ASSIGNMENT 1: 1D STEADY CONDUCTION HEAT TRANSFER CASE IN CYLINDRICAL WALLS.

Computational Methods in Heat and Mass Transfer

Submitted by-

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Problem Specification:

In this analysis, we have examined the temperature trend of a hollow cylinder of Carbon steel with an internal radius Rint, external radius Rext and height H by implementing a simulation of a steady, 1D conduction heat transfer by using numerical methods. We have assumed that the internal wall is exchanging energy by convection with boiling water at 100°C temperature and heat transfer coefficient of $2000 W/m^2K$ and

Figure 1: Hollow Cylinder Heat Conduction.

externally with air is at 25°C with heat transfer coefficient of $10W/m^2K$. The cylindrical wall has an internal energy source of $10W/m^3$. After that, the results of the numerical solution have been compared with the analytical (exact) solution to understand the accuracy of the numerical method as well as to verify it in the same process.

The Input Data chosen for the analysis are given below:

Physical Data	
Internal Radius (Rint)	5 (m)
Outer Radius (Rext)	10 (m)
Height (H)	1 (m)
Inner fluid Temperature (Ta)	100 (°C)
Outer Fluid Temperature (Tb)	25 (°C)
Inner fluid heat transfer coefficient (alpha_A)	2000 (W/m^2K)
Outer fluid heat transfer coefficient (alpha_B)	10 (W/m^2K)
Internal Heat generation (Qv)	10 (W/m^3)
Thermal Conductivity of carbon steel (λ)	51.9 (W/mK)
Numerical Data	
Number of Control Volumes (N)	200
Iteration limit	10^6
Tolerance	1^{-9}

Analytical Solution:

Since we are examining a cylinder in a steady state 1D condition, we need to use the heat conduction equation for a cylinder in 1D steady state, which is given below:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}_v}{\lambda}=0$$

or,

$$\frac{1}{r}\left[\left(r\frac{d^2T}{dr^2}\right)+\frac{dT}{dr}\right]+\frac{\dot{q}_v}{\lambda}=0$$

or,

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{\dot{q}_v}{\lambda}=0$$

Where, r is the radius of the cylinder from a reference point, T is the temperature of the cylinder with respect to r, λ is the thermal conductivity of the material (which is assumed constant throughout the material) and $\dot{q}_v$ is the internal heat generation per unit volume.

Now, integrating the whole equation, we get the following,

$$T=-\frac{\dot{q}_v r^2}{4\lambda}+C_1\ln(r)+C_2$$

This is the analytical equation of the temperature for a cylinder in a steady state 3D condition.

For our analysis, using the **natural boundary conditions** and assuming the inner and outer wall temperature for the cylindrical wall are Tw1 and Tw2, we can write,

At r=Rint, $T_{w1}=-\frac{\dot{q}_v(Rint)^2}{4\lambda}+C_1\ln(Rint)+C_2$

At r=Rext, $T_{w2}=-\frac{\dot{q}_v(Rext)^2}{4\lambda}+C_1\ln(Rext)+C_2$

From the following equations, we can write,

$$C_1=\frac{(T_{w1}-T_{w2})-\frac{\dot{q}_v}{4\lambda}[(Rext)^2-(Rint)^2]}{\ln\left(\frac{Rint}{Rext}\right)}$$

Thus,

$$C_2=T_{w1}+\frac{Q_v}{4\lambda}(Rext)^2-\ln(Rext)\frac{(T_{w1}-T_{w2})-\frac{Q_v}{4\lambda}[(Rext)^2-(Rint)^2]}{\ln\left(\frac{Rint}{Rext}\right)}$$

Thus, the temperature distribution equation can be rewritten as,

$$T = -\frac{Q_v r^2}{4\lambda} + \left[\frac{(T_{w1} - T_{w2}) - \frac{Q_v}{4\lambda}[(Rext)^2 - (Rint)^2]}{\ln\left(\frac{Rint}{Rext}\right)}\right]\ln(r) + T_{w1} + \frac{Q_v}{4\lambda}(Rext)^2$$
$$- \ln(Rext)\frac{(T_{w1} - T_{w2}) - \frac{Q_v}{4\lambda}[(Rext)^2 - (Rint)^2]}{\ln\left(\frac{Rint}{Rext}\right)}$$

Or,

$$T(r) = T_{w1} + \frac{\dot{q}_v}{4\lambda}((Rext)^2 - r^2) + \frac{(T_{w2} - T_{w1}) - \frac{\dot{q}_v}{4\lambda}[(Rext)^2 - (Rint)^2]}{\ln\left(\frac{Rint}{Rext}\right)}\ln\left(\frac{r}{Rext}\right)$$

This is the temperature profile that can give the value of the temperature for any position of r from the central axis. Using Fourier's Law of conduction, we get,

$$Q = -\lambda\frac{dT}{dr} = \frac{Q_v r}{2} - \frac{\lambda(T_{w2} - T_{w1}) + \frac{Q_v}{4}[(Rext)^2 - (Rint)^2]}{\mathrm{r}\ln\left(\frac{Rext}{Rint}\right)}$$

However, the edge wall temperature values Tw1 and Tw2 are still unknown. We need to apply the **specific boundary conditions** using the energy balance equation at each wall. The conditions are as follows,

At r=Rint, $\alpha_A(Ta - T_{w1}) = \frac{Q_v(\mathrm{Rint})}{2} - \frac{\lambda(T_{w2}-T_{w1})+\frac{Q_v}{4}[(Rext)^2-(Rint)^2]}{\mathrm{Rint}\times\ln\left(\frac{Rext}{Rint}\right)}$

At r=Rext, $\alpha_B(T_{w2} - Tb) = \frac{Q_v(\mathrm{Rext})}{2} - \frac{\lambda(T_{w2}-T_{w1})+\frac{Q_v}{4}[(Rext)^2-(Rint)^2]}{\mathrm{Rext}\times\ln\left(\frac{Rext}{Rint}\right)}$

From these two equations, we can the temperature values of Tw1 and Tw2. And now that we have the values of the temperatures at the wall, we can use the equation of the temperature to get temperature at any distance inside the cylinder.

