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Exam 3 Math 10 20183

Directions: All work must be shown to receive credit. No notes or study aids may be used.

Chapter 6 reminder

- Give a brief explanation of your approach.

Chapter 10 reminder

Let G be an undirected graph and n a nonnegative integer

- *path* of length n from vertex u to vertex v: is a sequence of n edges $e_1, e_2, \ldots e_n$ for which there exists a sequence of n+1 vertices $u = x_0, x_1, \ldots x_n = v$ where edge e_1 has endpoints x_0 and x_1, edge e_2 has endpoints x_1 and x_2, etc
- *circuit or cycle* of length n for a positive integer n: A path from vertex u back to itself using n edges.
- A path or circuit is called *simple* if it does not contain the same edge more than once.

In problems 1 to 7 there are actually 13 problems. Do 10 of them.

1) Words of length 7 over the standard 26 English letters

a) How many words are there if letters can be repeated?

Task 1 choose letter 26 ways ✓

⋮

Task 7 choose letter 26 ways

$\boxed{\text{By Prod. Rule } 26^7 \text{ words}}$

b) How many words are there if letters cannot be repeated?

Task 1 choose letter 26 ways

Task 2 choose letter 25 ways

⋮

Task 7 choose letter 20 ways

$\boxed{\text{By Prod. Rule } 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \text{ words}}$

c) How many words are there that begin with an A or end with a Z?

A $\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{\text{choose 6 letters, 26 ways each}}$ OR $\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{\text{choose 6 letters, 26 ways each}}$ Z

} assume w/ repetition

are you overcounting?

-2

$26^6 + 26^6$

$\boxed{2 \cdot 26^6}$

2) How many permutations of the letters ABCDEFG contain the string BCD?

treat BCD as one object

so count # of 5-permutations no repetition

$\boxed{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

3) How many bit strings of length 12 contain at least three 1's?

Total bit strings length 12 $= 2^{12}$

Need to subtract bit strings w/ 0 1's, 1 1's, and 2 1's

0 1's - 1^{12}

1 1's - $\binom{12}{1}$ place the 1

2 1's - $\binom{12}{2}$ place the 2 1's

$$\boxed{2^{12} - 1 - \binom{12}{1} - \binom{12}{2}}$$

4) Donuts: A store carries 20 different types of donuts. A group of 12 donuts is called a dozen.

a) How many ways are there to choose a dozen so that no two are the same? (all are different)

order doesn't matter, no repetition (combs.)

$$\boxed{\binom{20}{12}}$$

b) How many ways are there to choose a dozen if all the donuts in that dozen are of the same variety?

choose 1 type for the 12 donuts, 20 ways to do this

$$\boxed{\binom{20}{1}}$$

c) How many ways are there to choose a dozen donuts if there are no restriction?

combinations w/ repetition

want # solutions to $x_1 + x_2 + \dots + x_{20} = 12$

19 bars + 12 objects = 31 items

$$\boxed{\binom{31}{12}}$$

inj 1-1

surj onto

5) Counting functions

a) How many functions from an 8-set to a 12-set are there?

Task 1 choose image of 1st element - 12 ways

⋮

$\boxed{\text{By Prod. Rule } 12^8 \text{ functions}}$

Task 8 choose image of 8th element - 12 ways

b) How many onto functions are there from an 8-set to a 12-set?

Let S = 8-set, T = 12-set

$\boxed{0 \text{ b/c } |S| < |T|}$

c) How many injective functions are there from an 8-set to a 12-set?

Task 1 choose image 1st element - 12 ways

Task 2 choose image 2nd element - 11 ways

$\boxed{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}$

⋮

Task 8 choose image 8th element - 5 ways

6) A school has 900 lockers and 3500 students so students must share lockers. Assume that the students are assigned as uniformly as possible to the lockers. The goal is to make the fewest students share a locker. In this case what is the largest number of students that must necessarily share?

