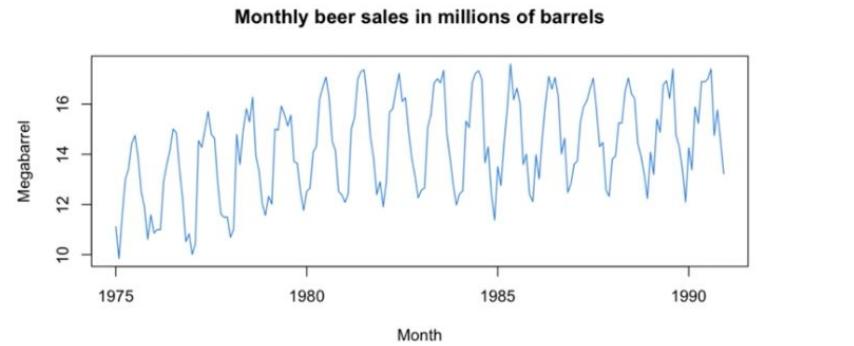


# Time series Analysis

# What is a time series

- Complex topic :
  - information theory
  - signal
  - probability/statistics
  - artificial intelligence/machine learning
  - optimization
  - Computer science (python)
  - Financial mathematics
- <http://www.laurentoudre.fr/ast.html>
- <http://www.laurentoudre.fr/signalm.html>
- Code
  - <https://scikit-learn.org/stable/>
  - [https://github.com/sduprey/initiation\\_python\\_finance](https://github.com/sduprey/initiation_python_finance)
  - [https://github.com/sduprey/timeseries\\_ressources](https://github.com/sduprey/timeseries_ressources)
- Books
  - <https://www.amazon.fr/Probabilit%C3%A9s-pour-Licence-exercices-corrig%C3%A9s/dp/2807332234/>
  - <https://www.amazon.fr/Probabilit%C3%A9s-Statistiques-Mod%C3%A9lisation-Ing%C3%A9nierie-Math%C3%A9matiques/dp/2340013062>
  - <https://www.amazon.fr/Analyse-s%C3%A9ries-temporelles-5e-%C3%A9dition/dp/2100835866>
  - <https://www.amazon.fr/dp/2340032997>
- Online resources :
  - <https://djalil.chafai.net/docs/M1/> (<https://djalil.chafai.net/docs/M1/m1-series-temporelles-cours-2017-2018.pdf>)
  - <http://www.laurentoudre.fr/ast.html>
  - [https://www.ceremade.dauphine.fr/~roche/Enseignement/Series\\_temp/](https://www.ceremade.dauphine.fr/~roche/Enseignement/Series_temp/)
  - <https://github.com/fastai/course22>
  - <https://github.com/fastai/fastbook>
  - <https://www.imo.universite-paris-saclay.fr/~antoine.levitt/fourier/cours.pdf>



# Proper mathematics development

[https://github.com/sduprey/timeseries\\_ressources/blob/main/m1-series-temporelles-cours.pdf](https://github.com/sduprey/timeseries_ressources/blob/main/m1-series-temporelles-cours.pdf)

# **Modeling vs Predicting : narrowing down the determinism**

**A framework which encompasses AI and econometry**

$$y_t = f(y_{t-1}, \dots, y_{t-p}; \theta) + \varepsilon_t$$

- AI predilection domain: complex function with embeddings, features engineering and a small pure random part (see for example the Rossmann sales kaggle competition)
- Econometrics predilection domain : function very simple (just a scalar for a random walk) (try for example to predict the bitcoin price)

# Modeling vs Predicting : narrowing down the determinism

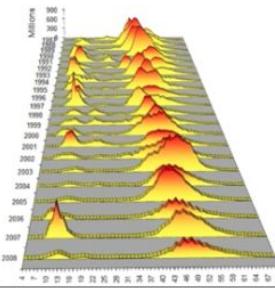
A framework which encompasses AI and econometry

$$y_t = f(y_{t-1}, \dots, y_{t-p}; \theta) + \varepsilon_t$$

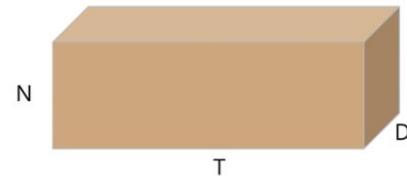
- Gives a functional form to your time series
- Gives you insight into the behavior of the time series
- “Why is the time series mean-reverting?”
- “Why does the time series grow unbounded?”
- “Is the time series predictable?”
- Someone who doesn’t know how to model might waste their time trying to predict something which is unpredictable! (e.g. a coin flip)

# Dataframe (R, Pandas, PyTorch/TensorFlow tensors)

	New York City	London	Tokyo	Paris
1990-01-01	1	2	3	4
1990-01-02	5	6	7	8
1990-01-03	9	10	11	12
1990-01-04	13	14	15	16



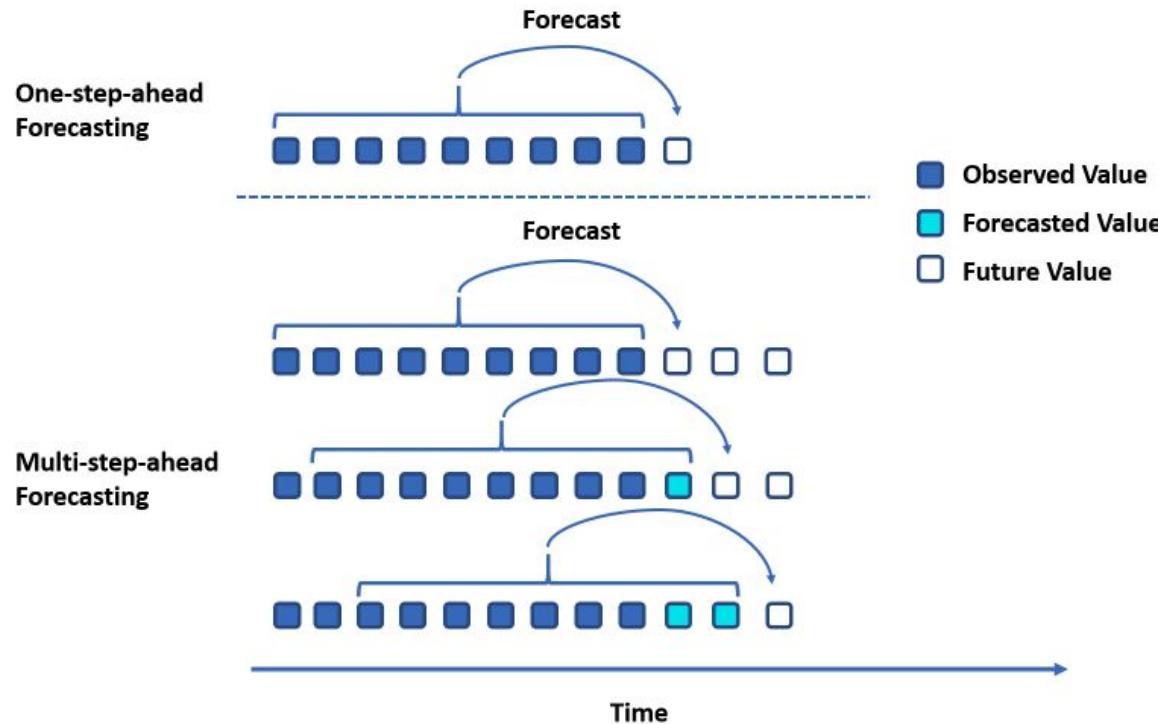
- $N \times T \times D$  (a box in 3-D space)
- Automatically think of this, whenever you see / hear “ $N \times T \times D$ ”
- It really helps, and makes it less abstract



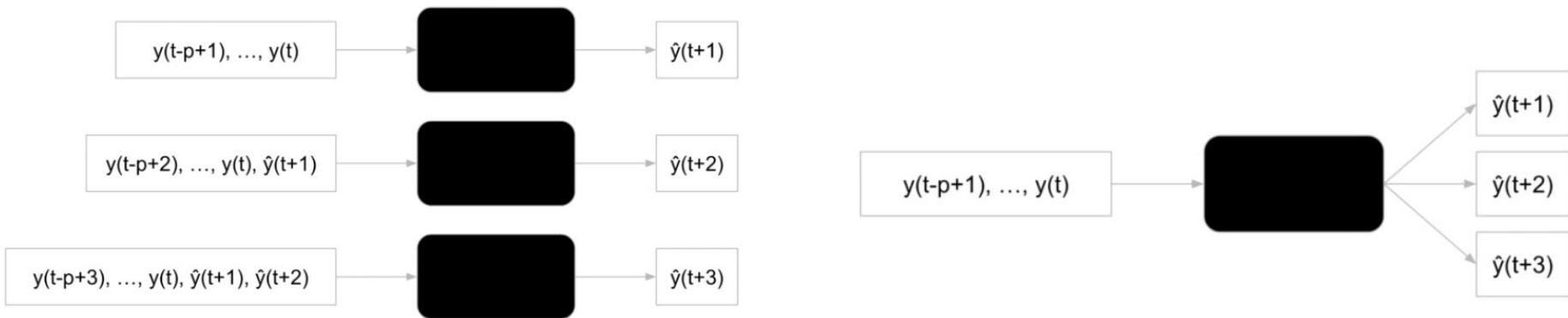
# Generate random tensors/time series

- `np.random.randn(3,3,3)`
- Generate from different probability distributions
- Distribution fitting with scikit-learn

# 1-step forecast versus multi-steps forecast



# Incremental multi-steps forward versus multi-output



# Stochastic processes in a nutshell

- Stochastic processes are processes that proceed randomly in time.
- Rather than consider fixed random variables  $X$ ,  $Y$ , etc. or even sequences of i.i.d random variables, we consider sequences  $X_0, X_1, X_2, \dots$  Where  $X_t$  represent some random quantity at time  $t$ .
- In general, the value  $X_t$  might depend on the quantity  $X_{t-1}$  at time  $t-1$ , or even the value  $X_s$  for other times  $s < t$ .
- Example: simple random walk .

Going more into maths:

- **Markov process**
- **Martingales**
- **Discretization of continuous processes**

Given an Itô process

$$dX(t) = \mu(t, X(t)) dt + \sigma(t, X(t)) dW(t),$$

and a time discretization  $\{t_i \mid i = 0, \dots, n\}$  with  $0 = t_0 < \dots < t_n$ , then the time-discrete stochastic process  $\tilde{X}$  defined by

$$\tilde{X}(t_{i+1}) = \tilde{X}(t_i) + \mu(t_i, \tilde{X}(t_i)) \Delta t_i + \sigma(t_i, \tilde{X}(t_i)) \Delta W(t_i)$$

is called an *Euler-Maruyama scheme* of the process  $X$  (where  $\Delta t_i := t_{i+1} - t_i$  and  $\Delta W(t_i) := W(t_{i+1}) - W(t_i)$ ).

# Random walk for instance

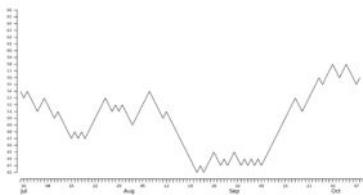
## Discrete random walk

$p_0 = \text{some initial value}$

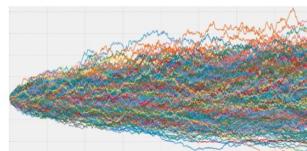
$p_1 = p_0 + e_1, \text{ where } e_1 \in \{-1, +1\}$

$p_2 = p_1 + e_2$

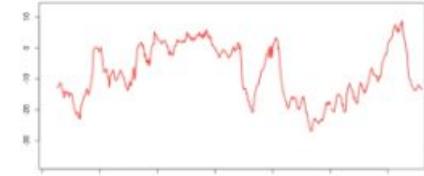
...



- Imagine yourself walking - you take one step **left** or **right** based on a coin flip - that's this random walk!
- Can't predict the future (50% chance of being correct)



## Gaussian random walk



$p_0 = \text{some initial value}$

$p_1 = p_0 + e_1, \text{ where } e_i \sim \mathcal{N}(0, \sigma^2)$

$p_2 = p_1 + e_2$

...

## Log Prices

- Consider a random walk with drift

$$p_t = p_{t-1} + \mu + e_t, \quad e_t \sim \mathcal{N}(0, \sigma^2)$$

- Take  $p(t-1)$  to the LHS - this is now the log return

$$r_t = p_t - p_{t-1} = \mu + e_t$$

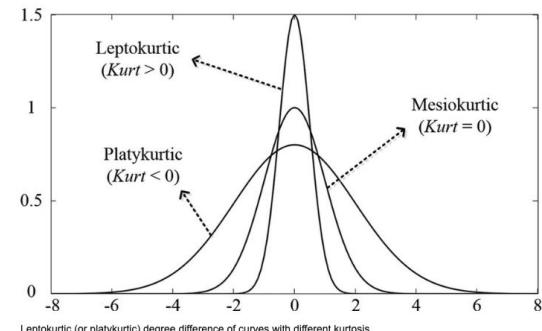
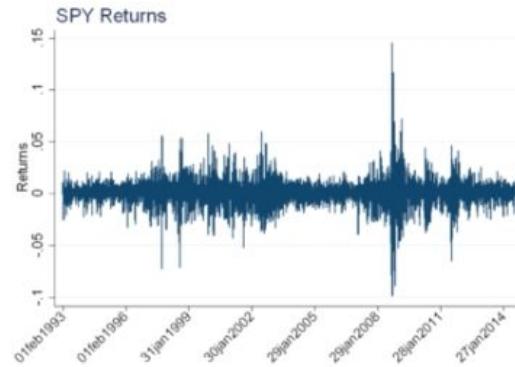
- The log return is therefore distributed as follows

$$r_t \sim \mathcal{N}(\mu, \sigma^2)$$



# Stylized facts about financial time-series which invalidate the random walk hypothesis

- Volatility clustering
- Shock asymmetry
- Leptokurtic residuals



Markov property       $p(w_t \mid w_{t-1}, w_{t-2}, \dots, w_0) = p(w_t \mid w_{t-1})$

## Gaussian random walk

$$x(t) = x(t-1) + e(t), \quad e(t) \sim \mathcal{N}(0, \sigma^2)$$

$$x(t) \sim \mathcal{N}(x(t-1), \sigma^2)$$

$$\text{var}\{x(t+\tau)\} = ?$$

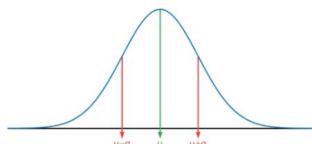
$$x(t+1) = x(t) + e(t+1)$$

$$x(t+2) = x(t+1) + e(t+2) = x(t) + e(t+1) + e(t+2)$$

...

$$x(t+\tau) = x(t) + e(t+1) + \dots + e(t+\tau)$$

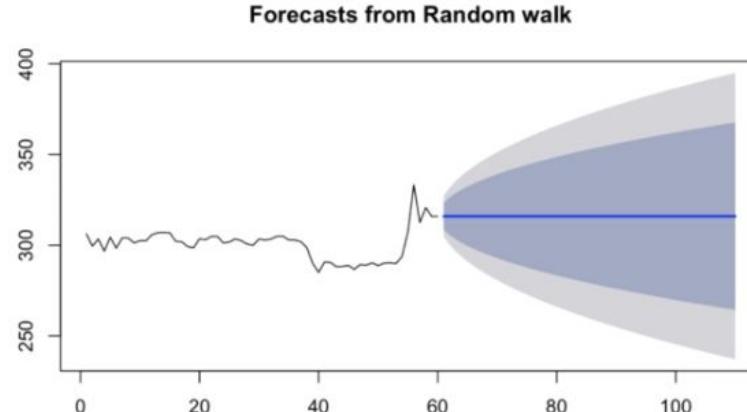
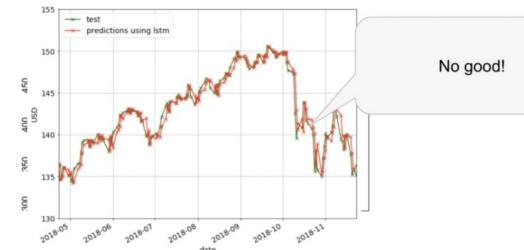
**Square root of time**  
**growing confidence interval**  
**Central limit theorem :**  
**Converges to a gaussian**