Numerical Solution:

For the numerical solution of heat conduction problem in steady state and 1D problem, we can write the equation like this,

$$\frac{d}{dx}\left[\lambda\frac{dT}{dx}\right] + \int QvdV = 0$$

Where, Qv is the internal heat generation per unit volume.

To derive the discretization equations, we needed to make several control-volume sections. For our current analysis, we assumed a constant mesh property, which means the control volume sizes are all uniform in our case.

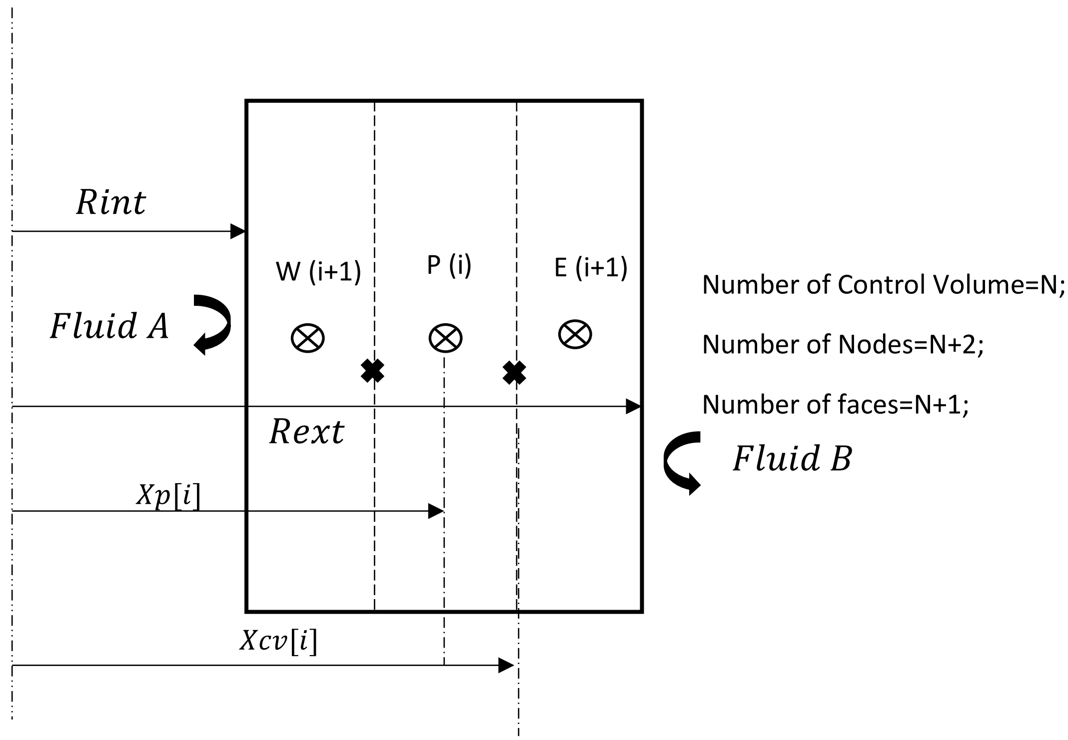


Figure 2: The discretization Process.

We need to use proper indexing for the convenience of the code writing during the numerical solution. For that purpose, a small discretization has been demonstrated to explain how the discretization takes place. Here, the node which is being analyzed is called the P node and the index for this node is [i]. The left node of the analysis node is called the W node with index of [i-1], whole the right node of the analysis node is called the E node with index of [i+1]. Similarly, the left face of the of analysis node is called 'w' face and the right face is called the 'e' face.

As we create the control volumes of equal thickness, each of the nodes are established in the middle of each control volumes. However, we have two extra nodes at the boundary of the cylinder walls. For this particular reason, for N number of control volume discretization, we have N+2 number of nodes, where N number of nodes are central nodes, and two nodes are boundary nodes.

For the analysis itself, the position of each node and the faces were recorded with proper indexing. The example of the indexing pattern has been shown in Figure 2. Notice that the indexing is 1D since the nodal distribution is only one dimensional due to the pattern of the problem.

Taking into consideration of the control volume with P node like Figure 2, we applied energy balance,

$$-\left[\lambda\frac{dT}{dx}\right]_w+\left[\lambda\frac{dT}{dx}\right]_e+\int QvdV=0$$

So far, the equation shown above should give the exact solution of the energy balance. However, we are going to discretize it using piecewise-linear profile, which would approximate the solution. The resulting equation would look like the following,

$$-\frac{\lambda_w(T_P-T_W)}{d_{PW}}+\frac{\lambda_e(T_E-T_E)}{d_{PE}}+Q_{v_P}V_P=0$$

Where, Vp is the volume of the control volume where the node P resides and Qvp is the internal heat generation per unit volume for that particular control volume. Manipulating the equation above, we can achieve the following equation for the control volumes which are the central nodes surrounded by other nodes at each direction.

$$a_P T_P = a_E T_E + a_W T_W + b_P$$

Or,

$$a_P[i]T[i] = a_E[i]T[i+1] + a_W[i]T[i-1] + b_P[i]$$

Here, ap, ae, aw, bp are all discretization coefficients.

For the boundary nodes, the energy balance must be done by considering the convection between the surrounding fluid at the boundaries and the heat conduction at the boundaries. For example, the left-most node adjacent to fluid A will have the following energy balance equation,

$$-\frac{\lambda_E(T_E - T_E)}{d_{PE}} = \alpha_A(T_A - T_P)$$

Which will result in the following equation,

$$a_P T_P = a_E T_E + b_P$$

$$a_P[i]T[i] = a_E[i]T[i+1] + b_P[i]$$

Notice that unlike the previous equation for the central nodes, the left-most node does not have any Tw term inside the equation. The absence of the term Tw is justified since for the left-most node, there is no neighbor node on the left. Similarly, for the right-most node, ae will be zero as well.

Now that we know the relationships between the discretization coefficients and temperatures of each node, we can use a solver to solve the set of equations.

