Lecture - 14

Eigenvalue and Eigenvector

Definition: Let V be a vector space over $\mathbb{F}$ and let $T: V \longrightarrow V$ be a linear operator. Then $\lambda \in \mathbb{F}$ is said to be an eigenvalue of T if there exists $v \in V$, $v \neq 0$ s.t.

$$T(v) = \lambda v.$$

Such a non-zero vector v is called an eigenvector of T associated to eigenvalue λ.

Example: ① Let $Z: V \longrightarrow V$ be the zero operator, i.e. $Z(v) = 0 \ \forall v \in V$ then 0 is the only eigenvalue of Z and any $v \in V$ and $v \neq 0$ is an eigenvector associated to the eigenvalue 0, since

$$Z(v) = 0 = 0 \cdot v$$

2. Let $I : V \longrightarrow V$ be the identity operator, i.e. $I(v) = v \ \forall v \in V$.

Then 1 is the only eigenvalue of I and any $v \neq 0$ is an eigenvector of I associated to the eigenvalue 1, since

$$I(v) = v = 1 \cdot v$$

Remark: 1. If 0 is the only eigenvalue of an operator $T : V \to V$ then T need not be the zero operator.

e.g. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$T(x, y) = (0, x).$$

If λ is an eigenvalue then $\exists\, (x,y) \neq (0,0)$ with $T(x,y) = \lambda(x,y)$

$$\Rightarrow \quad (0, x) = (\lambda x, \lambda y)$$

$$\Rightarrow \quad \lambda x = 0 \ \& \ \lambda y = x$$

If $\lambda \neq 0$ then $x = 0$ & $y = 0$ i.e. $(x,y) = (0,0)$

which is a contradiction since $(x,y) \neq (0,0)$.

Thus $\lambda = 0$ is the only possibility.
In fact, $T(1,0) = (0,0) = 0(1,0)$.
In other words, 0 is the only eigenvalue of T but $T \neq 0$ since

$$T(0,1) = (1,0) \neq (0,0).$$

2. If 1 is the only eigenvalue of an operator $T: V \to V$ then T need not be the identity operator.

Example: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$T(x,y) = (x+y, y)$$ has 1 its only eigenvalue but $T \neq I$, since $T(0,1) = (1,1) \neq (0,1)$.

Question: For a given linear operator, how to find eigenvalue and eigenvectors?

Remark: If v is an eigenvector of T associated to an eigenvalue λ of T, then for any $a \neq 0$ $a\in\mathbb{F}$, av is also an eigenvector associated to eigenvalue λ.

Example: Let $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

$$T(x,y)=(2x+3y,\ 3x+2y).$$

We want to find $\lambda\in\mathbb{R}$ and $(x,y)\neq(0,0)$

s.t. $\quad T(x,y)=\lambda(x,y)$

Then $\quad (2x+3y, 3x+2y)=(\lambda x,\lambda y)$

$$\Rightarrow \quad 2x+3y=\lambda x \quad \& \quad 3x+2y=\lambda y$$

$$\Rightarrow \begin{cases}(2-\lambda)x+3y=0\\ 3x+(2-\lambda)y=0\end{cases}$$

This system of equations has a non-trivial i.e. $(x,y)\neq(0,0)$ solution if and only if $\det\begin{pmatrix}2-\lambda & 3\\ 3 & 2-\lambda\end{pmatrix}=0$

or $\quad (2-\lambda)^2 - 9 = 0$

or $\quad 2-\lambda = \pm 3$

or $\quad \lambda = -1, 5$

For $\lambda = -1$ we get $3x + 3y = 0 \Rightarrow y = -x$
i.e. $(a, -a)$ for $a \neq 0$ is an eigenvector associated to $\lambda = -1$. In particular, $(1, -1)$ is an eigenvector associated to $\lambda = -1$.

For $\lambda = 5$ we get $3x - 3y = 0 \Rightarrow y = x$
i.e. (a, a) for $a \neq 0$ is an eigenvector associated to $\lambda = 5$. In particular, $(1, 1)$ is an eigenvector associated to $\lambda = 5$.

Observation: For the possible eigenvalue we had to solve a polynomial in λ arising from determinant of a matrix.

Once an eigenvalue is given then we had to solve a system of linear equations for eigenvector.

