<u>SOLUTION</u>

Q1. Find two linearly independent series solutions of

(a) $y'' - xy' + 2y = 0$
(b) $y'' + 3x^2y' - 2xy = 0$
(c) $y'' + x^2y' + x^2y = 0$
(d) $(1+x^2)y'' + y = 0$

Solution:

(a) Assuming the solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^k$ and differentiating term by term, we get

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1},$$
$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

Substituting these two relations in the given ODE, we get

$$y'' - xy' + 2y = \sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} - k a_k + 2a_k] x^k = 0$$

Then by Corollary 1(see the notes), we have

$$(k+2)(k+1)a_{k+2} - k a_k + 2a_k = 0; \; k = 0, 1, 2, 3, \cdots.$$

That is, we get a recurrence relation

$$a_{k+2} = \frac{(k-2)}{(k+2)(k+1)} a_k, \quad k = 0, 1, 2, 3, \cdots.$$

Therefore,

$$\begin{aligned}
k = 0: &\implies a_2 = -a_0 \\
k = 1: &\implies a_3 = -\frac{1}{3\,.\,2} a_1 = -\frac{1}{3!} a_1 \\
k = 2: &\implies a_4 = 0 \\
k = 3: &\implies a_5 = \frac{1}{4\,.\,5} a_3 = -\frac{1}{3!\,.\,4\,.\,5} a_1 = -\frac{1}{5!} a_1 \\
k = 4: &\implies a_6 = \frac{1}{15} a_4 = 0 \\
k = 5: &\implies a_7 = \frac{3}{6\,.\,7} a_5 = -\frac{3}{5!\,.\,6\,.\,7} a_1 = -\frac{3}{7!} a_1 \\
k = 6: &\implies a_8 = 0 \\
k = 7: &\implies a_9 = \frac{5}{8\,.\,9} a_7 = -\frac{3\,.\,5}{9!} a_1.
\end{aligned}$$

Iterating in this way, we obtain

$$a_{2k} = 0, \quad k = 2, 3, 4, \cdots.$$

Thus, we get

$$y(x) = a_0(1 - x^2) + a_1(x - \frac{1}{3!}x^3 - \frac{1}{5!}x^5 - \frac{1\,.\,3}{7!}x^7 - \frac{3\,.\,5}{9!}x^9 - \cdots).$$

The two solutions are

$$y_1(x) = (1 - x^2), \quad y_2(x) = x - \frac{1}{3!}x^3 - \frac{1}{5!}x^5 - \frac{1\,.\,3}{7!}x^7 - \frac{3\,.\,5}{9!}x^9 - \cdots .$$

For the convergence: from the above recurrence relation see that

$$\lim_{k\to\infty}\left|\frac{a_{k+2}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{k-2}{(k+2)(k+1)}\right| = 0,$$

hence by ratio test, the radius of convergence of $y_2(x)$ is ∞ and thus $y(x)$ converges in the whole of $\mathbb{R}$. Also, see that $W(y_1, y_2)(0) = 1$, hence y_1, y_2 are L.I.

(b) Let us consider $y(x) = \sum_{k=0}^{\infty} a_k x^k$ as the possible solution of the given ODE. Then differentiating term by term, we get

$$\begin{aligned} x^2 y'(x) &= \sum_{k=1}^{\infty} k a_k x^{k+1} = \sum_{k=0}^{\infty} k a_k x^{k+1}, \\ y''(x) &= \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = \sum_{k=-1}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} \\ &\qquad\qquad\qquad\qquad\quad = 2a_2 + \sum_{k=0}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1}. \end{aligned}$$

Substituting these two relations in the given ODE, we get

$$y'' + 3x^2 y' - 2xy = 2a_2 + \sum_{k=0}^{\infty} [(k+3)(k+2)a_{k+3} + 3k a_k - 2a_k] x^{k+1} = 0$$

Then comparing the coefficients, we have

$$a_2 = 0, \;\; (k+3)(k+2)a_{k+3} + 3k a_k - 2a_k = 0; \;\; k = 0, 1, 2, 3, \cdots .$$

That is

$$a_{k+3} = \frac{(3k-2)}{(k+3)(k+2)} a_k, \;\; k = 1, 2, 3, \cdots .$$

Therefore,

$$
\begin{aligned}
k=0: &\implies a_3 = -\frac{1}{3}a_0 = -\frac{2}{3!}a_0\\
k=1: &\implies a_4 = \frac{1}{3\,.\,4}a_1 = \frac{2}{4!}a_1\\
k=2: &\implies a_5 = \frac{1}{5}a_2 = 0\\
k=3: &\implies a_6 = \frac{7}{5\,.\,6}a_3 = -\frac{7\,.\,2}{3!\,.\,5\,.\,6}a_0 = -\frac{56}{6!}a_0\\
k=4: &\implies a_7 = \frac{10}{6\,.\,7}a_4 = \frac{20}{4!\,.\,6\,.\,7}a_1 = \frac{100}{7!}a_1\\
k=5: &\implies a_8 = 0\\
k=6: &\implies a_9 = \frac{16}{8\,.\,9}a_6 = -\frac{56\,.\,16}{6!\,.\,8\,.\,9}a_0 = -\frac{7\,.\,16\,.\,56}{9!}a_0\\
k=7: &\implies a_{10} = \frac{19}{9\,.\,10}a_7 = \frac{19\,.\,100}{7!\,.\,9\,.\,10}a_1 = \frac{8\,.\,19\,.\,100}{10!}a_1.
\end{aligned}
$$

Iterating in this way, we obtain

$$y(x) = a_0\left(1 - \frac{2}{3!}x^3 - \frac{56}{6!}x^6 - \cdots\right) + a_1\left(x + \frac{2}{4!}x^4 + \frac{100}{7!}x^7 + \cdots\right).$$

Therefore, the two solutions are

$$y_1(x) = 1 - \frac{2}{3!}x^3 - \frac{56}{6!}x^6 - \cdots, \quad y_2(x) = x + \frac{2}{4!}x^4 + \frac{100}{7!}x^7 + \cdots.$$

For the convergence: from the above recurrence relation, for the adjacent terms of the both the series y_1, y_2, we see that

$$\lim_{k\to\infty}\left|\frac{a_{k+3}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{3k-2}{(k+3)(k+2)}\right| = 0,$$

hence by ratio test, the radius of convergence for both the series $y_1(x), y_2(x)$ are ∞. Also, see that y_1, y_2 are L.I.