Steps to solve the set of equations:

1. We evaluated the values of ae, aw, ap and bp for each node of i=1 to N+2.
2. For N+2 nodes, we needed to set initial temperature values for each node for the iteration.
3. Using Gauss-Seidal Method, we used the initial temperature values to initialize the Gauss-Seidal solver. We selected a tolerance; thus, the solver would stop when the error of the calculated updated temperature with respect to the previous initial temperature is less than the tolerance.
4. If the calculated error is less than the tolerance, the solution has converged, and the temperature attained from the iteration is the final temperature. However, if the calculated error is higher than the tolerance, the calculated temperature is set as an initial temperature for the next iteration and the iteration continues till the solution has been reached.

Code Structure:

After storing all the data regarding the position of the nodes Xp[i] and Xcv[i], the following algorithm was used to solve the set of equations,

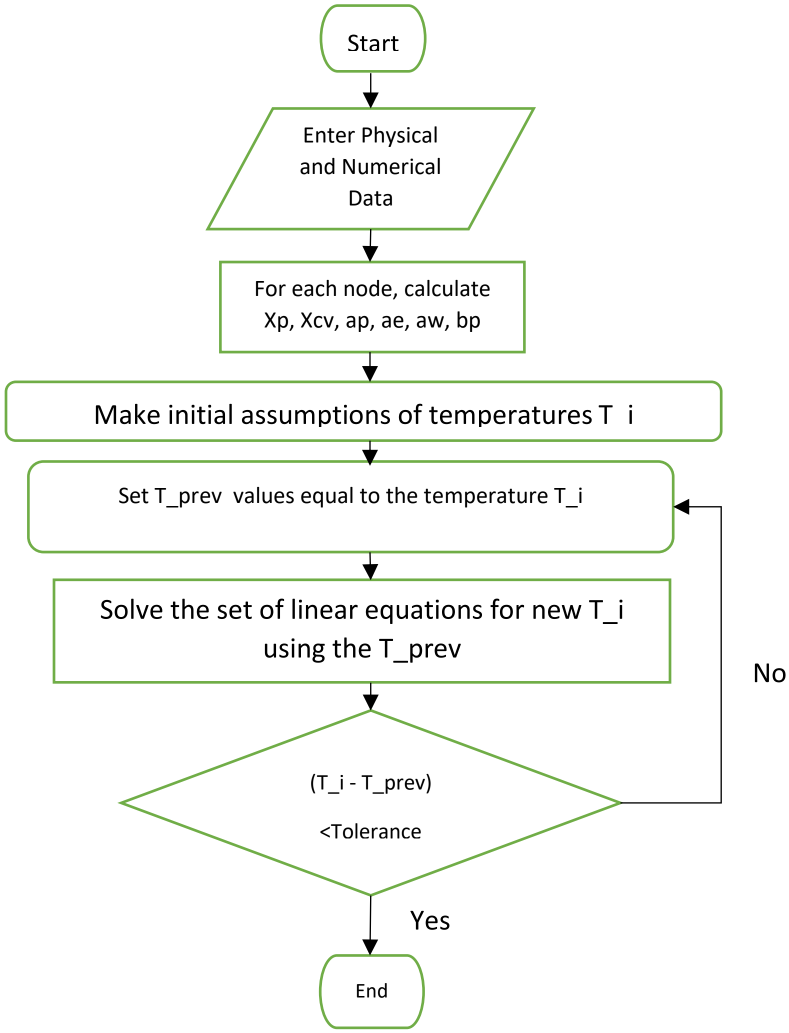


Figure 3: Flow chart of the Code Structure.

To see the main code written on MATLAB, please check the annex section of this report.

Code Verification tests:

To verify the code, we have compared the numerical solution with the analytical solution of our problem. As we can see in

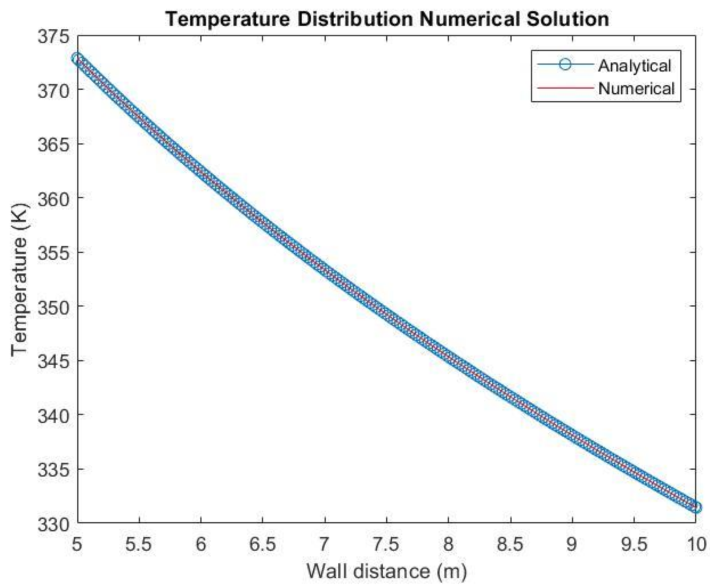


Figure 4: Analytical and Numerical Temperature Distribution.

Other than that, we had several ways to confirm the solution. These are the following ways we tried to verify the code:

1. We set the inside fluid temperature Ta and outside fluid temperature Tb to be equal and set the internal heat generation as zero. What we noticed from the result is that the analytical solution gives a perfectly straight temperature distribution, which shows that temperature at the left and right boundary is same, and T(x)=Ta=Tb. However, for the numerical solution, there is a slight temperature variation, but the variation is low as seen in Figure 5, which is expected since it depends on the tolerance value which was selected. Thus, we can say that the condition of having equal value in all the nodes is being satisfied to the order of the tolerance which was assigned.

