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# GRDzhadzha: A code for evolving relativistic matter on analytic metric backgrounds

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## Summary

Strong gravity environments such as those around black holes provide us with unique opportunities to study questions in fundamental physics (see e.g Barack et al., 2019; Barausse et al., 2015; Bertone et al., 2020; Macedo et al., 2013), such as the existence and properties of dark matter and dark energy. Characterising the behaviour of new fields and other types of matter in highly relativistic environments usually necessitates numerical simulations unless one imposes significant symmetries. Therefore we need to turn to numerical methods to study the dynamics and evolution of the complex systems of black holes and other compact objects in different environments, using numerical relativity (NR). These methods allow us to split the four-dimensional Einstein equations into three-dimensional spatial hypersurfaces and a time-like direction. Then if a solution is known at the initial spatial hypersurface, it can be numerically evolved in time, where an analytic solution no longer exists. Whilst the tools of NR provide the most complete (i.e., approximation free) method for evolving matter in such environments, in many cases of interest, the density of the matter components is negligible in comparison to the curvature scales of the background spacetime metric (Clough, 2021). In such cases it is a reasonable approximation to neglect the backreaction of the matter environment onto the metric and treat it as fixed (assuming the background itself is stationary or otherwise has an analytic form).

In such cases, one does not need to evolve all the metric degrees of freedom as in NR, but only the additional matter ones. It is possible to do this using any NR code in a trivial way by setting the evolution of the metric variables to zero, but this is clearly rather inefficient. This code, GRDzhadzha, directly evolves the matter variables on an analytically specified background. This significantly speeds up the computation time and reduces the resources needed (both in terms of CPU hours and storage) to perform a given simulation. The code is based on the publicly available NR code GRChombo (Andrade et al., 2021; Clough et al., 2015), which itself uses the open source Chombo framework (Adams et al., 2015) for solving PDEs.

In the following sections we discuss the key features and applications of the code, and give an indication of the efficiencies that can be achieved compared to a standard NR code.

## Key features of GRDzhadzha

GRDzhadzha inherits many of the features of GRChombo and Chombo, but avoids the complications introduced when evolving the metric. The key features are:

- Background metrics: The currently available backgrounds in the code are a static Kerr black hole in horizon-penetrating Kerr–Schild coordinates and a boosted black hole in isotropic Schwarzschild coordinates. As the code is templated over the background, it can easily be changed or adapted to other coordinate systems for different problems without major code modification.
- Matter evolution: The code calculates the evolution for the matter variables on the metric background using an ADM decomposition (Arnowitt et al., 2008; York, 1978) in space and time. Currently, we have implemented a real and a complex scalar field as examples of matter types. Again the code is templated over the matter class so that the matter types can be exchanged with minimal modification.
- Accuracy: The metric values and their derivatives are calculated exactly at each point, whereas the matter fields are evolved with a 4th order Runge–Kutta time integration and their derivatives calculated with the same finite difference stencils used in GRChombo (4th and 6th order are currently available).
- Boundary Conditions: GRDzhadzha inherits all the available boundary conditions in GRChombo, namely, extrapolating (extrapolating the field value radially from values within the numerical grid), Sommerfeld (radiative; Sommerfeld, 1912), reflective and periodic.
- Initial Conditions: The current examples provide initial data for real and complex scalar field matter. Since backreaction is ignored, there are no constraint equations to satisfy in the case of a scalar field, and the initial data can be freely specified.
- Diagnostics: GRDzhadzha has routines for verifying the conservation of matter energy densities, angular and linear momentum densities, and their fluxes, as discussed in (Clough, 2021; Croft, 2023).
- C++ class structure: Following the structure of GRChombo, GRDzhadzha is also written in C++ and uses object oriented programming (OOP) and templating.
- Parallelism: GRChombo uses hybrid OpenMP/MPI parallelism with explicit vectorisation of the evolution equations via intrinsics, and is AVX-512 compliant.
- Adaptive Mesh Refinement: The code inherits the flexible AMR grid structure of Chombo, which provides Berger–Oliger style (M. J. Berger & Oliger, 1984) AMR with block-structured Berger–Rigoutsos grid generation (M. Berger & Rigoutsos, 1991). Depending on the problem, the user may specify the refinement to be triggered by the matter or the background spacetime (Radia et al., 2022). One nice feature is that one does not need to resolve the horizon of the black hole unless matter is present at that location, so for an incoming wave a lot of storage and processing time can be saved by only resolving the wave, and not the spacetime background.

