Laplace Transform and Differential Equations

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Consider a real valued function $f(t)$ on $[0, \infty)$. We define **Laplace transform** of f (denoted by $\mathcal{L}[f(t)](s)$) as,

$$\mathcal{L}[f(t)](s) = \int_0^{\infty} e^{-st} f(t) dt, \tag{1}$$

provided the integral makes sense.

Definition (**Exponential Order:)**

A function f is said to be of exponential order if there exist constants M and α such that,

$$|f(t)| \leq Me^{\alpha t}. \tag{2}$$

For example: Any polynomial is of exponential order. Similarly, any bounded function is also of exponential order. However $f(t) = e^{t^2}$ is not of exponential order.

Also Consider:

Definition

Piecewise Continuous: *A function f is said to be piecewise continuous on domain D if f is continuous on D except on a countable set S on which it has jump discontinuity.*

Combining both the definition, we have the following result:

Theorem

Suppose f is piecewise continuous on $[0, \infty)$ and satisfies (2). Then $\mathcal{L}[f](s)$ exists for $s > \alpha$.

(α, ∞)

$$\left| \int_0^\infty e^{-st} f(t)\, dt \right| \le \int_0^\infty e^{-st} |f(t)|\, dt \le M \int_0^\infty e^{-st} e^{\alpha t}\, dt = M \int_0^\infty e^{-(s-\alpha)t}\, dt$$

$\Rightarrow$ if $s > \alpha$ $\mathcal{L}[f](s)$ converges.

Remark

In the following, we will not explicitly specify that f is exponential order but we will always assume it, whenever we need to define its Laplace transform.

Example:

Consider $f(t) = e^{\alpha t}$ which is exponential order. Laplace transform of it is calculated as follows:

$$\mathcal{L}[e^{\alpha t}](s) = \int_0^\infty e^{-st} e^{\alpha t} dt = \frac{1}{s-\alpha}, \quad s > \alpha.$$

$s > \alpha$

For $\alpha = 0$ we have

$$\mathcal{L}[1](s) = \frac{1}{s}.$$

$s > 0$

Theorem

The operator $\mathcal{L}$ is linear i.e.,

$$\mathcal{L}[af(t)+bg(t)](s)=a\mathcal{L}[f](s)+b\mathcal{L}[g](s)$$

where $\mathcal{L}[f]$ & $\mathcal{L}[g]$ is defined

for all s and constants a and b.

Proof.

$$\mathcal{L}[af(t)+bg(t)](s)=\int_0^\infty e^{-st}(af(t)+bg(t))dt$$

$$=a\int_0^\infty e^{-st}f(t)dt+b\int_0^\infty e^{-st}g(t)dt=a\mathcal{L}[f](s)+b\mathcal{L}[g](s).$$

□

Linearity (cont.)

Example

Laplace transform of hyperbolic functions $\cosh(at)$ and $\sinh(at)$:

$$\cosh(at) = \frac{1}{2}(e^{at} + e^{-at}), \qquad \sinh(at) = \frac{1}{2}(e^{at} - e^{-at})$$

$$\mathcal{L}[\cosh(at)] = \frac{1}{2}(\underbrace{\mathcal{L}[e^{at}]}_{s>a} + \underbrace{\mathcal{L}[e^{-at}]}_{s>0}) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \boxed{\frac{s}{s^2 - a^2}},$$

$$\mathcal{L}[\sinh(at)] = \frac{1}{2}(\mathcal{L}[e^{at}] - \mathcal{L}[e^{-at}]) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \boxed{\frac{a}{s^2 - a^2}}.$$

Example

Laplace transform of Trigonometric function $\sin(at)$ and $\cos(at)$:
Define $L_c = \mathcal{L}[\cos(at)]$ and $L_s = \mathcal{L}[\sin(at)]$, then using integration by parts,

$$L_c = \mathcal{L}[\cos(at)] = \int_0^\infty e^{-st}\cos(at)dt$$

$s<0$ $\quad$ $s>0$

$$= \left[\frac{e^{-st}}{-s}\cos(at)\right]_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)dt = \frac{1}{s} - \frac{a}{s}L_s$$

