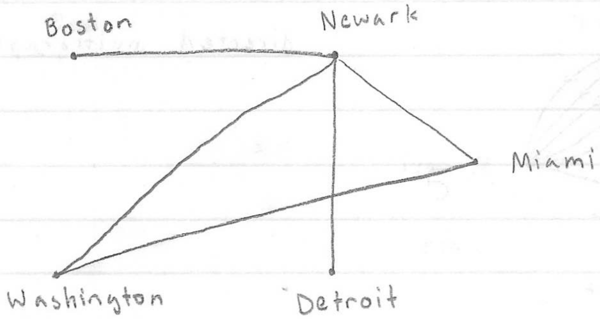
Stewart Dulaney
MATH 10
Section 2838
SID: 1545566

OT 15

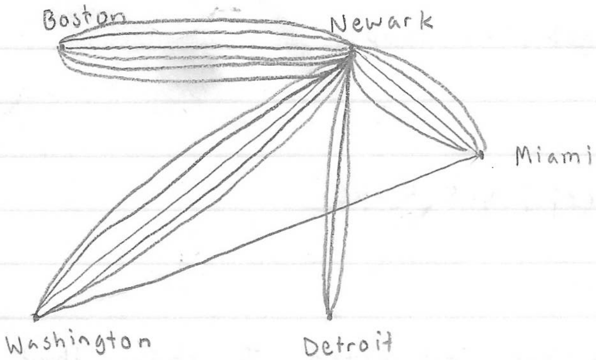
Textbook Ch 10 Problems

10.1 1. a)



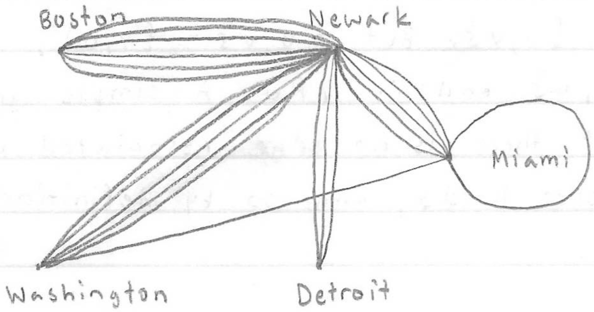
simple graph

b)



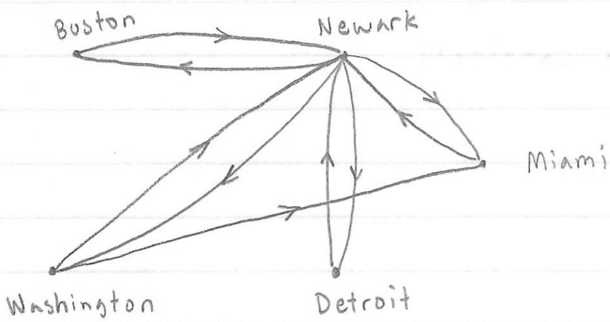
multigraph

c)



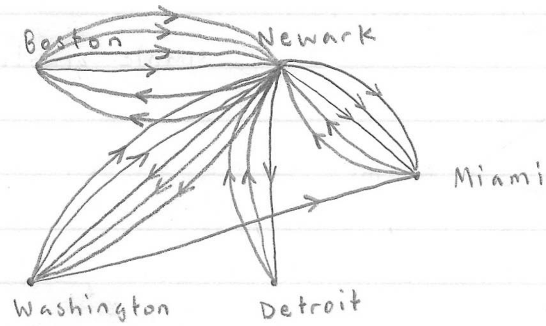
pseudograph

d)



directed graph

e)



