Math 10 Fall 2018 Exam 2 on selected sections from Ch 2, 5 and 9

Name:______________________________

Directions: All work must be shown to receive full credit. No notes, study aids or cell phones may be used. Scientific calculators are allowed but not needed.

1) Let $A=\{a,c,e,h,2,5,6\}$, $B=\{a,b,c,2,6,7\}$ and $C=\{\{a,c\},b,2,6\}$.

Rubric: 12 points - 2 points for each part – since these are fundamental concepts partial credit is not anticipated.

a) Find $A\cup B$

$A\cup B=\{a,c,e,h,2,5,6\}\cup\{a,b,c,2,6,7\} =\{a,c,e,h,2,5,6,a,b,c,2,6,7\}=\{a,a,b,c,e,h,2,2,5,6,6.7\}$
$A\cup B=\{a,b,c,e,h,2,5,6,7\}$

Or if Universe is $U=\{a,b,c,e,h,2,5,6,7\}$ then subsets can be represented by 9-bit strings.

$A=1011\ 1111\ 0$

$B=1110\ 0101\ 1$

$A\vee B=1111\ 1111\ 1$

So all of U

so $A\cup B=\{a,b,c,e,h,2,5,6,7\}$

b) Find $A\cap B$

$A\cup B=\{a,a,b,c,e,h,2,2,5,6,6.7\}$ so $A\cap B$ is the elements in common which would be ones listed more than once so

$A\cap B=\{a,c,2,6\}$

Or

$A=1011\ 1111\ 0$

$B=1110\ 0101\ 1$

$A\wedge B=1010\ 0101\ 0$

So 1st, 3rd, 6th, and 8th element of list of U

$A\cap B=\{a,c,2,6\}$

c) Find $B-A$

Elements in B that are not in A (with small sets can just stare at it) or can think $B\cap\overline{A}$

$\overline{A}=\overline{1011\ 1111\ 0}=0100\ 0000\ 1$

$B=1110\ 0101\ 1$

$\overline{A}=0100\ 0000\ 1$

$B\wedge\overline{A}=0100\ 0000\ 1$

so $B - A = B \cap \overline{A} = \{b, 7\}$

d) Determine $|A|$

Cardinality of A = number of elements in A. Since A is a small finite set just count = 7

e) Determine $|\mathcal{P}(A)|$

The power set of A. Since A is finite the cardinality is $2^{|A|} = 2^7$

As a reminder this is because when using A as a universal set it would need 7 bits. Any subset of A can then be represented using a 7-bit string where a 1is used for every bit that corresponds to an element in the subset and a 0 otherwise. So the number of subsets is equal to the number of words of length 7 over an alphabet {0,1}. So for every bit 2 choices for letter in alphabet which is independent of choice in next bit over 7 bits.

f) Which is the correct notation? $\{a, c\} \in C$ or $\{a, c\} \subseteq C$?

2) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x \bmod 10$. Determine if this is a surjection, injection, or bijection. Justify your answer.

Rubric: 12 points – must address all three parts. If it doesn't satisfy one of the definitions give specific counterexample – not handwaving. If it does satisfy, show details of explanation in proof - not handwaving.

Surjection: If "for every y is in the stated codomain then exists x in stated domain such that $f(x) = y$ " then the function is surjection. If instead can find any y is in the stated codomain where there is no solution x in stated domain to the equation $f(x) = y$, then not surjection.

Let $y \in \mathbb{Z}$ consider solving the equation $f(x) = y$ So solving $x \bmod 10 = y$. The left side can only be integers 0, 1,2,,,9. So cannot solve for $y = 10$ therefore this function is NOT A SURJECTION so NOT BIJECTION.

Injection: If "For all a, b in the domain, $f(a) = f(b)$ implies $a = b$. :" then the function is an injection. If instead can find a, b in the domain with $a \neq b$ with $f(a) = f(b)$ then the function is not injection.

Notice $0, 10 \in \mathbb{Z}$ and $f(0) = 0 \bmod 10 = 0 = 10 \bmod 10 = f(10)$. so NOT AN INJECTION.

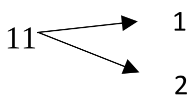
3) Let S be the set of bit strings $S = \{0, 1, 00, 11, ...\}$. Determine whether or not f is a function from S to the set of integers if $f(x)$ is the position of a 1 in the bit string x. Justify your answer.

Rubric: 6 points – A function f from A to B is a correspondence for which every element $a \in A$ is assigned a <u>unique element</u> in B called the image of a and denoted by $f(a)$. In other words given $a \in A$

can always find exactly one $b \in B$ so that f maps a to b. (Note: not all elements of B must have a preimage an the statement "given $b \in B$ can always solve $f(x) = b$" is not the same thing.