(c) We seek for the solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^k$. Then differentiating term by term, we get

$$
\begin{aligned}
x^2y'(x) &= \sum_{k=1}^{\infty} k a_k x^{k+1} = \sum_{k=0}^{\infty} k a_k x^{k+1} = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^{k+2},\\
y''(x) &= \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{k=-2}^{\infty}(k+4)(k+3)a_{k+4}x^{k+2}\\
&\qquad\qquad = 2a_2 + 6a_3x + \sum_{k=0}^{\infty}(k+4)(k+3)a_{k+4}x^{k+2}.
\end{aligned}
$$

Substituting these two relations in the given ODE, we get

$$
\begin{aligned}
0 &= y'' + x^2y' + x^2y\\
&= 2a_2 + 6a_3x + \sum_{k=0}^{\infty}[(k+4)(k+3)a_{k+4} + (k+1)a_{k+1} + a_k]x^{k+2}
\end{aligned}
$$

Comparing the coefficients, $a_2, a_3 = 0$.

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(d) Let us consider the solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^k$. Then differentiating term by term, we get

$$
\begin{aligned}
(1+x^2)y''(x) &= (1+x^2)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} \\
&= \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=0}^{\infty} k(k-1)a_k x^k \\
&= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} k(k-1)a_k x^k
\end{aligned}
$$

Substituting this in the given ODE, we get

$$(1+x^2)y'' + y = \sum_{k=0}^{\infty}[(k+1)(k+2)a_{k+2} + k(k-1)a_k + a_k]x^k = 0$$

Then comparing the coefficient of x^k, we have the recurrence relation

$$(k+1)(k+2)a_{k+2} + [k(k-1)+1]a_k = 0;\ k = 0,1,2,3,\cdots.$$

That is

$$a_{k+2} = -\frac{k(k-1)+1}{(k+1)(k+2)}a_k,\quad k = 0,1,2,3,\cdots.$$

Therefore,

$$
\begin{aligned}
k = 0: &\implies a_2 = -\frac{1}{2}a_0 \\
k = 1: &\implies a_3 = -\frac{1}{6}a_1 \\
k = 2: &\implies a_4 = -\frac{1}{4}a_2 = \frac{1}{8}a_0 \\
k = 3: &\implies a_5 = -\frac{7}{20}a_3 = \frac{7}{120}a_1 \\
k = 4: &\implies a_6 = -\frac{13}{30}a_4 = -\frac{13}{240}a_0 \\
k = 5: &\implies a_7 = -\frac{1}{2}a_5 = -\frac{7}{240}a_1
\end{aligned}
$$

Iterating in this way, we obtain

$$y(x) = a_0(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6 + \cdots) + a_1(x - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{7}{240}x^7 + \cdots).$$

Therefore, the two solutions of the given ODE are

$$
\begin{aligned}
y_1(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6 + \cdots; \\
y_2(x) &= x - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{7}{240}x^7 + \cdots.
\end{aligned}
$$

For the convergence: from the above recurrence relation, for the adjacent terms of the both the series y_1, y_2, we see that

$$\left|\frac{a_{k+2}}{a_k}\right| = \left|\frac{k(k-1)+1}{(k+1)(k+2)}\right| \to 1, \quad \text{as } k \to \infty$$

hence by ratio test, the radius of convergence for both the series $y_1(x), y_2(x)$ are 1, that is they both converge in $|x| < 1$. Also, see that y_1, y_2 are L.I.

□

Q2. Consider the Chebyshev equation

$$(1-x^2)y'' - xy' + \alpha^2 y = 0, \quad \alpha \in \mathbb{R}$$

(a) Compute two linearly independent series solutions for $|x| < 1$.
(b) Show that for each non-negative $\alpha = n$ there is a polynomial solution of degree n.

Solution:

(a) Here $-1, 1$ are regular singular points. Any point other that $-1, 1$ is ordinary point. So, we look for a solution in the form of the power series $y(x) = \sum_{k=0}^{\infty} a_k x^k$ around 0. Differentiating this power series term by term, we get

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1},$$
$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

Plugging these two equalities into the given ODE, we get

$$\begin{aligned} 0 &= (1-x^2)y'' - xy' - \alpha^2 y \\ &= \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=1}^{\infty} k a_k x^k - \alpha^2 \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} k a_k x^k - \alpha^2 \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} - (k^2 - \alpha^2)a_k] x^k. \end{aligned}$$

Then by Corollary 1(see the notes), we have the recurrence relation

$$(k+2)(k+1)a_{k+2} - (k^2 - \alpha^2)a_k = 0; \; k = 0, 1, 2, 3, \cdots.$$

That is

$$a_{k+2} = \frac{(k^2 - \alpha^2)}{(k+2)(k+1)} a_k, \quad k = 0, 1, 2, 3, \cdots.$$

