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MATH 10
SID: 1545566
Section 2838

Ch 5 Problems

5.1 3. a) $P(1)$ is the statement $1^2 = \dfrac{1(1+1)(2(1)+1)}{6}$

b) $$1 = \frac{1(2)(3)}{6}$$
$$1 = \frac{6}{6}$$
$$1 = 1$$

c) The inductive hypothesis is the statement that
$$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

d) For the inductive step, we want to show for each $k \geq 1$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis we can show
$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

e) By the Inductive hypothesis, the left hand side of the eq. from (d) equals $\dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$

$$(k+1)\left(\frac{k(2k+1)}{6} + (k+1)\right)$$
$$(k+1)\left(\frac{2k^2 + k + 6k + 6}{6}\right)$$
$$(k+1)\left(\frac{2k^2 + 7k + 6}{6}\right)$$
$$\frac{(k+1)(k+2)(2k+3)}{6}$$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

4. a) $P(1)$ is the statement $1^3 = \left(\frac{1(1+1)}{2}\right)^2$

b)

$$1 = \left(\frac{1(2)}{2}\right)^2$$

$$1 = \left(\frac{2}{2}\right)^2$$

$$1 = 1$$

c) The inductive hypothesis is the statement that

$$1^3 + 2^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

d) For the inductive step, we want to show for each $k \geq 1$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis we can show

$$1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

e) By the inductive hypothesis, the left hand side of the eq. from (d) equals

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$\frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$\frac{(k+1)^2(k^2 + 4(k+1))}{4}$$

$$\frac{(k+1)^2(k^2 + 4k + 4)}{4}$$

$$\frac{(k+1)^2(k+2)^2}{4}$$

$$\left(\frac{(k+1)(k+2)}{2}\right)^2$$

f) We have completed both the basis and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

5. To construct the proof, let $P(n)$ denote the proposition

$$1^2 + 3^2 + 5^2 + \dots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Basis Step: The statement $P(0)$ is true because

$$(2(0)+1)^2 = \frac{(0+1)(2(0)+1)(2(0)+3)}{3}$$

$$1^2 = \frac{(1)(1)(3)}{3}$$

$$1 = 1$$

This completes the basis step.

Inductive Step: For the inductive hypothesis we assume $P(k)$ is true, namely, that

$$1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$$

We must show that $P(k+1)$ is also true, namely, that

$$1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 + (2k+3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$$

Notice that the left side is equal to (by the inductive hypothesis):

$$= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$$

$$= \frac{(k+1)(2k+1)(2k+3) + 3(2k+3)^2}{3}$$

$$= \left(\frac{2k+3}{3}\right)(2k^2 + k + 2k + 1 + 6k + 9)$$

$$= \left(\frac{2k+3}{3}\right)(2k^2 + 9k + 10)$$

$$= \frac{(k+2)(2k+3)(2k+5)}{3}$$

as required. This completes the inductive step.

Basis Step: The statement P(1) is true b/c

$$1 \cdot 1! = (1+1)! - 1$$
$$1 = 2! - 1$$
$$1 = 2 \cdot 1 - 1$$
$$1 = 1$$

Inductive Step: For the inductive hypothesis we assume P(k) is true, namely, that

$$1 \cdot 1! + 2 \cdot 2! + \dots + k \cdot k! = (k+1)! - 1$$

We must show that P(k+1) is also true, namely, that

$$1 \cdot 1! + 2 \cdot 2! + \dots + k \cdot k! + (k+1)(k+1)! = (k+2)! - 1$$

By the inductive hypothesis, the LHS is equal to

$$= (k+1)! - 1 + (k+1)(k+1)!$$
$$= (k+1)!\,(1 + (k+1)) - 1$$
$$= (k+1)!\,(k+2) - 1$$
$$= (k+2)! - 1$$

7. To construct the proof, let P(n) denote the proposition

$$3 + 3 \cdot 5 + 3 \cdot 5^2 + \dots + 3 \cdot 5^n = \frac{3(5^{n+1} - 1)}{4}$$

Basis Step: The statement P(0) is true b/c

$$3 \cdot 5^0 = \frac{3(5^{0+1} - 1)}{4}$$
$$3 \cdot 1 = \frac{3(5-1)}{4}$$
$$3 = 3$$

Inductive Step: For the inductive hypothesis we assume P(k) is true, namely, that

$$3 + 3 \cdot 5 + 3 \cdot 5^2 + \dots + 3 \cdot 5^k = \frac{3(5^{k+1} - 1)}{4}$$

We must show that P(k+1) is also true, namely, that

$$3 + 3 \cdot 5 + 3 \cdot 5^2 + \dots + 3 \cdot 5^k + 3 \cdot 5^{k+1} = \frac{3(5^{k+2} - 1)}{4}$$

