ASSIGNMENT 1: 1D HEAT CONDUCTION PROBLEM.

Turbulence: Phenomenology, Simulation and Aerodynamics

Submitted by-

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Problem Specification:

In this analysis, we have examined the temperature trend of a plane wall by implementing a simulation of a steady, 1D conduction heat transfer by using numerical methods. However, we also calculated the transient case of the system for a specific time interval to see the change of the system. We have assumed that the temperature of the left and right wall is kept constant at 350K and 300K and the thermal diffusivity of the material is kept constant at 20 m^2/s. The cylindrical wall has an internal energy source of 50 W/m^3. After that, the results of the numerical solution have been compared with the analytical (exact) solution to understand the accuracy of the numerical method as well as to verify it in the same process. Also, there were several given data and plots at specific time which were generated, and we compared the values of the reference values with our solution.

Figure 1: Plane wall Heat Conduction.

The input data chosen for the analysis are given below:

Physical Data	
Wall Thickness (L)	5 (m)
Left Wall Temperature (Tleft)	350 (K)
Right Wall Temperature (Tright)	300 (K)
Thermal Diffusivity (alpha)	20 (m^2/s)
Internal Heat generation (Qt)	50 (W/K)
Transient Analytical Coefficient (A)	50
Numerical Data	
Number of Grid Points (N)	16
Iteration limit	10^6
Tolerance	1^{-6}

Analytical Solution Procedure (Steady Case):

An example of plane wall in a steady state 1D condition was given to us, and the heat conduction equation for a plane wall in 1D steady state is:

$$\left(\alpha\frac{\partial^2 T}{\partial x^2}\right) - Q_t T = 0;\ \frac{\partial T}{\partial t} = 0\ (Steady\ State)$$

The exact solution of this equation is given below:

$$T(x) = C_1 \mathrm{e}^{\mathrm{kx}} + C_2 \mathrm{e}^{-\mathrm{kx}}$$

Where,

$$k = \sqrt{\frac{Q_t}{\alpha}}$$

This is the analytical equation of the temperature distribution for a place wall in a steady state 1D case.

Using the boundary condition using the imposed left and right temperature of the wall boundary, the values of the coefficients C1 and C2 were calculated as given below:

$$C1 = \frac{\left(T_{right-}T_{left} * \exp(kL)\right)}{\exp(-kL) - \exp(kL)}$$

$$C2 = T_{left} - C1$$

Analytical Solution Procedure (Transient Case):

For the transient case of the same plane wall with the previous characteristics, the equation of the heat conduction is as follows,

$$\left(\alpha\frac{\partial^2 T}{\partial x^2}\right) - Q_t T = \frac{\partial T}{\partial t}\ (Transient\ State)$$

The exact solution of this equation is given below:

$$T(x) = C_1 \mathrm{e}^{\mathrm{kx}} + C_2 \mathrm{e}^{-\mathrm{kx}} + A\ exp(-\mathrm{\alpha\beta}^2 - Q_t)\mathrm{sin}\ (\mathrm{\beta}x)$$

Where,

$$k = \sqrt{\frac{Q_t}{\alpha}}\ ;$$

$$A = 50;$$

$$\beta = \frac{2\pi n}{L} = 12.56637062; (for\ n = 3)$$

Numerical Solution Algorithm:

The heat conduction for a plane wall in 1D problem, we can write the equation like this,

$$\left(\alpha\frac{\partial^2 T}{\partial x^2}\right) - Q_t T = \frac{\partial T}{\partial t}\ (Transient\ State)$$

$$\left(\alpha\frac{\partial^2 T}{\partial x^2}\right) - Q_t T = 0\ (Steady\ State)$$

Or,

$$(\lambda\nabla^2 T) - Q_t T = \frac{\partial T}{\partial t}\ (Transient\ State)$$

$$(\alpha\nabla^2 T) - Q_t T = 0\ (Steady\ State)$$

For solving this equation using Finite volume Method, we applied the Gauss-Ostrogradsky therorem, which is as follows,

$$\int \nabla.\vec{F}dV = \int \vec{F}.\vec{n}\,dS$$

The $\vec{F}.\vec{n}$ term signifies the flux due to force F crossing the boundary of a system. If this value is positive, it means that the flux is exiting the boundary.