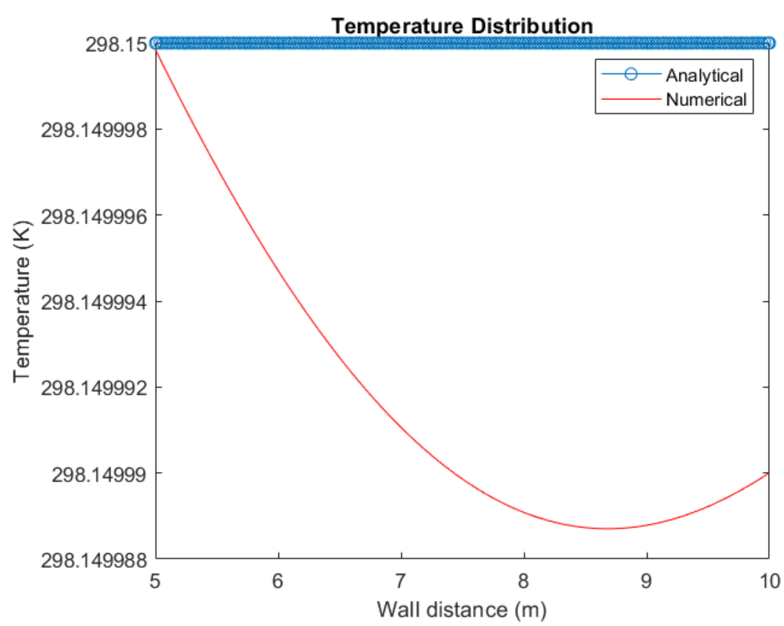


Figure 5: Temperature distribution of numerical and analytical solution.

2. The second verification method was used considering the energy balance. We calculated the energy entered, energy left, and energy generation and we did a global energy balance. As, we applied finite volume method, this method conserves energy balance. The value of the energy balance we got from the calculation was 0.0157, which is almost zero. Since the temperature was calculated with a specified tolerance, we got an energy balance value with a same order of error according to the tolerance.

Mesh Refinement studies:

For the mesh refinement studies, created a loop with respect to number of control volumes. For this analysis, we defined a loop which increases the number of control volumes from 1 to 200, and for each situation, the error has been calculated.

As expected, the error is gradually decreased when the number of control volumes are increased from 1 to 200 as seen in Figure 6. Thus, it is clearly visible that the more nodal points we have, the less error the numerical study will have.

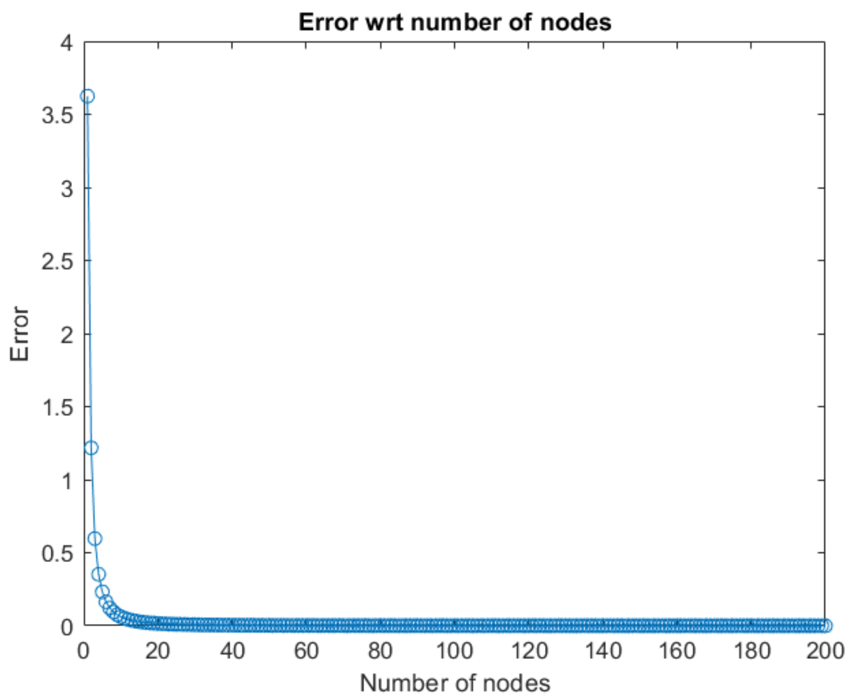


Figure 6: Error trend with increasing control volumes.

Physic Verification Studies:

a) Changing Thermal Conductivity:

For studying the physics of the problem and understanding the consequences for the changing physical parameters, first we examined the temperature distribution for 3 values of thermal conductivity. As our

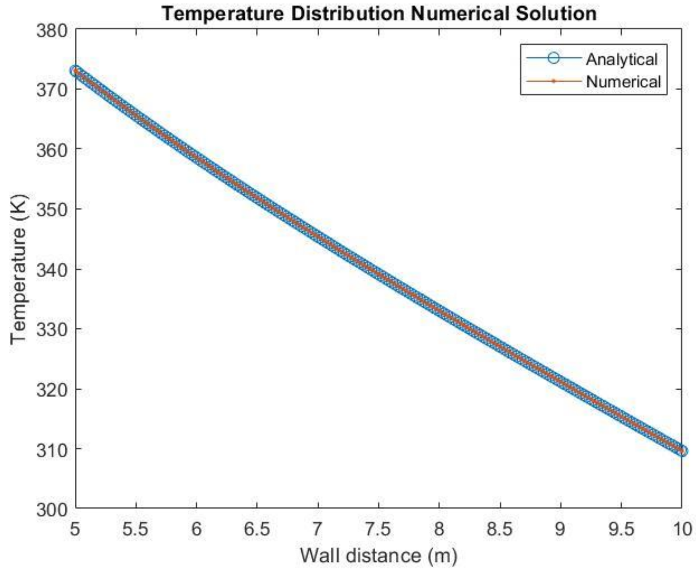


Figure 7: Temperature Distribution for 10 W/m2K.

original case had 51.9 W/m2k value of the thermal conductivity, we chose two different values which are both higher and lower than the previous analysis. The values that we selected were 10 and 100 W/m2K to understand the influence of both increasing and decreasing thermal conductivity.

