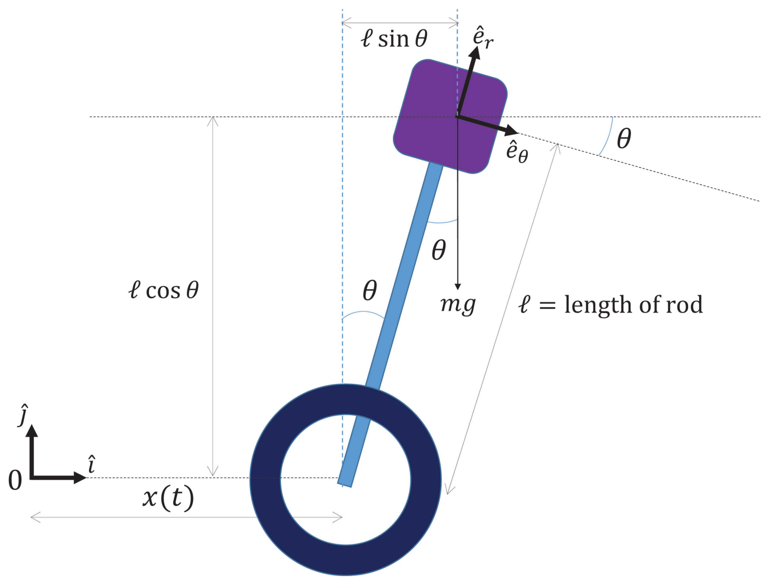

Figure 1: Inverted Pendulum on a Cart

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1 Governing Equation

Consider an inverted pendulum on a cart as shown in Figure 1. The origin is the point marked 0, and the unit vectors in the $\hat{\imath}$ and $\hat{\jmath}$ direction are shown. We assume the mass at the end of the pendulum has mass m, and the rod is massless. Additionally, we define two other unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ which will aid us in deriving the governing equations of the system. These vectors are as illustrated in the figure, with $\hat{e}_\theta$ orthogonal to $\hat{e}_r$. The cart moves in the $\hat{\imath}$ direction in order to keep the pendulum upright.

We start by writing the position of the mass in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ as follows:

$$\vec{r}_{\text{mass}} = (x + \ell \sin\theta)\hat{\imath} + (\ell \cos\theta)\hat{\jmath} \tag{1}$$

By taking the derivative with respect to time (and with reference to the origin and $\hat{\imath}$ and $\hat{\jmath}$), we have the following for the velocity in the $\hat{\imath}$ and $\hat{\jmath}$ directions:

$$\text{velocity in direction } \hat{\imath} = \dot{x} + \ell\dot{\theta}\cos\theta \tag{2}$$

$$\text{velocity in direction } \hat{\jmath} = -\ell\dot{\theta}\sin\theta \tag{3}$$

Note that θ here is also a function of time, so we need to use the chain rule when taking the derivatives of $\cos\theta$ and $\sin\theta$.

By taking the derivative with respect to time again we have the following for the acceleration in the $\hat{\imath}$ and $\hat{\jmath}$ directions:

$$\text{accelaration in direction } \hat{\imath} = \ddot{x} + \ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta \tag{4}$$

$$\text{acceleration in direction } \hat{\jmath} = -\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta \tag{5}$$

Since we have the accelerations in the $\hat{\imath}$ and $\hat{\jmath}$ directions, we could, in principle, apply $\sum \overrightarrow{F} = m\overrightarrow{a}$ in these directions and derive governing equations. The forces acting on the mass are gravity, which acts in the $-\hat{\jmath}$ direction, and the tension of the rod. In order to avoid dealing with the tension of the rod, we can look at the acceleration the $\hat{e}_r$ and $\hat{e}_\theta$ directions as shown in the figure. Since the tension in the rod acts in the $\hat{e}_r$ direction, it has no component in the $\hat{e}_\theta$ direction. So the tension in the rod has no effect on the acceleration of the mass in the $\hat{e}_\theta$ direction. The component of acceleration in the $\hat{e}_\theta$ direction has contributions from the accelerations in both the $\hat{\imath}$ and $\hat{\jmath}$ directions, and can be found as follows, by substituting (4) and (5):