Examine the potential function. Where would it send 0? It should send it to the position of a 1 in the string 0, but there isn't a 1 in the string 0 so we don't know where to send it; therefore, this is not a function.

Alternatively, where would it send the string 11? Since there is a 1 in position 1 and in position 2 then



4) Let $A = \{1, 2, 3, 12, 15\}$ and R be a binary relation on A defined by $\forall m, n \in A \quad (m, n) \in R \ iff \ \ m \mid n.$

a) Show R is a Poset.
b) Draw the corresponding Hasse diagram.
c) Give the minimal and maximal elements.
d) Give the least element and the greatest element, if they exist.

Rubric: 10 points ab) 7 pts cd) 3 pts

a) To show R is a Poset must show it is reflexive, antisymmetric and transitive

Reflexive: Show aRa for all $a \in A$

Notice for any integer m we have

$m = 1m$ now since if $m \in A$ we know $m \neq 0$ so we can divide both sides by m. So we can say $m \mid m \rightarrow mRm.$ Therefore the relation is reflexive.

Antisymmetric: Show "If aRb and bRa then $a = b.$"

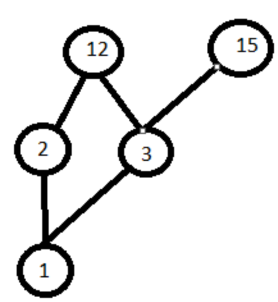
Assume aRb and bRa then $a \mid b$ and $b \mid a$ there exists integers $k, m \in \mathbb{Z}^+$ (positive because a and b are positive) such that $b = ma$ and $a = kb.$ Using substitution $b = m(kb) = (mk)b$. Since b is nonzero, multiplication by the reciprocal of b yields $mk = 1$. Since $k, m \in \mathbb{Z}^+$ and $mk = 1$ then $m = k = 1.$ This implies $a = b.$ Therefore the relation is antisymmetric.

Transitive" Show "If aRb and bRc then $aRc.$"

Assume aRb and bRc then $a \mid b$ and $b \mid c$ there exists integers $k, m \in \mathbb{Z}^+$ (positive because a, b and c are positive) such that $b = ma$ and $c = kb.$ Using substitution $c = k(ma) = (km)a$. Thus $a \mid b$ so $aRc.$ Therefore the relation is transitive.

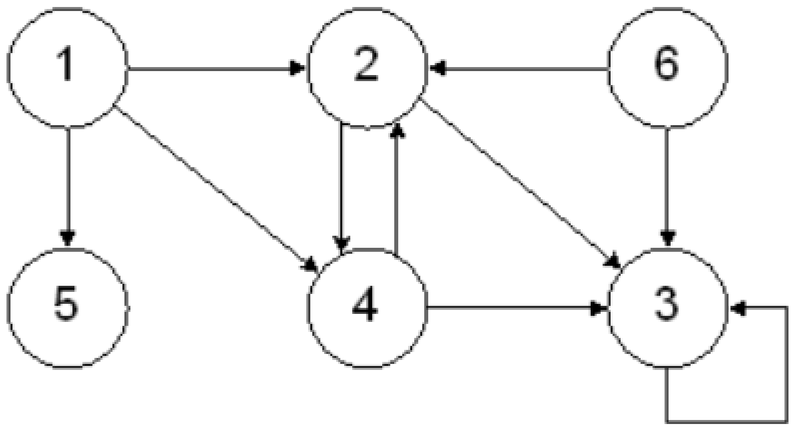
Since the relation is reflexive, antisymmetric and transitive it is a poset.

$R = \{(1,2), (1,3), (1,12), (1,15), (2,12), (3,12), (3,15)\}$



A minimal element is an element that has nothing “lower”: 1
A maximal element is an element that has nothing “higher”: 12, 15
The least element is an element that is related to all others and is minimal: 1
The greatest element is an element that is related to all others and is maximal: None

5) Let R be a relation on {1,2,3,4,5,6} represented by the directed graph below.



Rubric: 10 points

a) Is this relation reflexive? Why or why not?

No reflexive requires aRa for all elements in set. But there is not a loop at every vertex; ie $1\not R 1$. So it is not reflexive.

b) Is this relation irreflexive? Why or why not?

No irreflexive requires $a\not R a$ for all elements in set. But we have 3R3 so this is not irreflexive.

c) Is this relation symmetric? Why or why not?

No symmetric requires “If aRb then bRa.” But we have 1R5 and $5\not R 1$.

d) Is this relation transitive? Why or why not?

No transitive requires “If aRb and bRc then aRc." But we have 1R4 and 4R3 but $1\not R 3$ so this is not transitive.

Rubric: 6 and 7 together 10 points

6) To determine if a relation is an equivalency relation explain what must be checked.