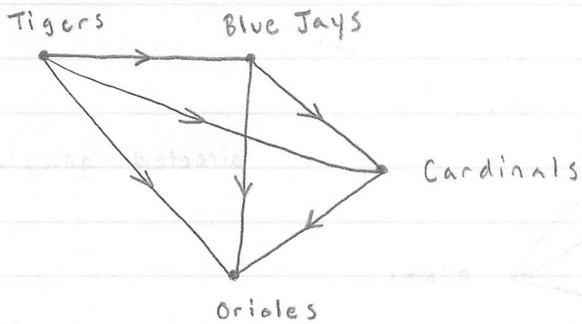
directed multigraph

3. undirected, no multiple edges, no loops
$\therefore$ simple graph

5. undirected, yes multiple edges, yes loops
$\therefore$ pseudograph

11. In a simple graph, edges are undirected. To show that R is symmetric we must show that if uRv, then vRu. If uRv, then there is an edge associated with $\{u,v\}$. But $\{u,v\} = \{v,u\}$, so this edge is associated w/ $\{v,u\}$ and $\therefore$ vRu. A simple graph does not allow loops, that is if there is an edge associated w/ $\{u,v\}$, then $u \neq v$. Thus uRu never holds, and so by definition R is irreflexive.

21.



27. The vertices represent the people at the party. B/c it is possible that a knows b's name but not vice versa, we need a directed graph. We will include an edge associated with (u,v) if and only if u knows v's name. There is no need for multiple edges (either a knows b's name or he doesn't). One could argue that we should not clutter the model w/ loops, b/c obviously everyone knows her own name. On the other hand, it certainly would not be wrong to include loops, especially if we took the instructions literally.

26. The vertices represent e-mail addresses in a network. It would be a directed graph with an edge associated w/ $(u,v) \iff u$ has sent an e-mail to v. You'd expect mass e-mail lists to have a distinctive connectivity pattern, you'd have a single address with an edge from it to thousands of others of vertices instead of a more typical point to point pattern.

29. We should use a directed graph, w/ the vertices being the courses and the edges showing the prerequisite relationship. Specifically, an edge from u to v means that course u is a prerequisite to course v. Courses that do not have any prerequisites are the courses with in-degree 0, and courses that are not the prerequisite for any other courses have out-degree 0.

10.2 1. # vertices = 6

edges = 6

v	deg (v)	
a	2	
b	4	
c	1	pendant vertex
d	0	isolated vertex
e	2	
f	3	

3. # vertices = 9

edges = 12

v	deg(v)	
a	3	
b	2	
c	4	
d	0	isolated
e	6	
f	0	isolated
g	4	
h	2	
i	3	

$3 + 2 + 4 + 0 + 6 + 0 + 4 + 2 + 3$

$5 + 10 + 6 + 3$

$15 + 9$

$24 = 2 \cdot$ # edges ✓

5. By theorem 2 the # of vertices of odd degree must be even for an undirected graph. Hence there cannot be a graph w/ 15 vertices of odd degree 5.

7. # vertices = 4

edges = 7

v	$deg^-(v)$	$deg^+(v)$
a	3	1
b	1	2
c	2	1
d	1	3

As a check we see that the sum of the in-degrees and the sum of the out-degrees are equal (both are equal to 7).

13. Since a person is joined by an edge to each of his or her collaborators, the degree of v is the # of collaborators v has. Similarly, the neighborhood of a vertex is the set of coauthors of the person represented by that vertex. An isolated vertex represents a person who has no coauthors (he or she has published only single-authored papers), and a pendant vertex represents a person who has published w/ just one other person.

We can use a proof by contradiction.

18. Let $G(V, E)$ be a simple graph w/ at least two vertices and $|V| = n$ where $n \geq 2$.

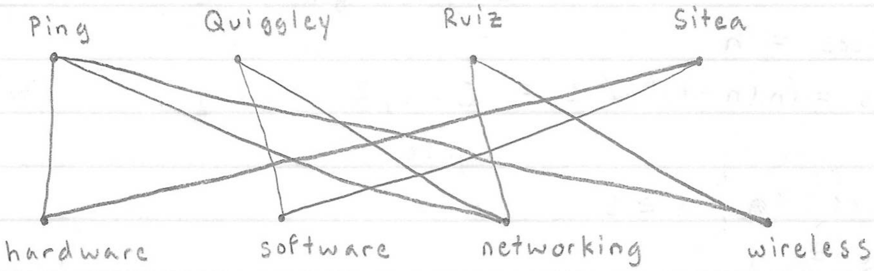
□ other methods to prove this? induction?

