Generalized Tikhonov Regularization

& Error Analysis

Report

Mathematics for Imaging and Signal Processing

A.A. 2025/2026

1. Visual Comparison of L2, H1, H2 Reconstructions

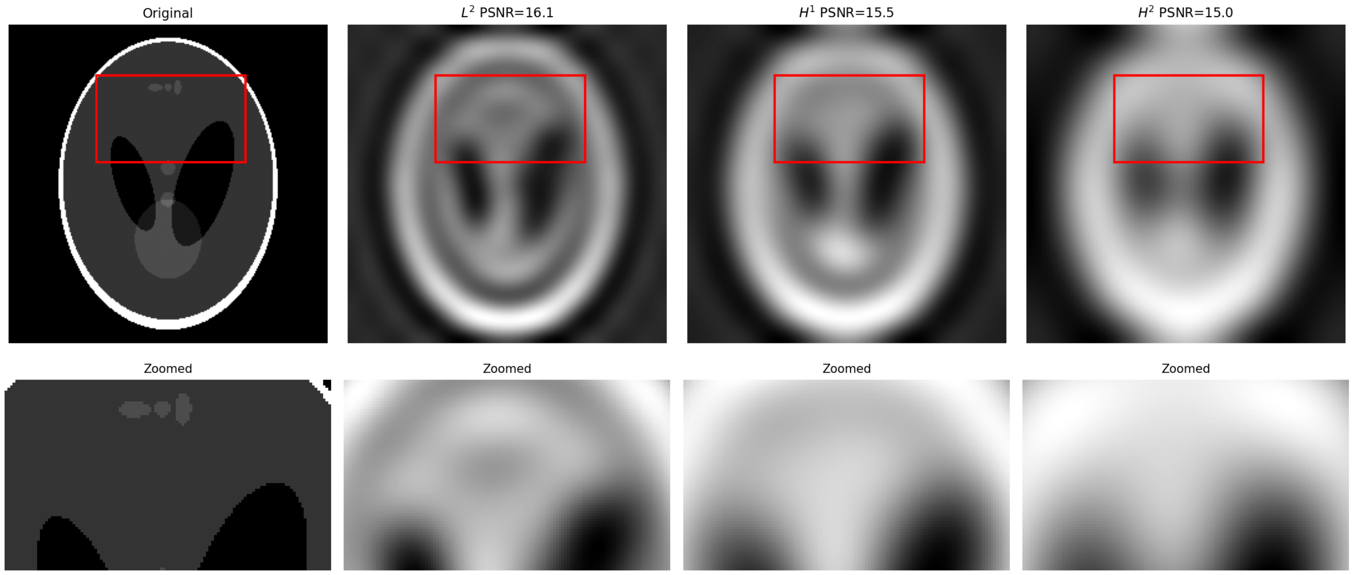
The Shepp-Logan phantom (256x256) was chosen as the test image because its sharp elliptical boundaries at various angles make it ideal for evaluating edge preservation under different regularization penalties. Three blur models were implemented: Gaussian, Linear Motion (with phase factor), and Out-of-Focus (Bessel J1). Observations were generated at SNR = 40 dB and 20 dB.

1.1 Gaussian Blur -High SNR (40 dB)

At 40 dB the noise is barely visible. The differences between penalties are subtle but clear in the zoomed crops (red boxes mark the zoom region):

- **L2:** treats all frequencies with the same weight. Edges are uniformly smoothed.
- **H1:** penalizes higher frequencies proportionally to |w|^2, so low/mid frequencies (where edges live) pass through with less damping. Edges are noticeably sharper.
- **H2:** the |w|^4 penalty crushes high frequencies very aggressively. The image is very smooth but fine detail is lost.

