June 17, 2025

Binary Orbit Model Summary

Selim Kalici

Yale University

We present a short summary of & context for the binary orbit model in this github repo.

1 Binary Stellar Orbits

$$\overbrace{v_r}^{\text{Barycentric radial velocity}} = \underbrace{\frac{q}{\sqrt{1-e^2}}\left[\frac{2\pi G m_1}{P(1+q)^2}\right]^{1/3}}_{\text{Orbital dynamics}} \overbrace{\sin(i)\Big[\cos(\theta+\omega)+e\cos(\omega)\Big]}^{\text{Orientation-based term}} \tag{1}$$

The orbital radial velocity, or line-of-sight velocity, equation is shown above in Equation 1. There are many derivations of the radial velocity equation, for reference see []. The parameters entering this equation are as follows:

v_r : Orbital radial velocity.

q : The mass ratio, $q = m_2/m_1$.

e : eccentricity of the orbit, $e = \sqrt{1 - b^2/a^2}$.

P : Period of the orbit.

i : Inclination of the orbit with respect to the plane of the sky.

θ : True anomaly.

ω : Argument of periastron or periapsis.

Implicitly, the following parameters are required as well.

E : Eccentric anomaly.

M : Mean anomaly.

The true anomaly, θ, is calculated from the eccentric anomaly, E:

$$\theta = E + 2\arctan(\beta\sin(E)/(1-\beta\cos(E)),$$

where,

$$\beta = e/\left(1+\sqrt{1-e^2}\right).$$

In turn, the eccentric anomaly is found via Kepler's equation for the mean anomaly, M,

$$M = E - e\sin(E).$$

Kepler's equation is famously *not* solvable analytically, so we solve for the eccentric anomaly numerically, via fixed point iteration. Then, propagating the value for the eccentric anomaly through, we may calculate the true anomaly. The code snippet used to calculated the true anomaly in the

```python
def get_true_anomaly(M: jax.Array, e: jax.Array, max_iter=100) ->
    jax.Array:
    @jax.jit
    def body_fun(carry):
        E_prev, _ = carry
        E_next = M + e * jnp.sin(E_prev)
        return E_next, E_next

    E_init = jnp.ones_like(M) * jnp.pi
    E_final, _ = lax.scan(
        lambda carry, _: (body_fun(carry), None),
        (E_init, E_init),
        xs=None,
        length=max_iter
    )
    E = E_final[0]

    beta = e / (1 + jnp.sqrt(1 - e**2))
    true_anomaly = E + 2 * jnp.arctan(
        beta * jnp.sin(E) / (1 - beta * jnp.cos(E))
    )
    return true_anomaly
```

Figure 1: Batch-friendly jit-compiled fixed point iteration for solving Kepler's equation, then calculating the true anomaly, θ.

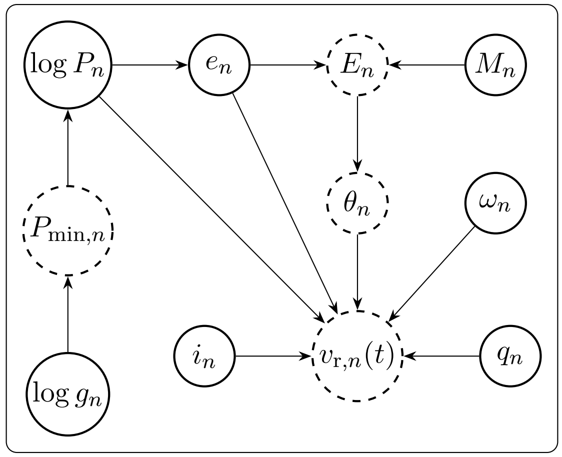

Figure 2: PGM of the statistical model. Solid nodes correspond to random variables and dashed nodes correspond to deterministically produced variables.