If  $\text{var}\{e(t)\} = \sigma^2$ , then  $\text{var} \sum_{k=1}^{\tau} e(t+k) = \tau \sigma^2$ , or  $\text{sd}\{x(t+\tau)\} = \sqrt{\tau} \sigma$

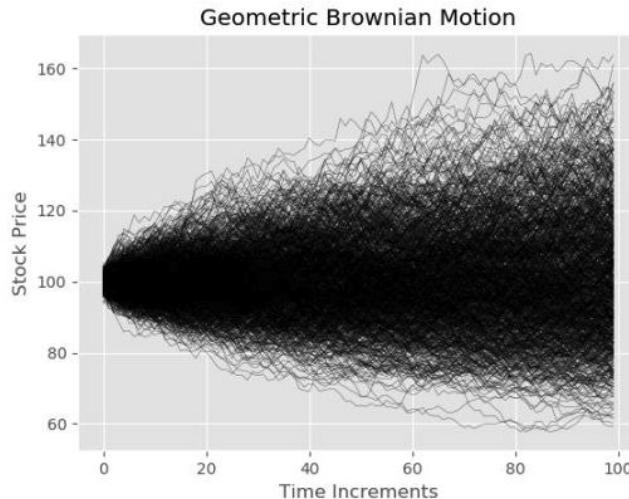
# Naive forecast or the importance of a baseline

- Compare your results against existing state of the art
- Be careful :
  - the naïve forecast looks good for martingale processes
  - In-sample versus out-of-sample forecasts
  - You can beat the naive forecast in-sample but not out-sample
- **The naive forecast is the best for a random walk (for any martingales)**



# Option pricing through Monte-Carlo simulation

- [https://github.com/sduprey/timeseries\\_ressources/Black\\_Scholes\\_option\\_pricing\\_through\\_Monte\\_Carlo\\_simulation.ipynb](https://github.com/sduprey/timeseries_ressources/Black_Scholes_option_pricing_through_Monte_Carlo_simulation.ipynb)



$$dS = S\mu dt + S\sigma dW(t)$$

$$dG = \left( \frac{\partial G}{\partial S} S\mu + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} S^2 \sigma^2 \right) dt + \frac{\partial G}{\partial S} S\sigma dW(t)$$

$$\frac{\partial G}{\partial S} = \frac{1}{S} \quad , \quad \frac{\partial G}{\partial t} = 0 \quad , \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$$

$$\begin{aligned} dG &= \left( \frac{1}{S} S\mu + 0 - \frac{1}{2} \frac{1}{S^2} S^2 \sigma^2 \right) dt + \frac{1}{S} S\sigma dW(t) \\ &= (\mu - \frac{\sigma^2}{2}) dt + \sigma dW(t) \end{aligned}$$

$$lookback = e^{-rT} \left[ \frac{1}{N} \sum_{i=1}^N (S_{max} - K)^+ \right]$$

# Forecasting metrics (like regression)

- Sum of squared errors (max likelihood when the errors are normally distributed)  $E = \sum_{i=1}^N (y_i - \hat{y}_i)^2$
- Mean squared error  $E = \frac{1}{N} \sum_{i=1}^N (y_i - \hat{y}_i)^2 \approx E((y - \hat{y})^2)$
- Root mean squared error (same unit)  $E = \sqrt{\frac{1}{N} \sum_{i=1}^N (y_i - \hat{y}_i)^2}$
- Mean absolute error  $E = \frac{1}{N} \sum_{i=1}^N |y_i - \hat{y}_i|$

# R squared

- Not an error - we want it to be bigger, not smaller
- If your model makes perfect predictions, then  $MSE=0, R^2=1$
- $R^2=0$  means your model does no better than predicting  $\bar{y}$

$$E = 1 - \frac{SSE}{SST} = 1 - \frac{MSE}{Var(Y)}$$

where  $SST = \sum_{i=1}^N (y_i - \bar{y})^2$ ,  $Var(Y) = \frac{1}{N} SST$

# Scikit-learn

- For classification models: `model.score(X, Y)` returns accuracy
- For regression models: `model.score(X, Y)` returns  $R^2$



# Problems of relativity

- Accuracy is not the best metrics for an imbalanced classifier

$$E = \frac{1}{N} \sum_{i=1}^N \left| \frac{y_i - \hat{y}_i}{y_i} \right|$$

- MAPE (mean absolute percentage error)

$$E = \frac{1}{N} \sum_{i=1}^N \frac{|y_i - \hat{y}_i|}{(|y_i| + |\hat{y}_i|) / 2}$$

- sMAPE

# Common transformations : analog to features engineering in artificial intelligence

- Power transform       $y'(t) = y(t)^\gamma$
- Log transform       $y'(t) = \log y(t)$  or  $\log(y(t) + 1)$
- Box-Cox transform      
$$y'(t) = \frac{y(t)^\lambda - 1}{\lambda} \quad \text{if } \lambda \neq 0$$
$$y'(t) = \log y(t) \quad \text{if } \lambda = 0$$

Since:

$$\lim_{\lambda \rightarrow 0} \frac{x^\lambda - 1}{\lambda} = \ln x$$

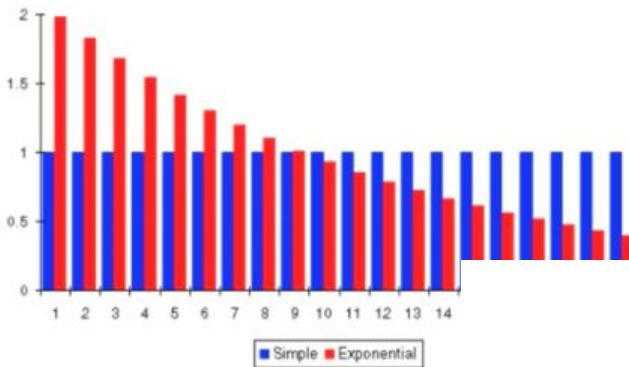
# Why the log transform is fundamental

- Watch fast.ai on random forest : log uniformises large tail distributions
- In more standard econometrics, it is all about stationarity
- Rescaling data to more linear trends (decibels are the best example)
- It is part of a broader trend : features engineering for tabular data

[https://github.com/fastai/fastbook/blob/master/09\\_tabular.ipynb](https://github.com/fastai/fastbook/blob/master/09_tabular.ipynb)

# Simple/Exponentially weighted moving average

- The SMA is equally weighted, since for a window of size N, each point has a weight 1/N
- The EWMA gives each point a weight that decays exponentially



$$\text{Arithmetic mean : } \bar{x} = \frac{1}{T} \sum_{t=1}^T x_t$$

$$\text{EWMA : } \bar{x}_t = \alpha x_t + (1 - \alpha)\bar{x}_{t-1}, \text{ where } 0 \leq \alpha \leq 1$$

# EWMA is tailoring the importance you give to more recent samples

We make  $1/t$  which dwindle constant !

$$\bar{x}_t = \left(1 - \frac{1}{t}\right) \bar{x}_{t-1} + \frac{1}{t} x_t$$

$$\bar{x}_t = (1 - \alpha) \bar{x}_{t-1} + \alpha x_t$$

$$\begin{aligned}\bar{x}_t &= (1 - \alpha) \bar{x}_{t-1} + \alpha x_t \\ &= (1 - \alpha)[(1 - \alpha) \bar{x}_{t-2} + \alpha x_{t-1}] + \alpha x_t \\ &= (1 - \alpha)^2 \bar{x}_{t-2} + (1 - \alpha)\alpha x_{t-1} + \alpha x_t \\ &= (1 - \alpha)^2[(1 - \alpha) \bar{x}_{t-3} + \alpha x_{t-2}] + (1 - \alpha)\alpha x_{t-1} + \alpha x_t \\ &= (1 - \alpha)^3 \bar{x}_{t-3} + (1 - \alpha)^2 \alpha x_{t-2} + (1 - \alpha)\alpha x_{t-1} + \alpha x_t \\ &\dots \\ &= (1 - \alpha)^t \bar{x}_0 + \alpha \sum_{k=0}^{t-1} (1 - \alpha)^k x_{t-k}\end{aligned}$$

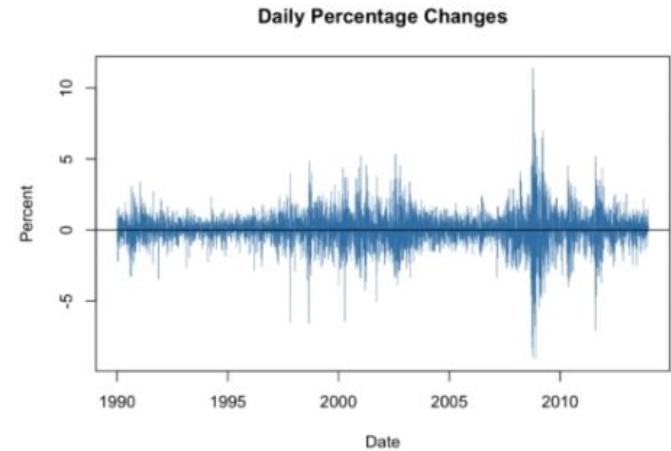
# Why use moving average ?

Local assessment of mean/variance of a time serie (volatility clustering)

```
# returns a Rolling object
rolling_window = df['GOOG'].rolling(window_size)

# returns a series/dataframe of rolling means
# can also calculate min, max, sum, var, etc.
rolling_window.mean()

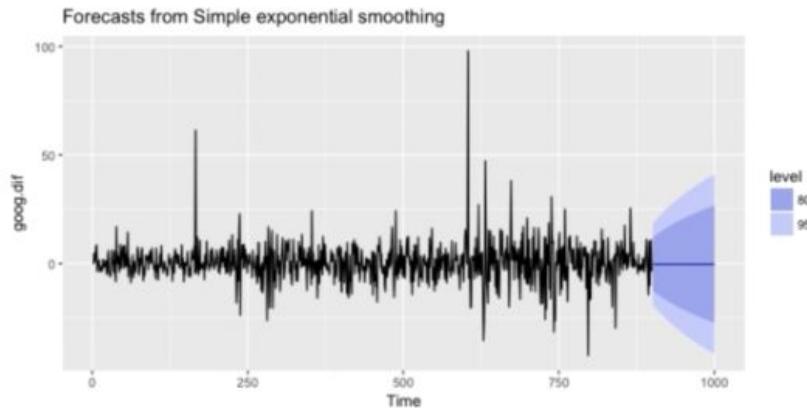
# multi-dimensional
covariance = df[['GOOG', 'AAPL']].rolling(50).cov()
```



# Simple Exponential Smoothing

It makes the assumption that the process fluctuates around a mean value, which we assess through an EWMA (very efficient for martingales !)

$$\text{Level}(t+h) = \text{EWMA}(\text{Time series from } 1 \dots t)$$



$$\hat{y}_{t+1|t} = \alpha y_t + (1 - \alpha) \hat{y}_{t|t-1}$$

# SES multi step forecast

*Forecast Equation :*  $\hat{y}_{t+h|t} = l_t, h = 1, 2, 3\dots$

*Smoothing Equation :*  $l_t = \alpha y_t + (1 - \alpha)l_{t-1}$

- Just the EWMA
- Notice: the original time indices are back!

```
from statsmodels.tsa.holtwinters import SimpleExpSmoothing  
  
# make the model - data is univariate  
ses = SimpleExpSmoothing(data)  
  
# 'fit' the model - returns a HoltWintersResults object  
result = ses.fit(smoothing_level=alpha, optimized=False)  
  
# in-sample prediction or out-of-sample forecast  
result.predict(start=start_date, end=end_date)
```

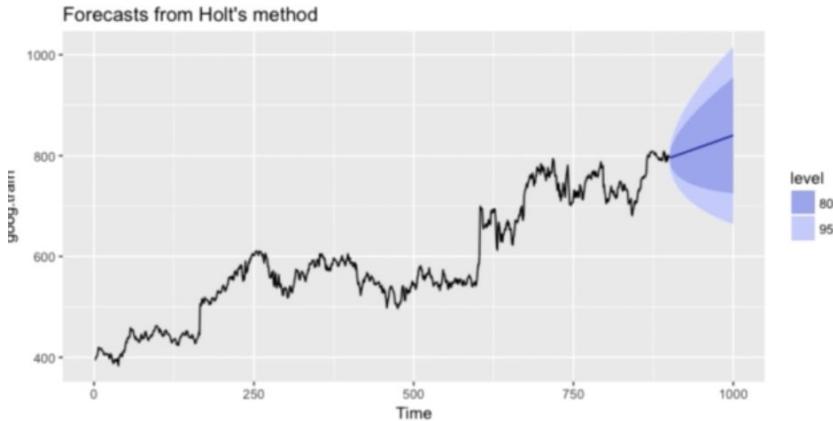
```
# get all in-sample predictions  
result.fittedvalues  
  
# forecast n steps ahead  
result.forecast(n)
```

If your data goes from  $t=1\dots T$ , then  
the first forecast will be at time  $T+1$

# Holt's linear trend model

It makes the assumption that the process has a linear trend (level is the intercept, trend is the slope)

$$\text{Level}(t+h) = \text{EWMA}(\text{Level of time series from } 1 \dots t)$$
$$\text{Trend}(t+h) = \text{EWMA}(\text{Trend of time series from } 1 \dots t)$$



$$y = mx + b$$

$$y_t = \text{slope} \times t + y_0$$

# Holt's linear trend model

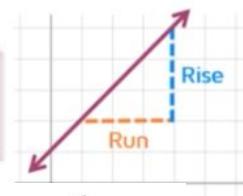
*Forecast Equation :*  $\hat{y}_{t+h|t} = l_t + h b_t$

*Level Equation :*  $l_t = \alpha y_t + (1 - \alpha)(l_{t-1} + b_{t-1})$

*Trend Equation :*  $b_t = \beta(l_t - l_{t-1}) + (1 - \beta)b_{t-1}$

$$\text{slope in one step} = \frac{l_t - l_{t-1}}{1}$$

$$\text{Slope} = \frac{\text{Vertical Change}}{\text{Horizontal Change}} = \frac{\text{Rise}}{\text{Run}}$$