Similarly,

$$L_s = \frac{a}{s}L_c$$

Solving these two linear equations for L_c and L_s we get,

$$\boxed{L_c = \frac{s}{s^2+a^2}, \qquad L_s = \frac{a}{s^2+a^2}.}$$

Differentiation of Laplace transform

Theorem

Suppose f is of exponential order, then in its region of convergence $(s>\alpha)$
$F(s) = \mathcal{L}[f(t)](s)$ *is differentiable infinitely many times at each point and*

$$F^{(n)}(s) = \boxed{(-1)^n}\mathcal{L}[t^n f(t)](s).$$

Proof.

$$= -\int_0^\infty \boxed{f(t)\, t}\, e^{-st}\, dt$$

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty f(t)\frac{d}{ds}(e^{-st})dt = -\mathcal{L}[tf(t)](s).$$

Then the result follow from induction argument. □

Theorem

If $F = \mathcal{L}[f]$ then,

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s_1)ds_1.$$

Proof.

$$\int_s^\infty F(s_1)ds_1 = \int_s^\infty \left[\int_0^\infty e^{-s_1 t} f(t)dt\right] ds_1 = \int_0^\infty f(t)\left[\int_s^\infty e^{-s_1 t} ds_1\right] dt$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}dt = \mathcal{L}\left[\frac{f(t)}{t}\right].$$

□

Theorem

Suppose f is differentiable for $t \geq 0$ and of exponential order. If $\mathcal{L}[f']$ is well defined then,

$$\mathcal{L}[f'(t)](s) = s\mathcal{L}[f(t)](s) - f(0).$$

$$\mathcal{L}[f'](s) \stackrel{?}{=} sF(s) - f(0) \qquad F = \mathcal{L}[f]$$

Proof.

By definition we have,

$$\mathcal{L}[f'(t)](s) = \int_0^{\infty} e^{-st} f'(t)dt$$

$$\mathcal{L}[x'(t)] = s\underbrace{X(s)}_{} - x(0), \quad x(0) = 0$$

$$x(t) = \mathcal{L}^{-1}[X(s)] = \left[\frac{x(0)}{s}\right]$$

Integrating by parts we get,

$$\mathcal{L}[f'](s) = \left[e^{-sr} f(r)\right]_0^{\infty} + s\int_0^{\infty} e^{-st} f(t)dt = s\mathcal{L}[f](s) - f(0).$$

□

Laplace Transform of Derivatives of a Function (cont.)

Corollary

Now repeating this process n times we get,

$$\mathcal{L}[f^{(n)}(t)](s) = s^n\mathcal{L}[f] - \overbrace{s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}^{(n-1)}.$$

Off course we have to assume that $\mathcal{L}[f^{(n)}]$ exists.

Theorem

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{\mathcal{L}[f]}{s}$$

$g(t)$

Proof.
Define $g(t) = \int_0^t f(\tau)d\tau$, then $g(0) = 0$ and g is differentiable with $g' = f$. Furthermore if f satisfies (2), then

$$|g(t)| \leq \int_0^t |f(\tau)|d\tau \leq M\int_0^t e^{\alpha\tau}d\tau = \frac{M}{\alpha}(e^{\alpha t} - 1) \leq \frac{M}{\alpha}e^{\alpha t}.$$

So, g is also exponential order. So, $\mathcal{L}[g]$ exists. Now,

$$\mathcal{L}[g'] = s\mathcal{L}[g] - g(0),$$

gives the desired equality. □

From these result we will now derive the following properties of Laplace transform:

$\mathcal{L}[f] = F(s)$

Theorem

$\mathcal{L}[e^{\alpha t} f(t)] = F(s-\alpha)$

$$\mathcal{L}[e^{\alpha t} f](s) = \mathcal{L}[f](s-\alpha)$$

Proof.

