Solving and Simulating Stochastic Differential Equations

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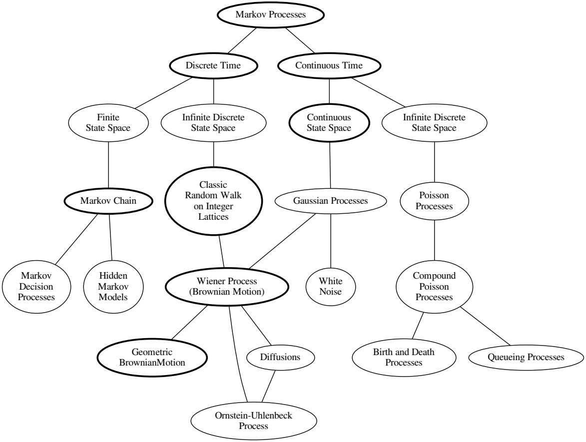
Rule of Four

Mathematical concepts should be presented:

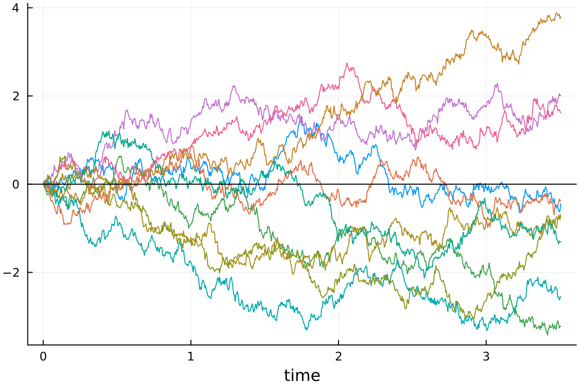
- Symbolically
- Visually
- Numerically
- Verbally

Definition 1.4:



Brownian Motion Paths

Definition 1.11

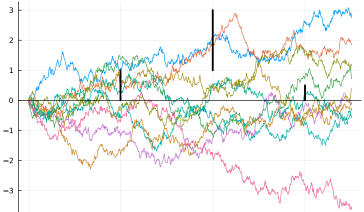


Calculating Probabilities 1

Definition 1.8 (2) and Proposition 1.1 (3):

Let $B(t)$ be standard Brownian motion.

- Find the probability that $0 < B(1) < 1$ (= 0.34134).
- Find the probability that $0 < B(1) < 1$ and $1 < B(2) < 3$ (= 0.10046).
- Find the probability that $0 < B(1) < 1$ and $1 < B(2) < 3$ and $0 < B(3) < 1/2$ (= 0.00861).



Calculating Probabilities 2

Definition 1.8 (2) and Proposition 1.1 (3):

Write the joint probability density function for $B(t_1) = x_1$ and $B(t_2) = x_2$ explicitly

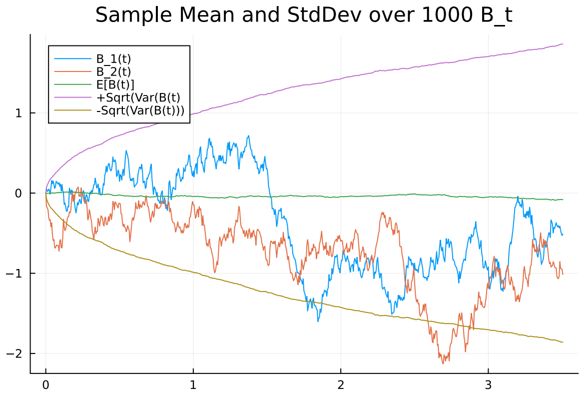
$$f(x_1, t_1; x_2, t_2) = \frac{1}{2\pi\sqrt{t_1 \cdot (t_2 - t_1)}} \mathrm{e}^{\frac{-x_1^2}{2t_1} - \frac{(x_2 - x_1)^2}{2(t_2 - t_1)}}$$

and integrate (with $t_1 = 1$, $t_2 = 2$)

$$\int_0^1 \int_1^3 \frac{1}{2\pi\sqrt{t_1 \cdot (t_2 - t_1)}} \mathrm{e}^{\frac{-x_1^2}{2t_1} - \frac{(x_2 - x_1)^2}{2(t_2 - t_1)}} \, \mathrm{d}x_1 \, \mathrm{d}x_2 = 0.10046$$

Properties of Brownian Motion

Proposition 1.1 (1) and (2)



Euler-Maruyama Method: 1

The sample path that the Euler-Maruyama method produces numerically is the analog of using the Euler method.

The Euler-Maruyama (EM) method for

$$\mathrm{d}X_t = G(t, X_t)\,\mathrm{d}t + H(t, X_t)\,\mathrm{d}B_t, \qquad X(t_0) = X_0$$

is based on the definition of the Ito stochastic integral:

$$X_{j+1} = X_j + G(t_j, X_j)\,\mathrm{d}t + H(t_j, X_j)(B(t_j + \mathrm{d}t) - B(t_j))$$
$$t_{j+1} = t_j + \mathrm{d}t\,.$$

Note that the initial conditions X_0 and t_0 set the starting point.

