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# Cheat Sheet



## 1 Tricks

- Proceed in this order to show measurability
  1. indicator is measurable
  2. simple function is measurable
  3. nonnegative function is measurable
  4. $X = X^+ - X^-$ is measurable
- Union bound: $[|X-Y| \geq 2\varepsilon] \subset \big([|X-Z| \geq \varepsilon] \cup [|Z-Y| \geq \varepsilon]\big)$. Apply monotonicity and countable subadditivity of a measure to get a result eerily similar to the triangle inequality.
- $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty \bigcup_{k=1}^n A_k = \bigcup_{n=1}^\infty B_n$ where $B_n \nearrow$
- Use set minus operation to construct nonoverlapping sequences
- Borrow results from real analysis
- Draw a picture
- By construction
- Use an inequality
- Translate math symbols
  - $\bigcap \Leftrightarrow$ inf
  - $\bigcup \Leftrightarrow$ sup
  - $\bigcup \Leftrightarrow$ there exists
  - $\bigcap \Leftrightarrow$ for all
- Without loss of generality
  - For finite measures work with probability measures instead by scaling down
  - Redefine on null sets for a.e. based arguments because the mapped value on the null set does not matter in Lebesgue integration
- $\liminf \leq \lim \leq \limsup$ can be useful in showing convergences
- Results about $\lambda$-systems, $\pi$-systems, and monotone classes can be useful in showing that a collection of sets is a $\sigma$-field
- Problems that assume integrability can be well set up to apply the dominated convergence theorem
- Find the singularity part first for a Lebesgue decomposition

# 2 Definitions

- $\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \{\omega : \omega \text{ is in all but finitely many } A_n\text{'s}\}$
- $\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \{\omega : \omega \text{ is in an infinite number of } A_n\text{'s}\}$
- $\liminf a_n = \sup_{n \geq 1} \inf_{k \geq n} a_k$ and $\limsup a_n = \inf_{n \geq 1} \sup_{k \geq n} a_k$
- If $\liminf A_n = \limsup A_n$, we call it $\lim A_n$
- Symmetric difference: $A \Delta B = AB' + A'B$
- Set minus: $A \setminus B = AB'$
- A $\pi$-system is closed under finite intersections
- A $\lambda$-system contains the whole space and is closed under monotone increasing limits and proper differences
- A field is a class of sets closed under complements and finite unions/intersections
- A $\sigma$-field is a class of sets closed under complements and arbitrary unions/intersections. It defines measurability.
- A measure is a nonnegative, countably additive set function with $\sigma$-field support
- A premeasure has the same properties as a measure, but on field support
- A measure is finite if the measure of the whole space is finite
- A measure $\mu$ is $\sigma$-finite if the whole space can be decomposed into disjoint sets of finite measure
- A signed measure $\phi$ is a set function measuring sets in a $\sigma$-algebra that measures the empty set as zero, is countably additive, and maps onto $(-\infty, \infty]$
- An outer measure is a nonnegative, countably subadditive set function with power set support
- A set is $\mu^*$-measurable with outer measure $\mu^*$ if for all $T$ in the whole space

$$\mu^*(T) = \mu^*(TA) + \mu^*(TA')$$

- A (Caratheodory) covering of $A$ is $\{A_n\}$, where each $A_n \in$ (some field), such that

$$A \subset \bigcup_{n=1}^{\infty} A_n$$

- An outer extension $\mu^*$ is

$$\inf\left\{\sum_{n=1}^{\infty}\mu(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n\right\}$$

where $\{A_n\}$ is a covering, the $A_n$ are in a field, and $\mu$ is a premeasure.

- Borel sets
  - $\sigma$-field generated by the field of intervals in $\mathbb{R}$
  - $\sigma$-field generated by class of all open sets with respect to a topology

- A null set has measure 0

- A measure space $(\Omega, \mathcal{A}, \mu)$ is a triplet with a space, a $\sigma$-field, and a measure

- A complete measure space includes all subsets of a null set in its $\sigma$-field

- A Lebesgue-Stieltjes measure measures finite intervals as finite

- A generalized distribution function is finite, nondecreasing, and right-continuous.

- A map $X$ is $\mathcal{A}' - \mathcal{A}$ measurable if $X^{-1}(\mathcal{A}') \subset \mathcal{A}$. We drop $\mathcal{A}' - \mathcal{A}$ if $\mathcal{A}' = \bar{\mathcal{B}}$, simply saying measurable.