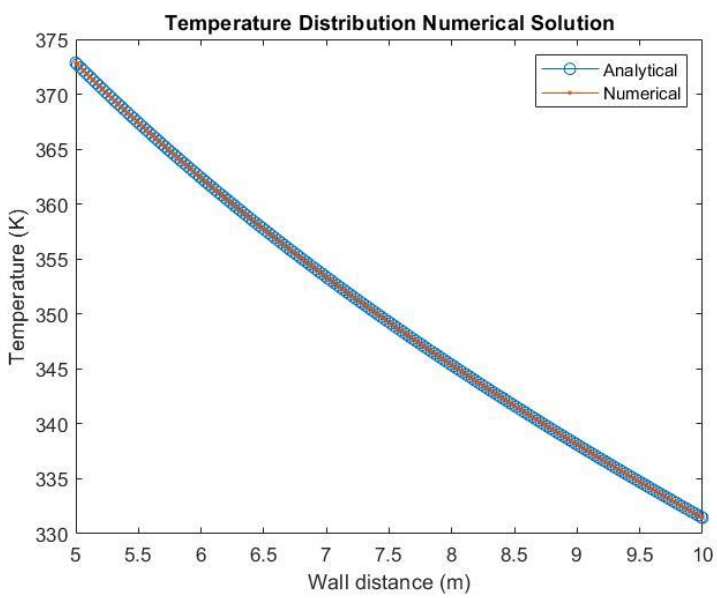


Figure 8: Temperature Distribution for 51.9 W/m^2K.

We noticed that decreasing the thermal conductivity increases the temperature difference between each node. We can clearly observe that the temperature difference between the boundaries is clearly increasing.

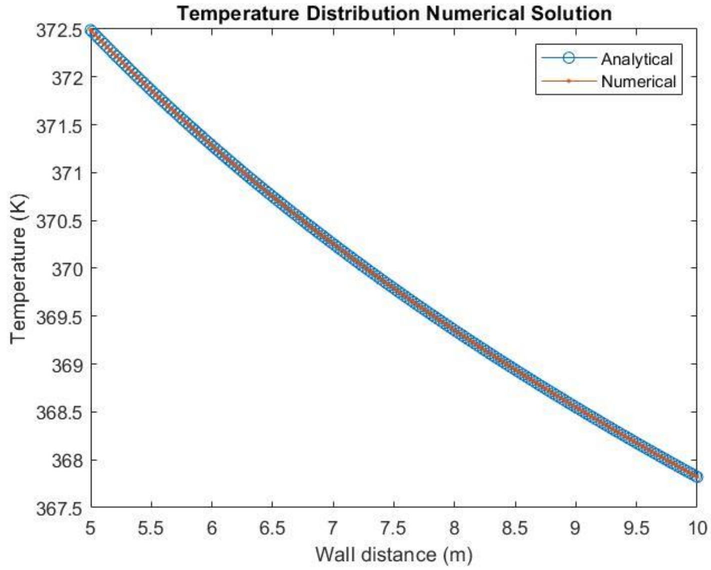


Figure 9: Temperature Distribution for 100 W/m^2K.

Similarly, when we increased the thermal conductivity of the material to 100 W/m2, the temperature difference significantly decreased. This happened because increasing the thermal conductivity increases the amount of heat flow according to Fourier's Law. If the heat flow is increased, then the temperature difference will decrease. Thus, our solution complies with the basic heat conduction physics.

Table 1: Thermal Conductivity analysis.

Thermal Conductivity	Energy Balance	Maximum local Temperature Error
10	0.0028	1.6185×10^{-4}
51.9 (Original)	0.0157	1.6226×10^{-4}
1000	0.3003	3.9802×10^{-5}

b) Changing Internal Heat generation:

Since we are analyzing the temperature distribution of the cylinder, we know that for uniform mesh, the volume and the surface area of each control volume will increase with increasing radius. Since the volume and surface will increase, the same amount of heat will not increase the equal amount of temperature according to the Fourier's Law. With increasing surface area, the temperature difference will decrease for the same heat flux. Which is why, for no heat generation or lower value of heat generation, we see that the temperature distribution is a nonlinear trend with a negatively decreasing slope.

However, when the value of heat generation is significant, the temperature gives a positively decreasing trend. Because, compared to the heat transfer coefficient at the boundaries, the heat generation is significantly higher, which increases the temperature of the internal nodes.

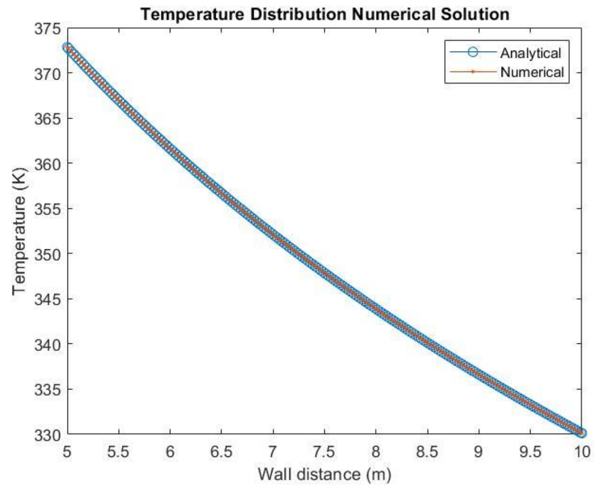


Figure 10: Temperature Distribution with 10 W/m^3 Heat generation.

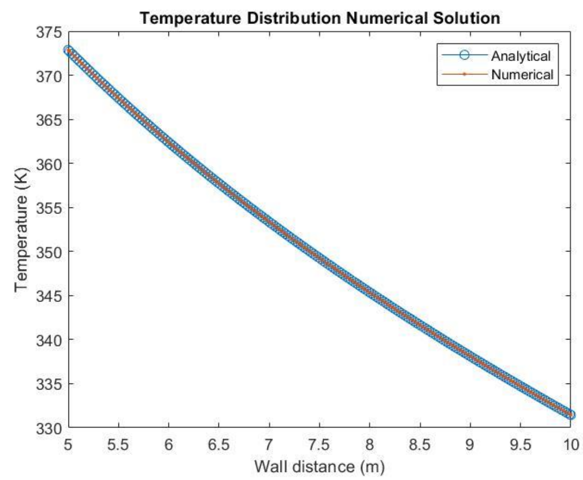


Figure 11: Temperature Distribution with 10 W/m^3 Heat generation.