## Statement of Need

As mentioned in the introduction, any numerical relativity code like GRChombo can undertake these simulations. Examples of these include the Einstein Toolkit, with its related Cactus (Löffler et al., 2012; Schnetter et al., 2004), and Kranc (Husa et al., 2006) infrastructure used by LEAN (Sperhake, 2007; Zilhao et al., 2010) and Canuda (Witek et al., 2019). Other notable but non-public codes include BAM (Bruegmann et al., 2008; Marronetti et al., 2007), AMSS-NCKU (Galaviz et al., 2010), PAMR/AMRD and HAD (East et al., 2012; Neilsen et al., 2007). Codes such as SPeC (Pfeiffer et al., 2003) and bamps (Hilditch et al., 2016) implement the generalised harmonic formulation of the Einstein equations using a pseudospectral method, and discontinuous Galerkin methods are used in SpECTRE (Cao et al., 2018; Deppe et al., 2021; Kidder et al., 2017). NRPy (Ruchlin et al., 2018) is a code aimed for use on non-HPC systems, which generates C code from Python, and uses adapted coordinate systems to minimise computational costs. CosmoGRaPH (Mertens et al., 2016) and GRAMSES (Barrera-Hinojosa & Li, 2020) are among several NR codes targeted at cosmological applications (see Adamek et al., 2020 for a comparison) and which also employ particle methods. Simflowny (Palenzuela et al., 2018), like CosmoGRaPH, is based on the SAMRAI infrastructure, and has targeted fluid and MHD applications. GRAthena++ (Daszuta et al., 2021) makes use of oct-tree AMR to maximise scaling.

Whilst there exist many NR codes (both public and private), which can in principle be used to perform simulations of fundamental fields on a fixed BH background, most do not have the efficiency advantages of GRDzhadzha[1]. In particular, the fact that the ADM variables and their derivatives are not evolved or stored on the grid saves both a lot of simulation run time, as well as output file storage space. To get a rough idea of the improvement in storage and CPU hours one can achieve, we performed a short test simulation using GRDzhadzha and compared it to a simulation performed using the full NR capabilities of GRChombo. We find that on average GRDzhadzha is 15–20 times faster than GRChombo and requires about 3 times less file storage. An additional advantage of this code versus using a full NR code, for problems with negligible backreaction, is that here the metric variables are calculated analytically at every point on the grid, which significantly decreases the margin for numerical error, and means that resolution can be focussed on the matter location, and not the spacetime curvature.

It is important to note that whilst backreaction is neglected in the metric calculation, this does not mean that the backreaction effects cannot be calculated. Fixed background simulations provide a first order (in the density) estimate of the gravitational effects caused by the matter, taking into account their relativistic behaviour. This is discussed further in Clough (2021) and some examples using the approach are Bamber et al. (2021), Traykova et al. (2021) and Traykova et al. (2023).

Since the interface and structure of the code is very close to the GRChombo numerical relativity code, it is possible for the results of these fixed background simulations to be used as initial data in full numerical relativity simulations (and vice versa), as was done in Bamber et al. (2023). Therefore if the backreaction is found to be significant due to some growth mechanism, the simulation can be continued in full NR.

# Key research projects using GRDzhadzha

So far the code has been used to study a range of fundamental physics problems:

- Studying the interference patterns in neutrino flavour oscillations around a static black hole (Alexandre & Clough, 2018)
- Growth of scalar hair around a Schwarzschild (Clough et al., 2019) and a Kerr (Bamber et al., 2021) black hole
- Determining the relativistic drag forces on a Schwarzschild black hole moving through a cloud of scalar field dark matter (Traykova et al., 2021, 2023)

[1] As far as we are aware, only NRPy and Canuda offer the same functionality. Some private codes also have such capabilities (see e.g. Traykova et al., 2018; based on Braden et al., 2015). Other codes may have similar features that are not explicitly separated out, so this makes it difficult to identify them.

