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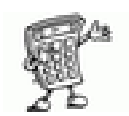
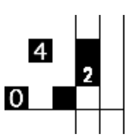
Topics in
Probability Theory and Stochastic Processes
Steven R. Dunbar

Lévy's Construction of Brownian Motion



Study Tip



Rating

Mathematicians Only: prolonged scenes of intense rigor.

Section Starter Question

If Z_1 and Z_2 are independent $N(0, \sigma^2)$ random variables, then what can be said about $Z_1 + Z_2$ and $Z_1 - Z_2$?

Key Concepts

1. Brownian Motion can be defined in a natural way by first constructing a set of independent normal random variables on the dyadic rationals and then combining them inductively to have the right incremental distributions for Brownian Motion.

2. The resulting construction of a continuous function B_t on [0,1] has the properties:

   (a) $B_0 = 0$,

   (b) for all $t_1 < t_2 < \dots t_{k-1} < t_k$ in $[0, 1]$, $B_{t_k} - B_{t_{k-1}}, \dots B_{t_2} - B_{t_1}$ are independent with joint distribution $(N(0, t_k - t_{k-1}), \dots, N(0, t_2 - t_1)$,

   (c) and almost surely $t \mapsto B_t$ is continuous on $[0, 1]$.

Vocabulary

1. A one-dimensional **standard Brownian Motion** is a stochastic process $B : \Omega \times [0, \infty) \to \mathbb{R}$ (alternatively a collection of random variables (indexed by $x \in [0, \infty)$) with the following properties:

   (a) $B(0) = 0$ almost surely.

   (b) For all $t_1 < t_2 < \dots t_{k-1} < t_k$ in $[0, 1]$, $B_{t_k} - B_{t_{k-1}}, \dots B_{t_2} - B_{t_1}$ are independent with joint distribution $(N(0, t_k - t_{k-1}), \dots, N(0, t_2 - t_1)$,

   (c) Almost surely, the path $x \to B(x)$ (called a sample path) is a continuous function on $[0, \infty)$.

2. $D_n = \{k/2^n : 0 \leq k \leq 2^n\}$ and $D = \bigcup_n D_n$ are the **dyadic rationals**, a countable dense set in$[0, 1]$.

Notation

1. $Z \sim N(0, 1)$ – A normally distributed random variable with mean 0 and variance 1.

2. $\{A_i\}_{i=0}^{\infty}$ – A sequence of events in some probability space.

3. $\limsup_{i \to \infty} A_i = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} A_i = \{A_i, \text{i.o. }\}$ – An event which occurs "infinitely often".

4. $B : \Omega \times [0, \infty) \to \mathbb{R}$ or B_t – The standard one-dimensional Brownian Motion as a stochastic process.

5. $D_n = \{k/2^n : 0 \leq k \leq 2^n\}$ and $D = \bigcup_n D_n$ are the **dyadic rationals**, a countable dense set in $[0, 1]$.

6. $\{Z_d\}$ – A countable collection of $N(0, 1)$ independent random variables indexed by D.

7. B_d – Random variables derived from Z_d, indexed by D with the independent increments property on D.

8. $d^+ = d + 2^{-n}$ and $d^- = d - 2^{-n}$ – The neighbors of d, which are actually members of D_{n-1}.
9. $r < s < t$ – Arbitrary points in D.
10. $F_0(t)$ for $n = 0, 1, 2, \dots$ – Piecewise linear functions defined in terms of the random variables Z_d.
11. $t_1 < t_2 < t_3$ – Points in $[0, 1]$.
12. $t_i = \lim_{n\to\infty} t_{i,n}$ for $i = 1, 2, 3$ – Sequences of dyadics from D with limit t_i.
13. $B_t^{(n)}$ – Independent standard Brownian Motion stochastic processes on the unit interval.

Mathematical Ideas

Preliminary Lemmas

Lemma 1. *If* $Z \sim N(0,1)$ *then* $\mathbb{P}[Z \geq x] \leq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot \mathrm{e}^{-x^2/2}$.

Proof.

$$\begin{aligned}
\mathbb{P}[Z \geq x] &= \int_x^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-u^2/2} \, \mathrm{d}u \\
&\leq \int_x^\infty \frac{1}{\sqrt{2\pi}} \frac{u}{x} \mathrm{e}^{-u^2/2} \, \mathrm{d}u \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \int_x^\infty u \mathrm{e}^{-u^2/2} \, \mathrm{d}u \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot \mathrm{e}^{-x^2/2}
\end{aligned}$$

□

Remark. For $x \geq 1/(\sqrt{2\pi}) \approx 0.398$, the simpler but cruder estimate is useful: $\mathbb{P}[Z \geq x] \leq \mathrm{e}^{-x^2/2}$.

