

1 Problems

1. Let $X \sim \text{Bernoulli}(p)$ and ϕ be the (signed) measure corresponding to the distribution function F_X .
 - (a) Find the Lebesgue decomposition of ϕ with respect to the counting measure.
 - (b) Find the Lebesgue decomposition of ϕ with respect to the Lebesgue measure.
 - (c) Find the Lebesgue decomposition of ϕ with respect to the Dirac measure δ_1 .
2. Let $X \sim \text{Poisson}(\lambda)$ and $Y_1, Y_2, \dots \sim \text{Gamma}(\alpha, \beta)$ random variables. Consider the random variable $Z = \sum_{i=0}^X Y_i$ and let ϕ be the (signed) measure corresponding to the distribution function F_Z .
 - (a) Find the Lebesgue decomposition of ϕ with respect to the Lebesgue measure.
 - (b) Find the Lebesgue decomposition of ϕ with respect to the Dirac measure δ_0 .
3. Let $X \sim \text{Exponential}(\theta)$ and let $Y \sim \text{Bernoulli}(p)$ random variables. Consider the random variable $Z = (-1)^Y \cdot X$ and let ϕ be the (signed) measure corresponding to the distribution function F_Z .
 - (a) Find the Lebesgue decomposition of ϕ with respect to measure μ that satisfies $\mu((-\infty, 0)) = 0$ and $\mu([0, \infty)) = 1$.
 - (b) Find the Lebesgue decomposition of ϕ with respect to measure μ that satisfies $\mu((-\infty, 0]) = 1$ and $\mu((0, \infty)) = 0$.
4. Let $\Omega = \{-1, 1\}$ and consider the measurable space $(\Omega, 2^\Omega)$. As well, let ϕ be the signed measure that satisfies $\phi(\{-1\}) = -1$ and $\phi(\{1\}) = 1$.
 - (a) Find the Lebesgue decomposition of ϕ with respect to the Dirac measure δ_1 .
 - (b) Find the Lebesgue decomposition of ϕ with respect to the Dirac measure δ_{-1} .

2 Solutions

1. Note that $\Omega = \{0, 1\}$ and the σ -field in question is 2^Ω . Moreover, ϕ and the measures in (a-c) are defined on the measurable space $(\Omega, 2^\Omega)$.
 - (a) Verify that $\phi_{ac} = \phi$ and ϕ_s is the zero measure.
 - (b) Verify that $\phi_s = \phi$ and ϕ_{ac} is the zero measure.
 - (c) Verify that $\phi_{ac} = \delta_1$ and $\phi_s = \delta_0$.
2. See [“Tweedie Index Parameter”](#).
 - (a) The random variable Z has a point mass at zero and has continuous density on the positive reals. Verify that $\phi_s = \delta_0 \cdot \exp\{-\lambda\}$ and $\phi_{ac} = \phi - \phi_s$.
 - (b) Verify that $\phi_{ac} = \delta_0 \cdot \exp\{-\lambda\}$ and $\phi_s = \phi - \phi_{ac}$.
3. See [double exponential random variable](#) article.
 - (a) With respect to this measure, we bifurcate the double exponential random variable at 0. Be cautious in handling the boundary.
 - (b) Reverse of (a).
4.
 - (a) Verify that $\phi_{ac} = \delta_1$ and $\phi_s = -\delta_{-1}$.
 - (b) Verify that $\phi_{ac} = -\delta_{-1}$ and $\phi_s = \delta_1$.