As this method is applicable for any kind of vector quantity, we can convert the term $\nabla^2 T$ using the Gauss-Ostrogradsky theorem.

a) **Steady-State:**

For calculating the steady state condition, we needed to apply the Gauss-Ostrogradsky method to the $\boldsymbol{\nabla^2 T}$ term of the steady state equation above, which resulted in the following equation,

$$\int \alpha\nabla^2 T dV = \int \alpha\nabla T dS = \left(\int \alpha\frac{\partial T}{\partial x}\,dS\right)_E - \left(\int \alpha\frac{\partial T}{\partial x}\,dS\right)_W\ (1D\ case)$$

Which clearly indicates that the heat fluxes that are crossing the boundary of our system are summed together.

Taking into consideration of the control volume with P node like Figure 2, we applied energy balance,

$$-\left[\alpha\frac{dT}{dx}\right]_w + \left[\alpha\frac{dT}{dx}\right]_e + Q_t T_P \Delta x = 0$$

So far, the equation shown above should give the exact solution of the energy balance. However, we are going to discretize it using piecewise-linear profile, which would approximate the solution. The resulting equation would look like the following,

$$-\frac{\alpha(T_P - T_W)}{d_{xW}} + \frac{\alpha(T_E - T_P)}{d_{xE}} - Q_t T_P \Delta x = 0$$

After organizing this equation, the new equation we get can be written like this,

$$a_P T_P + a_E T_E + a_W T_W = b$$
$$a_P T_P = b - a_E T_E - a_W T_W$$

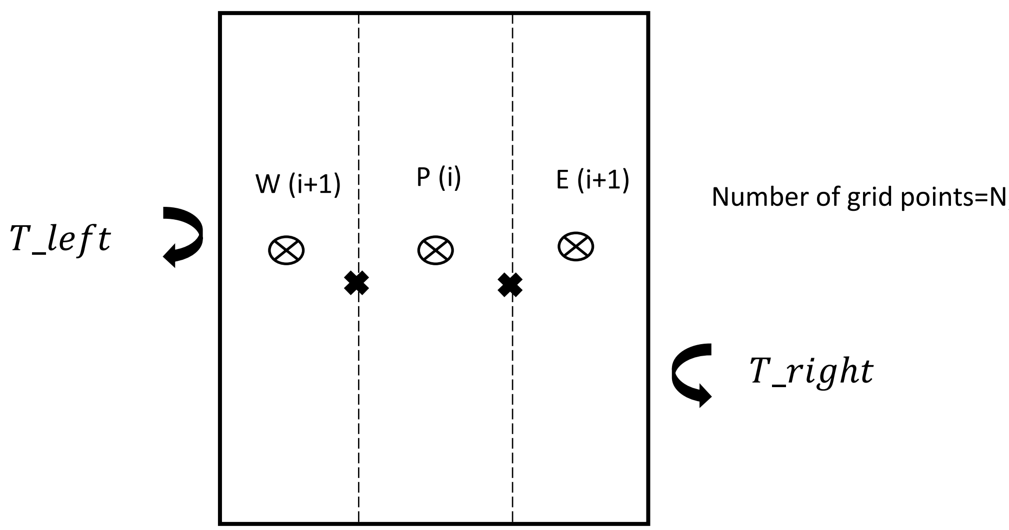

Figure 2: The discretization Process.

Where,

$$a_w = \frac{\alpha}{d_{xw}};\ a_E = \frac{\alpha}{d_{xE}};\ a_p = -(a_w + a_E + Q_t \Delta x);\ b = 0$$

This coefficient values are valid for the central nodes. However, for the boundary nodes, the temperature values were imposed and kept constant at that specific value. The coefficients for the boundary nodes were assigned accordingly.

Now that we know the relationships between the discretization coefficients and temperatures of each node, we can use a solver to solve the set of equations.

Steps to solve the set of equations for steady state problem:

1. We evaluated the values of ae, aw, ap and bp for each node of i=1 to N+2.
2. For N+2 nodes, we needed to set initial temperature values for each node for the iteration.