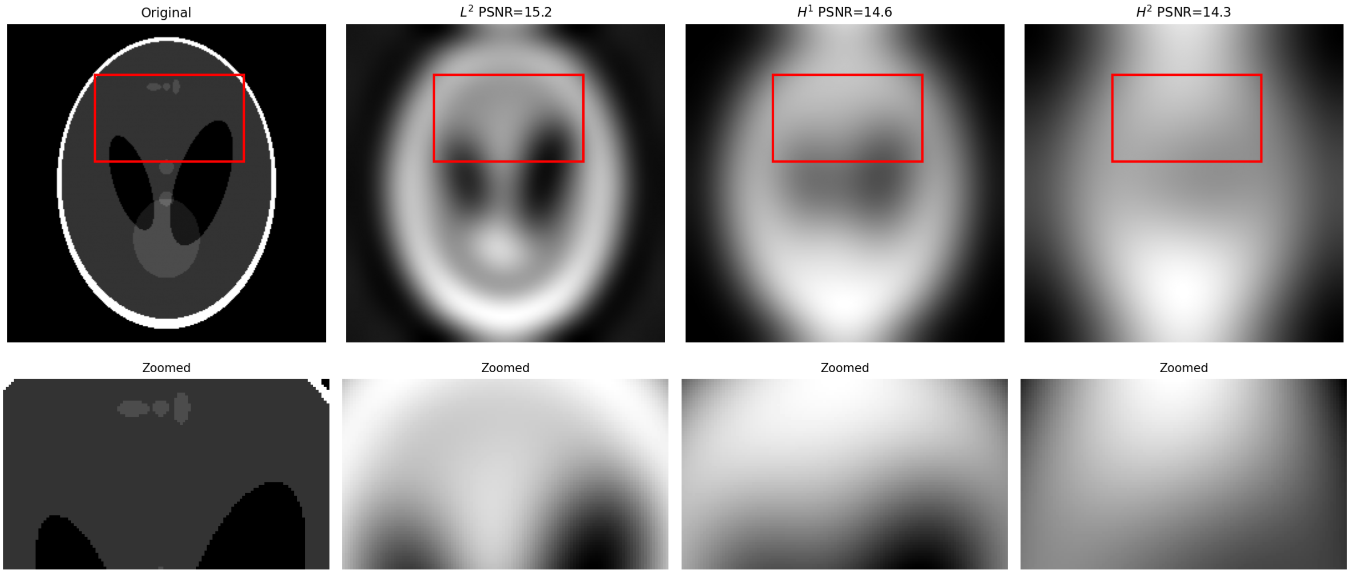
Figure 1: L^2, H^1, H^2 **Reconstructions -SNR = 40 dB (Gaussian Blur)**



1.2 Gaussian Blur -Low SNR (20 dB)

At 20 dB the differences are much more dramatic. L2 lets a lot of granular noise through (visible in flat regions). H1 still keeps edges reasonably sharp while damping most noise. H2 produces the cleanest image but looks over-smoothed -all fine structure is gone.