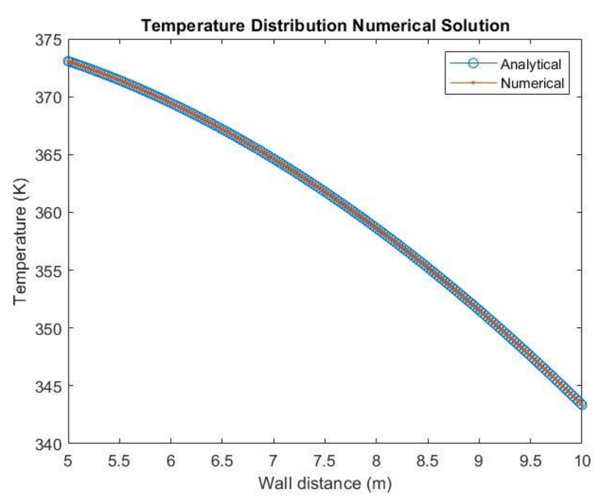


Figure 12: Temperature Distribution with 10 W/m^3 Heat generation.

Table 2: Heat Generation Analysis.

Qv	Energy Balance	Maximum local Temperature Error
0	0.0157	1.9183×10^{-4}
10 (Original)	0.0157	1.6226×10^{-4}
100	0.0157	1.0379×10^{-4}

c) Changing internal and external radius of cylinder:

For the last analysis of physics verification, we changed the thickness of our material by increasing and decreasing the thickness of the cylinder. According to the Fourier's Law, the temperature difference would increase with decreasing thickness and vice versa. This law gets verified in Figure 13, 14 and 15 accordingly. We can clearly notice that the temperature difference is lower when the thickness is lower, and it increases with increasing thickness.

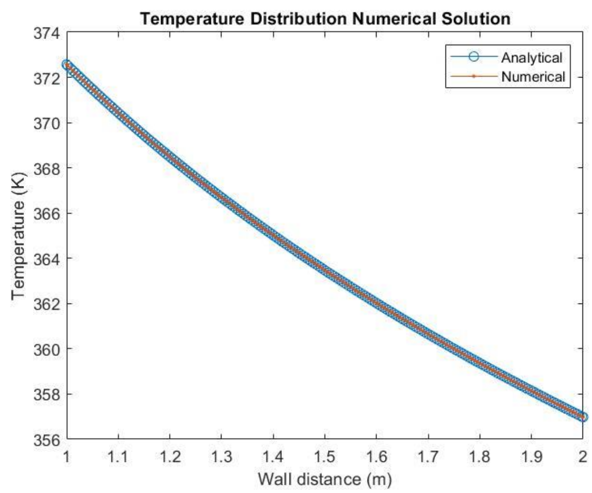


Figure 13: Temperature *distribution with Rint=1, Rext=2.*

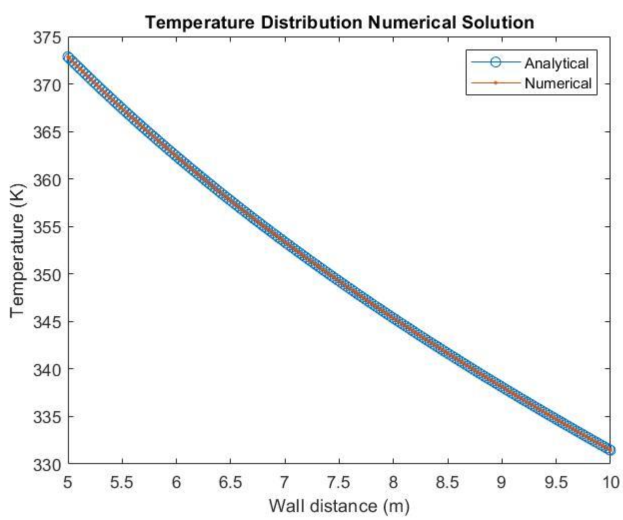


Figure 14: Temperature distribution with Rint=5, Rext=10.

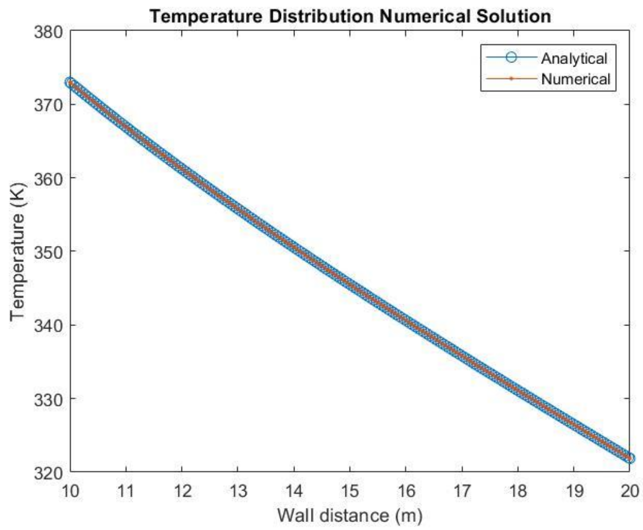


Figure 15: Temperature distribution with Rint=10, Rext=20.

