Math 10 Fall 2018 Exam 1 on selected sections from Ch 1, 4 and 12 Key

Name:______________________________

Directions: All work must be shown to receive full credit. No notes, study aids or cell phones may be used. Scientific calculators are allowed but needed.

1) State the negation of the statement: "Some box contains at least 12 items"

For the statement to be true all we need is one box that has 12 or more items in it. So to be false every box must have 11 or less.

Answer: No box contains at least 12 items. OR Every box contains at most 11 items. OR All boxes have less than 12 items.

2) Given the propositions

U: A person enters Utopia. D: A person shows a Driver's license. P: A person shows a passport.

and the statement

"To enter Utopia, you must show a driver's license or a passport."

a) Represent the statement symbolically

Step 1. Try to rephrase into standard form of "and", "or" or "if-then".
Most common wrong answer: *If you show a driver's license or a passport then you entered Utopia.*
This is wrong because you have had to show one for another reason like getting on a plane.

If you enter Utopia, then you must have shown a driver's license or a passport.
This fits the given statement better. If you want to enter, it is going to be required to show id.

Alternates" It is necessary to enter Utopia that you show DL or P. OR It is necessary that you show DL or P to enter Utopia.

Step 2: What symbols will be involved and do you need ()?
Since If-then type know it will be $* \to *$ but it also has an "or" so will have $* \vee *$

$U \to (D \vee P)$

Alternate answer: You did not enter Utopia or you showed your id. ID here is (D or P)
$\sim U \vee (D \vee P)$

b) Determine its negation.

$\sim[U \to (D \vee P)] \equiv \sim[\sim U \vee (D \vee P)] \equiv \sim\sim U \wedge \sim(D \vee P) \equiv U \wedge (\sim D \wedge \sim P)$

You enter Utopia, and you don't show a driver's license and you don't show an id.
Alternate: You enter Utopia AND you show neither a driver's license or an id.

3) Proof strategies

a) A student is asked to prove the statement "If a function f is continuous on [a,b] then it is integrable on [a,b]" using either proof by contradiction or proof by contraposition. Where do they start? In other words what is the first sentence or two on the students paper.

This is of the form If C then I.
Using proof by contradiction: Assume the negation of goal and find contradiction
Assume $\sim(C \to I) \equiv C \wedge \sim I$ and then find a contradiction.

In English could say. Assume a function is continuous on [a,b] and that it is not integrable……….
Goal would be to find somewhere a statement P and not P that both have to be true which would be impossible so this would be a contradiction.

Using proof by contraposition: Proof the contrapositive which is $\sim I \to \sim C$ instead.
Assume $\sim I$ and show $\sim C$.
In English could say. Assume a function is not integrable on [a,b]……. Goal would be to conclude that it is not continuous.

b) What is a counterexample to the statement " If n is a positive integer and n^2 is divisible by 4 then n is divisible by 4."

A counterexample is something that makes the statement false. So to make an if-then statement false you must find something that makes the hypothesis true and the conclusion false.

Want to find a positive integer n whose square is divisible by 4 but it has the property that it itself is not divisible by 4.

n =1 square is not divisible for 4 so won't be a counterexample.

n =2 has $n^2 = 4$ which is divisible by 4 and the number itself n =2 is not divisible by 4 So this is a counterexample

Answer: n =2 is a counterexample to the statement

4) Show $(P \wedge Q) \to P$ is a tautology.

Two strategies
Method 1: Complete a table and show all trues

P	Q	PandQ	If (PandQ) then P
T	T	T	If T then T True
T	F	F	If F then T True
F	T	F	If F then F True
F	F	F	If F then F True

Conclude: Since all rows of the truth table are "true" then the statement is always true so this is a tautology.

Method 2: Use rules

$(P \wedge Q) \rightarrow P$

$\equiv \left[\neg (P \wedge Q) \vee P \right]$ Using DeMorgan's Laws transform to

$\equiv \left[(\neg P \vee \neg Q) \vee P \right]$ Order of operations are implied by rules so parenthesis not needed

$\equiv \neg P \vee \neg Q \vee P$ Order doesn't matter for ands so rearrange

$\equiv \neg P \vee P \vee \neg Q$

$\equiv (\neg P \vee P) \vee \neg Q$

$\equiv True \vee \neg Q$

$\equiv True$

Since always true it is a tautology.