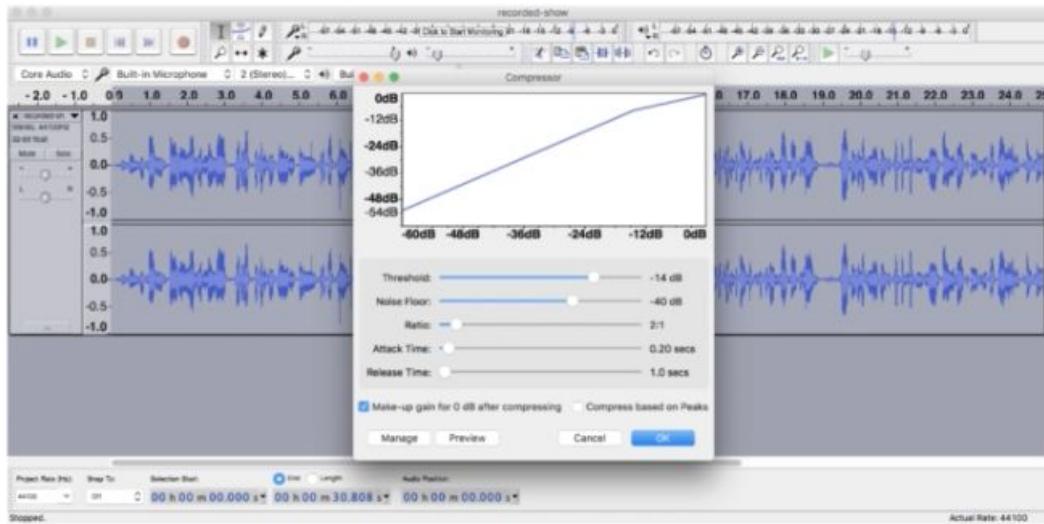


$$\text{one step increase due to trend} = l_{t-1} + 1 \times b_{t-1}$$

$$\text{Error} = \sum_{t=1}^T (y_t - \hat{y}_{t|t-1})^2$$

$$\alpha, \beta = \arg \min_{\alpha, \beta} \text{Error}$$

Alpha is a hyper parameter : like a frequency cut-off parameter in a signal filter

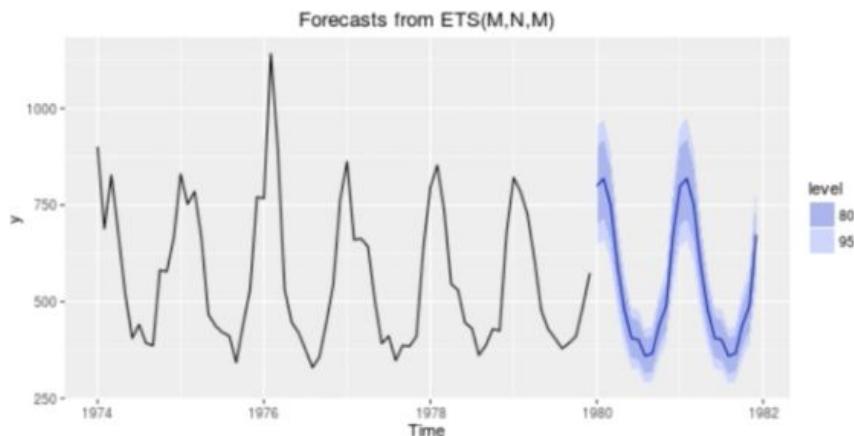


# Holt-Winter model

$\text{Level}(t+h) = \text{EWMA}(\text{Level of time series from } 1 \dots t)$

$\text{Trend}(t+h) = \text{EWMA}(\text{Trend of time series from } 1 \dots t)$

$\text{Seasonal}(t+h) = \text{EWMA}(\text{Seasonal of time series from } 1 \dots t)$



- Additive:  $y = \text{level} + \text{trend} + \text{seasonal}$
- Multiplicative  $y = (\text{level} + \text{trend}) * \text{seasonal}$

# Holt-Winter model (additif)

$$Forecast : \hat{y}_{t+h|t} = l_t + h b_t + s_{t+h-mk}$$

$$Level : \quad l_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)(l_{t-1} + b_{t-1})$$

$$Trend : \quad b_t = \beta(l_t - l_{t-1}) + (1 - \beta)b_{t-1}$$

What is 'm'? The period of the cycle

$$Seasonality : \quad s_t = \gamma(y_t - l_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m}$$

What is 'k'?

$$\text{The formula: } k = \text{floor}\left(\frac{h-1}{m}\right) + 1$$

The intuition: finds the latest matching seasonal component (e.g. for March forecast, find the last-known March)

```
from statsmodels.tsa.holtwinters import ExponentialSmoothing

# make the model - data is univariate
# trend/seasonal args can be 'add' or 'mul'
model = ExponentialSmoothing(
    data, trend='add', seasonal='add', seasonal_periods=12)

# 'fit' the model - returns a HoltWintersResults object
result = model.fit()

# in-sample prediction or out-of-sample forecast
result.fittedvalues
result.forecast(n)
```

# Explaining the k formula

$$\hat{y}_{t+h|t} = l_t + h b_t + s_{t+h-mk} \quad k = \text{floor}\left(\frac{h-1}{m}\right) + 1$$

- $t = 36, m = 12, h = 15$
- $k = \text{floor}[(15 - 1) / 12] + 1 = 2$
- Index for s:  $t + h - mk = 36 + 15 - 12 * 2 = 27$

Month	Mar	...	Dec	Jan	...	Jan	Feb	Mar
t	27	...	36	37	...	49	50	51

## Holt-Winter model (multiplicatif)

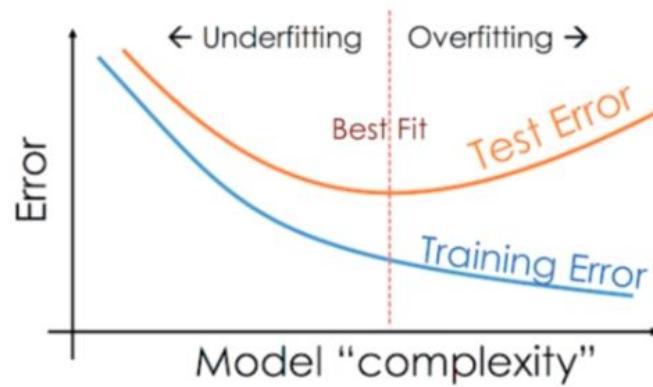
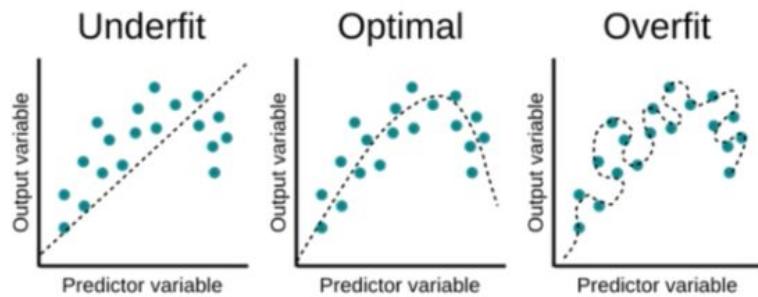
$$Forecast : \hat{y}_{t+h|t} = (l_t + hb_t)s_{t+h-mk}$$

$$Level : \quad l_t = \alpha \frac{y_t}{s_{t-m}} + (1 - \alpha)(l_{t-1} + b_{t-1})$$

$$Trend : \quad b_t = \beta(l_t - l_{t-1}) + (1 - \beta)b_{t-1}$$

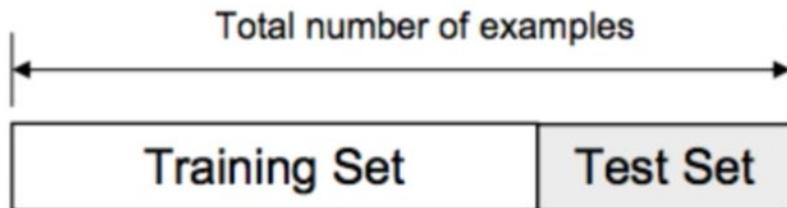
$$Seasonal : \quad s_t = \gamma \frac{y_t}{l_{t-1} + b_{t-1}} + (1 - \gamma)s_{t-m}$$

# Walk-Forward validation



# Single train-test splitting is wrong

- What happens when you try to optimize your model (via its configuration) by picking the best configuration for your test set?
- Your test set has effectively become “in-sample data”



# Cross-validation to mitigate overfitting the validation set

- How about multiple validation sets? (Note: I'm conflating validation/test)
- Think of your true test set as real future data
- E.g. Private data in Kaggle contest, new customers to your website
- K times: train on K-1 parts, evaluate on the remaining part

5-fold CV		DATASET			
Estimation 1	Test	Train	Train	Train	Train
Estimation 2	Train	Test	Train	Train	Train
Estimation 3	Train	Train	Test	Train	Train
Estimation 4	Train	Train	Train	Test	Train
Estimation 5	Train	Train	Train	Train	Test

# For time-series ?

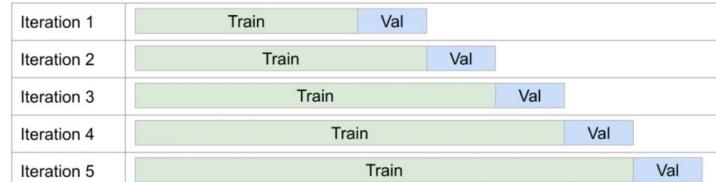
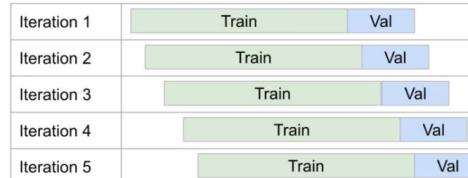
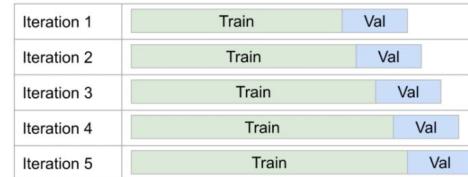
- Since the data is split randomly, you'll mix past and future data
- In the real-world, your model cannot be trained on future data!
- We can only use past data to predict future data

5-fold CV

	DATASET				
Estimation 1	Test	Train	Train	Train	Train
Estimation 2	Train	Test	Train	Train	Train
Estimation 3	Train	Train	Test	Train	Train
Estimation 4	Train	Train	Train	Test	Train
Estimation 5	Train	Train	Train	Train	Test

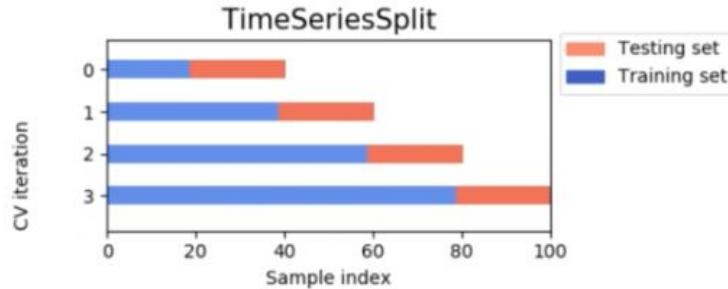


- Start with some minimum data to train on
- ... and validation forecast horizon  $h \geq 1$
- Walk forward one step (train set becomes +1 bigger)
- Validate on the next  $h$  data points, etc etc. until you reach the end



# TimeSeriesSplit (Scikit-Learn)

- Might save you from a **bit** of work, under the right circumstances
- To use with cross validation functions, your model must conform to scikit-learn interface (i.e. not statsmodels)
- Another limitation: not flexible - must use non-overlapping blocks
- All blocks are equal size - first train set is teeny tiny
- Do these reflect real-world operation?



Do your own code according to your needs and intuition !

We won't delve too much time using this approach into the different models we use : it complicates the script. We will just see an example script where you can plug your model afterwards.

# Statistical tests & linear regression

**1. Magnitude and Sign:** The magnitude of the t-statistic indicates the strength of evidence against the null hypothesis (typically, that the coefficient is zero, meaning no effect). A higher absolute value of the t-statistic suggests stronger evidence. The sign indicates the direction of the relationship.

**2. Threshold or Critical Value:** Compare the t-statistic to a critical value from the **t-distribution**, which depends on the desired level of significance (commonly 0.05 for a 95% confidence level) and the degrees of freedom. If the absolute value of the t-statistic is greater than the critical value, the coefficient is considered statistically significant, and the null hypothesis can be rejected. There are several online t-distribution table which can be used to determine critical t-values for both one-tailed and two-tailed tests. Here is one of the them: [T-Distribution Table of Critical Values](#).

**3. P-Value Association:** The **t-statistic is associated with a p-value**, which helps in determining significance. If the p-value is less than the chosen significance level (e.g., 0.05), the coefficient is statistically significant. This means there is sufficient evidence to suggest that the coefficient is different from zero. The null hypothesis can thus be rejected. There are several online **t-statistics / p-value calculator** for finding p-value from t-statistics value. Here is one of them: [P-value from T-statistics – Calculator](#)

```
1 import statsmodels.api as sm
2 # Load the dataset
3 data = sm.datasets.get_rdataset('cars', 'datasets').data
4 # Fit the linear regression model
5 model = sm.formula.ols('dist ~ speed', data=data).fit()
6 # Print the t-test results
7 print(model.summary())
```

```
print(model.summary())
```

OLS Regression Results							
Dep. Variable:	dist	R-squared:	0.651	Model:	OLS	Adj. R-squared:	0.644
Method:	Least Squares	F-statistic:	89.57	Date:	Thu, 13 Apr 2023	Prob (F-statistic):	1.49e-12
Time:	11:54:22	Log-Likelihood:	-206.58	No. Observations:	50	AIC:	417.2
Df Residuals:	48	BIC:	421.0	Df Model:	1	Covariance Type:	nonrobust
	coef	std err	t	P> t	[0.025	0.975]	
Intercept	-17.5791	6.758	-2.601	0.012	-31.168	-3.990	
speed	3.9324	0.416	9.464	0.000	3.097	4.768	
Omnibus:	8.975	Durbin-Watson:	1.676				
Prob(Omnibus):	0.011	Jarque-Bera (JB):	8.189				
Skew:	0.885	Prob(JB):	0.0167				
Kurtosis:	3.893	Cond. No.	50.7				

**1. Estimate (Coefficient):** The coefficient for **speed** is 3.9324. This means that for each additional unit of speed, the stopping distance (**dist**) increases by approximately 3.9324 units, assuming all other factors remain constant.

**2. Standard Error:** The standard error of the estimate for **speed** is 0.4155. This measures the accuracy of the coefficient (3.9324) estimate and indicates a relatively low level of uncertainty or variability in the estimate of the effect of **speed** on **dist**.

**3. t-value:** The t-value for **speed** is 9.464. This value is used to test the null hypothesis that the coefficient (effect of **speed** on **dist**) is zero. A larger absolute t-value indicates a greater deviation of the coefficient from zero, suggesting that it is highly unlikely to observe such a large t-value if the true coefficient were zero.

**4. p-value (Pr(>|t|)):** The p-value for **speed** is extremely small (1.49e-12), which is far below any standard significance level (like 0.05 or 0.01). This indicates that the effect of **speed** on **dist** is statistically significant, meaning that there is a very low probability that the observed relationship between **speed** and **dist** in the sample occurred by random chance.

