Stewart Dulaney
MATH 10
Section 2838
SID: 1545566

OT 14

Textbook Problems 6.1 - 6.5, 5.3

6.1

1. a) $18 \cdot 325 = 5850$ (Product Rule)
   b) $18 + 325 = 343$ (Sum Rule)

2. $27 \cdot 37 = 999$ offices (Product Rule)

3. a) $4 \cdot 4 \cdots 4 = 4^{10} = 1{,}048{,}576$ (Product Rule)
   b) $5 \cdot 5 \cdots 5 = 5^{10} = 9{,}765{,}625$ (Product Rule)

4. $12 \cdot 2 \cdot 3 = 72$ different types of shirt (Product Rule)

5. $6 \cdot 7 = 42$ (Product Rule)

6. $4 \cdot 6 = 24$ (Product Rule)

7. $26 \cdot 26 \cdot 26 = 17{,}576$ (Product Rule)

8. $26 \cdot 25 \cdot 24 = 15{,}600$ (Produlct Rule)

9. $1 \cdot 26 \cdot 26 = 676$ (Product Rule)

16. $\underbrace{26 \cdot 26 \cdot 26 \cdot 26}_{\text{\# of strings of length 4 of lowercase letters}} - \underbrace{25 \cdot 25 \cdot 25 \cdot 25}_{\text{\# of strings of length 4 of lowercase letters other than x}} = 26^4 - 25^4 = 66{,}351$

17. $128^5 - 127^5 = 1{,}321{,}368{,}961$

21. a) There are $\lfloor \frac{100}{7} \rfloor = 14$ integers less than 100 that are divisible by 7, and $\lfloor \frac{50}{7} \rfloor = 7$ of them are less than 50 as well. This leaves $14 - 7 = 7$ numbers between 50 and 100 that are divisible by 7. They are 56, 63, 70, 77, 84, 91, 98.

b) There are $\left\lfloor \frac{100}{11} \right\rfloor = 9$ integers less than 100 that are divisible by 11 and $\left\lfloor \frac{50}{11} \right\rfloor = 4$ of them are less than 50 as well. This leaves $9 - 4 = 5$ numbers between 50 and 100 that are divisible by 11. They are 55, 66, 77, 88, 99.

c) A number is divisible by both 7 and 11 if and only if it is divisible by their least common multiple, which is 77. Obviously there is only one such # between 50 and 100, namely 77. We could also work this out as we did in the previous parts: $\left\lfloor \frac{100}{77} \right\rfloor - \left\lfloor \frac{50}{77} \right\rfloor = 1 - 0 = 1$. Note also that the intersection of the sets we found in the previous two parts is precisely what we are looking for here.

23. a) $\left\lfloor \frac{999}{7} \right\rfloor - \left\lfloor \frac{99}{7} \right\rfloor = 142 - 14 = 128$

b) $\left\lfloor \frac{999}{2} \right\rfloor = 499$ even #'s ≤ 999, so $999 - 499 = 500$ odd #'s. $\left\lfloor \frac{99}{2} \right\rfloor = 49$ even #'s ≤ 99, so $99 - 49 = 50$ odd #'s. So $500 - 50 = 450$ odd #'s in range.

c) 9

d) $900 - \left(\left\lfloor \frac{999}{4} \right\rfloor - \left\lfloor \frac{99}{4} \right\rfloor \right) = 900 - (249 - 24) = 900 - 225 = 675$

e) $\left(\left\lfloor \frac{999}{3} \right\rfloor - \left\lfloor \frac{99}{3} \right\rfloor \right) + (225) - \left(\left\lfloor \frac{999}{12} \right\rfloor - \left\lfloor \frac{99}{12} \right\rfloor \right)$

$(333 - 33) + 225 - (83 - 8)$

$300 + 225 - 75$

$524 - 75$

450

f) $900 - 450 = 450$

g) $300 - 75 = 225$

h) 75

25. a) $10^3 - 10 = 990$

b) $5 \cdot 10^2 = 500$

c) $\underbrace{3}_{\substack{\text{choose}\\\text{position of}\\\text{digit that is}\\\text{not a four}}} \cdot \underbrace{9}_{\text{choose that digit}} = 27$

27. $3^{50} \approx 7.2 \times 10^{23}$ (Product Rule)

28. $10^3 \cdot 26^3 + 26^3 \cdot 10^3 = 35,152,000$

29. $26^2 \cdot 10^4 + 10^2 \cdot 26^4 = 52,457,600$

32. a) 26^8

b) $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19$

c) $1 \cdot 26^7$

d) $1 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19$

e) $1 \cdot 26^6 \cdot 1$

f) $1 \cdot 1 \cdot 26^6$

g) $1 \cdot 1 \cdot 26^4 \cdot 1 \cdot 1$

h) $1 \cdot 1 \cdot 26^6 + 26^6 \cdot 1 \cdot 1 - 1 \cdot 1 \cdot 26^4 \cdot 1 \cdot 1$

