Moment distribution

Introduction

While the advancement of computer based analysis continues to grow exponentially within the field of structural engineering, the tools that are used to analyse structures by hand are no less relevant. Many would argue that such tools are even more vital today than they have ever been if we are to fully understand the output of analysis applications. With this in mind, this Technical Guidance Note describes one of the most powerful analysis tools available: moment distribution.

Moment distribution is a method by which statically indeterminate structures are analysed elastically. It's based on the relative stiffness of elements that make up a structure and shifts bending moments from one section of the structure to another until they become balanced. Once this balance has been achieved, the forces and bending moments within the structure are modelled.

ICON LEGEND

- Analysis principles
- Worked example
- Further reading
- Web resources

Analysis principles

The origins of moment distribution analysis method date back to 1932 when a paper published by Professor Hardy Cross described a means by which indeterminate structures could be analysed by hand. This was driven by the increase in popularity of reinforced concrete structures as opposed to steel framed buildings. The former are made up of statically indeterminate sub-frames and thus a quick and easy method of analysing them was needed. The principle of moment distribution is based on creating fixed end moments at joints in a structure and then releasing them sequentially in order to derive the bending moments within it. This is done via an iterative process that relies on achieving equilibrium as the joints in the structure are released.

Consider Figure 1, which illustrates a 2 span beam that has fully fixed supports at each

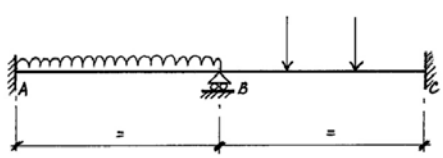
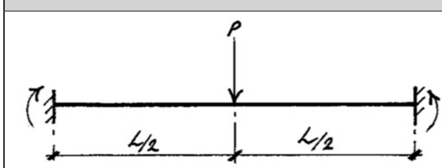
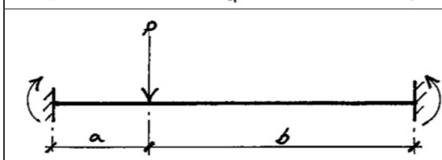
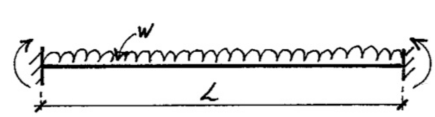
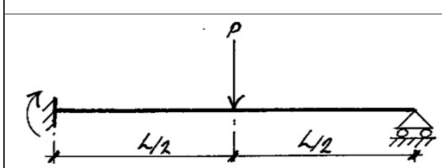
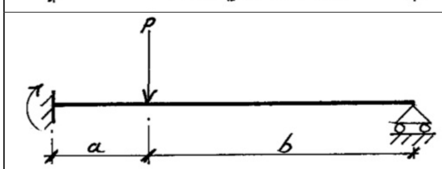
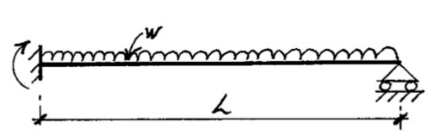


Figure 1
2 span beam with fixed end moment connections

This is an indeterminate structure, which can quite easily be analysed using the moment distribution method. This is done by placing fixed end moment connection at the point where the structure can rotate. The

Table 1: Fixed end moment solutions

Load condition	Fixed end moment
	$\frac{PL}{8}$
	$\frac{Pa^2b}{L^2}$
	$\frac{wL^2}{12}$
	$\frac{3PL}{16}$
	$\frac{Pab(2L-a)}{2L^2}$
	$\frac{wL^2}{8}$

This version 1.1 published October 2016.

additional bending moment generated at the fixed end is distributed between both of the spans. These additional moments are then distributed again until they are dissipated to the point where equilibrium is achieved. See Figure 2 for further explanation of this:

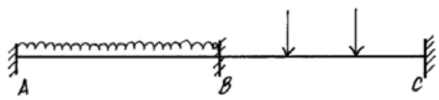


Figure 2
Fixed end moments applied to a 2 span beam

Sign conventions and fixed end moment solutions

Moment distribution does employ conventions and short-hand in order to simplify the method of analysis. The most important convention is the direction of the bending moments at each joint. Figure 3 explains what this convention is:

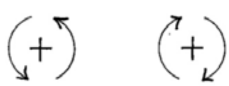


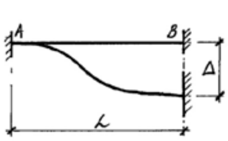
Figure 3
Moment distribution sign convention

Table 1 is a list of some of the most common fixed end moment solutions for typical load conditions. These are used when determining the moments at locked joints prior to them being released and distributed during the moment distribution analysis.