Remark:

1. For an invertible operator T, 0 cannot be an eigenvalue. (otherwise $\ker(T) \neq \{0\}$).

2. If $\lambda \neq 0$ is an eigenvalue of an invertible operator T then λ^{-1} is an eigenvalue of T^{-1}.

Proof: Suppose $\lambda \neq 0$ is an eigenvalue of T. Then $\exists\ v \neq 0$ s.t. $T(v) = \lambda v$.

Since T^{-1} exists and is linear trans, we get $T^{-1}(T(v)) = T^{-1}(\lambda v)$

$$\Rightarrow \quad I(v) = \lambda T^{-1}(v)$$

$$\Rightarrow \quad v = \lambda T^{-1}(v)$$

$$\Rightarrow \quad T^{-1}(v) = \lambda^{-1} v.$$

3) Suppose that λ is an eigenvalue of an operator $T : V \to V$.

Then $\exists\, v \neq 0$ s.t. $T(v) = \lambda v$.

Consider $T^2 = T \circ T : V \longrightarrow V$.

We get
$$\begin{aligned} T^2(v) &= T(T(v)) \\ &= T(\lambda v) \\ &= \lambda T(v) \\ &= \lambda(\lambda v) \\ &= \lambda^2 v \end{aligned}$$

i.e. λ^2 is an eigenvalue of T^2.

You can use induction and prove that λ^n is an eigenvalue of the operator $T^n = T \circ T \circ \cdots \circ T$ (n-times) for any integer $n \geq 1$.

4) If λ is an eigenvalue of T then $a\lambda$ is an eigenvalue for aT for all $a \in \mathbb{F}$.

Assumption: From now onwards V is finite dim. vector space over $\mathbb{F}$.

Definition: Let $T: V \to V$ be a lin. op. Let B be an ordered basis of V. Define trace and determinant of T by

$$\mathrm{tr}(T) := \mathrm{tr}([T]_B) \ \& \ \det(T) := \det([T]_B).$$

Remark: $\mathrm{tr}(T)$ and $\det(T)$ do not depend on the choice of the ordered basis B of V, because trace and determinant of conjugate matrices are the same.

Definition: (characteristic polynomial)

For a lin. op. $T: V \to V$ the polynomial

$$\det(XI - T) \in \mathbb{F}[X]$$ is called the characteristic polynomial of T.

Remark: The characteristic polynomial of $T: V \to V$ is a monic polynomial of degree equal to $\dim V$.

The equation $\det(XI - T) = 0$ is called the characteristic equation.

Definition: Let $T: V \to V$ be a lin. op. For an eigenvalue λ of T,

$$V_\lambda := \{ v \in V : T(v) = \lambda v \}$$

is a subspace which is called the eigenspace associated to λ.

Theorem: Let $T: V \to V$ be a lin. op. & $\lambda \in \mathbb{F}$.
Then the following are equivalent:

(a) λ is an eigenvalue of T.

(b) The operator $\lambda I - T : V \to V$ is NOT injective.

(c) $\det(\lambda I - T) = 0$.

Proof:

The statement (a)

$\iff$ $\exists\, v \neq 0$ in V s.t. $T(v) = \lambda v$

$\iff$ $\exists\, v \neq 0$ in V s.t. $(\lambda I - T)(v) = 0$.

$\iff$ $\ker(\lambda I - T) \neq \{0\}$.

$\iff$ $\lambda I - T$ is not injective

$\iff$ The statement (b).

$\iff$ For any ordered basis B, $[\lambda I - B]_B$ is not invertible.

$\iff$ $\det\left([\lambda I - T]_B\right) = 0$

$\iff$ $\det(\lambda I - T) = 0$

$\iff$ The statement (c)

Remark: The number of possible eigenvalues is $\dim(V)$ because degree of char. poly is $\dim(V)$.

Example: ($\mathbb{F}=\mathbb{C}$) Consider $T: \mathbb{C}^2 \to \mathbb{C}^2$ given by $T(z_1, z_2) = (z_1 - z_2, z_1 + z_2)$.