5) Consider the argument

"Either I wear a red tie or I wear blue socks. I am wearing blue socks. Therefore, I am not wearing a red tie."

a) Translate this argument into propositional form. Make sure to clearly state the meaning of each of your propositions, and use logical connectors and mathematical operators

b) Determine if this argument is valid. Justify your choice.

Comment: Consider the sentence "I want to be either rich or happy." The person is probably not saying wants to be one or the other but not both. "Either" is a soft word in English and content is used to determine whether or not it is an inclusive or and exclusive or.

Let R represent the statement "I wear a red tie."
Let B represent the statement " I wear blue socks."

if interpret as inclusive or

1. $R \vee B$
2. B

$\therefore \sim R$

b) Determine if this argument is valid. Justify your choice.

premises
$(R \vee B) \wedge B$ is true

$\boxed{\text{invalid}}$

if $R = True$
$B = True$

then
$R \vee B = True$
$B = true$

So premises hold but conclusion $\sim R = False$ is false

CAN SHOW INSTEAD
$(A \vee B) \wedge B \Rightarrow \sim A$ is not a tautology

if interpret as exclusive or

1. $R \oplus B$
2. B

$\therefore \sim R$

premises
$(R \oplus B) \wedge B$

$\boxed{\text{Valid}}$

1. $R \oplus B$ premise
2. $[(\neg R \wedge B) \vee (R \wedge \neg B)] \wedge B$ defn
3. $[(\neg R \wedge B) \wedge B] \vee [(R \wedge \neg B) \wedge B]$ distribution
4. $[\neg R \wedge (B \wedge B)] \vee [R \wedge (\neg B \wedge B)]$ Assoc
5. $[\neg R \wedge B] \vee [R \wedge False]$ idempotent, negation
6. $[\neg R \wedge B] \vee False$ Domination
7. $[\neg R \wedge B]$ identity
8. $\neg R$ simplification

6) Prove that $2m^2+3n^2=40$ has no solution for integers m and n.

Method 1: Exhaustive/methodical search.

Try to solve equation. Try to find two integers m and n such that $2m^2+3n^2=40$. Since m and n are real, then $0 \le 2m^2 \to 3n^2 \le 40$ and $0 \le 3n^2 \to 2m^2 \le 40$. But m and n are actually integers so $m \in \{\pm4,\pm3,\pm2,\pm1,0\}$ and $n \in \{\pm3,\pm2,\pm1,0\}$.

sum	$n=\pm3$	$n=\pm2$	$n=\pm1$	$n=0$
$m=\pm4$	59	44	35	32
$m=\pm3$	45	30	21	18
$m=\pm2$	35	20	11	8
$m=\pm1$	29	14	5	2
$m=0$	18	12	3	0

None of these work so there is no solution.

Method 2: Using modular ideas from chapter 4

Assume a solution exists. So there are two integers m and n such that $2m^2+3n^2=40$. If n is odd then so is $3n^2$ since all three factors are odd. But $2m^2$ is even. It is impossible that first term even added to second term odd can give even answer of 40. Thus n is even. Let $n=2k$ for some integer k.

$2m^2+3n^2=40$ become $2m^2+3(2k)^2=40$ which simplifies to $m^2+6k^2=20$. Repeating similar logic m must be even so there exists an integer t such that $m=2t$. Substituting and simplifying gives $2t^2+3k^2=10$. Repeat again and get $k=2u$ and $t^2+6u^2=5$. This then breaks into cases just like above but just a lot fewer. $t \in \{\pm2,\pm1,0\}$ $u \in \{0\}$ none of which work so there is no solution.

Other Methods: There are other ways to do this using more topics from number theory but these were the two we covered.

7) Linear Congruential Method for creating Pseudorandom numbers (numbers generated from a weak random source that are "random enough"): Use four integers satisfying particular inequalities, m, a, c, and x_0, to define an infinite sequence $\{x_n\}_{n=1}^{\infty}$ of these Pseudorandom numbers by $x_{n+1} \equiv (ax_n + c) \bmod m.$ Given $m = 17,\ a = 5,\ c = 2,$ and $x_0 = 3$ to compute the first five numbers generated by this method. Usually choose the Pseudorandom numbers to be between 0 and m-1.

