The Tolman-Oppenheimer-Volkoff Equation

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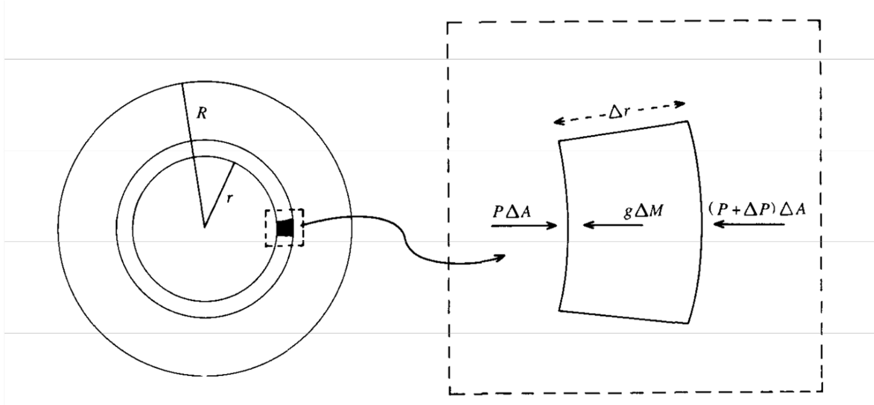

Figure 1: A cross section of a spherical body of fluid

I. Introduction

The equation of hydrostatic equilibrium describes the balance of forces that keep stars stable. The inward forces due to gravity must be balanced by the outward pressure generated mostly in the cores of these stars. Stars remain in hydrostatic equilibrium for the vast majority of their lifetime one the main sequence; and when this hydrostatic equilibrium is disturbed the star continues its evolution off the main sequence for many possible reasons.

In this work we discuss the general relativistic equation of hydrostatic equilibrium, the Tolman-Oppenheimer-Volkoff equation. First, we derive the nonrelativistic equation of hydrostatic equilibrium in Section II. Then in Sections III, IV, V, and VI we discuss and formulate the different components involved in the Einstein field equations. Then in Section VII we solve the field equations and derive the equation of hydrostatic equilibrium.

II. Hydrostatic Equilibrium

To derive the equation of hydrostatic equilibrium, we consider the balance of forces in a spherical body. We do this by using Newtons 2nd law, and letting the acceleration be equal to zero. Then we may see that

$$
\begin{aligned}
&[P(r+\delta r) - P(r)]\,\delta A + g\delta M \\
&= P(r+\delta r)\delta A - P(r)\delta A + g\delta M \\
&= P(r)\delta A - P(r)\delta A + \frac{dP}{dr}\delta r\delta A + g\delta M
\end{aligned}
$$

Now we use the definition of density

$$\delta M = \rho\delta r\delta A \tag{1}$$

to see that

$$\frac{GM}{r^2} + \frac{1}{\rho}\frac{dP}{dr} = 0 \tag{2}$$

Which leads us to our result,

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho \tag{3}$$

III. The Field Equations

We seek to determined the equation of hydrostatic equilibrium for a star in a general relativistic sense. To do this we must solve Einstein's field equations for a star.

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{4}$$

Here $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor. Expanded, this equation would look like,

$$\begin{pmatrix} G_{00} & G_{01} & G_{02} & G_{03} \\ G_{10} & G_{11} & G_{12} & G_{13} \\ G_{20} & G_{21} & G_{22} & G_{23} \\ G_{30} & G_{31} & G_{32} & G_{33} \end{pmatrix} = \frac{8\pi G}{c^4}\begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} \tag{5}$$

Where the full Einstein field equations are a set of 16 coupled partial differential equations. We first must choose a metric to describe the Einstein tensor on the left, then we make certain assumptions about the strucutre of our star to define a suitable stress energy tensor on the right. From there, we solve the set of partial differential equations to relate the inward force of gravity to the outward pressure generated in the star.

IV. The Metric Tensor

To define the Einstein tensor we must first choose our metric. The canonical metric tensor for a static spherically symmetric coordinate system has components

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -e^{2\Phi}\,dt^{2} + e^{2\Lambda}\,dr^{2} + r^{2}\,d\theta^{2} + r^{2}\sin^{2}\theta\,d\phi^{2} \tag{6}$$

We may rewrite this as

$$g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} = -e^{2\Phi}\,dt^{2} + e^{2\Lambda}\,dr^{2} + r^{2}\,d\theta^{2} + r^{2}\,d\Omega^{2} \tag{7}$$

where $d\Omega^2$ represents the typical spherical metric.

V. The Einstein Tensor

The Einstein tensor describes the curvature of a psuedo-Riemmanian manifold and is defined as the trace-reversed Ricci Tensor

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{8}$$

where $g_{\mu\nu}$ is the chosen metric, $R_{\mu\nu}$ is the Ricci tensor, and $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar. We may expand the Einstein Tensor as follows,

$$\begin{aligned}
G_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \\
&= R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g^{\gamma\zeta}R_{\gamma\zeta} \\
&= \left(\delta^{\gamma}_{\alpha}\delta^{\zeta}_{\beta} - \frac{1}{2}g_{\alpha\beta}g^{\gamma\zeta}\right)R_{\gamma\zeta} \\
&= \left(\delta^{\gamma}_{\alpha}\delta^{\zeta}_{\beta} - \frac{1}{2}g_{\alpha\beta}g^{\gamma\zeta}\right)\left({\Gamma^{\epsilon}}_{\gamma\zeta,\epsilon} - {\Gamma^{\epsilon}}_{\gamma\epsilon,\zeta} + {\Gamma^{\epsilon}}_{\epsilon\sigma}{\Gamma^{\sigma}}_{\gamma\zeta} - {\Gamma^{\epsilon}}_{\zeta\sigma}{\Gamma^{\sigma}}_{\epsilon\gamma}\right)
\end{aligned}$$