Therefore,

$$
\begin{aligned}
k=0: &\implies a_2 = -\frac{\alpha^2}{2!}a_0\\
k=1: &\implies a_3 = \frac{1^2-\alpha^2}{3!}a_1\\
k=2: &\implies a_4 = \frac{2^2-\alpha^2}{4\,.\,3}a_2 = -\frac{\alpha^2(2^2-\alpha^2)}{4!}a_0\\
k=3: &\implies a_5 = \frac{3^2-\alpha^2}{5\,.\,4}a_3 = \frac{(1^2-\alpha^2)(3^2-\alpha^2)}{5!}a_1\\
k=4: &\implies a_6 = \frac{4^2-\alpha^2}{6\,.\,5}a_4 = -\frac{\alpha^2(2^2-\alpha^2)(4^2-\alpha^2)}{6!}a_0\\
k=5: &\implies a_7 = \frac{5^2-\alpha^2}{7\,.\,6}a_5 = \frac{(1^2-\alpha^2)(3^2-\alpha^2)(5^2-\alpha^2)}{7!}a_1
\end{aligned}
$$

Iterating in this way, we obtain

$$
\begin{aligned}
y(x) = a_0&\left(1-\frac{\alpha^2}{2!}x^2-\frac{\alpha^2(2^2-\alpha^2)}{4!}x^4-\frac{\alpha^2(2^2-\alpha^2)(4^2-\alpha^2)}{6!}x^6-\cdots\right)\\
&+a_1\left(x+\frac{1^2-\alpha^2}{3!}x^3+\frac{(1^2-\alpha^2)(3^2-\alpha^2)}{5!}x^5+\frac{(1^2-\alpha^2)(3^2-\alpha^2)(5^2-\alpha^2)}{7!}+\cdots\right).
\end{aligned}
$$

Therefore, the two L.I. solutions are

$$
\begin{aligned}
y_1(x) &= 1-\frac{\alpha^2}{2!}x^2-\frac{\alpha^2(2^2-\alpha^2)}{4!}x^4-\frac{\alpha^2(2^2-\alpha^2)(4^2-\alpha^2)}{6!}x^6-\cdots;\\
y_2(x) &= x+\frac{1^2-\alpha^2}{3!}x^3+\frac{(1^2-\alpha^2)(3^2-\alpha^2)}{5!}x^5+\frac{(1^2-\alpha^2)(3^2-\alpha^2)(5^2-\alpha^2)}{7!}+\cdots.
\end{aligned}
$$

From the above recurrence relation we get

$$
\lim_{k\to\infty}\left|\frac{a_{k+2}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{k^2-\alpha^2}{(k+2)(k+1)}\right| = 1,
$$

then by the ratio test, we have the radius of convergence of both the series y_1, y_2 as 1. Thus, y_1, y_2 converge for $|x|<1$. Also, y_1, y_2 are L.I.

(b) If $\alpha = n$ is even then $n = 2m$ for some $m \in \mathbb{N}_0$. Then all terms in $y_1(x)$ after the m^{th} term vanish since they contain the factor $((2m)^2-\alpha^2)$. Now by choosing $a_1 = 0$, we get a solution of the given ODE that contains finitely many terms (m number of terms) with the highest power of x as $2m = n$.

Similarly, if $\alpha = n$ is odd then $n = 2m+1$ for some $m \in \mathbb{N}_0$. Then all terms in $y_2(x)$ after the m^{th} term vanish since they contain the factor $((2m+1)^2-\alpha^2)$. Now by choosing $a_0 = 0$, we get a solution of the given ODE that contains finitely many terms (m number of terms) with the highest power of x as $2m+1 = n$. Therefore, in both the cases we can get a polynomial of degree n.

□

Q3.Consider the Hermite equation

$$
y'' - 2xy' + 2\alpha y = 0, \quad \alpha \in \mathbb{R}.
$$

(a) Compute two linearly independent series solutions.
(b) Show that for each non-negative $\alpha = n$ there is a polynomial of degree n.

Solution:

(a) We look for a solution in the form of the power series $y(x) = \sum_{k=0}^{\infty} a_k x^k$ around 0. Differentiating this power series term by term, we get

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1},$$
$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

Plugging these two equalities into the given ODE, we get

$$\begin{aligned}
0 &= y'' - 2xy' + 2\alpha y \\
&= \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2\sum_{k=1}^{\infty} k a_k x^k + 2\alpha \sum_{k=0}^{\infty} a_k x^k \\
&= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k - 2\sum_{k=0}^{\infty} k a_k x^k + 2\alpha \sum_{k=0}^{\infty} a_k x^k \\
&= \sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} - 2(k-\alpha)a_k] x^k.
\end{aligned}$$

Then by Corollary 1(see the notes), we have the recurrence relation

$$(k+2)(k+1)a_{k+2} - 2(k-\alpha)a_k = 0; \; k = 0, 1, 2, 3, \cdots .$$

That is

$$a_{k+2} = \frac{2(k-\alpha)}{(k+2)(k+1)} a_k, \quad k = 0, 1, 2, 3, \cdots .$$

Therefore,

$$\begin{aligned}
k = 0: &\implies a_2 = -\frac{2\alpha}{2!} a_0 \\
k = 1: &\implies a_3 = \frac{2(1-\alpha)}{3!} a_1 \\
k = 2: &\implies a_4 = \frac{2(2-\alpha)}{4\,.\,3} a_2 = -\frac{2^2\alpha(2-\alpha)}{4!} a_0 \\
k = 3: &\implies a_5 = \frac{2(3-\alpha)}{5\,.\,4} a_3 = \frac{2^2(1-\alpha)(3-\alpha)}{5!} a_1 \\
k = 4: &\implies a_6 = \frac{2(4-\alpha)}{6\,.\,5} a_4 = -\frac{2^3\alpha(2-\alpha)(4-\alpha)}{6!} a_0 \\
k = 5: &\implies a_7 = \frac{2(5-\alpha)}{7\,.\,6} a_5 = \frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!} a_1
\end{aligned}$$