$$\mathcal{L}[e^{\alpha t} f](s) = \int_0^\infty e^{\alpha t} e^{-st} f(t)dt = \int_0^\infty e^{-(s-\alpha)t} f(t)dt = \mathcal{L}[f](s-\alpha).$$

□

Laplace Transform of Various Functions

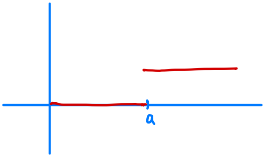
S. No.	$f(t)$	$\mathcal{L}[f]$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$\frac{2!}{s^3}$
4	t^n	$\frac{n!}{s^{n+1}}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\frac{1}{s-a}$
7	$\cos(at)$	$\frac{s}{s^2+a^2}$
8	$\sin(at)$	$\frac{a}{s^2+a^2}$
9	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11	$e^{at}\sin(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+a^2}$

$\to \cos(\omega t)$

Table 1: Laplace transform of some functions

Consider the *Heaviside function* at a,

$$H_a(t) = \begin{cases} 0, & \text{if } t < a, \\ 1, & \text{if } t \geq a. \end{cases}$$



then

$$\mathcal{L}[H_a(t)] = \int_0^\infty e^{-st} H_a(t)dt = \int_a^\infty e^{-st} dt = \frac{e^{-as}}{s}$$

$a = 0$

$H_0(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

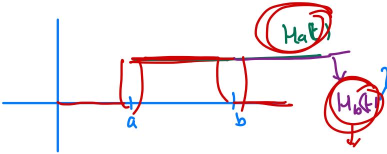
$H_0(t) = 1$

Characteristic function

Similarly, consider the *Characteristic function*,

$$\chi_S(t) = \begin{cases} 1 & t \in S \\ 0 & t \notin S \end{cases}$$

$$\chi_{[a,b]}(t) = \begin{cases} 0, & \text{if } t < a, \\ 1, & \text{if } a \le t \le b, \\ 0, & \text{if } t > b. \end{cases}$$



$$\chi_{[a,b]} = H_a(t) - H_b(t)$$

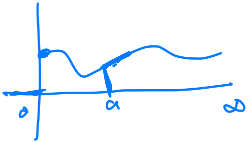
$a < b$

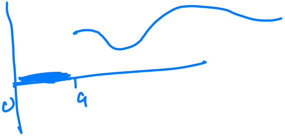
Laplace transform:

$$\mathcal{L}[\chi_{[a,b]}(t)](s) = \int_0^\infty e^{-st}\chi_{[a,b]}dt = \int_a^b e^{-st}dt = -\frac{e^{-bs}}{s} + \frac{e^{-as}}{s}$$

Shifted function

$$\tilde{f}(t) = f(t-a)H_a(t) = \begin{cases} 0, & \text{if } t < a, \\ f(t-a) & \text{if } t \geq a, \end{cases}$$





$f(t-a)$ $f(0)$

$t =$

Theorem

Let $f(t)$ be an function which satisfied (2) and $F(s) = \mathcal{L}[f](s)$. Consider the shifted function

$$\tilde{f}(t) = f(t-a)H_a(t) = \begin{cases} 0, & \text{if } t < a, \\ f(t-a) & \text{if } t \geq a, \end{cases}$$

has Laplace transform $\mathcal{L}[\tilde{f}] = e^{-as}F(s)$, i.e.

$$\mathcal{L}[f(t-a)H_a(t)] = e^{-as}F(s)$$

Proof.

$$\mathcal{L}[f(t-a)H_a(t)] = \int_0^\infty e^{-as} f(t-a)\boxed{H_a(t)}dt = \int_a^\infty e^{-st} f(t-a)dt.$$

Now introduce variable $t_1 = t - a$ and change the variable in the integral. We get,

$$\mathcal{L}[f(t-a)H_a(t)] = \int_0^\infty e^{-s(t_1+a)} f(t_1)dt_1 = e^{-as}\int_0^\infty e^{-st_1} f(t_1)dt = e^{-as}F(s).$$

□

Shifted function (cont.)