33. a) 21^8

b) $21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14$

a, e, i, o, u c) $5 \cdot 26^7$

d) $5 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19$

e) $26^8 - 21^8$

f) $\binom{5}{1}\binom{8}{1} 21^7$

g) $26^7 - 21^7$

h) $26^6 - 21^6$

34. a) 2^{10}

b) 3^{10}

c) 4^{10}

d) 5^{10}

35. a) $0 = 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0$

b) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

c) $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$

d) $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$

36. 2^n

37. a) No 1-1 fxns if $n > 2$.

$\boxed{\text{Case 1}}$ $n = 1$

2 fxns

$\boxed{\text{Case 2}}$ $n = 2$

$2 \cdot 1$ fxns

b) $n = 2$ $\{1, 2\}$ to $\{0, 1\}$

$n = 1$ $\{1\}$ to $\{0, 1\}$

If the fxn assigns 0 to both 1 and n, then there are $n-2$ fxn values free to be chosen. Each can be chosen in 2 ways. ∴ by the product rule there are 2^{n-2} such fxns, as long as $n > 1$. If $n = 1$, then clearly there is just one such fxn.

c) If $n = 1$, then there are no such fxns, since there are no positive integers less than n. So assume $n > 1$. In order to specify such a fxn, we have to decide which of the #'s from 1 to $n-1$, inclusive, will get sent to 1. There are $n-1$ ways to make this choice. There is no choice for the remaining #'s from 1 to $n-1$, inclusive, since they all must get sent to 0. Finally, we are free to specify the value of the fxn at n, and this may be done in 2 ways. By prod. rule final answer is $2(n-1)$.

44. There are $10 \cdot 9 \cdot 8 \cdot 7$ ways to order the 4 people

B/c you can rotate the people around the table in 4 ways and still get the same seating arrangement, by the division rule there are $\frac{5040}{4} = 1260$ arrangement.

46. a) $\overbrace{6}^{\text{first place bride in any of 6 positions}} \cdot \overbrace{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}^{\text{choose other 5 people for remaining positions}}$

b) $6 \cdot 5 \cdot 8 \cdot 7 \cdot 6 \cdot 5$

c) Let A = bride only

Let B = groom only

$|A| = \underbrace{90720}_{\text{ways bride}} - \underbrace{50400}_{\text{ways bride and groom}} = 40320$

Symmetrically, $|B| = 40320$

$\therefore$ answer is $40320 + 40320 = 80640$

47. a) $\overbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}^{\text{arrange 5 units (b/g one unit)}} \cdot \overbrace{2}^{\text{arrange b/g}}$

b) $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 - 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2$

c) $\frac{720}{2}$ by symmetry

52. $38 - 7 = 31$ CS only

$23 - 7 = 16$ Math only

$31 + 16 + 7 = 54$ Total

□ clarify

54. $26^2 + 26^3 + 26^4 + 26^5 = 12356604$

55. a) $68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12}$

b) $68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12} - (62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12})$

c) $\frac{9,920,671,339,261,325,541,376 \cdot 1\text{ns}}{60 \cdot 60 \cdot 24 \cdot 365.2425} = 314,374$ years

59. $10(8 \cdot 10^2 \cdot 8 \cdot 10^2 \cdot 10^4) + 10^2(8^2 \cdot 10^8) + 10^3(8^2\ 10^8)$

$= 7,104,000,000,000$

□ clarify

70. There are 2^n lines in the truth table, since each of the n propositions can have 2 truth values. Each line can be filled in w/ T or F, so there are a total of 2^{2^n} possibilities.

PP = pigeonhole principle

6.2

1. There are six classes: these are the pigeons. There are five days on which classes may meet (M - F): these are the pigeonholes. Each class must meet on a day (each pigeon must occupy a pigeonhole). By the pigeonhole principle at least one day must contain at least two classes.

2. Let 30 students be the pigeons
Let letters (26) A - Z be the pigeonholes ($k = 26$)
By the pigeonhole principle at least two have last names that begin w/ the same letter.

3. a) Let socks taken out be the pigeons
Let 2 colors be the pigeonholes ($k = 2$)
By the PP need to take out at least 3 socks to isure that at least two have the same color.
b) By the PP need to take out 14 socks, b/c at most 12 can be brown so at least two are black.

4. a) Let balls selected be pigeons
Let 2 colors be pigeonholes
By the Gen. PP need to select 5 balls then at least one box contains at least $\lceil \frac{N}{K} \rceil = \lceil \frac{5}{2} \rceil = 3$ balls
b) By the PP need to take out 13 balls, b/c at most 10 can be red so at least 3 are blue. On the other hand, if she selects 12 or fewer balls, then 10 of them could be red, and she might not get her 3 blue balls.

