1: A subset U of the set $W = \{0, 1, 2...\}$ of whole numbers is called ultimately periodic if there exists a number $M \geqslant 0$ and a number $p > 0$ such that for all integers $n \geqslant M$ the set U has the property that $n \in U \iff n + p \in U$.

Explain why every finite set of W is ultimately periodic.

Since this problem is of the form $P \Leftrightarrow Q$, we must show $P \Rightarrow Q$ and $Q \Rightarrow P$.

Answer:

Let $U = \{x_1, x_2, ..., x_n\}$ such that $x_1 \leqslant x_2 \leqslant ... \leqslant x_n$, and $m = x_n + 1$. Notice we can say x_n is the maximum element in U because U is finite.

(Step 1) Show $n \in U \Rightarrow n + p \in U$

Since $n \geqslant M$, it follows that $n > x_n$ and $n \in U$ is false because n is not in the range $x_1 \leqslant k \leqslant x_n$. Therefore, p can be any integer $p > 0$ and this conditional statement will always be true.

(Step 2) Show $n + p \in U \Rightarrow n \in U$

To prove this conditional statement is true, we use a proof by contraposition. We assume $n \notin U$, namely, that n has the values $m, m+1, m+2, ... = x_n + 1, x_n + 2, x_n + 3, ...$ because $n \geqslant m$. Therefore, p can be any integer $p > 0$ and $n + p \notin U$ because $n + p > m > x_n$ is not in the range $x_1 \leqslant k \leqslant x_n$.

2: Which is uncountable? Explain reasoning.

(a) Set of functions f from the Natural numbers to the set $\{1, 2\}$
Hint: can view EACH f as sending some subset of the natural numbers to 1 and the rest to 2. Use this viewpoint to explain why there is a one to one correspondence between the functions in question and the subsets of the natural numbers. What is true about the subsets of the natural numbers?

(b) Set T of functions g from the set $\{1, 2\}$ to the set of Natural numbers.
Hint: consider the set $A = \{(g(1), g(2))$ for g in $T\}$ What is the relation between $|T|$ and $|A|$? In what superset does A reside?

(a) Let $U = \{f | f : \mathbb{N} \to \{1, 2\}\}$ and $\tau : U \to \mathcal{P}(\mathbb{N})$.

We know that $\{x | f(x) = 1\} \cup \{y | f(y) = 2\} = \mathbb{N}$.

These two sets are disjoint so $\{y | f(y) = 2\} = \overline{\{x | f(x) = 1\}}$. Let $A_f = \{y | f(y) = 2\}$ and $\tau(f) = A_f$.

Let $B \subseteq \mathcal{P}(\mathbb{N})$ and we define f as follows: if $x \in B$, $f(x) = 2$ and if $x \notin B$, $f(x) = 1$. So $\tau(f) = B$. Therefore, τ is onto.

Let $\tau(f) = \tau(g)$. It follows that the preimage of 2 is identical for f and g. Because the subset of the natural numbers not being sent to 2 must be sent to 1, it follows that the preimage of 1 is also identical for f and g. Therefore, τ is one-to-one.

τ is a bijection, so $|U| = |\mathcal{P}(\mathbb{N})|$. Since $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$ and $\mathbb{N}$ is countable, $\mathcal{P}(\mathbb{N})$ is uncountable. Thus, it follows that U is also uncountable.

(b) Given $T = \{g|g : \{1,2\} \to \mathbb{N}\}$ and $A = \{(g(1), g(2))$ for g in $T\}$.

We define $\tau : T \to A$ and $\tau(g) = (g(1), g(2))$.

Let $(g(1), g(2)) = (h(1), h(2))$. Then $g(1) = h(1)$ and $g(2) = h(2)$. Therefore, $g = h$ because there are only two points on the graph and they are equal. Thus, τ is one-to-one.

Let $x \in A$. Then there exists n, m in $\mathbb{N}$ with $x = (n, m)$. Notice $g(1) = n$ and $g(2) = m$ is a function from $\{1,2\} \to \mathbb{N}$ and $\tau(g) = x$. Thus, τ is onto.

$|T| = |A|$ because there is a bijection between T and A.

A resides in the superset $\mathbb{N} \times \mathbb{N}$ because by the definition of cartesian product $\mathbb{N} \times \mathbb{N}$ is the set of all ordered pairs (a, b) where $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

$\mathbb{N} \times \mathbb{N}$ is known to be countable and $A \subseteq \mathbb{N} \times \mathbb{N}$, so A is also countable. Because $|T| = |A|$, T is also countable.