By the inductive hypothesis, the LHS is equal to

$$= \frac{3(5^{k+1}-1)}{4} + 3 \cdot 5^{k+1}$$

$$= \frac{3 \cdot 5^{k+1} - 3 + 12 \cdot 5^{k+1}}{4}$$

$$= 5^{k+1}\left(\frac{3}{4} + \frac{12}{4}\right) - \frac{3}{4}$$

$$= 5^{k+1}\left(\frac{15}{4}\right) - \frac{3}{4}$$

$$= 5^{k+2}\left(\frac{3}{4}\right) - \frac{3}{4}$$

$$= \frac{3(5^{k+2}-1)}{4}$$

8. To construct the proof, let $P(n)$ denote the proposition

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = \frac{(1-(-7)^{n+1})}{4}$$

<u>Basis Step:</u> The statement

$P(0)$ is true b/c

$$2(-7)^0 = \frac{(1-(-7)^{0+1})}{4}$$

$$2 = \frac{(1+7)}{4}$$

$$2 = 2$$

<u>Inductive Step:</u> For the inductive hypothesis we assume $P(k)$ is true, namely, that

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k = \frac{(1-(-7)^{k+1})}{4}$$

We must show that $P(k+1)$ is also true, namely, that

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k - 2(-7)^{k+1} = \frac{(1-(-7)^{k+2})}{4}$$

$$4$$

$$= \frac{1 - (-7)^{k+1} + 8(-7)^{k+1}}{4}$$

$$= (-7)^{k+1} \frac{(-1 + 8)}{4} + \frac{1}{4}$$

$$= \frac{(-7)^{k+2}(-1) + 1}{4}$$

$$= \frac{\left(1 - (-7)^{k+2}\right)}{4}$$

9. a) We can obtain a formula for the sum of the first n even positive integers from the the formula for the sum of the first n positive integers, since $2 + 4 + 6 + \dots + 2n = 2(1 + 2 + 3 + \dots + n)$. Therefore, using the result of Example 1, the sum of the first n even positive integers is $2\left(\frac{n(n+1)}{2}\right) = n(n+1)$

b) To construct the proof, let $P(n)$ denote the proposition $2 + 4 + 6 + \dots + 2n = n(n+1)$.

Basis step: The statement $P(1)$ is true b/c

$$2(1) = 1(1+1)$$

$$2 = 1(2)$$

$$2 = 2$$

Inductive step: For the inductive hypothesis we assume $P(k)$ is true, namely, that

$$2 + 4 + 6 + \dots + 2k = k(k+1)$$

We must show that $P(k+1)$ is also true, namely, that

$$2 + 4 + 6 + \dots + 2k + 2k + 2 = (k+1)(k+2)$$

By the inductive hypothesis, the LHS is equal to

$= k(k+1) + 2k + 2$

$= k^2 + k + 2k + 2$

$= k^2 + 3k + 2$

$= (k+1)(k+2)$

10. a)

n	$\frac{1}{n(n+1)}$	$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)}$
1	$\frac{1}{1(1+1)} = \frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2(2+1)} = \frac{1}{6}$	$\frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$
3	$\frac{1}{3(3+1)} = \frac{1}{12}$	$\frac{4}{6} + \frac{1}{12} = \frac{8}{12} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$
4	$\frac{1}{4(4+1)} = \frac{1}{20}$	$\frac{9}{12} + \frac{1}{20} = \frac{45}{60} + \frac{3}{60} = \frac{48}{60} = \frac{4}{5}$
5	$\frac{1}{5(5+1)} = \frac{1}{30}$	$\frac{48}{60} + \frac{1}{30} = \frac{48}{60} + \frac{2}{60} = \frac{50}{60} = \frac{5}{6}$

Notice the summation is equal to $\frac{n}{n+1}$

b) To construct the proof, let $P(n)$ denote the proposition

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

<u>Basis Step</u>: The statement $P(1)$ is true b/c

$$\frac{1}{1(1+1)} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2}$$

<u>Inductive Step</u>: For the inductive hypothesis we assume $P(k)$ is true, namely, that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

We must show that $P(k+1)$ is also true, namely, that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

By the inductive hypothesis the LHS is equal to

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{\cancel{(k+1)}(k+1)}{\cancel{(k+1)}(k+2)}$$

$$= \frac{k+1}{k+2}$$

11. a)

n	$\frac{1}{2^n}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$
~~0~~	~~$\frac{1}{2^0} = 1$~~	
1	$\frac{1}{2^1} = \frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2^2} = \frac{1}{4}$	$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
3	$\frac{1}{2^3} = \frac{1}{8}$	$\frac{3}{4} + \frac{1}{8} = \frac{6}{8} + \frac{1}{8} = \frac{7}{8}$
4	$\frac{1}{2^4} = \frac{1}{16}$	$\frac{7}{8} + \frac{1}{16} = \frac{14}{16} + \frac{1}{16} = \frac{15}{16}$

Notice the summation is equal to $\frac{2^n - 1}{2^n}$

b) To construct the proof let P(n) denote the proposition

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2n} = \frac{2^n - 1}{2^n}$$

Basis Step: The statement P(1) is true b/c

$$\frac{1}{2^1} = \frac{2^1 - 1}{2^1}$$

$$\frac{1}{2} = \frac{1}{2}$$

Inductive Step: For the inductive hypothesis we assume P(k) is true, namely, that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$

