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Cheat Sheet

1 Matrices

- Have rank equal to the number of linearly independent columns/rows
- Have rank equal to the dimension of their column space
- Have singular value decompositions $\mathbf{UDV}'$
- Have a generalized inverse
- Have a column space and a row space
- Have null space $\{\mathbf{x} : \mathbf{Ax} = \mathbf{0}\}$

1.1 Orthogonal projections

- Map vectors onto a subspace
- Are symmetric
- Are idempotent
- Are positive semidefinite
- Have eigenvalue 1 with multiplicity equal to their rank; all other eigenvalues are zero
- For any $\mathbf{A}$, $\mathbf{P_A} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}'$ is an orthogonal projection onto $\mathcal{C}(\mathbf{A})$
- $\mathbf{P_A A} = \mathbf{A}$
- $\text{rank}(\mathbf{P_A}) = \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{U})$ where $\mathbf{U}$ comes from the SVD of $\mathbf{A}$
- $\mathbf{I} - \mathbf{P}$ is an orthogonal projection matrix as well

1.2 Square matrices

- Are invertible if they have full rank
- Are invertible if they have empty null space
- Have rank equal to the number of nonzero eigenvalues
- Have trace equal to the sum of their diagonal elements

1.3 Symmetric matrices

- Have a spectral decomposition $\mathbf{UDU}'$
- Have trace equal to the sum of their eigenvalues
- Have determinant equal to the product of their eigenvalues
- Are positive definite if all their eigenvalues are positive
- Are positive semidefinite if all their eigenvalues are nonnegative

1.4 Matrix identities

- $\mathbf{AXX}' = \mathbf{BXX}'$ implies $\mathbf{AX} = \mathbf{BX}$
- $\mathbf{AA}' = 0$ implies $\mathbf{A} = 0$
- Block matrix inversion

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{C} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A} - \mathbf{BC}^{-1}\mathbf{B}')^{-1} & -(\mathbf{A} - \mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1} \\ -\mathbf{C}^{-1}\mathbf{B}'(\mathbf{A} - \mathbf{BC}^{-1}\mathbf{B}')^{-1} & \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}'(\mathbf{A} - \mathbf{BC}^{-1}\mathbf{B}')^{-1}\mathbf{BC}^{-1} \end{bmatrix}$$

- Invertible $\mathbf{A}$ implies $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$
- For $m \times n$ matrix A, $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$
- For invertible $\mathbf{A}$ and $\mathbf{C}$, $\text{rank}(\mathbf{AB}) = \text{rank}(\mathbf{B}) = \text{rank}(\mathbf{BC})$

2 Linear model

- $\mathbf{Y} = \mathbf{X}\beta + \varepsilon$ where $\mathbb{E}[\varepsilon|\mathbf{X}] = \mathbf{0}$
- Sometimes we assume $\varepsilon|\mathbf{X} \sim (\mathbf{0}, \sigma^2\mathbf{I})$

2.1 Least squares

- $\arg\min_\beta (\mathbf{Y} - \mathbf{X}\beta)'(\mathbf{Y} - \mathbf{X}\beta)$
- Solve the normal equations $\mathbf{X}'\mathbf{X}\beta = \mathbf{X}'\mathbf{Y}$
- Fitted values are the same for all least squares estimates
- Is unique for full rank design matrix
- Can be of the form $(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y}$
- Is unique under identifiability constraints

2.2 Statistical properties

- $a'\beta$ is estimable if a is in the row space of $\mathbf{X}$
- Estimable $a'\beta$ implies that $a'\hat{\beta}$ is BLUE
- For full rank design matrix, $a'\hat{\beta}$ is BLUE
- θ is identifiable if $\theta \neq \theta_0$ implies $f_\theta \neq f_{\theta_0}$
- For $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y}$, $a'\hat{\beta}$ is unbiased
- Additionally, if errors are uncorrelated, $\text{var}(a'\hat{\beta}|\mathbf{X}) = \sigma^2 a'(\mathbf{X}'\mathbf{X})^{-}a$
- $\hat{\sigma}^2 = \frac{1}{n-\text{rank}(\mathbf{X})}(\mathbf{Y} - \mathbf{X}\hat{\beta})'(\mathbf{Y} - \mathbf{X}\hat{\beta})$ is unbiased
- $\hat{\beta} \xrightarrow{p} \beta$ if $\lambda_{\min}(\mathbf{X}'\mathbf{X}) \to \infty$

2.3 Examples

- $\mathbf{X} = \mathbf{1}$
  - $\mathbf{P_X} = \frac{1}{n}\mathbf{1}\mathbf{1}'$
  - $\bar{Y}$ is BLUE
  - $\mathbf{P_X Y} = (\bar{Y}, \ldots, \bar{Y})'$
- Balanced one-way ANOVA
  - $\mathbf{X}$ is rank-deficient
  - $\mathbf{P_X} = \frac{1}{J}\begin{bmatrix} \mathbf{1}\mathbf{1}' & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\mathbf{1}' \end{bmatrix}_{2J \times 2J}$
  - $\mathbf{P_X Y} = (\bar{Y}_1, \ldots, \bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_2)'$
- Simple linear regression with intercept
  - $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$
  - $\hat{\beta}_1 = \frac{\sum((X_i - \bar{X})Y_i)}{\sum(X_i - \bar{X})^2}$
- Column-centered linear regression with intercept
  - $\hat{\alpha}_0 = \bar{Y}$
  - $\hat{\alpha} = (\mathbf{X}'(\mathbf{P_X} - \mathbf{P_1})\mathbf{X})^{-1}\mathbf{X}'(\mathbf{P_X} - \mathbf{P_1})\mathbf{Y}$