Exam 3 Math 10 20183 KEY

Directions: All work must be shown to receive credit. No notes or study aids may be used.

Chapter 6 reminder

- Give a brief explanation of your approach.

Chapter 10 reminder

Let G be an undirected graph and n a nonnegative integer

- *path* of length n from vertex u to vertex v: is a sequence of n edges $e_1, e_2, \ldots e_n$ for which there exists a sequence of n+1 vertices $u = x_0, x_1, \ldots x_n = v$ where edge e_1 has endpoints x_0 and x_1, edge e_2 has endpoints x_1 and x_2, etc
- *circuit or cycle* of length n for a positive integer n: A path from vertex u back to itself using n edges.
- A path or circuit is called *simple* if it does not contain the same edge more than once.

In problems 1 to 7 there are actually 13 problems. Do 10 of them. 4 each

1) Words of length 7 over the standard 26 English letters

   a) How many words are there if letters can be repeated?

      Since words of length 7 order matters
      Task 1: Pick first letter. Since 26 choices there are 26 ways to do this.
      Task 2: Pick second letter. Since letters can be repeated there are 26 ways to do this
      Etc so there are 26*26*26*26*26*26*26=26^7 words if letter can be repeated.

   b) How many words are there if letters cannot be repeated?

      Since words of length 7 order matters
      Task 1: Pick first letter. Since 26 choices there are 26 ways to do this.
      Task 2: Pick second letter. Since letters cannot be repeated there are only 25 letters unused so 25 ways
      …
      Task 7: Pick last letter -there are 20 ways to do this
      Etc so there are 26*25*24*23*22*21*20=26!/19! words if letters cannot be repeated.

   c) How many words are there that begin with an A or end with a Z?

      Let P1 be the set of words that begin with the letter A.
      Let P2 be the set of words that end with the letter Z.
      We want the set of words in the union of these two sets $|P_1 \cup P_2| = |P_1| + |P_2| - |P_1 \cap P_2|$
      In P1: there are there then only 6 free spots and using the logic like in 1a) there are 26^6 words in this set.
      In P2: using the same logic 26^6 words.
      In $P_1 \cap P_2$ must start with the letter A and end with the letter Z so there are only 5 free spots so 26^5.
      Answer: $26^6+26^6-26^5$

2) How many permutations of the letters ABCDEFG contain the string BCD?

   So essentially treat BCD as a block and it is permuted with the other 4 letters so 5! ways

3) How many bit strings of length 12 contain at least three 1's?

So want 3 ones, or 4 ones….. up to 12 ones.

- A bit string of length 12 that contains exactly k ones. Must first choose where ones go and then remaining spots are filled with 0's so $\binom{12}{k}$ ways
- If a bit string contains exactly k ones then it can't contain m ones where $k \neq m$. This is important because signifies no overlap in counting.

So can either answer as $\sum_{k=3}^{12}\binom{12}{k}$

Or can say want all bit strings minus ones that contain 0, 1, or 2 ones. So $2^{12}-\binom{12}{0}-\binom{12}{1}-\binom{12}{2}$

4) Donuts: A store carries 20 different types of donuts. A group of 12 donuts is called a dozen.

a) How many ways are there to choose a dozen so that no two are the same? (all are different)

If all are different than you must pick from the 20 individual types which 12 you want. Order doesn't matter so there are $\binom{20}{12}$ ways.

b) How many ways are there to choose a dozen if all the donuts in that dozen are of the same variety?

If all the donuts in the box are the same, then you simple must pick from the 12 individual types which 1 you want. Again, no ordering so $\binom{12}{1}=12.$

c) How many ways are there to choose a dozen donuts if there are no restriction?

So here we have to choose a total of 12 donuts but we don't know how many of each type. So let x_i be the number of donuts of type $i,\ 1\le i\le 20.$ Notice that these are nonnegative integers x_i and $\sum_{i=1}^{20}x_i=12.$ So we can view this first as looking at 20 boxes, one for each type, and putting donuts in those boxes until we have put in a total of 12. For ease of notation the donuts are going to be called "stars = *". Now for the boxes. We view boxes as 20 adjacent rooms; technically this involves a left wall, 19 middle walls and the right wall. The left and right wall are implied so they are not part of the problem. So for example ||* would be none in room 1 (leftmost room), none in room two (the empty room) between the two walls, and one in the third room. So this question then changes to how do you count different arrangements of 19 walls and 12 *. So there are 31 objects total. Task 1: Choose where to put the 19 walls in this 31 string. $\binom{31}{12}$ ways Task 2: Stars in other spots. Only 1 way.