We must show that P(k + 1) is also true, namely, that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$

By the inductive hypothesis the LHS is equal to

$$= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}}$$

$$= \frac{2(2^k - 1) + 1}{2^{k+1}}$$

$$= \frac{2^{k+1} - 2 + 1}{2^{k+1}}$$

$$= \frac{2^{k+1} - 1}{2^{k+1}}$$

14. To construct the proof let P(n) denote the proposition

$$\Sigma_{k=1}^{n} k 2^k = (n-1) 2^{n+1} + 2$$

Basis Step: The statement P(1) is true b/c

$$(1) 2^1 = (1-1) 2^{1+1} + 2$$

$$2 = 0 + 2$$

$$2 = 2$$

Inductive Step: For the inductive hypothesis we assume $P(j)$ is true, namely, that

$$\sum_{k=1}^{j} k2^k = (j-1)2^{j+1} + 2$$

We must show $P(j+1)$ is also true, namely, that

$$\sum_{k=1}^{j+1} k2^k = (j)2^{j+2} + 2$$

By the inductive hypothesis the LHS is equal to

$$= (j-1)2^{j+1} + 2 + (j+1)2^{j+1}$$

$$= 2^{j+1}(j-1+j+1) + 2$$

$$= 2^{j+1}(2j) + 2$$

$$= (j)2^{j+2} + 2$$

15. To construct the proof, let $P(n)$ denote the proposition

$$1\cdot 2 + 2\cdot 3 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Basis Step: The statement $P(1)$ is true b/c

$$1(1+1) = \frac{1(1+1)(1+2)}{3}$$

$$2 = \frac{6}{3}$$

$$2 = 2$$

Inductive Step: For the inductive hypothesis we assume $P(k)$ is true, namely, that

$$1\cdot 2 + 2\cdot 3 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

We must show that $P(k+1)$ is also true, namely, that

$$1\cdot 2 + 2\cdot 3 + \dots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

By the inductive hypothesis the LHS is equal to

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

16. To construct the proof, let $P(n)$ denote the proposition

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

Basis Step: The statement $P(1)$ is true b/c

$$1(1+1)(1+2) = \frac{1(1+1)(1+2)(1+3)}{4}$$

$$1(2)(3) = \frac{1(2)(3)(\cancel{4})}{\cancel{4}}$$

$$6 = 6$$

Inductive Step: For the inductive hypothesis we assume $P(k)$ is true, namely, that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

We must show that $P(k+1)$ is also true, namely, that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

By the inductive hypothesis the LHS is equal to

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4}$$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

18. a) $P(2)$ is the statement $2! < 2^2$

b) $2! < 2^2$

$$2 \cdot 1 < 4$$

$$2 < 4$$

c) For the inductive hypothesis we assume $P(k)$ is true, namely, that $k! < k^k$ for the positive integer k with $k > 1$

d) We must show that $P(k+1)$ is also true, namely, that $(k+1)! < (k+1)^{k+1}$

e) $(k+1)! = k! \cdot (k+1)$ by def'n of factorial

$< k^k \cdot (k+1)$ by inductive hypothesis

$< (k+1)^k \cdot (k+1)$ b/c $k^k < (k+1)^k$

$< (k+1)^{k+1}$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n greater than 1.

19. a) $\frac{1}{1^2} + \frac{1}{2^2} < 2 - \frac{1}{2}$

b) $1 + \frac{1}{4} < \frac{3}{2}$

$\frac{5}{4} < \frac{6}{4}$

c) $1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{k^2} < 2 - \frac{1}{k}$ where k positive integer w/ $k > 1$

d) We must show that $P(k)$ implies $P(k+1)$, namely,

$$1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$$

e) by the inductive hypothesis

$$1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right)$$

$$= 2 - \left(\frac{k^2 + 2k + 1 - k}{k(k+1)^2}\right)$$

$$= 2 - \frac{k^2 + k}{k(k+1)^2} - \frac{1}{k(k+1)^2}$$

$$= 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2} < 2 - \frac{1}{k+1}$$

f) We have completed both the basis step and the inductive step, so by the principal of mathematical induction, the statement is true for every positive integer n greater than 1.