- $X^+$ is $X$ when positive and 0 otherwise. $X^-$ is $-X$ when negative and 0 otherwise. Note that any map $X$ can be decomposed $X = X^+ - X^-$.

- Lebesgue integral of measurable map $X$
  1. (simple) $\int X\, d\mu = \sum_{i=1}^{n} c_i \cdot \mu(A_i)$
  2. (nonnegative) $\int X\, d\mu = \sup\{\int Y d\mu : 0 \leq Y \leq X,\ Y \,\text{simple}\}$
  3. (general) $\int X\, d\mu = \int X^+\, d\mu - \int X^-\, d\mu$

- The Riemann-Stieltjes integral is a further generalization of the Riemann integral.

$$\int_a^b f(x)\, dg(x) = \lim \sum_{i=1}^{n} f(x_{n,i}) \times [g(x_{n,i}) - g(x_{n,(i-1)})]$$

where the interval $[a, b]$ is partitioned into smaller and smaller subintervals. The Riemann integral where integrator $g$ is the identity function. See page 64 of *The Tweedie Index Parameter* for some practice exercises solving Riemann Stieltjes integrals.

- A norm for measurable maps

$$||X||_r := \begin{cases} (\mathbb{E}[|X|^r]^{1/r}) & r \geq 1 \\ \mathbb{E}[|X|^r] & 0 < r < 1 \end{cases}$$

- $\mathcal{L}_r$ space is $\{X : \int |X|^r d\mu < \infty\}$
- Convergences
  1. (almost everywhere / almost surely) $X_n(\omega) \to X(\omega)$ for all except a null set
  2. (in measure / in probability) $\mu([|X_n - X| \geq \varepsilon]) \to 0$ for all $\varepsilon > 0$
  3. (in distribution) For random variables, $F_n(x) \to F(x)$ at each continuity point $x$
  4. (in mean) $\mathbb{E}[|X_n - X|^r] \to 0$ for $X_n, X \in \mathcal{L}_r$
- A collection of measurable maps $\{X_t\}$ is integrable if $\sup_t \mathbb{E}[|X_t|] < \infty$
- A collection of measurable maps $\{X_t\}$ is uniformly integrable if

$$\sup_t \left\{ \mathbb{E}[|X_t| \cdot 1([|X_t| \geq \lambda]])] \right\} \to 0$$

- A signed measure $\phi_{ac}$ is absolutely continuous with respect to a measure $\mu$ if whenever $\mu$ measures a set to be zero $\phi_{ac}$ measures that same set to be zero
- A signed measure $\phi_s$ is singular with respect to a measure $\mu$ if there exists some set for which $\mu$ measures it to be zero and the $\phi_s$ measures the complement of that set to be zero
- The product $\sigma$-field $\mathcal{A} \times \mathcal{A}'$ is generated by $\sigma(\cdot)$ applied to a field that contains all finite disjoint unions of measurable rectangles $A \times A'$ where $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$.

# 3 Theorems

- Monotone property of measures. For measure $\mu$ and $A_n \nearrow$, the measure of the arbitrary union of is the limit of the individual measures. We also have the measure of the arbitrary intersection being the limit of individual measures if there is some point at which the measures become finite and the sets are nonincreasing.
- **Extension theorem.** A premeasure on a field can be extended to be a measure on the $\sigma$-field generated by that field using the (restricted) outer extension. If the premeasure on the field is $\sigma$-finite, then the extension is unique and $\sigma$-finite.
- Correspondence theorem. There is a 1-1 correspondence between Lebesgue-Stieltjes measures on the Borel sets and (representative members of the equivalence classes of) generalizd distribution functions.
- Measurability of common functions. Common functions of measurable maps, when well-defined, are measurable. This list includes $\pm, \times, \div, \inf, \sup, \liminf, \limsup, \lim$, compositions of continuous or measurable maps, and negative and positive parts.
- **Measurability via simple functions.**
  1. Simple functions are measurable
  2. Maps are measurable if and only if they are the limit of a sequence of simple functions.
  3. If a measurable map is nonnegative, then it is the limit of a sequence of nonnegative, nondecreasing simple functions.
- Elementary properties of the Lebesgue integral are linearity, scaling, and monotonicity.
- When we can permute lim and $\int$
  1. **(Monotone convergence theorem)** Suppose we have a sequence of nonnegative measurable maps $X_n$ increasing to measurable map $X$ almost everywhere. This assumption implies
  $$0 \leq \lim \int X_n \, d\mu = \int X \, d\mu$$
  2. (Fatou's lemma) For measurable maps $X_1, \ldots, X_n$, provided that $X_n \geq 0$ almost everywhere for all $n$,
  $$\int \liminf X_n \ \leq \liminf \int X_n \, d\mu$$
  3. **(Dominated convergence theorem)** Suppose $|X_n| \leq Y$ almost everywhere for such measurable maps $X_n$ and dominating measurable map $Y \in \mathcal{L}_1$. We further suppose that we have a convergence almost everywhere or in measure. Then,

