ASSIGNMENT 3: BURGER’S EQUATION.

Turbulence: Phenomenology, Simulation and Aerodynamics

Submitted by-

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DECEMBER 23, 2022

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Problem Specification

Solve the Burger's equation with Re=40. As initial condition we took $\hat{u}_k = \frac{1}{k}$. Since the mode k=0 has no interactions with other modes, we assumed $\hat{u}_0 = 0$ (no mean flow). We had plotted the energy spectrum of the steady state for N=20 as the initial case, which is under-resolved, and later with higher N resolution to get DNS solution. Later, we produced results for the same case using LES solver to compare the results with the DNS solver.

Introduction

The Navier-Stokes equations are a set of differential equations that describe the motion of the fluid substances, such as liquid and gases. These equations are used to predict the flow of the fluids under various conditions and can be used to model a wide range of phenomena, including the flow of air around an airplane wing, flow inside a pipe and the flow of blood through the human body. It is thus a model for the non-linear dynamics of turbulence inside the fluid flow.

$$\frac{\partial u}{\partial t} + (u.\nabla)u = \frac{1}{Re}\Delta u - \nabla p$$

$$\nabla . u = 0$$

Where, Re is the dimensionless Reynold's number defined as,

$$Re = \frac{\rho V L}{\mu}$$

Where ρ and μ are the density and dynamic viscosity of the working fluid and L and V are the characteristic length and velocity, respectively.

The burger's equation is a partial differential equation that describe the one-dimensional motion of a viscous fluid. It is a simplified version of the Navier-Stokes equations that assumes that the fluid is incompressible and has a constant viscosity. The Burger's equation is often used to model the flow of fluids in simple systems, such as the flow of a fluid through a tube or the flow of a fluid over a flat surface.

The Burger's equation can be written as,

$$\partial_t u + u\partial_x u = \frac{1}{Re}\partial_{xx} u + f$$

The connection between the Navier-Stokes equations and the Burger's equation is that the Burger's equation is a simplified version of the Navier-Stokes equation that is used to model the flow of viscous fluids in simple systems. While the Navier-stokes equations can be used to model the flows of the fluids in more complex systems, the Burger's equation is often used to model the flow of fluids in simpler systems where the assumptions of constant viscosity and incompressibility are reasonable. Thus, the burger's equation shares many of the aspects of the NS equation.

Since the Burger equation can be used as a simpler version which includes basic aspects of turbulence such as the energy cascade, the inter-scale interaction, and the roles of the convective and diffusive terms, in this report we are going to use the Burger's equation to do these investigations. To do that, you must solve the Burgers' equation in the Fourier space. Firstly, without any turbulence model. We started with Re=40 with N=20 and N=100; then, we tried other configurations. In a second step, we also included the proposed LES model [1].

Numerical Methods

DNS Method

The analysis of the Burger's equation in Fourier's space involves representing the function that describes the fluid flow as a series of sine and cosine functions, known as Fourier's series. This allows us to express the function in terms of its frequency components, which can be used to analyze the behavior of the fluid flow at different scales.

Once we have expressed the Burger's equation in Fourier space, we can use mathematical techniques to analyze the behavior of the fluid flow at different frequencies. This can help us understand how the fluid behaves at different scales and how it responds to different external forces.

For example, we can use the Fourier transform to analyze the stability of the fluid flow and to predict the formation of shock waves. We can also use it to study the behavior of the fluid at different wavelengths, which can be useful for understanding the formation of patterns in the fluid flow.

To perform the analysis, we first express the Burger's equation in Fourier space by taking the Fourier transformation of both sides of the equation. The Fourier transformation of the Burger's equation can be written as,

$$\partial_t \hat{u}_k + \sum_{k=p+q} \hat{u}_p \, iq\hat{u}_q = -\frac{k^2}{Re}\hat{u}_k + F_k$$

Or,

$$\frac{\hat{u}_k{}^{n+1} - \hat{u}_k{}^{n}}{\Delta t} + \sum_{k=p+q} \hat{u}_p \, iq\hat{u}_q = -\frac{k^2}{Re}\hat{u}_k + F_k$$

$$\hat{u}_k{}^{n+1} = \hat{u}_k{}^{n} + \Delta t\left(-\sum_{k=p+q} \hat{u}_p \, iq\hat{u}_q - \frac{k^2}{Re}\hat{u}_k + F_k\right)$$

Where,

$$u(x,t) = \sum_{k=-N}^{k=+N} \hat{u}_k e^{ikx}$$

Where, N is the total number of Fourier modes.

