Importance Sampling for Option Pricing

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January 22, 2016

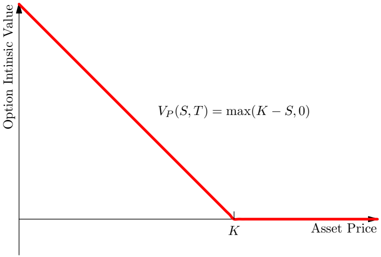
Outline

Put Option

A put option is the right to sell an asset at an established price at a certain time.

The established price is the strike price, K. The certain time is the exercise time T.

At the exercise time, the value of the put option is a piecewise linear, decreasing function of the asset value.



What is the price?

For an asset with a random value at exercise time:
What is the price to buy a put option before the exercise time?

Six factors affect the price of a asset option:

- the current asset price S;
- the strike price K;
- the time to expiration $T - t$ where T is the expiration time and t is the current time;
- the volatility of the asset price;
- the risk-free interest rate; and
- (the dividends expected during the life of the option.)

Geometric Brownian Motion

How do asset prices vary randomly?

Approximate answer is Geometric Brownian Motion: Stock prices can be mathematically modeled with a stochastic differential equation

$$\mathrm{d}S(t) = rS \, \mathrm{d}t + \sigma S \, \mathrm{d}W(t), \quad S(0) = S_0.$$

The solution of this stochastic differential equation is Geometric Brownian Motion:

$$S(t) = S_0 \exp((r - \frac{\sigma^2}{2})t + \sigma W(t)).$$

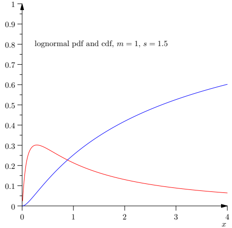
Simplest case

$$S(t) = \mathrm{e}^{W(t)}.$$

Log-Normal Distribution

At time t Geometric Brownian Motion has a lognormal probability density with parameters $m = (\ln(S_0) + rt - \frac{1}{2}\sigma^2 t)$ and $s = \sigma\sqrt{t}$.

$$f_X(x; m, s) = \frac{1}{\sqrt{2\pi}s} \exp\left(\frac{-1}{2} \left[\frac{\ln(x) - m}{s} \right]^2 \right).$$



Statistics of Log-Normal Distribution

The mean stock price at any time is

$$\mathbb{E}\left[S(t)\right] = S_0 \exp(rt).$$

The variance of the stock price at any time is

$$\operatorname{Var}\left[S(t)\right] = S_0^2 \exp(2rt)[\exp(\sigma^2 t) - 1].$$

Monte Carlo Sample Mean

Assume a security price is modeled by Geometric Brownian Motion, with lognormal pdf.

Draw n (pseudo-)random numbers $x_1, \dots, x_n$ from the lognormal distribution modeling the stock price S.

Approximate a put option price as the (present-value of the) expected value of the function $g(x) = \max(K - x, 0)$, with the **sample mean**

$$V_P(S,t) = \mathrm{e}^{-r(T-t)} \mathbb{E}\left[g(S)\right] \approx \mathrm{e}^{-r(T-t)} \left[\frac{1}{n} \sum_{i=1}^{n} g(x_i) \right] = \mathrm{e}^{-r(T-t)} \bar{g}_n .$$

Central Limit Theorem

The Central Limit Theorem implies that the sample mean $\bar{g}_n$ is approximately normally distributed with mean $\mathbb{E}\left[g(S)\right]$ and variance $\operatorname{Var}\left[g(S)\right]/\sqrt{n}$,

$$\bar{g}_n \sim N(\mathbb{E}\left[g(S)\right], \operatorname{Var}\left[g(S)\right]).$$

Recall that for the standard normal distribution $\mathbb{P}\left[|Z| < 1.96\right] \approx 0.95$

A 95% confidence interval for the estimate $\bar{g}_n$ is

$$\left(\mathbb{E}\left[g(S)\right] - 1.96\frac{\operatorname{Var}\left[g(S)\right]}{\sqrt{n}}, \mathbb{E}\left[g(S)\right] + 1.96\frac{\operatorname{Var}\left[g(S)\right]}{\sqrt{n}}\right).$$

Estimating the Mean and Variance

A small problem with obtaining the confidence interval: The mean $\mathbb{E}\left[g(S)\right]$ and the variance $\operatorname{Var}\left[g(S)\right]$ are both unknown.

