Project 4 Report

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1 Running Model

A spring-loaded inverted pendulum (SLIP) model was used to analyze the characteristics of running, as shown in Figure 1.

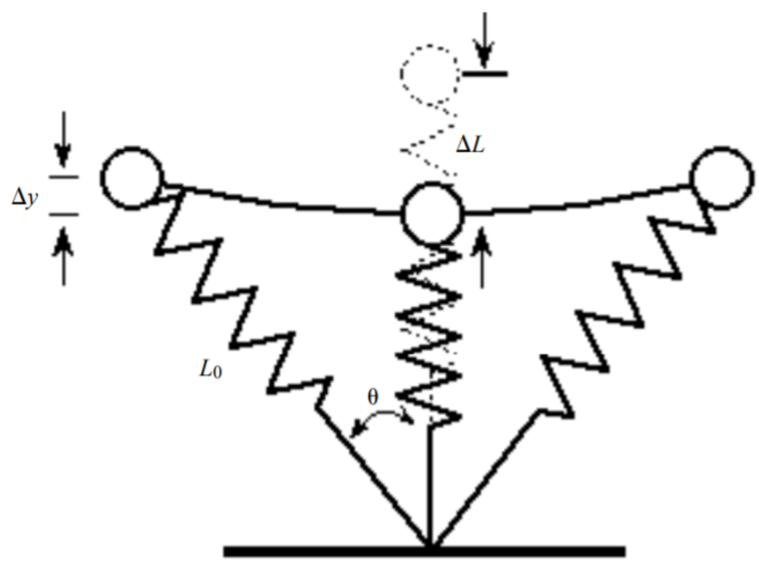

Figure 1: Spring-loaded inverted pendulum model of leg [1]. ΔL is the magnitude of spring compression, L_0 is the unstretched length, Δy is the vertical compression, and θ is the contact angle to the ground.

1.1 Model Assumptions

- All motion is parallel to the sagittal plane of the body.

- The leg links are rigid and massless.
- The mass of the body will be approximated as a point mass atop a simple inverted pendulum.
- Air resistance is negligible, there is no slipping at the point of ground contact.
- The spring is linear.

2 Findings

Part I

Response of Model Characteristics to Running Speed

The author's own total body weight M and total leg length L_0 were used for the analysis of the model. From these parameters, the leg stiffness, k_{leg}, was determined from the work of [1] and [2] respectively, given below as

$$k_{leg} = 715M^{0.615}, \quad (2.1)$$

$$k_{leg} = \frac{K_{LEG}Mg}{L_0}, \quad (2.2)$$

where K_{LEG} is the non-dimensionalized leg stiffness, found to be generally constant and equal to 15 in [2], and g is the acceleration of gravity. The values of k_{leg} were calculated using both equations and averaged for the value to be used in subsequent calculations, and was found to be **11527 N/m**. From the work of [1] and [2], it was also assumed that the leg stiffness was *constant*, regardless of running speed.

A range of average running speeds was determined, over which the SLIP model would be analyzed. The Froude number Fr is a ratio of the kinetic to potential energies experienced when walking, and generally, a value of 0.5 is a good prediction of the walk-to-run transition speed, and can be given as

$$Fr = \frac{u^2}{gL_0}, \quad (2.3)$$

where u is the average running velocity. For the author's own characteristics, the Froude number was plotted over a range of speeds to determine a minimum running speed of **2.215 m/s**, as shown in Figure 2.