And the energy of each mode is computed as,

$$E_k = \hat{u}_k \overline{\hat{u}_k}$$

Where, $\overline{\hat{u}_k}$ is the conjugate of $\hat{u}_k$.

LES Method

Large Eddy Simulation (LES) is a numerical method used to simulate fluid flow in complex systems, such as turbulent flow. In LES, the flow is simulated by solving the Navier-Stokes equations for the large-scale eddies in the flow, while the small-scale eddies are modeled using a sub-grid scale (SGS) model.

This approach allows LES to capture the effects of the large-scale eddies on the overall flow, while the SGS model accounts for the effects of the small-scale eddies. One advantage of LES is that it can accurately simulate the effects of large-scale structures, such as vortexes and coherent structures, on the overall flow. It is also able to capture the effects of the small-scale eddies on the flow, which is important for accurately predicting the behavior of turbulent flow.

For the LES modelling, the Burger's equation in Fourier space can be written as,

$$\partial_t \hat{u}_k + \sum_{k=p+q} \hat{u}_p \, iq\hat{u}_q = -(v + v_t)k^2 \hat{u}_k + F_k$$

Where,

$$v = \frac{1}{Re}$$

$$v_t = {v_t}^{+\infty} (\frac{E_{kn}}{k_N})^{\frac{1}{2}} \, {v_t}^*$$

$$ {v_t}^{+\infty} = 0.315 \frac{5-m}{m+1} \sqrt{3-m} \, {C_k}^{-\frac{3}{2}}$$

$$ {v_t}^* = 1 + 34.5 e^{-3.03 k_N / k}$$

Where, m is the slope of the energy spectrum, E_{kn} is the energy at the cutoff frequency, k_N and C_K are the Kolmogorov constant. ${v_t}^*$ is a non-dimensional eddy-viscosity. For our case, the energy spectrum was measured for m=2 and Ck=0.4523[1].

Note that, we will set $F_k = 0$, $\hat{u}_1 = 1$ after each loop so that the energy input is always from the first mode k=1.

Results and Discussion

Here, we have plotted the energy spectrum vs the wave number in logarithmic scale to see how the spectrum is behaving for different N and Re number.

According to the problem specification, the original case is for N=20 and Re=40. For this case, the number of modes N=20 is clearly under-resolved. According to the definition of LES and DNS, the DNS method should be more accurate.

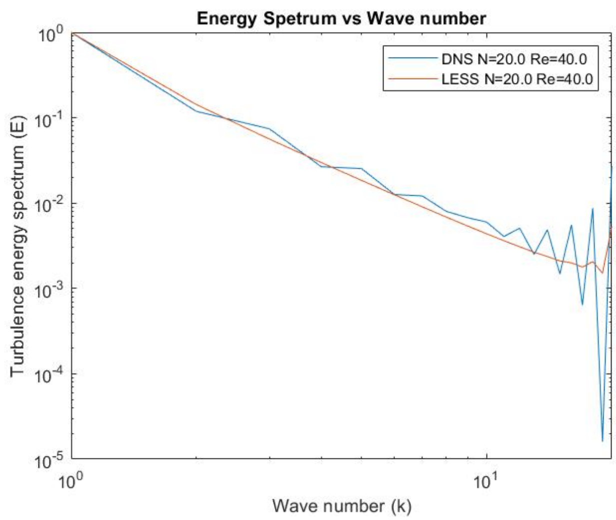

Figure 1: Original case (Re=40, N=20).

As we can see, since the case is under-resolved, we can see the energy cascade is far away from the Kolmogorov scale and ends the simulation before as N=20. The Energy fluctuation tries to compensate for the Kolmogorov scale energy dissipation. Since, the energy did not get the chance to dissipate. In the LES solution, since the simulation considers the effective viscosity, the curve is smoothed down compared to the more accurate case in DNS.