$$\begin{aligned}
\text{acceleration in direction } \hat{e}_\theta &= (\text{accel. in direction } \hat{\imath})\cos\theta - (\text{accel. in direction } \hat{\jmath})\sin\theta \\
&= (\ddot{x} + \ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta)\cos\theta - (-\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta)\sin\theta \qquad (6)\\
&= \ddot{x}\cos\theta + \ell\ddot{\theta}\cos^2\theta - \ell\dot{\theta}^2\sin\theta\cos\theta + \ell\ddot{\theta}\sin^2\theta + \ell\dot{\theta}^2\cos\theta\sin\theta \\
&= \ddot{x}\cos\theta + \ell\ddot{\theta}(\cos^2\theta + \sin^2\theta) \\
&= \ddot{x}\cos\theta + \ell\ddot{\theta} \qquad (7)
\end{aligned}$$

Now let's look at the total force acting on the mass in the $\hat{e}_\theta$ direction. Since the tension of the rod does not have a component in the $\hat{e}_\theta$ direction, the only force acting in that direction is gravity which points directly downwards. Thus the sum of forces acting in the $\hat{e}_\theta$ direction is

$$\text{Force in direction } \hat{e}_\theta = mg\cos(\pi/2 - \theta) = mg\sin\theta \qquad (8)$$

Applying $F = ma$ in the $\hat{e}_\theta$ direction, we just set the previous equation to equal (7)

$$mg\sin\theta = m(\ddot{x}\cos\theta + \ell\ddot{\theta}) \qquad (9)$$

$$g\sin\theta - \ell\ddot{\theta} = \ddot{x}\cos\theta \qquad (10)$$

Equation (10) is the governing equation of the inverted pendulum on a cart, assuming that acceleration of the cart can be controlled.

1.1 Linearization

Equation (10) is nonlinear, in the sense that the coefficient of $\ddot{x}$ is $\cos\theta$ which is a nonlinear function, and there is also a $\sin\theta$ function which is also nonlinear. Fortunately, we can linearize this differential equation, by assuming that θ is small. When θ is small we can use the following approximations for $\sin\theta$ and $\cos\theta$:

$$\sin\theta \approx \theta$$
$$\cos\theta \approx 1$$

Applying these approximations to (10) and treating them as exact yields the following differential equation

$$g\theta - \ell\ddot{\theta} = \ddot{x} \qquad (11)$$

$$g\theta - \ell\ddot{\theta} = \dot{v} \qquad (12)$$

$$g\theta - \ell\ddot{\theta} = a \qquad (13)$$

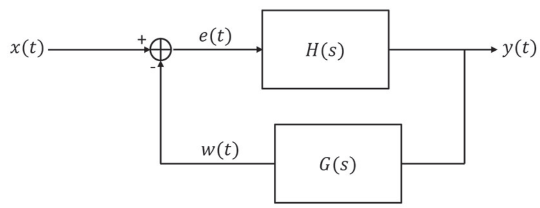
where v and a are the velocity and acceleration of the cart respectively.

2 Feedback Control Review

Here, we review some concepts from feedback control which will help us in designing our system to control the inverted pendulum on a cart.

2.1 Nuts and Bolts

Black's Formula. Consider the feedback system shown below.



The transfer function relating $Y(S)$ to $X(s)$ is

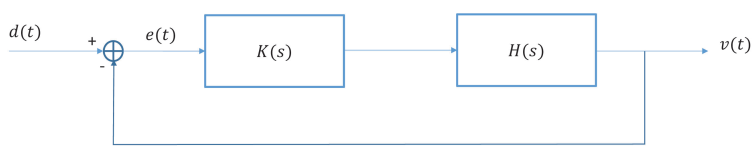
$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)} \tag{14}$$

Final-Value Theorem. Suppose that it is known that $y(t)$ approaches a finite steady state value (as opposed to oscillating indefinitely, or blowing up to infinity). The steady-state value can be found using:

$$y(\infty) = \lim_{s \to 0} sY(s) \tag{15}$$

2.2 Zero Steady State Error

In many control systems, we want to set some output function $v(t)$ in the example below to equal a desired function $d(t)$ as shown below.

