# Lecture 1.

Matrices: is a rectangular array of numbers or functions.

numbers: Real numbers $\mathbb{R}$ Complex numbers $\mathbb{C}$.

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad a_{ij} \in \mathbb{R} \text{ or } \mathbb{C}$$

row (i), column (j)

Size of matrix $A$ = (# of rows) $\times$ (# of columns)

$m \times n$

Matrices with single row $(m=1)$ or a single col $(n=1)$ is call a vector.

Matrices with single rows are called row vectors

cols — column vectors.

Ex:

$$\begin{bmatrix} 1 & 0 \\ \pi & e \end{bmatrix} \quad 2\times 2 \text{ matrix.}$$

$\begin{bmatrix} 1 \\ \pi \end{bmatrix}$ is a column vector $\quad [1 \ 0]$ is a row vector

- If $m = n$ then $A$ is a square matrix
- The diagonal entries $a_{11}, a_{22}, \dots a_{nn}$ are called the main diagonal of $A$.

Ex:

$$\begin{array}{cc} x & f(x) = e^x \\ \left[\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\right. & \left.\begin{array}{c} e \\ e^2 \\ e^3 \\ e^4 \end{array}\right] \end{array}$$

- Two matrices are equal $A = (a_{ij})$ $B = (b_{ij})$
  if and only if
  1. they have same size
  2. corresponding entries are equal i.e., $a_{ij} = b_{ij}$ for all $i, j$.

- The sum of two matrices $A = (a_{ij})$ $B = (b_{ij})$ of the same size

  $A + B = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$
  ↳ entries obtained by adding the corresponding entries of $A$ & $B$.

- Matrices of different sizes cannot be added.

Ex: $A = \begin{bmatrix} -4 & 6 & 2 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 6 & 2 \end{bmatrix}_{2\times 3}$ $A + B = \begin{bmatrix} -4+1 & 6+2 & 2+3 \\ 1-4 & 2+6 & 2+3 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 8 & 5 \\ -3 & 8 & 5 \end{bmatrix}$$

- scalar multiplication (Multiplication by a number)

The product of $m \times n$ matrix $A = (a_{ij})$ and a scalar $c \in \mathbb{R}$ (or $\mathbb{C}$) written as $cA$ is a matrix $(ca_{ij})$ obtained by multiplying each entry of $A$ by the scalar $c$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ e & \pi \end{bmatrix}_{3\times 2}$ $\quad c = \sqrt{2}$ $\quad \sqrt{2}A = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 3\sqrt{2} & 4\sqrt{2} \\ e\sqrt{2} & \pi\sqrt{2} \end{bmatrix}$

- negative of $A$ is defined as $-A = (-1)\cdot A$.

$A = (a_{ij})$ then $-A = (-a_{ij})$.

- $A+B$, $A-B$.

Rules for matrix addition & scalar multiplication.

1. $A+B = B+A$
2. $(A+B)+C = A+(B+C)$
3. $A+O = A$ $\qquad O_{m\times n}$
4. $A+(-A) = O$

$O$ is the zero matrix having the same size as that of $A$.

If $A$ is $3\times 2$ matrix then $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\quad A+O = A$.

If $m=1$ or $n=1$ then the $O$ is called a zero row or zero col.

(5) $c(A+B) = cA + cB$ — c is a scalar.

(6) $(c+d)A = cA + dA$

(7) $c(dA) = (cd)A$.

(8) $1 \cdot A = A$ — scalar number 1.

$I$ is the identity matrix

§ Matrix multiplication.

$A = (a_{ij})$ $\qquad$ $B = (b_{ij})$

Size $\quad m \times n$ $\qquad$ $r \times p$.

The product $c = AB$ is defined if and only if $r = n$ and (1) $AB$ is an $m \times p$ matrix.

(2) $C = AB = (c_{ij})$

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

$i = 1 \ldots m$
$j = 1 \ldots p$.

$4 \times 3$ $\qquad$ $3 \times 2$ $\qquad$ $4 \times 2$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}$$

$$c_{11} = [a_{11}\ a_{12}\ a_{13}]\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

Ex:
$$\begin{bmatrix} 3 & 5 & 1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Rules for matrix multiplication

1. $(dA)B = d(AB) = A(dB)$ — $d$ is a scalar
2. $A(BC) = (AB)C$
3. $(A+B)C = AC + BC$
4. $C(A+B) = CA + CB$.