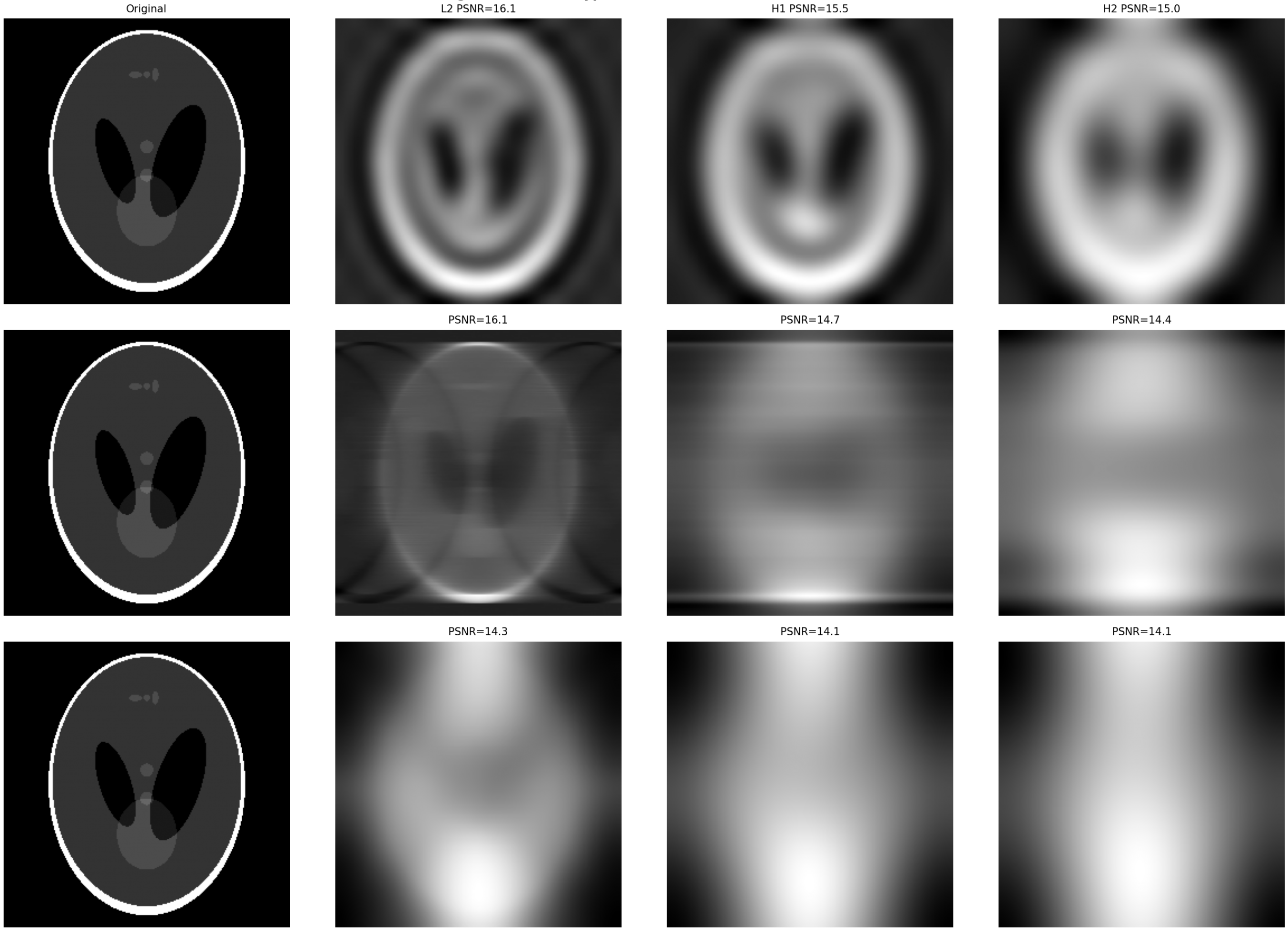
Figure 2: L^2, H^1, H^2 Reconstructions -SNR = 20 dB (Gaussian Blur)



1.3 Across All Blur Types

The same pattern holds across Gaussian, Motion, and Out-of-Focus blurs. H1 consistently gives the best balance between sharpness and noise removal. The out-of-focus case is the hardest to reconstruct because its transfer function has actual zeros (ring-shaped in the Fourier plane), meaning some frequency content is irrecoverably lost regardless of the penalty chosen.

