Lecture-18 (orthogonal basis, Best approximation)

Orthogonal basis: Let

$V \leftarrow$ i.p.s.

$\beta \subseteq V$ subset.

Then β is an orthogonal basis of V if

- β is a basis of V
- β is orthogonal.

Theorem: Every f.d. i.p.s. has an orthogonal basis.

Proof:

$V \leftarrow$ f.d.i.p.s.

$n = \dim(V)$

$S = \{v_1, \ldots, v_n\} \leftarrow$ basis of V

Apply Gram-Schmidth on S.

$S' = \{q_1, q_2, \ldots, q_n\} \leftarrow$ orthogonal basis of V. //.

Advantage of an orthogonal basis:

$V \leftarrow$ i.p.s.

$\beta \leftarrow$ basis of V.

$\beta = \{v_1, \ldots, v_n\}$

$v \in V \Rightarrow v = \sum_{i=1}^{n} \alpha_i v_i$ for some $\alpha_1, \dots \alpha_n \in \mathbb{F}$.

(finding these α_i's is not always easy).

Now suppose $\beta' = \{q_1, \dots q_n\}$ orthogonal basis of V.

$v \in V \Rightarrow v = \sum_{i=1}^{n} \gamma_i q_i$ for some $\gamma_i \in \mathbb{F}$.

but since q_i's are orthogonal, we have

$$\frac{\langle v | q_j \rangle}{\langle q_j | q_j \rangle} = \gamma_j \quad \text{for each } j.$$

$$\Rightarrow \quad v = \sum_{j=1}^{n} \frac{\langle v | q_j \rangle}{\langle q_j | q_j \rangle} . q_j \, .$$

Example: $V = \mathbb{R}^2$

Consider the inner product

$$\langle (x_1, y_1) | (x_2, y_2) \rangle = x_1 x_2 + 2 x_1 y_2 + 2 x_2 y_1 + 5 y_1 y_2$$

$\beta = \{e_1, e_2\}$ basis of $\mathbb{R}^2$.

but β is not an orthogonal basis of $\mathbb{R}^2$ w.r.t. the above i.p., because $\langle e_1 | e_2 \rangle = 2 \neq 0$.

Apply Gram-Schmidth orthogonalization process on $\beta = \{e_1, e_2\}$ to find an orthogonal basis w.r.t. the given i.p.

$$q_1 = e_1$$

$$q_2 = e_2 - \frac{\langle e_2 | q_1 \rangle}{\langle q_1 | q_1 \rangle} q_1 = e_2 - 2e_1$$

$\{e_1, e_2 - 2e_1\}$ is an orthogonal basis of $\mathbb{R}^2$.

Best approximation: let

$V \leftarrow$ i.p.s.

$W \subseteq V \leftarrow$ subspace of V

$v \in V$.

Then a vector $w_0 \in W$ is said to be a best approximation of v by a vector to W if

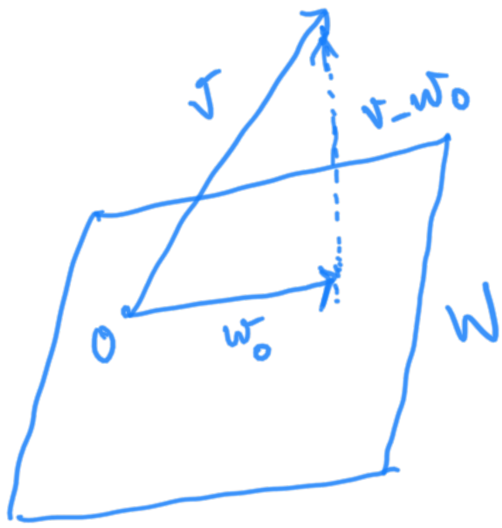
$$\|v - w_0\| \leq \|v - w\| \quad \forall\, w \in W$$

[we can say that w_0 is the nearest vector to v in W]

Recall. $\|x\|^2 = \langle x | x \rangle$

Observation:

Clearly, the vector $w_0 \in W$, where the perpendicular from v meets W, is the closest vector in W to v.



In other words, $v - w_0 \perp w \quad \forall\, w \in W$

or $\langle v - w_0 \mid w \rangle = 0 \quad \forall\, w \in W$.