   (a) $\int |X_n - X| \, d\mu \to 0$
   (b) $\int X_n \, d\mu \to \int X \, d\mu$
   (c) $\sup_{A \in \mathcal{A}} | \int_A X_n \, d\mu - \int_A X \, d\mu| \to 0$

- We can permute $\sum$ and $\int$ if the measurable maps $X_n \geq 0$ almost everywhere for all $n$
- Absolute continuity of the integral. For $X \in \mathcal{L}_1$, as $\mu(A) \to 0$ we get that

$$\int_A |X| \, d\mu \to 0$$

- Unconscious statistician. Let $X$ be a random variable and $g : (\Omega', \mathcal{A}') \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$ be any measurable function.
  (a) The induced measure $\mu_X$ determines the induced measure $\mu_{g(X)}$
  (b) $\int_{X^{-1}(A')} g(X(\omega)) d\mu(\omega) = \int_{A'} g(x) d\mu_X(x)$ for all $A'$ in $\mathcal{A}'$
  (c) $\int_{X^{-1}(g^{-1}(B))} g(X(\omega) \, d\mu(\omega) = \int_{g^{-1}(B)} g(x) \, d\mu_X(x) = \int_B y \, d\mu_Y(y)$ for all $B \in \bar{\mathcal{B}}$.

  This theorem says that we can work with (induced) probability models from elementary statistics and not worry about the original measure space.
- When Lebesgue is Riemann-Stieltjes. Let $g$ be continuous on a closed interval $[a, b]$. Then the Lebesgue-Stieltjes integral $\int_a^b g \, dF$ equals the Riemann-Stieltjes integral.
- We can permute partial derivative $\frac{\partial}{\partial t}$ and $\int$ for measurable map $X$ with partial derivatives existing for all $t$ in nondegenerate $[a, b]$ if $|\frac{\partial}{\partial t} X(t, \omega)| \leq Y(\omega)$ for all $t \in [a, b]$ and dominating measurable $Y \in \mathcal{L}_1$
- Skorokhod construction. If we have convergence in distribution for random variables $X, X_1, \ldots, X_n$, then we can talk in terms of $Y_n = F_n^{-1}(U) \cong X_n$ and $F^{-1}(U) \cong X$ where $U$ in a Unif(0,1) random variable and $Y_n \to_{a.s.} Y$.
- **Helly-Bray theorem.** Let there be a probability measure space $(\Omega, \mathcal{A}, P)$, $X_n \to_d X$, and bounded, continuous $g$ almost surely $F$. Then,
  (a) $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$
  (b) (Mann-Wald) We can relax the boundedness condition above.
     (a) $g(X_n) \to_{a.s.} g(X)$
     (b) $g(X_n) \to_p g(X)$
     (c) $g(X_n) \to_d g(X)$
  (c) Conversely, $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$ for all bounded, continuous $g$ implies $X_n \to_d X$

- Slutsky's theorem. Let $X_n \to_d X$, $Y_n \to_b a$, and $Z_n \to_p b$.

$$Y_n X_n + Z_n \to_d aX + b$$

- de la Vallée Poussin. Let $\mu(\Omega) < \infty$. A family of $\mathcal{L}_1$ -integrable functions $X_t$ is uniformly integrable if and only if there exists a convex function $G$ on $[0, \infty)$ for which
  1. $G(0) = 0$
  2. $G(x)/x \to \infty$ as $x \to \infty$
  3. $\sup_t \mathbb{E}[G(|X_t|)] < \infty$

  This theorem provides a criterion for uniform integrability.

