Lecture 7.

lec15

7.1 Lecture 15

7.1.1 bilinear forms

Bilinear forms on fin. dim. vec. space V over k for

$$b : V \times V \to k \longleftrightarrow \varphi_b \in \mathrm{Hom}(V, V^*)$$

which gives an isom. $B(V) \cong \mathrm{Hom}(V, V^*)$ $(\varphi_b(v) = b(v, \cdot))$

Say be in non degenerate if φ_b is injective.

In a basis $(e_1, \ldots e_n)$ of V, rep. b by a mat. A w ent $a_{ij} = b(e_i, e_j)$.

$$b(\sum_i x_i e_i, \sum_j x_j e_j) = \sum_{i,j}^{a_{ij} x_i y_j} = X^{\perp} A y.$$

where col. vectors are X and y that correspond to the summations in b.

Something something

i. b is symm. $\iff$ A is symm. $(b(v, w) = b(w, v) \iff A^{\perp} = A, a_{ij} = a_{ji})$

ii. b is nondegenerate $\iff$ A is invertible.

"Two vectors paired together to zero"

Definition 7.1: I

$S \subset V$ is a subspace of V, $b : V \times V \to k$ w/ is bilinear form, the **orthogonal compliment** is $S^{\perp} = \{v \in V \mid \forall w \in S, b(v, w) = 0\}$

Beware: if b is skew symmetric or symm. then $(*)$ $\iff$ $\forall w \in S, b(w, v) = 0$, otherwise (not skew symm or symm) we we have a left orthogonal and a right orthogonal (such that they are not the same), buttttt we want to have them be equal so we just will ignore this for now lol.

lemma 7.1 I

b is nondegenerate then $dim(S^{\perp}) = dim(V) - dim(S)$, else ineq.

Proof.

$$S^{\perp} = \ker\left(V \to S^*, (v \mapsto \varphi_b(v)_{|S} = b(v, \cdot)_{|S})\right)$$

which is the composition of $\varphi_b V \to V^*$ and restriction $r : V^* \to S^*$ and $l \mapsto l_{|S}$ which is surjective. So by rank thm. $\dim S^{\perp} \dim V - \mathrm{rank}(r \circ \varphi_b) \leq \dim(S^*) = \dim(S)$ is b is nondegenerate then $r \circ \varphi_b$ is surjective so $rk = \dim S$. ∎

Note: write less don't merely copy. Perhaps only do definitions and misc. info? proofs seem to be something you can go over post lecture, just try to understand for now.

Ex: $V = \mathbb{R}^n$ with standard dot prod. so $b(v, w) = \sum v_i w_i$ Then $V = S \oplus S^{\perp}$

Definition 7.2: A

inner product space is a vector space V over $\mathbb{R}$ together with a symmetric positive definite bilinear form $\langle *, * \rangle : V \times V \to \mathbb{R}$.

Definition 7.3:

u,v $\rangle = \langle v, u \rangle$

Definition 7.4:

u,u $\rangle \geq 0 \forall u \in V$, and $\langle u, u \rangle = 0 \iff u = 0$.

Definition 7.5: T

e **norm** of a vector is $||v|| = \sqrt{\langle v, v \rangle}$. $v, w \in V$ are orthogonal if $\langle v, w \rangle = 0$.

Theorem 7.1 y

what this is lol, just do it with norms $|\langle u, v \rangle| \leq ||u||||v||$

Theorem 7.2 E

ery finite dim inner product space $/\mathbb{R}$ has an ortho basis.

$$\frac{\mathrm{d/V}}{\mathrm{dx}}.$$