Answer: $\binom{31}{12}$

5) Counting functions

a) How many functions from an 8-set to a 12-set are there?

Functions require that for each element of domain assign it to an element of 12-set.
Order 8 set. for sake of typing will let the ith element be called *i*. Task *i*: Define $f(i)$. There are 12 choices.
So there are 8 tasks each with 12 choices so 8^{12}

b) How many onto functions are there from an 8-set to a 12-set?

Functions require that each element in domain set can go to only one element in the codomain. Onto requires that Codomain = Range. 8 elements in domain can only cover at most 8 elements in codomain so Range has at most 8 elements. Since codomain has 12 elements, this is impossible.
Answer: 0

c) How many injective functions are there from an 8-set to a 12-set?

Injective requires that if $a \neq b$ then $f(a) \neq f(b)$. This is very similar to 5a but now once f(1) is chosen f(2) must be chosen from the other 11 possible outcomes. f(3) now only has 10 choices……f(8) has only
Answer: 12*11*10*9*8*7*6*5=12P8

6) A school has 900 lockers and 3500 students so students must share lockers. Assume that the students are assigned as uniformly as possible to the lockers. The goal is to make the fewest students share a locker. In this case what is the largest number of students that must necessarily share?

Let the pigeons be the students and the holes be the lockers. We are trying to assign the pigeons into the holes.
First notice that $\left\lceil \frac{3500}{900} \right\rceil = 4$ So when try to evenly distribute the lockers, Pigeonhole Theorem tells us that some locker must have 4 or more.

Comments on language: Reread the question we have “uniformly as possible”, “largest number that must necessarily share” seem to cause the most confusion. What does it mean? Can’t say 5 or more MUST share because in fact 3500 = 900*3+800, So you could start by putting 3 per locker that assigns 2700 students now the next 800 have to go somewhere. To keep it as uniformly as possible put 1 additional student in 800 lockers. So 100 lockers have 3 students and 800 have 4. So 4 is the largest number that must necessarily share lockers. This is all summed up by just quoting the Pigeonhole Theorem and answer above is sufficient. Other arrangements could be all 3500 to one locker and 0. This does have a larger number of students in one locker but doesn’t meet the requirements of uniformly and also doesn’t meet the requirement that it is necessary to have so many share.

7) How many different strings can be made using the letters in the word GOOGLEO?

We want to permute the letters of this 7-string word but there are indistinguishable letters. So Task 1. Pick where in the 7 spots the two G's go: $\binom{7}{2}$

Task 2: Pick in the remaining 5 spots where the three O's go: $\binom{5}{3}$

Task 3: Pick where E goes into the remaining 2 spots. $\binom{2}{1}$

Task 4: Place L in the only remaining spot.

Answer: $\binom{7}{2}\binom{5}{3}\binom{2}{1}\binom{1}{1} = \dfrac{7!}{2!3!1!1!}$

Comment it is okay to go right to the final factorial answer once words are used to describe why this is the correct memorized formula to use.

8) Use a combinatorial argument to prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$ where $n, r,$ and k are nonnegative integers with $k \le r \le n.$

A combinatorial argument requires that a set of object be identified that is to be counted. The expression on one side will represent one way to count it and the expression on the right should represent another. A detailed explanation of what is being counted and how it is being counted should be included.

Consider a group of n people from which you want to choose a committee of r people <u>in which</u> there will be identified k leaders. The<u> in which</u> is critical. Another way to look at this is to let A be the set of n-people. We are looking for a subset B of A of r people. AND then from this subset B choose a subset C. So $C \subseteq B \subseteq A$ where $|C| = k,\ |B| = r, \text{ and } |A| = n$.

Method 1 (Left hand side): Pick Committee first then leaders. Pick B first then its subset C. First choose the subset B from the set A. $\binom{n}{r}$ ways to do this. Then second from B choose your k leaders. $\binom{r}{k}$ ways to do this. So total of $\binom{n}{r}\binom{r}{k}$ ways to get a group of r people from an n set in which there are k leaders.

Method 2 (Right hand side): Pick Leaders first then fill out the rest of committee. Pick C first $\binom{n}{k}$ ways to do this. Now have to fill out the remainder of the committee. But you have fill it out with people who are not the leaders so only n-k people to choose from and only need r-k more people. $\binom{n-k}{r-k}$ ways to do this. So total of $\binom{n}{k}\binom{n-k}{r-k}$ ways to get a group of r people from an n set in which there are k leaders.