20. Let $P(n)$ be the statement $3^n < n!$ where n is an integer greater than 6.

<u>Basis Step:</u> $P(7)$ is true b/c

$$3^7 < 7!$$
$$2187 < 5040$$

<u>Inductive Step:</u> For the inductive hypothesis we assume $P(k)$ is true where k is an integer w/ $k > 6$, namely, that

$$3^k < k!$$

We must show that $P(k+1)$, namely, that

$$3^{k+1} < (k+1)!$$

$$\begin{aligned} 3^{k+1} &= 3 \cdot 3^k \\ &< 3 \cdot k! && \text{by ind. hyp.} \\ &< (k+1) \cdot k! && \text{b/c } 3 < k+1 \\ &= (k+1)! && \text{by def'n factorial} \end{aligned}$$

21. Let $P(n)$ be the statement $2^n > n^2$ if n is an integer greater than 4.

<u>Basis Step:</u> $P(5)$ is true b/c

$$2^5 > 5^2$$
$$32 > 25$$

<u>Inductive Step:</u> For the inductive hypothesis we assume $P(k)$ is true where k is an integer w/ $k > 4$, namely, that

$$2^k > k^2$$

We must show that $P(k+1)$ is also true, namely, that

$$2^{k+1} > (k+1)^2$$

$$\begin{aligned} (k+1)^2 &= k^2 + 2k + 1 \\ &< k^2 + 2k + k && \text{b/c } 1 < k \\ &= k^2 + 3k \\ &< k^2 + k^2 && \text{b/c } 3 < k \\ &= 2k^2 \\ &< 2 \cdot 2^k && \text{by ind. hyp} \\ &= 2^{k+1} \end{aligned}$$

20. Let $P(n)$ be the statement $3^n < n!$ where n is an integer greater than 6.

Basis Step: $P(7)$ is true b/c

$$3^7 < 7!$$

$$2187 < 5040$$

Inductive Step: For the inductive hypothesis we assume $P(k)$ is true where k is an integer w/ $k > 6$, namely, that

$$3^k < k!$$

We must show that $P(k+1)$, namely, that

$$3^{k+1} < (k+1)!$$

$$\begin{aligned} 3^{k+1} &= 3 \cdot 3^k \\ &< 3 \cdot k! && \text{by ind. hyp.} \\ &< (k+1) \cdot k! && \text{b/c } 3 < k+1 \\ &= (k+1)! && \text{by def'n factorial} \end{aligned}$$

21. Let $P(n)$ be the statement $2^n > n^2$ if n is an integer greater than 4.

Basis Step: $P(5)$ is true b/c

$$2^5 > 5^2$$

$$32 > 25$$

Inductive Step: For the inductive hypothesis we assume $P(k)$ is true where k is an integer w/ $k > 4$, namely, that

$$2^k > k^2$$

We must show that $P(k+1)$ is also true, namely, that

$$2^{k+1} > (k+1)^2.$$

$$\begin{aligned} (k+1)^2 &= k^2 + 2k + 1 \\ &< k^2 + 2k + k && \text{b/c } 1 < k \\ &= k^2 + 3k \\ &< k^2 + k^2 && \text{b/c } 3 < k \\ &= 2k^2 \\ &< 2 \cdot 2^k && \text{by ind. hyp} \\ &= 2^{k+1} \end{aligned}$$

25. We can assume $h > -1$ is fixed, and prove the proposition by induction on n. Let $P(n)$ be the proposition $1 + nh \leq (1+h)^n$.

Basis Step: $P(0)$ is true b/c

$$1 + 0(0) \leq (1+0)^0$$
$$1 \leq 1$$

Inductive Step: We assume the inductive hypothesis $1 + kh \leq (1+h)^k$

We want to show that $1 + (k+1)h \leq (1+h)^{k+1}$

Since $h > -1$, it follows that $1 + h > 0$ so we can multiply both sides of the ind. hyp. by $1 + h$:

$$(1+h)(1+kh) \leq (1+h)^{k+1}$$

Thus to complete the proof it is enough to show

$$1 + (k+1)h \leq (1+h)(1+kh)$$

The RHS $= 1 + kh + h + kh^2$

$= 1 + (k+1)h + kh^2$

which is $\geq 1 + (k+1)h$ b/c $kh^2 \geq 0$

28. Let $P(n)$ be the statement $n^2 - 7n + 12 \geq 0$ where n is an integer $n \geq 3$.

Basis Step: $P(3)$ true b/c

$$3^2 - 7(3) + 12 \geq 0$$
$$9 - 21 + 12 \geq 0$$
$$-12 + 12 \geq 0$$
$$0 \geq 0$$

Inductive Step: We assume the inductive hypothesis:

$$k^2 - 7k + 12 \geq 0$$

We want to show that $(k+1)^2 - 7(k+1) + 12 \geq 0$

$(k+1)^2 - 7(k+1) + 12 = k^2 + 2k + 1 - 7k - 7 + 12$

$= k^2 - 7k + 12 + 2k - 6$

We know that $2k - 6 \geq 0$ b/c $k \geq 3$

so RHS ≥ 0 by ind. hyp.

31. Let $P(n)$ be the statement "2 divides $n^2 + n$" where n is a positive integer.

Basis Step: $P(1)$ is true b/c

$1^2 + 1 = 2$ is divisible by 2

Inductive Step: we assume the inductive hypothesis 2 divides $k^2 + k$

we want to show $(k+1)^2 + (k+1)$ is divisible by 2

$$(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1$$
$$= k^2 + k + 2(k+1)$$

$2 \mid k^2 + k$ by ind. hyp.