Lemma 2 (Borel-Cantelli)**.** *Let $\{A_i\}_{i=0}^{\infty}$ be a sequence of events in some probability space. Let $\limsup_{i\to\infty} A_i = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} A_i = \{A_i, i.o.\ \}$, where i.o. stands for* infinitely often*.*

1. *If $\sum_{i=0}^{\infty} \mathbb{P}[A_i] < \infty$, then $\mathbb{P}[\{A_i, i.o.\ \}] = 0$.*
2. *If the events $\{A_i\}_{i=0}^{\infty}$ are pairwise independent, and $\sum_{i=0}^{\infty} \mathbb{P}[A_i] = \infty$, then $\mathbb{P}[\{A_i, i.o.\ \}] = 1$.*

Proof. Omitted here. □

Lévy's Construction of Brownian Motion

Definition. A one-dimensional **standard Brownian Motion** is a stochastic process $B : \Omega \times [0, \infty) \to \mathbb{R}$ (alternatively a collection of random variables indexed by $x \in [0, \infty)$) with the following properties:

1. $B_0 = 0$,
2. for all $t_1 < t_2 < \dots t_{k-1} < t_k$ in $[0, 1]$, $B_{t_k} - B_{t_{k-1}}, \dots B_{t_2} - B_{t_1}$ are independent with joint distribution $(N(0, t_k - t_{k-1}), \dots, N(0, t_2 - t_1)$,
3. and almost surely $t \mapsto B_t$ is continuous on $[0, 1]$.

Theorem 3 (Wiener, 1923)**.** *One-dimensional standard Brownian Motion B_t exists.*

Remark. Although Norbert Wiener mathematically described Brownian Motion first in 1923, the following construction is due to Paul Lévy in 1948.

Proof. Step 1: Construct a set of random values on the dyadics in $[0, 1]$ satisfying the independent normal increments property.

(a) Let $D_n = \{k/2^n : 0 \leq k \leq 2^n\}$ and $D = \bigcup_n D_n$ be the **dyadic rationals**, a countable dense set in$[0, 1]$.

(b) Let $\{Z_d\}$ be a countable collection of $N(0, 1)$ independent random variables indexed by D.

(c) Inductively define a some new values B_d indexed by D with the independent increments property on D. Begin by defining $B_0 = 0$ and $B_1 = Z_1$.

(d) Claim: It is possible to make a collection B_d, with $d \in D$, such that for $r < s < t$ in D, $B_t - B_s \sim N(0, t-s)$ is independent of $B_s - B_r \sim N(0, s-r)$ and furthermore, that the B_d with $d \in D_n$ are globally independent of Z_d for $d \in D \setminus D_n$.

(e) Proof of Claim by Induction: The base case for $n = 0$ with D_0 holds since $r = 0$, $t = 1$ and there is no s. Then $B_1 - B_0 = Z_1 - 0 \ = Z_1 \sim N(0,1)$. $B_1 = Z_1$ is independent of all other B_d by constructions and $B_0 = 0$ is independent of all other B_d.

The induction step assumes the claim is true for $n-1$. Define for $d \in D_n \setminus D_{n-1}$

$$B_d = \frac{B_{d^-} + B_{d^+}}{2} + \frac{Z_d}{2^{(n+1)/2}}$$

where $d^+ = d + 2^{-n}$ and $d^- = d - 2^{-n}$ are the neighbors of d, which are actually members of D_{n-1}. Since $B_{d^-} - B_{d^+} \sim N(0, 1/2^{n-1})$ by the induction hypothesis, then $\frac{B_{d^-} - B_{d^+}}{2} \sim N(0, 1/2^{n+1})$. Furthermore, since $Z_d \sim N(0,1)$, then $\frac{Z_d}{2^{(n+1)/2}} \sim N(0, 1/2^{n+1})$. Together, these mean $B_d - B_{d^-} = \frac{B_{d^+} - B_{d^-}}{2} + \frac{Z_d}{2^{(n+1)/2}} \sim N(0, 1/2^n)$ and $B_{d^+} - B_d = \frac{B_{d^+} - B_{d^-}}{2} - \frac{Z_d}{2^{(n+1)/2}} \sim N(0, 1/2^n)$. This is enough to establish the first part of the induction claim. Note that B_d depends only on values from D_n so it is independent of all other values in $D \setminus D_n$. This establishes the second part of the induction claim.

Step 2: Using the B_d values, define some functions by linear interpolation between some values in D_n.