$x_{n+1} \equiv (5x_n + 2) \bmod 17,\quad n \ge 1.\quad x_0 = 3 \to x_1 \equiv 17 \bmod 17 \equiv 0 \bmod 17$ so can use $x_1 = 0$ Since I did not specify to keep it small answers are not unique.

$x_2 \equiv (5x_1 + 2) \bmod 17 \equiv \big((5x_1) \bmod 17 + 2 \bmod 17\big) \bmod 17$

$\equiv \big((5 \bmod 17)(x_1 \bmod 17) \bmod 17 + 2\big) \bmod 17$

$\equiv \big((5)(0) \bmod 17 + 2\big) \bmod 17$

$\equiv \big(0 \bmod 17 + 2\big) \bmod 17$

$\equiv (0 + 2) \bmod 17$

$\equiv (2) \bmod 17$

Continuing in this same vein

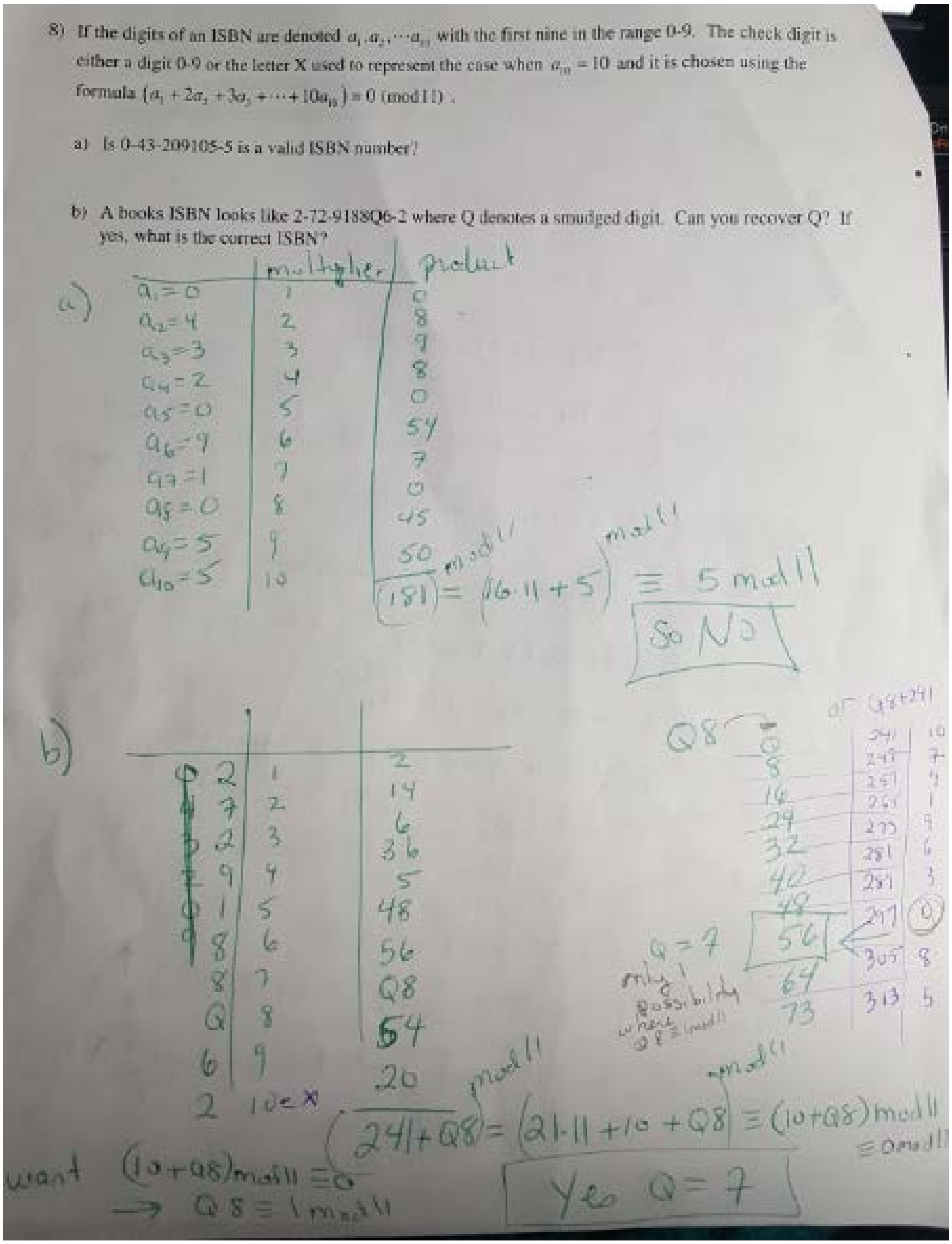
$x_3 \equiv (12) \bmod 17,\ x_4 \equiv (11) \bmod 17,$ and $x_5 \equiv (6) \bmod 17,$