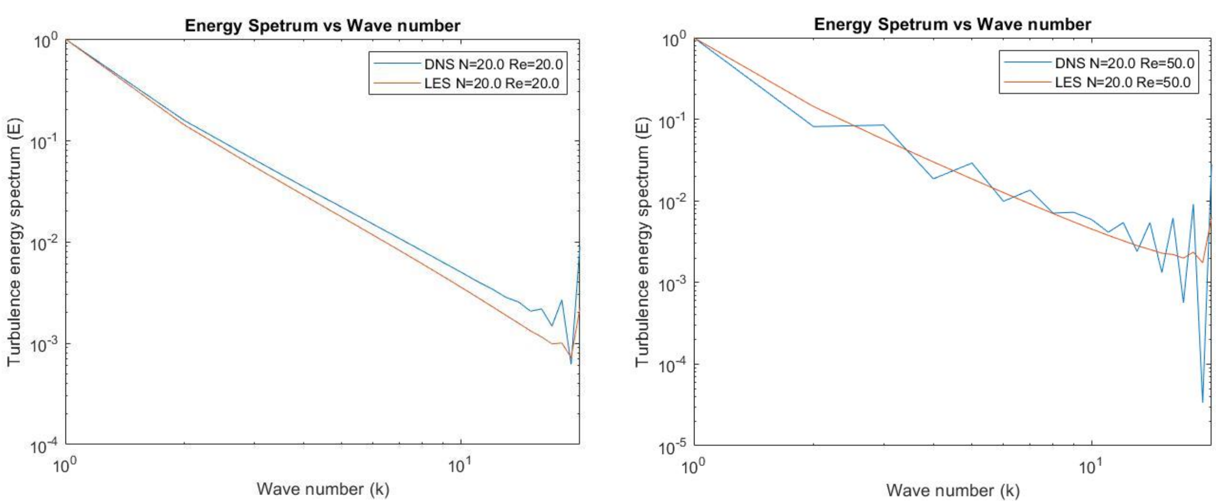

Figure 2: Cases for changed Reynolds number.

Now to compare the original case after parametric changes, first we increased and decreased the values of the Reynolds number to see what kind of change we can see. As we can see that the value of the increased Re number causes more fluctuations while the opposite smoothed down the trend.

In the context of the Burger's equation, the Reynolds number is a measure of the relative importance of the viscous and inertial forces in the fluid flow. It is defined as the ratio of the inertial forces to the viscous forces.

If the Reynolds number is increased, this means that the inertial forces are becoming more important relative to the viscous forces. In this case, the fluid flow may become more turbulent, with eddies and vortices forming at smaller scales. This can lead to an energy cascade, where energy is transferred from larger scales to smaller scales through the formation of these eddies and vortices.

On the other hand, if the Reynolds number is decreased, the viscous forces become more important, and the fluid flow may become more laminar. In this case, there may be less energy transfer between scales, and the flow may be more predictable and stable.

Since, the increased value of Re means the kinematic viscosity is decreasing, it means that there are more range of scales for eddies and the energy is being dissipated at range much higher than the range we set, which is N=20. For this reason, at k=20 a large portion of energy dissipates after Kolmogorov range. At N=20, our case becomes more under-resolved than the original and we can see further fluctuation.

However, when the Re value is decreased, the viscosity value is increased, and the dissipation term is more prominent. Thus, energy is dissipating at larger eddy scales, thus we see relatively smoother curve since our resolution at N=20 can capture dissipation range.

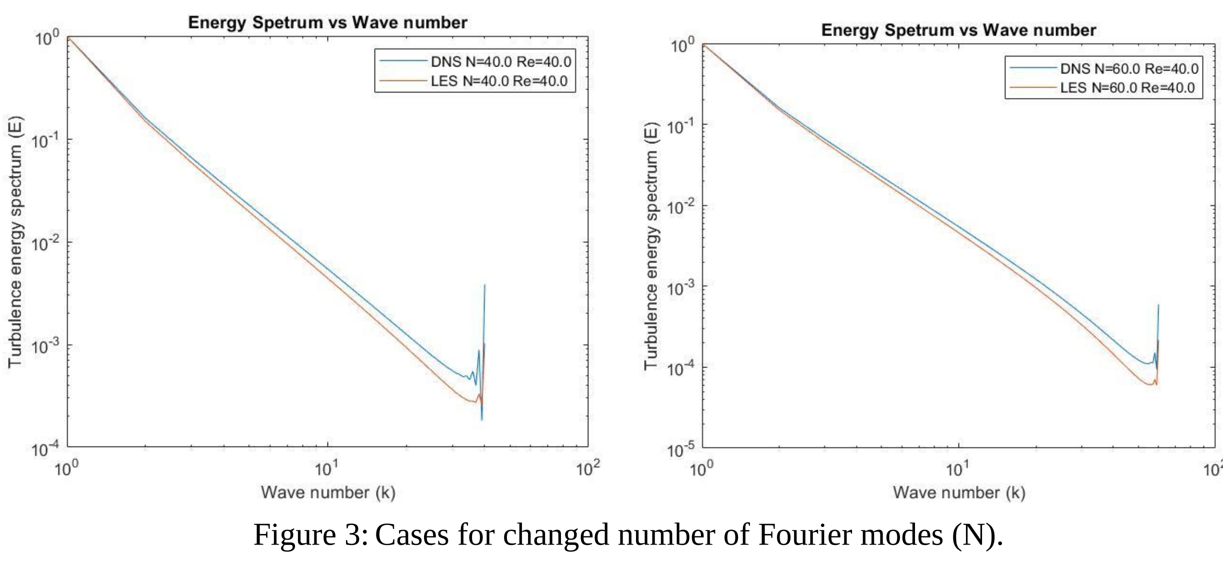

Figure 3: Cases for changed number of Fourier modes (N).