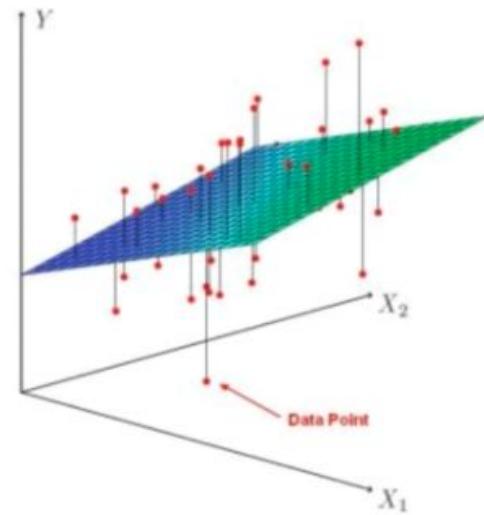
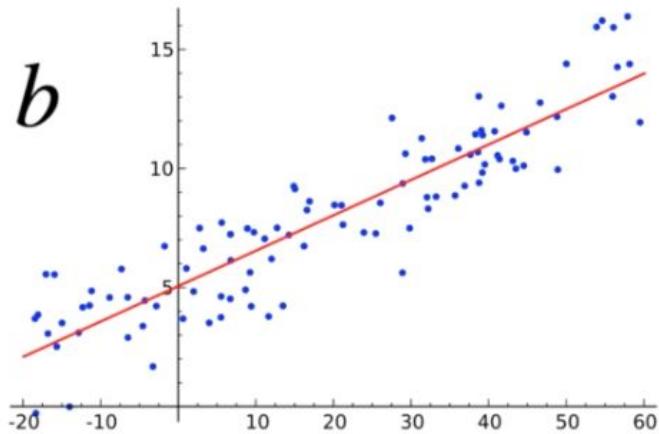
# ARIMA vs Exponential smoothing

- Exponential smoothing is very specific (linear trends, seasonality)
- ARIMA imposes no such structure
- It is more “machine learning”-like

# Autoregressive models

$$\hat{y} = w_1x_1 + w_2x_2 + b$$

$$\hat{y} = mx + b$$



# AR(p) : auto-regressive processus

$$\hat{y}_t = b + \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \dots + \varphi_p y_{t-p}$$

- Suppose our data is:  $y_1, y_2, \dots, y_{10}$
- In ML, we say X has shape N x D, but for ARIMA we'll stick with D == p

y1	y2	y3
y2	y3	y4
y3	y4	y5
y4	y5	y6
y5	y6	y7
y6	y7	y8
y7	y8	y9

This is our "X"

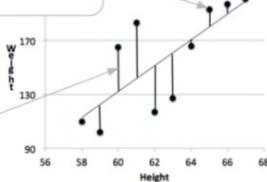
y4
y5
y6
y7
y8
y9
y10

This is our "Y"

$$y_t = b + \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \dots + \varphi_p y_{t-p} + \varepsilon_t$$
$$\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$$
$$\hat{y}_t = E(y_t)$$

y (no hat) is the true data point

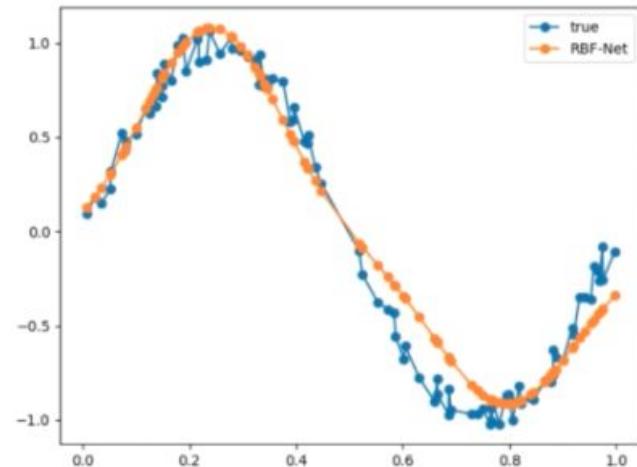
These are the errors



# Machine Learning is the next step

- Linear models aren't that powerful (only lines or planes)
- Why stick to linear regression?
- ARIMA helps us understand the modeling and statistical properties

```
model = NeuralNetwork()  
model.fit(X, Y)  
  
model = RandomForest()  
model.fit(X, Y)
```



MA(q)

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q}$$

Moving around the average c:

.

The diagram illustrates the decomposition of the expected value of  $y_t$ . A box labeled "Zero!" has arrows pointing to each term in the equation  $E(y_t) = E(c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q}) = c$ . The term  $c$  is also explicitly labeled at the end of the equation.

$$\begin{aligned} E(y_t) &= E(c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q}) \\ &= c \end{aligned}$$

## ARMA(p,q)

- ARMA(p, q) = AR(p) + MA(q)

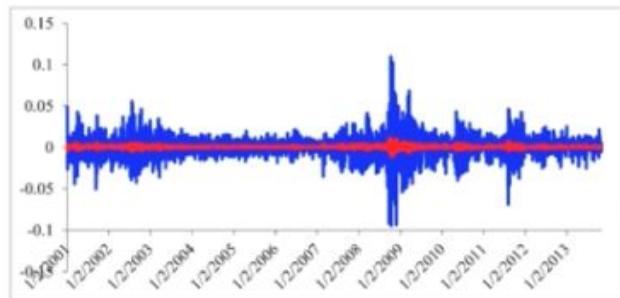
$$y_t = b + \boxed{\varphi_1 y_{t-1} + \dots + \varphi_p y_{t-p}} + \boxed{\theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}} + \varepsilon_t$$

The diagram illustrates the decomposition of an ARMA(p,q) process into its AR(p) and MA(q) components. Two arrows originate from the text 'ARMA(p, q) = AR(p) + MA(q)' and point to the two boxed terms in the equation below. The first arrow points to the term  $\varphi_1 y_{t-1} + \dots + \varphi_p y_{t-p}$ , which represents the AR(p) component. The second arrow points to the term  $\theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$ , which represents the MA(q) component.

## Differencing : detrending

- Where have we seen this?

- Log returns - the difference of log prices:  $r_t = p_t - p_{t-1}$
- Holt's Linear Trend Model:  $b_t = \beta(l_t - l_{t-1}) + (1-\beta)b_{t-1}$

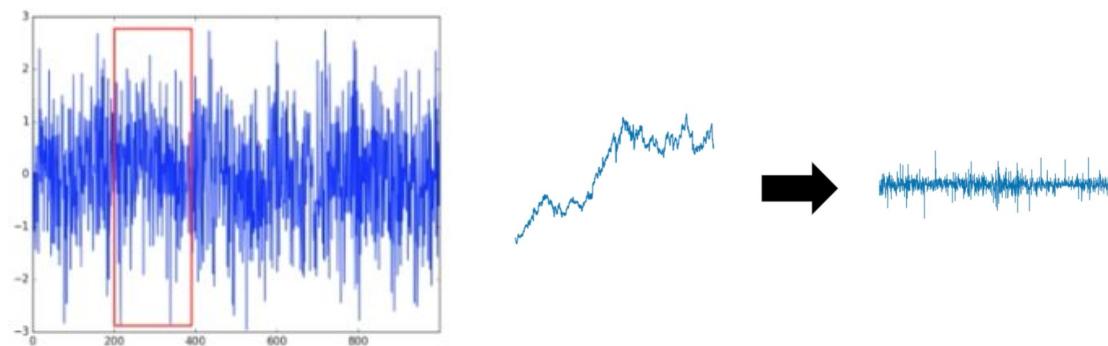


Given :  $\{y_t\} = \{y_1, y_2, \dots, y_T\}$  (some time series)

Differenced Series :  $\Delta y_t = y_t - y_{t-1}$

# ARMA modelling needs stationary time series

- When fitting an ARMA model, we want the data to be close to stationary
- Stationary = Does not change over time
- Stationarity is nice: mean, variance, autocorrelation, ... will be constant over time
  - Recall: linear models fit well when there is strong correlation between inputs / output
- Each “window” of the time series is like a “training point” for fitting the model



# I(d) and ARIMA(p,d,q)

- An I(d) process is a process that is stationary after differencing d times
- We say it's integrated to order d
- ARIMA(p, d, q) is just a model where we've differenced d times before applying ARMA(p, q)

## Random Walk

- ARIMA(p, 0, 0) is AR(p)
- It's also ARMA(p, 0)
- ARIMA(0, 0, q) is ARMA(0, q) and MA(q)
- ARIMA(0, d, 0) is I(d)

- ARIMA(0, 1, 0) is I(1) and this is a random walk

Differenced time series

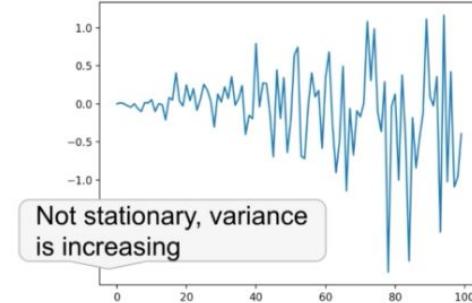
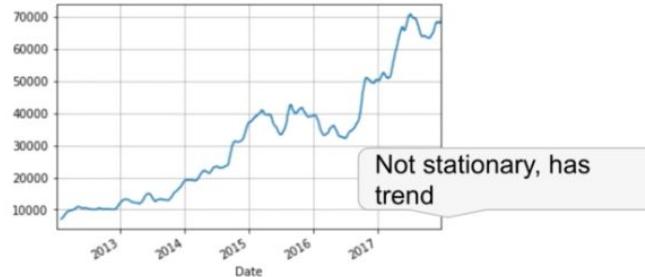
Absence of AR and MA components

$$\Delta y_t = \varepsilon_t$$

$$y_t - y_{t-1} = \varepsilon_t$$

$$y_t = y_{t-1} + \varepsilon_t$$

# Stationarity

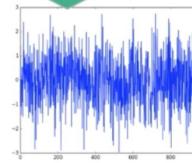
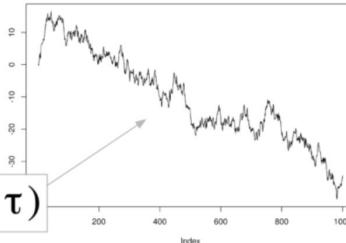


- Loosely, the distribution of the random variables in the time series does not change over time
- E.g. mean and variance will always be the same

- If a time series is nonstationarity, then you might need different models at different points in time!
- For nonstationary time series, you can't treat data points at different times like "samples" (e.g. computing the mean, variance)

You can't compute "the mean" from the data because the mean is changing!

$$\mu_Y(t) \neq \mu_Y(t + \tau)$$



# White noise

## Définition 1.9 : Bruit blanc

Un processus stationnaire  $(Z_t)_{t \in \mathbb{Z}}$  de moyenne  $\mu$  et d'autocovariance  $\gamma_Z$  est un...

- **bruit blanc faible** ou **bruit blanc** si  $\text{Cov}(Z_s, Z_t) = 0$  pour tous  $s \neq t$  ;
- **bruit blanc moyennement fort** si  $Z_s$  et  $Z_t$  sont indépendantes pour tous  $s \neq t$ , c'est-à-dire que  $(Z_t)_{t \in \mathbb{Z}}$  sont deux à deux indépendantes ;
- **bruit blanc fort** lorsque les variables  $(Z_t)_{t \in \mathbb{Z}}$  sont indépendantes ;
- **bruit blanc très fort** lorsque les variables  $(Z_t)_{t \in \mathbb{Z}}$  sont i.i.d.

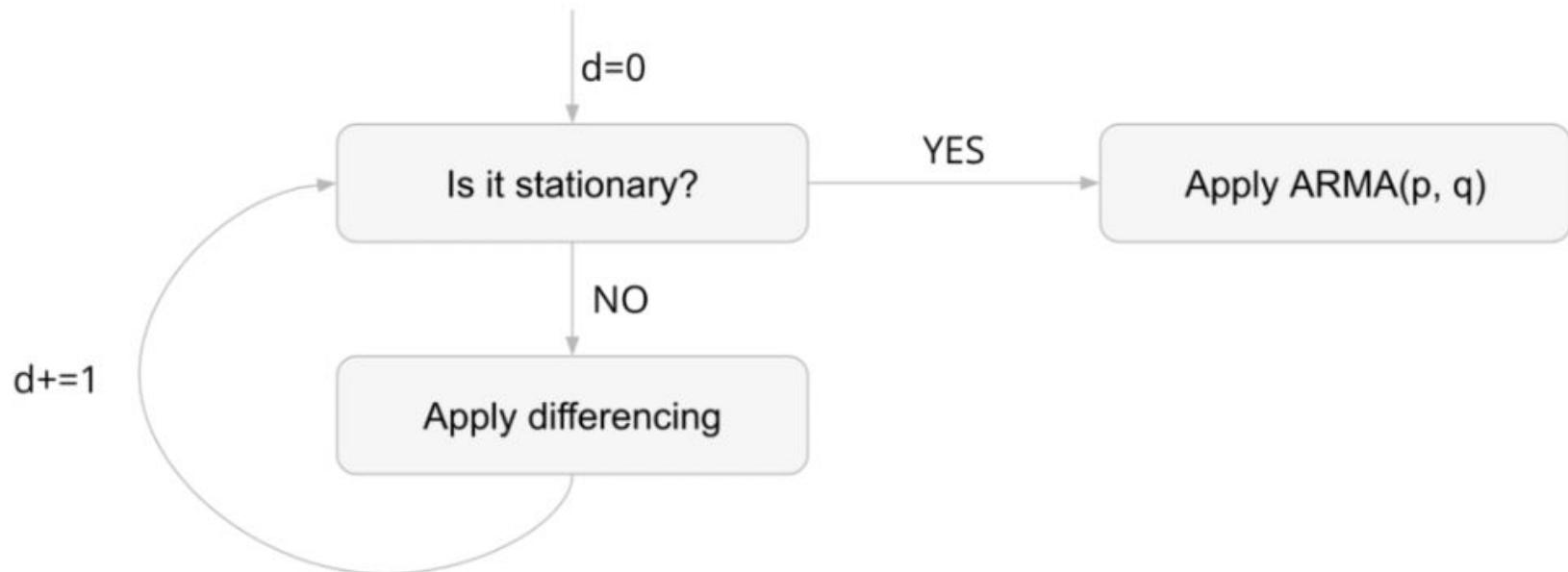
Dans tous les cas  $\gamma_Z(h) = \sigma^2 \mathbf{1}_{h=0}$  et on note  $\text{BB}(\mu, \sigma^2)$ . Dans ces notes de cours, lorsque la moyenne du BB n'est pas précisée, elle vaut zéro par convention.