Conclude: Since both methods count the same set they must be numerically equal.

9) Consider an undirected graph G on 5 vertices numbered 1, 2, 3, 4, 5 without loops or parallel edges.
NOTE: This is a simple undirected graph on 5 vertices

a) If the graph is connected, what is the minimum number of edges of G?
This is easily answered by quoting a homework problem assigned in 10.4 so answer number of vertices minus 1 = so answer is 4

If you don't remember the theory can logic it out:
*_*_*_*_* is a graph with 5 points and 4 edges. So can do it with 4 edges. Visually you should be able to picture how you could extend this idea to a graph with n vertices to have n-1 edges.
Can you do it with less? Can you construct a connected graph with 3 or less edges? Since connected the degree of every vertex is 1 or more. Now remember that $5=\sum_{i=1}^{5}1\le\sum_{i=1}^{5}\deg(v_i)=2(\#edges)\le 6$. So this means the only possibility is to find a degree sequence of 5 positive integers that sum to six. 2+1+1+1+1=6 is the only possibility. Starting with the vertex of degree 2. Drawing this vertex and its two edges accounts for three vertices and degree sequence 2,1,1 There is no way to connect another of our vertices with this component without increasing one of the degrees which means the other three vertices are a disconnected component which contradicts given.

The argument above is horrible once we get past 5 vertices. Instead we use induction for larger values of n. So assume that the above is the base case but in reality can start with n=1.
Assume a simple connected undirected graph with n vertices has at least n-1 edges for some integer $n\ge 5$.
Now must prove what happens if have a simple connected undirected graph with n+1 vertices. Suppose there exists a simple undirected connected graph with n+1 vertices and <u>less than n edges</u>. If all vertices have degree 2 or more than $2(n+1)=\sum_{i=1}^{n+1}2\le\sum_{i=1}^{n+1}\deg(v_i)=2(\#edges)\le 2n$. This is impossible so there must exist at least one vertex of degree 1. Since you have a degree one vertex, can create a new simple graph with n vertices by removing that degree one vertex and its lone corresponding edge. This new graph will still be connected and simple so by the induction hypothesis will have at least n-1 edges. That means that the original graph will have at least (n-1)+1=<u>n edges</u>. This contradicts our supposition. So simple undirected connected graph with n+1 vertices cannot have less than n edges so it has n or more. The result then follows by the theory of mathematical induction.

b) What is the maximum number of edges of G?

Complete simple graph on vertices would be the most could have: $\binom{n}{2}$ so 10

c) Draw an example of a graph G using at least 6 edges. – answers will vary

Any picture that used 5 vertices and included 6 edges was acceptable K_5 was the most common answer

d) Let S ={1,2,3,4,5}. Explain a way to use a subset of SxS to represent the set of edges of your graph.

Each edge has a vertex at each end say "a" and "b" this is in correspondence with subset {a,b}. The graph can then be represented by this collection of subsets.

e) Explain a way to use bit strings to represent the edges of your graph. You must include how long the bit string is and how to determine where the 1's are located, but you need not actually give the bit string.

So the adjacency matrix is a way to represent graph by matrix. 5x5 so 25 locations
Answer 1: Use 25 bit strings where first 5 are row 1 of matrix, next 5 are row 2…..
Answer 2: Graph has no loops so diagonals are 0 so don't need. If edge between 1 and 5 then a 1 in location (1,5) and in location (5,1) So really have duplicated the information. So in row 1 of matrix only really need to know information if there is edge 1 to 2, 1 to 3, 1 to 4, and 1 to 5. Similarly in row 2 only need 2 to 3, 2 to 4 and 2 to 5 etc. Essentially only using information in matrix right of the diagonal so 10 bits needed and the information comes from the matrix.

f) How many simple graphs on these 5 vertices are there?

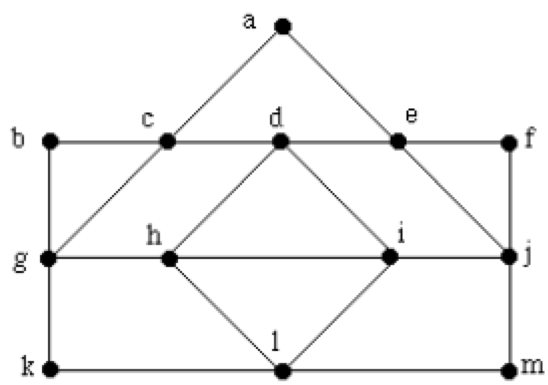
Method 1: Any graph you have will be a subgraph of K_5 which has 10 edges: Graphs with 0 edges + graphs with 1 edge+….+graphs with 10 edges. So $\sum_{k=1}^{10}\binom{10}{k}$ you should actually recognize this sum and write = 2^{10}

Method 2: Use 9e: Way to form 10 bit strings: 2^{10}

g) If we allow loops or parallel edges, are there finite or infinite number of graphs on these 5 vertices?