$2 \mid 2(k+1)$ by def'n

$\therefore\ 2 \mid k^2 + k + 2(k+1) = (k+1)^2 + (k+1)$

34. Let $P(n)$ be the statement "6 divides $n^3 - n$" where n is a nonnegative integer

Basis Step: $P(0)$ true b/c

$0^3 - 0 = 0$ is divisible by 6

Inductive Step: We assume the ind. hyp. $k^3 - k$ is divisible by 6

We want to show $(k+1)^3 - (k+1)$ is divisible by 6

$$(k+1)^3 - (k+1) = (k+1)(k+1)(k+1) - (k+1)$$
$$= (k^2 + 2k + 1)(k+1) - (k+1)$$
$$= k^3 + 2k^2 + k + k^2 + 2k + \cancel{1} - (k + \cancel{1})$$
$$= (k^3 - k) + 3k^2 + 3k$$
$$= (k^3 - k) + 3k(k+1)$$

$6 \mid (k^3 - k)$ by ind. hyp.

$2 \mid k(k+1)$ b/c k or $k+1$ even

$2 \mid 3k(k+1)$ "

$3 \mid 3k(k+1)$ by def'n

$6 \mid 3k(k+1)$

$6 \mid (k^3 - k) + 3k(k+1) = (k+1)^3 - (k+1)$

35. Let P(n) be the statement "$(2n-1)^2 - 1$ is divisible by 8" where n is a positive integer, b/c an odd positive integer can be written as $2n-1$.

Basis Step: P(1) true b/c

$$(2(1)-1)^2 - 1 = 0$$

$$8 \mid 0$$

Inductive Step: We assume the inductive hypothesis $8 \mid ((2k-1)^2 - 1)$

We want to show $8 \mid ((2k+1)^2 - 1)$

Difference: $((2k+1)^2 - 1) - ((2k-1)^2 - 1)$

$$(\cancel{4k^2} + 4k + \cancel{1} - \cancel{1}) - (\cancel{4k^2} - 4k + \cancel{1} - \cancel{1})$$

$$8k$$

$$8 \mid 8k$$

$8 \mid ((2k-1)^2 - 1)$ by ind. hyp.

$$8 \mid ((2k+1)^2 - 1)$$

36. Let P(n) be the statement "21 divides $4^{n+1} + 5^{2n-1}$" where n is a positive integer

Basis Step: P(1) true b/c

$$4^{1+1} + 5^{2(1)-1} = 4^2 + 5 = 21$$

$$21 \mid 21$$

Inductive Step: We assume the inductive hypothesis $21 \mid 4^{k+1} + 5^{2k-1}$

We want to show $21 \mid 4^{k+2} + 5^{2k+1}$

$$4^{k+2} + 5^{2k+1} = 4 \cdot 4^{k+1} + 5^2 \cdot 5^{2k-1}$$

$$= 4 \cdot 4^{k+1} + (4 + 21) \cdot 5^{2k-1}$$

$$= 4(4^{k+1} + 5^{2k-1}) + 21 \cdot 5^{2k-1}$$

$21 \mid 4(4^{k+1} + 5^{2k-1})$ by ind hyp

$21 \mid 21 \cdot 5^{2k-1}$ by def'n

$$21 \mid 4(4^{k+1} + 5^{2k-1}) + 21 \cdot 5^{2k-1} = 4^{k+2} + 5^{2k+1}$$

39. **Basis Step:** $P(1)$ true b/c

$A_1 \subseteq B_1$ implies $\bigcap_{j=1}^{1} A_j \subseteq \bigcap_{j=1}^{1} B_j$ b/c intersection of one set is itself

Inductive Step: We assume the inductive hypothesis that if $A_j \subseteq B_j$ for $j = 1, 2, \ldots, k$ then $\bigcap_{j=1}^{k} A_j \subseteq \bigcap_{j=1}^{k} B_j$.

☐ Review

We want to show that if $A_j \subseteq B_j$ for $j = 1, 2, \ldots, k+1$ then $\bigcap_{j=1}^{k+1} A_j \subseteq \bigcap_{j=1}^{k+1} B_j$. To show that one set is a subset of another we show that an arbitrary element of the first set must be an element of the second set. So let $x \in \bigcap_{j=1}^{k+1} A_j = \left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}$. Because $x \in \bigcap_{j=1}^{k} A_j$, we know by the ind. hyp. that $x \in \bigcap_{j=1}^{k} B_j$. B/c $x \in A_{k+1}$, we know from the given fact that $A_{k+1} \subseteq B_{k+1}$ that $x \in B_{k+1}$. $\therefore$ $x \in \left(\bigcap_{j=1}^{k} B_j\right) \cap B_{k+1} = \bigcap_{j=1}^{k+1} B_j$.