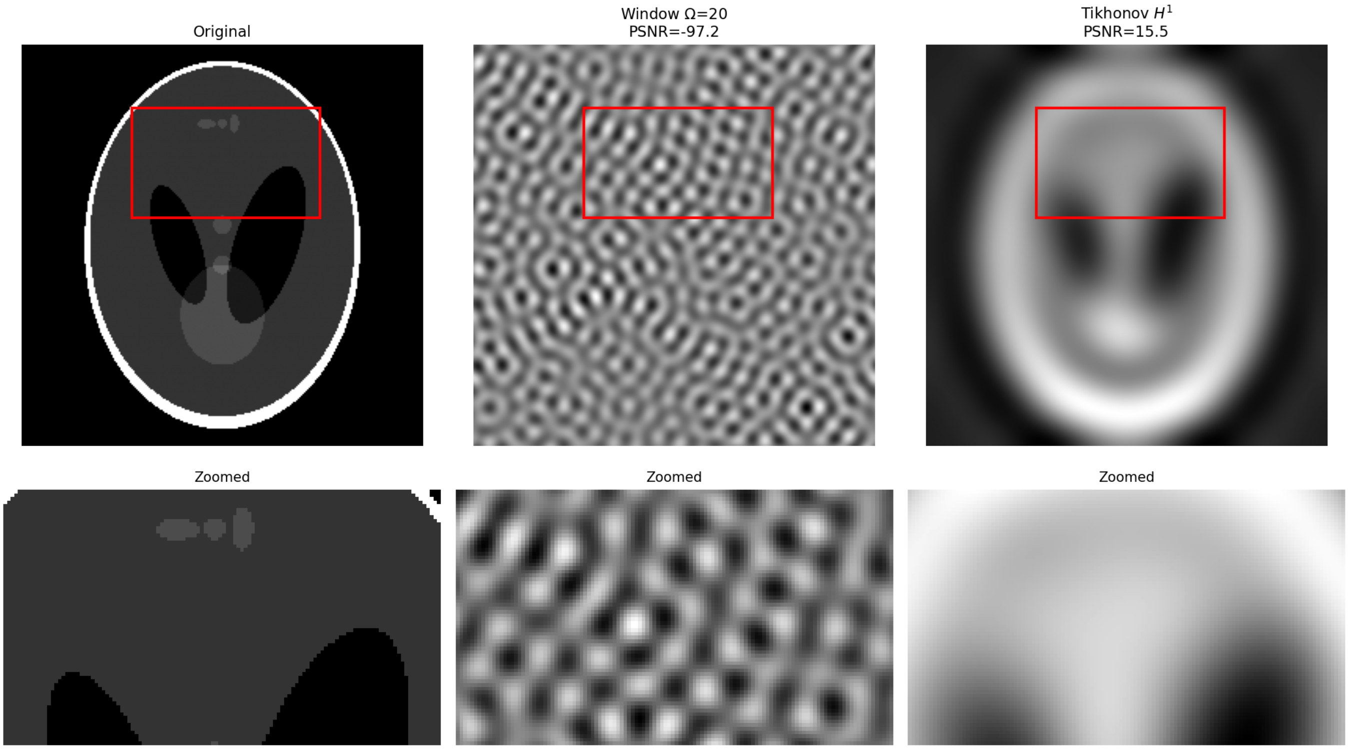
Figure 3: All Blur Types x All Penalties (SNR = 40 dB)



1.4 Spectral Windowing vs Tikhonov

Comparing the hard rectangular cutoff against Tikhonov H1, the windowed reconstruction shows clear Gibbs ringing (oscillations) near every sharp edge. This is because a rectangular window in frequency is equivalent to convolving with a sinc in space, and the sinc's sidelobes create those ripples. Tikhonov avoids this entirely through its smooth roll-off.

Figure 4: Spectral Windowing vs Tikhonov H^1 (SNR = 40 dB)