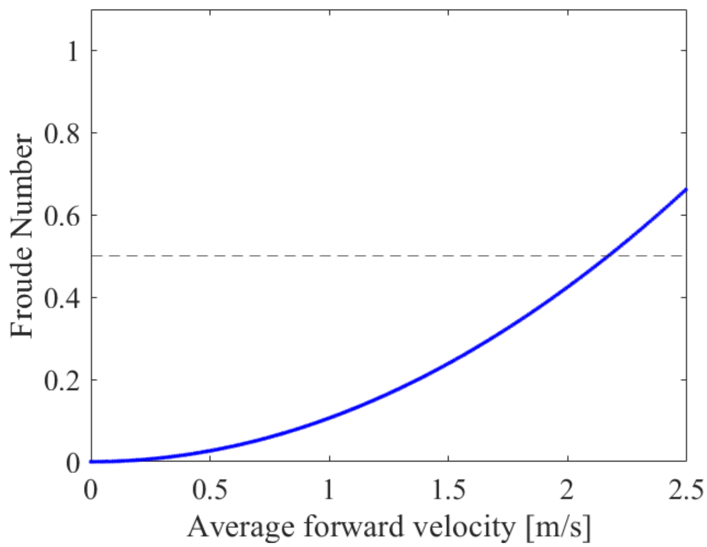

Figure 2: Plot of the Froude number as a function of average running velocity. The dotted line indicates $Fr = 0.5$.

Since the world record for the 100 m sprint is around 10 seconds, a reasonable upper limit for running speed was estimated to be around 7 m/s. Comparing this to average running speeds found online, the upper limit of running velocity was set as **7.15 m/s**.

For the SLIP model, an *effective* vertical stiff k_{vert} can be defined as

$$k_{vert} = \frac{R_y}{\Delta y}, \tag{2.4}$$

$$\Delta L = \Delta y + L_0(1 - \cos(\theta)), \tag{2.5}$$

where R_y is the vertical ground reaction force, Δy is the vertical change in the center of mass, and ΔL is the change in length of the leg. This allows the SLIP model shown in Figure 1 to be further simplified to a one-dimensional spring model with spring constant k_{vert}, with an analytical solution to its governing equation given as

$$y = \frac{mg}{k_{vert}}\left[\cos(\sqrt{\frac{k_{vert}}{M}}t) - 1\right] - v\sqrt{\frac{M}{k_{vert}}}\sin(\sqrt{\frac{k_{vert}}{M}}t). \tag{2.6}$$

The stride frequency f in strides per second can be approximated independent of mass and leg length as

$$f = 0.65 + 0.2u. \tag{2.7}$$

From this approximation, an average stride period T can also be calculated as the reciprocal of f.

The stride period can also be expressed as twice the sum of the durations of flight and contact, given as

$$T = 4\sqrt{\frac{M}{k_{vert}}}\left[\pi - \tan^{-1}(\frac{v}{g}\sqrt{\frac{k_{vert}}{M}})\right] + \frac{4v}{g}, \tag{2.8}$$

where v is the vertical velocity of the center of mass.

By combining expressions for the contact angle, contact time, and Δy, the following equation can be determined:

$$\frac{k_{vert}}{k_{leg}} = 1 + \frac{L_0 k_{vert} - \sqrt{L_0^2 k_{vert}^2 - M k_{vert} u^2 \left[\pi - \tan^{-1}(\frac{v}{g}\sqrt{\frac{k_{vert}}{M}})\right]^2}}{Mg + \sqrt{M^2 g^2 + M k_{vert} v^2}}. \tag{2.9}$$

The details of these derivations can be seen in [3].

With the approximated stride period T from Equation 2.7, Equations 2.8 and 2.9 were numerically solved for the unknowns, k_vert and v at running speeds in the range defined previously.

From [3], a number of walking characteristics can be calculated from k_{vert} and v for the SLIP model. The time of ground contact t_c can be calculated as twice the time taken for the body to move from its initial height to its minimum height, where $\dot{y}(t) = 0$. This gives that

$$t_c = 2\sqrt{\frac{M}{k_{vert}}}\left[\pi - \tan^{-1}(\frac{v}{g}\sqrt{\frac{k_{vert}}{M}})\right]. \tag{2.10}$$

Ground contact time was plotted over the range of running speeds, and is shown in Figure 3.

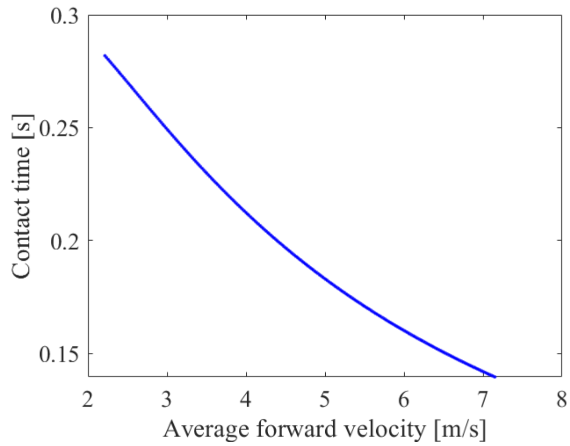

Figure 3: Ground contact time plotted over various running speeds.

With the time of contact known, the contact angle θ could also be found as

$$\theta = \sin^{-1}(\frac{ut_c}{2L_0}). \tag{2.11}$$

The results were likewise plotted over the tunning speeds, and is shown in Figure 4.

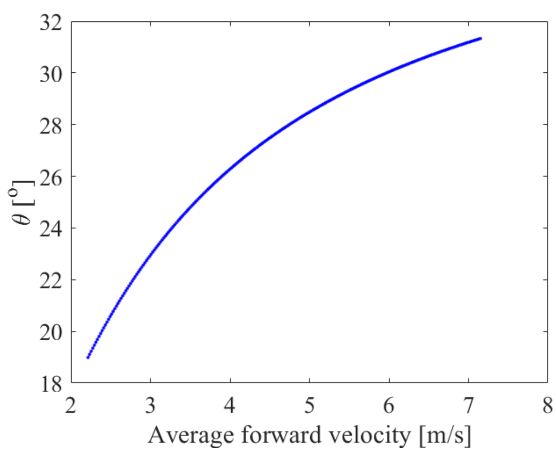

Figure 4: Ground contact angle plotted over range of running speeds.

The stride length of running L_s can be calculated as

$$L_s = \frac{u}{f}, \tag{2.12}$$

and for each velocity, L_s is shown in Figure 5.

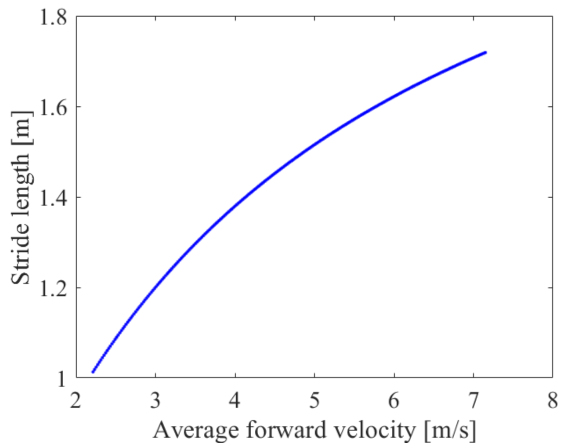

Figure 5: Stride length plotted over range of running speeds.

Lastly, the total vertical displacement of the mass Δy_t was determined as the sum of the displacements during contact and flight, given as

$$\Delta y_t = \Delta y + \Delta y_f, \tag{2.13}$$

where Δy_f is the maximum vertical displacement during flight. Δy during contact could be evaluated as the magnitude of y at $t_c/2$, and is given as

$$\Delta y = \frac{Mg + \sqrt{M^2 g^2 + M k_{vert}^2 v^2}}{k_{vert}}, \tag{2.14}$$

and Δy_f could be determined from simple projectile motion calculations at half the duration of flight $t_f = \frac{2v}{g}$. Over the range of running speeds, the total vertical displacement was then plotted, and is shown in Figure 6.

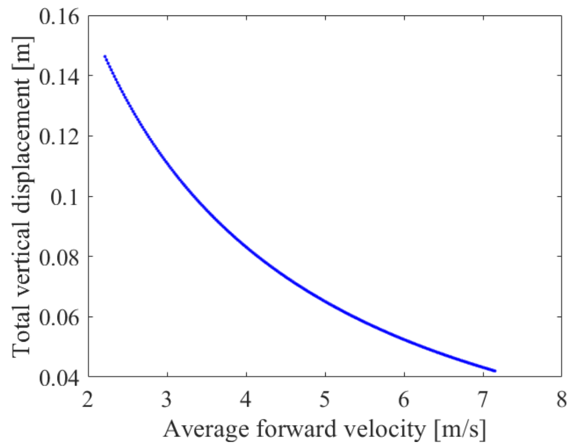

Figure 6: Total vertical displacement plotted over range of running speeds.

Part I Discussion

Many of the results in this section are intuitive to what one would expect as a person's running velocity increases. Figure 3 shows that contact time with the ground becomes shorter and shorter as running speed increases. At constant velocities, a person would need to strike the ground at higher frequencies to maintain a higher average speed, and so ground contact time is expected to decrease. Interestingly, the period of ground contact seems to remain roughly the same percentage of total period time regardless of speed, around **25-35 %**.

Similarly, ground contact angle seems to generally increase with running speed, but the effect diminishes at faster velocities. Again, as a person runs faster, it is more efficient to take larger steps to propel oneself forward. The contact angle seems to have an upper limit of around 40-45 degrees, at which point it seems like running could become unstable or possibly too physically strenuous.

As discussed previously regarding Figure 4, Figure 5 shows that stride length is closely related to the contact angle of the ground, and increases with respect to running velocity but with an undefined limit. As before, strides too large may be physically impossible or energy inefficient, but generally, a larger velocity seems to imply longer strides. This effect can also be consistently seen in the stride lengths of sprinters and marathon runners.

Lastly, Figure 6 seems to show distinctly the clear differences between walking and running. As running speed increases, the total vertical displacement decreases, demonstrating that running does not rely as much, if at all, on the transfer of gravitational to kinetic energy. Vertical displacement seems to be minimized as speeds increase, as all the energy spent by the body is directed to pushing it forwards, rather than upwards.

Part II

Work Done by Compound Pendulum Model

From the previous project, the leg was approximated as a compound pendulum with characteristics inertia $I =$**2.455 kgm^2**, length $L_C =$**0.358 m**, and mass $m =$**12.075 kg**. The equations for the work W of a compound pendulum can be given as

$$W = I(2\theta)^2(\omega_n^2 - \omega^2), \tag{2.15}$$

$$\omega_n = \sqrt{\frac{mgL_c}{I}}, \tag{2.16}$$

where ω_n is the natural frequency and ω is the driven frequency. The driven frequency was found as $\omega = 2\pi f$, and the work done over the range of running speeds is shown below in Figure 7.

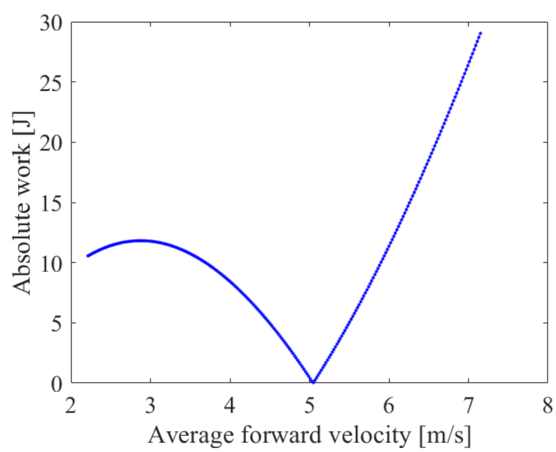

Figure 7: Work done by compound pendulum model as a function of constant running velocity.

To determine the effect of changing the leg rotation angle and frequency, the partial derivatives of Equation 2.15 were taken with respect to these variables, and are given as

$$\frac{\partial W}{\partial \theta} = 4I\theta(\omega_n^2 - \omega^2), \tag{2.17}$$

$$\frac{\partial W}{\partial f} = 4I\theta^2 w_n^2 - 8I\theta^2\omega. \tag{2.18}$$

These were plotted over the same interval and is shown in Figure 8.

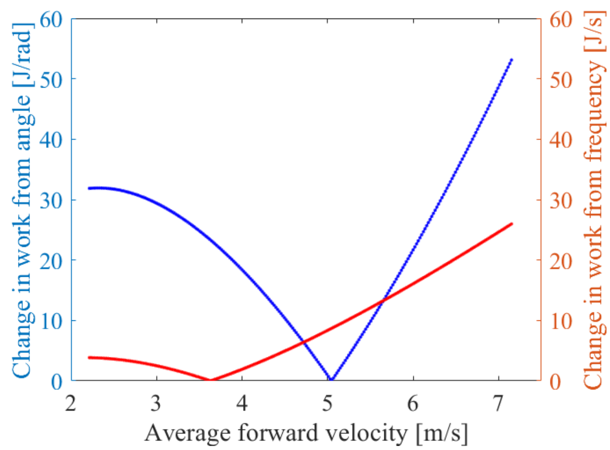

Figure 8: Change in work done from changing frequency and leg rotation angle, plotted over running velocities.

Part II Discussion

From Figure 7, there seems to exist a running speed at which minimial work is done. Mathematically, this point seems to exist when the driven frequency is equal to the natural frequency, and physically it seems like the most efficient speed at which to run would be at an individual's natural swinging frequency. An analysis into the stride frequencies of marathon runners would be interesting for comparative purposes.

Looking at Figure 8, it seems that the change in work from angular frequency is generally less than that from changing the total swing angle, except for the neighborhood near the natural frequency. This results seems to make sense, as the natural frequency is independent of the swing angle. At all other points however, it seems that significantly more work is done by moving the mass of the leg at larger swing angles than higher frequencies.

Since the inertia and length to the center of gravity were calculated as averages over a walking cycles, these values could be significantly different if derived from a running gait. Since running involves phases of flight and longer strides, the average inertia can be expected to increase significantly. While it is difficult to ascertain how the distance to the center of gravity might change, it does not seem like it would change greatly given that the legs are also similarly stretched out in walking. It is possible that with greater knee flexion, the center of mass may be pulled in closer to the body during running, but without a more thorough analysis, it is difficult to say conclusively.

Part III

Equations B1 and B2 in [2] give the full nonlinear dynamics of the SLIP model without assuming that horizontal velocity is constant. These were converted to dimensional form, and can be expressed as

$$\begin{aligned} \ddot{x} &= \frac{k_{leg}}{M}(L_0 - \sqrt{x^2+y^2})\sin(\theta), && (2.19)\\ \ddot{y} &= \frac{k_{leg}}{M}(L_0 - \sqrt{x^2+y^2})\cos(\theta) - g. && (2.20) \end{aligned}$$

While McMahon numerically integrates these equations and varies k_{leg} until a steady state solution is reached to find a constant value for the leg stiffness, here the initial angle of ground contact was varied until steady state solutions were seen. For a single running speed of $u_0 = \mathbf{4.673\ m/s}$, these equations were numerically integrated with MATLAB's `ode45` function over the interval [0, 3] s. At this running speed, the associated velocity in the vertical direction, v_0 was also calculated to be **-0.607 m/s**. To be in steady state, the following conditions were expected to be seen in the solution over one cycle:

$$\begin{aligned} y(t_0) &= y(t_f) = y_0, && (2.21)\\ \dot{x}(t_0) &= \dot{x}(t_f) = u_0, && (2.22)\\ \dot{y}(t_0) &= -v_0, && (2.23)\\ \dot{y}(t_0) &= v_0, && (2.24) \end{aligned}$$

where t_0 is the start time and t_f is the time for one cycle. The initial values passed into `ode45` were x_0, y_0, u_0, and v_0. The initial angle of ground contact θ_0 was implied with the relation that

$$\theta_0 = \tan^{-1}(\frac{x_0}{y_0}). \tag{2.25}$$

t_f was determined by finding at what time the contact angle would equal θ_0 reflected across the vertical. A plot of this is shown in Figure 9.

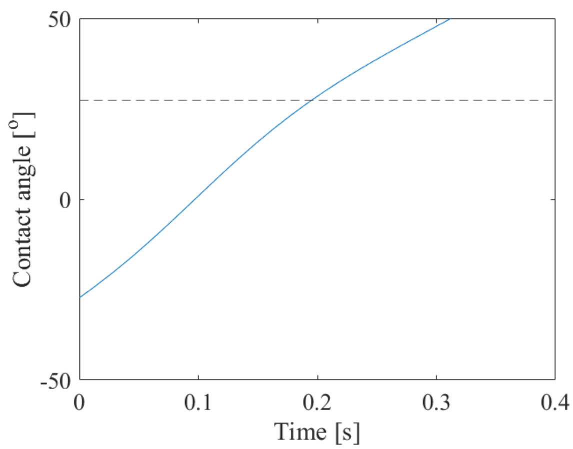

Figure 9: Angle vs. time plot for numerical simulation of Equations 2.19 and 2.20. The dotted line is θ_0.

With trial and error, an initial angle of 27.25° was found to exhibit the boundary conditions in Equations 2.21-2.24. The solutions were plotted over time, and are shown in Figures 10, 11, and 12.

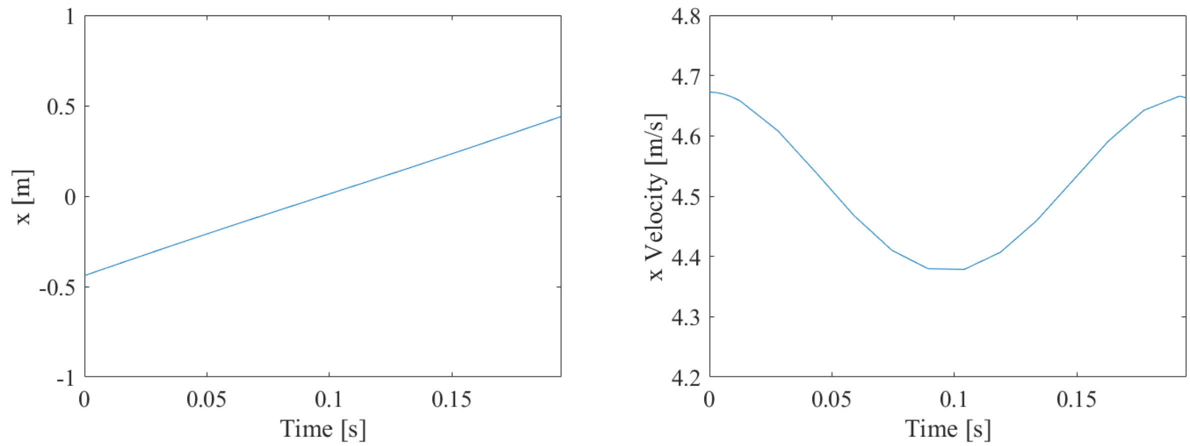

Figure 10: Horizontal position and velocity over time for the SLIP pendulum.

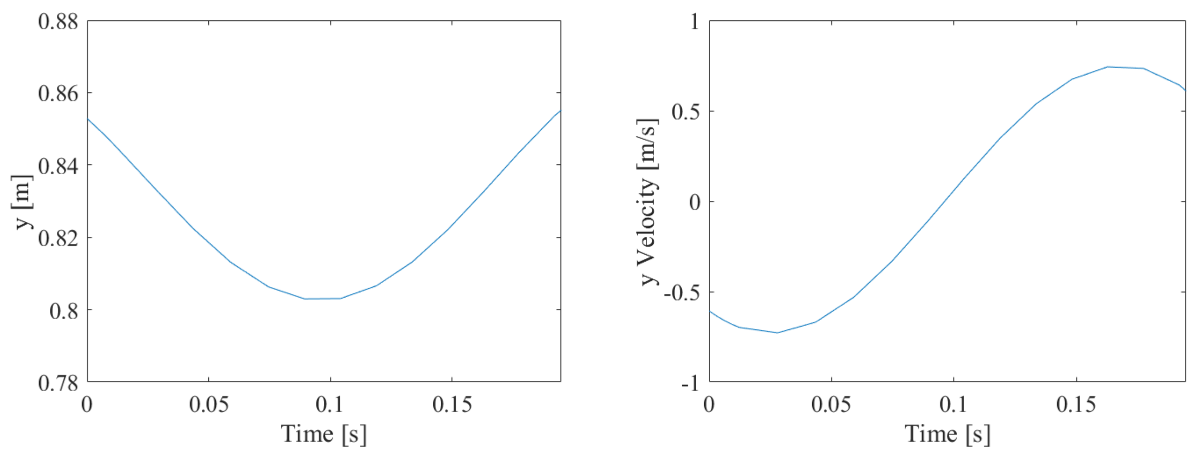

Figure 11: Vertical position and velocity over time for the SLIP pendulum.

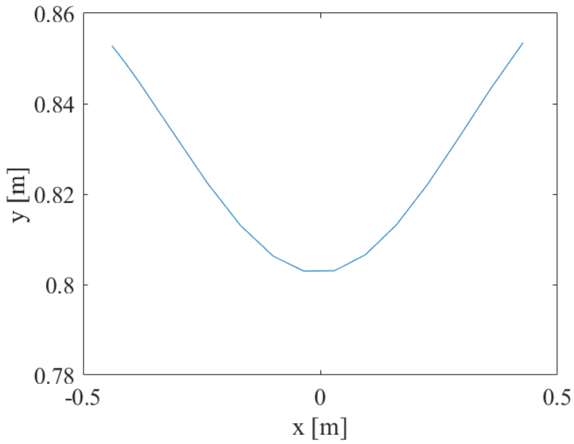

Figure 12: Motion of the center of mass from t_0 to t_f.

Part III Discussion

As a whole, the solutions obtained from Part I are fairly similar to those obtained from the full nonlinear dynamics of the model. This is not unexpected, as looking at Figure 10 shows that the actual variation in the horizontal velocity is not relatively large, showing a maximum difference of around **5-6 %**. Comparing the other figures show that the solutions are similar as well. At $u = 4.6$ m/s, the contact angle in Figure 4 is very close to the determined value of 27° as well. Similarly, the determined t_f of around 0.23 seconds agrees with the plot in Figure 3.

3 Results and Conclusions

The SLIP model of running was analyzed to determine how the contact angle, stride length, total vertical displacement, and contact time were affected by the running speed of the individual. It was determined that the full nonlinear dynamics of this model do not differ greatly from the assumption that v is constant, and solutions to either should be relatively accurate to real-world results. From [2], it was found that the vertical distance traveled by SLIP model in stance was around 6.2 cm at a speed of 5 m/s, and Figure 12 shows very similar results.

While the compound pendulum model was derived from walking gaits, analyses of the work done by the pendulum show that running with a frequency that matches the natural frequency of the leg is the most efficient gait. Furthermore, it seems that stride length costs more work at all speeds except for near the neighbourhood the natural frequency. Near this region, increasing the swing frequency of your leg seems to increase the amount of work done to the leg.

References

[1] C. T. Farley, J. Glasheen, and T. A. McMahon. Running springs: Speed and animal size. Journal of Experimental Biology, 185:71–86, 1993.

[2] T. A. McMahon and G. C. Cheng. The mechanics of running: How does stiffness couple with speed? Journal of Biomechanics, 23(Supp. 1):65–78, 1990.

[3] J. P. Schmiedeler. SLIP Model Overview. University of Notre Dame, 2020.