To prove that a relation R is an equivalency relation on a set A it must be verified that

- It is reflexive For all $a \in A, \ aRa$.
- It is symmetric "If aRb then bRa."
- It is transitive "If aRb and bRc then aRc."

7) Let R be the set of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ consisting of all pairs (f,g) such that $f'(x)=g'(x)$ for all real numbers x . In homework you showed this was an equivalency relation. Which functions are in the same equivalency class as the function f where $f(x)=2x+3$?

$$
\begin{aligned}
[f]_R &= \{g \mid g \text{ a differentiable function from } \mathbb{R} \text{ to } \mathbb{R} \text{ satisifiying } f'(x)=g'(x)\} \\
&= \{g \mid g \text{ a differentiable function from } \mathbb{R} \text{ to } \mathbb{R} \text{ satisifiying } 2=g'(x)\} \\
&= \{g \mid g(x)=2x+C \text{ where } C \text{ is a constant}\}
\end{aligned}
$$

8) Show (0,1) has the same cardinality as (0,2).

Rubric: 10 points

Notice $(0,1) \subseteq (0,2)$ so $|(0,1)| \le |(0,2)|$ But this does not answer the questionis of their cardinalities being equal. Additionally, saying both are infinitely uncountable does not address their cardinalities.

Answer: Define a function $f:(0,1) \to (0,2)$ by $f(x)=2x.$ Notice that if $y \in (0,2)$ then $x = y/2 \in (0,1)$ and $f(x) = f(y/2) = 2(y/2) = y$ so the function is onto. If $f(a)=f(b)$ then $2a=2b \to a=b.$ so the function is one to one. Since there is a bijection between the corresponding sets we have $|(0,1)|=|(0,2)|$. Thus the cardinalities are equal.

9) Let T be the set of all positive rational numbers that can be written with denominators less than 3. Determine if T is finite countable, infinite countable or uncountable. Justify your answer.

Rubric: 10 points

Given $T = \left\{\frac{a}{b} \mid a \in \mathbb{Z}^+, b \in \{1,2\}\right\}$ so $T = \left\{\frac{a}{1} \mid a \in \mathbb{Z}^+\right\} \cup \left\{\frac{a}{2} \mid a \in \mathbb{Z}^+\right\} = \mathbb{Z}^+ \cup \left\{\frac{a}{2} \mid a \in \mathbb{Z}^+\right\}$

Method 1: Notice $\mathbb{Z}+ \subset T$ it is infinite. Since $T \subset \mathbb{Q}$ it is a subset of a countable set. Put together it is infinite countable.

Method 2: $\mathbb{Z}^+$ is infinitely countable. The set $\left\{\frac{a}{2} \mid a \in \mathbb{Z}^+\right\}$ is in one to one correspondence with $\mathbb{Z}^+$ using the function $f(a/2) = a$. This is a bijection using an argument similar to #8. Thus $\left|\left\{\frac{a}{2} \mid a \in \mathbb{Z}^+\right\}\right| = \left|\mathbb{Z}^+\right|$. So this set is also infinitely countable. Thus T is a union of infinitely countable sets so it is infinitely countable.

Method 3: $T = \mathbb{Z}^+ \cup \left\{\frac{a}{2} \mid a \in \mathbb{Z}^+\right\}$ Notice if a is even $\frac{a}{2} \in \mathbb{Z}^+$ so $\mathbb{Z}^+ \subset \left\{\frac{a}{2} \mid a \in \mathbb{Z}^+\right\}$. Thus we really have $T = \left\{\frac{a}{2} \mid a \in \mathbb{Z}^+\right\}$. Then continue by constructing function as in Method 2 to prove this set is infinitely countable.

Method 4: Notice $T = \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4....\right\}$. This is a sequencing of T which doesn't terminate thus T is infinitely countable.

10) Use the principal of mathematical induction to prove: $2n + 3 \leq 2^n$ for $n \geq 4$.
Rubric: 10 points

Basis step: Notice for $n = 4$ that $2n + 3 = 11$ and $2^n = 16$ thus $2n + 3 \leq 2^n$ for $n = 4$.
Assume $2n + 3 \leq 2^n$ for some $n \geq 4$. Goal is to show the pattern holds for the next increment. Goal is to show $2(n+1) + 3 \leq 2^{n+1}$.

Notice $2(n+1) + 3 = 2n + 2 + 3 = (2n+3) + 2 \leq 2^n + 2$. The function $f(x) = 2^x$ is an increasing function it preserves inequalities. Since $n \geq 4$ then $2^n \geq 2^4 > 2$. Combining these two inequalities using the transitive property of inequalities gives
$2(n+1) + 3 = 2n + 2 + 3 = (2n+3) + 2 \leq 2^n + 2 < 2^n + 2^n = 2^n(1+1) = 2^n 2 = 2^{n+1}$. (which was the goal)
The result follows by the principal of mathematical induction.