Iterating in this way, we obtain

$$\begin{aligned}
y(x) &= a_0 \left(1 - \frac{2\alpha}{2!} x^2 - \frac{2^2\alpha(2-\alpha)}{4!} x^4 - \frac{2^3\alpha(2-\alpha)(4-\alpha)}{6!} x^6 - \cdots \right) \\
&\quad + a_1 \left(x + \frac{2(1-\alpha)}{3!} x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!} x^5 + \frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!} + \cdots \right).
\end{aligned}$$

Therefore, the two L.I. solutions are

$$y_1(x) = 1 - \frac{2\alpha}{2!}x^2 - \frac{2^2\alpha(2-\alpha)}{4!}x^4 - \frac{2^3\alpha(2-\alpha)(4-\alpha)}{6!}x^6 - \cdots ;$$
$$y_2(x) = x + \frac{2(1-\alpha)}{3!}x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!}x^5 + \frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!} + \cdots .$$

From the above recurrence relation, using ratio test one can check the radius of convergence of both y_1, y_2 are ∞. Thus y_1, y_2 converge in the whole of $\mathbb{R}$. Also, it is evident that $y_1, y_2$are L.I.

(b) If $\alpha = n$ is even then $n = 2m$ for some $m \in \mathbb{N}_0$. Then all the terms, present in $y_1(x)$ after the m^{th} term, vanish since they contain the factor $(2m - \alpha)$. Now by choosing $a_1 = 0$, we get a solution of the given ODE that contains finitely many terms (m number of terms) with the highest power of x as $2m = n$.

Similarly, if $\alpha = n$ is odd then $n = 2m + 1$ for some $m \in \mathbb{N}_0$. Then all terms in $y_2(x)$ after the m^{th} term vanish since they contain the factor $((2m+1)-\alpha)$. Now by choosing $a_0 = 0$, we get a solution of the given ODE that contains finitely many terms (m number of terms) with the highest power of x as $2m + 1 = n$. Therefore, in both the cases we can get a polynomial of degree n.

□

Q4. Show that

$$\int_{-1}^{1} P_n(x)P_m(x) = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m. \end{cases}$$

Solution:
The Legendre's differential equation is given as

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0.$$

This equation can be rewritten as

$$\left[\left(1 - x^2\right) y'\right]' + n(n + 1)y(x) = 0.$$

For $l = n \in \mathbb{Z}^+$, the equation has polynomial solution, denoted as $P_n(x)$.

Case I: First we consider the case $n \neq m$. For $l = n, m \in \mathbb{Z}^+$, let P_n, P_m be two solutions of the corresponding Legendre's equations, respectively. Therefore,

$$\frac{d}{dx}\left[\left(1 - x^2\right) \frac{dP_n(x)}{dx}\right] + n(n + 1)P_n(x) = 0 \tag{1}$$

and

$$\frac{d}{dx}\left[\left(1 - x^2\right) \frac{dP_m(x)}{dx}\right] + m(m + 1)P_m(x) = 0\,. \tag{2}$$

Then multiplying (1) by P_m and multiplying (2) by P_n and then subtracting, we obtain

$$\begin{aligned} &P_m \frac{d}{dx}\left\{\left(1 - x^2\right) \frac{dP_n(x)}{dx}\right\} - P_n \frac{d}{dx}\left\{\left(1 - x^2\right) \frac{dP_m(x)}{dx}\right\} \\ &\quad + \{n(n + 1) - m(m + 1)\}P_nP_m = 0. \end{aligned} \tag{3}$$

Integrating the above between -1 to 1, we get

$$\int_{-1}^{1}\left[P_m\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n(x)}{dx}\right\}-P_n\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_m(x)}{dx}\right\}\right]dx$$

$$+\{n(n+1)-m(m+1)\}\int_{-1}^{1}P_nP_m\,dx=0. \tag{4}$$

Applying the integration by parts formula, we have

$$\left[P_m\left(1-x^2\right)\frac{dP_n(x)}{dx}\right]_{-1}^{1}-\int_{-1}^{1}\frac{dP_m}{dx}\left\{\left(1-x^2\right)\frac{dP_n(x)}{dx}\right\}dx$$

$$-\left[P_n\left(1-x^2\right)\frac{dP_m(x)}{dx}\right]_{-1}^{1}+\int_{-1}^{1}\frac{dP_n}{dx}\left\{\left(1-x^2\right)\frac{dP_m(x)}{dx}\right\}dx$$

$$+\{n(n+1)-m(m+1)\}\int_{-1}^{1}P_nP_m\,dx=0. \tag{5}$$

This implies

$$\{n(n+1)-m(m+1)\}\int_{-1}^{1}P_nP_m\,dx=0.$$

Since $n\neq m$, we get

$$\int_{-1}^{1}P_nP_m\,dx=0.$$

Case II: Now we consider the case $n=m$. The explicit formula for the Legendre's polynomial is given by

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)].$$

Using this we can have that $(1-2xt+t^2)^{-1}$ is a generating function for $P_n(x)$, that is, $(1-2xt+t^2)^{-1}=\sum_{n=0}^{\infty}P_n(x)t^n$ (expand the binomial expansion of the LHS and compare the co-efficient of t^n). Therefore,

$$\frac{1}{1-2xt+t^2}=\left(\sum_{n=0}^{\infty}P_n(x)t^n\right)^2=\sum_{n,m\geq 0}P_n(x)P_m(x)t^{n+m}.$$