5. Let pigeons be the remainders when divided by 4 of group of 5 integers
Let pigeonholes be the 4 possible remainders 0, 1, 2, 3 when dividing by 4
By the PP at least 2 must have have the same remainder ($k=4$)
when dividing by 4.

7. Let the n consecutive integers be denoted $x+1, x+2, \ldots, x+n$, where

clarify x is some integer. We want to show that exactly one of these is

divisible by n. There are n possible remainders when an integer is divided by n, namely $0, 1, 2, \ldots, n-1$. There are two possibilities for the remainders of our collection of n numbers: either they cover all the possible remainders, or they do not. If they do not, then by the PP, since there are fewer than n pigeonholes (remainders) for n pigeons (the #'s in our collection), at least one remainder must occur twice. In other words, it must be the case that $x+i$ and $x+j$ have the same remainder when divided by n for some pair of #'s i and j w/ $0 < i < j \leq n$. Since $x+i$ and $x+j$ have the same remainder when divided by n, if we subtract $x+i$ from $x+j$, then we will get a # divisible by n. This means $j-i$ is divisible by n. But this is impossible, since $j-i$ is a positive integer strictly less than n. $\therefore$ the first possibility must hold, that exactly one of the #'s in our collection is divisible by n (all remainders covered so one has remainder 0).

9. Let pigeons be the students
Let pigeonholes be the 50 states
We want $\lceil \frac{N}{50} \rceil \geq 100$ by Gen. PP. This will be the case as long as $N \geq 99 \cdot 50 + 1 = 4951$.

12. Let pigeons be the ordered pairs of integers
Let pigeonholes be the 25 possible ordered pairs reduced modulo 5
$(0,0), (0,1), \ldots, (4,4)$
By the PP, if we have 26 ordered pairs (a,b) then there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

13. a) We can group the first 8 positive integers into four subsets of two integers each, each subset adding up to 9: $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, and $\{4,5\}$. If we select 5 integers from this set, then by the PP (at least) two of them must come from the same subset. These two integers have a sum of 9, as desired.

b) No. If we select one element from each of the subsets in part (a), then no sum will be 9. For example, we can select 1, 2, 3, and 4.

15. We can apply the PP by grouping the #'s cleverly into pairs (subsets) that add up to 7, namely $\{1,6\}$, $\{2,5\}$, and $\{3,4\}$. If we select 4 numbers then at least two of them must fall within the same subset by the PP, since there are only 3 subsets.

18. a) Let pigeons = students

Let pigeonholes = 2 genders (M/F)

By the Gen. PP at least one gender must have at least $\lceil \frac{9}{2} \rceil = 5$ students

b) If not, then there would be 2 or fewer male students and 6 or fewer female students, so there would be $2 + 6 = 8$ or fewer students in all, contradicting the assumption that there are 9 students in the class.

□ clarify

19. a) If this statement were not true, then there would be at most 8 from each class standing, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class.

b) If this statement were not true, then there would be at most 2 freshmen, at most 18 sophomores, and at most 4 juniors, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class.

27. We can prove these statements using both the result and the method of Example 13. First note that the role of "mutual friend" and "mutual enemy" is symmetric, so it is really enough to prove one of these statements; the other will follow by interchanging the roles. So let us prove that in every group of 10 people, either there are 3 mutual friends or 4 mutual enemies. Consider one person; call this person A. Of the other 9 people, either there must be 6 enemies of A, or there must be 4 friends of A (if there were 5 or fewer enemies and 3 or fewer friends, that would only account for 8 people). We need to consider the two cases separately.

☐ clarify

$\boxed{\text{Case 1}}$ A has 6 enemies.

Among them either there are 3 mutual friends or there are 3 mutual enemies. If there are 3 mutual friends, then we are done. If there are 3 mutual enemies, then these 3 people, together w/ A, form a group of 4 mutual enemies and again we are done.

$\boxed{\text{Case 2}}$ A has 4 friends

If some pair of these people are friends, then they, together w/ A, form the desired group of 3 mutual friends. Otherwise, these 4 people are the desired group of 4 mutual enemies.

Thus in either case we have found either 3 mutual friends or 4 mutual enemies.

31. First we need to figure out how many distinct combinations of initials and birthdays there are. The product rule tells us that since there are 26 ways to choose each initial and 366 ways to choose the birthday, there are $26^3 \cdot 366 = 6{,}432{,}816$ such combinations. By the Gen. PP, w/ these combinations as the pigeonholes and the 37 million people as the pigeons, there must be at least $\left\lceil \frac{37{,}000{,}000}{6{,}432{,}816} \right\rceil = 6$ people w/ the same combination.