binary orbit model is shown above in Figure 1. We also note that the true observed (i.e. **not** in the center-of-mass frame) radial velocity is offset by a systemic velocity from the orbital motion of the binary around its host galaxy, so the true observed radial velocity, $\tilde{v}_r$ is given by,

$$\tilde{v}_r = v_r + \gamma,$$

where γ is the aforementioned systemic velocity. Now, to evolve the binary orbit forward by some timestep δt, we must evolve the true anomaly forward in time. Again, we may not do this analytically, so we instead evolve the mean anomaly, M, calculate the eccentric anomaly, E, then again find the true anomaly θ. Evolving M is quite simple:

$$M(t + \delta t) = M(t) + \frac{2\pi \delta t}{P}.$$

Then, we again solve Kepler's equation via fixed point iteration to find E, and find θ. Succinctly, if we define the radial velocity semi-amplitude as

$$K \equiv \frac{q \sin(i)}{\sqrt{1 - e^2}} \left[\frac{2\pi G m_1}{P(1+q)^2} \right]^{1/3},$$

then the radial velocity equation that generates the radial velocity curve is

$$v_r(t) = K[\cos(\theta(t) + \omega) + e\cos(\omega)].$$

So we must only evolve the true anomaly over time to generate the radial velocity curve.

2 Statistical Model

The values that we may observe are the mass ratio, q, eccentricity, e, and period, P. The probability density function for a truncated normal distribution is given by,[1]

$$\psi(\mu, \sigma, a, b; x) \equiv \frac{\phi(\mu, \sigma; x)}{\Phi(\mu, \sigma, b) - \Phi(\mu, \sigma, a)} \ \forall x \in [a, b],$$

[1] see e.g. https://people.sc.fsu.edu/~jburkardt/presentations/truncated_normal.pdf

where Φ is the error function, and ϕ is the standard normal distribution. Here it is important to clarify that μ and σ do not represent the mean and standard deviation of the truncated normal distribution in general, but the mean and standard deviation of the normal distribution in the numerator. Furthermore, we refer to the uniform distribution as $\mathcal{U}(a, b)$. There are also several studies that attempt to survey parameter distributions for binary stars. For reference, see Duquennoy and Mayor 1991; Fischer and Marcy 1992; Kroupa et al. 1993; Raghavan et al. 2010; Marks and Kroupa 2011; Moe and Di Stefano 2017. In this work we choose to use the distributions of Duquennoy and Mayor 1991, as they are still commonly used throughout the literature. Although, a more robust statistical model would include the distributions of Moe and Di Stefano 2017. The period is distributed as a truncated log-normal,

$$\log_{10} P \sim \begin{cases} \psi(4.8, 2.3, \log P_{\min}, \log P_{\max}), & P \in [P_{\min}, P_{\max}]; \\ 0, & \text{otherwise.} \end{cases}$$

We follow the implementation of the Phd thesis of Meghin Spencer (Spencer 2017) and define the minimum period via the primary stars' surface gravity, mass, and the mass ratio of the binary system,

$$\log_{10} P_{\min} = \log_{10} \left[\frac{2\pi}{86400} \sqrt{\frac{\left(\frac{G\, m_1}{10^{\log g}/10^5}\right)^{3/2}}{G\, m_1\, (1+q)}} \right].$$

This is found via assuming that the minimum period corresponds to when the semi-major axis of the system is given by the radius of the primary star. Of course, this approximation is invalid for the binaries with the shortest periods, though we assume that these have a negligible effect. To determine a maximum period, Spencer 2017 considers the number density of a typical dwarf galaxy, and calculates a corresponding maximum semi-major axis. Then, following the calculation above, reports a value of $\log P_{\max} = 6.51$. The mass ratio distribution is also log-normal, with,

$$q \sim \begin{cases} \psi(0.23, 0.42, 0.1, 1.0); & 0.1 \leq q \leq 1.0; \\ 0, & \text{otherwise.} \end{cases}$$

For binary orbits with period less than ~ 11 days, the orbit is assumed to be circularized over a very short timescale, leading to $e = 0$. For orbits with periods below 1000 days, but above 11 days, the distribution is roughly log-normal up to some maximum eccentricity,

$$e_{\max} = 1 - \left(\frac{1}{2} P\right)^{-2/3}.$$

Above 1000 days, the eccentricity distribution is a linear power law. Together, this results in:

$$e \sim \begin{cases} 0, & \log_{10} P \leq 1.08; \\ \psi(0.25, 0.12, 0, e_{\max}) & 1.08 \leq \log_{10} P \leq 3; \\ e \times \mathbb{1}_{[0,1]}, & \log_{10} P \geq 3. \end{cases}$$

The mean anomaly and argument of periapsis are both assumed to be distributed randomly and uniformly:

$$M \sim \mathcal{U}(0, 2\pi);$$
$$\omega \sim \mathcal{U}(0, 2\pi).$$

And finally, the distribution of inclination is well known, and can be found from spherical-polar coordinates,

$$i \sim \sin(i).$$

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