Eigenvalue / eigenvector, Independent subspaces

Recall: (Eigenvalue / eigenvector of a matrix)

field: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

Let $A \in M_{n\times n}(\mathbb{F})$. Then $\lambda \in \mathbb{F}$ is called an eigenvalue of A if $\exists$ a nonzero vector $v \in \mathbb{F}^n$ such that $Av = \lambda v$.

- v is called an eigenvector of A corresponding to eigenvalue λ.
- $\ker(\lambda I - A) = \{v \in \mathbb{F}^n : (\lambda I - A)v = 0\}$

Clearly, $v (\neq 0)$ is an eigenvector of A corresponding to eigenvalue λ if and only if $v \in \ker(\lambda I - A)$.

$\ker(\lambda I - A)$ is called the eigenspace of A corresponding to eigen value λ.

- The polynomial $\det(tI - A)$ is called the characteristic polynomial of A.

  [Note that this is a monic poly. of degree n].

- The equation $\det(tI-A) = 0$ is called the char. equation of A.

- λ is an e.value of A if and only if λ is a root of the char. poly. of A.

  ($\Rightarrow$) $\exists$ $v (\neq 0) \in \mathbb{F}^n$ such that $Av = \lambda v$

  $$\Rightarrow (\lambda I - A)v = 0$$
  $$\Rightarrow \det(\lambda I - A) = 0.$$

  ($\Leftarrow$) $\det(\lambda I - A) = 0 \Rightarrow \exists$ $v(\neq 0) \in \mathbb{F}^n$ such that $(\lambda I - A)v = 0$

  $$\Rightarrow Av = \lambda v.$$

- Thus e.values of A are nothing but the roots of the char. poly. of A.

  This implies "A can have at most n e.values".

- Method of finding e.values / e.vectors of A.

<u>steps</u> 1. Compute the char. poly. of A

$$\det(tI - A)$$

<u>2.</u> find the e.values of A by solving the char. equation

$$\det(tI - A) = 0 \quad \text{for } t.$$

<u>3.</u> for each e.value λ, find the e.space

$$\ker(\lambda I - A).$$

<u>4.</u> find a basis for each e.space.

Eigenvalue / eigenvector of a <u>linear operator</u>:

Let $V \leftarrow$ vector space over $\mathbb{F}$

$T : V \to V$ linear operator.

a scalar $\lambda \in \mathbb{F}$ is called an e.value of T if $\exists$ a <u>nonzero vector</u> $v \in V$ such that

$$T(v) = \lambda v.$$

- v is called an e.vector of T corresponding to e.value λ.

- Note that for any $z \in \mathbb{C}$,

$$zI - T : V \longrightarrow V \quad \text{is a linear operator}$$
$$(zI-T)(v) = zv - T(v).$$

Thus, if λ is an e.value of T, then

$$\ker(\lambda I - T) = \{ v \in V : (\lambda I - T)(v) = 0 \}$$

is called the e.space of T corresponding to e.value λ.

- char. poly. of T: $\det(tI - T) = \det\left([tI-T]_\beta\right)$ for any basis β of V.

- char. equation of T: $\det(tI-T) = 0$.

- Method of e.value / e.vector of T.

<u>Steps</u> 1. Compute the char. poly. of T.

$$\det(tI-T)$$

2. find the e-values of T by solving the char. equation

$$\det(tI - T) = 0 \quad \text{for } t$$

3. for each e-value λ, find the e-space $\ker(\lambda I - T)$

4. find a basis for each e-space.

Result: Let $T: V \to V$ linear operator
let β be a basis of V.
Then λ is an e-value of T if and only if λ is an e-value of $[T]_\beta$.
Moreover, for each e-value λ,

$$\dim(\ker(\lambda I - T)) = \dim(\ker(\lambda I - [T]_\beta)).$$

Proof: ($\Rightarrow$) Suppose λ is an e-value of T.

$\Rightarrow \exists\ v\ (\neq 0) \in V$ such that

$$T(v) = \lambda v$$

$$\Rightarrow \quad [T(v)]_\beta = [\lambda v]_\beta$$

$\therefore$

$$\Rightarrow [T]_\beta [v]_\beta = \lambda [v]_\beta$$

$$\Rightarrow \lambda \text{ is an e-value of } [T]_\beta \quad (\because [v]_\beta \neq 0)$$

($\Leftarrow$) Let λ be an e-value of $[T]_\beta$.

Suppose $\beta = \{v_1, \ldots, v_n\}$.

$\exists\ x (\neq 0) \in \mathbb{F}^n$ such that $[T]_\beta x = \lambda x$.

let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and consider $v = \sum_{i=1}^{n} x_i v_i$.

Then $v \neq 0$, $[v]_\beta = x$.

