Project 3 Report

Se Hwan Jeon

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1 Planar Leg Model

A planar model of the human leg was developed during the swing phase, shown in Figure 1. The relationships between the hip and knee trajectories, torques, and work done for the leg complex were investigated.

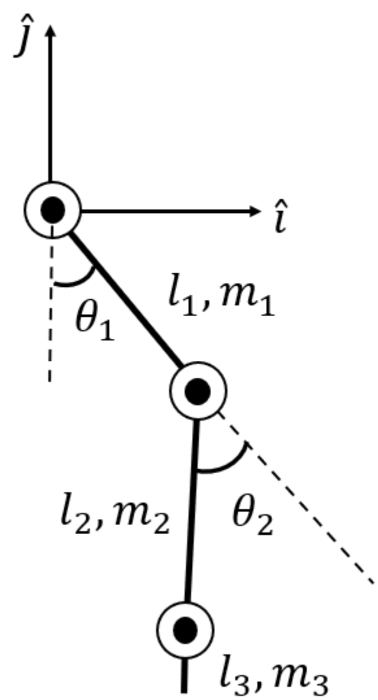

Figure 1: Sagittal plane model of human leg.

1.1 Model Assumptions

A number of assumptions were made for the two-link model:

- All motion is solely parallel to the sagittal plane of the body.
- Each link (thigh, shank, foot) is rigid.
- The ankle is locked and has no flexion, but its length, inertia, and mass contributions were accounted for.
- The origin of the frame is rigidly attached to the hip, and rotation of the trunk was in the anterior-posterior direction was not considered.

2 Findings

Part I

Hip and Knee Joint Trajectory

The data in Figure 3 of [1] was approximated with Engauge Digitizer^{TM} software, for an average walking speed of 1.42 m/s. The relative joint angles were originally plotted as a function of the percentage of swing completed. This was converted to time in seconds by multiplying the percentage by the total duration of swing, found in Appendix A [1]. The plots of relative joint angle and joint velocities versus time are shown in Figures 2 and 3. MATLAB's `interp1` function was used to restructure the approximated data to ensure that the values for each plot were determined at the same instants in time for subsequent calculations.

Hip and Knee Torques

The values in the "Young Mean" category were chosen for the analysis, as the relatively young author's weight and height (75 kg and 181 cm respectively) were used in subsequent calculations. Using Winter's Tables, the relevant body segment lengths and inertia properties were determined for each of the links in Figure 1 [2]. The expression for the inertia of any segment about the hip can be given as

$$I_{i,hip} = m_i r_c^2 + m_i d^2, \tag{2.1}$$

where I_{hip} is the inertia of link i about the hip, m_i is the mass of link i, r_c is the radius of gyration of link i, and d is the distance from the center of mass of that link to the hip. Note that for the shank and foot links, d would change as a function of the joint angles.

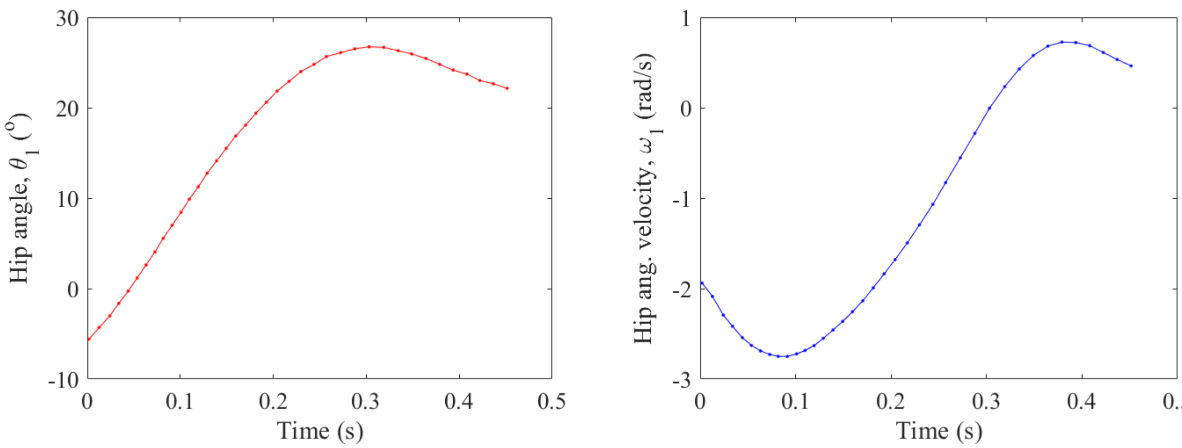

Figure 2: Hip joint displacement and angular velocity over time [1].

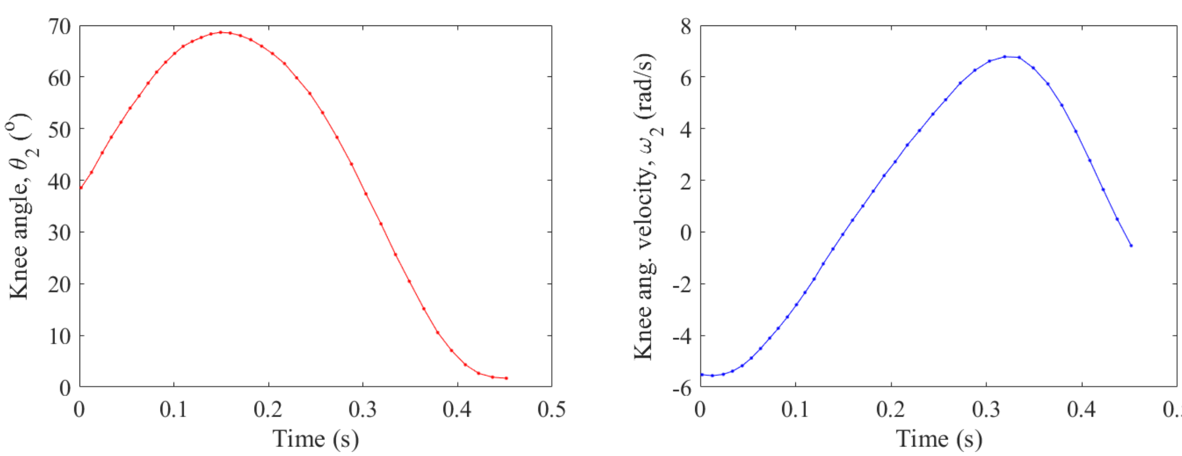

Figure 3: Knee joint displacement and angular velocity over time [1].

The angular acceleration for the hip and knee joints were determined over time with a simple linear approximation over the swing duration as shown in Figure 4. With the inertia, masses, link lengths, trajectories, and accelerations determined at each instant in time, the applied torques on the hip and knee joints were calculated with

$$\begin{aligned}
\tau_1 &= \ddot{\theta}_1 \left[I_1 + I_c + m_l l_1 l_c \cos(\theta_2) + \frac{m_1 l_1^2 + m_c l_c^2}{4} + m_c l_1^2 \right] \\
&\quad + \ddot{\theta}_2 \left[I_c + \frac{m_c l_c^2}{4} + \frac{m_c l_1 l_c}{2} \cos(\theta_2) \right] - \frac{m_c l_1 l_c}{2} \dot{\theta}_2^{\,2} \sin\theta_2 \\
&\quad - m_c l_1 l_c \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2) + \left[\frac{m_c l_c}{2} \cos(\theta_2 + \theta_1) + l_1 (\frac{m_1}{2} + m_c) \cos(\theta_1) \right] g,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\tau_2 &= \ddot{\theta}_1 \left[I_c + \frac{m_l l_1 l_c}{2} \cos(\theta_2) + \frac{m_c l_c^2}{4} \right] + \ddot{\theta}_2 \left[I_c + \frac{m_c l_c^2}{4} \right] + \frac{m_c l_1 l_c}{2} \dot{\theta}_1^{\,2} \sin\theta_2 \\
&\quad + \frac{m_c l_c g}{2} \cos(\theta_1 + \theta_2),
\end{aligned} \tag{2.3}$$

where 1 and 2 denote the hip and knee respectively, c denotes the combined knee and foot, τ is the applied torque at the joints, I is the inertia about the hip of the link(s), l is the length of the link, θ is the joint angle, and g is the gravitational acceleration constant.

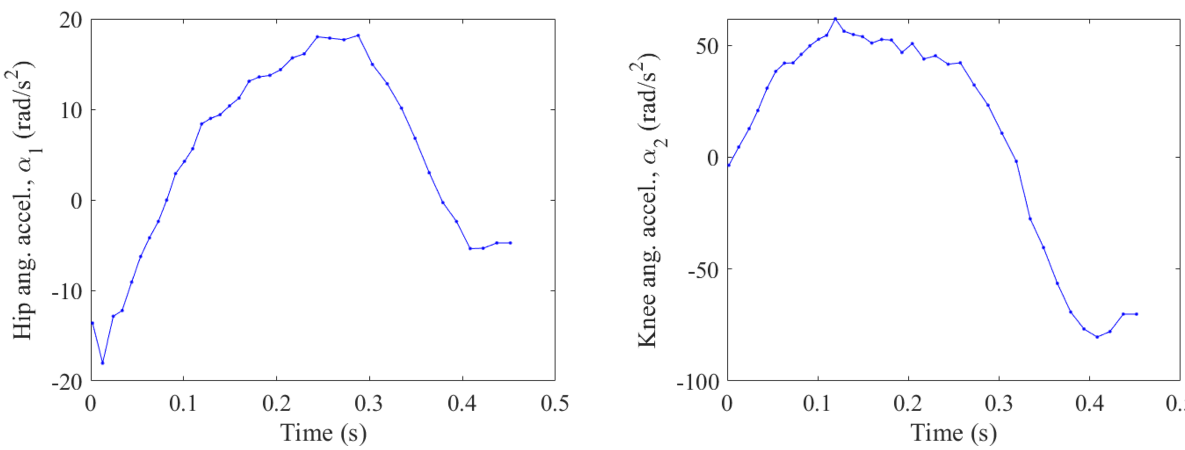

Figure 4: Numerical differentiation of obtained angular velocity data for hip (left) and knee (right) joint angle accelerations.

To produce the trajectories in Figures 2 and 3, the required torques at the hip and knee over time is shown in Figure 5.

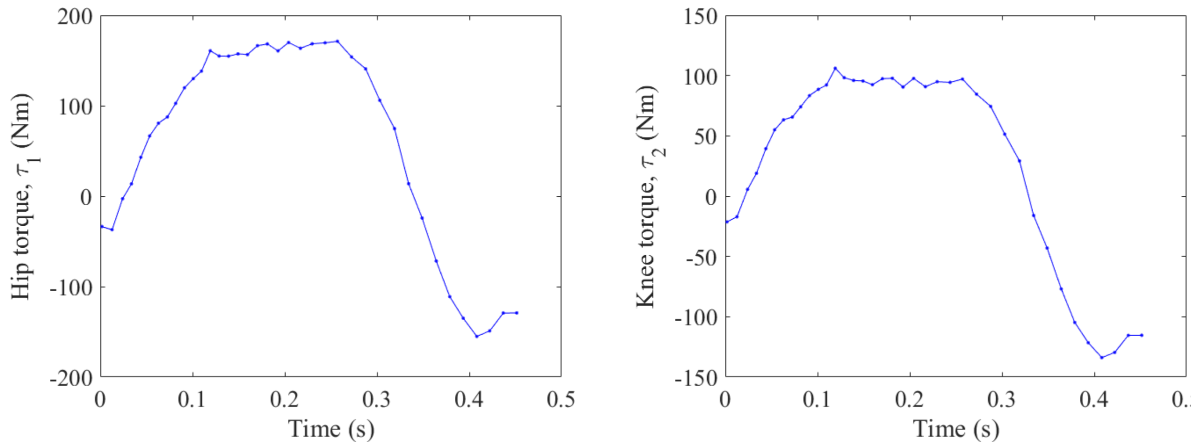

Figure 5: Required torque of hip (left) and knee (right) for walking at specified velocity of 1.42 m/s.

Work at Joints

The work done at the hip and knee joints was calculated with

$$W = \tau \Delta\theta, \tag{2.4}$$

where W is work and $\Delta\theta$ is the change in angle of the joints between the sampled instants of time. By multiplying the calculated torques found above by the changes in angle for the hip and knee joints, the work done could be calculated over time, and is shown in Figure 6. The total work done was estimated by summing the absolute values of the work done by the hip and knee joints.

The average work done by the knee joint was found to be **2.43 J**, the average done by the knee joint was found to be **-0.91 J**, and the average done by the absolute sum of the two was **7.121 J**.

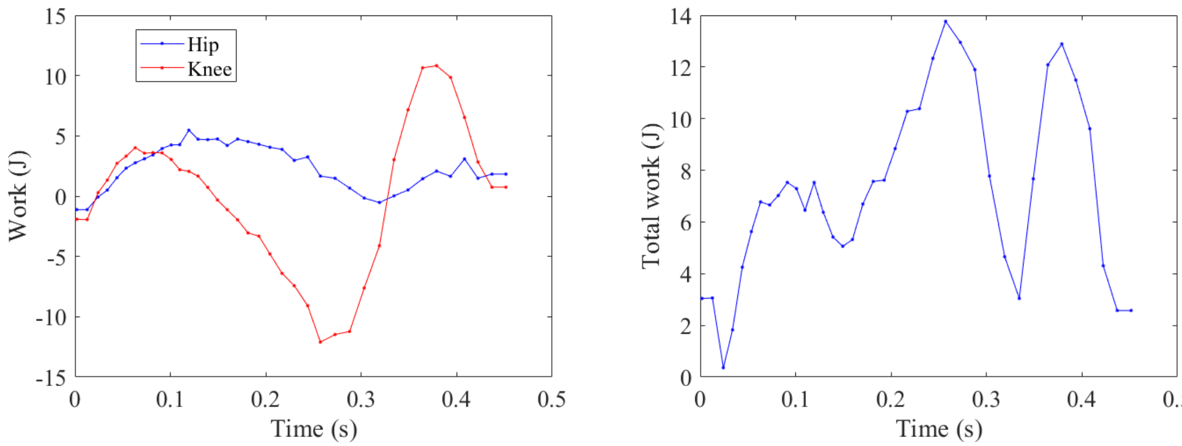

Figure 6: Work done by knee and hip while walking (left). Absolute value of total work done by both joints (right).

Step Length

The step length was estimated to be the distance traveled by the distal end of the shank during swing. The position p_s of this point at time t_1 can be found as

$$\boldsymbol{p_s} = \begin{bmatrix} l_1 \sin(\theta_1) + l_2 \sin(\theta_2 - \theta_1) \\ -l_1 \cos(\theta_1) - l_2 \cos(\theta_2 - \theta_1). \end{bmatrix} \tag{2.5}$$

The total distance traveled by the shank was estimated by using the distance formula between each sampled instant of time, so that

$$d = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}, \tag{2.6}$$

where d is the total distance traveled, i is the sampled instant in time, n is the total number of sampled points from [1], and x and y are the position of the end of the shank at time i.

This resulted in an estimated step length of **0.421 m**, or a stride length of **0.842 m**.

Part I Discussion

From the calculations done with the approximated data, a number of observations can be made about walking gait. Knee extension appears to peak early in swing before straightening for heel strike, and the hip enters heel strike still significantly offset from the vertical. Both angular velocities approach zero near heel strike, which seems intuitive, as one would want to avoid as much sliding contact with the ground as possible.

It is interesting that the torques at the knee and the hip appear similar in shape over time and only differ in their scale. During the swing phase, both seem to experience two instances of "free swing", where no torque is applied before applying a torque in the opposite direction.

From the work calculations, it seems that on average, the work done by the knee joint is actually negative. During swing, the position of the center of mass of the shank changes little, and the result seems to imply that the mass of the shank at push off is actually more elevated than the mass of the knee at heel strike, which seems intuitive as well given the additional height from the length of the foot at push off. Furthermore, when walking, little to no effort is actually used to extend the knee forwards. Instead, a natural swing seems to be most optimal.

With the step estimation, a step length of around 40 centimeters seems similar to averages reported for a variety of people. This could be modified by estimating only the horizontal distance traveled by the distal end of the shank for more accurate results.

Part II

Inertia Calculations

Over the swing phase, the total inertia about the hip was calculated for the trajectory shown in Figure 2 and 3 at each instant in time. Equation 2.1 was used to calculated the inertia of each link about the hip, and summed to find the total inertia. As mentioned previously, the distance between the centers of mass of the foot and shank would change as a function of θ_1 and θ_2. The inertia of the leg was plotted over time, and is shown in Figure 7.

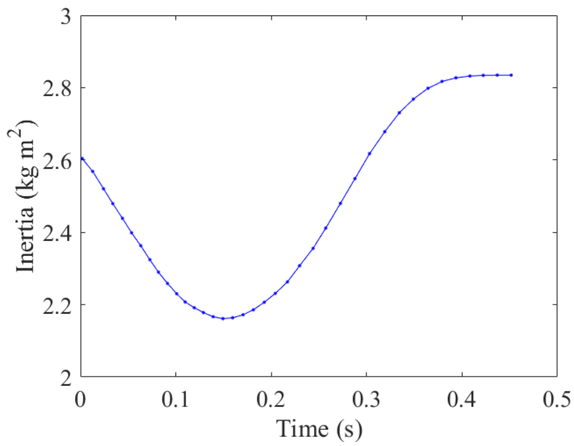

Figure 7: Inertia of the leg about the hip over swing phase of walking.

The average inertia over this interval was calculated to be **2.455 kg m**^{2}.

Center of Gravity

The location of the center of gravity was also tracked over the swing phase. The weighted average of the foot, thigh, and shank was determined to the find the position of the center of gravity, and can be given as

$$q_i = \frac{\sum_{j=1}^{n} m_i q_i}{\sum_{j=1}^{n} m_i}, \tag{2.7}$$

where j designates the links in the system, n is the total number of links, and q_i is the coordinate (x, y, or z).

The location of the center of mass was plotted parametrically as a function of time, and is shown in Figure 8 below.

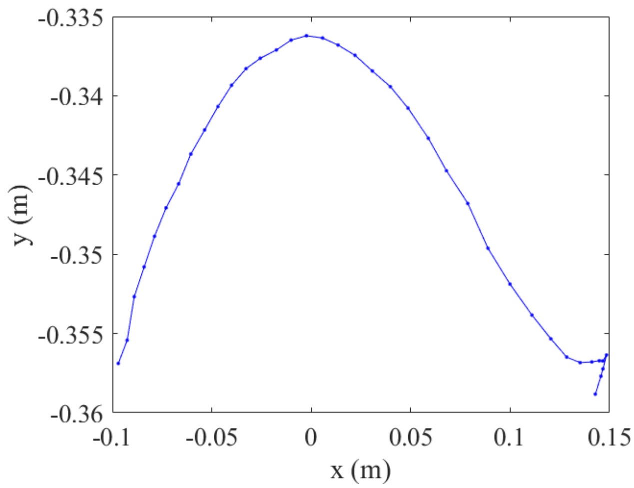

Figure 8: Center of gravity location relative to the hip over swing phase.

This average distance was determined by taking the magnitude of each point in Figure 8 and dividing it by the number of points, resulting in an average distance of **0.358 m** from the hip.

The angle between the location of the center of gravity, ϕ and the vertical was also determined by taking the arctangent of the y and x components of the position of the center of gravity. The result was plotted over time as shown in Figure 9.

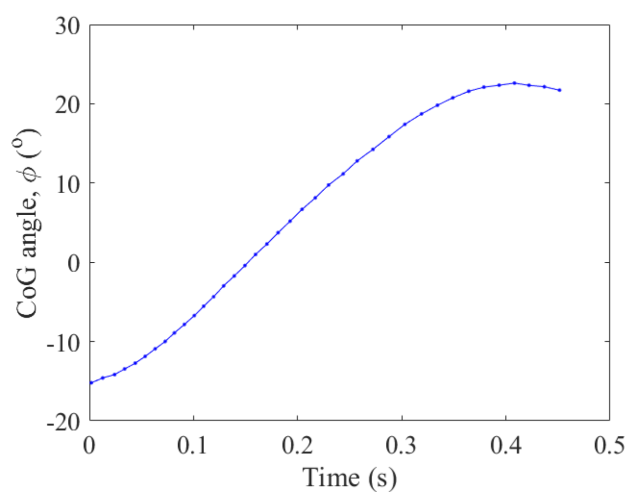

Figure 9: Center of gravity angle relative to vertical over time.

Compound Pendulum Model

For a compound pendulum, pictured in Figure 10, the governing equations for the system can be expressed as

$$I\ddot{\phi} = \tau \sin(\omega t) - mgl \sin(\phi), \tag{2.8}$$

where I is the inertia of the pendulum, ϕ is the angle to its center of mass from the vertical, and ω is the forcing frequency. By making the small angle approximation (which is generally valid for angles smaller than 30°, as seen in Figure 9), the equation can be solved explicitly for $\phi(t)$, and integrating its steady state behaviour over a step gives that

$$|W| = \left| I\phi^2(\omega_n^2 - \omega^2) \right|, \tag{2.9}$$

where W is the work required to move the pendulum and $\omega_n = \sqrt{\frac{mgl}{I}}$ is the natural frequency of the pendulum where l is the distance to the center of mass.

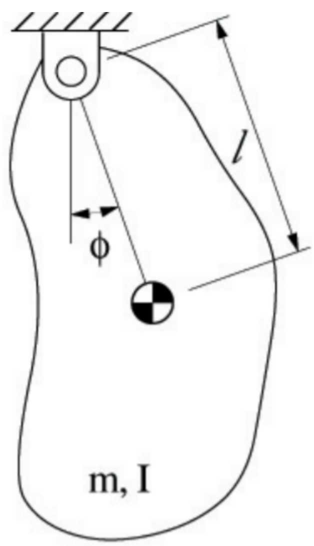

Figure 10: Compound pendulum model.

The leg can be approximated as a compound pendulum where I is the calculated average inertia, l is the average distance to the center of gravity, and ω is the frequency of oscillation of the swing leg, which was determined by taking the total change in ϕ and dividing it by the duration of swing of 0.46 seconds, from Appendix A. The resulting work done to the leg over time is shown in Figure 11.

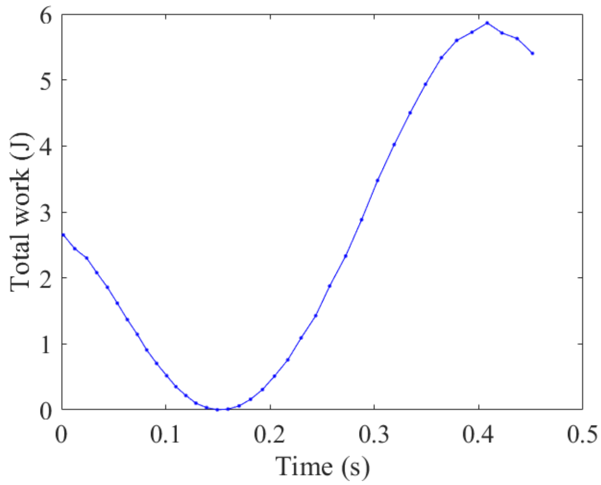

Figure 11: Work required to move the leg during the swing phase.

This resulted in an average work of **2.261 J** applied to the leg over the swing interval.

Step Length

As was done previously, the distance traveled by the end of the shank was approximated to be the step length. With the compound pendulum model, this point's position p_s can be given as

$$\boldsymbol{p_s} = \begin{bmatrix} (l_1 + l_2)\sin(\phi) \\ -(l_1 + l_2)\cos(\phi). \end{bmatrix} \tag{2.10}$$

Equation 2.6 was similarly used to determine the total distance traveled, and was found to be **0.601 m**.

Part II Discussion

Many of the results seem intuitive. Inertia seems smallest midway through swing, when the knee is bent, and greatest at push off and heel strike, when the leg is nearly fully extended.

From Figure 8, the center of gravity seems almost parabolic. The conversion from kinetic to potential back to kinetic energy seems most apparent here. While the relative change in vertical position is small, the average horizontal position of the center of gravity seems to be slightly in front of the hip. This again seems intuitive given that the data is for walking forwards. The discontinuities at heel strike are most likely due to the assumptions made about the foot angle being locked throughout swing.

The plot of the angle of the center of gravity looks somewhat like a combination of the angle plots in Figures 2 and 3. As confirmed by Figure 8, the center of gravity seems to always be in front of the hip, and by heel strike, returns closer to the hip again, similar to the behaviour of the knee joint.

With the compound pendulum model, the work done on the leg during the swing phase is in the same order of magnitude as the work calculated from the empirical data shown in Figure 6, but seem to different significantly in shape. Both peak at around 10 J midway through the swing phase, and demonstrate that less work is needed during push off and heel strike, but Figure 6 shows significant changes in the absolute value of the work done throughout. Their averages of 4.890 J and 5.409 J are similar however, and suggest that the compound pendulum model may be a reasonable approximation of the leg dynamics.

Notably, the work seems to be zero midway through swing. This is most likely when both the hip and knee are in free swing when the kinetic energy is maximum.

Lastly, the estimated step lengths from the two models are also similar, resulting in around 42 cm for the two link model and 60 cm for the compound pendulum model. Because the compound pendulum does not take into account flexion of the knee, an overshoot is expected, and neither value greatly differs from a "true" average step length of around 40 cm.

Part III

Heel Strike Work

From [3], the negative work at each step during heel strike can be estimated as

$$W^{-} = \frac{1}{2} M v_{post}^{2} \alpha^{2} (1-\rho)^{2}, \tag{2.11}$$

where W^{-} is the negative work, M is the total mass of the body, v_{post} is the velocity of the center of mass the instant after heel strike, α is the angle of the leg at heel strike with respect to the vertical, and ρ is the radius of curvature of the foot normalized by leg length.

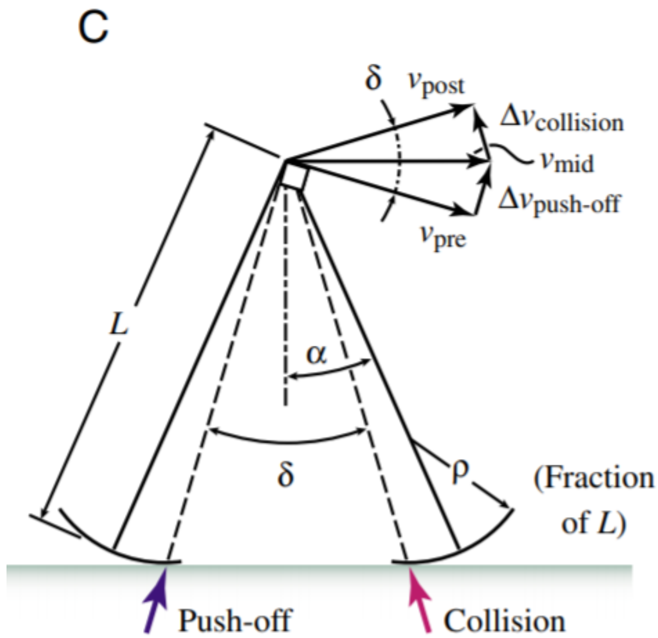

Figure 12: Rolling foot contact model of walking [3].

From the results of Part II, α was taken to be the angle of the center of gravity at the end of the swing phase, as can be seen in Figure 12. ρ was estimated to be 0.3 [4], and as v_{post} is perpendicular to the heel strike leg, it was approximated to be similar to the average horizontal velocity of 1.42 m/s. It was assumed that α is roughly half of δ in Figure 12, as the foot does not protrude significantly from the heel, and the length of the foot is small relative to the length of the leg. From this, v_{post} can be calculated from geometric relations, and can be expressed as

$$v_{post} = v_{mid} \cos(\delta/2), \tag{2.12}$$

where v_{mid} is the average horizontal velocity of 1.42 m/s. From this, v_{post} was estimated to be equal to .This resulted in a negative work of **4.065 J** at heel strike.

Part III Discussion

From Figures 6 and 11, the work done at the end of the swing phase during heel strike is roughly 2 and 5 J respectively. The averages seem roughly similar. It seems that the addition of curved feet to the model reduce the impact energy losses slightly compared to the model in Part III, but because of massless legs used in Figure 12, they are difficult to compare.

Part IV

Work Rate

Using the expressions for work rate from [5], the optimal step length can be determined. They are given as

$$\begin{aligned} WR &= \frac{s^2 f}{4\sqrt{4-s^2}}, && (2.13)\\ WR &= \frac{1}{8} f^3 s^4, && (2.14) \end{aligned}$$

where s is the total step length normalized by the length of the leg, f is the step frequency normalized by the natural frequency of the leg, and WR is the work rate.

The step frequency can be expressed as the velocity divided by the step length, $f = \frac{v}{s}$ with the appropriate normalizations. These equations can then be re-expressed as

$$\begin{aligned} WR &= \frac{sv}{4\sqrt{4-s^2}}, && (2.15)\\ WR &= \frac{1}{8} v^3 s. && (2.16) \end{aligned}$$

The optimal step length can be found by finding the minimum of the work rate functions for s, but a number of observations can be made about Equations 2.13 and 2.14 before determining their derivatives. First, it is assumed the work rate cannot be complex or be divisible by zero, and so for Equation 2.13, the domain is limited such that $s\epsilon(-2, 2)$. Plotting over this range shows that the function increases monotonically, as shown in Figure 12, and therefore no local or global minimum exists. Secondly, Equation 2.14 can be seen to be a linear function in s that likewise increases monotonically and has no minimum.

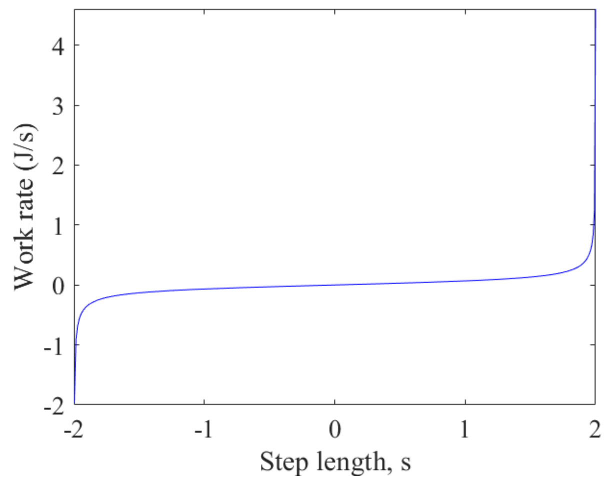

Figure 13: Work rate for normalized step length.

This implies that for the work rate expressions given, there is no "optimal" step length, and if one had to be chosen to minimize the energy used, s would trivially equal zero.

Part IV Discussion

Limitations in the model formulated by Kuo is what leads to this conclusion. To formulate Equation 2.13, the model of the leg assumed that the center of mass moved only horizontally, and to compensate, required the horizontal velocity to vary instead. Furthermore, the legs are assumed massless with instantaneous, straight transitions to double support. Clearly, this does not present a realistic model of walking when the work rate would clearly be affected by factors such as the swaying center of mass, mass of the leg, work lost to heel strike impact velocity, and so on, which are not accounted for here.

Similarly, for Equation 2.14, Kuo assumes that all the work done in push off redirects the velocity of the center of mass purely in a horizontal direction. If every step taken was entirely redirected to the desired forward direction, it is understandable that there is no optimal step length because there is no associated undesired energy expenditure.

Part V

Effect of Velocity

Lastly, the effect of changing the velocity to faster and slower speeds on the step length and frequency was investigated. Assuming that the work would always be minimized, Parts II-IV were revisited for a different average horizontal velocity.

For the analysis performed in Part II, a faster walking speed would imply a shorter duration of swing, and therefore a higher frequency of oscillation ω in Equation 2.9. This similarly implies higher average joint velocities, which would cause the angle to the center of mass of the leg in the compound pendulum model to swing out wider, resulting in longer strides. Similarly, the opposite can be expected if the average velocity was reduced - a longer forced frequency would result in narrower angles of swing, and producing smaller steps. Intuitively, taking more, longer steps produces a faster average velocity, and the opposite is true as well. However, a higher frequency entails rapid changes in the velocity of the center of mass of the swinging leg, a large drain on energy. For this model, increasing step length seems to be more optimal to minimize the work done.

For the negative work calculated in Part III, with a constant velocity assumed, a minimum can be found by differentiating Equation 2.11 with respect to time, giving

$$\frac{dW^-}{dt} = Mv_{post}^2(1-\rho)^2\alpha\dot{\alpha}. \tag{2.17}$$

From this, it is seen that minimizing the work rate depends on the rate of change of α, the swing angle of the leg. Physically, this means that increasing velocity could best be achieved by increasing step frequency, and a slower speed could be achieved with a slower step frequency as well. Because the model used in Part III does not consider the legs to have a distributed mass, but rather considers a point mass of the entire body at the hip, it is understandable that the energy losses discussed in Part II are not present.

Lastly, changes to the average velocity in Kuo's govern the step length for Equations 2.15 and 2.16. To have the minimal absolute work rate, the optimal step length would be to have a step size of near zero. At any velocity, this would result in a nearly infinite frequency. As before, because this model neglects the masses in the legs, an extremely high step frequency with small steps seems to be the solution to minimize the work done. Due to the deficiencies in the model discussed previously, an accurate balance between step length and frequency is not present.

3 Discussion and Conclusion

These approaches to analyzing walking have each emphasized different aspects of its overall motion. From empirical data, Mills's work can estimate the acting torques on the hips and knees from a two-link, planar model of the leg, allowing insights into how muscles may be activated and controlled to achieve a walking gait. It seems that the work done by the knee is actually negative overall, or at least close to zero, possibly for energy efficiency while traveling. By simplifying the model of the leg to a compound pendulum, the work done on the leg to drive it at walking frequency can be more easily calculated, peaking midway through swing and settling to relatively small values at push off and heel strike. Since this value found was the absolute value of work done, it is possible that as much of the work done at push off is recaptured with the conversion from kinetic to potential energy. Estimating step length as the distance traveled by the distal end of the shank also created fairly accurate results to what an average step length would be.

By adding curved feet to the model, a negative work of around 4 J was estimated for the heel strike. The total work done to the leg from Part II seems to be similar at roughly 5 J at heel strike. Despite the assumptions made in the model in Figure 12, the additions of curved feet seems to reduce energy losses compared to a compound pendulum model.

While no optimal step length was found for the work rate equations from Kuo, it seems that the massless leg model used implies that short step length and high frequency could minimize work most efficiently.

Overall, the analyses of walking done in this report illustrate a number of concepts about walking. The change in the center of gravity of the leg seems to be roughly sinusoidal with torques on the order of 100-200 Nm, and during swing, seems to have instances where little to no work is needed to maintain its angular velocity. This results in an extremely energy efficient mechanism where the majority of energy expended at push off seems to be translated into forward movement, with minimal losses at heel strike from rolling contacts.

References

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[5] Kuo, A. D. (2007). The six determinants of gait and the inverted pendulum analogy: A dynamic walking perspective. Human Movement Science, 26, 617-656.

Appendix A: Walking Characteristics

Spatial–temporal variables of the young and elderly subjects

Variable	Young Mean (SEM)	Elderly Mean (SEM)
Gait velocity (m/s)	1.41 (0.04)	1.44 (0.03)
Stride length (m)	1.70 (0.03)	1.68 (0.03)
Stride duration (s)	1.18 (0.02)	1.13 (0.03)
Support duration (s)	0.72 (0.02)	0.70 (0.02)
Swing duration (s)	0.46 (0.01)	0.43 (0.01)*
Double support duration (s)	0.27 (0.02)	0.28 (0.01)
Support period (% stride)	61.3 (0.7)	62.3 (0.6)
Swing period (% stride)	38.7 (0.7)	37.7 (0.6)
Double support (% stride)	22.5 (1.3)	24.6 (1.2)

* Significantly different from young (MANOVA), $p < 0.05$.

Figure 14: Experimental data from walking gait tests, obtained from [1].