This theorem is very important as this allows us to calculate Laplace transform of functions with jump discontinuities. Furthermore, from this we can also deduce that

$f(t-a)$

$$\boxed{\mathcal{L}[f(t)H_a(t)] = e^{-as}\mathcal{L}[f(t+a)],}$$

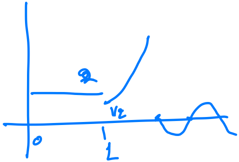
which is more useful form of the above result.

$f \quad \frac{t^2}{2} H_2(t)$

Example:

Find the Laplace transform of the following function:

$$f(t) = \begin{cases} 2, & \text{if } 0 < t < 1, \\ \frac{1}{2}t^2, & \text{if } 1 \leq t < \pi/2, \\ \cos(t), & \text{if } \quad t \geq \pi/2. \end{cases}$$



Using the Heaviside function we will rewrite $f(t)$ as,

$\chi_{[0,1]}$ $\qquad$ $\chi_{[1,\pi/2]}$ $\qquad$ $\chi_{[\pi/2,\infty)}$

$$f(t) = 2(1 - H_1(t)) + \frac{1}{2}t^2(H_1(t) - H_{\pi/2}(t)) + H_{\pi/2}(t)\cos(t)$$

$[0,1]$ $\qquad$ $[1, \pi/2]$

Now using the Linearity of Laplace transform, we get,

$$\mathcal{L}[f] = 2\mathcal{L}\left[1 - H_1(t)\right] + \frac{1}{2}\mathcal{L}\left[t^2 H_1(t)\right] - \frac{1}{2}\mathcal{L}\left[t^2 H_{\pi/2}(t)\right] + \mathcal{L}\left[H_{\pi/2}\cos(t)\right]$$

Now,

$$2\mathcal{L}\left[1 - H_1(t)\right] = \frac{2(1 - e^{-s})}{s}$$

and

$$\frac{t^2}{2}$$

$$\frac{1}{2}\mathcal{L}\left[t^2 H_1(t)\right] = e^{-s}\mathcal{L}\left[\frac{1}{2}(t+1)^2\right] = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s^2} + \frac{1}{2s}\right).$$

Similarly,

$$\frac{1}{2}\mathcal{L}\left[H_{\pi/2} t^2\right] = e^{-\pi s/2}\left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right).$$

$$\frac{1}{2}\mathcal{L}\left[\left(t - \pi/2\right)^2\right.$$

Now,

$$\mathcal{L}[H_{\pi/2}(t)\cos(t)] = e^{-\pi s/2}\mathcal{L}[\cos(t + \pi/2)] = -e^{-\pi s/2}\frac{1}{1+s^2}.$$

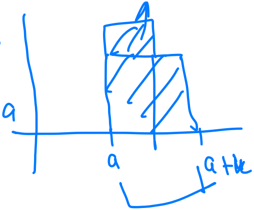
Combing all the terms we get the Laplace transform of f.

Dirac Delta "Function"

Consider the function,

$$f_k(t-a) = \begin{cases} \frac{1}{k} & \text{if } a \le t \le a+k, \\ 0, & \text{otherwise.} \end{cases}$$

$$\lim_{k \to 0} f_k(t-a) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}$$



$$\lim_{k \to 0} \int f_k(t-a) = 1$$

Note that,

$$I_k = \int_0^\infty f_k(t-a)dt = 1, \quad \forall k$$

We define Dirac delta function $\delta(t-a)$ as follows,

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a),$$

which gives,

$$\delta(t-a) = \begin{cases} \infty, & \text{if} \quad t=a, \\ 0, & \text{otherwise.} \end{cases}$$

X

and

$$\int_0^\infty \delta(t-a)dt = 1.$$

$\int \delta(t-a) f(t) dt$

$= \lim_{k\to 0} \int f_k(t-a) f(t) dt$

Dirac Delta "Function" (cont.)