35. The 38 time periods are the pigeonholes and the 677 classes are the pigeons. By Gen. PP there is some time period in which at least $\lceil 677/38 \rceil = 18$ classes are meeting. Since each class must meet in a different room, we need 18 rooms.

36. Let $K(x)$ be the # of other computers that computer x is connected to. The possible values for $K(x)$ are $1, 2, 3, 4, 5$ (pigeonholes). Since there are 6 computers (pigeons), the PP guarantees that at least two of the values $K(x)$ are the same.

44. Let the pigeons be the addresses of the 51 houses
Let the pigeonholes be the 50 sets of consecutive integers $\{1000, 1001\}$, $\{1002, 1003\}$, $\{1004, 1005\}$, ..., $\{1098, 1099\}$.
By the PP if we have 51 numbers in the range 1000 to 1099 inclusive, then at least two of them must come from the same set.

6.3 1. a, b, c ; a, c, b ; b, a, c ; b, c, a ; c, a, b ; c, b, a

2. $P(7,7) = \dfrac{7!}{(7-7)!} = 7! = 5040$

3. $P(6,6) = \dfrac{6!}{(6-6)!} = 6! = 720$

5. a) $P(6,3) = \dfrac{6!}{(6-3)!} = \dfrac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}} = 120$

$= (6)(6-1)(6-3+1)$

$= 6 \cdot 5 \cdot 4 = 120$

7. $P(9,5) = \frac{9!}{(9-5)!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 15120$

8. $P(5,5) = 5!$

9. $P(12,3) = \frac{12!}{(12-3)!} = \frac{12!}{9!}$

13. Task 1 arrange men $P(n,n) = n!$

Task 2 arrange women $P(n,n) = n!$

Task 3 decide whether row starts w/ man or woman 2

$n! \cdot n! \cdot 2$

15. $C(26,5) = \frac{26!}{5!(26-5)!} = \frac{26!}{5!21!}$

18. a) 2^8

b) $C(8,3) = 56$

c) $C(8,3) + C(8,4) + C(8,5) + C(8,6) + C(8,7) + C(8,8)$

d) $C(8,4)$

21. a) $P(5,5)$

b) $P(4,4)$

c) $P(5,5)$

d) $P(4,4)$

e) $P(3,3)$

f) 0

23. Task 1 position men relative to each other $P(8,8)$

Task 2 choose 5 of 9 slots between men to position women $P(9,5)$

$P(8,8) \cdot P(9,5)$

25. a) $P(100,4)$
b) $P(99,3)$
c) $4 \cdot P(99,3)$
d) $P(99,4)$
$= 90{,}345{,}024$
e) $4 \cdot 3 \cdot P(98,2)$
f) $4 \cdot 3 \cdot 2 \cdot P(97,1)$
g) $P(4,4)$
h) $P(96,4)$
i) $4 \cdot P(99,3)$
j) $4 \cdot 3 \cdot P(96,2)$

26. a) $C(13,10)$
b) $P(13,10)$
c) $C(13,10) - C(10,10)$

27. a) $C(25,4)$
b) $P(25,4)$

□ clarify why comb.

28. $C(40,17)$

31. a) $5 \cdot 6 \cdot 21^5$
b) $5^2 \cdot C(6,2) \cdot 21^4$
c) $26^6 - 21^6$
d) $26^6 - \underbrace{21^6}_{\text{no vowels}} - \underbrace{(5 \cdot 6 \cdot 21^5)}_{\text{exactly one vowel}}$

33. $C(10,3) \cdot C(15,3)$

38. $C(45,3)\ C(57,4)\ C(69,5)$

39. $P(26,3) \cdot P(10,3)$

clarify 40. $\frac{P(5,3)}{3} = 20$

41. Designate the head of the table and seat the people clockwise. Clearly there are $P(n,r)$ ways to do this. B/c rotations of the table do not make a different seating, this overcounts by a factor of r. $\therefore$ the answer is $\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}$

6.4 1. a) $(x+y)^4$

$x^4 \quad \binom{4}{4} = 1$

$x^3 y \quad \binom{4}{3} = 4$

$x^2 y^2 \quad \binom{4}{2} = 6$

$x y^3 \quad \binom{4}{1} = 4$

$y^4 \quad \binom{4}{0} = 1$

$x^4 + 4x^3 y + 6x^2y^2 + 4xy^3 + y^4$

b) $(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$

$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$

$= x^4 + 4x^3 y + 6x^2y^2 + 4xy^3 + y^4$

4. $x^5 y^8 \qquad (x+y)^{13}$

$j = 8$

$\binom{13}{8} = \frac{13!}{8!(13-8)!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot \cancel{8!}}{\cancel{8!}\ 5!} = 1287$