Like the previous case, we also change the value of the number of the Fourier modes which we consider as N=20 in the original case.

Since from the previous cases it was clear that the case was under-resolved for N=20, we slowly increased to see the change in the results. It was clearly visible that when we increased the number of Fourier mode in our calculation, the curves were clearly smoother in both the LES and DNS solutions. The reason is that we more modes being considered, the Kolmogorov scales are getting considered more and more. Thus, the DNS and the LES solution are not being forced to compensate for the energy dissipation by displaying the fluctuation at the end.

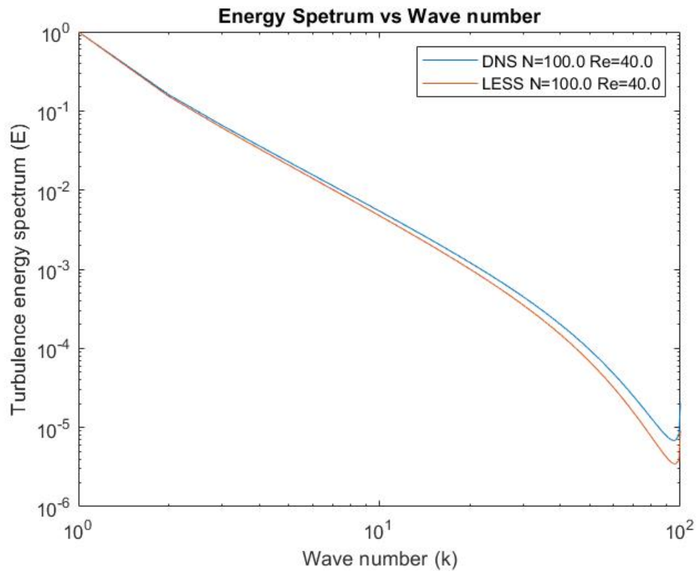

Figure 4: Energy Spectrum for N=100 and Re=40.

Finally, the energy spectrum vs wave number for N=100 and Re=40 shows that the total Energy cascade is much more reasonable and accurate. The reason is like the previous explanation. Even though the Reynolds number is not that high, we are considering a large amount of Fourier modes by considering the case for N=100, which gives are the most accurate resolution of the energy case theory.

Conclusion

Burgers' equation is often used as a simple model for various physical phenomena, such as traffic flow, sonic booms, and gravitational waves. It is particularly useful for studying the behavior of inviscid fluids, which are fluids that have zero viscosity and do not experience any resistance to flow. Burgers' equation is a nonlinear partial differential equation, which means that it includes terms that are nonlinear in the dependent variables. This makes it useful for modeling phenomena that exhibit nonlinear behavior, such as shock waves or turbulence. However, Burgers' equation has some limitations. It does not accurately describe the behavior of viscous fluids, which are fluids that have a non-zero viscosity and experience resistance to flow. In addition, it does not account for other physical effects such as heat transfer or mass transfer, which are important in many real-world situations. Overall, Burgers' equation is a useful model for understanding the behavior of inviscid fluids in one spatial dimension, but it should not be used blindly without carefully considering its limitations.

By solving Burger's equation, we can study the balance between the convective term and the diffusive term and how they affect the energy cascade in the flow. For example, if the convective term is dominant, we would expect to see energy cascading from large scales to small scales. If the diffusive term is dominant, we would expect to see energy dissipating at small scales and flowing back to large scales. In general, the energy cascade in turbulence is a complex process that is influenced by a variety of factors, including the Reynolds number, the nature of the flow, and the presence of other physical effects such as heat transfer or mass transfer. Understanding the inter-scale interaction and the roles of the convective and diffusive terms can provide insight into the behavior of turbulent flows and help us better predict and control them.

References

[1] CTTC (UPC), “Burgers’ equation in Fourier space,” vol. 4, no. 4, pp. 1–8, 2014.