These are respectively estimated with the sample mean $\bar{g}_n$ and the sample variance

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (g(x_i) - \bar{g}_n)^2$$

Using Student's t-distribution

The sample quantity

$$\frac{(\bar{g}_n - \mathbb{E}\left[g(X)\right])}{s/\sqrt{n}}$$

has a probability distribution known as the Student's t-distribution, so the 95% confidence interval limits of ± 1.96 must be modified with the corresponding 95% confidence limits of the appropriate Student-t distribution.

Monte Carlo Confidence Interval

The **95% level Monte Carlo confidence interval** for $\mathbb{E}\left[g(X)\right]$

$$\left(\bar{g}_n - t_{n-1,0.975} \frac{s}{\sqrt{n}}, \bar{g}_n + t_{n-1,0.975} \frac{s}{\sqrt{n}} \right).$$

Example

Confidence interval estimation to calculate a simplified put option price for a simplified security. The simplified security has a

- risk-free interest rate $r = \sigma^2/2$,
- a starting price $S = 1$,
- a standard deviation $\sigma = 1$.
- $K = 1$,
- time to expiration is $T - t = 1$.

$$V_P(S,t) = \mathrm{e}^{-r(T-t)} \int_0^\infty \max(0, K - x) \mathbb{P}\left[\mathrm{e}^{W(T-t)} \in \mathrm{d}x\right]$$

R Program for Estimation

```
#+name Rexample

n <- 10000
S <- 1
sigma <- 1
r <- sigma^2/2
K <- 1
Tminust <- 1
x <- rlnorm(n)  #Note use of default meanlog=0,
y <- sapply(x, function(z) max(0, K - z ))
t.test(exp(-r*Tminust) * y)  # all simulation re

One Sample t-test

data:  exp(-r * Tminust) * y
```

Problems with Monte Carlo

Applying Monte Carlo estimation to a random variable with a large variance creates a confidence interval that is correspondingly large.

Increasing the sample size, the reduction is $\frac{1}{\sqrt{n}}$.

Variance reduction techniques increase the efficiency of Monte Carlo estimation. Reduce variability with a given number of sample points, or for efficiency achieve the same variability with fewer sample points.

Importance Sampling

Importance sampling is a variance reduction technique.

Some values in a simulation have more influence on the estimation than others. The probability distribution is carefully changed to give "important" outcomes more weight. If "important" values are emphasized by sampling more frequently, then the estimator variance can be reduced.

The key to importance sampling is to choose a new sampling distribution that "encourages" the important values.

Choosing a new PDF

Let $f(x)$ be the density of the random variable, so we are trying to estimate

$$\mathbb{E}\left[g(x)\right] = \int_{\mathbb{R}} g(x) f(x) \, \mathrm{d}x \,.$$

We will attempt to estimate $\mathbb{E}\left[g(x)\right]$ with respect to another strictly positive density $h(x)$. Then easily

$$\mathbb{E}\left[g(x)\right] = \int_{\mathbb{R}} g(x) \frac{f(x)}{h(x)} h(x) \, \mathrm{d}x \,.$$

or equivalently, we are now trying to estimate

$$\mathbb{E}_Y\left[\frac{g(Y)f(Y)}{h(Y)}\right] = \mathbb{E}_Y\left[\tilde{g}(Y)\right]$$

where Y is a new random variable with density $h(y)$.