(a) Let $F_0(x) = xZ_1$. Let

$$F_n(x) = \begin{cases} 2^{-(n+1)/2} Z_d & x \in D_n \setminus D_{n-1} \\ 0 & x \in D_{n-1} \\ \text{linear} & \text{between adjacent points in } D_n \end{cases}$$

These functions are continuous on $[0,1]$. A graph of $F_i(t)$ for $i = 0, \ldots 6$ for some random normal values selected with a pseudo random number generator are in Figure 1.

(b) Claim: $B_d = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d)$.

(c) Proof of Claim by Induction: The base case is $n = 0$ with $F_0(0) = 0 = B_0$ and $F_0(1) = Z_1 = B_1$. The induction step supposes the claim holds for $n-1$. Let $d \in D_n \setminus D_{n-1}$ which is where new values of $\sum_{i=0}^{n} F_i(d)$ need to be checked. For $0 \le i \le n-1$, F_i is linear on $[d^-, d^+]$ so

$$\sum_{i=0}^{n-1} F_i(d) = \sum_{i=0}^{n} \frac{F_i(d^-) + F^+(d^+)}{2} = \frac{B_{d^-} + B_{d^+}}{2}.$$

Additionally, $F_n(d) = 2^{-(n+1)/2} Z_d$, so together

$$\sum_{i=0}^{n} F_i(d) = \frac{B_{d^-} + B_{d^+}}{2} + 2^{-(n+1)/2} Z_d = B_d$$

establishing the claim.

Step 3: Apply the Borel-Cantelli Lemma to show that $|Z_d| \sim c\sqrt{n}$ cannot happen infinitely often.

(a) Using the estimate for normal random variables in the Lemma, $\mathbb{P}\left[|Z_d| \ge c\sqrt{n}\right] \le \frac{2}{c\sqrt{n}} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-c^2 n/2} < \mathrm{e}^{-c^2 n/2}$ for all $n \ge 1$ (as long as $c > \sqrt{\frac{2}{\pi}} \approx 0.798$).

(b) Then

$$\sum_{n=0}^{\infty} \sum_{d \in D_n} \mathbb{P}\left[|Z_d| \ge c\sqrt{n}\right] \le \sum_{n=0}^{\infty} 2^n \mathrm{e}^{-c^2 n/2} = \sum_{n=0}^{\infty} \mathrm{e}^{(\log 2 - c^2/2)n}.$$

(c) So if $c > \sqrt{2 \log 2} \approx 1.177$, the series converges.

(d) By the Borel-Cantelli Lemma, then there is a random but finite N so that for $n > N$, $d \in D_n$, $|Z_d| \le c\sqrt{n}$.

(e) This means $\max_{t \in [0,1]} |F_n(t)| \le c\sqrt{n} 2^{-n/2}$.

(f) Then by the Weierstrass M-test, $\sum_{n=0}^{\infty} F_n(t)$ is uniformly and absolutely convergent on $[0, 1]$ to a continuous function, say $B_t = \sum_{n=0}^{\infty} F_n(t)$. A suggestive graph of $\sum_{n=0}^{\infty} F_n(t)$ approximating B_t in in Figure 2.

Step 4: Check that all increments have the correct finite dimensional joint distributions.

(a) Let $t_1 < t_2 < t_3$ be 3 points in $[0,1]$. Each is the limit of sequences of dyadics from D, $t_i = \lim_{n\to\infty} t_{i,n}$ for $i = 1, 2, 3$.

(b) Then
$$B_{t_3} - B_{t_2} = \lim_{n\to\infty} (B_{t_{3,n}} - B_{t_{2,n}})$$
is the limit of Gaussian sequences with mean 0 and variance $\lim_{n\to\infty}(t_{3,n} - t_{2,n}) = t_3 - t_2$. Likewise for $B_{t_2} - B_{t_1}$.

(c) The sequential increments are independent since $t_{1,n} < t_{2,n} < t_{3,n}$ (for sufficiently large n), and therefore the limit increments $B_{t_3} - B_{t_2}$ and $B_{t_2} - B_{t_1}$ are independent.

Step 5: Finally construct Brownian Motion on $[0, \infty)$.

(a) By the preceding construction, for $n \geq 1$ let $B_t^{(n)}$ be independent standard Brownian Motion processes on the unit interval.

(b) Let $B_t = B_{t-\lfloor t\rfloor}^{(\lfloor t\rfloor)} + \sum_{0\leq i\leq \lfloor t\rfloor} B_1^{(i)}$

$\square$

Remark. The slopes of the linear parts of $F_n(x)$ are

$$\frac{\pm 2^{-(n+1)/2} Z_d}{1/2^n} = \pm 2^{(n-1)/2} Z_d = O(2^{n/2})$$

which suggests that the limit function $\sum_{i=0}^{\infty} F_i(d)$ will be non-differentiable. See Figure 2. This is not enough to prove non-differentiability, but is suggestive.