Now we integrate from -1 to 1:

$$\int_{-1}^{1}\frac{1}{1-2xt+t^2}dx=\sum_{n,m\geq 0}\left(\int_{-1}^{1}P_n(x)P_m(x)dx\right)t^{n+m}.$$

By using the orthogonality of P_n and P_m for $n\neq m$ (see Case I), we get

$$\frac{-1}{2t}\left[\ln|1-2xt+t^2|\right]_{-1}^{1}=\sum_{n=0}^{\infty}\left(\int_{-1}^{1}[P_n(x)]^2dx\right)t^{2n},$$

and after simplifying the left-hand side, we have

$$\frac{1}{t}\ln\left|\frac{1+t}{1-t}\right|=\sum_{0}^{\infty}\left(\int_{-1}^{1}[P_n(x)]^2dx\right)t^{2n}. \tag{6}$$

Now recall that for $|s|<1$, the Taylor series of $\ln(1+s)$ is

$$\ln(1+s)=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}s^j.$$

Hence for $|t| < 1$, we have

$$\frac{1}{t}\ln\left|\frac{1+t}{1-t}\right| = \frac{1}{t}[\ln(1+t) - \ln(1-t)]$$
$$= \frac{1}{t}\left(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}t^j - \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}(-t)^j\right)$$
$$= \frac{1}{t}\left(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}+1}{j}t^j\right)$$
$$= \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}. \tag{7}$$

From (6) and (7), we obtain,

$$= \sum_{n=0}^{\infty}\frac{2}{2n+1} = \sum_{0}^{\infty}\left(\int_{-1}^{1}[P_n(x)]^2dx\right)t^{2n}.$$

By identifying the coefficient of t^{2n} from the above, we get

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{2}{2n+1}.$$

□

Q5. Show that there are constants $c_0, c_1, c_2, \cdots c_n$ such that

$$x^n = c_0P_0(x) + c_1P_1(x) + \cdots + c_nP_n(x).$$

Solution:
First we show that the set of Legendre's polynomials $\{P_0, P_1, P_2, \cdots, P_n\}$ form a basis for the vector space $\mathbb{P}_n$, where $\mathbb{P}_n$ is the set of all polynomials of degree $\leq n$. Let us take any linear combination of $P_i's$, $i = 0, 1, \cdots, n$,

$$a_0P_0(x) + a_1P_1(x) + \cdots + a_nP_n(x) = 0, \quad \text{where } \ a_i \in \mathbb{R}.$$

Then multiplying the above with $P_i(x)$, $i = 0, 1, \cdots, n$, and integrating from -1 to 1, using the Q5., we get $a_i = 0$ for each $i = 0, 1, \cdots, n$. Thus $\{P_0, P_1, P_2, \cdots, P_n\}$ is L.I. Since $dim(\mathbb{P}_n) = n + 1$, hence $\{P_0, P_1, P_2, \cdots, P_n\}$ forms a basis for $\mathbb{P}_n$. Since $x^n \in \mathbb{P}_n$, there exists a unique set of $c_i's$ such that

$$x^n = c_0P_0(x) + c_1P_1 + \cdots + c_nP_n(x).$$

□

Q6. Find all solutions of the following equations for $x > 0$:

(a) $x^2y'' + 2xy' - 6y = 0$
(b) $2x^2y'' + xy' - y = 0$
(c) $x^2y'' - 5xy' + 9y = 0$

Solution:
Here for all the three problems, 1 is the only regular singular point. Any point other that 1 is ordinary point. Thus, we look for the solution for $x > 0$.

(a) Let us assume the solutions of the form $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$ and by differentiating term by term, we get

$$y'(x) = \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1},$$
$$y''(x) = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}$$

Substituting these two relations in the given ODE, we get

$$x^2y'' + 2xy' - 6y = \sum_{k=0}^{\infty}[(k+r)(k+r-1)a_k + 2(k+r)a_k - 6a_k]x^{k+r} = 0$$

Comparing the coefficient of x^{r+k}

$$[(k+r)(k+r-1) + 2(k+r) - 6]a_k = 0$$

and since $a_0 \neq 0$ implies

$$r(r-1) + 2r - 6 = 0, \quad \text{(Indicial Polynomial)}$$

which has two roots

$$r_1 = 2,\, r_2 = -3.$$

Now by assuming $a_k \neq 0$, we get

$$(k+r)(k+r-1) + 2(k+r) - 6 = 0.$$

For $r_1 = 2$, the above implies

$$(k+2)(k+1) + 2(k+1) - 6 = 0$$

which gives us $k = 0, -5$. Now by discarding $k = -5$, for $r_1 = 2$, we get $a_0 \neq 0$. If y_1 is a solution corresponding to $r_1 = 2$, then

$$y_1(x) = a_0x^2.$$

Similarly, for $r_2 = -3$, we get

$$(k-3)(k-4) + 2(k-3) - 6 = 0$$

which gives us $k = 0, 5$. That is for $r_2 = -3$, $a_0, a_5 \neq 0$. Thus, if y_2 is a solution corresponding to $r_2 = -3$, then

$$y_2(x) = a_0x^{-3} + a_5x^2.$$

Observe that y_1, y_2 are L.I. Thus any general solution can be written as

$$y(x) = C_1y_1(x) + C_2y_2(x) \text{ for some arbitrary constants } C_1, C_2.$$

(b) We seek for the solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$. Now by differentiating this term by term, we get

$$y'(x) = \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1},$$
$$y''(x) = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}$$

Substituting these two relations in the given ODE, we get

$$2x^2y'' + xy' - y = \sum_{k=0}^{\infty}[2(k+r)(k+r-1)a_k + (k+r)a_k - a_k]x^{k+r} = 0$$