One of the important property of Dirac delta function is,

$$= \lim_{k \to 0} \int_0^\infty g(t) \, f_k(t-a) = g(a)$$

$$\int_0^\infty g(t)\delta(t-a)dt = g(a)$$

for any continuous function g. This follows from

$$\boxed{\int_0^\infty g(t)\delta(t-a)dt = \lim_{k\to 0}\int_0^\infty f_k(t-a)g(t)dt.}$$

$$\mathcal{L}[f_k(t-a)] = \int_0^\infty e^{-st} f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k} e^{-st}\,dt$$

Note that

$$\mathcal{L}[f_k(t-a)] = e^{-as}\frac{1-e^{-ks}}{ks}$$

Exercise

So, we define,

$$\boxed{\mathcal{L}[\delta(t-a)] = \lim_{k\to 0}\mathcal{L}[f_k(t-a)] = e^{-as}.}$$

Consider a IVP with constant coefficients,

$$x''(t) + ax'(t) + bx(t) = g(t), \qquad x(0) = x_0. \quad x'(0) = x_1.$$

Let $X(s) = \mathcal{L}[x(t)](s)$, then

$$s^2X(s) - sx_0 - x_1 + a(sX(s) - x_0) + bX(s) = G(s).$$

Solving it for X, we get,

$$X(s) = \frac{sx_0 + x_1 + ax_0 G(s)}{s^2 + as + b}.$$

well defined

$\mathcal{L}^{-1}[X] = x(t)$

if $\mathcal{L}[x(t)] = X(s)$

Solution $x(t)$ is,

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{sx_0 + x_1 + ax_0 G(s)}{s^2 + as + b}\right].$$

$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+as+b}\right]$

$= g(t) * f(t)$

$\mathcal{L}[f] = \mathcal{L}[g]$

$f \overset{?}{=} g$ if true, $\mathcal{L}^{-1}$ is well defined

Important Question:

Let f and g are two functions such that there Laplace transform is same. What can be wait about these functions ? are they equal. This question is answered in the following result:

Theorem

Let f, g are two piecewise continuous on $[0, \infty)$ and satisfy (2). If $\mathcal{L}[f] = \mathcal{L}[g]$ on their common region of convergence, then $f(t) = g(t)$, for all $t \geq 0$ except at countable number of points. Furthermore, if f and g are continuous then $f(t) = g(t)$ for all $t \geq 0$.

we assume $\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} g(t)\,dt$

$\Rightarrow \int_0^\infty (f(t) - g(t))\, e^{-st}\, dt = 0 \Leftrightarrow f = g$

(Assuming f and g are continuous)

Definition

Let $F(s)$ is a given function. If there exists a function $f(t)$ such that $\mathcal{L}[f](s)$ exists and $F(s) = \mathcal{L}[f](s)$, we say ~~F~~ f is inverse Laplace transform of ~~f~~ F and denote, $f(t) = \mathcal{L}^{-1}[F(s)](t)$ or simply $\mathcal{L}^{-1}[F] = f$. .

Example:

f(t)

As $\mathcal{L}[H_a(t)] = e^{-as}/s$, implies $\mathcal{L}^{-1}[e^{-as}/s] \doteq H_a(t)$.

upto point of discontinuity

Linearity of Inverse L.T.

Note that as $\mathcal{L}$ is linear operator, it follows immediately that $\mathcal{L}^{-1}$ is also linear i.e.

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G].$$

Also note the following results:

- $$\mathcal{L}^{-1}[F(s-\alpha)] = e^{\alpha t} f(t)$$

- $$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^{\tau} f(\tau) d\tau$$

- $$\mathcal{L}^{-1}[e^{-\alpha s} F(s)] = H_{\alpha}(t) f(t-a)$$

Some useful Inverse Laplace Transform:

S. No.	$F(s)$	$\mathcal{L}^{-1}[F]$
1	$\frac{1}{(s^2+\alpha^2)^2}$	$\frac{1}{2\alpha^2}(\sin(\alpha t) - \alpha t\cos(\alpha t))$
2	$\frac{s}{(s^2+\alpha^2)^2}$	$\frac{t}{2\alpha}\sin(\alpha t)$
3	$\frac{s^2}{(s^2+\alpha^2)^2}$	$\frac{1}{2\alpha}(\sin(\alpha t) + \alpha t\cos(\alpha t))$

Table 2: Inverse Laplace transform of some functions

Partial fraction

Exercise

When solving for linear ODEs we will encounter several functions of the form $P(s)/Q(s)$ where P and Q are polynomial such that degree of P is less then degree of Q. We need to take their inverse Laplace transform. This can be done via the following results:

Characteristic polynomial

Theorem

Let $P(s)$ and $Q(s)$ be polynomial of degree n and m respectively such that $n < m$. If Q has m simple roots $\lambda_1, \cdots \lambda_m$, then

$$\boxed{\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \sum_1^m \frac{P(\lambda_i)}{Q'(\lambda_i)} e^{\lambda_i t}}$$

Definition

The convolution of $f(t)$ and $g(t)$, denoted by $f * g$, is the function define as,

$$(f * g)(t) = \int_0^t f(t-\theta)g(\theta)d\theta$$

Some properties of convolution are given here:

1. $$f * g = g * f$$

2. $$f * (g_1 + g_2) = f * g_1 + f * g_2$$

3. $$(f * g) * h = f * (g * h)$$

Convolution (cont.)

4.

$$f * 0 = 0 * f = 0$$

5.

$$1 * f = f * 1 \neq f$$

Theorem

Let f, g be piecewise continuous function of exponential order and $F(s) = \mathcal{L}[f]$ and $G(s) = \mathcal{L}[g]$, then

$$\mathcal{L}[f * g(t)](s) = F(s) \cdot G(s)$$

and this implies,

$$\mathcal{L}^{-1}[F(s)G(s)] = \boxed{(f * g)}(t).$$

Proof:

$$F(s)G(s) = \left[\int_0^\infty f(\sigma)e^{-s\sigma}\,d\sigma\right]\left[\int_0^\infty g(\tau)e^{-s\tau}\,d\tau\right]$$

As σ and τ are independent variables, we can rewrite $F(s)G(s)$ as,

$$F(s)G(s) = \int_0^\infty \left[\int_0^\infty f(\sigma)e^{-s(\sigma+\tau)}\,d\sigma\right] g(\tau)\,d\tau.$$

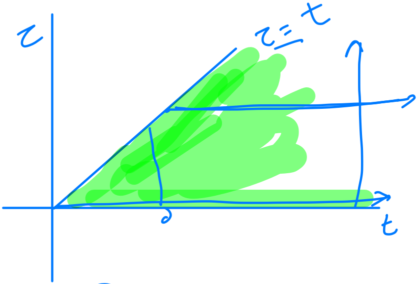
We fix τ and introduce the change of variable for internal integral as $t = \sigma + \tau$, then $dt = d\sigma$ and we get,

$$F(s)G(s) = \int_0^\infty \left[\int_\tau^\infty f(t-\tau)e^{-st}\,dt\right] g(\tau)\,d\tau$$

$$= \int_0^\infty \left[\int_\tau^\infty f(t-\tau)g(\tau)e^{-st}\,dt\right] d\tau$$

Convolution and L.T. (cont.)

We reverse the order of integration to get,



$$F(s)G(s) = \int_0^\infty \left[\int_0^t f(t-\tau)g(\tau)e^{-st}d\tau \right] dt$$

$$= \int_0^\infty \underbrace{\left[\int_0^t f(t-\tau)g(\tau)d\tau \right]}_{f*g} e^{-st}dt = \mathcal{L}[f*g]$$