If all the degrees of the vertices are different they must be exactly $\{0, 1, 2, \ldots, n-1\}$ b/c each vertex has degree $\leq n-1$. But it is impossible to have a vertex of degree 0 (connected to no other vertex) and one of degree $n-1$ (connected to every other vertex) simultaneously. $\Rightarrow \Leftarrow$

$\therefore$ There must be two vertices in G w/ the same degree.

19. Model the friendship relation w/ a simple undirected graph in which the vertices are people in the group, and two vertices are adjacent if those two people are friends. The degree of a vertex is the # of friends in the group that person has. By Exercise 18, there are two vertices w/ the same degree, which means that there are two people in the group w/ the same # of friends in the group.

27. a)



The bipartite graph has vertices h, s, n, and w representing the support areas and P, Q, R, and S representing the employees. The qualifications are modeled by the bipartite graph w/ edges Ph, Pn, Pw, Qs, Qn, Rn, Rw, Sh, and Ss.

b) $V_1 = \{P, Q, R, S\}$
$V_2 = \{h, s, n, w\}$

Hall's Marriage Theorem
complete matching $\Leftrightarrow$ $|N(A)| \geq |A|$ for all subsets A of V_1

□ ask for diff. explanation

Since every vertex representing an employee? area has at least degree 2, the condition in Hall's theorem is satisfied for sets of size less than 3. We can easily check that the # of employees qualified for each of the four subsets of size 3 is at least 3, and clearly the # of employees qualified for each of the subsets of size 4 has size 4.

c) The answer is not unique; one complete matching is $\{Ph, Qn, Rw, Ss\}$ which is easily found by inspection.

31. We model this w/ an undirected bipartite graph, w/ the men and the women represented by the vertices in the two parts and an edge between two vertices if they are willing to marry each other. By Hall's theorem, it is enough to show that for every set S of women, the set $N(S)$ of men willing to marry them has cardinality at least $|S|$. A clever way to prove this is by counting edges. Let m be the # of edges between S and $N(S)$. Since every vertex in S has degree k, it follows that $m = k|S|$. B/c these edges are incident to $N(S)$, it follows that $m \leq k|N(S)|$. Combining these two facts gives $k|S| \leq k|N(S)|$, so $|N(S)| \geq |S|$ as desired.

35. a) K_n complete graph

vertices $= n$

edges $= (n(n-1))/2 = C(n,2) = {}_nC_2$

b) C_n cycle

vertices $= n$, $n \geq 3$

edges $= n$

c) W_n wheel

vertices $= n+1$, $n \geq 3$

edges $= n + n = 2n$

d) $K_{m,n}$ complete bipartite graph

vertices $= m + n$

edges $= mn$

e) Q_n n-dimensional hypercube, or n-cube

vertices $= 2^n$

edges $= \dfrac{n2^n}{2} = n2^{n-1}$

Q_1	Q_2	Q_3
1	4	12

Each vertex has degree n since there are n strings that differ from any given string in exactly one bit (any one of the n different bits can be changed). Thus sum of the degrees in $n2^n$. By handshaking theorem, # edges is

Degree sequence:

36. 21)

v	deg(v)
a	1
b	1
c	1
d	1
e	4

4, 1, 1, 1, 1

22)

v	deg(v)
a	3
b	2
c	3
d	2
e	2

3, 3, 2, 2, 2

23)

v	deg(v)
a	3
b	3
c	4
d	2
e	2
f	2

4, 3, 3, 2, 2, 2

24)

v	deg(v)
a	2
b	2
c	4
d	2
e	2
f	4

4, 4, 2, 2, 2, 2

25)

v	deg(v)
a	2
b	3
c	2
d	3
e	3
f	3

3, 3, 3, 3, 2, 2

graphic

42\. a) 5, 4, 3, 2, 1, 0

Not graphic b/c sum is 15 which not multiple of 2 req'd by handshake theorem

42\. b) 6, 5, 4, 3, 2, 1

$6 + 5 + 4 + 3 + 2 + 1$

$11 + 7 + 3$

$18 + 3$

21

Not graphic b/c sum is 15 which not multiple of 2 req'd by handshake theorem

☐ ask if have to know this 45.

□ ask if 46.
have to know this

64. Let $G = (V, E)$ be a bipartite graph

□ ask about book sol that finds the max of $f(x)$

Let $V = V_1 \cup V_2$ where V_1, V_2 are disjoint sets

We know that the maximum # of edges $|V_1| \cdot |V_2|$ will occur when $|V_1|$ and $|V_2|$ are closest in value.

<u>Case 1:</u> $|V| = v$ is even

Then let $|V_1| = |V_2| = \frac{v}{2}$

so max. # of possible edges $= \left(\frac{v}{2}\right)^2 = \frac{v^2}{4}$

$\rightarrow e \leq \frac{v^2}{4}$

<u>Case 2:</u> $|V| = v$ is odd

Then let $|V_1| = \frac{v+1}{2}$ and $|V_2| = \frac{v-1}{2}$

so max. # of possible edges $= \left(\frac{v+1}{2}\right)\left(\frac{v-1}{2}\right)$

$\rightarrow e \leq \left(\frac{v+1}{2}\right)\left(\frac{v-1}{2}\right) = \frac{v^2-1}{4} \leq \frac{v^2}{4}$

$\rightarrow e \leq \frac{v^2}{4}$

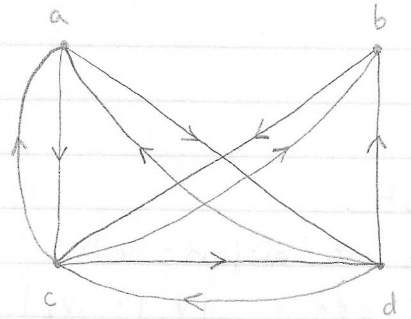
10.3 3. Adjacency List: (directed graph)

Initial Vertex	Terminal Vertices
a	a, b, c, d
b	d
c	a, b
d	d, c, b

11.

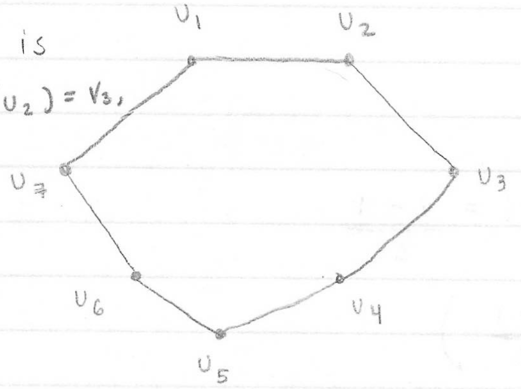
$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & & & & \\ b & & & & \\ c & & & & \\ d & & & & \end{array}$$

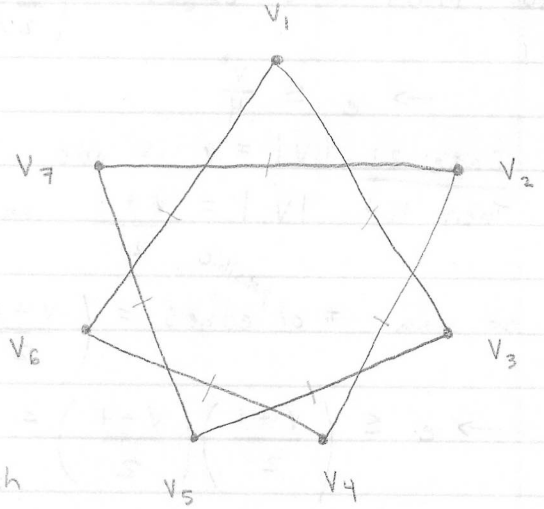
$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



