Lecture-13

Matrix representation of a linear transformation

Recall: V, W are vector spaces over $\mathbb{F}$.

A map $T: V \to W$ is said to be linear transformation if

$$T(au + bv) = a\,T(u) + b\,T(v)$$
$$\forall\, u, v \in V \text{ and } a, b \in \mathbb{F}.$$

In this Lecture all vector spaces are assumed to be finite dimensional.

Let $T: V \longrightarrow W$ be a linear trans.

Let $B = \{v_1, \ldots, v_m\}$, $B' = \{w_1, \ldots, w_n\}$ be ordered bases of V and W respectively.

Let

$$T(v_j) = \sum_{i=1}^{n} t_{ij}\, w_i$$

for $j = 1, \ldots, m$ and $t_{ij} \in \mathbb{F}$.

This gives us an $n \times m$ matrix A whose (i,j)-th entry is t_{ij}.

The coefficients of $T(v_j)$ above constitutes the j-th column of A.

This matrix is called the matrix representation of T with respect to the bases B, B' and it is denoted by $[T]_B^{B'}$.

When $V = W$ and $B = B'$ then we write the matrix representation by $[T]_B$.

Example! Consider the following lin. trans.

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \quad \text{given by}$$

$$T(x,y) = (x+y,\ 2x+y,\ 4x+11y)$$

Consider the following ordered bases

$$B = \{(1,0), (1,1)\} \text{ of } \mathbb{R}^2 \quad \text{and}$$

$$B' = \{(1,0,0), (0,1,0), (0,1,1)\} \text{ of } \mathbb{R}^3.$$

We will find $[T]_B^{B'}$.

$$T(1,0) = (1,2,4)$$

$$= 1(1,0,0) - 2(0,1,0) + 4(0,1,1)$$

$$T(1,1) = (2,3,15)$$

$$= 2(1,0,0) - 12(0,1,0) + 15(0,1,1)$$

We get

$$[T]_B^{B'} = \begin{bmatrix} 1 & 2 \\ -2 & -12 \\ 4 & 15 \end{bmatrix}$$

Lemma: 1 Let $T: V \longrightarrow W$ be a lin. trans.

Let B, B' be orderderd bases of V, W resp.

Then $\forall v \in V$

$$[T(v)]_{B'} = [T]_B^{B'} [v]_B$$

Recall: The notation $[v]_B$ is for the coordinate vector (which is a column matrix) of v w.r.t. the ordered basis B.

Proof: Write $B = \{v_1, ---, v_m\}$ and

$$B' = \{w_1, ---, w_n\}.$$

If $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ then $v = \sum_{i=1}^{m} a_j v_j$

We have

$$T(v) = T\left(\sum_{i=1}^{m} a_j v_j\right)$$

$$= \sum_{i=1}^{m} a_j T(v_j)$$

$$T(v) = \sum_{j=1}^{m} a_j \left(\sum_{i=1}^{n} t_{ij} w_i \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} t_{ij} a_j \right) w_i$$

$$[T(v)]_{B'} = \begin{bmatrix} \sum_{j=1}^{m} t_{1j} a_j \\ \vdots \\ \sum_{j=1}^{m} t_{nj} a_j \end{bmatrix} = [T]_B^{B'} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

Clearly,

$$[T(v)]_{B'} = [T]_B^{B'} [v]_B .$$

Lemma: 2. Suppose $A, B \in M_{m\times n}(\mathbb{F})$.

If $AX = BX \;\forall\; X \in M_{n\times 1}(\mathbb{F})$ then $A = B$.

Proof: Let $X = e_i^t \in M_{n\times 1}(\mathbb{F})$

where $e_i \in \mathbb{F}^n$ is a vector with i-th entry 1 and others 0 for $i = 1, ---, n$.

Write $A = (a_{ij})$ and $B = (b_{ij})$.

Then $A e_i^T = B e_i^T$

$$\Rightarrow \begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix} = \begin{bmatrix} b_{1i} \\ \vdots \\ b_{mi} \end{bmatrix} \quad \forall\; i = 1, ---, n.$$

$$\Rightarrow \quad a_{ij} = b_{ij} \quad \forall\; i, j$$

$$\Rightarrow \quad A = B.$$

Lemma 3. Let V be an n-dimensional vector space and B an ordered basis of V. Then $V \longrightarrow \mathbb{F}^n$ given by

$$v \longmapsto [v]_B^t \text{ is an isomorphism}$$

Proof: The map is well defined (why?).

If $[v]_B = [w]_B$ then $v = w$, which makes the map one-to-one. $B = \{v_1, \ldots, v_n\}$

For any $(a_1, \ldots, a_n) \in \mathbb{F}^n$,

take $v = a_1 v_1 + \cdots + a_n v_n$

(where $B = \{v_1, \ldots, v_n\}$),

then $[v]_B^t = (a_1, \ldots, a_n)$.

The map is onto.

From the definition, it follows that

$$[av_1 + bv_2]_B = a[v_1]_B + b[v_2]_B$$

and hence the map is linear trans.

$m = \dim V$
$n = \dim W$

Observation: Let $T: V \to W$ be a lin. trans.

