studies-in-algebra-and-group-theory

Lecture Notes

Compiled: October 10, 2025

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Lecture 1. naïve-set-theory *Date: October 09, 2025*

0.1 naïve-set-theory

0.1.1 chapters 6-10

ordered pairs

We begin with a discussion of ordered pairs. Consider a set of numbers $\{1,2,3,4\}$. For any set there is no particular order that triumphs over another, i.e., $\{1,2,3,4\} = \{1,3,2,4\} = \{3,4,2,1\} = \ldots$, an obvious truth. Say we want $\{1,2,3,4\}$ to be in the order of $\{3,2,1,4\}$ for some arbitrary reason; we may then consider the collection,

$$\mathscr{C} = \{\{3\},\{3,2\},\{3,2,1\},\{3,2,1,4\}\}.$$

It becomes obvious what is happening here: the method by which we get order is dependent on the number of itterations in which a particular element appears in any number of sets in a given collection (in comparison to the other elements in the other sets of the respective collection).

Then consider this process but for a nested collection (a collection within a collection) which contains two sets, namely the arbitrary elements a, b in opposing orders, and notice that it gives us the definition of a 2-tuple (ordered pair),

$$\mathscr{C}_1 = \{\{a\},\{a,b\}\} \coloneqq (a,b) \text{ and } \mathscr{C}_2 = \{\{b\},\{b,a\}\} \coloneqq (b,a).$$

This, of course, naturally extends to greater n-tuples. There are some obvious concequences of this that will not be gone over here.

The **Cartesian product** is a set of ordered pairs which can be given by taking power sets.

Definition 1 (Cartesian product). The cartesian product is the cross product of two sets given by

$$A \times B \coloneqq \{a \in A, b \in B \mid (a,b) \in \mathcal{P}(\mathcal{P}(A \cup B))\}.$$

relations

The review or ordered pairs leads us to our next discussion on relations. A relation in more colloquial terms is a dynamic or characteristic of a given element shared with another given element such that they are arranged in the form of ordered pairs. Think of equality, then think of equality as equating $(2+3, 1+4)$, or perhaps something more interesting, the set of all (x, y) such that x is a man, y is an enby (slay), and x is married to y (many such cases). We can then think of relations as a set of ordered pairs $R \subset A \times B$. Consider the following examples:

1. $R \coloneqq \{(a, b) \in \mathbb{N}^2 \mid y - x > 0\} \Leftrightarrow aRb \equiv y > x$, the relation of greater than;
2. $R \coloneqq \{(a, B) \in A \times \mathcal{P}(A) \mid a \in B\} \Leftrightarrow aRB \equiv a \in B$, the relation of *belonging* (being an elemnt of);