Consider a IVP with constant coefficients,

$$x''(t) + ax'(t) + bx(t) = g(t), \qquad x(0) = x_0. \quad x'(0) = x_1.$$

Let $X(s) = \mathcal{L}[x(t)](s)$, then

$$s^2X(s) - sx_0 - x_1 + a(sX(s) - x_0) + bX(s) = G(s).$$

Solving it for X, we get,

$$X(s) = \frac{sx_0 + x_1 + ax_0 G(s)}{s^2 + as + b}.$$

Solution $x(t)$ is,

$$x(t) = \underline{\mathcal{L}^{-1}[X(s)] =} \mathcal{L}^{-1}\left[\boxed{\frac{sx_0 + x_1 + ax_0 G(s)}{s^2 + as + b}}\right].$$

Example: Consider the IVP,

$$x''(t) + x(t) = g(t), \qquad x(0) = 0, \quad x'(0) = k.$$

We get,

$$X(s) = \frac{k + G(s)}{s^2 + 1}.$$

So,

$$x(t) = \mathcal{L}^{-1}\left[\frac{k}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{G(s)}{s^2+1}\right] = k\sin(t) + \mathcal{L}^{-1}\left[\frac{G(s)}{s^2+1}\right].$$

g(t)

$\frac{1}{s^2+1}$

sin(t)

Inverse Laplace transform of $F(s) = 1/(s^2+1)$ is $\sin(t)$. Using the convolution property,

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+1}\right] = \int_0^t \sin(t-\theta)g(\theta)d\theta,$$

g(t) * sin t

which gives,

$$\boxed{x(t) = k\sin(t) + \int_0^t \sin(t-\theta)g(\theta)d\theta.}$$

Higher Order ODEs with Constant Coefficients

$$x^{(n)} + a_1 x^{(n-1)} \cdots + a_{n-1}x' + a_n x = g(t), \qquad x(0) = x_0, \cdots x^{(n-1)}(0) x_{n-1}.$$

Taking Laplace transform of the ODE we get,

$$s^n X(s) - s^{n-1}x_0 - s^{n-2}x_1 - \cdots - x_{n-1} + \cdots a_n X(s) = G(s),$$

which can be written as,

$$P(s)X(s) - Q(s) = G(s),$$

where

$$P(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

and

$$Q(s) = s^{n-1}x_0 + s^{n-2}x_1 + \cdots + a_{n-2}(sx_0 + x_1) + a_{n-1}x_0.$$

Solving for $X(s)$ gives,

$$X(s) = \frac{G(s) + Q(s)}{P(s)},$$

$\frac{Q(s)}{P(s)}$

and so the solution will be,

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{G(s)}{P(s)}\right] + \mathcal{L}^{-1}\left[\frac{Q(s)}{P(s)}\right].$$

$\mathcal{L}^{-1}\left[\frac{1}{P(s)}\right].$

Partial fraction

Linear ODE with variable coefficients:

Note that,

$$\mathcal{L}[tx'(t)] = -\frac{d}{ds}[sX - x_0] = -X - s\frac{dX}{ds}.$$

Similarly,

$$\mathcal{L}(tx'') = -2sX - s^2\frac{dX}{ds}.$$

In fact, if the coefficient are of the form $at + b$, taking Laplace transform results in first order ODE of $X(s)$. Solving this ODE and taking inverse Laplace transform results in the solution.

Consider the IVP,

$$x'' + tx = 0, \qquad x(0) = 0, \quad x'(0) = b.$$

Taking Laplace transform we get,

$$s^2X(s) - b + \mathcal{L}[tx(t)] = s^2X - b - X'(s) = 0$$

So, we get,

$$X'(s) - s^2X(s) = -b.$$

This a linear ODE which can be easily solved by method of integrating factor. Then taking the inverse Laplace transform results in the solution.

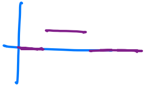
Generalized Solutions

- In all the theory discussed so far for ODE we have always assumed that coefficient and the forcing terms are atleast continuous.
- However, in several practical problems we have to consider the forcing which are not continuous or not even function (e.g. Dirac delta). These equations are still need to be solved.
- The method of Laplace transform is powerful enough to handle such situations.
- However, the solution we get may not be of desired smoothness. For example solution of second order may not be C^2. These solutions are called generalized solutions.