Now comparing the coefficient of x^{r+k}, we obtain

$$[2(k+r)(k+r-1) + (k+r) - 1]a_k = 0.$$

Now for $k = 0$,

$$a_0[2r(r-1) + r - 1] = 0.$$

Since $a_0 \neq 0$,

$$2r(r-1) + r - 1 = 0 \quad \text{(Indicial Polynomial)}$$

which on simplifying gives two roots

$$r_1 = 1,\ r_2 = -\frac{1}{2}.$$

Now by assuming $a_k \neq 0$, we get

$$2(k+r)(k+r-1) + (k+r) - 1 = 0.$$

For $r_1 = 1$, the above implies

$$2(k+1)k + (k+1) - 1 = 0$$

which gives us $k = 0, -\frac{3}{2}$. Now by discarding $k = -\frac{3}{2}$, for $r_1 = 1$, we have $a_0 \neq 0$. Thus, if y_1 is a solution corresponding to $r_1 = 1$, then

$$y_1(x) = a_0 x.$$

Similarly, for $r_2 = -\frac{1}{2}$, we have

$$2(k-\frac{1}{2})(k-\frac{3}{2}) + (k-\frac{1}{2}) - 1 = 0$$

which gives us $k = 0, -\frac{2}{3}$. Now by discarding $k = -\frac{2}{3}$, for $r_2 = -\frac{1}{2}$, we have $a_0 \neq 0$. Hence, if y_2 is a solution corresponding to $r_2 = -\frac{1}{2}$, then

$$y_2(x) = a_0 x^{-\frac{1}{2}}.$$

Since y_1, y_2 are L.I., any general solution can be written as

$$y(x) = C_1y_1(x) + C_2y_2(x) \text{ for some arbitrary constants } C_1, C_2.$$

(c) Let us assume $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$ and by differentiating term by term, we get

$$y'(x) = \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1},$$

$$y''(x) = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}$$

Substituting these two relations in the given ODE, we get

$$L(y) := x^2y'' - 5xy' + 9y = \sum_{k=0}^{\infty}[(k+r)(k+r-1)a_k - 5(k+r)a_k + 9a_k]x^{k+r} = 0$$

Then equating the coefficient of x^{r+k} to zero, we obtain

$$[(k+r)(k+r-1)-5(k+r)+9]a_k = 0.$$

Now for $k=0$,

$$a_0[r(r-1)-5r+9] = 0.$$

Since $a_0 \neq 0$,

$$p(r) := r(r-1)-5r+9 = 0 \quad \text{(Indicial Polynomial)}$$

which on simplifying gives $(r-3)^2 = 0$, that is the roots are

$$r_1 = r_2 = 3.$$

So, $p(3) =$ and $p'(3) = 0$. Now by assuming $a_k \neq 0$, we get

$$(k+r)(k+r-1)-5(k+r)+9 = 0.$$

For $r_1 = 3$, the above implies

$$(k+3)(k+2)-5(k+3)+9 = 0$$

which gives us $k=0$. That is, for $r_1 = 3$, $a_0 \neq 0$. Thus, if y_1 is a solution corresponding to $r_1 = 3$,

$$y_1(x) = a_0 x^3.$$

Due to the existence of repeated root, we assume the second solution to be

$$y_2(x) = (\ln(x))y_1(x) + x^3 \sum_{k=1}^{\infty} b_k x^k.$$

Therefore,

$$y_2'(x) = (\ln(x))y_1'(x) + \frac{y_1}{x} + \sum_{k=1}^{\infty}(k+3)b_k x^{k+2},$$

$$y_2''(x) = (\ln(x))y_1''(x) + \frac{2y_1'}{x} - \frac{y_1}{x^2} + \sum_{k=1}^{\infty}(k+3)(k+2)b_k x^{k+1}.$$

Plugging these into the ODE and using the solution $y_1 = a_0 x^3$, we get

$$\begin{aligned} 0 = L(y_2) &= x^2(\ln(x))y_1'' + 2xy_1' - y_1 + \sum_{k=1}^{\infty}(k+3)(k+2)b_k x^{k+3} \\ &- 5x(\ln(x))y_1' - 5y_1 - 5\sum_{k=1}^{\infty}(k+3)b_k x^{k+3} \\ &+ 9(\ln(x))y_1 + 9\sum_{k=1}^{\infty} b_k x^{k+3} \\ &= (\ln(x))L(y_1) + (2xy_1' - 6y_1) + \sum_{k=1}^{\infty}[(k+3)(k+2)-5(k+3)+9]b_k x^{k+3} \\ &= \sum_{k=1}^{\infty} k^2 b_k x^{k+3}. \end{aligned}$$

Comparing the coefficient of x^{k+3}, we get

$$k^2 b_k = 0$$

that is, $b_k = 0 \ \forall k = 1, 2, \cdots$. Hence the second solution y_2 is given as

$$y_2(x) = a_0 x^3(\ln(x)).$$

Since y_1, y_2 are L.I, any general solution can be written as

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \ \text{ for some arbitrary constants } C_1, C_2.$$

$\square$

Q6. Find all solutions of the following equations for $x > 0$:

(a) $3x^2 y'' + 5xy' + 3xy = 0$
(b) $x^2 y'' + 3xy' + (1 + x)y = 0$
(c) $x^2 y'' - 2x(x + 1)y' + 2(x + 1)y = 0$

Solution:
Here for all the three problems, 1 is the only regular singular point. Any point other that 1 is ordinary point. Thus, we investigate the solution for $x > 0$.