- Vitali theorem. Let $\mu(\Omega) < \infty$, $r > 0$, measurable maps $X_n \in \mathcal{L}_r$ and converging in measure to $X$. TFAE:
  1. $\{|X_n|^r : r \geq 1\}$ are uniformly integrable random variables
  2. $X_n \to_r X$
  3. $\mathbb{E}[|X_n|^r] \to \mathbb{E}[|X|^r]$
  4. $\limsup \mathbb{E}[|X_n|^r] \leq \mathbb{E}[|X|^r] < \infty$

- **Modes of convergence** Let $0 < r' \leq r$. Let $X_n$'s and $X$ be measurable and almost everywhere finite.
  1. almost everywhere convergence and finite measure implies convergence in measure
  2. convergence in measure implies that there is a subsequence such that we have almost everywhere convergence
  3. convergence in $r^{\text{th}}$ mean implies convergence in measure and $\{|X_n|^r : n \geq 1\}$ are uniformly integrable
  4. convergence in measure and uniformly integrable random variables implies convergence in $r^{\text{th}}$ mean
  5. $X_n \to_r X$ and finite measure implies $X_n \to_{r'} X$
  6. convergence in probability implies convergence in distribution
  7. for finite measure, convergence in measure if and only if every subsesequence has a further subsequence for which almost everywhere convergence holds
  8. convergence in distribution implies that we can find Skorokhod random variables converging almost surely and equal in distribution

  See Figure 3.5.1 in *Probability for Statisticians*, 2nd ed. (Shorack 2017).

- Jordan-Hahn decomposition. The space $\Omega$ for which a signed measure $\phi$ is defined can be decomposed as
$$\Omega = \Omega^+ + \Omega^-$$
where the signed measure of sets in the positive space is nonnegative and the signed measure of sets in the negative space is nonpositive.

- **Lebesgue decomposition theorem.** Let $\mu$ and $\phi$ be $\sigma$-finite measures defined on measurable space $(\Omega, \mathcal{A})$. Then,

  (a) $\phi = \phi_{ac} + \phi_s$

  (b) For some finite $\mathcal{A}$-measurable function $X$ unique almost everywhere for $\mu$,
  $$\phi_{ac}(A) = \int_A X \, d\mu$$

  (c) (Radon-Nikodym) Same as (b), but with the extension that $\phi$ is absolutely continuous with respect to $\mu$ if and only if
  $$\phi(A) = \int_A X \, d\mu = \int_A \frac{d\phi}{d\mu} \, d\mu$$
  where $\frac{d\phi}{d\mu}$ is the Radon-Nikodym derivative

- Cavalieri principle. Let $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \nu)$ be measure spaces. Let $C \in \mathcal{A} \times \mathcal{A}'$. Then,

  1. Every $C_{\omega'} = \{\omega : (\omega, \omega') \in \Omega \times \Omega'\} \in \mathcal{A}$ and every $C_\omega = \{\omega' : (\omega, \omega') \in \Omega \times \Omega'\} \in \mathcal{A}'$ whenever $C \in \mathcal{A} \times \mathcal{A}'$
  2. Product measure $\phi$ is defined s.t.
  $$\begin{aligned} \phi(C) &= \int_{\Omega'} \mu(C_{\omega'}) \, d\nu \\ &= \int_{\Omega} \nu(C_\omega) \, d\mu \end{aligned}$$

  Great visualizations are available in this playlist. Intuitively, we measure slices of a space using the $\mu$ measure and then add up the slices using the $\nu$ measure and vice versa.

- **Fubini's theorem.** Let $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \nu)$ be $\sigma$-finite measure spaces. Suppose we have a measurable map $X$ on the product measure space and $\int_{\Omega \times \Omega'} X \, d\phi$ is finite. Then,

  1. some tedious comments about measurable, $\mu$-integrable, and $\nu$-integrable functions

2.

$$\int_{\Omega\times\Omega'} X\, d\phi = \int_{\Omega'}\int_{\Omega} X\, d\mu d\nu$$
$$= \int_{\Omega}\int_{\Omega'} X\, d\nu\, d\mu$$

3. (Tonelli) It is not immediate for most problems that the measurable map is $\phi$-integrable. We get 2. and can apply Fubini's theorem if either
   - $X \geq 0$
   - $\int\int |X|\, d\nu\, d\mu$ is finite
   - $\int\int |X|\, d\mu d\nu$ is finite