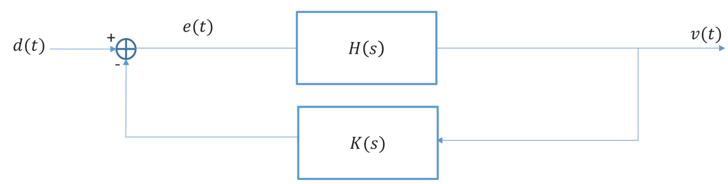


A controller of the form $K(s) = K_p + \frac{K_i}{s}$, called a proportional-integral (PI) controller can remove the error, $e(t)$, in steady state (provided the output converges to some value). This can be shown for specific examples of $H(s)$ by applying the final-value theorem.

Intuitively, the system above will not let any steady-state error to persist with an integral term (i.e. $\frac{K_i}{s}$) in the controller. This is because the integral term will accumulate any steady state error which will eventually cause the output to change. And so, if the output converges to some value, it must be the case that the steady-state error is zero.

2.3 Zero Steady State Output

In certain systems, we may want the output signal to go to zero. In that case, putting a PI controller in the feedback path will make the output go to zero (if the output converges). Consider the following system



If $K(s) = K_p + \frac{K_i}{s}$, then there is an integral in the block marked $K(s)$. So if $v(t)$ converges in steady state (i.e. it does not oscillate indefinitely, or blow up to infinity), then the steady state value of $v(t)$ must be zero. If it were non-zero, the integral in $K(s)$ will accumulate the values of $v(t)$, and cause a change in $e(t)$ (assuming $d(t)$ is constant in steady state). This change will cause $v(t)$ to change. Therefore, the only way for $v(t)$ to converge is for $v(t)$ to be zero in steady state. Therefore, in applications where the output signal needs to be zero, putting a PI controller in the feedback path can be helpful.

3 Controlling the System

Assuming zero initial conditions, we can take the Laplace Transform of both sides of (12). We are making a huge assumption that the velocity can be directly controlled here. But more on that later...

$$g\Theta(s) - \ell s^2\Theta(s) = sV(s) \tag{16}$$

Treating v as the input and θ as the output, we have the transfer function relating the velocity of the cart to the angle as follows

$$H_{v\theta}(s) = \frac{s}{g - \ell s^2} = \frac{-s/\ell}{s^2 - \frac{g}{\ell}} = \frac{-s/\ell}{(s - \sqrt{\frac{g}{\ell}})(s + \sqrt{\frac{g}{\ell}})} \tag{17}$$

From the transfer function above, it is clear that the system is unstable since there is a pole at $s = \sqrt{\frac{g}{\ell}}$. In other words, there may be some velocities that result in θ growing rapidly, which is expected from physical properties of an inverted pendulum. Therefore, we will need to do something to stabilize the system.

Let's start by drawing a simple block diagram relating $v(t)$ and $\theta(t)$ as shown in Fig. 2.

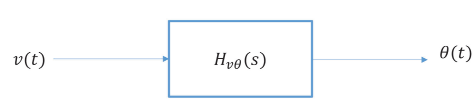


Figure 2: Block diagram illustrating relationship between velocity and angle

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One way to control the system is to make the velocity a function of the angle. You can do this by using a sensor to measure angle and then actuating the motors accordingly. Assuming that the relationship between the angle and the velocity is represented by another transfer function $K(s)$, we have the block diagram in Figure 3. Figure 3 shows that the velocity is related to the angle measurement by transfer function $K(s)$ (which will have to be implemented by the designer), and the dynamics of the inverted pendulum on a cart is represented by $H_{v\theta}(s)$. If the system starts with all initial conditions being zero, i.e. $\theta(0) = \dot{\theta}(0) = \ddot{\theta}(0) = 0$, this system will not change over time, and it will remain balanced forever.

In real life however, a physical pendulum will be subject to disturbances, where the angle $\theta(t)$ could be perturbed, e.g. due to the wind, vibrations of the floor, etc. The block diagram shown in Figure 3 does not include any external disturbance (which

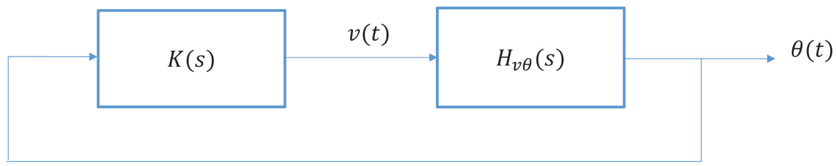

Figure 3: Block diagram with feedback

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is why a pendulum starting with zero initial conditions can remain balanced forever). So we need to incorporate an external disturbance into our system, $d(t)$.

A possible block diagram that incorporates disturbance is shown in Figure 4, where the signal $d(t)$ represents an external disturbance which perturbs $\theta(t)$. One possible example for an external disturbance is the impulse function $\delta(t)$ which can be used to model a very short-lived input such as a sudden jerk.

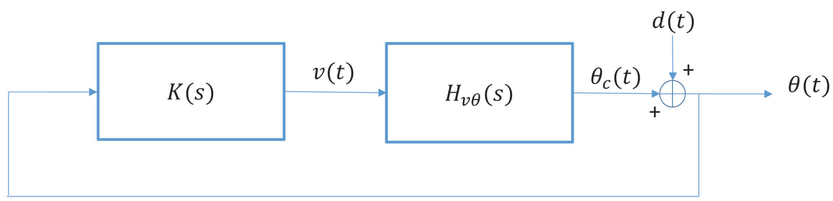

Figure 4: Block diagram of feedback system incorporating disturbance

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We can rearrange the block diagram in Figure 4 into the more familiar form in Figure 5, with the input on the left.

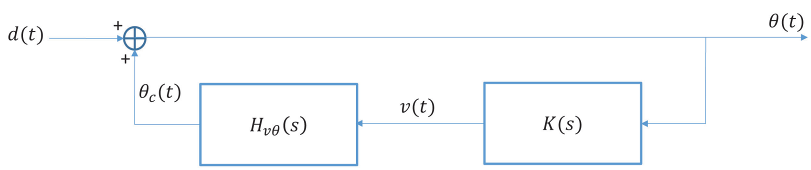

Figure 5: Block diagram of feedback system incorporating disturbance as an input

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Next we use Black's formula to write the transfer function relating $d(t)$ to $\theta(t)$ is:

$$\frac{\Theta(s)}{D(s)} = \frac{1}{1 - H_{v\theta}(s)K(s)} \tag{18}$$

Note that there is a minus sign in the denominator because the feedback signal is added in the summing junction in Figure 5, as opposed to subtracted in the traditional configuration. Let's use a Proportional-Integral controller to balance the system. In this case,

$$K(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s}. \tag{19}$$

Substituting this $K(s)$ into (18) yields,

$$\frac{\Theta(s)}{D(s)} = \frac{1}{1 - \frac{s/\ell}{s^2 - g/\ell}\frac{K_p s + K_i}{s}} = \frac{1}{1 - \frac{\frac{1}{\ell}(K_p s + K_i)}{s^2 - g/\ell}} = \frac{s^2 - g/\ell}{s^2 - g/\ell - \frac{1}{\ell}(K_p s + K_i)} = \frac{s^2 - \frac{g}{\ell}}{s^2 - \frac{K_p}{\ell}s - \frac{K_i}{\ell} - \frac{g}{\ell}}. \tag{20}$$

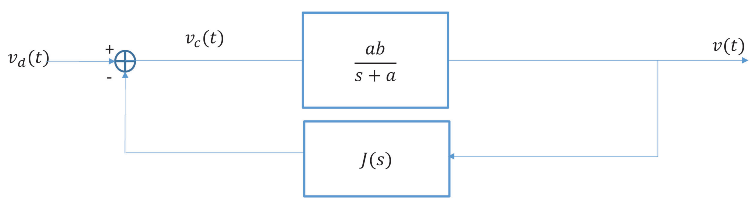
4 Inverted Pendulum with Motor Model

In the previous section, we assumed that the velocity of the robot could be controlled instantaneously. In real life, this is not possible since velocity cannot instantaneously change, i.e. the robot will have to accelerate up to the desired velocity. A simple model relating the control velocity which is sent to the robot, to the actual velocity of the robot is captured by the following transfer function (this is called a first-order motor model):

$$\frac{V(s)}{V_c(s)} = \frac{ab}{s+a} \tag{21}$$

where a and b are parameters of the system.

Ultimately, when the pendulum is balanced, we want the steady state value of $v(t)$ to equal zero as we want our pendulum to be have zero velocity in steady state. To ensure this, we can place a PI controller in a feedback loop, as discussed in Section 2.3. This is shown below, with $J(s) = J_p + \frac{J_i}{s}$

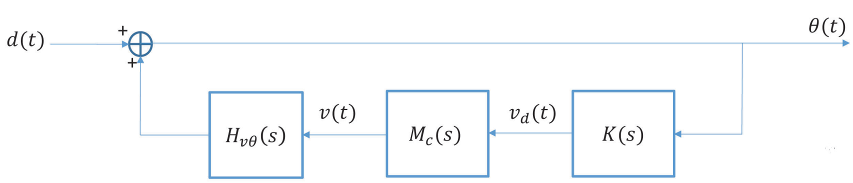


The transfer function relating the actual velocity of the robot to the desired velocity with this controller is

$$M_c(s) = \frac{V(s)}{V_d(s)} = \frac{\frac{ab}{s+a}}{1 + \left(J_p + \frac{J_i}{s}\right)\frac{ab}{s+a}} = \frac{ab}{s + a + abJ_p + \frac{abJ_i}{s}} = \frac{abs}{s^2 + as + abJ_p s + abJ_i}. \tag{22}$$

5 Putting it All Together

Combining the last two sections, we have the following block diagram for the controlled inverted pendulum on a cart



where $\theta(t)$ is the angle, $v_d(t)$ is the desired velocity produced by the angle controller (block marked $K(s)$), $v(t)$ is the actual velocity of the base, and $d(t)$ is an external disturbance.

If we can ensure that the system above is stable and responds fast enough, then we can balance the pendulum.

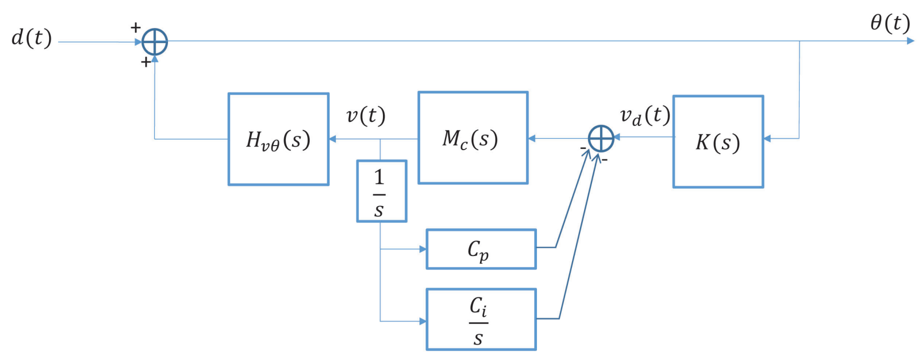
There is, however, a subtle issue with the system above. If we examine the transfer function relating the disturbance to the velocity, applying Black's Formula we get

$$\frac{V(s)}{D(s)} = \frac{M_c(s)K(s)}{1 - M_c(s)K(s)H_{v\theta}(s)}. \tag{23}$$

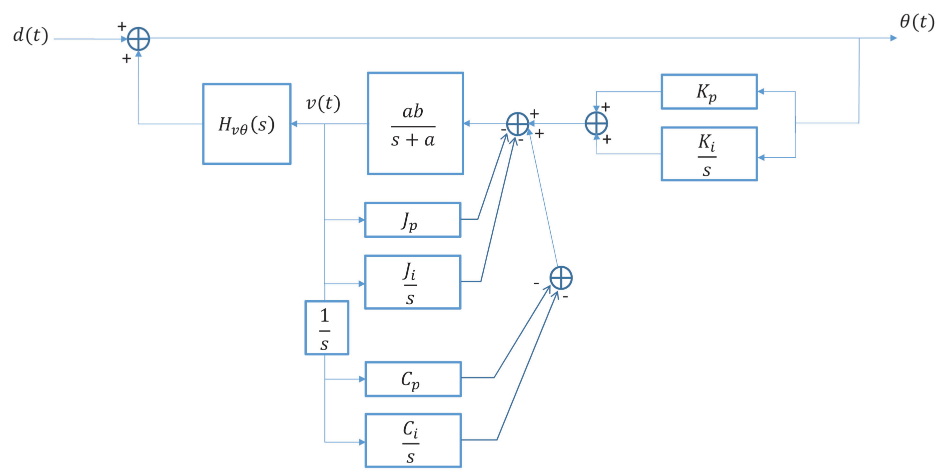
If we look at the position $x(t)$, which is the integral of the velocity, i.e. $X(s) = \frac{1}{s}V(s)$, we get

$$\frac{X(s)}{D(s)} = \frac{M_c(s)K(s)}{s(1 - M_c(s)K(s)H_{v\theta}(s))}. \tag{24}$$

which has a pole at the origin. This pole results in a marginally stable system (since it is not in the left half of the s-plane). As a result, the position of the pendulum could drift significantly over time. Fortunately, we know what to do to force $x(t)$ to have a steady state value of zero. We can apply a PI controller to the signal $X(s) = \frac{1}{s}V(s)$ as discussed in Section 2.3, to make $X(s)$ have a steady state value of zero. This results in a block diagram as follows



If we substitute in for both the $K(s)$ and $J(s)$ controllers, we get



Notice that the block marked C_p and $\frac{J_i}{s}$ are redundant, in that they produce an output that is proportional to $\frac{1}{s}V(s)$ in the s domain. Therefore, we can combine the effects of C_p and J_i which yields the block diagram for the system as shown in Figure 6, where the parameter $J_{iC_p} = J_i + C_p$.

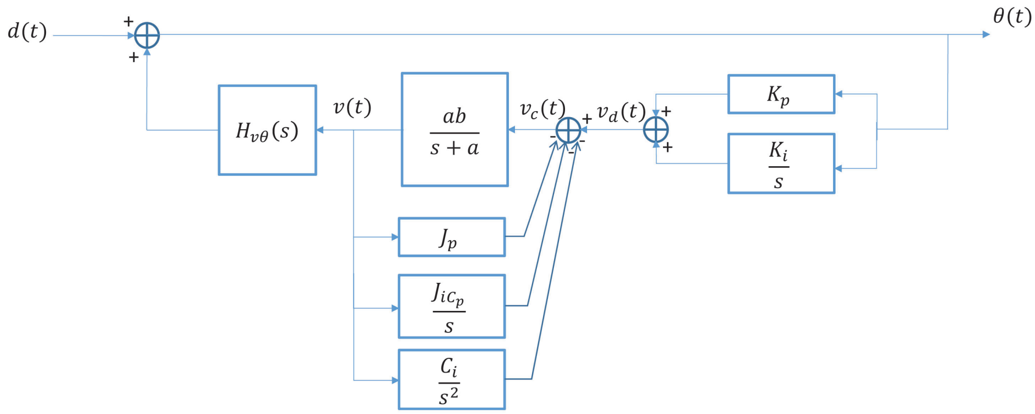


Figure 6: Complete Control System for Inverted Pendulum on a Cart

Thus, with appropriately selected values for the parameters $K_p, K_i, J_p, J_{iC_p}, C_i$, the poles will reside in the left-half of the s-plane resulting in a system in which the *angle, velocity and position* of the cart all approach zero in response to a disturbance to the robot's body (*i.e.*, angle of the pendulum).

5.1 Implementing this Control System on Rocky

There are three general steps to implementing and demonstrating this control system on your Rocky robot.

1. Decide upon the response behavior that you would like Rocky to exhibit. Choose locations for the poles of the system that would result in that behavior.
2. Calculate values for the control parameters that correspond to the chosen poles. A MATLAB script has been provided that you can modify for use for this case.
3. Modify the existing control algorithm sketch by setting the control parameters to the values that you found and provide the equations for motor velocity control inputs, v_c, to both motors.

In the stationary balancing code sketch, you will need to add equations for computing $v_c(t)$ using $v(t)$ and $\theta(t)$ following Figure 6 as

$$v_d(t) = K_p\theta(t) + K_i \int \theta(t)dt \quad \text{and} \tag{25}$$

$$\begin{aligned} v_c(t) &= v_d(t) - J_p v(t) - J_{iC_p} \int v(t)dt - C_i \int \left(\int v(t)dt \right) dt \\ &= v_d(t) - J_p v(t) - J_{iC_p} x(t) - C_i \int x(t)dt \end{aligned} \tag{26}$$

where $x(t)$ is the distance travelled.

6 Acknowledgements

This document is based on the version written by Siddhartan Govindasamy, one of the co-developers of the ESA course.