# Testing for stationarity

- We use the Augmented Dickey-Fuller Test (ADF Test)
- Think of it like an API:
  - Given: null hypothesis, alternative hypothesis
  - Input: time series, Output: p-value
  - Action: accept or reject the null hypothesis
- For ADF test:
  - Null: time series is non-stationary
  - Alternative: time series is stationary



# How to use ADF test in selecting d in ARIMA



# Strong vs Weak

- Strong: the entire distribution does not change over time

$$F_Y(y_{t_1+\tau}, y_{t_2+\tau}, \dots, y_{t_n+\tau}) = F_Y(y_{t_1}, y_{t_2}, \dots, y_{t_n}), \forall \tau, t_1, t_2, \dots, t_n$$

- Weak : First order (mean) and second-order statistics (covariance) stay the same

- The mean does not change over time

$$\mu_Y(t) = \mu_Y(t + \tau) \text{ for all } \tau$$

- The autocovariance does not change over time

$$K_{YY}(t_1, t_2) = K_{YY}(t_1 - t_2, 0) \text{ for all } t_1, t_2$$

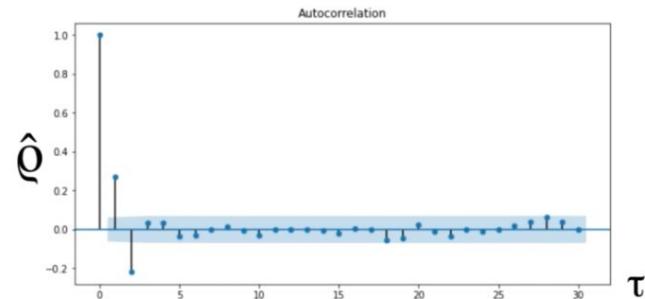
# Autocorrelation function

*Stationary* :  $\rho(Y(t_1), Y(t_2)) = \rho(t_1 - t_2) = \rho(\tau)$

- Also known as correlogram
- Autocorrelation is to autocovariance as correlation is to covariance
- Auto = Self (both RVs come from the same time series)

Autocovariance :  $cov(Y_{t_1}, Y_{t_2})$

Autocorrelation :  $\frac{cov(Y_{t_1}, Y_{t_2})}{\sigma_Y(t_1)\sigma_Y(t_2)}$



# How to determine q in MA(q)

- Assign q to be the maximum non-zero lag
- E.g. in below chart, q = 2
- Usually, the ACF for lags < q are also non-zero

- This can be derived mathematically! (we won't do it right now)
- For MA(1):

$$y_t = c + \theta_1 \varepsilon_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \sim N(0, \sigma^2)$$

## Why does it work ?

$$\varrho(1) = \frac{\theta_1}{1+\theta_1^2}, \quad \varrho(\tau) = 0 \text{ for } \tau > 1$$

- For MA(2):

$$\varrho(1) = \frac{\theta_1 + \theta_1 \theta_2}{1+\theta_1^2+\theta_2^2}, \quad \varrho(2) = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}, \quad \varrho(\tau) = 0 \text{ for } \tau > 2$$

## PACF of order p of AR(p)

- Definition: The PACF at lag  $\tau$  is the autocorrelation between  $Y(t)$  and  $Y(t+\tau)$ , *conditioned on*  $Y(t+1), Y(t+2), \dots, Y(t+\tau-1)$



$$\varphi(\tau, \tau) = \text{corr}(Y_{t+\tau} - \hat{Y}_{t+\tau}, Y_t - \hat{Y}_t)$$

$$\hat{Y}_{t+\tau} = \beta_0 + \beta_1 Y_{t+1} + \beta_2 Y_{t+2} + \dots + \beta_{\tau-1} Y_{t+\tau-1}$$

$$\hat{Y}_t = \beta_0' + \beta_1' Y_{t+1} + \beta_2' Y_{t+2} + \dots + \beta_{\tau-1}' Y_{t+\tau-1}$$

## AIC & BIC criteria for hyperparameters selection

- Difference is in the penalty term
- Auto ARIMA uses AIC by default (but usually both give same answer)

$$AIC = 2(\#params) - 2 \log L$$

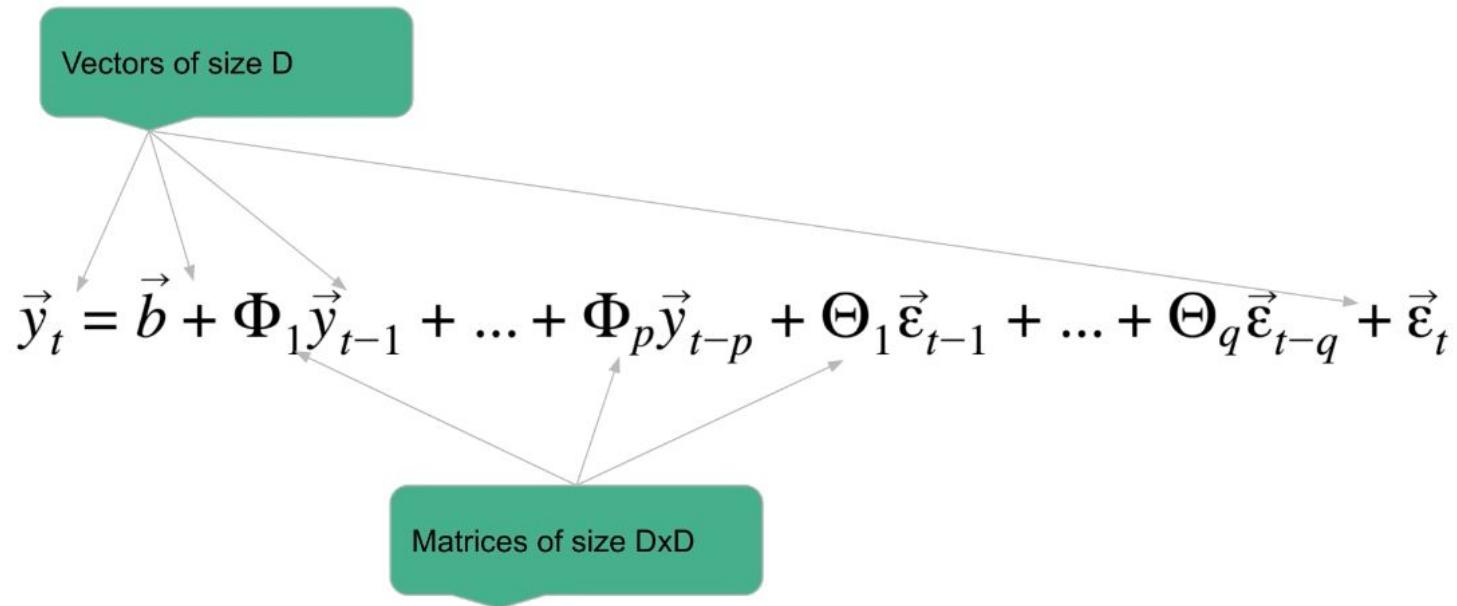
$$BIC = (\#params) \log T - 2 \log L$$

$L$  = likelihood

$T$  = length of time series (number of data points)

# Vector Autoregressive Moving Average

VARMA



# Cross terms from multivariate series

- VAR(1)

$$y^{(1)}_t = b^{(1)} + \varphi_{11}y^{(1)}_{t-1} + \varphi_{12}y^{(2)}_{t-1} + \varepsilon^{(1)}_t$$

$$y^{(2)}_t = b^{(2)} + \varphi_{21}y^{(1)}_{t-1} + \varphi_{22}y^{(2)}_{t-1} + \varepsilon^{(2)}_t$$

- 2 separate AR(1)'s

$$y^{(1)}_t = b^{(1)} + \varphi_{11}y^{(1)}_{t-1} + \varepsilon^{(1)}_t$$

$$y^{(2)}_t = b^{(2)} + \varphi_{22}y^{(2)}_{t-1} + \varepsilon^{(2)}_t$$

# Model Identifiability is a complex topic

- Fact: VARMA models are not unique
- Cannot uniquely determine p and q
- We'll see this warning in statsmodels

## Sparse Identification and Estimation of Large-Scale Vector AutoRegressive Moving Averages

Ines Wilms<sup>a\*</sup>, Sumanta Basu<sup>b\*</sup>, Jacob Bien<sup>c†</sup> and David S. Matteson<sup>b</sup>

<sup>a</sup> Department of Quantitative Economics, Maastricht University, Maastricht, The Netherlands

<sup>b</sup> Department of Statistics and Data Science, Cornell University, Ithaca, NY, USA

<sup>c</sup> Data Sciences and Operations, University of Southern California, Los Angeles, CA, USA

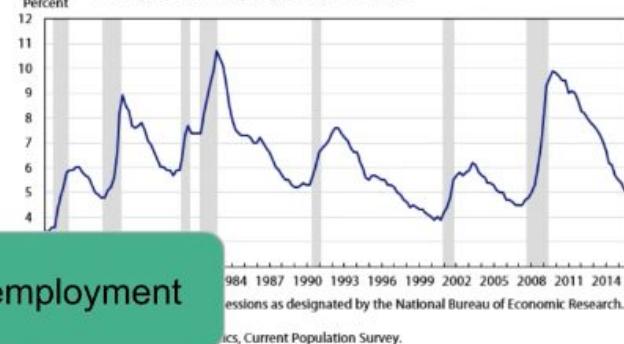
**Abstract.** The Vector AutoRegressive Moving Average (VARMA) model is fundamental to the theory of multivariate time series; however, identifiability issues have led practitioners to abandon it in favor of the simpler but more restrictive Vector AutoRegressive (VAR) model.

# No VARIMA as I will change according to univariate series

- We only want to use VARMA / ARMA if the time series is stationary
- Can find some packages / discussion for VARIMA, but not in statsmodels
- Each component of your time series may need to be differenced a different number of times than the others! (i.e. there isn't one "d" value)



Figure 1. Unemployment rate for people 16 years and older, quarterly averages, seasonally adjusted, 1969–2015



## VARMA(p,q) code + exogenous data

T x D

```
model = VARMAX(train, order=(p, q))  
res = model.fit(maxiter=100)  
fcast = res.get_forecast(Ntest)  
train_pred = res.fittedvalues  
test_pred = fcast.predicted_mean # df with 'col1', ... 'colD'  
ci = fcast.conf_int() # df with 'lower col1', 'upper col1', ...
```

# Granger causality

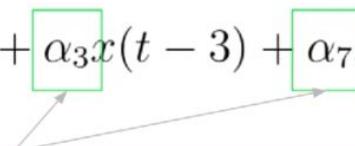
- Recall: in regression analysis, we determine whether it's statistically significant by testing if it's "big enough" (

- Model 1:

$$y(t) = b + \phi_1 y(t-1) + \dots \phi_p y(t-p) + \varepsilon(t)$$

- Model 2:

$$y(t) = b + \phi_1 y(t-1) + \dots \phi_p y(t-p) + \alpha_3 x(t-3) + \alpha_7 x(t-7) + \varepsilon(t)$$



If any of these cross-coefficients are statistically significant, then we say  $x(t)$  "Granger causes"  $y(t)$  – note: you can pick any lags, these are just examples

```
> summary(m)

Call:
lm(formula = y ~ u + v + w)

Residuals:
    Min      1Q  Median      3Q     Max 
-3.3965 -0.9472 -0.4708  1.3730  3.1283 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 1.4222    1.4036  1.013  0.32029  
u            1.0359    0.2811  3.685  0.00106 **  
v            0.9217    0.3787  2.434  0.02211 *  
w            0.7261    0.3652  1.988  0.05744 .  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.625 on 26 degrees of freedom
Multiple R-squared:  0.4981,    Adjusted R-squared:  0.4402 
F-statistic: 8.603 on 3 and 26 DF,  p-value: 0.0003915
```

# PROPHET

Business oriented

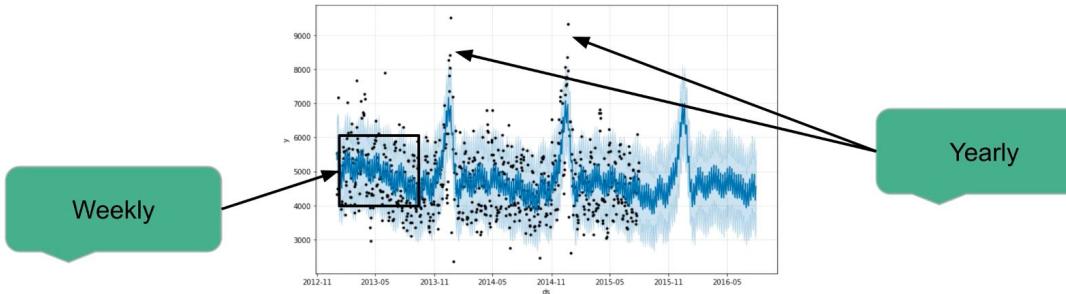
- Built with business datasets in mind
- Model sales, CPU usage, adspend, etc.
- Do you run a business? Do you count anything? That's a time series!
- Easy to account for weekends, holidays, special events
- Easy to add custom exogenous inputs (e.g. NFL Sundays)
- Excellent choice for consultants and freelancers

Pragmatic

- Just a few lines of code and relatively fast
- Recall: ARIMA can struggle with large T
- Most things (even cross-validation) are just 1 line of code
- Very interpretable
- Reminiscent of ETS methods
- Plot components with a function call
- Seasonality at multiple scales
  - E.g. weekly / yearly
- E.g. decrease on Sunday, but increase on Black Friday

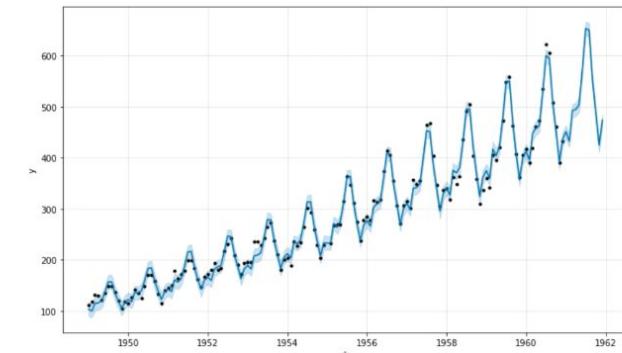
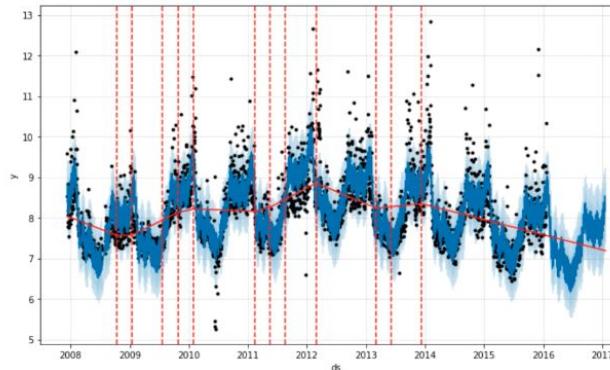
Interpretable, bayesian  
for client interaction

- Seasonality at multiple scales
- Weekly: less customers on Sundays
- Yearly: Christmas, New Years, Valentine's Day, etc.
- Cyclical but not exactly periodic: Thanksgiving, Black Friday, Easter, etc.
- Lunar Calendar: Lunar New Year



- Changes in trend
- Might be due to new website design, new product release, etc.

