1: Prove x is odd if and only if $x^2 + 6x + 9$ is even.

Strategy 1: Since this problem is of the form $P \Leftrightarrow Q$. You must show $P \Rightarrow Q$ and $Q \Rightarrow P$.

Strategy 2: Using contrapositives (pg 8) You can instead prove $\neg P \Leftrightarrow \neg Q$ by showing $\neg P \Rightarrow \neg Q$ and $\neg Q \Rightarrow \neg P$.

Strategy 3: You can sometimes start with a known biconditional and connect ideas: $P \Leftrightarrow A_1$ but $A_1 \Leftrightarrow A_2$ etc...... $A_k \Leftrightarrow Q$ thus $P \Leftrightarrow Q$.

Answer: (using Strategy 1)

Let $P =$ "x is odd" and $Q =$ "$x^2 + 6x + 9$ is even".

(Step 1) Show $P \Rightarrow Q$

We assume that P is true. By the definition of an odd integer, it follows that $x = 2k + 1$, where k is some integer. This implies that:

$$\begin{aligned}
x^2 + 6x + 9 &= (2k+1)^2 + 6(2k+1) + 9 \\
&= (2k+1)(2k+1) + 12k + 6 + 9 \\
&= (4k^2 + 2k + 2k + 1) + 12k + 15 \\
&= 4k^2 + 16k + 16 \\
&= 2(2k^2 + 8k + 8)
\end{aligned}$$

Because $x^2 + 6x + 9$ is $2t$, where t is some integer $2k^2 + 8k + 8$, $x^2 + 6x + 9$ is even. This proves $P \Rightarrow Q$.

(Step 2) Show $Q \Rightarrow P$

We use a proof by contraposition and show $\neg P \Rightarrow \neg Q$. We assume $\neg P$, namely, that x is even. By the definition of an even integer, it follows that $x = 2m$, where m is some integer. This implies that:

$$\begin{aligned}
x^2 + 6x + 9 &= (2m)^2 + 6(2m) + 9 \\
&= 4m^2 + 12m + 9 \\
&= 4m^2 + 12m + 8 + 1 \\
&= 2(2m^2 + 6m + 4) + 1
\end{aligned}$$

Because $x^2 + 6x + 9$ is $2n + 1$, where n is some integer $2m^2 + 6m + 4$, $x^2 + 6x + 9$ is odd. This proves $\neg P \Rightarrow \neg Q$, which tells us that $Q \Rightarrow P$ is also true.

Because we have shown that both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, we have shown that $P \Leftrightarrow Q$ is true.