ODE

Lecture 3

Recall

Existence and uniqueness of Solution

consider

$$\left.\begin{aligned} t\, x' &= 2x \\ x(t_0) &= x_0 \end{aligned}\right\}$$

Here $f(t, x) = \dfrac{2x}{t}$

Note that f is continuous around (t_0, x_0) if $t_0 \neq 0$.

$$\frac{\partial f}{\partial x} = \frac{2}{t}$$

Clearly $\frac{\partial f}{\partial x}$ is continuous in a small enough rectangle around (t_0, x_0) if $t_0 \neq 0$ and hence $\frac{\partial f}{\partial x}$ is bounded

Consequently f is Lipschitz in x-variable

Hence by the existence and uniqueness theorem the given IVP has a unique solution in an interval about t_0, for $t_0 \neq 0$.

For $t_0 = 0$

Nothing can be said from the existence and uniqueness theorem.

Fortunately, we can solve the actual problem and find

$x = At^2$ is the general solution.

When $t_0 = 0$, there exists no solution when $x_0 \neq 0$.

When $t_0 = 0$ and $x_0 = 0$, then we have infinite number of solutions

$x(t) = At^2$ for given IVP.

Continuous dependence

Consider $\left\{ \begin{array}{l} x'(t) = f(x,t) \\ x(t_0) = x_0 \end{array} \right\}$ Actual IVP

The value x_0 at t_0 may not be known exactly in practice, but that itself may be an approximation $\tilde{x}_0$.

$$\left.\begin{aligned} x'(t) &= f(x,t) \\ x(t_0) &= \tilde{x}_0 \\ &= x_0 + \delta \end{aligned}\right\} \text{IVP considered}$$

<u>Qn:</u> Whether the Solution depends continuously on the initial data.

i.e, a small change in the initial data should not give rise to a big change in the solution.

<u>example</u>

$x'(t) = K\, x(t)$? actual IVP I

$x'(t) = \ldots$
$x(0) = x_0$ } IVP I

$x'(t) = k\, x(t)$
$x(0) = x_0 + \delta$ } IVP considered $\underline{\mathrm{II}}$

Suppose $x(t)$ is a solution of $\underline{\mathrm{I}}$, Then

$$x(t) = x_0\, e^{kt}$$

If $\tilde{x}(t)$ is a solution of $\underline{\mathrm{II}}$, then

$$\tilde{x}(t) = (x_0 + \delta)\, e^{kt}$$

$$x(t) - \tilde{x}(t) = \delta\, e^{kt}$$

continuous

This shows

dependence on the initial data.

Well-posedness

Suppose the mathematical model is given by

$$\begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

Then the IVP is well-posed if the following properties are satisfied.

1. There exists a Solution — existence
2. Solution is unique for the initial data — uniqueness

3. Solution depends continuously on the initial data – Stability

Picard's iteration

Consider the IVP

$$(*)_1 \begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

We solve $(*)_1$ by a process of iteration.

– The Picard's method of successive approximations.

This method generate a sequence of Solutions

$x_0(t), \quad x_1(t), \quad x_2(t) \ldots$

Observation

Let $x(t)$ be a solution of $(*)$;

Then $x(t)$ is automatically continuous.

$$x'(t) = f(t, x).$$

$$\Rightarrow \int_{t_0}^{t} x'(s)\, ds = \int_{t_0}^{t} f(s, x(s))\, ds$$

$$\Rightarrow x(t) - x(t_0) = \int_{t_0}^{t} f(s, x(s))\, ds$$

$$\Rightarrow x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds$$

$$\Rightarrow \boxed{x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds}$$

t_0 —— $(*)_2$

Thus $x(t)$ is a continuous solution of the integral eq^n $(*)_2$

Conversely, let $x(t)$ be a continuous solution of $(*)_2$

Then

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

$$\Rightarrow \quad x'(t) = f(t, x(t))$$

ie $x(t)$ satisfies the de in $(*)_1$

$$x(t_0) = x_0 + \int_{t_0}^{t_0} f(s, x(s))\, ds$$

$$= x_0$$

Key observation.

The IVP $(*)_1$ and integral eqn in $(*)_2$ are equivalent in the sense that, the solutions of $(*)_1$ — if any exist — are precisely the continuous solutions of $(*)_2$

In particular, we automatically obtain a solution for $(*)_1$ if we can construct a continuous solution for $(*)_2$