37. Determine whether pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

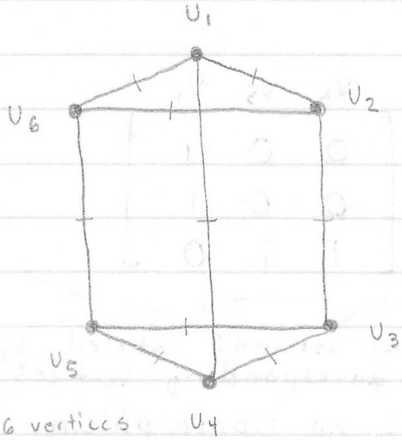
Formally, one isomorphism is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_7$, $f(u_5) = v_2$, $f(u_6) = v_4$, $f(u_7) = v_6$.

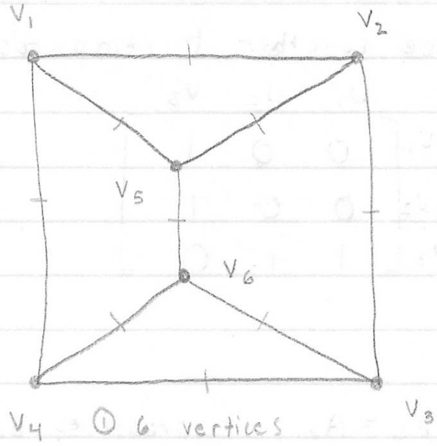




These graphs are isomorphic b/c each has the same # of vertices (7), same # of edges (7), and # of vertices of each degree (7 vertices w/ degree 2). Each is the 7-cycle (C_7).

39.





① 6 vertices

② 9 edges

③

v	$\deg(v)$
u_1	3
u_2	3
u_3	3
u_4	3
u_5	3
u_6	3

① 6 vertices

② 9 edges

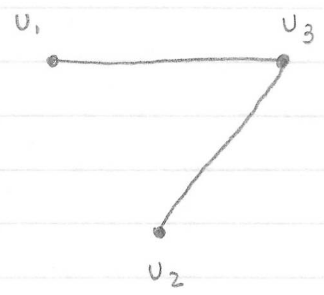
③

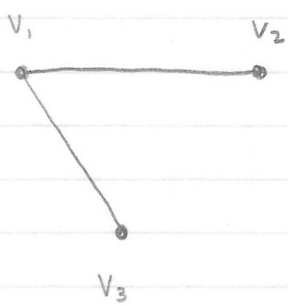
v	$\deg(v)$
v_1	3
v_2	3
v_3	3
v_4	3
v_5	3
v_6	3

These two graphs are isomorphic. One can see this visually — just imagine "moving" vertices u_1 and u_4 into the inside of the rectangle, thereby obtaining the picture on the right. Formally, one isomorphism is $f(u_1) = v_5$, $f(u_2) = v_2$, $f(u_3) = v_3$, $f(u_4) = v_6$, $f(u_5) = v_4$, and $f(u_6) = v_1$.

57. a)
$$\underset{A_G}{\begin{array}{c|ccc} & u_1 & u_2 & u_3 \\ \hline u_1 & 0 & 0 & 1 \\ u_2 & 0 & 0 & 1 \\ u_3 & 1 & 1 & 0 \end{array}}, \quad \underset{A_H}{\begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & 0 & 1 & 1 \\ v_2 & 1 & 0 & 0 \\ v_3 & 1 & 0 & 0 \end{array}}$$

Both graphs consist of two sides of a triangle, they are clearly isomorphic. Formally, one possible isomorphism is $f(u_1) = v_2$, $f(u_3) = v_1$, $f(u_2) = v_3$.