5. There is one term for each j from 0 to 100, so there are 101 terms.

7. $x^9 \qquad (2-x)^{19}$

$j = 9 \qquad (2 + (-x))^{19}$

term is $\binom{19}{9} 2^{10} (-x)^9$

coefficient is $\binom{19}{9} 2^{10} (-1)^9$

8. $x^8 y^9 \quad (3x + 2y)^{17}$

$j = 9$

term $\binom{17}{9}(3x)^8(2y)^9$

coeff. $\binom{17}{9} 3^8 \cdot 2^9$

20. It is easy to see that both sides equal

$$\frac{(n-1)!\, n!\, (n+1)!}{(k-1)!\, k!\, (k+1)!\, (n-k-1)!\, (n-k)!\, (n-k+1)!}$$

clarify

22. $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k} \qquad r \le n \quad k \le r$

n, r, k nonnegative integers

a) Suppose that we have a set w/ n elements, and we wish to choose a subset A w/ k elements and another, disjoint, subset w/ r − k elements. The LHS gives us the # of ways to do this, namely the product of the # of ways to choose the r elements that are to go into one or the other of the subsets and the # of ways to choose which of these elements are to go into the first of the subsets. The RHS gives us the # of ways to do this as well, namely the product of the # of ways to choose the first subset and the number of ways to choose the second subset from the elements that remain.

b) On the one hand,

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{\cancel{r!}(n-r)!} \cdot \frac{\cancel{r!}}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!}$$

On the other hand,

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!\cancel{(n-k)!}} \cdot \frac{\cancel{(n-k)!}}{(r-k)!(n-k-(r-k))!}$$

$$= \frac{n!}{k!(n-r)!(r-k)!}$$

27. $\displaystyle\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$ n, r positive integers

a) We will count the # of bit strings of length $n+r+1$ containing exactly r 0's and $n+1$ 1's. There are $\binom{n+r+1}{r}$ such strings, since a string is completely specified by deciding which r of the bits are to be the 0's. To see that the LHS counts the same thing, let $\ell+1$ be the position of the last 1 in the string. Since there are $n+1$ 1's, we know that ℓ cannot be less than n. Thus there are disjoint cases for each ℓ from n to $n+r$. For each such ℓ, we completely determine the string by deciding which of the ℓ positions in the string before the last 1 are to be 0's. Since there are n 1's in this range, there are $\ell - n$ 0's. Thus there are $\binom{\ell}{\ell-n}$ ways to choose the positions of the 0's. Now by the sum rule the total # of bit strings will be
$\displaystyle\sum_{\ell=n}^{n+r} \binom{\ell}{\ell-n} = \sum_{\ell=n}^{n+r} \binom{n+k}{k}$. $(k = \ell - n)$

b) We need to prove this by induction on r; Pascal's identity will enter at a crucial step. We let $P(r)$ be the statement to be proved. The basis step is clear, since the equation reduces to $\binom{n}{0} = \binom{n+1}{0}$, which is the true proposition $1 = 1$. Assuming the inductive hypothesis, we derive $P(r+1)$ in the usual way:

$$\sum_{k=0}^{r+1} \binom{n+k}{k} = \left(\sum_{k=0}^{r} \binom{n+k}{k}\right) + \binom{n+r+1}{r+1}$$

$$= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} \quad \text{by ind. hyp.}$$

$$= \binom{n+(r+1)+1}{r+1} \quad \text{by Pascal's Identity}$$

29. $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

We will follow the hint and count the # of ways to choose a committee w/ leader from a set of n people. Note that the size of the committee is not specified, although it clearly needs to have at least one person (its leader). On the one hand, we can choose the leader first, in any of n ways. We can then choose the rest of the committee, which can be any subset of the remaining n-1 people; this can be done in 2^{n-1} ways since there are this many subsets. ∴ the RHS of the proposed identity counts this. On the other hand, we can organize our count by the size of the committee. Let k be the # of people who will serve on the committee. The # of ways to select a committee w/ k people is clearly $\binom{n}{k}$, and once we have chosen the committee, there are clearly k ways in which to choose its leader. By the sum rule the LHS of the proposed identity therefore also counts the # of such committees. Since the two sides count the same quantity, they must be equal.

33. a) Clearly a path of the desired type must consist of m moves to the right and n moves up. ∴ each such path can be represented by a bit string consisting of m 0's and n 1's, w/ the 0's representing moves to the right and the 1's representing moves up. Note that the total length of this bit string is m+n.

b) We know from this section that the # of bit strings of length m+n containing exactly n 1's is $\binom{m+n}{n}$, since one need only specify the positions of the 1's. Note that this is the same as $\binom{m+n}{m}$.