Result: Let $V \leftarrow$ i.p.s.

$W \subseteq V$ subspace.

$v \in V$

Then $w_0 \in W$ is a best approximation of v by a vector in W if and only if

$$\langle v - w_0 \mid w \rangle = 0 \quad \forall\, w \in W.$$

Proof: First observe that for $w \in W$, we have

$$\|v - w\|^2 = \|(v - w_0) + (w_0 - w)\|^2$$

$$= \langle (v - w_0) + (w_0 - w) \mid (v - w_0) + (w_0 - w) \rangle$$

$$= \langle (v - w_0) \mid v - w_0 \rangle + \langle w_0 - w \mid v - w_0 \rangle$$

$$+ \langle v - w_0 \mid w_0 - w \rangle + \langle w_0 - w \mid w_0 - w \rangle$$

$$= \|v-w_0\|^2 + 2\,\mathrm{Re}\left(\langle v-w_0 | w_0-w\rangle\right) + \|w_0-w\|^2$$

So,

$$\|v-w\|^2 = \|v-w_0\|^2 + 2\,\mathrm{Re}\left(\langle v-w_0 | w_0-w\rangle\right) + \|w_0-w\|^2 \quad \forall w\in W \qquad (*)$$

($\Leftarrow$) Suppose $v-w_0 \perp w \ \ \forall w\in W$

Claim w_0 is a best approximation.

From $(*)$,

$$\|v-w\|^2 = \|v-w_0\|^2 + \|w_0-w\|^2 \quad \forall w\in W$$

$$\Rightarrow \|v-w_0\| \leq \|v-w\| \quad \forall w\in W$$

$\Rightarrow$ w_0 is a best approximation.

($\Rightarrow$) Conversely, let w_0 be a best approximation of v

$$\Rightarrow \|v-w_0\| \leq \|v-w\| \quad \forall w\in W.$$

by $(*)$,

$$\|w_0-w\|^2 + 2\,\mathrm{Re}\left(\langle v-w_0 | w_0-w\rangle\right) \geq 0 \quad \forall w\in W \qquad (1)$$

Since, every vector $u\in W$, can be written as w_0-w for some $w\in W$ as

$$u = w_0 - \underbrace{(w_0 - u)}_{w} \quad (\checkmark)$$

$\Rightarrow$ from (1)

$$\|u\|^2 + 2\,\mathrm{Re}\left(\langle v - w_0 | u \rangle\right) \geq 0 \quad \forall\, u \in W \qquad (2).$$

<u>Claim</u> $v - w_0 \perp w \quad \forall\, w \in W.$

Let $w \in W$,

choose $u = -\dfrac{\langle v - w_0 | w_0 - w \rangle}{\|w_0 - w\|^2}(w_0 - w)$

$\Rightarrow$ from (2), that

$$\frac{|\langle v - w_0 | w_0 - w \rangle|^2}{\|w_0 - w\|^{\cancel{4}\,2}} \cancel{\|w_0 - w\|^2} - 2\,\mathrm{Re}\underbrace{\left(\frac{|\langle v - w_0 | w_0 - w \rangle|^2}{\|w_0 - w\|^2}\right)} \geq 0$$

$$\Rightarrow \quad -\underbrace{\frac{|\langle v - w_0 | w_0 - w \rangle|^2}{\|w_0 - w\|^2}} \geq 0 \quad \forall\, w \in W$$

$$\Rightarrow \quad \langle v - w_0 | w_0 - w \rangle = 0 \quad \forall\, w \in W$$

$$\Rightarrow \quad \langle v - w_0 | u \rangle = 0 \quad \forall\, u \in W \quad //.$$

Uniqueness of <u>best approximation</u>:

Best approximation of

a vector by a vector in a subspace of an i.p.s is unique.

Proof: Let $V \leftarrow$ i.p.s.

$W \subseteq V \leftarrow$ subspace of V.

