
1: Canada has a two dollar coin known as the "toonie." What is wrong with the following argument, which purports to prove (by induction) that any debt of $n > 1$ Canadian dollars can be repaid (exactly) with only toonies?

Proof:

Step 1. This argument starts with $N = 2$. Notice that a two dollar debt can be repaid with a single toonie. Thus, the assertion is true for $n = 2 = N$.

Step 2: Now let $k \geqslant 2$ and suppose that the assertion is true for all l, $2 \leqslant l < k$. The goal is to show that the assertion is true for $n = k$. For this, apply the induction hypothesis to $k - 2$ and see that a $(k-2)$-dollar debt can be repaid with toonies. Adding one more toonie allows one to repay k dollars with only toonies, as required. By the Principal of Mathematical Induction, any debt of $n > 1$ dollars can be repaid with toonies.

Answer:

The problem with this argument is that in the inductive step, $l = k - 2$ is not necessarily in the range $2 \leqslant l < k$. For example, if $k = 3$, then $k - 2 = 3 - 2 = 1$ and the inductive hypothesis cannot be applied.

2: Use Mathematical induction to prove $3^{2n} - 1$ is divisible by 8 for every $n \geqslant 1$.

Answer:

To construct the proof, let $P(n)$ denote the proposition: "$3^{2n} - 1$ is divisible by 8".

(BASIS STEP:)

The statement $P(1)$ is true because $3^{2(1)} - 1 = 9 - 1 = 8$ is divisible by 8. This completes the basis step.

(INDUCTIVE STEP:)

For the inductive hypothesis we assume that $P(k)$ is true. That is, we assume $3^{2k} - 1$ is divisible by 8 for an arbitrary positive integer k. To complete the inductive step, we must show that when we assume the inductive hypothesis, it follows that $P(k+1)$, the statement that $3^{2(k+1)} - 1$ is divisible by 8, is also true. That is, we must show that $3^{2(k+1)} - 1$ is divisible by 8.

Note that:

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= 3^2 \cdot 3^{2k} - 1 \\ &= 9 \cdot 3^{2k} - 1 \\ &= 8 \cdot 3^{2k} + 3^{2k} - 1 \end{aligned}$$

We can now use the inductive hypothesis and parts (i) and (ii) of Theorem 1 from Section 4.1. By part (ii) of the theorem, we conclude the first term in this last sum is divisible by 8. By the inductive hypothesis, $3^{2k} - 1$ is divisible by 8. Hence, by part (i) of the theorem, we conclude that $8 \cdot 3^{2k} + 3^{2k} - 1 = 3^{2(k+1)} - 1$ is divisible by 8. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $3^{2n} - 1$ is divisible by 8 for every $n \geqslant 1$.