Lecture 9.

Hermitians, $\mathbb{C}$-spectrals, etc.

9.1 Hermitian inner products

We now wish to real with inner products on complex vector spaces (as opposed to our previous study on $\mathbb{R}$). Hermitian inner products are $\mathbb{C}$-linear, i.e., conjugate-linear in one of the inpts.

Definition 9.1: Hermitian forms

A **Hermitian form** on a $\mathbb{C}$-vector space V is $H : V \times V \to \mathbb{C}$ s.t.

1. H is ***sesquilinear***
   - $H(u, v + w) = H(u, v) + H(v, w)$
   - $H(u, \lambda v) = \lambda H(u, v)$
   - however, $H(\lambda u, v) = \overline{\lambda} H(u, v)$, where our $\overline{\lambda}$ is our complex conjugate (just $\overline{z}$ really).
2. H conjugate symmetric $H(u, v) = \overline{H(v, u)} \implies H(u, v) \in \mathbb{R}$

Definition 9.2: Hermitian inner products

A hermitian inner product on V is a positive definite ($H(u, u) \geq 0 \forall u \neq 0$)

Definition 9.1.1: limit

limiter

Remark 1

$$\varphi_H : V \to V^*,\ u \mapsto H(u, \cdot)$$

where H is $\mathbb{C}$-linear $V \to \mathbb{C}$ and H is linear in the second input. We say φ is a ***complex antilinear*** map, and $(\varphi(\lambda u) = \overline{\lambda}\varphi(u))$.

In the case that H is positive def. $\implies$ nondegenerate $\forall (u \neq 0) \in V, \varphi_H(u) = H(u, \cdot) \neq 0$. When we are given a subspace $S \subset V$, it must be orthogonal,

$$S^\perp = \{v \in V \mid H(v, w) = 0 \forall w \in S\}$$

s.t. it is a subspace. Of course, as with any finite dimensional conjugate $\dim V < \infty$, we have $(S^\perp)^\perp$. Continuing on, we note that when $V = S \oplus S^\perp$, $S \cap S^\perp = \{0\}$,

$$u \in S \cap S^\perp \implies H(u, u) \text{ where the first } u \in S, \text{ and the second } u \in S^\perp = 0 \implies u = 0,$$

so $\dim S^\perp = \dim V - \dim S$

Theorem 9.1 Th

V admits an orhtogonal basis in $\mathbb{C}$. The proof is the same as in $\mathbb{R}$.

Definition 9.3

An **orthonormal basis** of (V,H) is a basis $\{e_i\}$ s.t.

$$H(e_i,e_i) = \|e_i\|^2 = 1,\ H(e_i,e_j) = 0 \forall i \neq j, e_i \perp e_j.$$

some other stuff

... ... So, in matrix form, $H(z,w) = \overline{z}^{\mathrm{T}} w = z^* = (\overline{z}_1 \dots \overline{z}_n)$ as conjugate transpose.

Consider fourier series as a (sorta-kinda) example of $V = C^\infty(S^1, \mathbb{C})$ w/ is infinitely differentiable $S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{C}, (\iff 1)$-periodic functions $\mathbb{R} \to \mathbb{C}$. L^2-inner product,

$$H(f,g) = \int_{S^1} \overline{f(t)} g(t) dt = \int_0^1 \overline{f(t)} g(t) dt$$

and

$$H(f,f) = \int_0^1 |f(t)|^2 > 0 \text{ if } f \neq 0.$$

Note that then

$$f_n(t) = e^{2\pi i n t} = \cos(2\pi n t) + i \sin(2\pi n t), n \in \mathbb{Z}$$

is orthogonal! So, $H(f_n, f_m) = \delta_{n,m}$.

Definition 9.4

We let V be a complex space, H a Hermitian inner product, and $T : V \longrightarrow V$ a linear operator.

$$\text{the adjoint of } T \text{ is } T^* : V \longrightarrow V \text{ s.t. } H(T^* v, w) = H(v, Tw) \forall v, w \in V$$

$$(\iff H(Tv,w) = H(v, T^* w)).$$

Some immediate concequences:

- T is *self-adjoint* if $T^* = T$
- T is *unitary* if $H(Tv,Tw) = H(v,w)$
- T is normal if $TT^* = T^*T$ unitary operators form a group (omg!) $U(V) = U(V,H) \subset \mathrm{Aut}(V) = \mathrm{GL}(V)$, $U(n) \subset \mathrm{GL}(n,\mathbb{C})$, e.g., $U(1) = S^1 = \{z \mid |z| = 1\} \subset \mathbb{C}^*$.

Proposition 1

in an orthonom. basis $\mathcal{M}(T^*) = \mathcal{M}(T)^* = \overline{\mathcal{M}(T)^t}$ so $H(Tv,w) = (\mathcal{M}v)^* w = v^*(\mathcal{M}^* w) = H(v, T^* w)$, self-adjoint operators $\leftrightarrow$ Hermitian matricies

Definition 9.5: Complex spectral theorem

put it here

proof goes here

in such a basis $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

9.1.1 some number theory (non-degenerate symmetric bilinear forms)

this u can go over later via the notes just put it in here