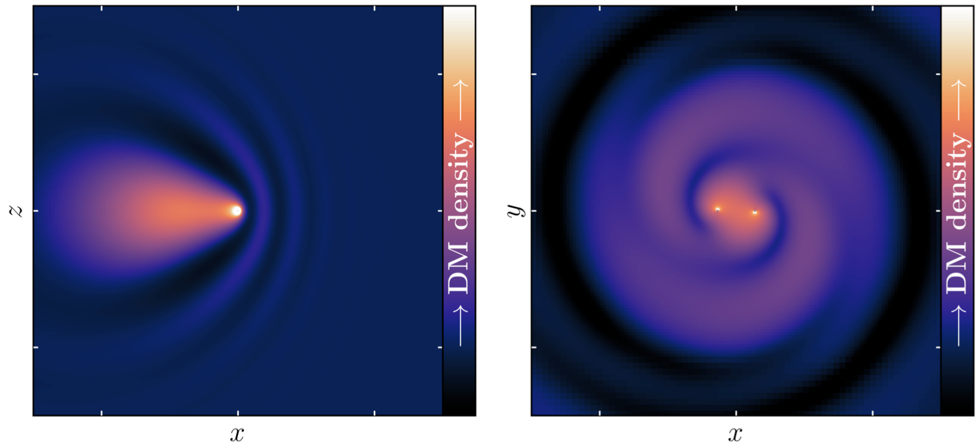


**Figure 1:** Examples of the physics studied with GRDzhadzha. The left image shows the formation of overdense tail of scalar dark matter behind a moving BH, due to dynamical friction, from Traykova et al. (2021). The right image is from a study of the scalar clouds around black hole binaries in Bamber et al. (2023), in which the initial conditions were generated with a modified version of GRDzhadzha.

- Studying the dynamical friction effects on a Kerr black hole (Magnus effect) (Wang et al., 2023)
- Superradiance with self-interacting vector field (Clough et al., 2022) and with spatially varying mass (Wang et al., 2022)
- BH mergers in wave dark matter environments (Bamber et al., 2023)


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# References


Adamek, J., Barrera-Hinojosa, C., Bruni, M., Li, B., Macpherson, H. J., & Mertens, J. B. (2020). Numerical solutions to Einstein's equations in a shearing-dust Universe: a code comparison. *Class. Quant. Grav.*, *37*(15), 154001. https://doi.org/10.1088/1361-6382/ab939b

Adams, M., Colella, P., Graves, D. T., Johnson, J. N., Keen, N. D., Ligocki, T. J., Martin, D. F., McCorquodale, P. W., Modiano, D., Schwartz, P. O., D., S. T., & Van Straalen, B. (2015). *Chombo software package for AMR applications - design document*. https://commons.lbl.gov/display/chombo

Alexandre, J., & Clough, K. (2018). Black hole interference patterns in flavor oscillations. *Phys. Rev. D*, *98*(4), 043004. https://doi.org/10.1103/PhysRevD.98.043004

Andrade, T., Salo, L., Aurrekoetxea, J., Bamber, J., Clough, K., Croft, R., de Jong, E., Drew, A., Duran, A., Ferreira, P., Figueras, P., Finkel, H., França, T., Ge, B.-X., Gu, C., Helfer, T., Jäykkä, J., Joana, C., Kunesch, M., … Wong, K. (2021). GRChombo: An adaptable numerical relativity code for fundamental physics. *The Journal of Open Source Software*, *6*(68), 3703. https://doi.org/10.21105/joss.03703

Arnowitt, R. L., Deser, S., & Misner, C. W. (2008). The Dynamics of general relativity. *Gen. Rel. Grav.*, *40*, 1997–2027. https://doi.org/10.1007/s10714-008-0661-1

Bamber, J., Aurrekoetxea, J. C., Clough, K., & Ferreira, P. G. (2023). Black hole merger simulations in wave dark matter environments. *Phys. Rev. D*, *107*(2), 024035. https://doi.org/10.1103/PhysRevD.107.024035

