MATH 55A PSET 3

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1. Let V be an n-dimensional vector space, and $T : V \to V$ a linear map; suppose that T is nilpotent, i.e. $T^N = 0$ for some $N > 0$.

(a) Show that $T^n = 0$.

(b) Show that $I + T$ is invertible (where I is the identity operator), and give a formula for its inverse.

Proof. We begin with (a). Note that this creates three cases,

I. $n = N$, trivial;

II. $n > N$, $\implies T^n = T^N T^{n-N} = 0T^n - N = 0$;

III. $n < N$, w

□

2. Let V be a vector space over a field k and $\phi : V \to V$ a linear operator. dimensions of the kernal and image of T? What changes if char $= 2$?

(a) Show that for all $m \geq 1$, $\operatorname{Im}(\phi^{m+1}) \subset \operatorname{Im}(\phi^m)$, and if $\operatorname{Im}(\phi^{m+1}) = \operatorname{Im}(\phi^m)$ then $\operatorname{Im}(\phi^n) = \operatorname{Im}(\phi^m)$ for all $n \geq m$.

(b) The *eventual image* of ϕ, denoted $\operatorname{evIm}(\phi)$, is the set of vectors which can be expressed as $\phi^m(v)$ for *all* $m \in \mathbb{N}$, i.e. $\operatorname{evIm}(\phi) = \bigcap_{m\geq 1} \operatorname{Im}(\phi^m)$. Show that $\operatorname{evIm}(\phi)$ is an invariant subspace for ϕ.

(c) Show that, if V is finite-dimensional, then the eventual image of ϕ and its generalized kernel $\operatorname{gKer}(\phi) = \{v \in V \mid \exists m \in \mathbb{N},\ \phi^m(v) = 0\}$ coincide with the image and kernel of ϕ^n where $n = \dim V$, and the restriction of ϕ to $\operatorname{evIm}(\phi)$ is surjective.

(d) Still assuming V is finite-dimensional, deduce from the above results that V decomposes into the direct sum of invariant subspaces $V = \operatorname{evIm}(\phi) \oplus \operatorname{gKer}(\phi)$, where ϕ is invertible on $\operatorname{evIm}(\phi)$ and nilpotent on $\operatorname{gKer}(\phi)$.

(e) Show that, if V is infinite-dimensional, then none of the statements in (d) need to hold: find an infinite-dimensional vector space V and two linear operators $\phi, \psi : V \to V$ for which:

(1) $\operatorname{evIm}(\phi) = \operatorname{gKer}(\phi) = V$, the restriction of ϕ to $\operatorname{evIm}(\phi)$ is not injective, and the restriction of ϕ to $\operatorname{gKer}(\phi)$ is not nilpotent;

(2) $\operatorname{evIm}(\psi) = \operatorname{gKer}(\psi) = 0$.

Proof. □

3.

Fix a field k, and consider the category Vect_k of all vector spaces over k.

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(a) Show that there exists a contravariant functor from the category Vect_k to itself, which on objects takes each vector space V to its dual $V^* = \mathrm{Hom}(V, k)$.

(b) Recall that for each vector space V we have a "natural" homomorphism $ev_V : V \to V^{**}$ taking every vector $v \in V$ to the element $ev_V(v)$ of $V^{**} = \mathrm{Hom}(V^*, k)$ which maps $\ell \in V^*$ to $\ell(v) \in k$. Show that these homomorphisms determine a natural transformation from the identity functor to the square of the functor of part (a).

4.

(a) Find a field $\mathbb{F}_4$ with 4 elements! Namely, denote the elements by $\{0, 1, \alpha, \beta\}$ and write out the tables for addition and multiplication in $\mathbb{F}_4$.

(b) If we forget the multiplicative structure of $\mathbb{F}_4$ and just think of it as an abelian group for addition, is it isomorphic to $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$?

(c) Show that this is the unique field with 4 elements up to isomorphism.