3. Using Gauss-Seidal Method, we used the initial temperature values to initialize the Gauss-Seidal solver. We selected a tolerance; thus, the solver would stop when the error of the calculated updated temperature with respect to the previous initial temperature is less than the tolerance.
4. In our case, the Gauss-Seidal method was introduced in the code structure. The code structure of Gauss-Seidal depends on the lower diagonal of iteration matrix, which translates to using the latest iterated value for each linear equation solving process even before the loop ends.
5. However, as per instruction, the code structure was changed to Jacobi method, which always uses the previous iteration values before the loop ends for all the linear equations.
6. If the calculated residual is less than the tolerance, the solution has converged, and the temperature attained from the iteration is the final temperature. However, if the calculated residual is higher than the tolerance, the calculated temperature is set as an initial temperature for the next iteration and the iteration continues till the solution has been reached.

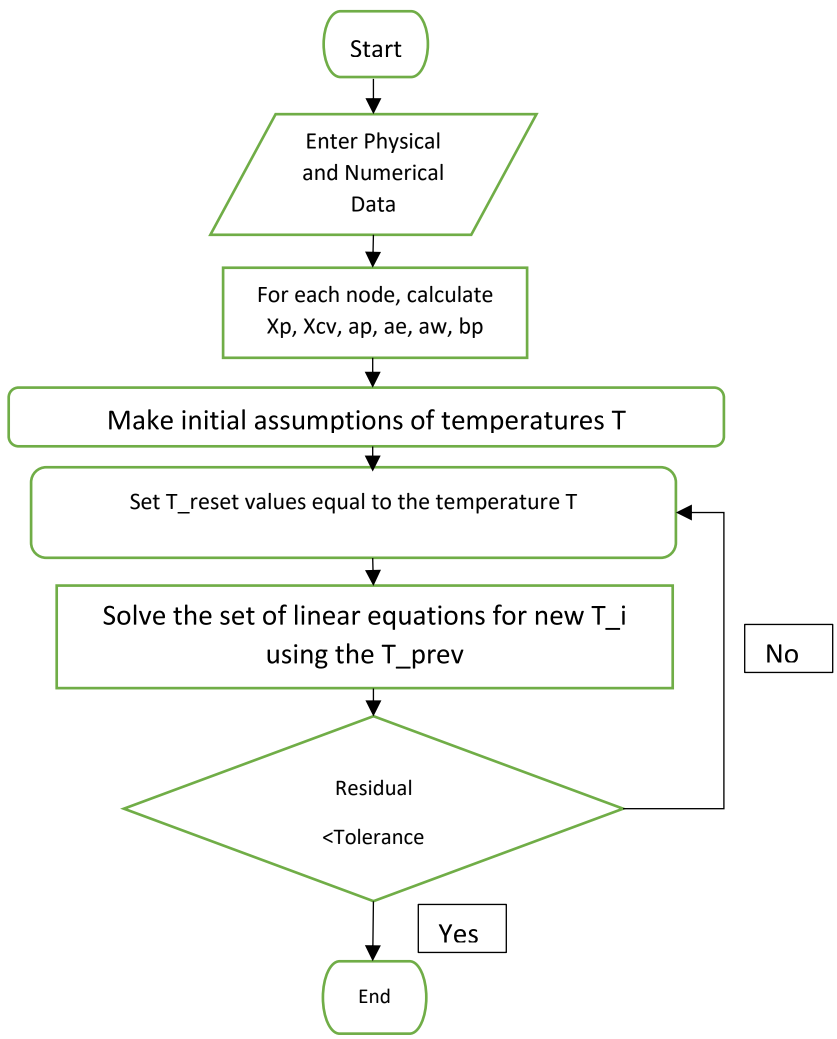

Figure 3: Flow chart of the code structure.

b) **Transient-State:**

Since we now have the steady state calculation, we can move forward to solve the transient state 1D conduction problem.

The equation of the transient state for unit area is,

$$\frac{\left(T_P{}^{n+1} - T_P{}^{n}\right)}{\Delta t}\Delta x = -\frac{\alpha(T_P{}^{n} - T_W{}^{n})}{d_{xW}} + \frac{\alpha(T_E{}^{n} - T_P{}^{n})}{d_{xE}} - Q_t T_P \Delta x$$

Which can be written as,

$$T_P{}^{n+1} = T_P{}^{n} + \frac{\Delta t}{\Delta x}\left[-\frac{\alpha(T_P{}^{n} - T_W{}^{n})}{d_{xW}} + \frac{\alpha(T_E{}^{n} - T_P{}^{n})}{d_{xE}} - Q_t T_P \Delta x\right]$$

Where, the n signifies the previous time-step values while the n+1 signifies the next iterated time-step values. Here, we can clearly see that the time integration scheme only depends on previous time step values of temperature. Thus, this equation is an Explicit scheme equation which can be written like,

$$\frac{\phi^{n+1} - \phi^{n}}{\Delta t} = f(\phi^{n}, \phi^{n-1}, \phi^{n-2}, \dots \dots)$$

However, an explicit simulation can blow up if certain conditions are not met during the simulation.

