OT 13

1) How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 20$ where

a) $x_1 \geq 2,\ x_2 \geq 5$ and the rest are nonnegative

let $y_1 = x_1 - 2$ $\quad y_2 = x_2 - 5$

$y_1 + y_2 + x_3 + x_4 = 13$

3 bars + 13 objects = 16 items

$\binom{16}{3}$

b) $0 \leq x_1 \leq 7,\ 1 \leq x_2 < 5$, and $x_3 \geq 6$ and other is nonnegative.

This can be done using the following equivalent conditions:

$\{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 6\} - \{x_1 \geq 8,\ x_2 \geq 0,\ x_3 \geq 6\} - \{x_1 \geq 0, x_2 \geq 5, x_3 \geq 6\}$

$- \{x_1 \geq 0,\ x_2 = 0,\ x_3 \geq 6\} + \{x_1 \geq 8,\ x_2 \geq 5,\ x_3 \geq 6\} + \{x_1 \geq 8,\ x_2 = 0,\ x_3 \geq 6\}$

$\boxed{\text{Case 1}}$ $\{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 6\}$

Let $x_3' = x_3 - 6$

$x_1 + x_2 + x_3' + x_4 = 14$

3 bars + 14 objects = 17 items

$\binom{17}{14} = 680$

$\boxed{\text{Case 2}}$ $\{x_1 \geq 8,\ x_2 \geq 0,\ x_3 \geq 6\}$

Let $x_1' = x_1 - 8,\ x_3' = x_3 - 6$

$x_1' + x_2 + x_3' + x_4 = 6$

3 bars + 6 objects = 9 items

$\binom{9}{6} = 84$

Case 3 $\{x_1 \geq 0,\ x_2 \geq 5,\ x_3 \geq 6\}$

Let $x_2' = x_2 - 5$, $x_3' = x_3 - 6$

$x_1 + x_2' + x_3' + x_4 = 9$

3 bars + 9 objects = 12 items

$\binom{12}{9} = 220$

Case 4 $\{x_1 \geq 0,\ x_2 = 0,\ x_3 \geq 6\}$

Let $x_3' = x_3 - 6$

$x_1 + x_3' + x_4 = 14$

2 bars + 14 objects = 16 items

$\binom{16}{14} = 120$

To solve this problem, we subtract Case 2, 3, and 4 from Case 1. However, notice the following two sets are subtracted twice and must be added back:

(i) $\{x_1 \geq 8,\ x_2 \geq 5,\ x_3 \geq 6\}$ (included in 2nd and 3rd terms)

Let $x_1' = x_1 - 8$, $x_2' = x_2 - 5$, $x_3' = x_3 - 6$

$x_1' + x_2' + x_3' + x_4 = 1$

3 bars + 1 object = 4 items

$\binom{4}{1} = 4$

(ii) $\{x_1 \geq 8,\ x_2 = 0,\ x_3 \geq 6\}$ (included in 2nd and 4th terms)

Let $x_1' = x_1 - 8$, $x_3' = x_3 - 6\}$

$x_1' + x_3' + x_4 = 6$

2 bars + 6 objects = 8 items

$\binom{8}{6} = 28$

The total number of solutions with the given constraints is

$680 - 84 - 220 - 120 + 4 + 28 = \boxed{288}$