Lecture 6. category theory *October 03, 2025*

1 Category Theory

1.1 Categories

Definition 1 (Category). A category $\mathscr{C}$ is a collection of **objects**, such that:

(i.) for each pair of objects $A, B \in \mathrm{Ob}(\mathscr{C})$;

(ii.) a collection of morphisms is given by $\mathrm{Mor}(A, B)$;

(iii.) there exists operation titled *composition of morphisms* given by

$$\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$$

such that $f, g \mapsto g \circ f$;

(iv.) Category Theory Axioms:

(a) every object A has an identity morphism $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ such that

$$\forall f \in \mathrm{Mor}(A, B),\ f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f;$$

(b) composition (of morphisms) is associative, i.e., $(f \circ g) \circ h = f \circ (g \circ h)$.

We can think of categories and sets as, for the most part, the same thing, except for the fact that a category may include the category of all sets; this is where we deliniate sets and categories in order to avoid pesky paradoxes.

Some examples of categories, for purposes of clarity:

1. category of sets, $\mathrm{Mor}(A, B) =$ all maps $A \to B$
2. Vect_k as a fin. dim. vector space over k, then $\mathrm{Mor} =$ linear maps
3. groups, group homomorphisms
4. topological spaces, continuous maps
5. and more...

There exists a relationship called **isomorphism** not too dissimilar from group-theoretic isomorphisms.

Definition 2 (Isomorphism). $f \in \mathrm{Mor}(A, B)$ is an *isomorphism* if has inverse, i.e., $\exists g \in \mathrm{Mor}(B, A)$ s.t. $g \circ f = id_A$ and $f \circ g = id_A \Rightarrow$ if $\exists g \Rightarrow !\exists g$ and that id_A is an isomorphism. $f \Rightarrow f^{-1}$ $(gf^{-1}$)are both called isomorphisms, individually.

Concequentially the automorphisms $Aut(A)$ is always a group.

sets :permutations, groups: automs, vec: invertible lin op.

Isomorphic objects of e have isomorphic automorphic groups. Given $f \in \mathrm{Mor}(A, B)$ as an isomorphism $c_f : Aut(A) \to Aut(B)$ which maps $g \mapsto f \circ g \circ f^{-1}$.

We may define sum, product, and quotient objects in categories. Consider a product $A \times B := \mathcal{C}$ of objects $A, B \in \mathscr{C}$ characterized by $\mathcal{P}$ projects $\pi_1 \in \mathrm{Mor}(\mathcal{P}, A), \pi_2 \in \mathrm{Mor}(\mathcal{P}, A)$ s.t. $\forall$ object T, $\forall f_1 \in \mathrm{Mor}(T, A), f_2 \in \mathrm{Mor}(T, B)$, $\exists$ unique $f \in \mathrm{Mor}(T, \mathcal{P})$ s.t. $\pi_1 \circ f = f_1$, $\pi_2 \circ f = f_2$.

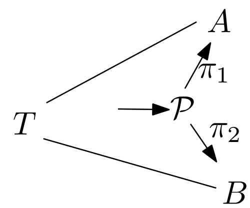

Figure 1

Definition 3 (functor). C, D categories, a (*covariant*) functor $F : C \to D$ is an assigment:

1. for every object X of C an object $F(X)$ of D
2. for each morphism $f \in \mathrm{Mor}_C(X, Y)$, a morphism $F(f) \in \mathrm{Mor}_D(F(X), F(Y))$ s.t.
   (a) $F(id_X) = id_{F(X)}$
   (b) $F(g \circ f) = F(g) \circ F(f)$

Consider Forgetful functors , on a vector space over k given vector sace V $F : Vect_k \to Vect_k$objects: $W \mapsto Hom(V, W)$ lin. maps $V \to W$, morphism $f : W \to W' \Rightarrow$??

Anywhos. He's going on about linear maps as morphisms for categories and functors but I don't care all that much at the moment. As a functor $F : \mathrm{Vect}_k \to \mathrm{Vect}_k$ can be shown as $Hom(V, \circ)$. What the mell does obj. $V \mapsto V_{\mathbb{C}} = V \oplus iV$

Definition 4. A *contravariant functor* $F : C \to D$ is the same except reverses direction of morphism:

$$f \in \mathrm{Mor}_C(X, Y) \mapsto F(f) \in \mathrm{Mor}_D(F(Y), F(X)) \mid F(g \circ f) = F(f) \circ F(g).$$

So for example, consider $V \to V^*$ s.t. $f \in Hom(V, W) \mapsto f^* \in Hom(W^*, V^*)$.

Definition 5. Given two functors $F, G : \mathscr{C} \to \mathscr{D}$ a natural transformation t from F to G is the data $\forall X \in Ob(\mathscr{C})$, a morphism $t_X \in \mathrm{Mor}_{\mathscr{D}}(F(X), G(X))$ s.t.

$$\forall X, Y \in Ob(\mathscr{C}), \forall f \in \mathrm{Mor}_{\mathscr{C}}(X, Y),$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "t_x"'] & F(Y) \arrow[d, "t_y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}.$$

Definition 6 (Bilinear forms). A ***bilinear form*** on a vector space V over a field k is a map $b : V \times V \to k$ (this is to say we take $(v, w) \mapsto b(v, w)$) that is linear in each variable, i.e., $\forall u, v, w \in V$, $\lambda \in k, b(\lambda v, w) = b(v, \lambda w) = \lambda b(v, w)$ ("multiplied" on both sides separately).

Definition 7 (symmetric)**.** Say b is symmetric if $b(v, w(= b(w, v)))\forall v, w \in V$ and is skew symmetric if $b(v, w) = -b(w, v)$.

I have mentally clocked out, I spent last night and this morning going over the notes for today and ngl i understood it better before he started explaining it

AWWWWWWW

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}.$$