Stability Criterion (CFL Condition):

For a differential equation,

$$\frac{d\phi}{dt} = c\,\phi$$

The solution would be as following,

$$\phi(t) = \phi(0)\exp{(ct)}$$

Here, c must be a negative value. The first equation can be written as,

$$\frac{\phi^{n+1} - \phi^{n}}{\Delta t} = c\,\phi^{n}\ (First\ order\ Euler\ explicit\ scheme)$$

$$\phi^{n+1} = (1 + c\Delta t)\,\phi^{n}$$

$$\phi^{n+1} = a\,\phi^{n}$$

Where,

$$a = (1 + c\Delta t)$$

For a stable simulation, the absolute value of a must be less than 1. Using calculations, the required time step for stable simulation was maintained using the following relationship,

$$\Delta t \leq min(0.25\ \frac{\Delta x^2}{\alpha})$$

Results:

Numerical vs Analytical Approach (Steady case):

For the numerical and analytical solution, the number of grid points that we selected was 16. After that, we implemented the Jacobi method using the algorithm which was presented in the previous section.

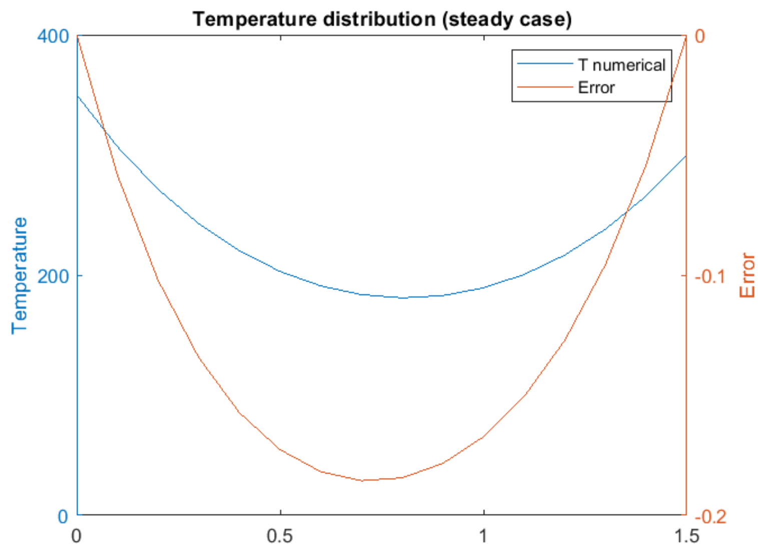

Figure 4: Numerical Solution and Error from Analytical Solution (steady).

Similarly, for the same positions of the grid points, we calculated the analytical solution for each point using the analytical solution equation. Later we calculated the error of the numerical solution with respect to the analytical solution (exact solution) result of the case to check the error for each grid points.

We noticed that the numerical solution only has zero error at the boundary values of the nodes. This is because the boundary temperatures were imposed in both analytical and numerical solutions. However, the further the node is from the boundaries, the higher the error becomes and at one point it reaches a maximum error. For this case of 16 grid points, the maximum absolute error was within a range of 0 to 0.2 value.

Grid convergence method (Steady case):

For the grid convergence study, we analyzed our case for 4 different scenarios with different number of grid points such as 4, 8, 12 and 16 grid points. Using different number of grid points for each study, we found a consistent trend of the truncation error for them.