To see whether f compares edges:

$$A_G = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \end{array} \begin{array}{c} \begin{array}{ccc} U_1 & U_2 & V_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \qquad A_H = \begin{array}{c} \\ V_2 \\ V_3 \\ V_1 \end{array} \begin{array}{c} \begin{array}{ccc} V_2 & V_3 & V_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

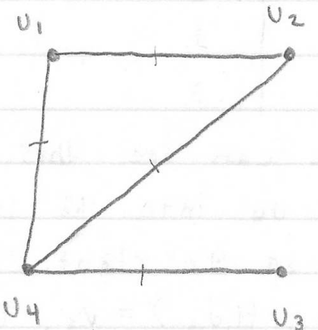
rows and columns labeled by the images of the corresponding vertices in G

B/c $A_G = A_F$, f preserves edges. $\therefore$ f is an isomorphism.

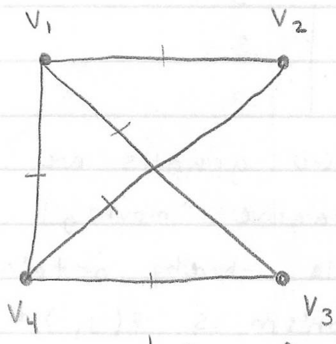
b)

$$A_G = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{array} \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad A_H = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

G



H



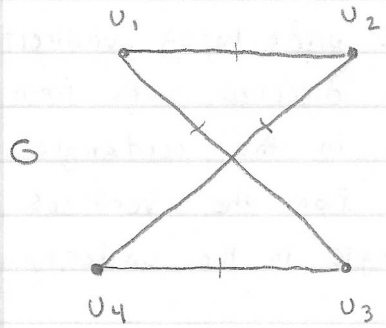
v	deg (v)
U_1	2
U_2	2
U_3	1
U_4	3

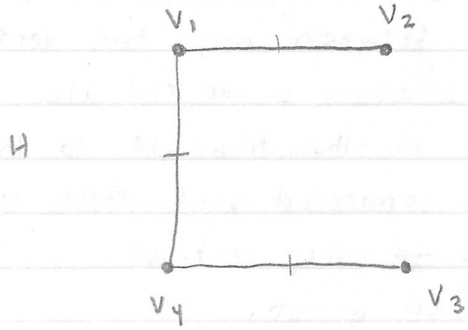
v	deg (v)
V_1	3
V_2	2
V_3	2
V_4	3

$\therefore$ not an isomorphism b/c G has a vertex of degree 1, U_3, while H does not. Also G has 4 edges and H has 5 edges.

c)

$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad A_H = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$





$\therefore$ Not isomorphic b/c G has 4 edges but H has 3 edges.

10.4 1. a) yes this is a path
not simple
not a circuit
length = 4

b) not a path, since there is no edge from c to a

c) not a path, since there is no edge from b to a

d) yes this is a path
yes simple (since no edge is repeated)
yes circuit (since it ends at the same vertex at which it began)
length = 5 (it has 5 edges in it)

3. This graph is Not connected - it has three components

5. This graph is not connected. There is no path from the vertices in one of the triangles to the vertices in the other.

Notice there is no path from a to any other vertex, b/c both edges involving a are directed toward a.

11. a) Not strongly connected b/c there is no directed path from a to b in this graph. However, this graph is weakly connected b/c there is a path between any two vertices in the underlying undirected graph. Notice there is no path from c to any other vertex, b/c both edges involving c are directed toward c.

b) Not strongly connected b/c there is no directed path from c to d in this graph. However, this graph is weakly connected b/c there is a path between any two vertices in the underlying undirected graph.

c) Not strongly connected b/c there is no directed path from the vertices in the triangle to the vertices in the rectangle. Not weakly connected since there is no path from the vertices in the triangle to the vertices in the rectangle in the underlying undirected graph.