(a) We look for the solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$ and by differentiating this term by term, we get

$$y'(x) = \sum_{k=0}^{\infty} (k + r)a_k x^{k+r-1},$$
$$y''(x) = \sum_{k=0}^{\infty} (k + r)(k + r - 1)a_k x^{k+r-2}$$

Substituting these two relations in the given ODE, we get

$$\begin{aligned} 0 = L(y) &= 3x^2 y'' + 5xy' + 3xy \\ &= \sum_{k=0}^{\infty} [3(k + r)(k + r - 1) + 5(k + r)]a_k x^{k+r} + 3\sum_{k=0}^{\infty} a_k x^{k+r+1} \\ &= \sum_{k=0}^{\infty} [3(k + r)(k + r - 1) + 5(k + r)]a_k x^{k+r} + 3\sum_{k=1}^{\infty} a_{k-1} x^{k+r} \\ &= [3r(r - 1) + 5r]a_0 + \sum_{k=1}^{\infty} [(3(k + r)(k + r - 1) + 5(k + r))a_k + 3a_{k-1}]x^{k+r} \\ &= p(r)a_0 x^r + \sum_{k=1}^{\infty} [\{p(k + r)\}a_k + 3a_{k-1}]x^{k+r}, \end{aligned}$$

where $p(r) := r(3r + 2)$ is the indicial polynomial. Now comparing the coefficients, we have

$$p(r) = r(3r + 2) = 0 \quad (\text{since } \ a_0 \neq 0)$$

whose roots are given as

$$r_1 = 0, \, r_2 = -\frac{2}{3}$$

and also, we get the recurrence relation

$$p(k + r)a_k + 3a_{k-1} = 0; \ k = 1, 2, 3, \cdots .$$

Then for $r_1 = 0$,

$$a_k = -\frac{3}{p(k)}a_{k-1};\ k = 1, 2, 3, \cdots .$$

Since $0, -\frac{2}{3}$ are the only roots of p, the above expression is well defined. Therefore by iterating, we get for $k = 1, 2, \cdots$,

$$a_k = \frac{(-3)^k a_0}{p(k)p(k-1)\cdots p(1)}.$$

Thus for $r_1 = 0$, the first solution is given as

$$y_1(x) = a_0 + \sum_{k=1}^{\infty} \frac{(-3)^k a_0}{p(k)p(k-1)\cdots p(1)}\, x^k.$$

Again, for $r_2 = -\frac{2}{3}$,

$$a_k = -\frac{3}{p(k-\frac{2}{3})}a_{k-1};\ k = 1, 2, 3, \cdots .$$

Since $0, -\frac{2}{3}$ are the only roots of p, the denominator in the above expression is non zero. Therefore by iterating, we get for $k = 1, 2, \cdots$,

$$a_k = \frac{(-3)^k a_0}{p(k-\frac{2}{3})p(k-\frac{5}{3})\cdots p(\frac{1}{3})}.$$

Thus for $r_2 = -\frac{2}{3}$, the second solution is given as

$$y_2(x) = a_0 x^{-\frac{2}{3}} + x^{-\frac{2}{3}} \sum_{k=1}^{\infty} \frac{(-3)^k a_0}{p(k-\frac{2}{3})p(k-\frac{5}{3})\cdots p(\frac{1}{3})}\, x^k.$$

Observe that y_1, y_2 are L.I. Thus any general solution can be written as

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \text{ for some arbitrary constants } C_1, C_2.$$

(b) We look for the solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$ and by differentiating this term by term, we get

$$y'(x) = \sum_{k=0}^{\infty} (k+r)a_k x^{k+r-1},$$
$$y''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1)a_k x^{k+r-2}$$

Substituting these two relations in the given ODE, we get

$$\begin{aligned}
0 = L(y) &= x^2y'' + 3xy' + (1+x)y \\
&= \sum_{k=0}^{\infty}[(k+r)(k+r-1) + 3(k+r)]a_kx^{k+r} + \sum_{k=0}^{\infty} a_kx^{k+r} + \sum_{k=0}^{\infty} a_kx^{k+r+1} \\
&= \sum_{k=0}^{\infty}[(k+r)(k+r-1) + 3(k+r) + 1]a_kx^{k+r} + \sum_{k=1}^{\infty} a_{k-1}x^{k+r} \\
&= [r(r-1) + 3r + 1]a_0x^r + \sum_{k=1}^{\infty}[((k+r)(k+r-1) \\
&\qquad\qquad + 3(k+r) + 1)a_k + a_{k-1}]x^{k+r} \\
&= (r+1)^2a_0x^r + \sum_{k=1}^{\infty}[(k+r+1)^2a_k + a_{k-1}]x^{k+r}.
\end{aligned}$$

Now comparing the coefficients, we have the indicial equation

$$p(r) := (r+1)^2 = 0 \quad (\text{since } \; a_0 \neq 0)$$

whose roots are given as

$$r := r_1 = r_2 = -1$$

and also, we get the recurrence relation

$$(k+r+1)^2a_k + a_{k-1} = 0; \; k = 1, 2, 3, \cdots.$$

Then for $r = -1$:

$$a_k = -\frac{1}{k^2}a_{k-1}; \; k = 1, 2, 3, \cdots.$$

Therefore by iterating, we get for $k = 1, 2, \cdots$,

$$a_k = \frac{(-1)^k a_0}{k^2(k-1)^2 \cdots 1^2} = \frac{(-1)^k a_0}{(k!)^2}.$$

Thus the first solution is given as

$$y_1(x) = a_0x^{-1} + x^{-1}\sum_{k=1}^{\infty}\frac{(-1)^k a_0}{(k!)^2}\,x^k.$$

Due to the existence of repeated root, we assume the second solution to be

$$y_2(x) = (\ln(x))y_1(x) + x^{-1}\sum_{k=1}^{\infty} b_kx^k.$$

Therefore,

$$\begin{aligned}
y_2'(x) &= (\ln(x))y_1'(x) + \frac{y_1}{x} + \sum_{k=1}^{\infty}(k-1)b_kx^{k-2}, \\
y_2''(x) &= (\ln(x))y_1''(x) + \frac{2y_1'}{x} - \frac{y_1}{x^2} + \sum_{k=1}^{\infty}(k-1)(k-2)b_kx^{k-3}.
\end{aligned}$$

