Lecture 7. lec15 *October 06, 2025*

1 Lecture 15

1.1 bilinear forms

Bilinear forms on fin. dim. vec. space V over k for

$$b : V \times V \to k \longleftrightarrow \varphi_b \in \mathrm{Hom}(V, V^*)$$

which gives an isom. $B(V) \cong \mathrm{Hom}(V, V^*)$ $(\varphi_b(v) = b(v, \cdot))$
Say be in non degenerate if φ_b is injective.
In a basis $(e_1, \dots e_n)$ of V, rep. b by a mat. A w ent $a_{ij} = b(e_i, e_j)$.

$$b(\sum_i x_i e_i, \sum_j x_j e_j) = \sum_{i,j}^{a_{ij}x_iy_j} = X^\perp Ay.$$

where col. vectors are X and y that correspond to the summations in b.
Something something

i. b is symm. $\Leftrightarrow$ A is symm. $(b(v, w) = b(w, v) \Leftrightarrow A^\perp = A, a_{ij} = a_{ji})$

ii. b is nondegenerate $\Leftrightarrow$ A is invertible.

"Two vectors paired together to zero"

Definition 1 (Orthogonality). If $S \subset V$ is a subspace of V, $b : V \times V \to k$ w/ is bilinear form, the **orthogonal compliment** is $S^\perp = \{v \in V \mid \forall w \in S, b(v, w) = 0\}$

Beware: if b is skew symmetric or symm. then $(*) \Leftrightarrow \forall w \in S, b(w, v) = 0$, otherwise (not skew symm or symm) we we have a left orthogonal and a right orthogonal (such that they are not the same), butttttt we want to have them be equal so we just will ignore this for now lol.

Lemma 1. If b is nondegenerate then $dim(S^\perp) = dim(V) - dim(S)$, else ineq.

Proof.

$$S^\perp = \ker\big(V \to S^*,\ (v \mapsto \varphi_b(v)_{|S} = b(v, \cdot)_{|S})\big)$$

which is the composition of $\varphi_b V \to V^*$ and restriction $r : V^* \to S^*$ and $l \mapsto l_{|S}$ which is surjective. So by rank thm. $\dim S^\perp \dim V - \mathrm{rank}(r \circ \varphi_b) \leq \dim(S^*) = \dim(S)$ is b is nondegenerate then $r \circ \varphi_b$ is surjective so $rk = \dim S$. ☐

Note: write less don't merely copy. Perhaps only do definitions and misc. info? proofs seem to be something you can go over post lecture, just try to understand for now.

Ex: $V = \mathbb{R}^n$ with standard dot prod. so $b(v, w) = \sum v_i w_i$ Then $V = S \oplus S^\perp$

Definition 2 (Inner Product Spaces). An **inner product space** is a vector space V over $\mathbb{R}$ together with a symmetric positive definite bilinear form $\langle *, * \rangle : V \times V \to \mathbb{R}$.

Definition 3 (ips symmetric). $\langle u, v \rangle = \langle v, u \rangle$

Definition 4 (positive definite). $\langle u, u \rangle \geq 0 \forall u \in V$, and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$.

Definition 5 (norm). The **norm** of a vector is $||v|| = \sqrt{\langle v, v \rangle}$. $v, w \in V$ are orthogonal if $\langle v, w \rangle = 0$.

Theorem 1 (Cauchy-Scwartz inequality). yk what this is lol, just do it with norms $|\langle u, v \rangle| \leq ||u||||v||$