Let B_1, B_1' be two (ordered) bases of V and
B_2, B_2' two (ordered) bases of W.

Recall how the coordinate of a vector changes when you change the ordered basis (Lecture 9).

For $v \in V$, $$[v]_{B_1'} = P\,[v]_{B_1} \quad (1)$$

and $$[T(v)]_{B_2'} = Q\,[T(v)]_{B_2} \quad (2)$$

where $P \in M_{m\times m}(\mathbb{F})$ and $Q \in M_{n\times n}(\mathbb{F})$ are base change (invertible) matrices.

In fact, $P = [Id_V]_{B_1}^{B_1'}$ & $Q = [Id_W]_{B_2}^{B_2'}$.

Use Lemma 1 above in Equation (2),

$$[T]_{B_1'}^{B_2'}[v]_{B_1'} = Q\,[T]_{B_1}^{B_2}[v]_{B_1} \quad (3)$$

Use (1) in (3),

$$[T]_{B_1'}^{B_2'} \, P \, [v]_{B_1} = Q \, [T]_{B_1}^{B_2} \, [v]_{B_1} \qquad (4)$$

Write $A = [T]_{B_1'}^{B_2'} P$, $B = Q \, [T]_{B_1}^{B_2}$

Using Lemma 3, the Equation (4) implies

$$AX = BX \quad \text{for all } X \in M_{n \times 1}(\mathbb{F}).$$

Then Lemma 2 gives us $A = B$ i.e.

$$[T]_{B_1'}^{B_2'} \, P = Q \, [T]_{B_1}^{B_2}.$$

<u>Conclusion:</u> (1) $[T]_{B_1}^{B_2} = Q^{-1} [T]_{B_1'}^{B_2'} P.$

(2) When $V = W$, $B_1 = B_2 = B$ & $B_1' = B_2' = B'$

we have $P = Q$ and

$$[T]_B = P^{-1} [T]_{B'} P.$$

i.e. $[T]_B$ and $[T]_{B'}$ are similar or conjugate.

<u>Recall:</u> Two matrices $A, B \in M_n(\mathbb{F})$ are called conjugate/similar if $\exists$ invertible P s.t. $B = P^{-1}AP$.

Example: Consider the following lin. trans.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \text{ given by}$$

$$T(x,y) = (x+y,\ 2x+y,\ 4x+11y)$$

Consider the following ordered bases

$$B_1 = \{(1,0), (1,1)\},\ B_1' = \{(1,0),(0,1)\} \text{ of } \mathbb{R}^2$$

$$B_2 = \{(1,0,0), (0,1,0), (0,1,1)\},$$

$$B_2' = \{(1,0,0), (0,1,0), (0,0,1)\} \text{ of } \mathbb{R}^3.$$

Then we know $P = [Id_{\mathbb{R}^2}]_{B_1}^{B_1'}$

$$Id_{\mathbb{R}^2}(1,0) = (1,0) = 1(1,0) + 0(0,1)$$

$$Id_{\mathbb{R}^2}(1,1) = (1,1) = 1(1,0) + 1(0,1)$$

Therefore $P = [Id_{\mathbb{R}^2}]_{B_1}^{B_1'} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$

Similarly, for Q consider

$$Id_{\mathbb{R}^3}(1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$

$$Id_{\mathbb{R}^3}(0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$$

$$Id_{\mathbb{R}^3}(0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

Therefore, $Q = [Id_W]_{B_2}^{B_2'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

We compute $[T]_{B_1}^{B_2}$:

$$T(1,0) = (1,2,4) = (1,0,0) - 2(0,1,0) + 4(0,1,1)$$

$$T(1,1) = (2,3,15) = 2(1,0,0) - 12(0,1,0) + 15(0,1,1)$$

$$\Rightarrow [T]_{B_1}^{B_2} = \begin{bmatrix} 1 & 2 \\ -2 & -12 \\ 4 & 15 \end{bmatrix}.$$

Similarly compute $[T]_{B_1'}^{B_2'} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 11 \end{bmatrix}$

Verify: $[T]_{B_1}^{B_2} = Q^{-1} [T]_{B_1'}^{B_2'} P.$

i.e.
$$\begin{bmatrix} 1 & 2 \\ -2 & -12 \\ 4 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Definition: Let $S, T : V \longrightarrow W$ be lin. trans.

We define

① a lin. trans. $S+T : V \longrightarrow W$ by

$$(S+T)(v) = S(v) + T(v) \quad \forall v \in V.$$

② a lin. trans $\lambda T : V \to W$ for $\lambda \in \mathbb{F}$

by $$(\lambda T)(v) = \lambda T(v) \quad \forall v \in V.$$

Exercises: Let $S, T : V \to W$ be lin. trans. and B, B' ordered bases of V, W resp. Then

ⓐ $$[S+T]_B^{B'} = [S]_B^{B'} + [T]_B^{B'}$$

ⓑ $$[\lambda T]_B^{B'} = \lambda [T]_B^{B'}$$

ⓒ For a lin. trans. $T' : W \to U$ and a bases B'' of U we have

$$[T' \circ T]_B^{B''} = [T']_{B'}^{B''} \cdot [T]_B^{B'}.$$