Bamber, J., Clough, K., Ferreira, P. G., Hui, L., & Lagos, M. (2021). Growth of accretion driven scalar hair around Kerr black holes. *Phys. Rev. D*, *103*(4), 044059. https://doi.org/10.1103/PhysRevD.103.044059

Barack, L., Cardoso, V., Nissanke, S., Sotiriou, T. P., Askar, A., Belczynski, C., Bertone, G., Bon, E., Blas, D., Brito, R., Bulik, T., Burrage, C., Byrnes, C. T., Caprini, C., Chernyakova, M., Chruściel, P., Colpi, M., Ferrari, V., Gaggero, D., … Zilhão, M. (2019). Black holes, gravitational waves and fundamental physics: a roadmap. *Classical and Quantum Gravity*, *36*(14), 143001. https://doi.org/10.1088/1361-6382/ab0587

Barausse, E., Cardoso, V., & Pani, P. (2015). Environmental Effects for Gravitational-wave Astrophysics. *J. Phys. Conf. Ser.*, *610*(1), 012044. https://doi.org/10.1088/1742-6596/610/1/012044

Barrera-Hinojosa, C., & Li, B. (2020). GRAMSES: a new route to general relativistic $N$-body simulations in cosmology. Part I. Methodology and code description. *JCAP*, *01*, 007. https://doi.org/10.1088/1475-7516/2020/01/007

Berger, M. J., & Oliger, J. (1984). Adaptive Mesh Refinement for Hyperbolic Partial Differential Equations. *J. Comput. Phys.*, *53*, 484. https://doi.org/10.1016/0021-9991(84)90073-1

Berger, M., & Rigoutsos, I. (1991). An algorithm for point clustering and grid generation. *IEEE Transactions on Systems, Man, and Cybernetics*, *21*(5), 1278–1286. https://doi.org/10.1109/21.120081

Bertone, G., Croon, D., Amin, M., Boddy, K. K., Kavanagh, B., Mack, K. J., Natarajan, P., Opferkuch, T., Schutz, K., Takhistov, V., Weniger, C., & Yu, T.-T. (2020). Gravitational wave probes of dark matter: challenges and opportunities. *SciPost Physics Core*, *3*(2), 007. https://doi.org/10.21468/SciPostPhysCore.3.2.007

Braden, J., Bond, J. R., & Mersini-Houghton, L. (2015). Cosmic bubble and domain wall instabilities I: parametric amplification of linear fluctuations. *JCAP*, *03*, 007. https://doi.org/10.1088/1475-7516/2015/03/007

Bruegmann, B., Gonzalez, J. A., Hannam, M., Husa, S., Sperhake, U., & Tichy, W. (2008). Calibration of Moving Puncture Simulations. *Phys. Rev. D*, *77*, 024027. https://doi.org/10.1103/PhysRevD.77.024027

Cao, Z., Fu, P., Ji, L.-W., & Xia, Y. (2018). Application of local discontinuous Galerkin method to Einstein equations. *Int. J. Mod. Phys. D*, *28*(01), 1950014. https://doi.org/10.1142/S0218271819500147

Clough, K. (2021). Continuity equations for general matter: applications in numerical relativity. *Class. Quant. Grav.*, *38*(16), 167001. https://doi.org/10.1088/1361-6382/ac10ee

Clough, K., Ferreira, P. G., & Lagos, M. (2019). Growth of massive scalar hair around a Schwarzschild black hole. *Phys. Rev. D*, *100*(6), 063014. https://doi.org/10.1103/PhysRevD.100.063014

Clough, K., Figueras, P., Finkel, H., Kunesch, M., Lim, E. A., & Tunyasuvunakool, S. (2015). GRChombo : Numerical Relativity with Adaptive Mesh Refinement. *Class. Quant. Grav.*, *32*(24), 245011. https://doi.org/10.1088/0264-9381/32/24/245011

Clough, K., Helfer, T., Witek, H., & Berti, E. (2022). Ghost Instabilities in Self-Interacting Vector Fields: The Problem with Proca Fields. *Phys. Rev. Lett.*, *129*(15), 151102. https://doi.org/10.1103/PhysRevLett.129.151102