# 4 Inequalities

- A function is convex on some interval if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

  for all $x, y$ in the interval and $0 \leq \alpha \leq 1$

- ($\mathcal{L}_s \subset \mathcal{L}_r$) For $\mu(\Omega) < \infty$, we have that $\mathcal{L}_s \subset \mathcal{L}_r$ whenever $0 < r < s$

- $C_r$ **inequality.** $\mathbb{E}[|X+Y|^r] \leq C_r(\mathbb{E}[|X|^r] + \mathbb{E}[|Y|^r])$ where

$$C_r := \begin{cases} 2^{r-1} & r > 1 \\ 1 & 0 < r \leq 1 \end{cases}$$

  This inequality is a triangle inequality for integrals.

- Young's inequality.

$$|ab| \leq \frac{|a|^r}{r} + \frac{|b|^s}{s}$$

- **Hölder's inequality.** For $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{\frac{1}{p}} \times \mathbb{E}[|Y|^q]^{\frac{1}{q}}$$

- Cauchy-Schwarz inequality.

$$\begin{aligned} \mathbb{E}[XY]^2 &\leq \mathbb{E}[|XY|]^2 \\ &\leq \mathbb{E}[|X|^2] \times \mathbb{E}[|Y|^2] \\ &= \mathbb{E}[X^2] \times \mathbb{E}[Y^2] \end{aligned}$$

- Lyapunov's inequality. For $\mu(\Omega) < \infty$, $||X||_r$ is increasing in $r$ for all $r > 0$

- Minkowski's inequality. For all $r \geq 1$,

$$\mathbb{E}[|X+Y|^r]^{\frac{1}{r}} \leq \mathbb{E}[|X|^r]^{\frac{1}{r}} + \mathbb{E}[|Y|^r]^{\frac{1}{r}}$$

  Using Minkowski's and $C_r$ inequalities, we give meaning to $|| \cdot ||_r$.

- Basic inequality. For even $g \geq 0$ increasing on $[0, \infty)$ and measurable $X$,

$$\mu(|X| > \lambda) \leq \frac{\mathbb{E}[g(X)]}{g(\lambda)}, \quad \forall \lambda > 0$$

  This inequality is more general than Markov's and Chebyshev's.

# Cheat Sheet

- **Markov inequality.** $\mu(|X| > \lambda) \leq \frac{\mathbb{E}[|X|^r]}{\lambda^r}$ for all $\lambda > 0$. This inequality provides upper bounds on probabilities.
- Chebyshev's inequality. $\mu(|X - \mathbb{E}[X]| \geq \lambda) \leq \frac{\text{Var}(X)}{\lambda^2}$ for all $\lambda > 0$. This inequality provides upper bounds on measures.
- Paley-Zygmund inequality. For nonnegative random variable $X$ with $\mathbb{E}[X] < \infty$,

$$P(X > \lambda) \leq \frac{(\max\{\mathbb{E}[X] - \lambda, 0\})^2}{\mathbb{E}[X]^2}, \quad \forall \lambda > 0$$

  This inequality provides lower bounds on tail probabilities.
- Jensen's inequality. For $g$ convex on $[a, b]$, $P(X \in [a, b]) = 1$, and $\mathbb{E}[X] \in (a, b)$,

$$g(\mathbb{E}[X]) \leq \mathbb{E}[g(X)]$$

  For strictly convex $g$ we achieve equality iff $X = \mathbb{E}[X]$ almost everywhere.
- Bonferroni inequality.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

- Littlewood inequality. Let $m_r = \mathbb{E}[|X|^r]$. Then, for $r \geq s \geq t \geq 0$,

$$m_r^{s-t} m_t^{r-s} \geq m_s^{r-t}$$

# 5 Examples

- $\lambda([a,b]) = b - a$ for all intervals is the Lebesgue measure
- Induced measure $\mu_X(B) = \mu(X^{-1}(B))$ borrows its measure.
- Probability measure is a finite measure such that $\mu(\Omega) = 1$
- $\mu(A) = 0$ for all $A \in \mathcal{A}$ is the trivial/zero measure. This measure can be useful when constructing (counter)examples, particularly when discussing completeness.
- $\mu(A) = |A|$ for all $A \in \mathcal{A}$ is the counting measure
- Dirac measure

$$\delta_x(A) := \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

- Typewriter sequence

$$X_n = 1_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]}, \quad k \geq 0,\, 2^k \leq n < 2^{k+1}$$

  This sequence of measurable maps converges in measure and in $L_1$ norm, but not it does not converge almost everywhere.