6.5 1. 3^5 - order matters

- repetition allowed

4. - order matters

- repetition allowed

6^7

5. Task 1 5 ways to assign 1st job

Task 2 5 ways to assign 2nd job

Task 3 5 ways to assign 3rd job

5^3

8. 20 bars + 12 objects = 32 items

$\binom{32}{20} = \binom{32}{12}$

9. 8 types

- order does not matter

- repetition allowed

Let $b_1, b_2, \ldots, b_8$ be the # of bagels of the 8 types listed

a) $b_1 + b_2 + \ldots + b_8 = 6$

7 bars + 6 objects = 13 items

$\binom{13}{6}$

b) $b_1 + b_2 + \ldots + b_8 = 12$

7 bars + 12 objects = 19 items

$\binom{19}{12}$

c) $b_1 + b_2 + \ldots + b_8 = 24$

7 bars + 24 objects = 31 items

$\binom{31}{24}$

d) $b_1 + b_2 + \ldots + b_8 = 12$ $b_i \geq 1$ for $i \in \{1, \ldots, 8\}$

Let $b_i' = b_i - 1$

$(b_1' + 1) + (b_2' + 1) + \ldots + (b_8' + 1) = 12$

$b_1' + b_2' + \ldots b_8' = 4$

7 bars + 4 objects = 11 items $\Rightarrow \binom{11}{4}$

e) $b_1 + b_2 + \dots + b_8 = 12$ $\quad b_3 \geq 3$

Let $b_3' = b_3 - 3$

$b_1 + b_2 + b_3' + \dots + b_8 = 9$

7 bars + 9 objects = 16 items

$\binom{16}{9}$

Next we need to account for $b_4 \leq 2$ by subtracting those that satisfy our 1st condition and violate our 2nd condition

$b_1 + b_2 + \dots + b_8 = 12$ $\quad b_4 \geq 3, \; b_3 \geq 3$

$b_1 + b_2 + b_3' + b_4' + \dots + b_8 = 6$

7 bars + 6 objects = 13 items

$\binom{13}{6}$

$\binom{16}{9} - \binom{13}{6} = \boxed{9724}$

11. - order does not matter

- repetition allowed

Let p = # pennies, n = # nickels

$p + n = 8$

1 bar + 8 objects = 9 items

$\binom{9}{8}$

14. 3 bars + 17 objects = 20 items

$\binom{20}{17}$ solutions

$x_1 \geq 1$

15. a) Let $x_1' = x_1 - 1$

$x_1' + x_2 + x_3 + x_4 + x_5 = 20$

4 bars + 20 objects = 24 items

$\binom{24}{20}$

b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$

Let $x_1' = x_1 - 2$, $x_2' = x_2 - 2$, $x_3' = x_3 - 2$, $x_4' = x_4 - 2$, $x_5' = x_5 - 2$

$x_1' + x_2' + x_3' + x_4' + x_5' = 11$

4 bars + 11 objects = 15 items

$\binom{15}{11}$

c) $0 \le x_1 \le 10$

4 bars + 21 objects = 25 items

$\binom{25}{21}$ total solutions

$x_1 \ge 11$

Let $x_1' = x_1 - 11$

$x_1' + x_2 + x_3 + x_4 + x_5 = 10$

4 bars + 10 objects = 14 items

$\binom{14}{10}$

$\binom{25}{21} - \binom{14}{10} = 11{,}649$

d) $0 \le x_1 \le 3$, $1 \le x_2 < 4$, $x_3 \ge 15$

$\{x_3 \ge 15\} - \{x_1 \ge 4, x_3 \ge 15\} - \{x_2 \ge 4, x_3 \ge 15\} - \{x_2 = 0, x_3 \ge 15\}$
$+ \{x_2 = 0, x_1 \ge 4, x_3 \ge 15\}$

$\boxed{\text{Case 1}}$ $\{x_3 \ge 15\}$.

Let $x_3' = x_3 - 15$

$x_1 + x_2 + x_3' + x_4 + x_5 = 6$

4 bars + 6 objects = 10 items

$\binom{10}{6}$

$\boxed{\text{Case 2}}$ $\{x_1 \ge 4, x_3 \ge 15\}$

Let $x_1' = x_1 - 4$

$x_1' + x_2 + x_3' + x_4 + x_5 = 2$

4 bars + 2 objects = 6 items

$\binom{6}{2}$

$\boxed{\text{Case 3}}$ $\{x_2 \ge 4, x_3 \ge 15\}$

Let $x_2' = x_2 - 4$

$x_1 + x_2' + x_3' + x_4 + x_5 = 2$

4 bars + 2 objects = 6 items

$\binom{6}{2}$

$\boxed{\text{Case 4}}$ $\{x_2 = 0, x_3 \ge 15\}$

$x_1 + x_3' + x_4 + x_5 = 6$

3 bars + 6 objects = 9 items

$\binom{9}{6}$

$\boxed{\text{Case 5}}$ $\{x_2 = 0, x_1 \ge 4, x_3 \ge 15\}$

$x_1' + x_3' + x_4 + x_5 = 2$

3 bars + 2 objects = 5 items

$\binom{5}{2}$

$\binom{10}{6} - \binom{6}{2} - \binom{6}{2} - \binom{9}{6} + \binom{5}{2} = 210 - 15 - 15 - 84 + 10 = 106$