Here we will try to find "Generalized Solutions"

$x'' + x = g(t)$ discontinuous

No C^2 cont



Example 1:
Consider the IVP,

$$x'' + 3x' + 2x = \boxed{H_1(t) - H_2(t)}, \qquad x(0) = 0, \quad x'(0) = 0.$$

Taking Laplace transform

$$s^2X + 3sX + 2X = \frac{1}{s}(e^{-s} - e^{-2s})$$

which gives,

$$X(s) = \boxed{\frac{e^{-s} - e^{-2s}}{s(s^2+3s+2)}}.$$

$e^{-s} \quad e^{-2s}$

$\dfrac{1}{s(s^2+3s+2)}$

Now using partial fractions,

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}.$$

Generalized Solutions (cont.)

So,

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

Using the shift theorem,

$$x = \mathcal{L}^{-1}[[e^{-s}\mathcal{L}[f] - e^{-2s}\mathcal{L}[f]] = f(t-1)H_1(t) - f(t-2)H_2(t)$$

$$= \begin{cases} 0, & \text{if } \quad 0 < t < 1, \\ \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} & \text{if } \quad 1 < t < 2, \\ -e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} + e^{-(t-2)} - \frac{1}{2}e^{-2(t-2)} & \text{if } \quad t > 2. \end{cases}$$

$t=1$

$t=2$

Note that the solution is not C^2 (Check !).

C^1

Generalized Solutions (cont.)

Example 2:
Instead of Heaviside function on right if we take $\delta(t-1)$ then we get,

$$x'' + 3x' + 2x = \boxed{\delta(t-1)}, \qquad x(0) = 0, \quad x'(0) = 0.$$

taking Laplace transform we get,

$$X = \frac{e^{-s}}{s^2 + 3s + 2} = e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right).$$

Inverse Laplace transform gives,

$$x(t) = \mathcal{L}^{-1}[X] = \begin{cases} 0, & \text{if} \quad 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)}, & \text{if} \quad t > 1. \end{cases}$$

$t = 1$
not

Note that the solution $x(t)$ is not even C^1.

$\int$ $\int$

Consider a system of ODE,

$$x_1' = a_{11}x_1 + a_{12}x_2 + g_1(t),$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + g_2(t).$$

Let us define $X_1 = \mathcal{L}[x_1]$ and $X_2 = \mathcal{L}[x_2]$, then assume $G_1 = \mathcal{L}[g_1]$ and $G_2 = \mathcal{L}[g_2]$ exists, we get,

$$(a_{11} - s)X_1 + a_{12}X_2 = -x_1(0) - G_1(s),$$
$$a_{21}X_1 + (a_{22} - s)X_2 = -x_2(0) - G_2(s).$$

which can be written as,

$$(A - sI)\vec{X} = -\vec{X}(0) - \vec{G}.$$

This is a linear system for X_1 and X_2. Solving this we get expression for X_1 and X_2 is the form of s. Then inverse Laplace transform gives the solution $x_1(t)$ and $x_2(t)$.

Example 1: Mixing problem involving two tanks
Consider the System:

$$x_1'(t) = -\frac{8}{100}x_1 + \frac{2}{100}x_2 + 6.$$

Similarly for tank T_2 we have,

$$x_2'(t) = \frac{8}{100}x_1 - \frac{8}{100}x_2.$$

with initial conditions $x_1(0) = 0$ and $x_2(0) = 150$. Taking Laplace transform of both the equations we get,

$$(-0.08 - s)X_1 + 0.02X_2 = -\frac{6}{s},$$
$$(0.08)X_1 + (-0.08 - s)X_2 = -150.$$

System of Equations (cont.)

Solving for X_1 and X_2 we get,

$$X_1(s) = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{375.5}{s+0.04},$$
$$X_2(s) = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04}.$$

Taking inverse Laplace transformation we get,

$$x_1(t) = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t},$$
$$x_2(t) = 100 + 125e^{-0.12t} - 75e^{-0.04t}.$$