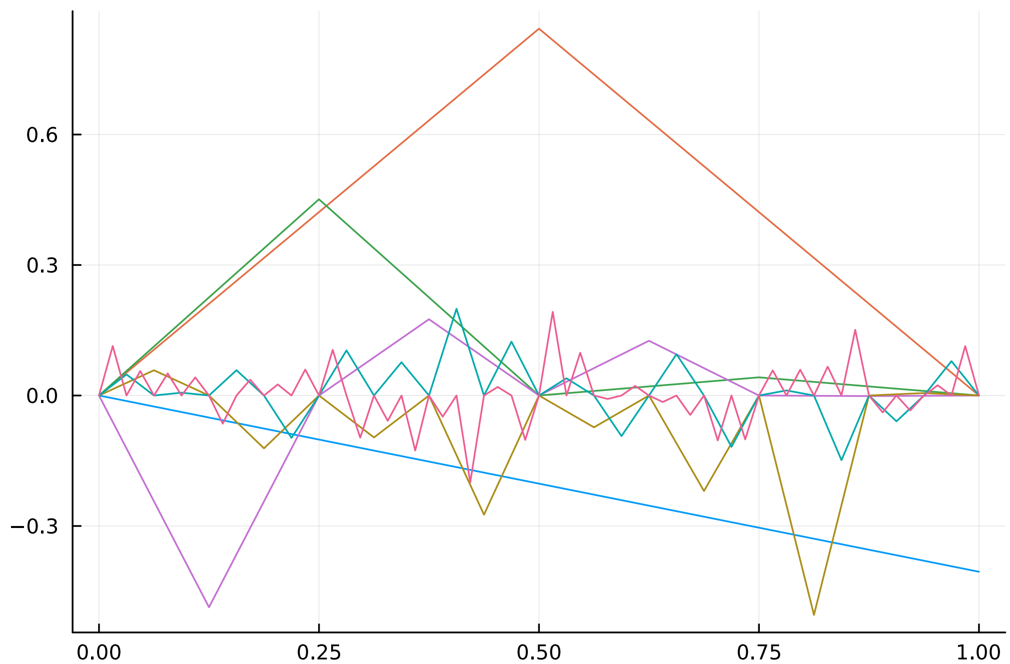


Figure 1: Graphs of $F_n(t)$ for $n = 0, \ldots, 6$.

Figure 2: Graph of $\sum_{n=0}^{6} F_n(t)$ suggesting the form of Brownian Motion and its non-differentiability.

Section Ending Answer

$Z_1 + Z_2$ and $Z_1 - Z_2$ and independent random variables each having the distribution $N(0, 2\sigma^2)$.

Sources

This section is adapted from: *An Invitation to Sample Paths of Brownian Motion*, Chapter 3, by Yuval Peres, An Invitation to Sample Paths of Brownian Motion.

Algorithms, Scripts, Simulations

Algorithm

Scripts

Data: Depth of approximation
Result: Plots of approximation of Brownian Motion

Initializations
Load interpolation and plotting packages
Set the depth of approximation
Set Z_1 and define function F_0
Create list of dyadic rationals to depth

Function Creation Define the values of F_i on D_i to depth
Use linear interpolation to create functions F_i

Create Plots
Plot the functions F_i and save the plot
Create the Brownian Motion function as a sum of F_i
Plot the Brownian Motion approximation and save the plot

Scripts

Julia

```julia
using Plots, DataInterpolations

N = 6

Z1 = randn()
F0 = t -> t * Z1

D = [ [j//2^i for j=0:2^i] for i = 1:N]
Fdata = [ [ iseven(j) ? 0 : randn()/2^((i+1)/2) for j=0:2^i] for i = 1:N]
Finterps =  [ LinearInterpolation(Fdata[i], D[i]) for i in 1:N ]

ts = range(0, 1, length=2^(N+1)+1)
F0s = F0.(ts)

Fs = [Finterps[i].(ts) for i in 1:6]
plot(ts, [F0s, Fs[1,:], Fs[2,:],  Fs[3,:], Fs[4,:], Fs[5,:], Fs[6,:] ], label=false)
savefig("individualFs.pdf")

sumF = F0s + Fs[1] + Fs[2] + Fs[3] + Fs[4] + Fs[5] + Fs[6]
plot(ts, sumF, label=false)
savefig("LevyBrownianMotion.pdf")
```

Problems to Work for Understanding

Reading Suggestion:

References

Outside Readings and Links:

1. *An Invitation to Sample Paths of Brownian Motion*, by Yuval Peres, An Invitation to Sample Paths of Brownian Motion
2. *Brownian Motion* by Peter Mörters and Yuval Peres, Brownian Motion

Solutions

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