17. 2 0's $\binom{10}{2}$

3 1's $\binom{8}{3}$

5 2's $\binom{5}{5}$

$\binom{10}{2}\binom{8}{3}\binom{5}{5} = 45 \cdot 56 \cdot 1 = 2520$

OR permutation w/ repetition, distinguishable groups, indistinguishable members, distinguishable boxes

$\frac{10!}{2!\,3!\,5!} = 2520$

19. $n = 14$

$n_1 = n_2 = 3 \qquad n_3 = n_4 = n_5 = 2 \qquad n_6 = n_7 = 1$

$\frac{14!}{3!\,3!\,2!\,2!\,2!\,1!\,1!}$

21. order doesn't matters (combs) w/ repetition

$n = 9 \quad r = 6$

$\binom{n+r-1}{r} = \binom{9+6-1}{6}\binom{14}{6}$

22. combs w/ repetition (indistinguishable objects / distinguishable boxes)

$n = 6 \quad r = 12$

$\binom{n+r-1}{n-1} = \binom{6+12-1}{6-1} = \binom{17}{5}$

23. $\underbrace{C(12,2)}_{\text{choose objs box 1}} \cdot C(10,2) \cdot C(8,2) \cdot C(6,2) \cdot C(4,2) \cdot \underbrace{C(2,2)}_{\text{choose objs box 6}}$

$= 7{,}484{,}400$

*review diff. methods (3)

25. Let $d_1, d_2, \ldots, d_6$ be the digits of a natural # less than 1,000,000; they can each be anything from 0 to 9. If want the sum of the digits to equal 19, then we are asking for the # of solutions to the equation $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 19$ w/ $0 \le d_i \le 9$ for each i.

5 bars + 19 objects = 24 items

So $\binom{24}{19}$ ignoring upper bound

$\shortparallel$

42504

We must subtract the # of solutions that violate the restriction. If digits are to add up to 19 and one or more of them is to exceed 9, then exactly one of them will have to exceed 9, since $10 + 10 > 19$. There are six ways to choose the digit that will exceed 9. Once we have made that choice (WLOG assume it is d_1 that is made to be greater than or equal to 10), then we count the # of solutions to the equation $d_1' + d_2 + d_3 + d_4 + d_5 + d_6 = 19 - 10 = 9$.
5 bars + 9 objects = 14 items so $\binom{14}{9} = 2002$ solns. $\therefore$ answer is $42504 - 6 \cdot 2002 = 30492$