- Easily deal with outliers and missing data





# Component of Prophet model

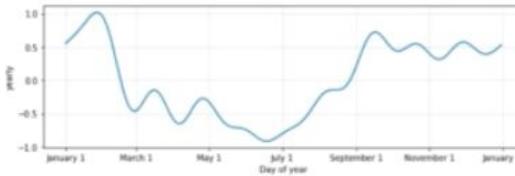
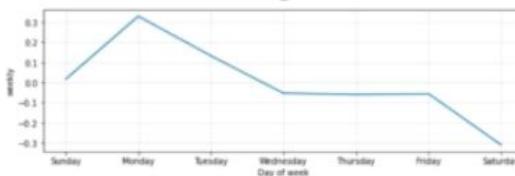
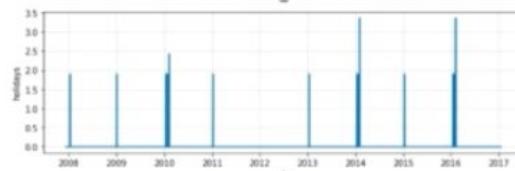
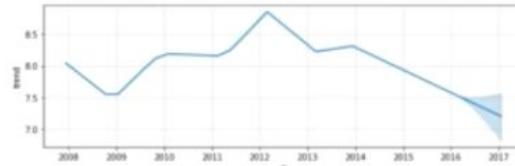
$$y(t) = g(t) + s(t) + h(t) + \varepsilon(t)$$

$g(t)$  : *trend*

$s(t)$  : *seasonal*

$h(t)$  : *holiday*

$\varepsilon(t)$  : *error*

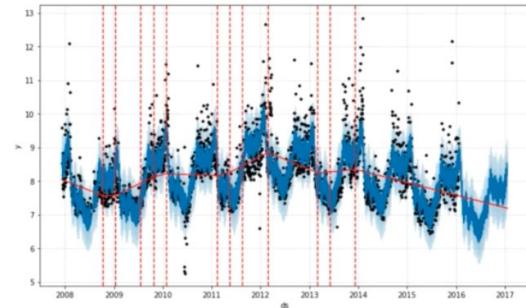


# Trend $g(t)$

- Linear

slope  $\times$  time + intercept

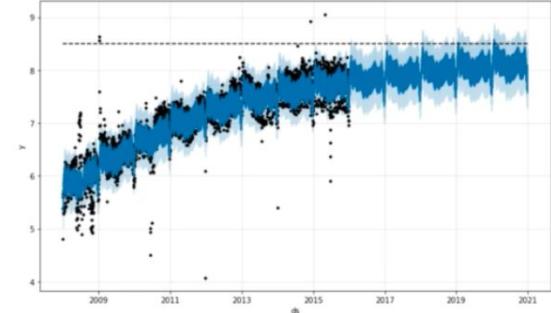
$$g(t) = (k + \mathbf{a}(t)^\top \boldsymbol{\delta})t + (m + \mathbf{a}(t)^\top \boldsymbol{\gamma})$$



- Logistic

$C(t) * \text{sigmoid}(\text{linear}(t))$

$$g(t) = \frac{C(t)}{1 + \exp(-(k + \mathbf{a}(t)^\top \boldsymbol{\delta})(t - (m + \mathbf{a}(t)^\top \boldsymbol{\gamma})))}$$



$$g(t) = \frac{C}{1 + \exp(-k(t - m))}$$

## Seasonal $s(t)$

- Fourier series - don't worry if you're not an engineer / physicist
- We know that sines and cosines are periodic
- $P = 365.25$  for yearly data,  $P = 7$  for weekly, when time scaled in days

$$s(t) = \sum_{n=1}^N \left( a_n \cos\left(\frac{2\pi nt}{P}\right) + b_n \sin\left(\frac{2\pi nt}{P}\right) \right)$$

## Holidays $h(t)$

- One-hot encode every holiday
- Influence of each holiday governed by components of  $\kappa$  (additive by default)
- Can also model surrounding days, but paper doesn't specify how

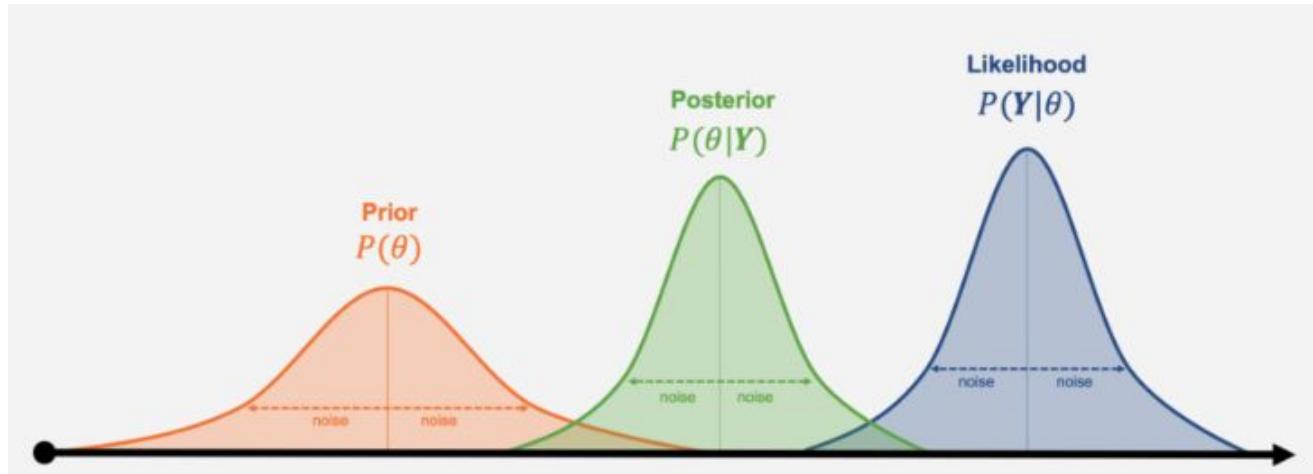
$$Z(t) = [\mathbf{1}(t \in D_1), \dots, \mathbf{1}(t \in D_L)]$$

$$h(t) = Z(t)\kappa.$$

we use a prior  $\kappa \sim \text{Normal}(0, \nu^2)$ .

# Bayesian model

- Hinted at on previous slide - all parameters have priors
- Behind the scenes, Prophet uses Stan (a Bayesian ML library)
- Allows for “human in the loop” optimization
- i.e. include your client on the modeling process (e.g. modify changepoint prior to yield the appropriate number of trend changes)



# Modeling heteroscedasticity

- ARCH
- GARCH
- NGARCH
- IGARCH
- EGARCH
- GARCH-M
- QGARCH
- GJR-GARCH
- TGARCH
- fGARCH
- COGARCH
- ZD-GARCH

Normally, we want to model this (e.g. ARIMA)

$$y_t = f(y_{t-1}, \dots, y_{t-p}; \theta) + \varepsilon_t$$

Ex: Gaussian white noise

Now we want to model this!

Assume :  $E[\varepsilon_t] = 0, E[\varepsilon_t^2] = \sigma^2$

## Example: ARIMA + GARCH

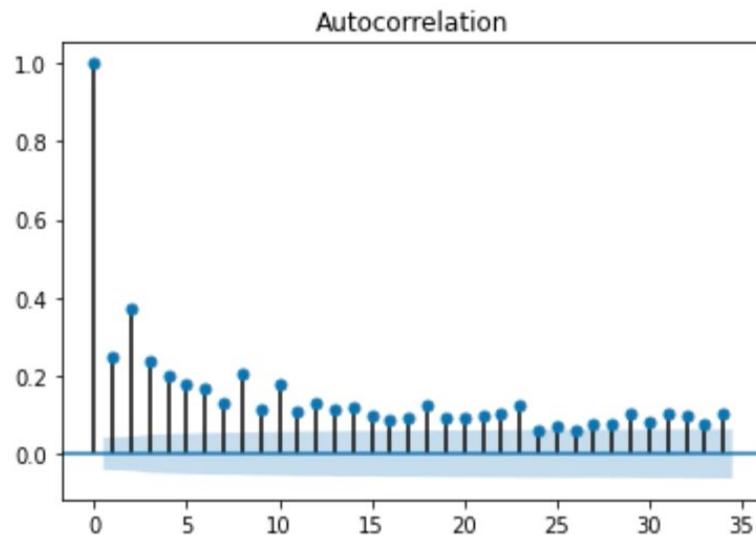
ARIMA: used to model time series mean (variance = 0)

$$y_t = f(y_{t-1}, \dots, y_{t-p}; \theta) + \varepsilon_t$$

GARCH: used to model time series variance (mean = 0)

# ACF of squared returns show auto-regressivity in variance

- For stocks, log returns ACF shows randomness ( $I(1)$  is the best model)
- The ACF of squared log returns does not look random at all



# ARCH(1)

## ARCH(1)

Could be  $N(0, 1)$ , but not necessary

$$E[z_t] = 0, E[z_t^2] = 1$$

Time series we want to model

$$\varepsilon_t = z_t \sqrt{\omega + \alpha_1 \varepsilon_{t-1}^2}$$

Bias term

Autoregressive coefficient

$$\varepsilon_t = z_t \sigma_t$$

=> epsilon has variance sigma squared

- There must be constraints, since variance cannot be negative

$$\varepsilon_t = z_t \sqrt{\omega + \alpha_1 \varepsilon_{t-1}^2}$$

- General constraints

$$\omega > 0, 1 \geq \alpha_1 \geq 0$$

- Constraint for stationarity and finite variance

$$\alpha_1 < 1$$

$$E [\varepsilon_t^2 | \varepsilon_{t-1}, \varepsilon_{t-2}, \dots, \varepsilon_{t-p}] = \sigma_t^2$$

$$E [\varepsilon_t^2] = ? \text{ (call it } \sigma^2)$$

$$E [\varepsilon_t^2] = E [z_t^2] E [\omega + \alpha_1 \varepsilon_{t-1}^2]$$

$$\sigma^2 = 1 \times (\omega + \alpha_1 \sigma^2)$$

$$(1 - \alpha_1) \sigma^2 = \omega$$

$$\sigma^2 = \frac{\omega}{1 - \alpha_1}$$

Conditional  
versus  
unconditional  
variance

$$\varepsilon_t^2 = z_t^2 (\omega + \alpha_1 \varepsilon_{t-1}^2)$$

$$\varepsilon_t = z_t \sigma_t$$

$$\frac{\varepsilon_t^2}{z_t^2} = \sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$$

We model the variance ! Mean or from an ARIMA

$$E [\varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \dots, \varepsilon_{t-p}] = 0$$

$$E [\varepsilon_t] = 0$$

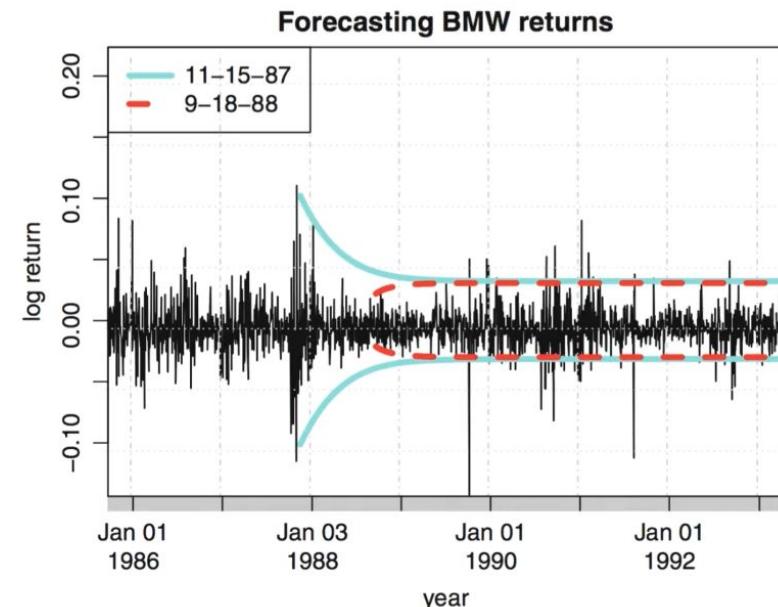
$$\text{cov}(\varepsilon_t, \varepsilon_{t-s}) = 0, \forall s \neq 0$$

$$\varepsilon_t \perp \varepsilon_{t-s} \rightarrow \text{cov}(\varepsilon_t, \varepsilon_{t-s}) = 0$$

$$\text{cov}(\varepsilon_t, \varepsilon_{t-s}) = 0 \not\rightarrow \varepsilon_t \perp \varepsilon_{t-s}$$

# Long term forecast converge to the unconditional variance

- Our forecast of variance will also converge to unconditional variance
- Meaning: ARCH and GARCH are useful for short-term forecasts
- Makes sense: we cannot predict  
Elon Musk making a tweet about  
Bitcoin, or some country banning  
a cryptocurrency exchange
- Obvious example: COVID-19
- Imagine if time series models could  
predict world-scale events!  
(of course, this is unreasonable)
- Aside from insider trading, you  
probably cannot predict these

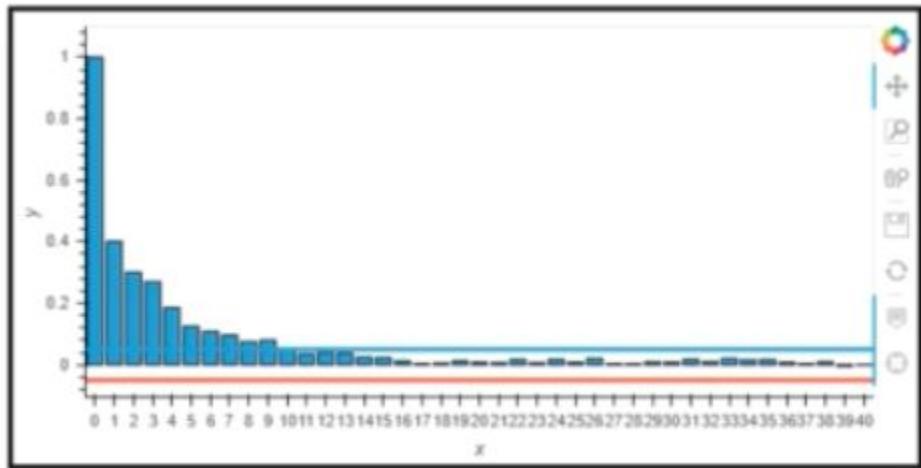


ARCH(p)

$$\varepsilon_t = z_t \sqrt{\omega + \sum_{\tau=1}^p \alpha_\tau \varepsilon_{t-\tau}^2}$$

# ACF of squared epsilons : geometric decay

$$\rho_{\varepsilon^2}(h) = \alpha_1^{|h|}$$



- Unconditional mean is zero
- Conditional mean is zero
- ACF of series shows no correlation with lags (just like stock returns)
- ACF of squared series does show correlation (just like stock returns)
- Allows us to model volatility clustering

GARCH(p,q) : persistent volatility, less HFT moves

ARCH(p) : AR(p)

GARCH(p,q) : ARMA(p,q)

There are other ways to form this analogy, but I think you get the idea...

$$\sigma_t^2 = \omega + \sum_{k=1}^p \alpha_k \varepsilon_{t-k}^2 + \sum_{k=1}^q \beta_k \sigma_{t-k}^2$$

$$\varepsilon_t = z_t \sigma_t$$

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

- Consider a GARCH(1,1) for simplicity (actually, this is the most common)
- $\varepsilon_{t-1}$  term is random,  $\sigma_{t-1}$  term is not random
- The “randomness” comes from  $z_t$
- Imagine  $\beta_1 = 0.9$  - then  $\sigma_t$  would decay slowly over time

GARCH(1,1) as ARMA(1,1)

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad \eta_t = \varepsilon_t^2 - \sigma_t^2$$

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 (\varepsilon_{t-1}^2 - \eta_{t-1})$$

$$\sigma_t^2 = \omega + (\alpha_1 + \beta_1) \varepsilon_{t-1}^2 - \beta_1 \eta_{t-1}$$

$$\sigma_t^2 + \eta_t = \omega + (\alpha_1 + \beta_1) \varepsilon_{t-1}^2 - \beta_1 \eta_{t-1} + \eta_t$$

$$\varepsilon_t^2 = \omega + (\alpha_1 + \beta_1) \varepsilon_{t-1}^2 - \beta_1 \eta_{t-1} + \eta_t$$

$$\eta_t = \varepsilon_t^2 - \sigma_t^2$$

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$$

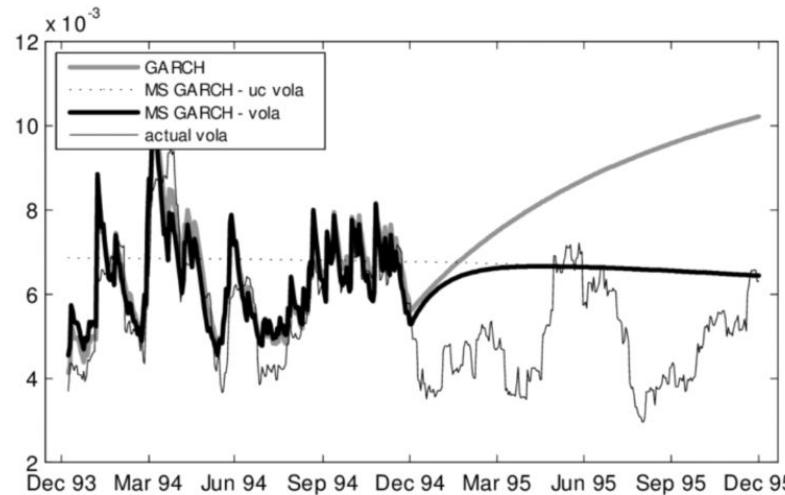
$$\sigma_t^2 + \eta_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \eta_t$$

$$\varepsilon_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \eta_t$$

$$y_t^2 = \omega + \alpha_1 y_{t-1}^2 + \eta_t, \text{ where } y_t = \varepsilon_t$$