Croft, R. (2023). Local continuity of angular momentum and noether charge for matter in general relativity. *Class. Quant. Grav.*, *40*(10), 105007. https://doi.org/10.1088/1361-6382/accc6a

Daszuta, B., Zappa, F., Cook, W., Radice, D., Bernuzzi, S., & Morozova, V. (2021). GR-Athena++: Puncture Evolutions on Vertex-centered Oct-tree Adaptive Mesh Refinement. *Astrophys. J. Supp.*, *257*(2), 25. https://doi.org/10.3847/1538-4365/ac157b

Deppe, N., Throwe, W., Kidder, L. E., Fischer, N. L., Armaza, C., Bonilla, G. S., Hébert, F., Kumar, P., Lovelace, G., Moxon, J., O'Shea, E., Pfeiffer, H. P., Scheel, M. A., Teukolsky, S. A., Anantpurkar, I., Boyle, M., Foucart, F., Giesler, M., Iozzo, D. A. B., … Wlodarczyk, T. (2021). *SpECTRE* (Version 2021.05.03). Zenodo. https://doi.org/10.5281/zenodo.4734670

East, W. E., Pretorius, F., & Stephens, B. C. (2012). Hydrodynamics in full general relativity with conservative AMR. *Phys. Rev. D*, *85*, 124010. https://doi.org/10.1103/PhysRevD.85.124010

Galaviz, P., Bruegmann, B., & Cao, Z. (2010). Numerical evolution of multiple black holes with accurate initial data. *Phys. Rev. D*, *82*, 024005. https://doi.org/10.1103/PhysRevD.82.024005

Hilditch, D., Weyhausen, A., & Brügmann, B. (2016). Pseudospectral method for gravitational wave collapse. *Phys. Rev. D*, *93*(6), 063006. https://doi.org/10.1103/PhysRevD.93.063006

Husa, S., Hinder, I., & Lechner, C. (2006). Kranc: A Mathematica application to generate numerical codes for tensorial evolution equations. *Comput. Phys. Commun.*, *174*, 983–1004. https://doi.org/10.1016/j.cpc.2006.02.002

Kidder, L. E., Field, S. E., Foucart, F., Schnetter, E., Teukolsky, S. A., Bohn, A., Deppe, N., Diener, P., Hébert, F., Lippuner, J., Miller, J., Ott, C. D., Scheel, M. A., & Vincent, T. (2017). SpECTRE: A task-based discontinuous Galerkin code for relativistic astrophysics. *Journal of Computational Physics*, *335*, 84–114. https://doi.org/10.1016/j.jcp.2016.12.059

Löffler, F., Faber, J., Bentivegna, E., Bode, T., Diener, P., Haas, R., Hinder, I., Mundim, B. C., Ott, C. D., Schnetter, E., Allen, G., Campanelli, M., & Laguna, P. (2012). The Einstein Toolkit: a community computational infrastructure for relativistic astrophysics. *Classical and Quantum Gravity*, *29*(11), 115001. https://doi.org/10.1088/0264-9381/29/11/115001

Macedo, C. F. B., Pani, P., Cardoso, V., & Crispino, L. C. B. (2013). Into the lair: gravitational-wave signatures of dark matter. *Astrophys. J.*, *774*, 48. https://doi.org/10.1088/0004-637X/774/1/48

Marronetti, P., Tichy, W., Bruegmann, B., Gonzalez, J., Hannam, M., Husa, S., & Sperhake, U. (2007). Binary black holes on a budget: Simulations using workstations. *Class. Quant. Grav.*, *24*, S43–S58. https://doi.org/10.1088/0264-9381/24/12/S05

Mertens, J. B., Giblin, J. T., & Starkman, G. D. (2016). Integration of inhomogeneous cosmological spacetimes in the BSSN formalism. *Phys. Rev. D*, *93*(12), 124059. https://doi.org/10.1103/PhysRevD.93.124059

Neilsen, D., Hirschmann, E. W., Anderson, M., & Liebling, S. L. (2007). Adaptive Mesh Refinement and Relativistic MHD. *11th Marcel Grossmann Meeting on General Relativity*, 1579–1581. https://doi.org/10.1142/9789812834300_0200