Table 3: Cylinder Thickness Analysis

Rint	Rext	Energy Balance	Maximum local Temperature Error
1	2	0.0152	7.0045×10^{-5}
5 (Original)	10 (Original)	0.0157	1.6226×10^{-4}
10	20	0.0156	1.3459×10^{-4}

Contents

Assignment 1: 1D steady conduction heat transfer case in cylindrical walls

```
clc
clear all
close all
```

Input Data

Physical Data

```
Rint=5; %Internal radius (m)
Rext=10; %External Radius (m)
H=1; %Height (m)
TA=100+273.15; %temperature of the left environment (K)
TB=25+273.15; %temperature on the right environment (K)
alpha_A= 2000; % Boiling water insdie the cylinder (W/m2K)
alpha_B= 10; % Air with free convection outside the cylinder (W/m2K)
qv=10; % internal heat generation (W/m3)
lemda=51.9; %Thermal conductivity of Carbon steel(W/mK)
e_t=Rext-Rint; %thickness (m)
```

Numerical Data

```
maxiter=1e6; %Limit for number of iterations
tolerance=1e-9; %Tolerance for Iterative process
N_e= 200; %Number of CVs
d=e_t/N_e; %Thickness of each elements (m)
```

Previous Calculations and vector definitions

Mesh Definition

```
loop_interval=1;
N_list=1:loop_interval:N_e; %Changing the number of control volumes
Error_change=zeros(1,length(N_list)); %Array for storing chagning error wrt CVs
cas=1;
for N_e=N_list %Loop for changing CVs
```

```
R_cv=linspace (Rint,Rext,N_e+1); %Positions of Xcv(m)
V=zeros(1,N_e+2); %Volume of control volume for each node
S=2*pi*R_cv*H; %Surface Area
R_nodes=[]; %Position of the nodes
```

```
%Positions of Xp and Volume calculations
for i=1:1:N_e+2
    if i==1
        R_nodes(i)=Rint;
        V(i)=0;
    elseif i<N_e+2
        R_nodes(i)=(R_cv(i-1)+R_cv(i))/2;
        V(i)=pi*(R_cv(i)^2-(R_cv(i-1)^2))*H;
    else
        R_nodes(i)=Rext;
        V(i)=0;
    end
end
```

Discretization coefficients

```
ap=ones(1,N_e+2);
bp=ones(1,N_e+2);
aw=ones(1,N_e+2);
ae=ones(1,N_e+2);
%Node i=1
ap(1)=(lemda/(R_nodes(2)-R_nodes(1)))+alpha_A; %first node on left edge
aw(1)=0; %first node on left edge
ae(1)=(lemda/(R_nodes(2)-R_nodes(1))); %first node on left edge
bp(1)=alpha_A*TA; %first node on left edge
%Node i=2:N+1 (Central Nodes)
for i=2:1:N_e+1
    aw(i)=(lemda*S(i-1))/(R_nodes(i)-R_nodes(i-1)); %central nodes
    ae(i)=(lemda*S(i))/(R_nodes(i+1)-R_nodes(i)); %central nodes
    ap(i)=aw(i)+ae(i); %central nodes
    bp(i)=qv*V(i); %central nodes
end
%Node i=N+2
ap(N_e+2)=(lemda/(R_nodes(N_e+2)-R_nodes(N_e+1)))+alpha_B;%last node on right edge
aw(N_e+2)=(lemda/(R_nodes(N_e+2)-R_nodes(N_e+1)));%last node on right edge
ae(N_e+2)=0;%last node on right edge
bp(N_e+2)=alpha_B*TB;%last node on right edge
```

Gauss-Seidal Solver

Initial Guess and Iterative Temperature vectors

```
T_i=zeros(1,N_e+2); %Numerical Temperature array
T_prev=ones(1,N_e+2); %Guess Temperature array
iter=1;
while iter<maxiter
    for i=1:1:N_e+2
        if i==1 %For left-most node
            T_i(i)=(ae(i)*T_i(i+1)+bp(i))/ap(i);
        elseif i<N_e+2 %For central nodes
            T_i(i)=(aw(i)*T_i(i-1)+ae(i)*T_i(i+1)+bp(i))/ap(i);
        else %For right-most node
            T_i(i)=(aw(i)*T_i(i-1)+bp(i))/ap(i);
        end
    end

    residual=max(abs(T_i-T_prev));
    if residual<tolerance
        break;
    else
        T_prev=T_i;
        iter=iter+1;
    end
end
```

Global Energy Balance

```
Energy_Balance=S(1)*alpha_A*(TA-T_i(1))-S(end)*alpha_B*(T_i(end)-TB)+qv*sum(V);
```

Analytical Solution Procedure

```
syms Tw1 Tw2 C1 C2
```

Applying wall and specific boundary conditions

```
equa01=C1==((Tw1-Tw2)-(qv/(4*lemda))*(Rext^2-Rint^2))/(log(Rint/Rext));
equa02=C2==Tw1+((qv*(Rint^2))/(4*lemda))-C1*log(Rint);
equa03=(alpha_A*(TA-Tw1))-lemda*(((qv*Rint)/(2*lemda))-(C1/Rint));
equa04=(alpha_B*(Tw2-TB))-lemda*(((qv*Rext)/(2*lemda))-(C1/Rext));
```

Determine wall temperatures

```
Sol_Ana=solve([equa01,equa02,equa03,equa04],[Tw1 Tw2 C1 C2]);
T_w1=double(Sol_Ana.Tw1); %Left boundary temperature
T_w2=double(Sol_Ana.Tw2); %Right boundary temperature
C1=double(Sol_Ana.C1);
C2=double(Sol_Ana.C2);
% fprintf('The temperature of the left wall: Tw1= %f\nThe temperature of the right wall:Tw2=%f\n',T_w1, T_w2);
```

Solving

```
T=[]; %Values of temperature
Q=[]; %Values of heat flux
R_ana=R_nodes;
i=1;
for r=1:1:N_e+2
    T(i)=T_w2+(qv/(4*lemda))*(Rext^2-R_nodes(i)^2)-((T_w2-T_w1)+(qv/(4*lemda))*(Rext^2-Rint^2))*((log(Rext/R_nodes(i))/(log(Rext./Rint))));
    Q(i)=((qv*R_nodes(i))/2)-lemda*(((T_w1-T_w2)+(qv/(4*lemda))*(Rext^2-Rint^2))/(R_nodes(i)*log(Rint/Rext)));
    i=i+1;
end

Error_change(N_e)=max(abs(T-T_i));

cas=cas+1;
```