Notice the underlying undirected graph is clearly not connected (one component has vertices b, f, and e), so this graph is neither strongly nor weakly connected.

26. a) 0

b) cfed

cbad

cbed

3

c) cbead

cbaed

cfbed

cfbad

cfead

5

d) cbfead

⋮

see next pg using adjacency matrices

26 cont'd.

$$A = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 1 & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 & 1 & 0 \\ e & 1 & 1 & 0 & 1 & 0 & 1 \\ f & 0 & 1 & 1 & 0 & 1 & 0 \end{array}$$

a)

$$A^2 = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 3 & 1 & 2 & 1 & 2 & 2 \\ b & 1 & 4 & 1 & 3 & 2 & 2 \\ c & 2 & 1 & 3 & 0 & 3 & 1 \\ d & 1 & 3 & 0 & 3 & 1 & 2 \\ e & 2 & 2 & 3 & 1 & 4 & 1 \\ f & 2 & 2 & 1 & 2 & 1 & 3 \end{array}$$

Since the $(3,4)^{th}$ entry is 0, there are no paths of length 2.

b) The $(3,4)^{th}$ entry of A^3 turns out to be 8, so there are 8 paths of length 3.

c) The $(3,4)^{th}$ entry of A^4 turns out to be 10, so there are 10 paths of length 4.

d) The $(3,4)^{th}$ entry of A^5 turns out to be 73, so there are 73 paths of length 5.

e) The $(3,4)^{th}$ entry of A^6 turns out to be 160, so there are 160 paths of length 6.

f) The $(3,4)^{th}$ entry of A^7 turns out to be 739, so there are 739 paths of length 7.

28. Show that every connected graph w/ n vertices has at least n-1 edges.

Let P(n): every connected graph w/ n vertices has at least n-1 edges
We show this by induction on n.

Basis Step: For $n = 1$ there is nothing to prove. $\Rightarrow$ basis step true

Inductive Step: Inductive hypothesis: Assume P(k), that every connected graph w/ k vertices has at least k-1 edges, where $k \geq 1$
Goal to show P(k+1):
We will do so using a proof by contradiction.
Let G be a connected graph w/ k+1 vertices and fewer than k edges, where $k \geq 1$.
Since the sum of the degrees of the vertices of G is equal to two times the # of edges, we know that the sum of the degrees is less than 2k, which is less than $2(k+1)$. $\therefore$ some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1. Remove this vertex and its edge. Clearly the result is still connected, and it has k vertices and fewer than k-1 edges, contradicting the inductive hypothesis. Therefore the statement holds for G. Thus $P(k) \Rightarrow P(k+1)$. The result follows by the principle of mathematical induction.

10.5

1. Since there are four vertices of odd degree (a, b, c, and e) and $4 > 2$, this graph has neither an Euler circuit nor an Euler path.

2.

v	deg(v)
a	2
b	4
c	2
d	4
e	4
f	4
g	2
h	4
i	2

This graph has at least two vertices and each of its vertices has even degree so it has an Euler circuit.

3.

v	deg(v)
a	3
b	4
c	4
d	3
e	6

This graph has an Euler path but not an Euler circuit b/c it has exactly two vertices of odd degree.

4.

v	deg(v)
a	4
b	4
c	3
d	4
e	4
f	3

This graph has an Euler path but not an Euler circuit b/c it has exactly two vertices of odd degree.

5.

v	deg (v)
a	4
b	4
c	4
d	4
e	6

This graph has an Euler circuit b/c it has at least two vertices and each of its vertices has even degree.

6.

v	deg (v)
a	4
b	3
c	3
d	6
e	2
f	2
g	2
h	2
i	6

This graph has an Euler path but not an Euler circuit b/c it has exactly two vertices of odd degree.

7.

v	deg (v)
a	4
b	4
c	6
d	4
e	4
f	2
g	4
h	4
i	4

This graph has an Euler circuit b/c it has at least two vertices and each of its vertices has even degree.