$v \in V$

Suppose $w_1, w_2 \in W \leftarrow$ two best approximations of v in W

claim: $w_1 = w_2$.

$$\langle v - w_1 | w \rangle = 0 \quad \forall w \in W \quad \text{--- (1)}$$

$$\langle v - w_2 | w \rangle = 0 \quad \forall w \in W \quad \text{--- (2).}$$

$$\Rightarrow \langle v - w_1 | w_1 - w_2 \rangle = 0 \quad \& \quad \langle v - w_2 | w_1 - w_2 \rangle = 0$$

$$\langle w_1 - w_2 | w_1 - w_2 \rangle = \langle (v - w_2) - (v - w_1) | w_1 - w_2 \rangle$$

$$= \underbrace{\langle v - w_2 | w_1 - w_2 \rangle}_{0} - \underbrace{\langle v - w_1 | w_1 - w_2 \rangle}_{0}$$

$$= 0$$

$$\Rightarrow \quad w_1 - w_2 = 0$$

$$\Rightarrow \quad w_1 = w_2.$$

//.

How to find the best approximation:

Result: Let $V \leftarrow$ i.p.s.

$W \subseteq V \leftarrow$ subspace of V

$\{w_1, \dots, w_K\} \leftarrow$ orthogonal basis of W.

Then the best approximation of a vector $v \in V$ in W is given by

$$w_0 = \sum_{j=1}^{K} \frac{\langle v | w_j \rangle}{\langle w_j | w_j \rangle} w_j .$$

Proof: Let $v \in V$.

Let $w_0 \in W$ be the best approximation in W.

$$\Rightarrow \quad w_0 = \sum_{j=1}^{K} \alpha_j w_j \quad \text{for some } \alpha_j \in F.$$

We know. $\langle v - w_0 | w \rangle = 0 \quad \forall w \in W$

$$\Rightarrow \langle v - w_0 | w_j \rangle = 0 \quad \forall j = 1, \dots K.$$

$$\| $$

$$\left\langle v - \sum_{i=1}^{K} \alpha_i w_i \,\middle|\, w_j \right\rangle$$

$$\| $$

$$\langle v | w_j \rangle - \sum_{i=1}^{k} \alpha_i \langle w_i | w_j \rangle$$

$$\parallel$$

$$\langle v | w_j \rangle - \alpha_j \langle w_j | w_j \rangle$$

$$\Rightarrow \quad \alpha_j = \frac{\langle v | w_j \rangle}{\langle w_j | w_j \rangle}$$ for each $j = 1, 2, \ldots k$.

$$\Rightarrow \quad w_0 = \sum_{i=1}^{k} \frac{\langle v | w_i \rangle}{\langle w_i | w_i \rangle} w_i$$ is the best approximation of v in W.

//.

Example: $V = \mathbb{R}^3$ with the inner product

$$\langle (x_1, y_1, z_1) | (x_2, y_2, z_2) \rangle = x_1 x_2 + 2x_1 y_2 + 2x_2 y_1 + 5 y_1 y_2 + z_1 z_2 .$$

find the shortest distance of $v = e_1 + e_2 + e_3 = (1,1,1)$ from the subspace $W = \{ (x, y, z) \in \mathbb{R}^3 \mid z = 0 \}$

Soln: Clearly $\dim(W) = 2$.

Step-1 Find an orthogonal basis of W.

for this $\beta = \{e_1, e_2\}$ basis of W.

Apply Gram-Schmidth on β to

find $\beta' = \{q_1, q_2\}$ $\leftarrow$ orthogonal basis

$$q_1 = e_1, \quad q_2 = e_2 - 2e_1 \quad \text{(verify)}$$

<u>Step 2</u> Best approximation is

$$w_0 = \frac{\langle v | q_1 \rangle}{\langle q_1 | q_1 \rangle} q_1 + \frac{\langle v | q_2 \rangle}{\langle q_2 | q_2 \rangle} q_2$$

$$= \frac{\langle e_1 + e_2 + e_3 | e_1 \rangle}{\langle e_1 | e_1 \rangle} e_1 + \frac{\langle e_1 + e_2 + e_3 | e_2 - 2e_1 \rangle}{\langle e_2 - 2e_1 | e_2 - 2e_1 \rangle} (e_2 - 2e_1)$$

$$= \frac{3}{1} \cdot e_1 + \frac{1}{1} \cdot (e_2 - 2e_1)$$

$$= e_1 + e_2$$

$$= (1, 1, 0) \quad //.$$