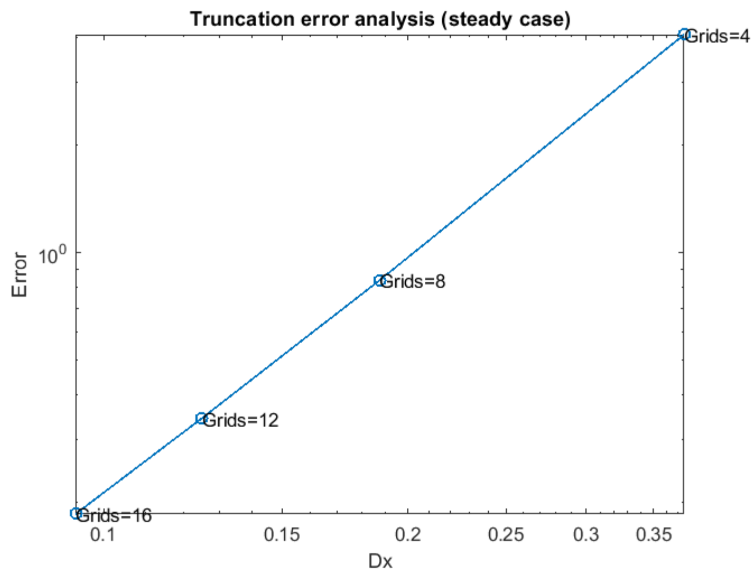

Figure 5: Truncation Error for different number of Grid points.

We can see from the Figure 5 that while we are increasing the number of grid points for each case, the truncation error is decreasing. This is what we expected since the truncation error should decrease when we refine our mesh to have finer mesh arrangement. Thus, our algorithm is consistent with the theoretical meshing properties.

Numerical vs Analytical Approach (Transient case):

For the numerical and analytical solution of the transient case, we tried to see the transition of the values for both solutions for specific time steps. Since the analytical solution is an exact solution, we would like to take it as a reference to calculate the accuracy of our numerical process.

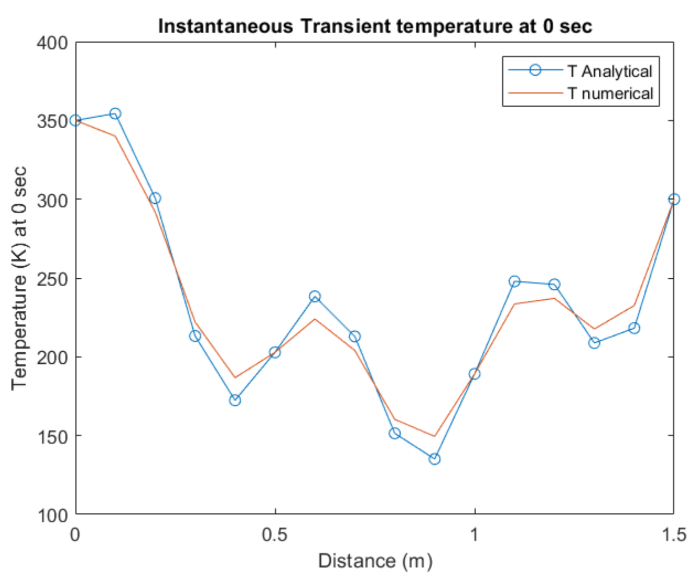

Figure 6: Temperature distribution at t=0s.

The visualization process of the transition towards steady state helped us realize that for the initial time step, the error was vastly significant among the nodal points compared to the analytical solution.

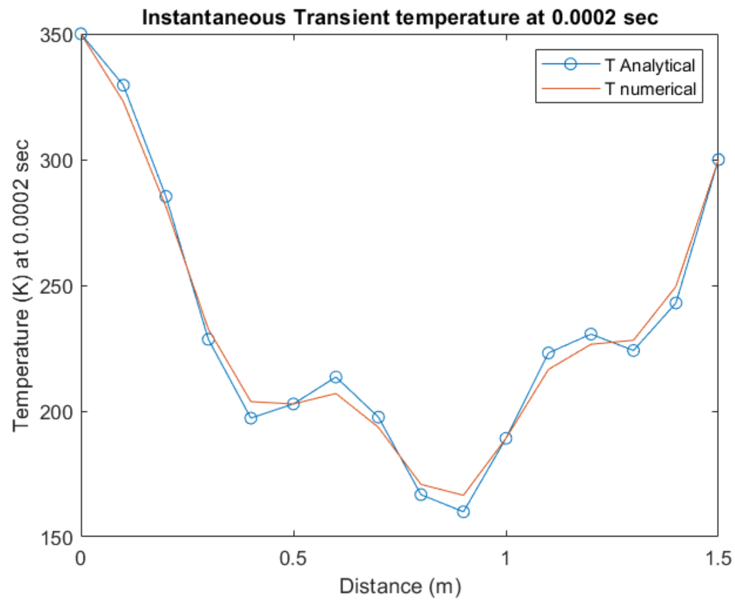

Figure 7: Temperature distribution at t=0.0002s.