Spring supports

There are many instances where the assumption that a support is infinitely stiff is not a conservative one. In such instances it is prudent to model the supports of the structure as a spring, rather than an immovable prop. Such conditions can be allowed for within moment distribution

Table 2: Fixed end moments for displaced supports

Support condition	Fixed end moment
	$\frac{6EI\Delta}{L^2}$
	$\frac{3EI\Delta}{L^2}$

analysis. This is achieved by defining the amount of vertical displacement of the support when loads are applied to it and then determining the resulting fixed end moments. Table 2 shows two typical support conditions which have dropped by a distance 'Δ' and the resulting fixed end bending moments as shown.

Relative stiffness

The proportions of the fixed end moments that are distributed are dependent upon the stiffness of the elements within the structure. The stiffness is defined as the ratio between the span of the element between supports and its inherent stiffness, expressed in the form of Young's Modulus (E) and second moment of area (I).

The relative stiffness of one element in a structure, to those it shares a joint with, is used to determine the distribution factor to the fixed end moments as they are distributed.

A member with continuous supports at both ends has a stiffness defined as:

$$\frac{EI}{L}$$

Where one end of the element has no continuity at the point of support, the value of its stiffness is defined as:

$$\frac{3}{4}\frac{EI}{L}$$

This equates to the relative stiffness of a member with a non-continuous support being ¾ that of a member with continuous supports.

Worked example

Figure 4 is of a multi-span beam with an E of 205 kN/mm² and an I of 87318x10⁴ mm⁴. Determine the bending moments in the structure using moment distribution.

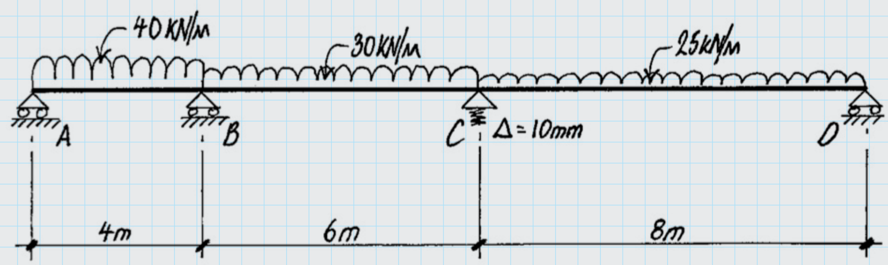


Figure 4
3 span beam with a spring support

The first step is to determine the relative stiffness of the beam. With EI being constant along its entire length, the stiffness is based on end support conditions and length of span between supports.

RELATIVE STIFFNESS:

$BA = \frac{3}{4} \times \frac{1}{4} = 0{\cdot}19$, $BC = \frac{1}{6} = 0{\cdot}17$

∴ DISTRIBUTION FACTOR ⇒ $BA = \frac{0{\cdot}19}{0{\cdot}19+0{\cdot}17} = 0{\cdot}53$, $BC = \frac{0{\cdot}17}{0{\cdot}19+0{\cdot}17} = 0{\cdot}47$

$CB = \frac{1}{6} = 0{\cdot}17$, $CD = \frac{3}{4} \times \frac{1}{8} = 0{\cdot}1$

∴ DISTRIBUTION FACTOR ⇒ $CB = \frac{0{\cdot}17}{0{\cdot}17+0{\cdot}1} = 0{\cdot}63$, $CD = \frac{0{\cdot}1}{0{\cdot}17+0{\cdot}1} = 0{\cdot}37$

With the distribution factors for the mid span supports calculated, the fixed end moments (FEMs) are derived.