# Making prediction of variance

- We want to predict the statistics of  $\varepsilon_t$  (e.g. its mean and variance)
- We care about variance (or standard deviation - i.e. volatility)



# A deep learning approach to GARCH

- 1) How does it make predictions? (i.e. go from input to output)  
Usually some equation involving model parameters ( $\omega, \alpha, \beta$  for GARCH)
- 2) How do we find those model parameters?  
E.g. Deep Learning: "call fit" / "gradient descent"

- For deep learning, ARIMA, and many other ML models, the objective is based on the log-likelihood
- Maximum Likelihood Estimation (MLE): maximize the likelihood wrt model parameters - this is the "training process"
- Regression: typically minimize  $\sum_i (y_i - \hat{y}_i)^2$  (squared error)
- Equivalent to maximizing likelihood when errors are normal
- This requires another assumption! The variance of errors is constant

# Maximum likelihood in case of homoscedasticity

*Assumption* :  $y_t \sim \mathcal{N}(\hat{y}_t, \sigma^2)$

*Equivalent* :  $y_t = \hat{y}_t + \varepsilon_t$ , where  $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$

$$\hat{y}_t(\theta) = f(y_{t-1}, \dots, y_{t-p}; \theta)$$

$$L(\theta) = \prod_{t=1}^T \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(y_t - \hat{y}_t)^2}{\sigma^2}\right)$$

$$l(\theta) = \log L(\theta) = \sum_{t=1}^T \left\{ -\frac{1}{2} \log(2\pi\sigma^2) - \frac{1}{2} \frac{(y_t - \hat{y}_t)^2}{\sigma^2} \right\}$$

$$\theta^* = \arg \max_{\theta} \sum_{t=1}^T \left\{ -\frac{1}{2} \log(2\pi\sigma^2) - \frac{1}{2} \frac{(y_t - \hat{y}_t)^2}{\sigma^2} \right\}$$

$$\theta^* = \arg \max_{\theta} \sum_{t=1}^T \left\{ -\frac{1}{2} \frac{(y_t - \hat{y}_t)^2}{\sigma^2} \right\}$$

$$\theta^* = \arg \max_{\theta} \sum_{t=1}^T \left\{ -(y_t - \hat{y}_t)^2 \right\}$$

$$\theta^* = \arg \min_{\theta} \sum_{t=1}^T \left\{ (y_t - \hat{y}_t)^2 \right\}$$

# Max likelihood in case of hetero scedasticity

$$\theta^* = \arg \max_{\theta} \sum_{t=1}^T \left\{ -\frac{1}{2} \log(2\pi\sigma_t^2) - \frac{1}{2} \frac{(y_t - \hat{y}_t)^2}{\sigma_t^2} \right\}$$

$$\theta^* = \arg \max_{\theta} \sum_{t=1}^T \left\{ -\frac{1}{2} \log(\sigma_t^2) - \frac{1}{2} \frac{(y_t - \hat{y}_t)^2}{\sigma_t^2} \right\}$$

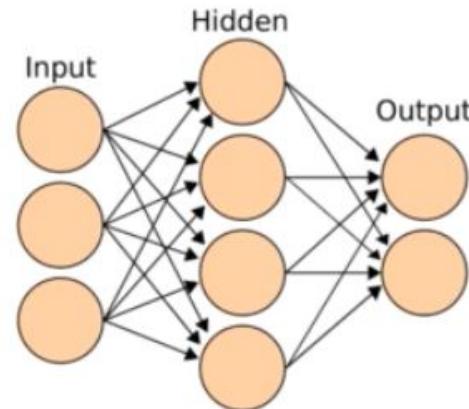
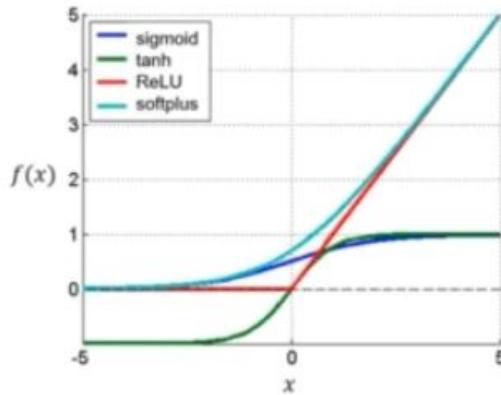
$$\theta^* = \arg \min_{\theta} \sum_{t=1}^T \left\{ \log(\sigma_t^2) + \frac{(y_t - \hat{y}_t)^2}{\sigma_t^2} \right\}$$

$$\theta^* = \arg \min_{\theta} \sum_{t=1}^T \left\{ \log(\sigma_t^2) + \frac{\varepsilon_t^2}{\sigma_t^2} \right\}$$

- Can plug into any stock optimizer (e.g. L-BFGS) - see Scipy docs
- We also need to include constraints on  $\omega, \alpha, \beta$
- Easy to plug in with any optimizer
- Note: Derivation based on normal, but similar steps can be done for t-distribution, skewed t-distribution, etc.

# The deep learning way

- Why stick with GARCH? (a linear model)
- We can parameterize  $\sigma_t$  using a neural network
- Just need to know how to make custom loss functions in TF2/PyTorch/...
- How to constraint output to be positive?
- Use positive-only activation (e.g. softplus)



# Prerequisites for machine learning

## Tensorflow 2.0



Tensorflow 2.0: Deep Learning and Artificial Intelligence

Lazy Programmer Inc., Lazy Programmer Team

## PyTorch



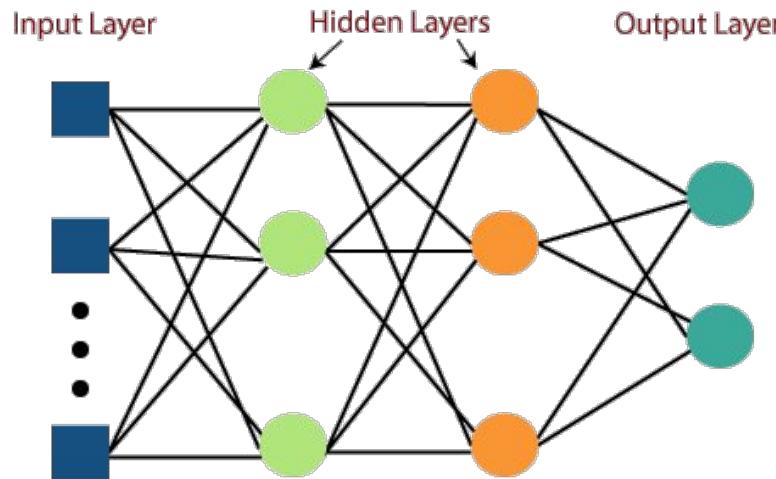
PyTorch: Deep Learning and Artificial Intelligence

Lazy Programmer Team, Lazy Programmer Inc.

In-depth deep learning series (up to CNNs and RNNs)



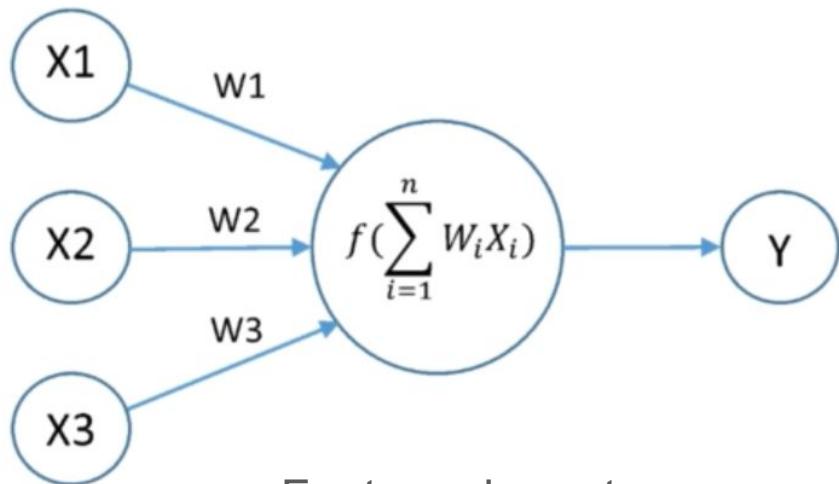
# Multi-Layer perceptron



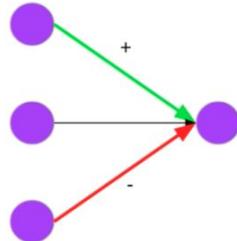
- <https://github.com/fastai/course22/blob/master/04-how-does-a-neural-net-really-work.ipynb>
- <https://developers.google.com/machine-learning/crash-course/introduction-to-neural-networks/video-lecture?hl=fr>

A line :  $ax + b$

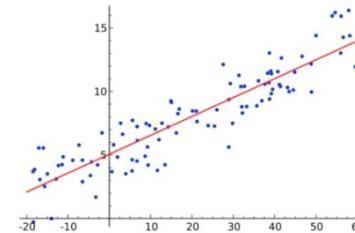
A neuron :  $\sigma(w^T x + b)$



Features importance



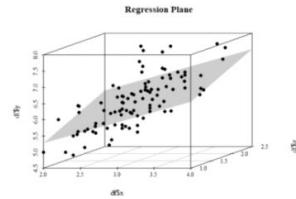
1d regression



$$\hat{y} = mx + b \text{ (or } ax + b\text{)}$$

2d regression

$$\hat{y} = w_1 x_1 + w_2 x_2 + b$$

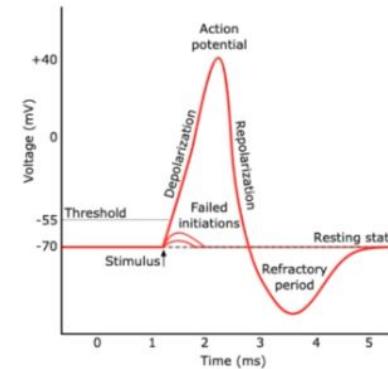
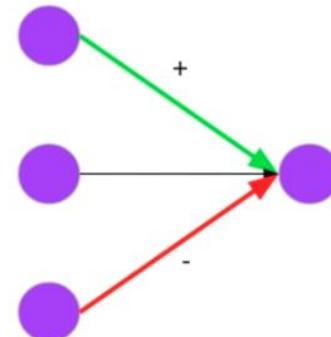


# On the importance of introducing non-linearities through activation

- <https://github.com/fastai/course22/blob/master/04-how-does-a-neural-net-really-work.ipynb>
- [Introduction aux réseaux de neurones](#) (google)

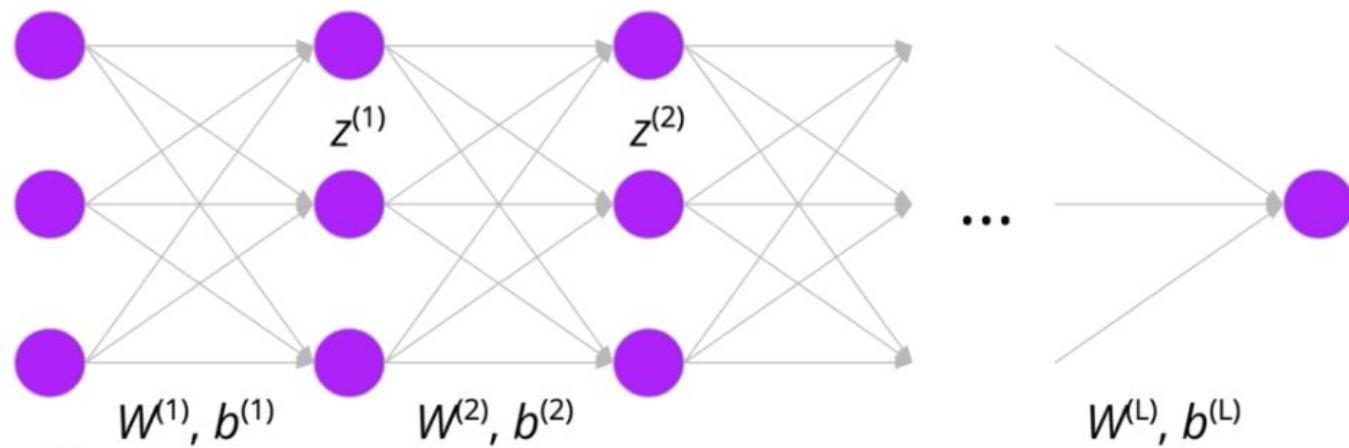
The “threshold” in this case is  $-b$  (negative)

As long as  $w^T x$  is  $> -b$ ,  
then  $w^T x + b > 0$ , and the prediction is 1



$$p(y = 1 \mid x) = \sigma(w^T x + b) = \sigma\left(\sum_{d=1}^D w_d x_d + b\right)$$

# Stacking neurons



$$z^{(1)} = \sigma(W^{(1)T}x + b^{(1)})$$

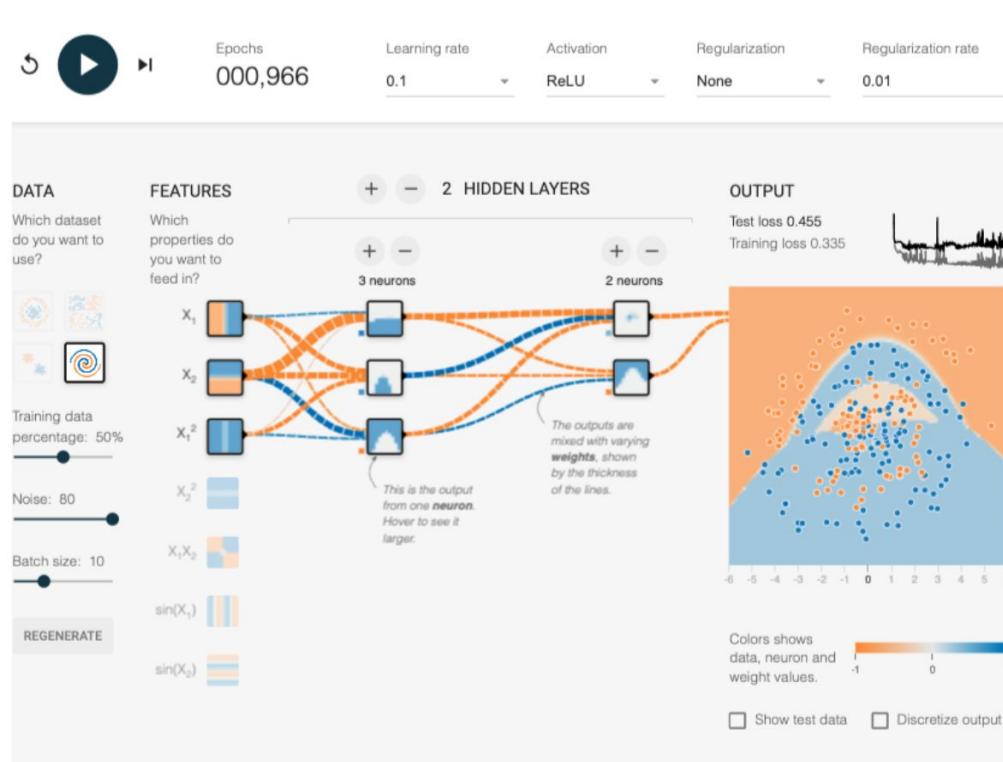
$$z^{(2)} = \sigma(W^{(2)T}z^{(1)} + b^{(2)})$$

$$z^{(3)} = \sigma(W^{(3)T}z^{(2)} + b^{(3)})$$

...