- Identity: $\mathbb{1}A = A$.

$$\mathbb{1}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- It is possible that $AB \neq BA$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 9 & -5 \\ 1 & 1 \end{bmatrix} \neq BA = \begin{bmatrix} 6 & -10 \\ -1 & 4 \end{bmatrix}$$

- $AB = 0_{m\times p} \not\Rightarrow A = 0$ or $B = 0$ 
$m\times n$ $n\times p$ matrix

$0\cdot 2 = 0$
$1\cdot 2 \neq 0$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \quad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- $AC = BC \not\Rightarrow A = B$ or $C = 0$

$2\cdot 3 = 2\cdot 4$
$3 \neq 4$

$$A = \begin{bmatrix} -3 & 3 \\ 11 & -3 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 \\ 8 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

$$AC = \begin{bmatrix} 3 & 9 \\ 5 & 15 \end{bmatrix} = BC = \begin{bmatrix} 3 & 9 \\ 5 & 15 \end{bmatrix}$$

- Transposition

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

The transpose of a $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix $A^T$ that has the first row of $A$ as its first column, second row of $A$ is its second colun ....

$$A^T = (a_{ji}) = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & & \ddots & \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}$$

Transposition laws ① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

(3) $(cA)^T = cA^T$ c is a scalar

(4) $\boxed{(AB)^T = B^T A^T}$

- Skew-Symmetric matrices: are square matrices s.t. transpose equals the negative of A

$$\begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & 6 \\ -5 & -6 & 3 \end{bmatrix}$$

$$A^T = -A.$$

- Symmetric matrices are those which equals its transpose

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

$$A = A^T.$$

- Diagonal matrices: Matrices which can have non zero entries only on the main diagonal

- Triangular matrices:
  - ↳ Upper Triangular matrices: Are square matrices that can have nonzero entries only on and above the main diagonal

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

  - ↳ lower Triangular matrices: Are square matrices that can have nonzero entries only on & below the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$$

- Power of Matrices: $A$ is a square matrix

$$A^2 = A \cdot A$$
$$A^3 = A^2 \cdot A$$
$$\vdots$$
$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}$$

Partitioned matrices.

$$A = \left[\begin{array}{ccc|cc} 2 & 0 & -3 & 1 & 7 \\ -1 & 4 & 2 & 0 & 4 \\ 6 & -1 & 1 & 8 & -3 \\ \hline 0 & 2 & 7 & -3 & 3 \\ 2 & 0 & -6 & 9 & 0 \\ \hline 1 & -1 & 8 & 5 & -1 \\ 4 & 6 & 9 & 7 & 8 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

Another matrix B of the same size as that A.

$$A+B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}+B_{11} & A_{12} \; B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \\ A_{31}+B_{31} & A_{32}+B_{22} \end{bmatrix}$$

$$\gamma A = \begin{bmatrix} \gamma A_{11} & \gamma A_{12} \\ \gamma A_{21} & \gamma A_{22} \\ \gamma A_{31} & \gamma A_{32} \end{bmatrix}$$

Ex:

$$A = \left[\begin{array}{cc|ccc} 3 & -1 & 2 & 4 & 0 \\ 0 & 2 & 1 & -3 & 1 \\ 2 & 3 & -4 & 0 & -4 \\ \hline 1 & 6 & 0 & 2 & -2 \end{array}\right] \qquad B = \left[\begin{array}{c|c|c} 3 & 1 & 2 \\ 4 & 0 & 3 \\ \hline -1 & 7 & 0 \\ 2 & 4 & 1 \\ 0 & -1 & -1 \end{array}\right]$$

$4 \times 5$ $\qquad$ $5 \times 3$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} & A_{11}B_{13} + A_{12}B_{23} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{23} \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} 11 & 33 & 7 \\ 1 & -4 & 4 \\ 22 & 30 & 9 \\ \hline -31 & 11 & 24 \end{array}\right]$$

- Inverse of matrix: If $A$ is a square matrix, the inverse is a matrix $B$ s.t. $A \cdot B = \underline{I}$ and $B \cdot A = I$

• Def: Trace: A trace of a square matrix A is the sum of the diagonal entries of A. It is denoted by tr(A)

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Properties: ① $tr(A+B) = tr(A) + tr(B)$

② $tr(A) = tr(A^T)$

③ $\boxed{tr(AB) = tr(BA)}$

• Def: Idempotent matrices: Square matrix A is idempotent if

$$A^2 = A$$

Ex: $tr\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1+4$

$tr\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1+5+9$

Ex: $A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$. check $A^2 = A$.

identity.

Property: If A is idempotent then so is $I - A$

$(I-A)^2 = (I-A)(I-A) = I - A - A + A^2 = A$.

$= I - A - A + A = I - A$