FIXED END MOMENTS

$$FEM_{BA} = \frac{40\,kN/m \times 4m^2}{8} = +80\,kNm$$

$$FEM_{BC} = -\left(\frac{30\,kN/m \times 6m^2}{12}\right) + \left(\frac{6 \times 205\,kN/mm^2 \times 87318 \times 10^4\,mm^4 \times 10mm \times 10^{-3}}{(6m \times 10^3)^2}\right) = -388\,kNm$$

$$FEM_{CB} = +\left(\frac{30\,kN/m \times 6^2}{12}\right) - \left(\frac{6 \times 205\,kN/mm^2 \times 87318 \times 10^4\,mm^4 \times 10mm \times 10^{-3}}{(6m \times 10^3)^2}\right) = -208\,kNm$$

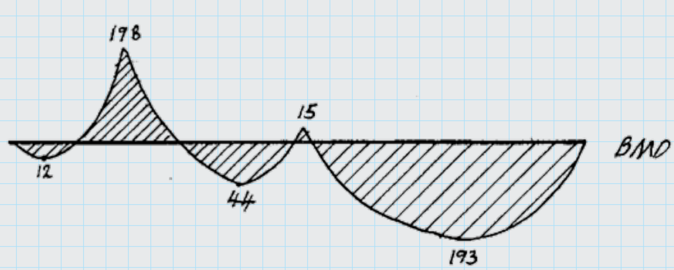
$$FEM_{CD} = -\left(\frac{25\,kN/m \times 8^2m}{8}\right) + \left(\frac{3 \times 205\,kN/m^2 \times 87318 \times 10^4\,mm^4 \times 10mm \times 10^{-3}}{(8m \times 10^3)^2}\right) = -116\,kNm$$

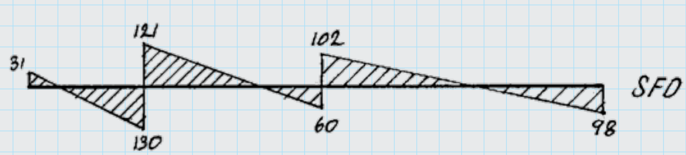
Notice that for the spring, the FEM is calculated by adding the moment due to the applied load to the one generated by the movement of the spring support. This is a form of super-position to ensure all of the bending moment generated within the structure is taken into account. Also notice that the magnitude of the bending moment is far greater for spring supports that it is for those that are infinitely stiff.

The final part of the analysis is the act of carrying out the moment distribution. The process for this is as follows:

1) Calculate the sum of the FEM at each internal support
2) Distribute this sum of bending moments in proportion to the relative stiffness of elements that connect to the support
3) Reverse the sign of the distributed moments to counter the effects of the out of balance bending moment and carry half of their magnitude to the adjacent continuous and fixed supports
4) Repeat Step 2 until the remaining distributed bending moments are close to 1% of the initially distributed bending moment
5) Sum the total bending moments for all internal supports; they should balance at each support
6) Draw bending moment and shear force diagram

	A	B		C		D
DF		0.53	0.47	0.63	0.37	
FEM	0	+80	-388	-208	-116	0
DIST		+163	+145	+204	+120	
CO			+102	+73		
DIST		-54	-48	-46	-27	
CO			-23	-24		
DIST		+12	+11	+15	+9	
CO			+8	+6		
DIST		-4	-4	-4	-2	
CO			-2	-2		
DIST		+1	+1	+1	+1	
FINAL	0	+198	-198	+15	-15	0





Application

Thus far reference has only been made to the use of moment distribution as a means to analyse continuous beams. It should be noted that this method of analysis can be applied to more complex frames, with the only major difference being the distribution of FEMs being spread through more than two elements. It is quite possible therefore to adopt moment distribution when analysing sway frames and sub-frames. For more information on this, the reader is directed to *Understanding Structural Analysis* by David Brohn.

"Such tools are even more vital today than they have ever been"

Glossary and further reading

Carry over – The carry over of fixed end moments as they are distributed along the structure.

Distribution – The act of distributing bending moments as supports are freed, in order to achieve equilibrium.

Fixed end moments (FEMs) – these are the bending moments calculated at each joint in a structure.

Relative stiffness – The relative stiffness between connected elements within a structure.

Further Reading

Cross. H.: (1932) Analysis of continuous frames by distributing fixed-end moments *Trans. Am. Soc. Civ. Eng.* Vol. 96 (1) pp.1-10

Brohn, D.M.: (2005) *Understanding Structural Analysis* 3rd Ed. New Paradigm Solutions

Web resources

The Institution of Structural Engineers library:
www.istructe.org/resources-centre/library