Studies in Algebra and Group Theory
Math 55a – Harvard University

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Contents

Lecture 1.

Naïve-set-theory

1.1 Ordered Pairs

We begin with a discussion of ordered pairs. Consider a set of numbers $\{1,2,3,4\}$. For any set there is no particular order that triumphs over another, i.e., $\{1,2,3,4\} = \{1,3,2,4\} = \{3,4,2,1\} = \ldots$, an obvious truth. Say we want $\{1,2,3,4\}$ to be in the order of $\{3,2,1,4\}$ for some arbitrary reason; we may then consider the collection,

$$\mathscr{C} = \{\{3\},\{3,2\},\{3,2,1\},\{3,2,1,4\}\}.$$

It becomes obvious what is happening here: the method by which we get order is dependent on the number of iterations in which a particular element appears in any number of sets in a given collection (in comparison to the other elements in the other sets of the respective collection).

Then consider this process but for a nested collection (a collection within a collection) which contains two sets, namely the arbitrary elements a, b in opposing orders, and notice that it gives us the definition of a 2-tuple (ordered pair),

$$\mathscr{C}_1 = \{\{a\},\{a,b\}\} \coloneqq (a,b) \text{ and } \mathscr{C}_2 = \{\{b\},\{b,a\}\} \coloneqq (b,a).$$

This, of course, naturally extends to greater n-tuples. There are some obvious concequences of this that will not be gone over here.

The **Cartesian product** is a set of ordered pairs which can be given by taking power sets.

Definition 1.1.1: Cartesian product

The cartesian product is the cross product of two sets given by

$$A \times B \coloneqq \{a \in A, b \in B \mid (a,b) \in \mathcal{P}(\mathcal{P}(A \cup B))\}.$$

1.2 Relations

The review or ordered pairs leads us to our next discussion on relations. A relation in more colloquial terms is a dynamic or characteristic of a given element shared with another given element such that they are arranged in the form of ordered pairs. Think of equality, then think of equality as equating $(2+3, 1+4)$, or perhaps something more interesting, the set of all (x,y) such that x is a man, y is an enby (slay), and x is married to y (many such cases). We can then think of relations as a set of ordered pairs $R \subset A \times B$. Consider the following examples:

1. $R \coloneqq \{(a,b) \in \mathbb{N}^2 \mid y - x > 0\} \iff aRb \equiv y > x$, the relation of greater than;
2. $R \coloneqq \{(a,B) \in A \times \mathcal{P}(A) \mid a \in B\} \iff aRB \equiv a \in B$, the relation of *belonging* (being an elemnt of);
3. $R \coloneqq \{(c,d)\} \in \{\text{all cats}\} \times \{\text{all dogs}\} \mid$ c and d live in the same house.

Naturally, there are an infinite number of possible relations, however there is a particular class of these which we are interested in for the time being—that which relates two equivalent elements (the definition of what equivalence means is, for now, notably ambiguous).

1.2.1 Equivalence relations and partitions

If a relation R exists s.t. xRx for any $x \in X$, then we say R is reflexive. If $xRy \iff yRx$, it is symmetric, and if $xRy \implies yRz \implies xRz$, it is transitive. If a given relation R satisfies all three of these properties (reflixivity, symmetry, transitivity), then we say it is an ***equivalence relation***.

Definition 1.2.1: Equivalence relations

An equivalence relation, usually denoted $\sim$, is a relation that is reflexive, symmetric, and transitive $\forall x_n \in X$.

Question 1

Consider each of the three qualifies (axioms) for the existence of an equivalnce relation. Find an explicit relation that has one property but not the other for each of the three axioms.

Solution

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Lecture 2.

Groups

2.1 Groups

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