Most typical answer: 3, 0, 2, 12, 11, 6

8) If the digits of an ISBN are denoted $a_1, a_2, \cdots a_{10}$ with the first nine in the range 0-9. The check digit is either a digit 0-9 or the letter X used to represent the case when $a_{10} = 10$ and it is chosen using the formula $(a_1 + 2a_2 + 3a_3 + \cdots + 10a_{10}) \equiv 0 \pmod{11}$.

a) Is 0-43-209105-5 is a valid ISBN number?

b) A books ISBN looks like 2-72-9188Q6-2 where Q denotes a smudged digit. Can you recover Q? If yes, what is the correct ISBN?



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a)

	multiplier	product
$a_1=0$	1	0
$a_2=4$	2	8
$a_3=3$	3	9
$a_4=2$	4	8
$a_5=0$	5	0
$a_6=9$	6	54
$a_7=1$	7	7
$a_8=0$	8	0
$a_9=5$	9	45
$a_{10}=5$	10	50

$181 \overset{\text{mod } 11}{=} (16\cdot 11+5) \overset{\text{mod } 11}{\equiv} 5 \text{ mod } 11$

$\boxed{\text{So NO}}$

b)

2	1	2
7	2	14
2	3	6
9	4	36
1	5	5
8	6	48
8	7	56
Q	8	Q8
6	9	54
2	10 = X	20

$241 + Q8 \overset{\text{mod } 11}{=} (21\cdot 11 + 10 + Q8) \overset{\text{mod } 11}{\equiv} (10+Q8) \text{ mod } 11 \equiv 0 \text{ mod } 11$

want $(10+Q8) \text{ mod } 11 \equiv 0$

$\rightarrow Q8 \equiv 1 \text{ mod } 11$

Q8	or Q8+241	
0	241	10
8	249	7
16	257	4
24	265	1
32	273	9
40	281	6
48	289	3
56	297	0
64	305	8
73	313	5

Q = 7 only possibility where $Q8 \equiv 1 \text{ (mod 11)}$

$\boxed{\text{Yes } Q = 7}$

9) Let $F(x,y,z) = \left(x + y\bar{z} \right)\overline{(yz)}$

a) Use a table to express the values of the Boolean function.
b) Find the disjunctive normal form of F.
c) Use a K-map to find a minimal expansion as a Boolean sum of Boolean products.
d) Construct the circuit of the minimal expansion from inverters, AND gates, and OR gates.

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a)

x	y	z	$\bar{z}$	$y\bar{z}$	$x+y\bar{z}$	yz	$\overline{yz}$	$(x+y\bar{z})\overline{(yz)}$	
1	1	1	0	0	1	1	0	0	
0	1	1	0	0	0	1	0	0	
1	0	1	0	0	1	0	1	1	$x\bar{y}z$
0	0	1	0	0	0	0	1	0	
1	1	0	1	1	1	0	1	1	$xy\bar{z}$
0	1	0	1	1	1	0	1	1	$\bar{x}y\bar{z}$
1	0	0	1	0	1	0	1	1	$x\bar{y}\bar{z}$
0	0	0	1	0	0	0	1	0	

b) $F(x,y,z) = x\bar{y}z + xy\bar{z} + \bar{x}y\bar{z} + x\bar{y}\bar{z}$

c)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		1	1	1
$\bar{x}$		1		

$x\bar{y}$

$y\bar{z}$

$F(x,y,z) = y\bar{z} + x\bar{y}$

d.