Let pigeons be the 3500 students

Let pigeonholes be the 900 lockers

By the Generalized Pigeonhole Principle there must be at least one locker shared by $\left\lceil \frac{3500}{900} \right\rceil = \boxed{4 \text{ students.}}$

7) How many different strings can be made using the letters in the word GOOGLEO?

2 G's

3 O's

1 L

1 E

distinguishable groups

indistinguishable members

distinguishable boxes

$$\boxed{\frac{7!}{2!\,3!\,1!\,1!}}$$

40/40

8) Use a combinatorial argument to prove the identity $\binom{n}{r}\binom{r}{k}=\binom{n}{k}\binom{n-k}{r-k}$ where n, r, and k are nonnegative integers with $k \leq r \leq n$.

We can prove the identity by showing the LHS and RHS count the same set of objects in different ways. In both cases we want to count the # of two disjoint subsets of n objects, the first of size k and the second of size r-k from the remaining objects.

For the RHS, we first choose k objects from n objects to give us the # of ways to choose the first subset. Next we choose r-k objects from the remaining n-k objects to give us the number of ways to choose the second subset.

For the LHS, we first choose r objects from n objects. Then we choose k objects from the r objects we've chosen, which gives us the # of ways to choose the first subset and the # of ways to choose the second subset b/c the remaining r-k objects form th second subset.

Because we count the # of the same two subsets for both the LHS and RHS, this forms a combinatorial proof for the identity.

6

9) Consider an undirected graph G on 5 vertices numbered 1, 2, 3, 4, 5 without loops or parallel edges.

a) If the graph is connected, what is the minimum number of edges of G?

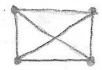
$n - 1 = 5 - 1 = \boxed{4}$ ✓

Scratch paper:

b) What is the maximum number of edges of G?

$4 + 3 + 2 + 1 = \boxed{10}$ ✓

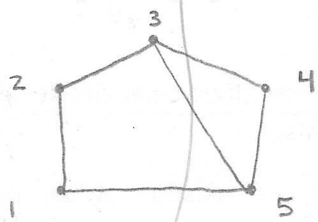
K_4



6 edges

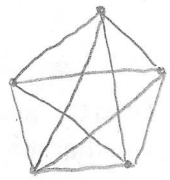
$3 + 2 + 1$

c) Draw an example of a graph G using at least 6 edges. – answers will vary



✓

K_5



$4 + 3 + 2 + 1$

d) Let S ={1,2,3,4,5}. Explain a way to use a subset of SxS to represent the set of edges of your graph.

Let $G = (V, E)$ then we can use ordered pairs $(x, y) \in S \times S$ to represent edges of the graph where $x \in V$ and $y \in V$ and there is an edge from x to y in E. Then $E \subseteq S \times S$.

e) Explain a way to use bit strings to represent the edges of your graph. You must include how long the bit string is and how to determine where the 1's are located, but you need not actually give the bit string.

We can use bit strings of length 5 to represent edges where a 1 in positions i and j means that edge is incident on vertices i and j and the rest of the bits are 0. The string would be 01010 for an edge incident on vertices 2 and 4. Note the positions numbered 1-5. what would be shortest bit string to represent whole G

f) How many simple graphs on these 5 vertices are there?

~~$10 + 1 = \boxed{11}$ b/c the simple graphs can have from 0 to 10 edges~~

$\underbrace{1}_{0 \text{ edges}} + \underbrace{\binom{5}{2}}_{1 \text{ edges}} + \binom{5}{2}^2 + \binom{5}{2}^3 + \binom{5}{2}^4 + \binom{5}{2}^5 + \binom{5}{2}^6 + \binom{5}{2}^7 + \binom{5}{2}^8 + \binom{5}{2}^9 + \underbrace{\binom{5}{2}^{10}}_{10 \text{ edges}}$