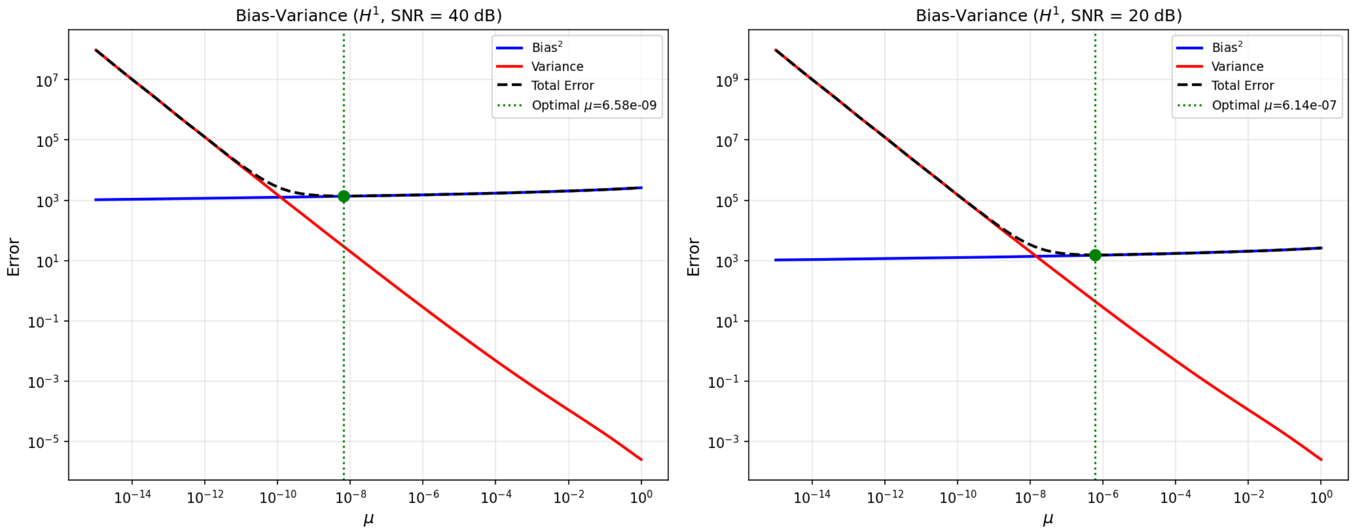
2. The Bias-Variance Trade-off

The total reconstruction error decomposes into two competing terms:

Total Error^2(mu) = ||(R_mu A - I) f_true||^2 + ||R_mu w||^2
[Bias (detail loss)] [Variance (noise)]

Both terms were computed in the frequency domain via Parseval's identity for the H1 penalty, sweeping mu from 10^-6 to 10^0.

Figure 5: Bias-Variance Trade-off



Key observations from the bias-variance plots:

- Bias increases monotonically with mu. More regularization means the reconstruction filter deviates further from the ideal inverse, and we lose more image content.
- Variance decreases monotonically with mu. Stronger regularization damps the filter, so less noise passes through.
- The total error is U-shaped on a log-log scale. Its minimum (the optimal mu*) sits roughly where the bias and variance curves intersect.
- Comparing 40 dB vs 20 dB: the bias curve is identical (it only depends on the true image and blur kernel, not on noise). The variance curve shifts upward proportionally to sigma^2. This pushes the optimal mu* to a larger value -noisier data needs more regularization.

2.1 Comparing L2, H1, H2 Total Error

Plotting the total error curves for all three penalties shows that H1 achieves the lowest minimum total error for this test image. This makes sense: the Shepp-Logan phantom is piecewise constant with sharp edges, and H1 (gradient penalty) is designed to preserve exactly that kind of structure.

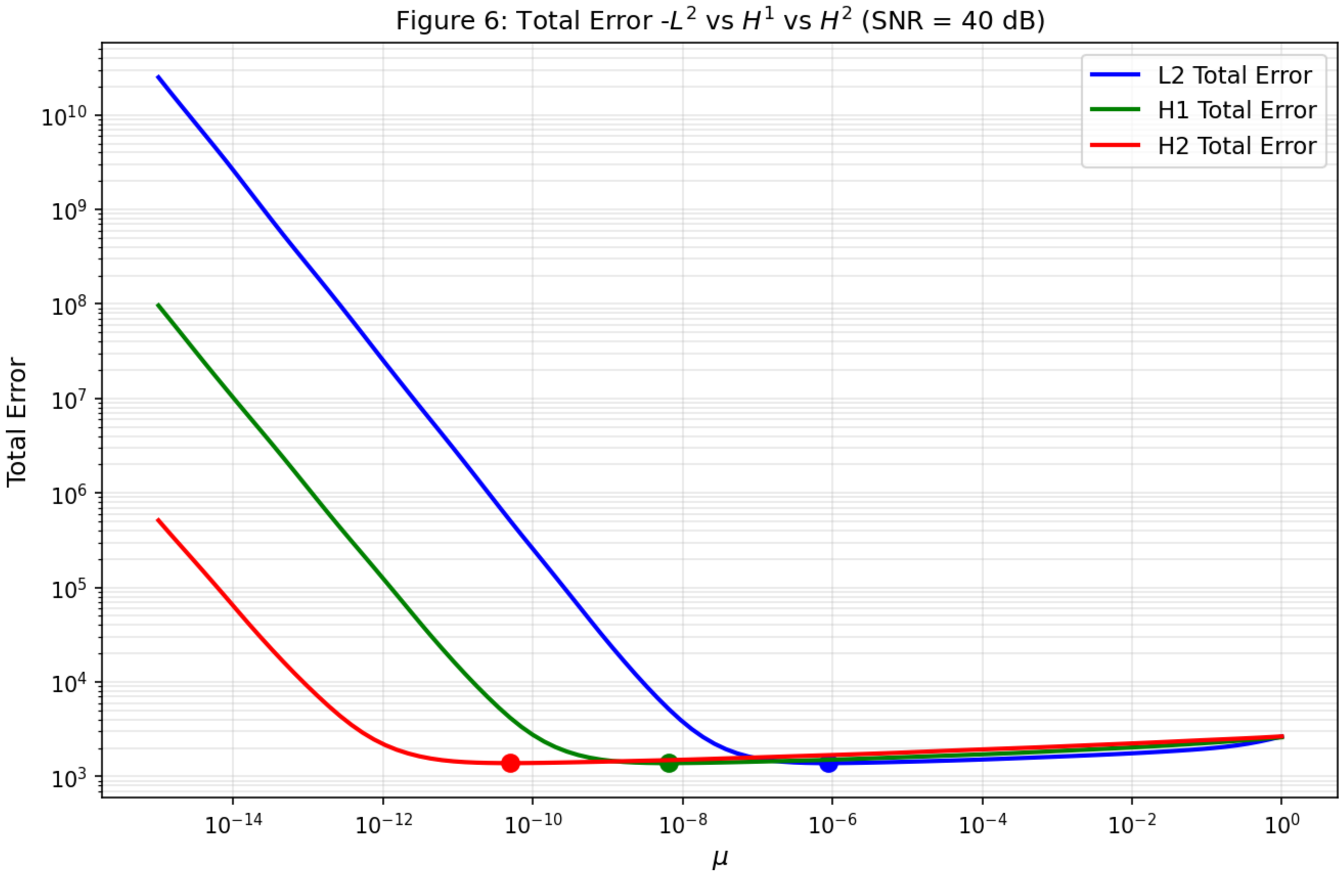


Figure 6: Total Error -L^2 vs H^1 vs H^2 (SNR = 40 dB)
L2 Total Error
H1 Total Error
H2 Total Error
Total Error
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{-14}
10^{-12}
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
10^{0}
μ

3. Effect of Smoothness Order on Edge Preservation and Noise Suppression

The key difference between H1 and H2 comes down to how fast the penalty grows with frequency:

H1 Penalty: |P|^2 = |w|^2

- DC (w = 0) is not penalized at all -the mean brightness is preserved exactly.
- Low frequencies get a small penalty -the large-scale structure comes through.
- High frequencies get a moderate penalty -noise is damped, but edges (which need mid-to-high frequencies) are still partially retained.
- Result: good compromise -edges are visible and noise is controlled.

H2 Penalty: |P|^2 = |w|^4

- Low frequencies are barely penalized (similar to H1).
- Mid-to-high frequencies are crushed much more heavily than in H1.
- Edge detail, which lives in the mid-to-high frequency range, gets suppressed along with the noise.
- Result: very smooth -noise is almost gone, but the image looks blurry and fine detail is lost.

In short: increasing the smoothness order from H1 to H2 improves noise suppression but hurts edge preservation. For images with sharp boundaries (like medical or scientific images), H1 is usually the better choice. H2 might be preferable only when noise is very strong and we are willing to sacrifice detail for a cleaner result.

The spectral windowing comparison reinforces these findings: the hard rectangular cutoff avoids the smooth penalty altogether and just chops frequencies at a threshold. This is conceptually simpler but produces Gibbs ringing near edges. Tikhonov with any of the three penalties avoids this because it uses a smooth roll-off instead of a sharp cutoff.

Summary Table

Penalty	Edge Preservation	Noise Suppression	Best Use Case
L2	Worst (uniform blur)	Moderate	Simple regularization baseline
H1	Good (edges sharp)	Good	Piecewise-constant images, edges
H2	Over-smoothed	Best	Very noisy data, smooth targets
Window	Gibbs ringing	Depends on cutoff	Quick & dirty, no mu tuning