Euler-Maruyama Method: 2

$\mathrm{d}X = 2X \,\mathrm{d}t + X \,\mathrm{d}B_t, X(0) = 1$ with $\mathrm{d}t = 0.1$ and $\mathrm{d}B_t = B(gt_{j-1} + \mathrm{d}t) - B(t_{j-1})$ (Proposition 1.1 (3))

j	t_j	X_j	$2X_j \,\mathrm{d}t$	$\mathrm{d}B_t$	$X_j \,\mathrm{d}B_t$	X_{j+1}
0	0	1	0.2	0.271	0.271	1.471
1	0.1	1.471	0.294	-0.049	-0.072	1.693
2	0.2	1.693	0.339	-0.448	-0.758	1.274
3	0.3	1.274	0.255	-0.21	-0.268	1.261
4	0.4	1.261	0.252	0.333	0.420	1.933
5	0.5	1.933	0.387	-0.683	-1.320	1.000
6	0.6	1.000	0.200	-0.292	-0.292	0.908
7	0.7	0.908	0.182	0.128	0.116	1.206
8	0.8	1.206	0.241	-0.34	-0.410	1.037
9	0.9	1.037	0.207	-0.147	-0.152	1.092
10	1.0	1.092				

Error Measures:

Let $X_t^{(h)}$ be the numerical solution on $[0, T]$ with step-size $h = \mathrm{d}t$.

Weak Error at time t (for a Lipschitz function f) is

$$\epsilon_w(h) = \left| \mathbb{E}\left[f(X_T)\right] - \mathbb{E}\left[f(X_T^{(h)})\right] \right|.$$

(This is the $L^1(\Omega)$ distance between $f(X_T)$ and $f(X_T^{(h)})$.)

Strong Error is:

$$\epsilon_s(h) = \mathbb{E}\left[\sup_{t \in [0,T]} \| X_t - X_t^{(h)} \|^2 \right]^{1/2}$$

Error Bounds:

Theorem (Euler-Maruyama Error Bounds)

$$\epsilon_w(h) = \left| \mathbb{E}\left[f(X_T)\right] - \mathbb{E}\left[f(X_T)\right] \right| = O(h)$$

$$\epsilon_s(h) = \mathbb{E}\left[\sup_{t \in [0,T]} \| X_t - X_t^{(h)} \|^2 \right]^{1/2} = O(h^{1/2})$$

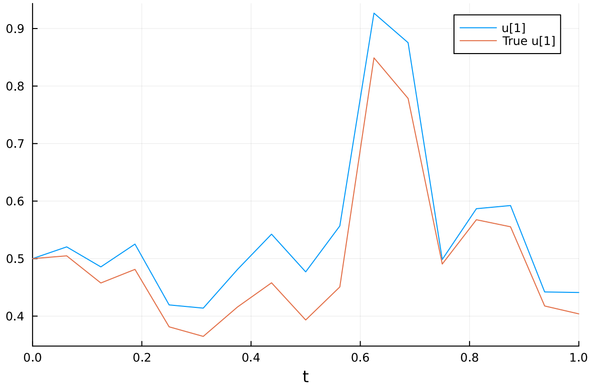
Proof: Uses Ito's Isometry (LN, footnote 2), Gronwall's Inequality, Jensen's Inequality and some simple inequalities.

Other Numerical Methods:

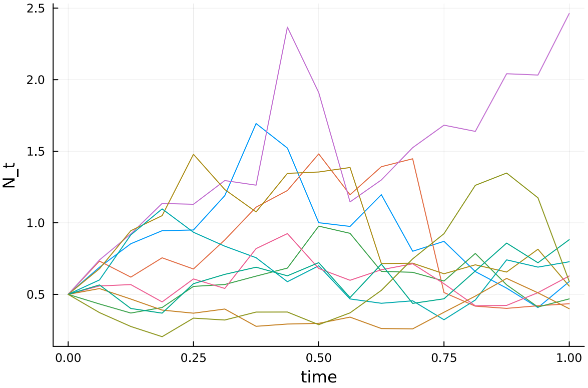
- Euler-Heun(Maruyama) - Adapts the Euler-Heun method. Strong Order 0.5 in the Stratonovich sense.
- Milstein (1974) - Adds a 2nd-order correction based on Ito integral definition. Various adaptations have $\epsilon_w(h) = \epsilon_s(h) = O(h)$.
- Stochastic Runge-Kutta (difficulty is iterated stochastic integrals).
- Others: Achieve $\epsilon_w(h) = O(h^2)$ and $\epsilon_s(h) = O(h^{3/2})$ but need special noise structure for systems.