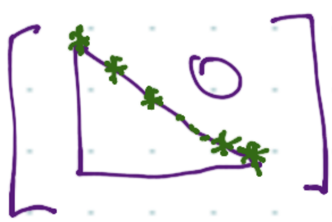
We have
$$[T]_\beta x = \lambda x$$
$$\Rightarrow [T]_\beta [v]_\beta = \lambda [v]_\beta$$
$$\Rightarrow [T(v)]_\beta = [\lambda v]_\beta$$
$$\Rightarrow [T(v) - \lambda v]_\beta = 0$$
$$\Rightarrow T(v) - \lambda v = 0$$
$$\Rightarrow T(v) = \lambda v.$$
$$\Rightarrow \lambda \text{ is an e-value of } T.$$

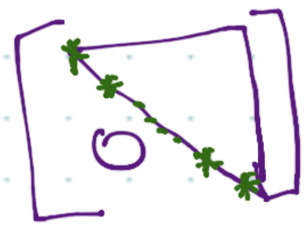
$\dim(\ker(\lambda I - T)) = \dim(\ker(\lambda I - [T]_\beta))$ (Exercise?)

Question How to find e-values of a large matrix?

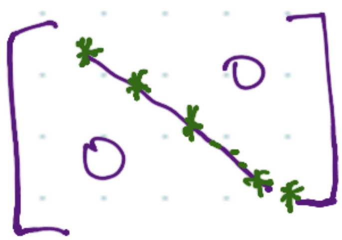
Abel's Theorem: There are no formulas for finding the roots of generic polynomial of degree greater than 4.

- e-values of a triangular matrix





- e-values of a diagonal matrix.



- Similar matrices have the same e-values

- **Question:** whether

Independent Subspaces: Let $V \leftarrow$ vector space over $\mathbb{F}$

$W_1, W_2, \dots, W_m \subseteq V$ subspaces

Then they are called independent if

$w_1 + w_2 + \dots + w_m = 0$ for $w_i \in W_i$

implies $w_i = 0$ for each $i = 1, 2, \dots m$.

Result: Let $V \leftarrow$ vector space over $\mathbb{F}$.

$W_1, \dots, W_m \subseteq V$ subspaces.

Then $W_1, \dots, W_m$ are independent if and only if

$$\dim(W_1 + \dots + W_m) = \dim(W_1) + \dots + \dim(W_m)$$

Proof: first note that

$$\dim(W_1 + \dots + W_m) \leq \sum_{i=1}^{m} \dim(W_i) \quad \text{--- (1)}$$

Verify? Hint: Proof by induction

let $k_i = \dim(W_i)$ and

$\beta_i = \{w_{i,1}, w_{i,2}, \dots w_{i,k_i}\}$ basis of W_i

Consider $\beta = \bigcup_{i=1}^{m} \beta_i$.

Then
- β spans $W_1 + \cdots + W_m$ (why?)
- $|\beta| \leq \sum_{i=1}^{m} |\beta_i|$ — (2)

($\Rightarrow$) Suppose that $W_1, \ldots, W_m$ are independent.

claim: $\dim(W_1 + \cdots + W_m) = \sum_{i=1}^{m} \dim(W_i)$

Observe that $\beta_i \cap \beta_j = \phi$ for $i \neq j$

Since $v \in \beta_1 \cap \beta_2$

$\Rightarrow$ $v + (-v) + 0 + \cdots + 0 = 0$

$\Rightarrow$ $v = 0$ but $v \neq 0$

If we show that β is linearly independent, then this proves our claim, because

$\Rightarrow$ β is a basis of $W_1 + \cdots + W_m$

$\Rightarrow$ $\dim(W_1 + \cdots + W_m) = |\beta| = \sum_{i=1}^{m} |\beta_i| = \sum_{i=1}^{m} \dim(W_i)$

Thus we show that β is linearly independent.

For this, let $\sum_{i=1}^{m} \sum_{j=1}^{k_i} \alpha_{i,j} w_{i,j} = 0$, $\alpha_{i,j} \in \mathbb{F}$

$$\Rightarrow \underbrace{\sum_{j=1}^{k_1} \alpha_{1,j} w_{1,j}}_{\in W_1} + \underbrace{\sum_{j=1}^{k_2} \alpha_{2,j} w_{2,j}}_{\in W_2} + \cdots\cdots + \underbrace{\sum_{j=1}^{k_m} \alpha_{m,j} w_{m,j}}_{\in W_m} = 0$$

Since $W_1, \ldots, W_m$ are independent, we have

$$\sum_{j=1}^{k_i} \alpha_{i,j} w_{i,j} = 0 \quad \text{for each } i = 1, 2, \ldots m.$$

but for each i, β_i is linearly independent.

$$\Rightarrow \alpha_{i,j} = 0 \quad \forall\, i, j.$$

$\Rightarrow \beta$ is linearly independent.

