Math 55a Homework 2

Due Wednesday September 17, 2025.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

Material covered: Subgroups, normal subgroups, quotients; modular arithmetic, permutations, etc. (most of Artin chapter 2).

1. Describe a polygon in $\mathbb{R}^2$ whose symmetry group is $\mathbb{Z}/3$. What is the fewest number of vertices that such a polygon can have?

2. Let G be the set of *affine transformations* of the real line, i.e. maps $f_{a,b} : \mathbb{R} \to \mathbb{R}$ of the form $f_{a,b} : x \mapsto ax + b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$.

(a) Show that G is a group, with group law given by composition.

(b) Show that the subsets $H = \{f_{a,b} \,|\, a = 1\}$ and $K = \{f_{a,b} \,|\, b = 0\}$ are subgroups of G. Which one is normal, and what is the quotient of G by that subgroup?

3. Let $G = S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ be the group of complex numbers of modulus 1, with multiplication. Let $H \subset G$ be the subgroup $\{\pm 1\}$. Show that the quotient G/H is isomorphic to G.

4. Show that, if G is a finite group and $H \subset G$ is a subgroup of index 2 (i.e., $|G|/|H| = 2$), then H is a normal subgroup of G.

> **Proof.** We are given $|G|/|H| = 2$. Both left and right cosets, then, must give the same partition of G; thus, for the left coset partition either $g \in H \Rightarrow gH = H$ or $g \notin H \Rightarrow gH = G \setminus H$, and, for the right coset partition, either $g \in H \Rightarrow g \notin H \Rightarrow Hg = G \setminus H$; thus both result in $\{H, G \setminus H\} \Rightarrow gH = Hg = G \setminus H, \forall g \notin H$. Thus $gHg^{-1} = (gH)g^{-1} = (Hg)g^{-1} = H \Rightarrow H \trianglelefteq G$. □

5. Let G be the group of rotations preserving a cube in $\mathbb{R}^3$. Show that G is isomorphic to the symmetric group S_4 (permutations of a four-element set). (Hint: which four-element set?)

> **Proof.** Let G be the octahedral group O_h for $\mathbb{R}^3$. We show that it is a homomorphism: let φ be a map $\varphi : O_h \to S_4$. We want to show $\varphi(ab) = \varphi(a)\varphi(b), \forall a, b \in O_h$. Consider the group of rotations perserving a cube in $\mathbb{R}^3$, notice if we have a cube of length 1, the coordinates of the verticies are $(1, 1, 1), (1, -1, 1), (-1, -1, 1), (-1, 1, 1), (1, 1, -1), (1, -1, -1), (-1, 1, -1), (-1, -1, -1)$; the rotations then across the diagonals that preserve the cube are:

$$
\begin{aligned}
d_1 &: (-1,-1,-1) \to (1,1,1)\\
d_2 &: (-1,-1,1) \to (1,1,-1)\\
d_3 &: (-1,1,-1) \to (1,-1,1)\\
d_4 &: (1,-1,-1) \to (-1,1,1).
\end{aligned}
$$

Let ρ be a rotations in O_h. Then $\varphi(\rho)$ is the permutations of the diagonal rotations $\{d_1, d_2, d_3, d_4\}$. Consider two diagonal rotations ρ_1 and $\rho_2 \in O_h$.

$$
\varphi(\rho_1\rho_2) \stackrel{?}{=} \varphi(\rho_1)\varphi(\rho_2)
$$

.

□

6*. The dihedral group D_n is the group of order $2n$ consisting of all symmetries (rotations and reflections) of the regular n-gon in the plane. Find all normal subgroups of D_n. (Hint: the answer is slightly different for even n vs. for odd n.)

7. Prove or provide a counterexample: if A is a normal subgroup of B and B is a normal subgroup of C, then A is a normal subgroup of C.

Proof. We will show that $A \coloneqq \{(), (12)(34)\}$ is a normal subgroup of $B \coloneqq V_4 = \{(), (12)(34), (13)(24), (14)(23)\}$, the normal Klein 4 group. The Klein 4 group will also be shown to be a normal subgroup of $C \coloneqq A_4$, where

$$
A_4 = \{(), (123), (132), (234), (243), (341), (314), (412), (421), (12)(34), (13)(34), (14)(23)\}.
$$

We end will end with the notion that while it is true that $A \trianglelefteq B$ and $B \trianglelefteq C$, it is not the case that $A \trianglelefteq C$.

