Tutorial Sheet - 3

1. Determine if the following limits exist:

$$\text{(a)} \lim_{x\to 0}[x] \qquad \text{(b)} \lim_{x\to 0}\operatorname{sgn}(x) \qquad \text{(c)} \lim_{x\to 0}\sin(\tfrac{1}{x}) \qquad \text{(d)} \lim_{x\to 0}\sqrt{x}\sin(\tfrac{1}{x}) \qquad \text{(e)} \lim_{x\to 0} x\cos(\tfrac{1}{x}).$$

2. Determine if the following limits exist: $\lim_{x\to 0}\frac{x-|x|}{x}$ and $\lim_{x\to\infty} x^{1+\sin x}$.

3. Show that the function f is continuous only at $x = 1/2$.

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational.} \end{cases}$$

4. Determine which of the following functions are uniformly continuous in the interval mentioned:

$$\text{(a) } e^{x^2}\sin(x^2) \text{ on } (0,1) \qquad \text{(b) } |\sin x| \text{ on } [0,\infty) \qquad \text{(c) } \sqrt{x}\sin x \text{ on } \mathbb{R} \qquad \text{(d) } \sin(x^2) \text{ on } \mathbb{R}.$$

5. Determine if the following functions are differentiable at 0. Also find $f'(0)$, if it exists

$$\text{(a) } f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0. \end{cases} \quad \text{(b) } f(x) = e^{-|x|},\ x \in \mathbb{R}. \quad \text{(c) } f(x) = \begin{cases} x\cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

6. Determine if $f'(x)$ is continuous at 0 for the following functions:

$$\text{(a) } f(x) = \begin{cases} x^3\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases} \quad \text{(b) } f(x) = \begin{cases} x^2\cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases} \quad \text{(c) } f(x) = \begin{cases} x^2\ln\frac{1}{|x|} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

7. Let f be differentiable on $\mathbb{R}$ and $\sup_{x\in\mathbb{R}}|f'(x)| < 1$. Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$. Prove that $\{s_n\}$ is a convergent sequence.

8. Let f be differentiable on $\mathbb{R}$ and $|f(x) - f(y)| \leq (x-y)^2$. Then show that f is constant.

9. Evaluate the following limits:

$$\text{(a)} \lim_{x\to 0}\frac{e^x - (1+x)}{x^2} \qquad \text{(b)} \lim_{t\to 0}\frac{1-\cos t - t^2/2}{t^4} \qquad \text{(c)} \lim_{x\to\infty} x^2(e^{-1/x^2} - 1).$$

10. Find an approximation of $\sin x$ when error is of magnitude no greater than 5×10^{-4} and $|x| < 3/10$.

11. Estimate the error in the approximation of $\sinh x = x + \dfrac{x^3}{3!}$ when $|x| < 0.5$.

12. Find the radius of convergence of the following power series

$$\text{(a)} \sum_{n=0}^{\infty}(n+1+2^n)x^n \ \text{(b)} \sum_{n=0}^{\infty}\frac{1}{a^n}x^{2n},\ a\neq 0 \ \text{(c)} \sum_{n=0}^{\infty}\frac{1}{n!n^n}x^n \ \text{(d)} \sum_{n=0}^{\infty}\frac{n!}{n^n}x^n \ \text{(e)} \sum_{n=1}^{\infty}\frac{n^{n^2}}{(n+1)^{n^2}}(x-1)^n.$$

13. Write the Taylor's series around 0 and find the radius of convergence.

$$\text{(a) } \frac{1}{1+x} \qquad \text{(b) } \sinh x \qquad \text{(c) } e^x\sinh x \qquad \text{(d) } x\sin x.$$

14. Obtain the Taylor's series around 0 for the following series using term by term differentiation/integration and calculate the radius of convergence. Is this the maximal interval of validity of the series?

$$\text{(a) } \tan^{-1}(x) \ \text{(b) } \sin^{-1}(x) \ \text{(c) } \sinh^{-1}(x) \ \text{(d) } \frac{1}{(1+x^2)^2}.$$