40. **Basis Step:** $P(1)$ true b/c

$(A_1) \cup B = A_1 \cup B$

Inductive Step: We assume the ind. hyp. for a positive integer k:

$$(A_1 \cap A_2 \cap \ldots \cap A_k) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_k \cup B)$$

We want to show:

$$(A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_k \cup B) \cap (A_{k+1} \cup B)$$

The RHS is equal to:

$= ((A_1 \cap A_2 \cap \ldots \cap A_k) \cup B) \cap (A_{k+1} \cup B)$ by ind. hyp.

$= (A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}) \cup B$ by distributive prop.

43. **Basis Step:** $P(1)$ true b/c

$\overline{A_1} = \overline{A_1}$

Inductive Step: We assume the inductive hypothesis:

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$$

▫ review

We want to show $\overline{\bigcup_{k=1}^{n+1} A_k} = \bigcap_{k=1}^{n+1} \overline{A_k}$

$$\overline{\bigcup_{k=1}^{n+1} A_k} = \overline{\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}}$$

↓

$$= \overline{\bigcup_{k=1}^{n} A_k} \cap \overline{A_{n+1}} \quad \text{by DeMorgan's Law}$$

$$= \left(\bigcap_{k=1}^{n} \overline{A_k} \right) \cap \overline{A_{n+1}} \quad \text{by Ind. hyp.}$$

$$= \bigcap_{k=1}^{n+1} \overline{A_k}$$

45. Basis Step: $P(2)$ true b/c

$\frac{2(2-1)}{2} = \frac{2}{2} = 1$ and a set w/ 2 elements has exactly one subset w/ 2 elements

□ review

Inductive Step: We assume the ind. hyp.:
a set w/ k elements has $\frac{k(k-1)}{2}$ subsets w/ exactly two elements.

We want to show that a set S w/ $k+1$ elements has $\frac{(k+1)(k)}{2}$ subsets w/ exactly two elements.
Fix an element a is S, and let T be the set of elements of S other than a. There are two varieties of subsets of S containing exactly two elements. First there are those that do not contain a. These are precisely the two-element subsets of T, and by the inductive hypothesis, there are $\frac{k(k-1)}{2}$ of them. Second, there are those that contain a together w/ one element of T. Since T has k elements, there are exactly k subsets of this type. ∴ the total number of subsets of S containing exactly two elements is

$$\frac{k(k-1)}{2} + k = \frac{k^2 - k + 2k}{2} = \frac{k(k+1)}{2}.$$

54. Let $P(n)$ be the proposition: given a set of $n+1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

□ review

Basis Step: $P(1)$ true b/c

$1+1=2$ positive integers none exceeding $2(1)$

$\{1, 2\}$

$1 \mid 2$

Inductive Step: We assume the ind. hyp. that $P(k)$ is true for some positive integer k. We want to show $P(k+1)$.
Let S be a set of $k+2$ positive integers none exceeding $2k+2$.
If $S \cap \{2k+1, 2k+2\}$ has cardinality 0 or 1 then we apply the ind. hyp. to $S \setminus \{2k+1, 2k+2\}$ to conclude that this set contains a dividing pair of integers.
We are left w/ the case that $2k+1$ and $2k+2$ are both in S and $S \setminus \{2k+1, 2k+2\}$ consists of k positive integers of size at most $2k$ that pairwise don't divide each other. If $k+1$ is in S then we are done b/c $k+1$ divides $2k+2$. Suppose $\therefore$ that $k+1 \notin S$. Then we replace S by the set $S' = (S \setminus \{2k+2\}) \cup \{k+1\}$. The new set S' is covered by the previous case, so it contains a divisible pair. If that pair does not involve $k+1$ then it is also in S. If it involves $k+1$ then this means that some $\ell \in S \setminus \{k+1\}$ divides ℓ. That ℓ must also divide $2k+2$ and hence S contains a divisible pair.

56. Let $P(n)$ be the statement $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ for every positive integer n.

□ review

Basis Step: $P(1)$ true b/c and $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$$A^1 = \begin{bmatrix} a^1 & 0 \\ 0 & b^1 \end{bmatrix}$$

Inductive Step: We assume the ind. hyp.:

$$A^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

We want to show that $P(k+1)$ is also true

$$A^{k+1} = A \cdot A^k = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$$

$$= \begin{bmatrix} a \cdot a^k + 0 \cdot 0 & 0 \cdot 0 + 0 \cdot b^k \\ 0 \cdot a^k + b \cdot 0 & 0 \cdot 0 + b \cdot b^k \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{bmatrix}$$

57. <u>Basis Step:</u> P(0) and P(1) true b/c

$$\frac{d}{dx} x^0 = \lim_{h \to 0} \frac{(x+h)^0 - x^0}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0 = 0 \cdot x^{-1}$$

□ review

$$\frac{d}{dx} x^1 = \lim_{h \to 0} \frac{(x+h)^1 - x^1}{h} = \lim_{h \to 0} \frac{h}{h} = 1 = 1 \cdot x^0$$

We are told to assume the product rule holds:

$$\frac{d}{dx} (f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Hence $\frac{d}{dx} x^{n+1} = \frac{d}{dx} (x \cdot x^n) = x \cdot \frac{d}{dx} x^n + x^n \cdot \frac{d}{dx} x$

$$= x \cdot nx^{n-1} + x^n \cdot 1$$
$$= nx^n + x^n$$
$$= (n+1)x^n$$

5.2 1. Let P(n) be the statement that you can run n miles. We want to prove that P(n) is true for all positive integers n.