Plugging these into the ODE and using the solution y_1, we get

$$\begin{aligned}
0 = L(y_2) = x^2(\ln(x))y_1'' &+ 2xy_1' - y_1 + \sum_{k=1}^{\infty}(k-1)(k-2)b_k x^{k-1} \\
&+ 3x(\ln(x))y_1' + 3y_1 + 3\sum_{k=1}^{\infty}(k-1)b_k x^{k-1} \\
&+ x(\ln(x))y_1 + \sum_{k=1}^{\infty} b_k x^k \\
&+ (\ln(x))y_1 + \sum_{k=1}^{\infty} b_k x^{k-1} \\
&= (\ln(x))L(y_1) + 2xy_1' + 2y_1 \\
&\quad + b_1 + \sum_{k=2}^{\infty}[\{(k-1)(k-2) + 3(k-1) + 1\}b_k + b_{k-1}]x^{k-1} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k a_0}{(k-1)!^2} x^{k-1} + b_1 + \sum_{k=2}^{\infty}[k^2 b_k + b_{k-1}]x^{k-1}.
\end{aligned}$$

Comparing the coefficients, we get $b_1 = -a_0$ and

$$k^2 b_k + b_{k-1} = \frac{(-1)^k a_0}{(k-1)!^2} \quad \forall k = 2, 3, 4 \cdots .$$

That is

$$b_k = -\frac{b_{k-1}}{k^2} + \frac{(-1)^k a_0}{(k!)^2} \quad \forall k = 2, 3, 4 \cdots .$$

Therefore,

$$\begin{aligned}
k = 2 : &\implies b_2 = -\frac{b_1}{4} + \frac{a_0}{4} = \frac{a_0}{2} \\
k = 3 : &\implies b_3 = -\frac{a_0}{12} \\
k = 4 : &\implies a_4 = \frac{a_0}{144}
\end{aligned}$$

Iterating in this way, we obtain the second solution y_2 as

$$y_2(x) = a_0 x^{-1}(\ln(x))[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} x^k] + [-a_0 x + \frac{a_0}{2}x^2 - \frac{a_0}{12}x^3 + \frac{a_0}{144}x^4 - \cdots .].$$

Observe that y_1, y_2 are L.I. Thus any general solution can be written as

$y(x) = C_1 y_1(x) + C_2 y_2(x)$ for some arbitrary constants C_1, C_2.

(c) We look for the solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$ and by differentiating this term by term, we get

$$\begin{aligned}
y'(x) &= \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1}, \\
y''(x) &= \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}
\end{aligned}$$

Substituting these two relations in the given ODE, we get

$$\begin{aligned}
0 = L(y) &= x^2y'' - 2x(x+1)y' + 2(x+1)y \\
&= \sum_{k=0}^{\infty}[(k+r)(k+r-1)]a_k x^{k+r} \\
&- 2\sum_{k=0}^{\infty}(k+r)a_k x^{k+r+1} - 2\sum_{k=0}^{\infty}(k+r)a_k x^{k+r} \\
&+ 2\sum_{k=0}^{\infty} a_k x^{k+r} + 2\sum_{k=0}^{\infty} a_k x^{k+r+1} \\
&= \sum_{k=0}^{\infty}[(k+r)(k+r-1) - 2(k+r) + 2]a_k x^{k+r} \\
&\quad - 2\sum_{k=0}^{\infty}(k+r-1)a_k x^{k+r+1} \\
&= [r(r-1) - 2r + 2]a_0 x^r \\
&+ \sum_{k=1}^{\infty}[((k+r)(k+r-1) - 2(k+r) + 2)a_k - 2(k+r-2)a_{k-1}]x^{k+r} \\
&= (r-1)(r-2)a_0 x^r + \sum_{k=1}^{\infty}[(k+r-1)(k+r-2)a_k - 2(k+r-2)a_{k-1}]x^{k+r}.
\end{aligned}$$

Now comparing the coefficients, we have the indicial equation

$$p(r) := (r-1)(r-2) = 0 \quad (\text{since } \ a_0 \neq 0)$$

whose roots are given as

$$r_1 = 2, r_2 = 1$$

and also, we get the recurrence relation

$$(k+r-1)a_k = 2a_{k-1}; \ k = 1, 2, 3, \cdots .$$

Then for $r_1 = 2$:

$$a_k = \frac{2}{(k+1)}a_{k-1}; \ k = 1, 2, 3, \cdots .$$

Therefore by iterating, we get for $k = 1, 2, \cdots,$

$$a_k = \frac{(2)^k a_0}{(k+1)k(k-1)\cdots 2} = \frac{(2)^k}{(k+1)!}a_0.$$

Thus the first solution is given as

$$y_1(x) = a_0x^2 + x^2\sum_{k=1}^{\infty}\frac{(2)^k a_0}{(k+1)!}\,x^k.$$

Again for $r_2 = 1$:

$$a_k = \frac{2}{k}a_{k-1}; \ k = 1, 2, 3, \cdots .$$

Therefore by iterating, we get for $k = 1, 2, \cdots,$

$$a_k = \frac{(2)^k a_0}{k(k-1)\cdots 1} = \frac{(2)^k}{k!} a_0.$$

Thus the second solution corresponding to $r_2 = 1$ is given as

$$y_2(x) = a_0 x + x \sum_{k=1}^{\infty} \frac{(2)^k a_0}{k!} x^k.$$

Since $x > 0$, the solutions are L.I.

$\square$