Thus, the real transition visualization was significantly different compared to the exact solution of the process. For example, Figure 6 and Figure 7 represents the transient condition of the temperature distribution for initial time. As we can see, the temperature pattern far from being similar compared to the analytical solution.

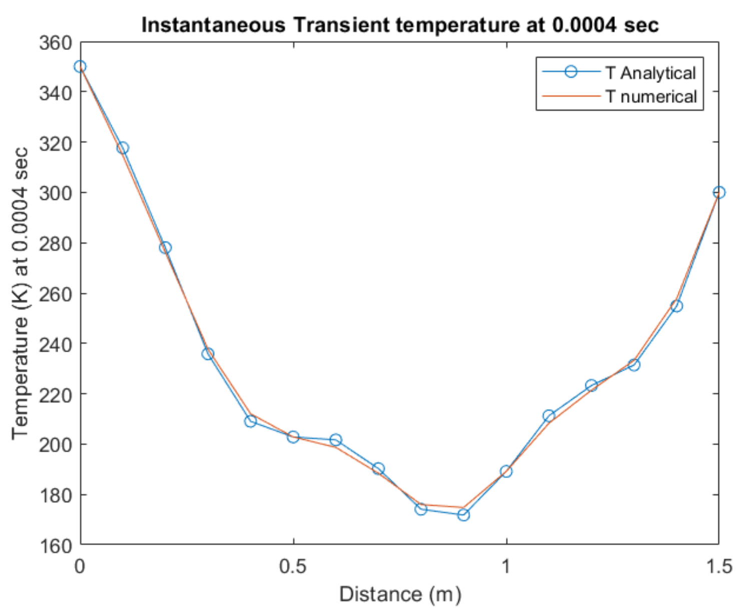

Figure 8: Temperature distribution at t=0.0004s.

However, as the time progresses, the analytical and the numerical solutions starts to converge slowly as both curves gets closer to each other gradually, which can be notices in Figure 8 and 9.

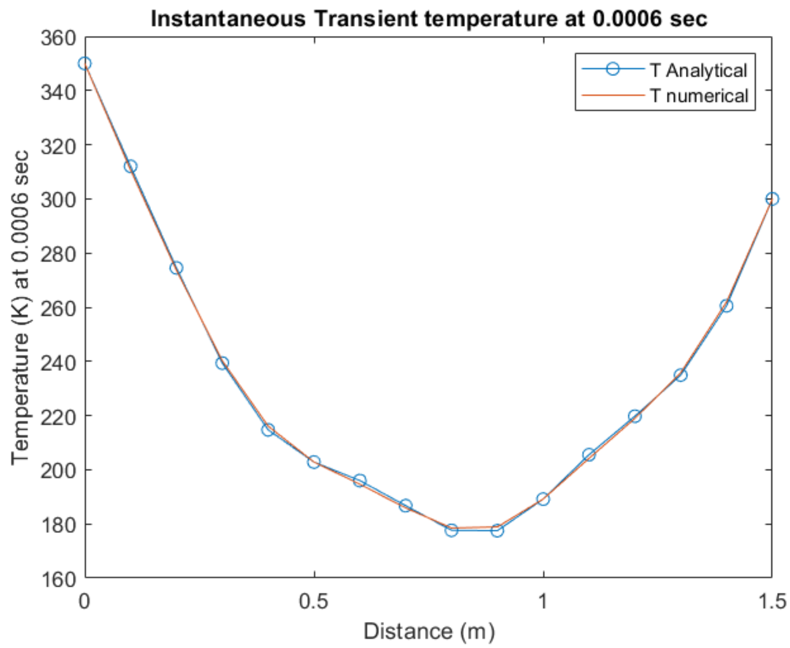

Figure 9: Temperature distribution at t=0.0006s.