Take $B = \{e_1 = (1,0), e_2 = (0,1)\}$.

$$\det(XI - T) = \det([XI - T]_B)$$
$$= \det \begin{pmatrix} X-1 & 1 \\ -1 & X-1 \end{pmatrix}$$
$$= (X-1)^2 + 1$$

Eigenvalues are roots of $\det(XI-T)$ which are $\lambda = 1+i, 1-i$.

For $\lambda = 1+i$, $T(z_1, z_2) = (1+i)(z_1, z_2)$

$$\Rightarrow (z_1 - z_2, z_1 + z_2) = (1+i)(z_1, z_2)$$

$$\Rightarrow \begin{cases} z_1 - z_2 = (1+i) z_1 \\ z_1 + z_2 = (1+i) z_2 \end{cases}$$

$$\Rightarrow z_2 = -i z_1$$

One eigenvector is $(1, -i)$ and the eigenspace $V_{1+i} = \{(z, -iz) : z \in \mathbb{C}\}$.

For $\lambda = 1-i$, $(z_1 - z_2, z_1 + z_2) = (1-i)(z_1, z_2)$

$$\Rightarrow \begin{cases} z_1 - z_2 = (1-i) z_1 \\ z_1 + z_2 = (1-i) z_2 \end{cases}$$

$$\Rightarrow \quad z_2 = i z_1$$

One eigenvector is $(1, i)$ and the eigenspace $V_{1-i} = \{(z, iz) : z \in \mathbb{C}\}$.

Observation: $B' = \{(1,-i), (1,i)\}$ is also a basis of $\mathbb{C}^2$ over $\mathbb{C}$.

$$[T]_{B'} = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

Remark: For an operator there need not be any eigenvalue.

e.g. $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$S(x,y) = (y, -x)$. Take $B = \{e_1, e_2\}$.

$$\det(XI - S) = \det \begin{pmatrix} X & -1 \\ 1 & X \end{pmatrix} = X^2 + 1.$$

The polynomial X^2+1 has no real root therefore S has no eigenvalue and eigenvector.

Definition: Let $A \in M_{n\times n}(\mathbb{F})$.

1. The polynomial $\det(XI - A) \in \mathbb{F}[X]$ is called the characteristic polynomial of A.

2. $\lambda \in \mathbb{F}$ is called an eigenvalue of A if $\det(\lambda I - A) = 0$.

3. A column matrix $v \in M_{n\times 1}(\mathbb{F})$, $v \neq 0$ is an eigenvector of A associated to an eigenvalue λ if

$$Av = \lambda v.$$

Convention: Since $M_{n\times 1}(\mathbb{F})$ and $\mathbb{F}^n$ are isomorphic as vector spaces over $\mathbb{F}$, by map

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longmapsto (a_1, a_2, \ldots, a_n),$$

many times $v \in \mathbb{F}^n$ is also treated as a column vector. This convention should not cause any confusion.

Theorem: Let V be a vector space over F and $\dim V = n$. Let $T: V \longrightarrow V$ be a linear operator. Let B be an ordered basis of V. Then the following statements are equivalent:

(a) $v \in V$ is an eigenvector of T associated to eigenvalue λ of T.

(b) $[v]_B \in M_{n\times 1}(\mathbb{F})$ is an eigenvector of $[T]_B \in M_{n\times n}(\mathbb{F})$ associated to eigenvalue λ of $[T]_B$.

Proof: statement (a)

$\iff \quad T(v) = \lambda v \qquad v \neq 0$

$\iff \quad [T(v)]_B = [\lambda v]_B$

$\iff \quad [T]_B [v]_B = \lambda [v]_B \qquad [v]_B \neq 0$

$\iff$ statement (b).

Example: $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \in M_3(\mathbb{C})$.

Find eigenvalues and eigenvectors of A.

The characteristic polynomial of A is

$$\det(XI - A) = \det \begin{pmatrix} X-2 & -1 & 0 \\ 0 & X-2 & -1 \\ 0 & 0 & X-2 \end{pmatrix}$$

$$= (X-2)^3.$$

Eigenvalues of A are roots of $(X-2)^3$, then 2 is the only eigenvalue of A.

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in M_{3\times 1}(\mathbb{C})$ be an eigenvalue, then

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x + y = 2x & \text{i.e. } y = 0 \\ 2y + z = 2y & \text{i.e. } z = 0 \\ 2z = 2z \end{cases}$$

i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

i.e. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of A.