Once we start allowing loops and parallel edges between the same vertices (this is like having multiple choices of airlines when flying from LAX to San Francisco) then there will be infinite number of graphs can just pick two vertices and just add infinite loops. This is totally unrealistic; I was more interested in that you were recognizing how complicated loops and multiple edges between two vertices makes counting.

10) Consider the graph:



a) Does this graph have an Euler Circuit? Construct such a circuit if one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if it exists.

   Remember there are two basic iff theorems that tell you whether or not you have a Euler Path or Euler Circuit. For this graph all the degrees are even meaning the answer is "Yes there is an Euler Circuit because all the degrees are even."

   Finding it: Please refer to text or website on how to construct a Euler Circuit. It is a fair final exam question to have a problem that shows you know this construction (by something other than lucky guess). So can give you a graph and give a path that starts at a vertex "a" ends at a vertex "a" but doesn't use all the edges. You can then be asked to finish this construction to a Euler Circuit. In the 7^{th} ed, this is explained on pages 695 and 696.

b) Is this graph bipartite? Justify your answer.
   Answer NO because

   Method 1: A graph is bipartite iff it is two-colorable. If a graph contains a complete graph on n vertices K_n than its chromatic number be n or larger. Since this graphs, contains a triangle, K_3 . it requires at least 3 colors so it can not be two-colorable so it is not bipartite.

   Method 2: Try to two color and either succeed or reach a problem. Start at one vertex call it red, then explicitly state what that forces to be blue and continue from there. This can get awful to write an explanation of:

   Ex.

   - Start at "a" call it red. Red ={a} Blue={}
   - This forces "c" and "e" to be blue. Red ={a} Blue={c,e}
   - "c" being blue forces "b" ,"d" and "g" to be red while "e" being blue forces "d", "J" and "f" red. Red ={a, b,d,g,j,f} Blue={c,e}

   Now we have a problem both "b" and "g" are red and they are joined by an edge. This violates rules for 2-coloring so it is not possible.

11) Suppose that you are responsible for scheduling times for lectures for a university.

Lecture	A	C	G	H	I	L	M	P	S
Astronomy		X	X	X			X		
Chemistry	X								X
Greek	X			X		X	X	X	
History	X		X			X			X
Italian						X	X		X
Latin			X	X	X		X	X	X
Music	X		X		X	X			
Philosophy			X			X			
Spanish		X		X	X	X			

The letter A in the top row is the abbreviation of Astronomy. An X denotes that two lectures have a common student.

a) How can a graph be used to represent this information? Clearly identify what should be used as vertices and how an edge between vertices will be determined. You can just explain what to do; you don't actually have to draw the graph.

The above is essentially an adjacency matrix where the vertices are the subjects and an edge is between two subjects if there is X representing a common student.

b) How can this graph be used to determine the number of scheduling times required? You can just explain what to do; you are not actually required to determine a numeric answer.

Determine the chromatic number of the graph. Each set of classes at the same time can be given the same scheduling time since they have no students in common. Scheduling times for the different colors must be nonoverlapping.

12) For each of the following, either give an example or prove there are none:

a) A simple graph with 6 vertices, whose degrees are 2, 2, 2, 3, 4, 4.

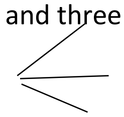
NO. This violates that sum of degree sequences must be even.

b) A simple graph with 8 vertices, whose degrees are 0, 1, 2, 3, 4, 5, 6, 7.

NO. Since it is simple no loops at one vertex are allowed and no vertices can be connected by 2 or more edges. That means all 7 edges from the vertex of degree 7 must each go to a distinct vertex from set of other 7 vertices. This is impossible because one of these other vertices has degree 0.

c) A simple graph with degrees 1, 2, 2, 3.

YES – so must draw it - Starting with the degree 3 vertex, draw its vertex and three edges which then accounts for all four vertices. This current picture has degree sequence 1,1,1,3 .



Simple connect two of the degree 1 vertices of this to get the answer. Example: