math55

Lecture Notes

Compiled: October 10, 2025

Contents

Lecture 1. naive set theory *Date: August 29, 2025*

0.1 Naïve Set Theory

1. Consider the sentence not$(x \in x)$, i.e., the subset by way of axiom of specification $B = \{x \in A : x \notin x\}$. Is it the case that $B \in A$?

   No. It is not the case that $B \in A$. If $B \in A$, then either $B \in B$, $B = \{B \in A : B \notin A\}$, but B cannot be in A by definition. Similarly, if $B \in A$ AND $B \notin B$, then by definition B is contained within the subset B, which once again, is a contradiction.

2. Given some set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\} = \mathfrak{E}$ prove or disprove that all sets in $\mathfrak{E}$ obtained this way are unique.

   For all $x \in \mathfrak{E}$ there exists another element, $(y \neq \emptyset) \in \mathfrak{E}$, such that $x \subset y$. We show that $x \neq y$. For contradiction, we say $y \subset x$. Then $x = \{\}$ then $y \subset \emptyset$; this cannot be, however, since the only subset of the empty set is the empty set, and, since $y \neq \emptyset$ this is a contradiction.

   Let $n, k \in \mathbb{N}$. Given $A_0 = \emptyset$ and $A_{n+1} = \{A_n\}$. We show that $A_n \neq A_k$ if $n \neq k$. Suppose then, for contradiction, when 5.

   Lie Groups - Group and manifold (locally like $\mathbb{R}^n$). an example may be $GL_n(\mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^n$

   dim $= 0$; Lie groups are discrete groups (classifcation of discrete groups are hopeless)

   we can reduce a lie group to a discrete group and a connector group.

Lecture 2. quicknotesbcipaddied *Date: September 10, 2025*

0.2 quicknotesbcipaddied

(3) $\Rightarrow$ (1). Assume G/K coset mult is a grp. s.t. G to G/K

Remark:

given a group homomorphism $\phi : G \to H$, then $K = \ker(\phi)$ is a normal subgroup of G, and ϕ factors as $G \to G/K \to \Im(\phi) \hookrightarrow H$ a maps to aK maps to phi (a). This is the same grp. isomorphism from earlier.

Ex: S_3 are perm. of 1,2,3. iso. to D_3. Three transportations of swapping elements these correspond to reflections of a triangle and have order 2; we have two 3-cycles (correspond to rotations and have order 3).

Lecture 3. lecc10 *Date: September 24, 2025*

0.3 lecc10

Suppose $\phi : V \to V$, $(v_1, \ldots, v_n) \mid \mathscr{M}(\phi) = A$, each $V_k =$)span$(v_1 \ldots v_k)$ invariant under ϕ.

Lecture 4. groups *Date: September 26, 2025*

0.4 Groups

0.4.1 Definitions

In this course we study one of the most fundamental objects in mathematics: groups.

Definition 1. A *group* is a set S together with a map

$$m : S \times S \to S$$

called a *law of composition*. Writing ab for $m(a, b)$, the law of comp. has to satisfy three axioms:

i. (existence of identity): $\exists e \in S$, called the identity, s.t. $ae = ea = a \forall a \in S$;

ii. (existence of inverses): for any element $a \in S \exists b \in S \mid ab = ba = e$, we usually denote this as a^{-1} or $-a$;

iii. (the associative law): for any three elements $a, b, c \in S$ we have

$$a(bc) = (ab)c.$$

This results in some immediate concequnces, namely the uniqueness of the identity element and the *cancellation law*. The proofs are trivial.

We usually represent groups in the form G, as a simplified for (G, m) or $\langle G, m \rangle$ (the latter being less common). The cardinality of the set of a given group is called the order of said group: i.e., the cardinality of the group $|G|$ is the cardinality of the set G (the use of the same notation for groups and sets can be confusing at times but are context dependent).

We have an additional type of group that naturally appears from any of the common number systems $\mathbb{R}, \mathbb{N}, \ldots$, called an abelian group.

Definition 2. An abelian group is a group that is commutative; that is to say:

$$\forall a, b \in G,\ ab = ba.$$

This in general is not true for all groups, as symmetric groups and dihedral groups do not satisfy this definition (you will see more on this later). Additionally, there are a few other structures that arise when we alter the axioms of a group: remove the second axiom and you have a *monoid*, remove the third axiom and you have a *semigroup*. They will appear a few times but not often (for now).

0.4.2 Constructions

Definition 3. The product of two groups is given by $G \times H$, where the set is the Cartesian product adn the law of composition is given termwise:

$$G \times H = \{(a,b) \mid a \in G, b \in H\} \,\&\, (a,b) \cdot (a',b') \coloneqq (a \cdot a', b \cdot b').$$

We also may use $G \oplus H$, but this is usually more case specific.

Products of greater than two groups, even infinite groups, is much the same; given the groups $G_1, \ldots, G_n$ we can define a product,

$$G_1 \times \ldots \times G_n \coloneqq \{(a_1, \ldots, a_n) \mid a_i \in G_i\},$$

with the law of composition given by

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) \coloneqq (a_1 \cdot b_2, \ldots, a_n \cdot b_n).$$

We ue products and sums to denote infinite and finite group products, respectively:

$$\prod_{i=1}^{\infty} G_i \,\&\, \bigoplus_{i=1}^{n} G_i.$$

At times we may have a group G^n such as $\mathbb{R}^n$, etc, where each G_n contains n-tuples of its elements.

The final note is on groups of polynomials with real coefficients, denoted $\mathbb{R}[x]$. We use

$$\mathbb{R}[x] \coloneqq \{a_0 + a_1 x + a_2^2 \ldots + a_n x^n \mid n \in \mathbb{N} \text{ and } a_i \in \mathbb{R}\}$$

for finite polynomials as a direct sum, and the following for infinite polynomials as a direct product (i.e., the group of polynomial power series),

$$\mathbb{R}[[x]] \coloneqq \{a_0 + a_1 x + a_2^2 \ldots \mid a_i \in \mathbb{R}\}.$$

Lecture 5. relations *Date: September 27, 2025*

0.5 Relations

0.5.1 Symmetry Groups

An alternative to number systems as the basis of groups is symmetries of systems/ objects. To construct these consider the set S and a group acting on the set S called the permutation group $Perm(S)$ which is the set of bijective maps $f : S \leftrightarrow S$ has a law of composition defined as function composition. The order of the group $Perm(S)$ is $|Perm(S)| = n!$ when S is finite (usually this is the case with a permutation group). If $S = \{1, 2, \ldots, n\}$ then $Perm(S)$ is denoted S_n is is known as the symmetry group.

> **Definition 4** (Symmetric Groups)**.** The symmetric group is a set $Perm(S)$ acting on a set S of order n, denoted S_n such that $Perm(S)$ is the set of bijective maps $f : S \leftrightarrow S$ with the law of composition of $Perm(S)$ being function composition. We denote the permutations via σ^n and τ^n, where σ is a permutation via shifts and τ is a permutation via swaping two given elements.

Problem (1.1.1)**.** We know that the group S_n is non-abelian $\forall n \geq 3$. Why is this? First we demonstrate by example. Consider the group with the set $S = \{1, 2, 3, 4\}$, notice that $|S_4| = 4! = 24$. The group is made of up a sequence of perumtations $e, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3$, where σ^n is a shift of order n such that $(1, 2, \ldots, m) \mapsto (\ldots, m, 1, 2, \ldots)$, so, for example, $(1, 2, 3)$ under σ^2 results in $\sigma^2(1, 2, 3) = (2, 3, 1)$ since $(1, 2, 3) \overset{\sigma}{\mapsto} (3, 1, 2) \overset{\sigma}{\mapsto} (2, 3, 1)$. The τ is a switch between any two elements: $(1, 2, 3) \overset{\tau}{\mapsto} (1, 3, 2)$, but τ^2 simply reverses whatever mapping we made by acting on the same elements we applied τ to, i.e., $(1, 2, 3) \overset{\tau}{\mapsto} (1, 3, 2) \overset{\tau}{\mapsto} (1, 2, 3)$. Given these facts, (note that we are going to drop the commas for the sequence of numbers) notice that $\tau\sigma(1234) \Rightarrow (1234) \overset{\sigma}{\mapsto} (4123) \overset{\tau}{\mapsto} (1423)$, but $\sigma\tau(1234) \Rightarrow (1234) \overset{\tau}{\mapsto} (2134) \overset{\sigma}{\mapsto} (4213)$. Notice, then, that $(1423) \neq (4213)$, thus it mustn't be the case that S_4 is abelian. This, in general, is true for all $S_n \mid n \geq 3$.

A similar notion that arises from symmetric groups is the dihedral group: the set of symmetries of S under rotation and reflection where S is the set of verticies of a regular n-gon in the plane (a plane). We call this D_n.

> **Definition 5** (Dihedral Groups)**.** A dihedral group, denoted D_n is the set of symmetries of S under rotation and reflection (function composition) as the law of composition, where S is the set of all verticies of a regular n-gon. This is a special type of symmetric

> group.

As a quick side-note we use μ and ρ is function notation for our relfections and rotations, respectively, in relation to dihedral groups. There is a good amount of similarity with τ and σ from symmetric groups.

Problem (1.1.2)**.** Consider the dihedral group D_3, notice that it is isomorphic to S_3. Notice, however, that this is not the case for D_4 and S_4. We use ρ and μ for rotations and reflections (LoC) under D_n (similar to σ and τ). For D_3 we have a 3-gon, which is simply a triangle. We may assign to each of its three verticies a number such that under the transformations of ρ and μ we achieve rotations and reflections. Below is an example of this:

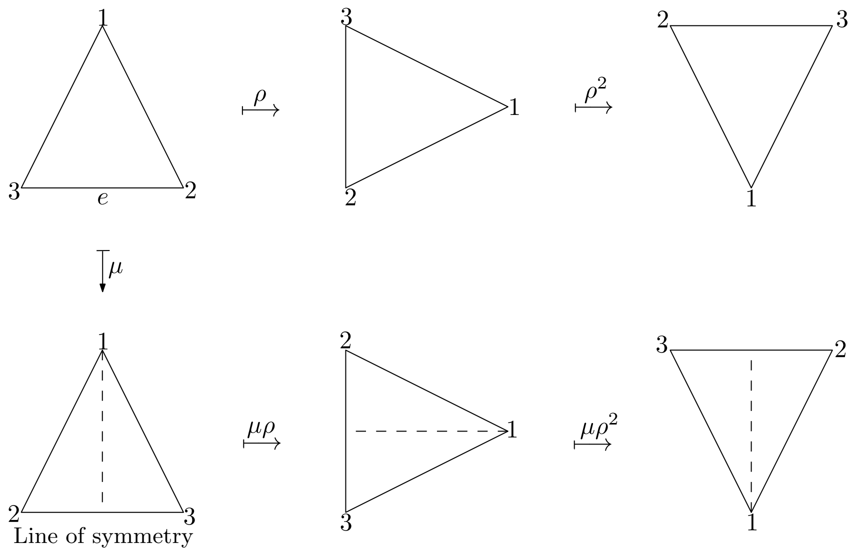

Figure 1

The observed triangle is a geometric representation of D_3. We can see that the order is given by $|D_3| = |\{e, \rho, \rho^2, \mu, \mu\rho, \mu\rho^2\}| = 6$, the same as S_3; however when observing D_4 and S_4, we notice that these are incapible of being isomorphic since they have different orders: $|D_4| = |\{e, \rho, \rho^2, \rho^3, \mu, \mu\rho, \mu\rho^2, \mu\rho^3\}| = 8$ and $|S_4| = |\{e, \sigma, \sigma^2, \sigma^3, \tau\sigma, \tau\sigma^2, \tau\sigma^3, \tau^2\sigma, \ldots\}| = 24$. Interesting! It seems, from observation that while S_n follows the rule of order $n!$, D_n follows the rule of order $2n$. Notice that the mappings that involve μ in D_3 are all reflections on a different axis of symmetry, the axis of symmetry under e, ρ and ρ^2. We can thus imagine D_3 as $2n$ due to the fact that it has n rotations, and therefore n axies of symmetry $\Rightarrow n + n = 2n$. □

Something to note is that, as we increase in dimensionality (recall that D_n is restricted to n-gons in $\mathbb{R}^2$), i.e., as n increases for $\mathbb{R}^n$, then the order of the group has more and more complex reflections (μs), which meet S_n's order and even exceeds it. A general rule is: for $\mathbb{R}^n$, $|G| = 2^n(n!)$. The number of rotations and reflections, however, can be infinite (for example consider the set of all symmetries of a circle).

Definition 6 (Linearity)**.** Linearity of a bijective function is determined by the rule that $\phi(v + w) = \phi(v) + \phi(w), \forall v, w \in \mathbb{R}^n$ and $\phi(\lambda v), \forall v \in \mathbb{R}^n, \lambda \in \mathbb{R}$. Notice our v, w quantities are vectors and our λ quantity is a scalar.

Definition 7 (General and Special Linear Groups). Consider the symmetries of $S = \mathbb{R}^n$ but restrict the set of bijections to be only those which are linear (as defined in the previous definition). This forms the group $GL_n(\mathbb{R})$, i.e., the group of $n \times n$ matrices with nonzero determinant. Similarly, a special case is $SL_n(\mathbb{R})$, which is the special linear group which is the same as the general linear group except the determinant is specified to be 1.

Definition 8 (Orthogonal Groups). An orthogonal group is a type of symmetric group that is the linear maps $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ that respect (preserve) distance.

0.5.2 Group Relations

Definition 9 (Subgroups). A subgroup of G is a subset $H \subset G$ such that H is closed under the law of composition of G (i.e., $\forall a, b \in H ab \in H$) and $\forall a \in H \exists a^{-1} \in H$ s.t. $aa^{-1} = e$.

Definition 10 (Generators and Generation). Group generation happens as a concequence of our definition of subgroups. Notice, that given $H \leq G$ (we use $\leq$ and $\geq$ notation to indicate subgroup-ness) we can construct the smallest subgrop of G by taking the intersection of all subgroups of G; notice that this intersection is indeed a subgroup as well. For generation of a group we want to take the samllest subset denoted $S \subset G$ such that there exist a samllest subgroup that contains S:

$$G_{smallest} = \bigcap_{H \leq G; H \geq S} H$$

we use the terminology that $S = \langle S \rangle$ is the generator of G if $\langle S \rangle = G$.

We can now use this bit of information to come up with a more formal definition of cyclic groups:

Definition 11 (Cyclic Groups). A group is a cyclic group $\Leftrightarrow$ all elements in the group are generated by a single element.

0.5.3 Homomorphisms

Definition 12 (Homomorphisms). If H and G are two groups, a homomorphism is given by $\varphi : H \to G$ as a map that respects the laws of composition in G and H, i.e., $\forall a, b \in H$,

$$\varphi(ab) = \varphi(a)\varphi(b).$$

We can denote this using a commutative diagram,

$$\begin{tikzcd} G \times G \arrow[r, "\varphi\times\varphi"] \arrow[d, "m_G"'] & H \times H \arrow[d, "m_H"] \\ G \arrow[r, "\varphi"] & H, \end{tikzcd}$$

where m_G, m_H are the laws of compsoition for G, H, respectively. What makes this diagram a homomorphism is thereby the very fact that it commutes.

Let's take two interesting cases: the inverse and the identity of a group under homomorphism: consider e_H and e_G. By homomorphism, it must be the case that $\varphi(e_H e_G) = \varphi(e_H)\varphi(e_G)$, however, notice that since this is the identity, in order to preserve group structure, we result in the identity (e_H or simply e). if this is the case we can say $\varphi(e_G) = e$ and $\varphi(e_H) = e$, thus, $\varphi(e_H) = e = \varphi(e_G)$. Similarly, take the inverse of some given $a \in G$, $a^{-1} \in G$, notice that $\varphi(a^{-1}) = \varphi(a)^{-1}$.

Lecture 6. category theory *Date: October 03, 2025*

0.6 Category Theory

0.6.1 Categories

Definition 13 (Category). A category $\mathscr{C}$ is a collection of **objects**, such that:

(i.) for each pair of objects $A, B \in \mathrm{Ob}(\mathscr{C})$;

(ii.) a collection of morphisms is given by $\mathrm{Mor}(A, B)$;

(iii.) there exists operation titled *composition of morphisms* given by

$$\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$$

such that $f, g \mapsto g \circ f$;

(iv.) Category Theory Axioms:

(a) every object A has an identity morphism $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ such that

$$\forall f \in \mathrm{Mor}(A, B),\ f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f;$$

(b) composition (of morphisms) is associative, i.e., $(f \circ g) \circ h = f \circ (g \circ h)$.

We can think of categories and sets as, for the most part, the same thing, except for the fact that a category may include the category of all sets; this is where we deliniate sets and categories in order to avoid pesky paradoxes.

Some examples of categories, for purposes of clarity:

1. category of sets, $\mathrm{Mor}(A, B) =$ all maps $A \to B$
2. Vect_k as a fin. dim. vector space over k, then $\mathrm{Mor} =$ linear maps
3. groups, group homomorphisms
4. topological spaces, continuous maps
5. and more...

There exists a relationship called **isomorphism** not too dissimilar from group-theoretic isomorphisms.

Definition 14 (Isomorphism). $f \in \text{Mor}(A, B)$ is an *isomorphism* if has inverse, i.e., $\exists g \in \text{Mor}(B, A)$ s.t. $g \circ f = id_A$ and $f \circ g = id_A \Rightarrow$ if $\exists g \Rightarrow !\exists g$ and that id_A is an isomorphism. $f \Rightarrow f^{-1}$ $(g \equiv f^{-1}$
)are both called isomorphisms, individually.

Concequentially the automorphisms $Aut(A)$ is always a group.

sets :permutations, groups: automs, vec: invertible lin op.

Isomorphic objects of e have isomorphic automorphic groups. Given $f \in \text{Mor}(A, B)$ as an isomorphism $c_f : Aut(A) \to Aut(B)$ which maps $g \mapsto f \circ g \circ f^{-1}$.

We may define sum, product, and quotient objects in categories. Consider a product $A \times B \coloneqq \mathcal{C}$ of objects $A, B \in \mathscr{C}$ characterized by $\mathcal{P}$ projects $\pi_1 \in \text{Mor}(\mathcal{P}, A), \pi_2 \in \text{Mor}(\mathcal{P}, A)$ s.t. $\forall$ object T, $\forall f_1 \in \text{Mor}(T, A)$, $f_2 \in \text{Mor}(T, B)$, $\exists$ unique $f \in \text{Mor}(T, \mathcal{P})$ s.t. $\pi_1 \circ f = f_1$, $\pi_2 \circ f = f_2$.

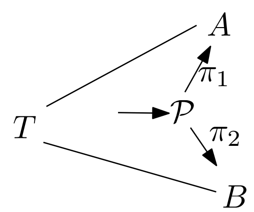

Figure 2

Definition 15 (functor). C, D categories, a (*covariant*) functor $F : C \to D$ is an assigment:

1. for every object X of C an object $F(X)$ of D
2. for each morphism $f \in \text{Mor}_C(X, Y)$, a morphism $F(f) \in \text{Mor}_D(F(X), F(Y))$ s.t.
   (a) $F(id_X) = id_{F(X)}$
   (b) $F(g \circ f) = F(g) \circ F(f)$

Consider Forgetful functors , on a vector space over k given vector sace V $F : Vect_k \to Vect_k$objects: $W \mapsto Hom(V, W)$ lin. maps $V \to W$, morphism $f : W \to W' \Rightarrow ??$

Anywhos. He's going on about linear maps as morphisms for categories and functors but I don't care all that much at the moment. As a functor $F : \text{Vect}_k \to \text{Vect}_k$ can be shown as $Hom(V, \circ)$. What the mell does obj. $V \mapsto V_{\mathbb{C}} = V \oplus iV$

Definition 16. A *contravariant functor* $F : C \to D$ is the same except reverses direction of morphism:

$$f \in \mathrm{Mor}_C(X,Y) \mapsto F(f) \in \mathrm{Mor}_D(F(Y),F(X)) \mid F(g \circ f) = F(f) \circ F(g).$$

So for example, consider $V \to V^*$ s.t. $f \in Hom(V,W) \mapsto f^* \in Hom(W^*,V^*)$.

Definition 17. Given two functors $F, G : \mathscr{C} \to \mathscr{D}$ a natural transformation t from F to G is the data $\forall X \in Ob(\mathscr{C})$, a morphism $t_X \in \mathrm{Mor}_{\mathscr{D}}(F(X), G(X))$ s.t.

$$\forall X, Y \in Ob(\mathscr{C}), \forall f \in \mathrm{Mor}_{\mathscr{C}}(X,Y),$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "t_x"'] & F(Y) \arrow[d, "t_y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}.$$

Definition 18 (Bilinear forms)**.** A ***bilinear form*** on a vector space V over a field k is a map $b : V \times V \to k$ (this is to say we take $(v,w) \mapsto b(v,w)$) that is linear in each variable, i.e., $\forall u, v, w \in V$, $\lambda \in k, b(\lambda v, w) = b(v, \lambda w) = \lambda b(v,w)$ ("multiplied" on both sides separately).

Definition 19 (symmetric)**.** Say b is symmetric if $b(v, w(= b(w,v)))\forall v, w \in V$ and is skew symmetric if $b(v,w) = -b(w,v)$.

I have mentally clocked out, I spent last night and this morning going over the notes for today and ngl i understood it better before he started explaining it

AWWWWWW

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}.$$

Lecture 7. lec15 *Date: October 06, 2025*

0.7 Lecture 15

0.7.1 bilinear forms

Bilinear forms on fin. dim. vec. space V over k for

$$b : V \times V \to k \longleftrightarrow \varphi_b \in \mathrm{Hom}(V, V^*)$$

which gives an isom. $B(V) \cong \mathrm{Hom}(V, V^*)$ $(\varphi_b(v) = b(v, \cdot))$
Say be in non degenerate if φ_b is injective.
In a basis $(e_1, \ldots e_n)$ of V, rep. b by a mat. A w ent $a_{ij} = b(e_i, e_j)$.

$$b(\sum_i x_i e_i, \sum_j x_j e_j) = \sum_{i,j}^{a_{ij} x_i y_j} = X^{\perp} A y.$$

where col. vectors are X and y that correspond to the summations in b.
Something something

i. b is symm. $\Leftrightarrow$ A is symm. $(b(v, w) = b(w, v) \Leftrightarrow A^{\perp} = A, a_{ij} = a_{ji})$

ii. b is nondegenerate $\Leftrightarrow$ A is invertible.

"Two vectors paired together to zero"

Definition 20 (Orthogonality). If $S \subset V$ is a subspace of V, $b : V \times V \to k$ w/ is bilinear form, the **orthogonal compliment** is $S^{\perp} = \{v \in V \mid \forall w \in S, b(v, w) = 0\}$

Beware: if b is skew symmetric or symm. then $(*) \Leftrightarrow \forall w \in S, b(w, v) = 0$, otherwise (not skew symm or symm) we we have a left orthogonal and a right orthogonal (such that they are not the same), buttttt we want to have them be equal so we just will ignore this for now lol.

Lemma 1. If b is nondegenerate then $dim(S^{\perp}) = dim(V) - dim(S)$, else ineq.

Proof.

$$S^{\perp} = \ker\left(V \to S^*, \, (v \mapsto \varphi_b(v)_{|S} = b(v, \cdot)_{|S})\right)$$

which is the composition of $\varphi_b V \to V^*$ and restriction $r : V^* \to S^*$ and $l \mapsto l_{|S}$ which is surjective. So by rank thm. $\dim S^{\perp} \dim V - \operatorname{rank}(r \circ \varphi_b) \le \dim(S^*) = \dim(S)$ is b is nondegenerate then $r \circ \varphi_b$ is surjective so $rk = \dim S$. □

Note: write less don't merely copy. Perhaps only do definitions and misc. info? proofs seem to be something you can go over post lecture, just try to understand for now.

Ex: $V = \mathbb{R}^n$ with standard dot prod. so $b(v, w) = \sum v_i w_i$ Then $V = S \oplus S^{\perp}$

Definition 21 (Inner Product Spaces). An **inner product space** is a vector space V over $\mathbb{R}$ together with a symmetric positive definite bilinear form $\langle *, * \rangle : V \times V \to \mathbb{R}$.

Definition 22 (ips symmetric). $\langle u, v \rangle = \langle v, u \rangle$

Definition 23 (positive definite). $\langle u, u \rangle \ge 0 \forall u \in V$, and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$.

Definition 24 (norm). The **norm** of a vector is $||v|| = \sqrt{\langle v, v \rangle}$. $v, w \in V$ are orthogonal if $\langle v, w \rangle = 0$.

Theorem 1 (Cauchy-Scwartz inequality). yk what this is lol, just do it with norms $|\langle u, v \rangle| \le ||u|| ||v||$

Theorem 2. Every finite dim inner product space $/\mathbb{R}$ has an ortho basis.

$$\frac{\mathrm{d/V}}{\mathrm{dx}}.$$

Lecture 8. lec16 *Date: October 08, 2025*

0.8 Orthogonal adjoint operators and Hermatians

Definition 25. T is self-adjoint if $T^* = T$, i.e.,

$$\langle Tu, v\rangle = \langle u, Tv\rangle$$

so $\mathcal{M}(T)$ in an orthogonal basis is symmetric ($\mathcal{M}_{ij} = \mathcal{M}_{ji}$).

Theorem 3 (spectral thm)**.** If $T : V \to V$ is self adjoint then T is diagonalizable, with real eigenvalues. Even more, T can be diagonalized in an orthogonal basis of $(V, \langle\cdot,\cdot\rangle)$!

Lecture 9. hermatians, $\mathbb{C}$-spectrals, etc. *Date: October 10, 2025*

0.9 hermatian inner products

We now wish to real with inner products on complex vector spaces (as opposed to our previous study on $\mathbb{R}$). Hermation inner products are $\mathbb{C}$-linear, i.e., conjugate-linear in one of the inpts.

Definition 26 (hermation forms). A **Hermation form** on a $\mathbb{C}$-vector space V is $H : V \times V \to \mathbb{C}$ s.t.

1. H is ***sesquilinear***
   - $H(u, v + w) = H(u, v) + H(v, w)$
   - $H(u, \lambda v) = \lambda H(u, v)$
   - however, $H(\lambda u, v) = \overline{\lambda} H(u, v)$, where our $\overline{\lambda}$ is our complex conjugate (just $\overline{z}$ really).
2. H conjugate symmetric $H(u, v) = \overline{H(v, u)} \Rightarrow H(u, v) \in \mathbb{R}$

Definition 27 (hermation inner products). A hermatian inner product on V is a positive definite ($H(u, u) \geq 0 \forall u \neq 0$)

Remark 1

$$\varphi_H : V \to V^*,\ u \mapsto H(u, \cdot)$$

where H is $\mathbb{C}$-linear $V \to \mathbb{C}$ and H is linear in the second input. We say φ is a ***complex antilinear*** map, and ($\varphi(\lambda u) = \overline{\lambda}\varphi(u)$).

In the case that H is positive def. $\Rightarrow$ nondegenerate $\forall (u \neq 0) \in V, \varphi_H(u) = H(u, \cdot) \neq 0$. When we are given a subspace $S \subset V$, it must be orthogonal,

$$S^\perp = \{v \in V \mid H(v, w) = 0 \forall w \in S\}$$

s.t. it is a subspace. Of course, as with any finite dimensional conjugate $\dim V < \infty$, we have $(S^\perp)^\perp$. Continuing on, we note that when $V = S \oplus S^\perp$, $S \cap S^\perp = \{0\}$,

$$u \in S \cap S^\perp \Rightarrow H(u, u) \text{ where the first } u \in S, \text{ and the second } u \in S^\perp = 0 \Rightarrow u = 0,$$

so $\dim S^\perp = \dim V - \dim S$

Theorem 4. V admits an orhtogonal basis in $\mathbb{C}$. The proof is the same as in $\mathbb{R}$.

Definition 28. An **orthonormal basis** of (V, H) is a basis $\{e_i\}$ s.t.

$$H(e_i, e_i) = \|e_i\|^2 = 1,\ H(e_i, e_j) = 0 \forall i \neq j, e_i \perp e_j.$$

some other stuff

... ... So, in matrix form, $H(z, w) = \overline{z}^{\mathrm{T}} w = z^* = (\overline{z}_1 \dots \overline{z}_n)$ as conjugate transpose.

Consider fourier series as a (sorta-kinda) example of $V = C^\infty(S^1, \mathbb{C})$ w/ is infinitely differentiable $S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, ($\Leftrightarrow$ 1)-periodic functions $\mathbb{R} \to \mathbb{C}$. L^2-inner product,

$$H(f, g) = \int_{S^1} \overline{f(t)} g(t) dt = \int_0^1 \overline{f(t)} g(t) dt$$

and

$$H(f, f) = \int_0^1 |f(t)|^2 > 0 \text{ if } f \neq 0.$$

Note that then

$$f_n(t) = e^{2\pi i n t} = \cos(2\pi n t) + i \sin(2\pi n t), n \in \mathbb{Z}$$

is orthogonal! So, $H(f_n, f_m) = \delta_{n,m}$.

Definition 29. We let V be a complex space, H a hermatian inner product, and $T : V \to V$ a linear operator.

$$\text{the adjoint of } T \text{ is } T^* : V \to V \text{ s.t. } H(T^* v, w) = H(v, Tw) \forall v, w \in V$$

$$(\Leftrightarrow H(Tv, w) = H(v, T^* w)).$$

Some immediate concequences:

- T is *self-adjoint* if $T^* = T$
- T is *unitary* if $H(Tv, Tw) = H(v, w)$
- T is normal if $TT^* = T^*T$ unitary operators form a group (omg!) $U(V) = U(V, H) \subset \mathrm{Aut}(V) = \mathrm{GL}(V)$, $U(n) \subset \mathrm{GL}(n, \mathbb{C})$, e.g., $U(1) = S^1 = \{z \mid |z| = 1\} \subset \mathbb{C}^*$.

Proposition 1

in an orthonom. basis $\mathcal{M}(T^*) = \mathcal{M}(T)^* = \overline{\mathcal{M}(T)^t}$ so $H(Tv, w) = (\mathcal{M}v)^* w = v^*(\mathcal{M}^* w) = H(v, T^* w)$, self-adjoint operators $\leftrightarrow$ hermatian matricies

Definition 30 (complex spectral theorem). put it here

proof goes here

in such a basis $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

0.9.1 some number theory (non-degenerate symmetric bilinear forms)

this u can go over later via the notes just put it in here