Reducing the variance

For variance reduction, determine a new density $h(y)$ so $\operatorname{Var}_Y[\tilde{g}(Y)] < \operatorname{Var}_X[g(X)]$.

Consider

$$\begin{aligned}\operatorname{Var}[\tilde{g}(Y)] &= \mathbb{E}\left[\tilde{g}(Y)^2\right] - (\mathbb{E}[\tilde{g}(Y)])^2 \\ &= \int_{\mathbb{R}} \frac{g^2(x) f^2(x)}{h(x)} \, \mathrm{d}x - \mathbb{E}[g(X)]^2 .\end{aligned}$$

By inspection, we can see that we can make $\operatorname{Var}[\tilde{g}(Y)] = 0$ by choosing $h(x) = g(x)f(x)/\mathbb{E}[g(X)]$. This is the ultimate variance reduction.

Need $\mathbb{E}[g(X)]$, what we are trying to estimate!

Educated Guessing

Importance sampling is equivalent to a change-of-measure from $\mathbb{P}$ to $\mathbb{Q}$ with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{f(x)}{h(x)}$$

Choosing a good importance sampling distribution requires educated guessing. Each instance of importance sampling depends on the function and the distribution.

Trivial Parameters

Calculate confidence intervals for a Monte Carlo estimate of a European put option price, where $g(x) = \max(K - x, 0)$ and S is distributed as a Geometric Brownian Motion.

To keep parameters simple

- risk free interest rate $r = \sigma^2/2$, the
- standard deviation $\sigma = 1$,
- the strike price $K = 1$ and
- time to expiration is 1

The quantity to estimate

$$\int_0^\infty \max(0, 1 - x)\, \mathbb{P}[\mathrm{e}^W \in dx]$$

$$= \int_0^\infty \max(0, 1 - x) \frac{1}{\sqrt{2\pi}\sigma x \sqrt{T}} \exp(\frac{-\ln(x)^2}{2T}\, \mathrm{d}x\,.$$

First Change of variable

Want

$$\int_0^\infty \max(0, 1-x)\, \mathbb{P}[e^{W(1)} \in dx].$$

After a first change of variable the integral is

$$\mathbb{E}\left[g(S)\right] = \int_{-\infty}^{0} (1 - e^{x}) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

Another Change of variable

$x = -\sqrt{y}$ for $x < 0$. Then $\mathrm{d}x = \frac{\mathrm{d}y}{2\sqrt{y}}$ and the expectation integral becomes

$$\int_0^\infty \frac{1 - \mathrm{e}^{-\sqrt{y}}}{\sqrt{2\pi}\sqrt{y}} \frac{\mathrm{e}^{-y/2}}{2} \, \mathrm{d}y \, .$$

Comparative results

	n	putestimate	confintleft	confintright
B-S		0.14461		
MC	100	0.15808	0.12060	0.19555
MC	1000	0.15391	0.14252	0.16531
MC	10000	0.14519	0.14167	0.14871
Norm	100	0.13626	0.099759	0.17276
Norm	1000	0.14461	0.133351	0.15587
Norm	10000	0.14234	0.138798	0.14588
Exp	100	0.14388	0.13614	0.15163
Exp	1000	0.14410	0.14189	0.14631
Exp	10000	0.14461	0.14392	0.14530

A real example

Put option on S & P 500, SPX131019P01575000

- $S = 1614.96$
- $r = 0.8\%$ (estimated, comparable to 3 year and 5 year T-bill rate)
- $\sigma = 18.27\%$ (implied volatility)
- $T - t = 110/365$ (07/01/2013 to 10/19/2013)
- $K = 1575$
- $n = 10{,}000$

Quoted Price: $44.20

Results

Method	Value	Confidence Interval
Black-Scholes	44.21273	
MonteCarlo	45.30217	(43.84440, 46.75994)
Importance Sample	44.13482	(43.91919, 44.35045)