$$p(y = 1 \mid x) = \sigma(W^{(L)T}z^{(L-1)} + b^{(L)})$$

# Fitting complex patterns through non-linear activation function and multiple layers

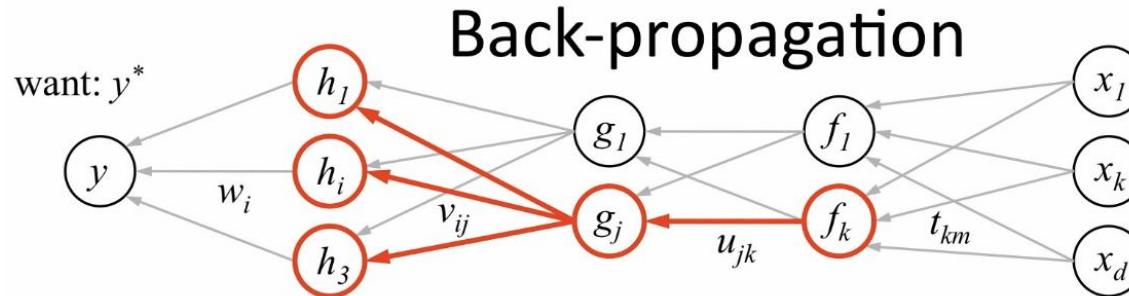


# Building MLP from scratch (tensorflow and pytorch)

- <https://github.com/fastai/course22/blob/master/05-linear-model-and-neural-net-from-scratch.ipynb>
- [https://colab.research.google.com/github/google/eng-edu/blob/main/ml/cc/exercises/linear\\_regression\\_with\\_synthetic\\_data.ipynb](https://colab.research.google.com/github/google/eng-edu/blob/main/ml/cc/exercises/linear_regression_with_synthetic_data.ipynb)

# The matrix calculus you need for deep learning

<https://explained.ai/matrix-calculus>



1. receive new observation  $\mathbf{x} = [x_1 \dots x_d]$  and target  $y^*$
2. **feed forward:** for each unit  $g_j$  in each layer 1...L  
compute  $g_j$  based on units  $f_k$  from previous layer:  $g_j = \sigma\left(u_{j0} + \sum_k u_{jk} f_k\right)$
3. get prediction  $y$  and error  $(y - y^*)$
4. **back-propagate error:** for each unit  $g_j$  in each layer L...1

(a) compute error on  $g_j$

$$\frac{\partial E}{\partial g_j} = \sum_i \underbrace{\sigma'(h_i)}_{\substack{\text{should } g_j \\ \text{be higher or lower?}}} \underbrace{v_{ij}}_{\substack{\text{how } h_i \text{ will} \\ \text{change as } g_j \text{ changes}}} \underbrace{\frac{\partial E}{\partial h_i}}_{\substack{\text{was } h_i \text{ too} \\ \text{high or too low?}}}$$

(b) for each  $u_{jk}$  that affects  $g_j$

<p>(i) compute error on <math>u_{jk}</math></p> $\frac{\partial E}{\partial u_{jk}} = \frac{\partial E}{\partial g_j} \underbrace{\sigma'(g_j)}_{\substack{\text{do we want } g_j \text{ to} \\ \text{be higher/lower?}}} \underbrace{f_k}_{\substack{\text{how } g_j \text{ will change} \\ \text{if } u_{jk} \text{ is higher/lower?}}}$	<p>(ii) update the weight</p> $u_{jk} \leftarrow u_{jk} - \eta \frac{\partial E}{\partial u_{jk}}$
--	--

# Multi-classification : soft-max and cross-entropy

A1	File	Edit	View	Insert	Format	Data	Tools	Help
1	entropy_example	.XLSX						
2	cat	-4.89	0.01	0.00	0	0		
3	dog	2.60	13.43	0.87	1	1		
4	plane	0.59	1.81	0.12	0	2		
5	fish	-2.07	0.13	0.01	0	3		
6	building	-4.57	0.01	0.00	0	4		
7			15.38	1.00				
8								
9								
10								
11								
12								
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14								
15								

$e^{z_i}$

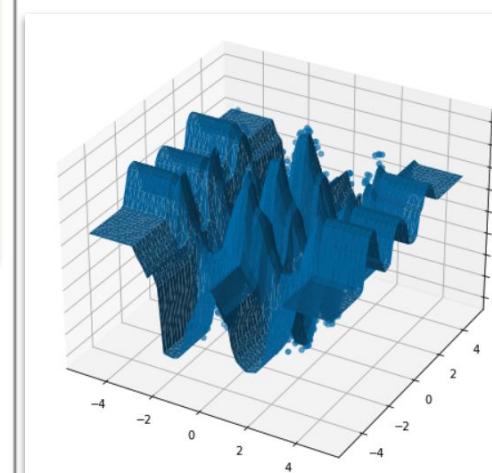
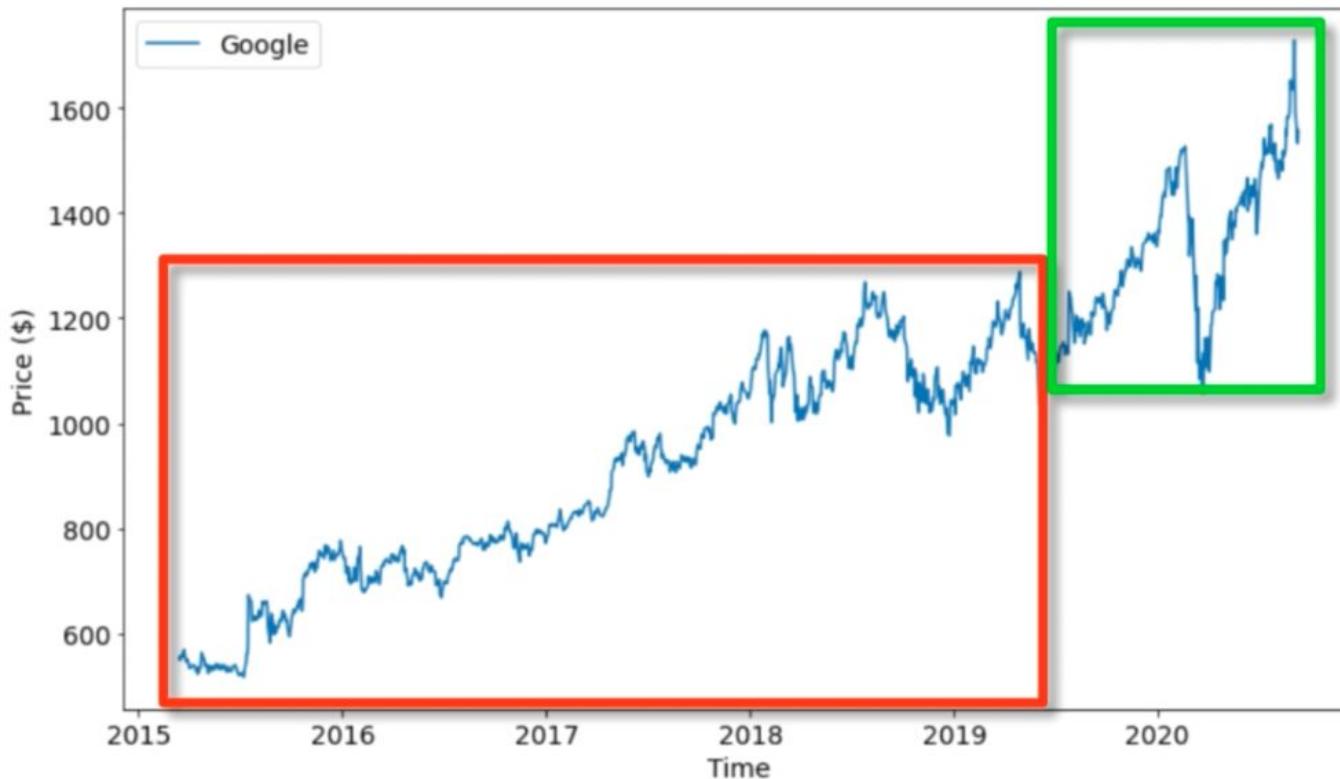
$$\frac{e^{z_i}}{\sum_{j=1}^K e^{z_j}}$$

$$H(P^* | P) = - \sum_i \underbrace{P^*(i)}_{\text{TRUE CLASS DISTRIBUTION}} \log \underbrace{P(i)}_{\text{PREDICTED CLASS DISTRIBUTION}}$$

# ML & time-series analysis

See ML\_ notebooks

# Extrapolation



# Training a model on difference or not ?

$$\Delta y_t = y_t - y_{t-1}$$

*Training data :*  $\{\Delta y_1, \Delta y_2, \Delta y_3, \dots, \Delta y_T\}$

*Train Predictions :*  $\{\hat{\Delta}y_1, \hat{\Delta}y_2, \dots, \hat{\Delta}y_T\}$

$$\begin{aligned}\hat{\Delta}y_t &\approx y_t - y_{t-1} \\ \hat{y}_t &= y_{t-1} + \hat{\Delta}y_t\end{aligned}$$

$$\hat{y}_{T+3} = y_T + \hat{\Delta}y_{T+1} + \hat{\Delta}y_{T+2} + \hat{\Delta}y_{T+3}$$

Our prediction of the  
first difference

# Fourier Transform

## Convolution pour les signaux finis

Soient  $f, h$  des signaux discrets  $f, h \in \mathbb{R}^{\mathbb{Z}}$ . On peut définir la convolution :

$$(f * h)[n] = \sum_{p \in \mathbb{Z}} f[p] h[n - p]$$

Pour des signaux finis  $\tilde{f}, \tilde{h} \in \mathbb{R}^N$ , on périodise :

$$f[n] = \tilde{f}[n \bmod N], \quad h[n] = \tilde{h}[n \bmod N]$$

On définit la **convolution circulaire** :

$$(f \circledast h)[n] = \sum_{p=0}^{N-1} f[n] h[n - p].$$

## Vecteur propres de la convolution circulaire

Soit  $L$  un opérateur de convolution circulaire :

$$Lf[n] = (f \circledast h)[n]$$

Les exponentielles complexes discrètes

$$e_k[n] = \exp\left(2i\pi k \frac{n}{N}\right).$$

sont les vecteurs propres de la convolution circulaire puisque :

$$\begin{aligned} (L e_k)[n] &= \exp\left(2i\pi k \frac{n}{N}\right) \sum_{p=0}^{N-1} h[p] \exp\left(-2i\pi k \frac{p}{N}\right) \\ &= \hat{h}[k] e_k[n] \quad \text{pour} \quad \hat{h}[k] = \sum_{p=0}^{N-1} h[p] \exp\left(-2i\pi k \frac{p}{N}\right). \end{aligned}$$

# Discrete Fourier Transform

On note  $\langle \cdot, \cdot \rangle$  le produit scalaire à droite sur  $\mathbb{C}^n$  :

$$\langle f, g \rangle = \sum_{n=0}^{N-1} f[n] \overline{g[n]}.$$

Les  $(e_k)_{k \in \{0, \dots, N-1\}}$  forment une base orthogonale de  $\mathbb{C}^n$ .

Par conséquent

$$f = \sum_{k=0}^{N-1} \frac{\langle f, e_k \rangle}{\|e_k\|_2^2} e_k.$$

Par définition

$$\hat{f}[k] = \langle f, e_k \rangle = \sum_{n=0}^{N-1} f[n] \overline{e_k[n]} = \sum_{n=0}^{N-1} f[n] \exp\left(-2i\pi k \frac{n}{N}\right).$$

On a  $\|e_k\|_2^2 = N$ , ce qui donne la formule de reconstruction :

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}[k] \exp\left(2i\pi k \frac{n}{N}\right)$$

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{i2\pi k \frac{n}{N}}$$

To find the energy at a particular frequency,  
your signal around a circle at that frequency,  
average a bunch of points along that path.

# Basics

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Box-Cox.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Box-Cox.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Price simulation.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Price simulation.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Black\\_Scholes\\_option\\_pricing\\_through\\_Monte\\_Carlo\\_simulation.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Black_Scholes_option_pricing_through_Monte_Carlo_simulation.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Stationarity.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Stationarity.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ACF\\_PACF.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ACF_PACF.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ACF\\_PACF\\_Stocks.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ACF_PACF_Stocks.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Granger\\_Causality.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Granger_Causality.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Naive Forecast.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Naive Forecast.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/WalkForward.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/WalkForward.ipynb)

# Exponential models

[https://github.com/sduprey/timeseries\\_ressources/blob/main/SMA.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/SMA.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/EMWA\\_HoltWinters.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/EMWA_HoltWinters.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ETS\\_CHAMPAGNE.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ETS_CHAMPAGNE.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ETS\\_Stock.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ETS_Stock.ipynb)

# ARIMA

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ARIMA.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ARIMA.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ARIMA\\_Sales\\_forecast.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ARIMA_Sales_forecast.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ARIMA\\_stock.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ARIMA_stock.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Auto\\_ARIMA.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Auto_ARIMA.ipynb)

# Prophet

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Prophet\\_Airline.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Prophet_Airline.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Prophet\\_Rossmann.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Prophet_Rossmann.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Prophet\\_stockprice.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Prophet_stockprice.ipynb)

# GARCH

[https://github.com/sduprey/timeseries\\_ressources/blob/main/GARCH.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/GARCH.ipynb)

# VARMA

[https://github.com/sduprey/timeseries\\_ressources/blob/main/VARMA.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/VARMA.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/VARMA\\_US\\_ECO.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/VARMA_US_ECO.ipynb)

# Machine Learning (SVM-RF-Logistic regression)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/Extrapolation.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/Extrapolation.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ML\\_Airline\\_No\\_Differencing.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ML_Airline_No_Differencing.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ML\\_Airline\\_with\\_differencing.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ML_Airline_with_differencing.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ML\\_Champagne.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ML_Champagne.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ML\\_Stocks\\_regression.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ML_Stocks_regression.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ML\\_Stocks\\_Classification.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ML_Stocks_Classification.ipynb)

# Deep learning models

[https://github.com/sduprey/timeseries\\_ressources/blob/main/ANN\\_Airline.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/ANN_Airline.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/CNN\\_Airline.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/CNN_Airline.ipynb)

[https://github.com/sduprey/timeseries\\_ressources/blob/main/RNN\\_Airline.ipynb](https://github.com/sduprey/timeseries_ressources/blob/main/RNN_Airline.ipynb)

# Project proposition

- Advanced regression theory and statistical test (T-test, F-test, R squared)
- Rewriting the AI time series notebook in PyTorch
- Price a barrier knocking call using Monte-Carlo simulation of a stochastic process
- Apply Prophet to the airline passengers dataset
- Deep dive in the maths of ARMA process
- Spectral analysis
- Cointegration and stationarity statistical tests
- VEC for multivariate time series modeling
- Pattern recognition/signal filtering (outliers detection/regime change detection)