□ review

<u>Basis Step:</u> Given P(1) and P(2) true.

<u>Inductive Step:</u> We assume the ind. hyp. P(j) is true for all $j \leq k$ and fix $k \geq 2$. We want to show that P(k+1) is true.

Since $k \geq 2$, $k-1$ is a positive integer less than or equal to k, so by the ind. hyp., we know that P(k−1) is true. That is, we know that you can run k−1 miles. We were told that "you can always run two more miles once you have run a specified number of miles" so we know that you can run $(k-1)+2 = k+1$ miles.

3. a) P(8) is true b/c we can form 8 cents of postage w/ 1 3-cent stamp and 1 5-cent stamp. P(9) is true, b/c we can form 9 cents of postage w/ 3 3-cent stamps. P(10) is true, b/c we can form 10 cents of postage w/ 2 5-cent stamps.

□ review

b) The inductive hypothesis is the statement that using just 3-cent and 5-cent stamps we can form j cents postage for all j w/ $8 \leq j \leq k$ where we assume $k \geq 10$.

c) In the inductive step we must show, assuming the ind. hyp., that we can form $k+1$ cents postage using just 3-cent and 5-cent stamps.

d) We want to form $k+1$ cents postage. Since $k \geq 10$, we know that $P(k-2)$ is true, that is, that we can form $k-2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed $k+1$ cents postage.

e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 8.

7. We can form the following amounts of money as indicated: $2 = 2$, $4 = 2 + 2$, $5 = 5$, $6 = 2 + 2 + 2$. By having considered all the combinations, we know that the gaps in this list (\$1 and \$3) cannot be filled. We claim that we can form all amounts of money greater than or equal to 5 dollars. Let $P(n)$ be the statement that we can form n dollars using just 2-dollar and 5-dollar bills where $n \geq 5$.

□ review

<u>Basis Step:</u> We already observed that the basis step is true for $n = 5$ and 6.

<u>Inductive Step:</u> We assume the ind. hyp., that $P(j)$ is true for all j w/ $5 \leq j \leq k$, where k is a fixed integer ≥ 6. We want to show $P(k+1)$ is true. B/c $k-1 \geq 5$, we know $P(k-1)$ is true, that is, that we can form $k-1$ dollars. Add another 2-dollar bill, and we have formed $k+1$ dollars.

8. Since both 25 and 40 are multiples of 5, we cannot form any amount that is not a multiple of 5. So let's determine for which values of n we can form $5n$ dollars using these gift certificates, the first of which provides 5 copies of \$5, and the second of which provides 8 copies.

□ review

We can achieve the following values of n:

$5 = 5$, $8 = 8$, $10 = 5 + 5$, $13 = 8 + 5$, $15 = 5 + 5 + 5$,
$16 = 8 + 8$, $20 = 5 + 5 + 5 + 5$, $21 = 8 + 8 + 5$, $23 = 8 + 5 + 5 + 5$,
$24 = 8 + 8 + 8$, $25 = 5 + 5 + 5 + 5 + 5$, $26 = 8 + 8 + 5 + 5$,

$28 = 8+5+5+5+5$, $29 = 8+8+8+5$, $30 = 5+5+5+5+5$,
$31 = 8+8+5+5+5$, $32 = 8+8+8+8$.

By having considered all the combinations, we know that the gaps in the list cannot be filled.

We claim that we can form total amounts of the form $5n$ for all $n \geq 28$ using these gift certificates.

To prove this by strong induction, let $P(n)$ be the statement that we can form $5n$ dollars in gift certificates using just 25-dollars and 40-dollars certificates.

We want to prove that $P(n)$ is true for all $n \geq 28$.

Basis Step: From our work above, we know that $P(n)$ is true for all $n = 28, 29, 30, 31, 32$.

Inductive Step: Assume the ind. hyp., that $P(j)$ is true for all j w/ $28 \leq j \leq k$, where k is a fixed integer ≥ 32.

We want to show $P(k+1)$ is true. B/c $k-4 \geq 28$, we know that $P(k-4)$ is true, that is, that we can form $5(k-4)$ dollars. Add one more 25-dollars certificate, and we have formed $5(k+1)$ dollars.