Palenzuela, C., Miñano, B., Viganò, D., Arbona, A., Bona-Casas, C., Rigo, A., Bezares, M., Bona, C., & Massó, J. (2018). A Simflowny-based finite-difference code for high-performance computing in numerical relativity. *Class. Quant. Grav.*, *35*(18), 185007. https://doi.org/10.1088/1361-6382/aad7f6

Pfeiffer, H. P., Kidder, L. E., Scheel, M. A., & Teukolsky, S. A. (2003). A Multidomain spectral method for solving elliptic equations. *Comput. Phys. Commun.*, *152*, 253–273. https://doi.org/10.1016/S0010-4655(02)00847-0

Radia, M., Sperhake, U., Drew, A., Clough, K., Figueras, P., Lim, E. A., Ripley, J. L., Aurrekoetxea, J. C., França, T., & Helfer, T. (2022). Lessons for adaptive mesh refinement in numerical relativity. *Class. Quant. Grav.*, *39*(13), 135006. https://doi.org/10.1088/1361-6382/ac6fa9

Ruchlin, I., Etienne, Z. B., & Baumgarte, T. W. (2018). SENR/NRPy+: Numerical Relativity in Singular Curvilinear Coordinate Systems. *Phys. Rev. D*, *97*(6), 064036. https://doi.org/10.1103/PhysRevD.97.064036

Schnetter, E., Hawley, S. H., & Hawke, I. (2004). Evolutions in 3-D numerical relativity using fixed mesh refinement. *Class. Quant. Grav.*, *21*, 1465–1488. https://doi.org/10.1088/0264-9381/21/6/014

Sommerfeld, A. (1912). Die greensche funktion der schwingungslgleichung. *Jahresbericht Der Deutschen Mathematiker-Vereinigung*, *21*, 309–352. http://eudml.org/doc/145344

Sperhake, U. (2007). Binary black-hole evolutions of excision and puncture data. *Phys. Rev. D*, *76*, 104015. https://doi.org/10.1103/PhysRevD.76.104015

Traykova, D., Braden, J., & Peiris, H. V. (2018). Accretion of a Symmetry Breaking Scalar Field by a Schwarzschild Black Hole. *Phil. Trans. Roy. Soc. Lond. A*, *376*(2114), 20170122. https://doi.org/10.1098/rsta.2017.0122

Traykova, D., Clough, K., Helfer, T., Berti, E., Ferreira, P. G., & Hui, L. (2021). Dynamical friction from scalar dark matter in the relativistic regime. *Phys. Rev. D*, *104*(10), 103014. https://doi.org/10.1103/PhysRevD.104.103014

Traykova, D., Vicente, R., Clough, K., Helfer, T., Berti, E., Ferreira, P. G., & Hui, L. (2023). *Relativistic drag forces on black holes from scalar dark matter clouds of all sizes*. https://arxiv.org/abs/2305.10492

Wang, Z., Helfer, T., Clough, K., & Berti, E. (2022). Superradiance in massive vector fields with spatially varying mass. *Phys. Rev. D*, *105*(10), 104055. https://doi.org/10.1103/PhysRevD.105.104055

Wang, Z., Helfer, T., Traykova, D., Clough, K., & Berti, E. (2023). The Magnus Effect on black holes in scalar dark matter environments. *In Preparation*.

Witek, H., Gualtieri, L., Pani, P., & Sotiriou, T. P. (2019). Black holes and binary mergers in scalar Gauss-Bonnet gravity: scalar field dynamics. *Phys. Rev. D*, *99*(6), 064035. https://doi.org/10.1103/PhysRevD.99.064035

York, J. W., Jr. (1978). Kinematics and Dynamics of General Relativity. *Workshop on Sources of Gravitational Radiation*, 83–126.

Zilhao, M., Witek, H., Sperhake, U., Cardoso, V., Gualtieri, L., Herdeiro, C., & Nerozzi, A. (2010). Numerical relativity for D dimensional axially symmetric space-times: formalism and code tests. *Phys. Rev. D*, *81*, 084052. https://doi.org/10.1103/PhysRevD.81.084052