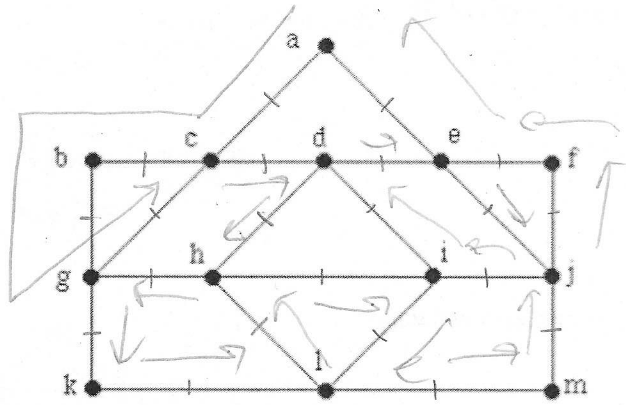
why squared ?

g) If we allow loops or parallel edges, are there finite or infinite number of graphs on these 5 vertices.

Infinite b/c there may be infinite # of loops on a vertex or infinite parallel edges between two vertices.

10/12

10) Consider the graph:



a) Does this graph have an Euler Circuit? Construct such a circuit if one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if it exists.

V	deg (v)
a	2
b	2
c	4
d	4
e	4
f	2
g	4
h	4
i	4
j	4
k	2
l	4
m	2

Graph has an Euler circuit b/c all vertices have even degree.

Will construct using sequence of vertices since graph simple.

a, c, b, g, c, d, h, g, k, l, h, i, l, m, j, i, d, e, j, f, e, a

10

b) Is this graph bipartite? Justify your answer.

13 vertices sum degrees $= 5(2) + 8(4) = 42 = 2m$

$\Rightarrow m = 21$ edges

Bipartite means can split vertex set into V_1, V_2 where no edges between vertices in V_1 or between vertices in V_2

No, b/c contains triangles so for example when you try to assign the vertices b, c, g to V_1 and V_2 you will always end up w/ adjacent vertices in one of the vertex sets, violating the restrictions for bipartite.

11) Suppose that you are responsible for scheduling times for lectures for a university.

Lecture	A	C	G	H	I	L	M	P	S
Astronomy		X	X	X			X		
Chemistry	X								X
Greek	X			X		X	X	X	
History	X		X			X			X
Italian						X	X		X
Latin			X	X	X		X	X	X
Music	X		X		X	X			
Philosophy			X			X			
Spanish		X		X	X	X			

The letter A in the top row is the abbreviation of Astronomy. An X denotes that two lectures have a common student.

a) How can a graph be used to represent this information? Clearly identify what should be used as vertices and how an edge between vertices will be determined. You can just explain what to do; you don't actually have to draw the graph.

We can represent the courses as the vertices. Two vertives have an edge between them if they have a student in common.

b) How can this graph be used to determine the number of scheduling times required? You can just explain what to do; you are not actually required to determine a numeric answer.

Determine the chromatic number, which will give the minimum # of lecture time slots required. If each vertex is assigned a different (lecture time slot) color, there won't be scheduling conflicts for students enrolled in more than one of the courses.

6

80/82

12) For each of the following, either give an example or prove there are none:

a) A simple graph with 6 vertices, whose degrees are 2, 2, 2, 3, 4, 4.

Sum degrees $= 2+2+2+3+4+4 = 6+3+8 = 17$

None b/c the sum of the degrees is odd.

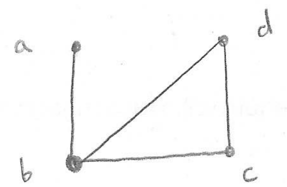
b) A simple graph with 8 vertices, whose degrees are 0, 1, 2, 3, 4, 5, 6, 7.

None b/c a vertex w/ degree 7 must have an edge to every other vertex so impossible to have a vertex w/ degree 0.

Since simple & only 7 other vertices

c) A simple graph with degrees 1, 2, 2, 3.

Sum degrees $= 8 = 2m \Rightarrow m = 4$ edges



v	deg (v)
a	1
b	3
c	2
d	2

8/8