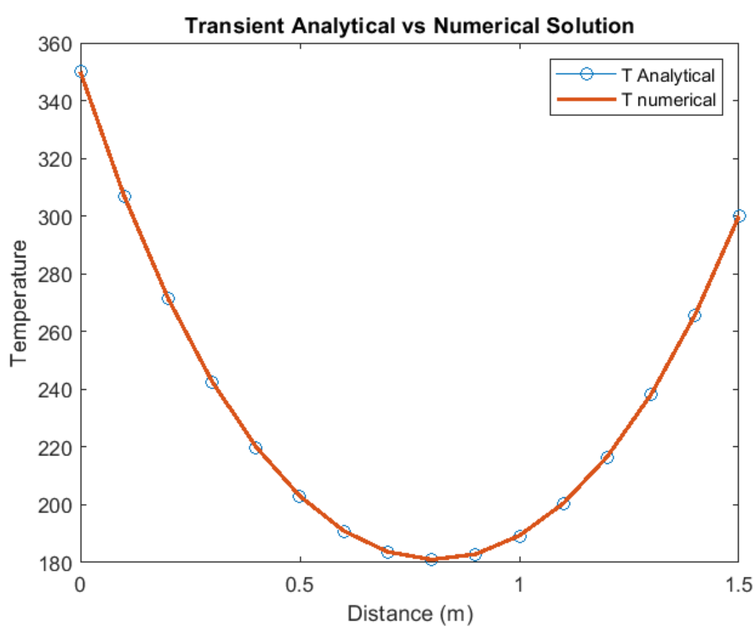

Figure 10: Temperature distribution at end of transition period.

Finally, after some time the solution reached a convergence with respect to the residual limit which was set in the code structure. From Figure 10, it can be said that the numerical and the analytical solution are almost like each other and the temperature distribution of the system after reaching a steady state.

However, the values are not exactly same, since there must be some truncation error due to the discretization, which can be seen from Figure 11 below. It is evident that the final stage of the transient process is almost like the steady state, where the error ranges from an absolute value of 0 to 0.2. Another reason for this error can be explained by the maximum residual error accepted during the process. Since the final steady state result is related to the limit set for the residual, there is another error introduced

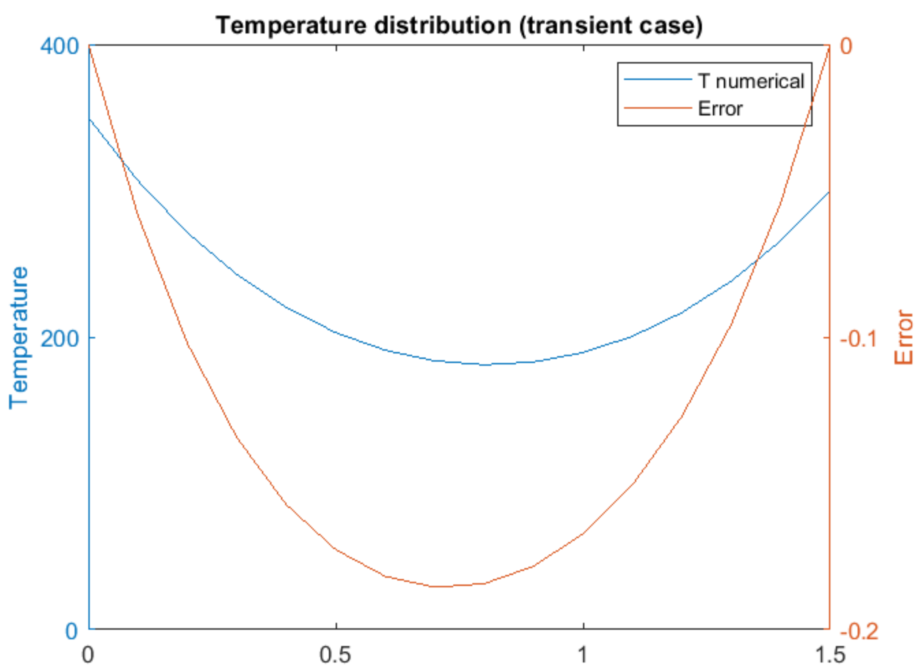

Figure 11: Numerical Solution and Error comparing Analytical (reaching steady state).

Discussion:

Overall, from the calculation, it was evident that the transient numerical solution might not represent the proper accurate graphical output, but the trend of the numerical solution follows the analytical solution quite well. In terms of the numerical solution, the greater number of grid point is taken, the better the solution converges and matches with the analytical solution, as seen during the truncation error analysis. After the transient solution reaches steady state, it is identical to the initial steady state result as well. Similarly, the error between the numerical and the analytical temperature trend is quite negligible.