31. 5 A's $\quad n = 11 \quad \dfrac{11!}{5!2!2!1!1!}$

2 B's

2 R's

1 C

1 D

32. 3 A's → treat as one letter $\quad n = 6 \quad \dfrac{6!}{1!2!1!1!1!}$

2 R's

1 D

1 V

1 K

35. $\boxed{\text{Case 1}}$ Length 9

$$\frac{9!}{4!\,2!\,1!1!1!} = 7560$$

$\boxed{\text{Case 2}}$ Length 8

Omitted

V/G/N $\quad \dfrac{8!}{4!2!1!1!} = 840$

R $\quad \dfrac{8!}{4!1!1!1!1!} = 1680$

E $\quad \dfrac{8!}{3!\,2!1!1!1!} = 3360$

EVERGREEN

4 E's

1 V

2 R's

1 G

1 N

$3(840) + 1680 + 3360 = 7560$

Case 3 Length 7

Omitting

VG $\dfrac{7!}{4!\,2!\,1!} = 105$

VN $\dfrac{7!}{4!\,2!\,1!} = 105$

GN $\dfrac{7!}{4!\,2!\,1!} = 105$

VR $\dfrac{7!}{4!\,1!\,1!\,1!} = 210$

GR $\dfrac{7!}{4!\,1!\,1!\,1!} = 210$

NR $\dfrac{7!}{4!\,1!\,1!\,1!} = 210$

RR $\dfrac{7!}{4!\,1!\,1!\,1!} = 210$

EV $\dfrac{7!}{3!\,2!\,1!\,1!} = 420$

EG $\dfrac{7!}{3!\,2!\,1!\,1!} = 420$

EN $\dfrac{7!}{3!\,2!\,1!\,1!} = 420$

ER $\dfrac{7!}{3!\,1!\,1!\,1!\,1!} = 840$

EE $\dfrac{7!}{2!\,2!\,1!\,1!\,1!} = 1260$

Total is 4515 of length 7

$\therefore$ answer is $7560 + 7560 + 4515 = 19635$

39. We can describe any such travel in a unique way by a sequence of 4 x's, 3 y's, and 5 z's.

$$\frac{12!}{4!3!5!}$$

41. $C(52,7)\,C(45,7)\,C(38,7)\,C(31,7)\,C(24,7) = \dfrac{52!}{7!7!7!7!7!\,17!}$

42. $$\frac{52!}{13!13!13!13!}$$

44. a) $x_1 + x_2 + x_3 + x_4 = 12$

□ clarify

3 bars + 12 objects = 15 items

$\binom{15}{12}$

b) $4 \cdot 5 \cdots 15 = 217{,}945{,}728{,}000$

45. a) $x_1 + x_2 + \dots + x_k = n$

□ clarify

$k-1$ bars + n objects = $k-1+n$ items

$$\binom{n+k-1}{n}$$

b) $k(k+1)\cdots(k+n-1) = \dfrac{(k+n-1)!}{(k-1)!}$

OR $\binom{n+k-1}{n} \cdot n!$

5.3 47. a) $P_{m,m} = P_m$ b/c a # exceeding m can never be used in a partition of m

b) We need to verify all five lines of this definition, show that the recursive references are to a smaller value of m or n and check that they take care of all cases and are mutually compatible. Let us do the last of these first. The first two lines take care of the case in which either m or n is equal to 1. They are consistent w/ each other in case $m=n=1$. The last 3 lines are mutually exclusive and take care of all the possibilities for m and n if neither is equal to 1, since, given any two numbers, either they are equal or one is greater than the other. Note finally that the 3rd line allows $m=1$; in that case the value is defined to be $P_{1,1}$ which is consistent w/ line one since $P_{1,n}=1$.

o clarity

Next let us make sure the logic of the def'n is sound, specifically that $P_{m,n}$ is being defined in terms of $P_{i,j}$ for $i \leq m$ and $j \leq n$, w/ at least one of the inequalities strict. There is no problem w/ the 1st two lines since these are not recursive. The 3rd line is okay, since $m<n$, and $P_{m,n}$ is being defined in terms of $P_{m,m}$. The 4th line is also okay, since here $P_{m,m}$ is being defined in terms of $P_{m,m-1}$. Finally, the last line is okay, since the subscripts satisfy the desired inequalities.

Finally, we need to check the content of each line. The first line says that there is only one way to write the # 1 as the sum of positive integers, which is true as $1=1$. The 2nd line says that there is only one way to write the # m as the sum of positive integers, none of which exceeds 1, and that is also true b/c $m = 1+1+\dots+1$. The 3rd line says that the # of ways to write m as the sum of integers not exceeding n is the same as the # of ways to write m as the sum of integers not exceeding m as long as $m<n$. This is true since we would never use a # from $\{m+1, m+2, \dots, n\}$ in such a sum anyway. The 4th line says that the # of ways to write m as the sum of positive integers not exceeding m is 1 plus the # of ways to write m as the sum of positive integers not exceeding $m-1$. Indeed, there is exactly one way to write m as the sum of positive integers not exceeding m that actually uses m, namely $m=m$; all the rest use only #'s less than

or equal to $m-1$. For the 5th line, we may use an n or we may not. There are exactly $P_{m,n-1}$ ways to form the sum w/o using n, since in that case each summand is less than or equal to $n-1$. If we do use at least one n, then we must have $m = n + (m-n)$. The # of ways this can be done, then, is the same as the # of ways to complete the partition by writing $(m-n)$ as the sum of positive integers not exceeding n. Thus there are $P_{m-n,n}$ ways to write m as the sum of numbers not exceeding n, at least one of which equals n. By the Sum Rule we have $P_{m,n} = P_{m,n-1} + P_{m-n,n}$.

c) $P_5 = P_{5,5} = 1 + P_{5,4} = 1 + P_{5,3} + P_{1,4} = 1 + P_{5,2} + P_{2,3} + 1$
$= 1 + P_{5,1} + P_{3,2} + P_{2,2} + 1 = 1 + 1 + P_{3,1} + P_{1,2} + 1 + P_{2,1} + 1$
$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$

$P_6 = P_{6,6} = 1 + P_{6,5} = 1 + P_{6,4} + P_{1,5} = 1 + P_{6,3} + P_{2,4} + 1 = 1 + P_{6,2} + P_{3,3} + P_{2,2} + 1$
$= 1 + P_{6,1} + P_{4,2} + 1 + P_{3,2} + 1 + P_{2,1} + 1 = 1 + 1 + P_{4,1} + P_{2,2} + 1 + P_{3,1} + P_{1,2} + 1 + 1 + 1$
$= 1 + 1 + 1 + 1 + P_{2,1} + 1 + 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 11$