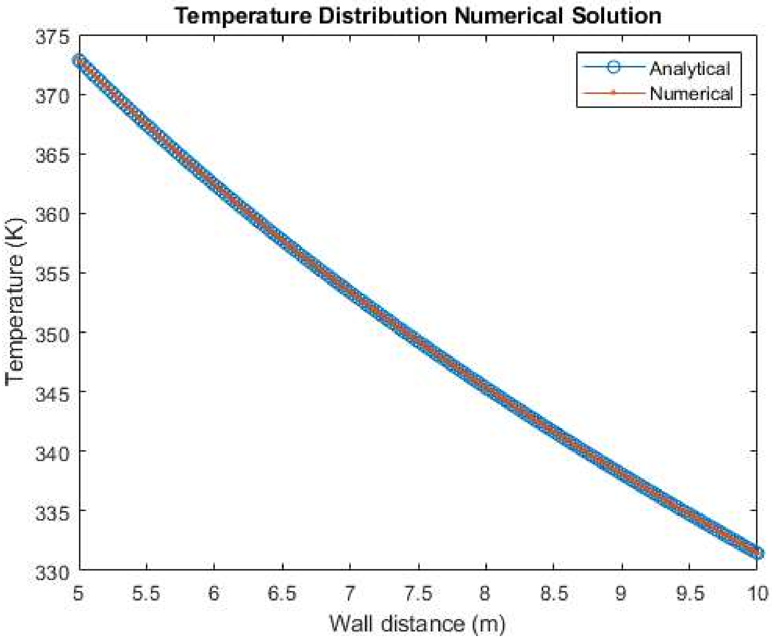
```
end
```

Error between Analytical and Numerical

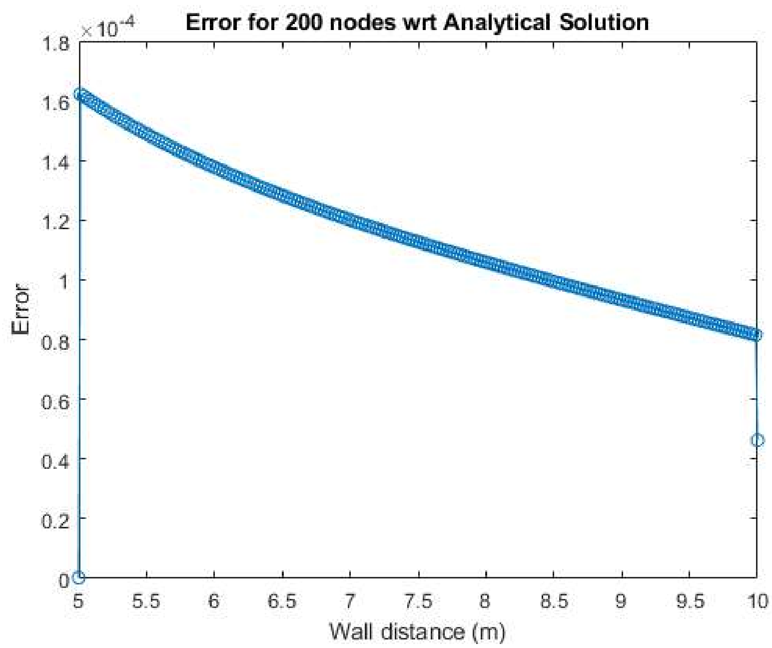
```
Final_Error=(abs((T-T_i)));
Max_Error=max(Final_Error);
```

plotting

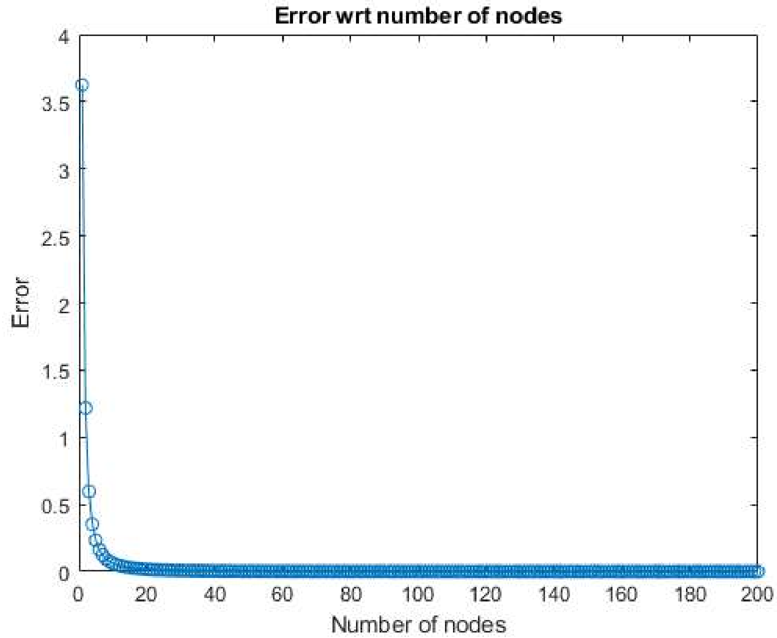
```
figure
plot (R_ana, T,'-o'); %Plotting the analytical temperature distribution
title ('Temperature Distribution Analytical Solution')
xlabel ('Wall distance (m)')
ylabel ('Temperature (K)')
hold on
%Plotting the Numerical temperature distribution
plot (R_nodes, T_i,'.-')
title ('Temperature Distribution Numerical Solution')
xlabel ('Wall distance (m)')
ylabel ('Temperature (K)')
legend ('Analytical','Numerical')
hold off
figure %Plotting the error between the Analytical and Numerical
plot (R_ana, Final_Error,'-o');
title ('Error for 200 nodes wrt Analytical Solution')
xlabel ('Wall distance (m)')
ylabel ('Error')
figure %Evolution of Error with increasing number of control volumes
plot (N_list, Error_change,'-o');
title ('Error wrt number of nodes')
xlabel ('Number of nodes')
ylabel ('Error')
```



Temperature Distribution Numerical Solution
Analytical
Numerical
Temperature (K)
Wall distance (m)
375
370
365
360
355
350
345
340
335
330
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10



Error for 200 nodes wrt Analytical Solution
×10^-4
Error
Wall distance (m)
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10


Error wrt number of nodes
Error
Number of nodes
4
3.5
3
2.5
2
1.5
1
0.5
0
0
20
40
60
80
100
120
140
160
180
200