First, B is a subgroup of C, since it is a subset consisting of all the disjoint cycles in C closed under the same operation and each element in B is its own inverse. Since it is closed under both and is a subset, it is obviously a subgroup. An additional note is that B is an abelian group, shown by the following Cayley table:

	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Second, to show that $B \trianglelefteq C$, consider a new set, say Π which is the set of possible ways to partition $\{1,2,3,4\}$ into two unordered pairs, so $\Pi = \{\pi_1, \pi_2, \pi_3\}$, where $\pi_1 = \{\{1,2\},\{3,4\}\}, \pi_2 = \{\{1,3\},\{2,4\}\}$, and $\pi_3 = \{\{1,4\},\{2,3\}\}$. Notice that, the permutations of A_4 are permutations from S_4, additionally notice that the composition law applies for any S_n, and since $|\Pi| = 3$ we can keep S_3 and its composition law in mind. If I take a mapping $\varphi : A_4 \to S_3$, since A_4, S_3, S_4 are closed under the same composition, notice that any $\forall a, b, \varphi(ab) = \varphi(a)\varphi(b)$, thus φ is homomophic. Given surjectivity, the image of the map is thereby the whole group S_3; thus the $\ker(\varphi)$ can be computed, to save you the visual trouble, I have already parsed it, and it should be, by now, clear to see that the only mappings of the permutations on the elements of Π to S_3 are $\ker(\varphi) = \{e, (12)(34), (13)(24), (14)(23)\}$, notice, however that this is the same as V_4! Since we have shown $\ker(\varphi) = V_4$, then, by first isomorphism theorem, $V_4 \trianglelefteq A_4 \Leftrightarrow B \trianglelefteq C$.

Recall that $A \subset B$ and B is abelian. Thus, for any $bAb^{-1} = A \Rightarrow Abb^{-1} = A \Rightarrow Ae = A \Rightarrow A = A$, and thus A is a normal subgroup of B.

To demonstrate that $A \ntrianglelefteq C$ a singular counterexample will suffice. Let $c = (123) \in C$ and $c^{-1} = (132) \in C$. Take $a = (12)(34) \in A$ and compute $cAc^{-1} = A$:

$$(123) \circ (12)(34) \circ (132) = (14)(23),\ (14)(23) \neq (12)(34).$$

Since $(14)(23) \in B$, but $(14)(23) \notin A$ we have $A \ntrianglelefteq C$.

$\square$

8. Find the order of the group $GL_2(\mathbb{Z}/2)$ of 2×2 matrices with entries in $\mathbb{Z}/2$ and nonzero determinant (mod 2). Is $GL_2(\mathbb{Z}/2)$ isomorphic to another group we've encountered?

9*. (a) Let p be a prime number, and suppose $H \subset S_p$ is a subgroup of the group of permutations of $\{1, 2, \ldots, p\}$. Show that if H contains the p-cycle $(123\ldots p)$ and a transposition (a permutation that exchanges two elements and fixes all others), then $H = S_p$. (In other words: no matter how we choose $1 \le i < j \le p$, S_p is generated by $(123\ldots p)$ and (ij)).

(b) Show that the conclusion of part (a) may be false if we don't assume p is prime (even though it is still true that S_p is generated by $(123\ldots p)$ and a *suitably chosen* transposition).

10*. Let G be a finitely generated group, and $H \subset G$ a subgroup of finite index. Show that H is finitely generated.

(Hint: Choose a finite subset $S \subset G$ containing one representative of each coset of H, so every element of G is the product of an element of S and an element of H. Given a word in the generators of G and their inverses, how do you rewrite it as the product of an element of S and an element of H?)

11*. (Optional, extra credit): Let G be a non-abelian finite group, and consider its center $Z(G) = \{a \in G \,|\, ax = xa \ \forall x \in G\}$.

(a) Show that $G/Z(G)$ is not a cyclic group.

(b) Show that at most $5/8^{\text{ths}}$ of the pairs of elements of G commute, i.e. the set $C = \{(a, b) \in G \times G \,|\, ab = ba\}$ satisfies $|C| \le \frac{5}{8}|G|^2$.

(c) Show that this bound is optimal, i.e. there exists a non-abelian finite group for which $|C| = \frac{5}{8}|G|^2$.

12*. (Optional, extra credit): Given a set S, let $\mathcal{E} \subset \mathcal{P}(S)$ be such that (1) $S \in \mathcal{E}$, (2) if $A \in \mathcal{E}$ then $S - A \in \mathcal{E}$, (3) if $A, B \in \mathcal{E}$ then $A \cup B \in \mathcal{E}$ and $A \cap B \in \mathcal{E}$.

Prove that if S is finite then there is a set T and a surjective map $f : S \to T$ such that $\mathcal{E} = \{f^{-1}(A),\ A \subset T\}$. What happens if S is infinite?

13. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?

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