Where $\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols which are defined via

$${\Gamma^{\alpha}}_{\beta\gamma} = \frac{1}{2}g^{\alpha\epsilon}\left(g_{\beta\epsilon,\gamma} + g_{\gamma\epsilon,\beta} - g_{\beta\gamma,\epsilon}\right) \tag{9}$$

So, the Einstein Tensor is defined entirely via the metric. Then directly after defining a metric, such as Equation 6, we may directly compute the Einstein tensor. One result of this is that since our metric is diagonal, our Einstein tensor will also be diagonal. The G_{00} term of the Einstein tensor is then found as

$$G_{00} = \frac{1}{r^2} e^{2\Phi} \frac{d}{dr} \left[r(1 - e^{-2\Lambda}) \right] \tag{10}$$

by using Equation 9 with our chosen metric in combination with the definition for the Einstein tensor chosen above. We may similarly find the G_{11} term of the Einstein tensor to be

$$G_{11} = -\frac{1}{r^2} e^{2\Lambda} \left(1 - e^{-2\Lambda}\right) + \frac{2}{r}\Phi' \tag{11}$$

VI. The Stress-Energy Tensor

The stress-energy tensor characterizes the energy/matter distribution and behaviour for our star. We will assume our star is a static, perfect fluid. For a perfect fluid, the stress-energy tensor has the form

$$T_{\mu\nu} = (\rho + P)\, U_\mu U_\nu + P g_{\mu\nu} \tag{12}$$

Since we're in a static fluid, $U^i = 0$ for all spacial directions and $U_0 = -ce^{\Phi}$, $U^0 = ce^{-\Phi}$ is implied by the normalization condition that $\langle U, U \rangle = U^0 U_0 = -c^2$. Then, for the metric indicated in Equation 6, it is straightforward to see that

$$T = \begin{pmatrix} \rho c^2 e^{2\Phi} & 0 & 0 & 0 \\ 0 & P e^{2\Lambda} & 0 & 0 \\ 0 & 0 & r^2 P & 0 \\ 0 & 0 & 0 & \sin^2(\theta) r^2 P \end{pmatrix} \tag{13}$$

This describes the energy of a static, perfect fluid.

VII. Solving the Field Equations

Now that we have defined both the components of the Einstein tensor and the stress-energy tensor, we have found a set of 4 (reduced from 16) coupled differential equations. For an additional constraint we look to the conservation of energy

$$T^{\mu\nu}_{;\nu} = \nabla_\nu T^{\mu\nu} = 0 \tag{14}$$

Since we assumed our fluid to be spherically symmetric and static it is justified to presume that $\partial_\theta T^{\mu\nu} = \partial_\phi T^{\mu\nu} = \partial_t T^{\mu\nu} = 0$. But, the distribution of energy was not assumed to be even, so the radial derivative is of most interest to us, and taking it yields the expression

$$\frac{dP}{dr} = -(\rho + P)\frac{d\Phi}{dr} \tag{15}$$

We will work from this equation to find the equation of Hydrostatic Equillibrium. Notice that we do not have to implement any further constraints to our star, since we already assumed it was in hydrostatic equilibrium in our derivations of the stress-energy and Einstein tensor. Equation 15 is simply an expression for the radial derivative of pressure in a static, spherically symmetric perfect fluid. Now, we must find an expression for Φ' to yield a solution. We first consider the $(0,0)$ term of the field equations,

$$\frac{8\pi G}{c^4}\rho e^{2\Phi} = \frac{1}{r^2}e^{2\Phi}\frac{d}{dr}\left[r(1-e^{-2\Lambda})\right] \tag{16}$$

Integrating both sides with respect to r yields,

$$e^{-2\Lambda} = 1 - \frac{2Gm}{rc^2} \tag{17}$$

This will be an auxiliary equation in solving the (r,r) term of the field equations for Φ'.

$$\frac{8\pi G}{c^4}Pe^{2\Lambda} = -\frac{1}{r^2}e^{2\Lambda}\left(1 - e^{-2\Lambda}\right) + \frac{2}{r}\Phi' \tag{18}$$

Dividing $e^{2\Lambda}$ from both sides and solving for Φ' we find that

$$\Phi' = \left(\frac{4\pi G}{c^4} Pr + \frac{1}{2r}(1 - e^{-2\Lambda})\right) e^{2\Lambda} \tag{19}$$

Then, substituting this into Equation 15 and using our identity for $e^{2\Lambda}$ in Equation 17 we find:

$$\frac{dP}{dr} = -\frac{G}{r^2}\left(\rho + \frac{P}{c^2}\right)\left(m + 4\pi r^3 \frac{P}{c^2}\right)\left(1 - \frac{2Gm}{c^2 r}\right)^{-1} \tag{20}$$

From the preceding derivations, we have the metric for a static, spherically symmetric, perfect fluid given by

$$ds^2 = e^{\nu} c^2\, dt^2 - \left(1 - \frac{2Gm}{rc^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{21}$$

VIII. Conclusion

We have provided a possible derivation for the Tolmnan-Oppenheimer-Volkoff equation, a relativistic solution for one of the most fundamental equations of state in stars. We did this by first assuming the form of the metric given our boundary conditions, then finding the components of the Einstein and stress-energy tensors. Finally, we solved the field equations by considering the (0,0) and (1,1) terms of the field equations. From here we found Equation 20, the general relativistic equation of hydrostatic equilibrium!

References

[Schutz] Shutz, Bernard (2009). A First Course in General Relativity (2nd ed.). Cambridge: Cambridge University Press *Cambridge, Massachusetts*