- $\hat{\mathcal{A}}_\mu \neq \hat{\mathcal{A}}_\nu$ for finite measures $\mu$ and $\nu$ on measurable space $(\Omega, \mathcal{A})$. See a discrete space example in Homework 2 Problem 4.
- We cannot use the Fubini theorem if $\mu$ or $\nu$ is not $\sigma$-finite or if the measurable map is not $\phi$-integrable, e.g.
  - counting measure on an interval
  - a measurable map with a discontinuity (divide by zero)

# 6 Motivations and Commentary

- The Banach Tarski paradox conjures up a situation that challenges our intuition about volume measurement. Crucial to the paradox is the assumption of the Axiom of Choice. This axiom says that, given an arbitrary collection of nonempty bins, we can pick an item from each bin. Resolving this paradox involves defining what can be measured.

- We introduce outer measure and $\mu^*$-measurable as technical devices to prove the extension theorem.

- Galen Shorack introduces the extension theorem is Chapter 1 in the context of defining the Lebesgue measure $\lambda$. However, we can extend any premeasure defined on a field. For example, we use the extension theorem to define the product measure.

- The Lebesgue-style procedure to proving measure theory results for indicator functions, simple functions, nonnegative functions, and finally general functions relies of linearity, MCT, and linearity for steps 2, 3, and 4. Therefore, in some cases, we only have to argue the case for indicator functions.

- "I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral." - Henri Lebesgue

- Lebesgue integral animation

- Lebesgue integration is helpful in making rigorous theoretical arguments in mathematics. Riemann integration is more practical. Based on a theorem aforementioned, the Lebesgue integral equals the Riemann integral for continuous functions on closed intervals, so we often default to Riemann integration.

# 7 Analysis

- A topology is a class of sets closed under finite intersections and arbitrary unions. It defines openness.
- A function is continuous if its preimages are open with respect to a topology
- **Heine-Borel covering.** An arbitrary collection of open sets that covers a compact set contains a finite subcollection of open sets that covers the compact set as well.
- **Convergent sequence.** We say for a sequence $(x_n)$ that $\lim x_n = x$ if for all $\varepsilon > 0$ there exists some $N$ such that $n > N$ implies $d(x_n, x) < \varepsilon$.
- **Cauchy sequence.** $\{x_n\}$ s.t. $\forall\, \varepsilon > 0$ there exists $N < m, n$ such that $d(x_m, x_n) < \varepsilon$.
- **Bolzano-Weierstrass.** Every bounded sequence has a convergent subsequence.
- **Completeness of $\mathbb{R}$.** Every Cauchy sequence is a convergent sequence and every convergent sequence is a Cauchy sequence.
- **Archimedean property.** There are no infinitely large or infinitely small elements. We can always find something bigger or smaller.
- **Derivatives.**
  - $f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$
  - $f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b-a}$
  - differentiable implies continuous, but continuous does not imply differentiable
  - $F(x) = \int_a^x f(t)\, dt$ is referred to as fundamental because it relates integral calculus with differential calculus
  - For continuous $f$ on closed interval $[a, b]$ there exists some $c$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

  - For $f$ continuous on $[a, b]$, $f$ maps onto all values between $f(a)$ and $f(b)$
- Taylor series.

$$f(x) = \frac{f^{(0)}(0)(x-0)^0}{0!} + \frac{f^{(1)}(0)(x-0)^1}{1!} + \frac{f^{(2)}(0)(x-0)^2}{2!} + \frac{f^{(3)}(0)(x-0)^3}{3!} + \dots$$

# 8 Extra

- **Borel-Cantelli lemmas.** Let $E_1, \dots, E_n$ be events
  1. $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ implies $\mu(\limsup E_n) = 0$
  2. For independent events, $\sum_{n=1}^{\infty} P(E_n) = \infty$ implies $P(\limsup E_n) = 1$
- Every countable set of reals has Lebesgue measure 0.
- **Froda's theorem.** Let $f$ be a real-valued, monotonic function on an open interval. Then the set of discontinuities of $f$ is countable.
- We use Froda's theorem and the fact that countable sets have Lebesgue measure 0 to justify an argument in the Skorokhod construction.