$(\Leftarrow)$ Conversely, let $\dim(W_1 + \cdots + W_m) = \sum_{i=1}^{m} \dim(W_i)$

Since β spans $W_1 + \cdots + W_m$, we have

$$\dim(W_1 + \cdots + W_m) \leq |\beta|$$

$$\sum_{i=1}^{m} \dim(W_i) = \sum_{i=1}^{m} |\beta_i|$$

$$\Rightarrow \quad |\beta| \geq \sum_{i=1}^{m} |\beta_i| \quad \text{—— (3).}$$

From (2) & (3), we have

$$|\beta| = \sum_{i=1}^{m} |\beta_i| = \dim(W_1 + \cdots + W_m).$$

Thus, $\begin{cases} \beta \text{ spans } W_1 + \cdots + W_m, \text{ and} \\ |\beta| = \dim(W_1 + \cdots + W_m). \end{cases}$

$\Rightarrow$ β is a basis of $W_1 + \cdots + W_m$.

<u>claim</u>: $W_1, \ldots, W_m$ are independent.

let $w_1 + w_2 + \cdots + w_m = 0$, where $w_i \in W_i$

$\Rightarrow$ for each i, $\displaystyle w_i = \sum_{j=1}^{k_i} b_{i,j}\, w_{i,j}$ for some $b_{i,j} \in \mathbb{F}$

$$\Rightarrow \sum_{i=1}^{m} w_i = \sum_{i=1}^{m} \sum_{j=1}^{k_i} b_{i,j}\, w_{i,j} = 0$$

$\Rightarrow \quad b_{i,j} = 0 \quad \forall\ i, j$, since β is a basis.

$\Rightarrow \quad w_i = 0$ for each i

$\Rightarrow \quad W_1, \ldots, W_m$ are independent.

//.

Theorem: Suppose $T: V \longrightarrow V$ linear operator. Then the eigenspaces corresponding to distinct e-values are independent subspaces of V.

Proof: Suppose $\lambda_1, \ldots, \lambda_m$ are distinct e-values of T.

Define $W_i = \ker(\lambda_i I - T) \leftarrow$ e-spaces

claim: $W_1, \ldots, W_m$ are independent.

Proof by induction on m:

when $m=1$, nothing to prove.

when $m=2$, let $w_1+w_2=0$, where $w_1 \in W_1$, $w_2 \in W_2$

$$\Rightarrow w_1 = -w_2$$

$$\Rightarrow w_1 \in W_1 \cap W_2$$

$\Rightarrow$ w_1 is an e.vector for both λ_1 and λ_2.

$\Rightarrow$ $T(w_1) = \lambda_1 w_1$ and $T(w_1) = \lambda_2 w_1$

$\Rightarrow$ $(\lambda_1 - \lambda_2) w_1 = 0$

$\Rightarrow$ $w_1 = 0$ as $\lambda_1 \neq \lambda_2$

$\Rightarrow$ $w_2 = 0$.

$\Rightarrow$ W_1, W_2 are independent.

So result is true when $m=2$.

Induction hypothesis: Suppose the result is true for $m-1$.

Let $w_1 + \cdots + w_m = 0$, where $w_i \in W_i$ — (1)

Applying T both sides in (1), we get

$$T(w_1) + \cdots + T(w_m) = 0$$

$$\lambda_1 w_1 + \cdots + \lambda_m w_m = 0 \quad \text{— (2)}$$

Again, multiplying λ_m both sides in (1), we get

$$\lambda_m w_1 + \lambda_m w_2 + \cdots + \lambda_m w_m = 0 \quad \text{—(3)}$$

from (2) − (3), we get

$$\underbrace{(\lambda_1 - \lambda_m) w_1}_{\in W_1} + \underbrace{(\lambda_2 - \lambda_m) w_2}_{\in W_2} + \cdots + \underbrace{(\lambda_{m-1} - \lambda_m) w_{m-1}}_{\in W_{m-1}} = 0$$

by induction hypothesis, $(\lambda_i - \lambda_m) w_i = 0$ for $i = 1, 2, \ldots m-1$

$\Rightarrow$ $w_i = 0$ as λ_i are distinct

$\Rightarrow$ from (1), $w_m = 0$.

$\Rightarrow$ $W_1, W_2, \ldots, W_m$ are independent. //.

<u>Corollary</u>: Eigenvectors corresponding to distinct e-values are linearly independent.

<u>Proof</u>: <u>Hint</u>: $\lambda_1, \ldots, \lambda_m \leftarrow$ distinct e-values.

$$\begin{matrix} \lambda_1, & \ldots, & \lambda_m \\ \downarrow & & \downarrow \\ v_1, & \ldots, & v_m \end{matrix} \leftarrow \text{corresponding e-vectors}$$

$$\underbrace{\alpha_1 v_1}_{W_1} + \cdots + \underbrace{\alpha_m v_m}_{W_m} = 0 \Rightarrow \alpha_i v_i = 0$$

$$\Rightarrow \alpha_i = 0 \quad \forall i.$$