Example 2.1

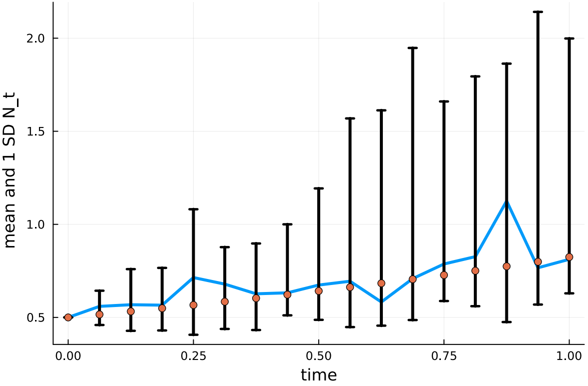
$dN_t = rN_t \, \mathrm{d}t + \alpha N_t \, \mathrm{d}B_t, N(0) = N_0$ with $r = 1$, $\alpha = 1$, $N_0 = 1/2$. Solve with EM, $h = \mathrm{d}t = 1/16$



Example 2.1 Ensembles



Example 2.1 Means, Variances



Example 2.2

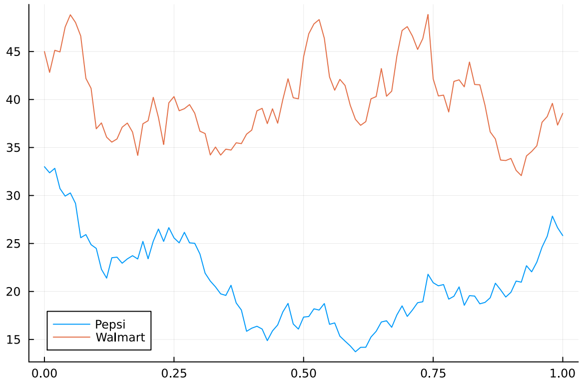
Financial Model 2.2, (4b)

Pepsi and Walmart Stock prices, 2006

$$
\begin{aligned}
dX_p &= -0.0545 X_p \, dt + 0.5 X_p \, dB_p(t) + 0.1 X_w \, dB_w(t) \\
dX_w &= +0.0125 X_w \, dt + 0.1 X_p \, dB_p(t) + 0.5 X_w \, dB_w(t)
\end{aligned}
$$

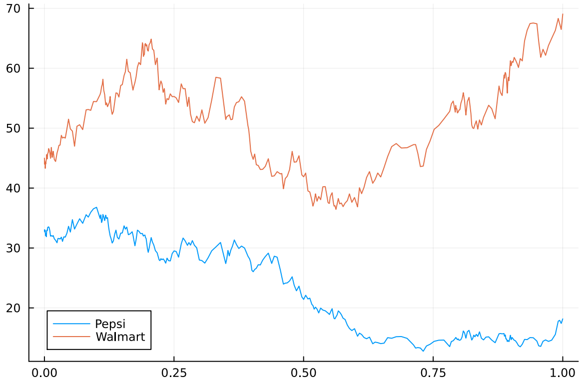
Example 2.2

Euler-Maruyama Method, dt = 0.01



Example 2.2

RKMilCommute - An explicit Runge-Kutta discretization of the strong order 1.0 Milstein method, adpative step size.



Milstein Method 1

The Milstein method increases the accuracy of the E-M approximation by adding a second-order "correction" term derived from the stochastic Taylor series expansion of $X(t)$ by applying Ito's lemma to the $G()$ and $H()$ functions.

$$
\begin{aligned}
\mathrm{d}X_t &= G(t, X_t)\,\mathrm{d}t + H(t, X_t)\,\mathrm{d}B_t, \qquad X(t_0) = X_0 \\
X_{j+1} &= X_j + G(t_j, X_j)\,\mathrm{d}t + H(t_j, X_j)\,\mathrm{d}B_j + \\
&\qquad \frac{1}{2} H(t_j, X_j) H'(t_j, X_j)((\mathrm{d}B_j)^2 - \mathrm{d}t) \\
t_{j+1} &= t_j + \mathrm{d}t\,.
\end{aligned}
$$

Milstein Derivation for Geometric Brownian Motion

$$\mathrm{d}X_t = rX \,\mathrm{d}t + \alpha X \,\mathrm{d}B_t$$

$$\mathrm{d}(\log X_t) = \left(r - \frac{\alpha^2}{2}\right) \mathrm{d}t + \alpha \,\mathrm{d}B_t$$ LN, Ex 2.1, p.3

$$X_{t+\mathrm{d}t} = X_t \exp\left(\int_t^{t+\mathrm{d}t} \left(r - \frac{\alpha^2}{2}\right) \mathrm{d}t + \int_t^{t+\mathrm{d}t} \alpha \,\mathrm{d}B_t\right)$$

$$X_{t+\mathrm{d}t} \approx X_t \left(1 + r \,\mathrm{d}t - \frac{\alpha^2}{2} \,\mathrm{d}t + \alpha \,\mathrm{d}B_t + \frac{\alpha^2}{2}(\mathrm{d}B_t)^2\right)$$

$$= X_t + rX_t \,\mathrm{d}t + \alpha X_t \,\mathrm{d}B_t + \frac{1}{2}\alpha X_t \cdot \alpha((\mathrm{d}B_t)^2 - \mathrm{d}t)$$