11. There are four base cases. If $n = 1 = 4 \cdot 0 + 1$, then clearly the first player is doomed, so the second player wins. If there are two, three, or four matches ($n = 4 \cdot 0 + 2$, $n = 4 \cdot 0 + 3$, or $n = 4 \cdot 1$), then the 1st player can win by removing all but one match. Now assume the strong inductive hypothesis, that in games w/ k or fewer matches, the 1st player can win if $k \equiv 0, 2,$ or $3 \pmod 4$ and the 2nd player can win if $k \equiv 1 \pmod 4$. Suppose we have a game w/ $k+1$ matches, w/ $k \geq 4$. If $k+1 \equiv 0 \pmod 4$, then the 1st player can remove 3 matches, leaving $k-2$ matches for the other player. Since $k-2 \equiv 1 \pmod 4$, by the ind. hyp., this is a game that the 2nd player at that point (who is the 1st player in our game) can win. Similarly, if $k+1 \equiv 2 \pmod 4$, then the 1st player can remove one match, leaving k matches for the other player. Since $k \equiv 1 \pmod 4$, by the ind. hyp., this is a game that

□ review

the 2nd player at that point (who is the 1st player in our game) can win. And if $k+1 \equiv 3 \pmod 4$, then the 1st player can remove two matches, leaving $k-1$ matches for the other player. Since $k-1 \equiv 1 \pmod 4$, by the ind. hyp., this is again a game that the 2nd player at that point (who is the 1st player in our game) can win. Finally, if $k+1 \equiv 1 \pmod 4$, then the 1st player must leave k, $k-1$, or $k-2$ matches for the other player. Since $k \equiv 0 \pmod 4$, $k-1 \equiv 3 \pmod 4$, and $k-2 \equiv 2 \pmod 4$, by the ind. hyp., this is a game that the 1st player at that point (who is the 2nd player in our game) can win. Thus the 1st player in our game is doomed, and the proof is complete.

12. Basis Step: $P(1)$ true b/c

$$1 = 2^0$$

Inductive Step: We assume for a positive integer k that $P(i)$ is true for all $1 \le i \le k$. We consider two cases, namely, when $k+1$ is even and when $k+1$ is odd. If $k+1$ is even, then $\frac{(k+1)}{2}$ is an integer, and by the ind. hyp., we can express $\frac{(k+1)}{2}$ by a sum of distinct powers of two. We can then multiply this sum by 2, which simply increases the exponents of each power of two by 1, so this is again a sum of distinct powers of two that is equal to $k+1$.

When $k+1$ is odd, we have that k is even. By the ind. hyp., we can express k as a sum of distinct powers of two. However, since k is even, the sum cannot contain $2^0 = 1$. Thus, we can add 2^0 to this sum, which remains a sum of distinct powers of two, and equals $k+1$.

review

25. a). The inductive step here allows us to conclude that $P(3)$, $P(5)$, ... are all true, but we can conclude nothing about $P(2)$, $P(4)$, ...

review

b) We can conclude that $P(n)$ is true for all positive integers n, using strong induction.

c) The inductive step here allows us to conclude that $P(2)$, $P(4)$, $P(8)$, $P(16)$, ... are all true, but we cannot conclude anything about $P(n)$ when n is not a power of 2.

d) This is mathematical induction. We can conclude that $P(n)$ is true for all positive integers n.

26. □ review

a) The statement $P(n)$ is true for all nonnegative integers n that are even.

b) The statement $P(n)$ is true for all nonnegative integers n that are divisible by 3.

c) The statement $P(n)$ is true for all nonnegative integers n.

d) The statement $P(n)$ is true for all nonnegative integers n w/ $n \neq 1$, since every such n is expressible as a sum of 2's and 3's.

5.3 43. □ review

This is similar to Theorem 2. For the full binary tree consisting of just the root r the result is true since $n(T) = 1$ and $h(T) = 0$, and $1 \geq 2 \cdot 0 + 1$. For the ind. hyp., we assume that $n(T_1) \geq 2h(T_1) + 1$ and $n(T_2) \geq 2h(T_2) + 1$ where T_1 and T_2 are full binary trees. By the recursive definitions of $n(T)$ and $h(T)$, we have $n(T) = 1 + n(T_1) + n(T_2)$ and $h(T) = 1 + \max(h(T_1), h(T_2))$.
$\therefore$ $n(T) = 1 + n(T_1) + n(T_2) \geq 1 + 2h(T_1) + 1 + 2h(T_2) + 1 \geq 1 + 2 \cdot \max(h(T_1), h(T_2)) + 2$ since the sum of two nonnegative numbers is at least as large as the larger of the two. But this equals $1 + 2(\max(h(T_1), h(T_2)) + 1) = 1 + 2h(T)$, and our proof is complete.

58. □ review

a) $F(n) = 1 + F(\lfloor n/2 \rfloor)$ for $n \geq 1$ and $F(1) = 1$

$$F(1) = 1 + F(\lfloor 1/2 \rfloor)$$
$$= 1 + F(0)$$

$F(0)$ undefined, $\therefore$ not a well-defined function

b) $F(n) = 1 + F(n-3)$ for $n \geq 2$, $F(1) = 2$, and $F(2) = 3$

$$F(2) = 1 + F(2-3)$$
$$= 1 + F(-1)$$

$F(-1)$ undefined, $\therefore$ not a well-defined function