Z - z}{-z}$$

We know the point of intersection will have the z-coordinate as d_s then we get the x and y coordinate of the 2D space as:

$$X = \frac{z - d_s}{z}(d_e - x) + x \quad Y = \frac{z - d_s}{z}(-y) + y$$

We can write down the formulae omitting the unwanted things as:

$$X = \frac{difference\ of\ x\ coordinate}{z} + k \quad Y = \frac{difference\ of\ y\ coordinate}{z} + k$$

We can infer from this formula why the objects at distinct are seen small compared to when they are closer, because they are inversely proportional to the z distance. The only remaining step is now to take samples of the object from different areas with equal distribution so that we get the entire image of the object in the 2D plane.

Figure 2: how we see the world