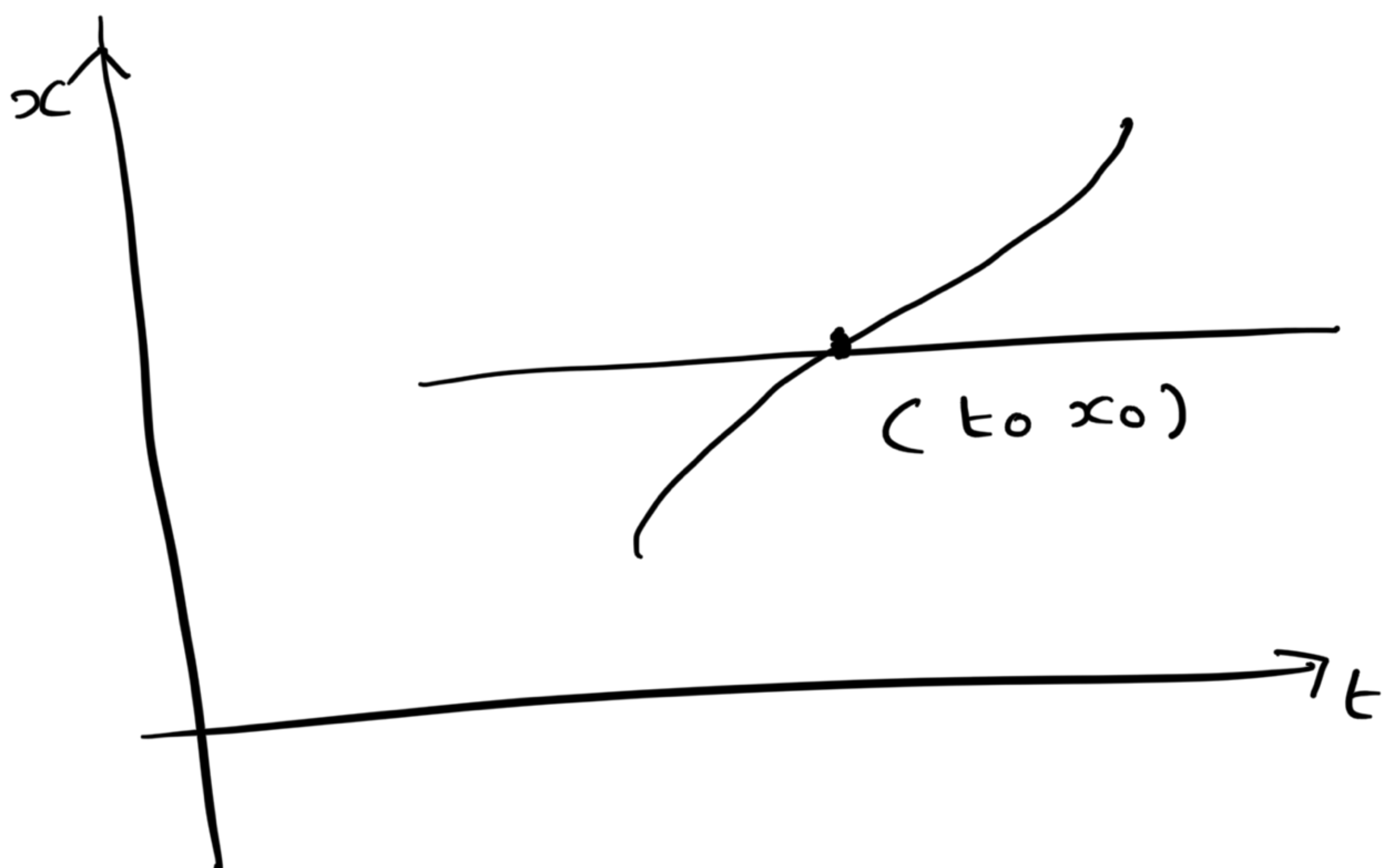
Consider

$$\begin{cases} x'(t) = f(t, x) \end{cases}$$

$$\circledast_1 \left\{ x(t_0) = x_0 \right.$$

A rough approximation to a solution of $\circledast_1$ is given by

$$x_0(t) = x_0 \quad \forall\, t$$



We insert this to RHS $\circledast_2$ to obtain a new and perhaps a better approximation to solution $x_1(t)$

Solution ...

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\,ds$$

$$= x_0 + \int_{t_0}^{t} f(s, x_0)\,ds$$

Next we use $x_1(t)$ to generate another, perhaps even better approxemation $x_2(t)$ in the same way

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\,ds$$

At the n^{th} stage of the process, we have

$$\boxed{x_n(t) = \underline{\underline{x_0}} + \int_{x_0}^{x} f(s, x_{n-1}(s))\,ds}$$

Example

$$\left.\begin{aligned} x' &= t + x \\ x(0) &= 1 \end{aligned}\right\}$$

$x_0 = \underline{\underline{1}}$ $\quad t_0 = 0 \quad$ $f(t, x(t))$
$= t + x.$

define

$$x_0(t) = 1 \quad \forall\ t$$

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\,ds$$

$$= 1 + \int_0^t (s+1)\,ds$$

$$= 1 + t + \frac{t^2}{2}$$

$$+ \int^{t} f(s, x(s))\,ds$$

$$x_2(t) = x_0 + \int_{t_0} f(s, x_1(s))\,ds$$

$$= 1 + \int_0^t \left(s + 1 + s + \frac{s^2}{2}\right) ds$$

$$= 1 + t + t^2 + \frac{t^3}{3!}$$

$$x_3(t) = x_0 + \int_{t_0}^{t} f(s, x_2(s))\,ds$$

$$= 1 + \int_0^t \left(s + 1 + s + s^2 + \frac{s^3}{3!}\right) ds$$

$$= 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{4!}$$

In general

$$x_n(t) = 1 + t + 2\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}\right) + \frac{t^{n+1}}{(n+1)!}$$

$$\longrightarrow 2(e^t - t - 1)$$

?

Note that $2e^t - t - 1$ is exact Solution of given IVP.

Theorem

Suppose the IVP

$$\left.\begin{aligned} x'(t) &= f(t, x) \\ x(t_0) &= x_0 \end{aligned}\right\}$$

be such that

$$f : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

satisfies conditions prescribed in existence and uniqueness theorem.

Then the successive approximations $x_n(t)$ converges to the unique solution $x(t)$