

HATCHER SOLUTIONS

MICHAŁ MĄDRALA, RUBA RUDZIK, AND KRZYSZTOF SZYMAŃSKI

ABSTRACT. We present the solutions to the excercises found in Allen Hatcher's „Algebraic Topology” [Hat01] with our own perspective and commentary. We also present proofs of selected theorems (with what we hope is a more detailed explanation).



Dedicated to Jan Dymara, Świątosław Gal & Jacek Świątkowski

1. INTRODUCTION

Blabla że co w ogóle robimy...

We denote a k -sphere by S^k , k -disk by D^k and a $[0, 1]$ segment by \mathcal{I} . By \simeq we mean homotopy or, when placed between spaces, homotopy equivalence when placed between spaces. We denote homeomorphisms by \approx .

2. THE FUNDAMENTAL GROUP

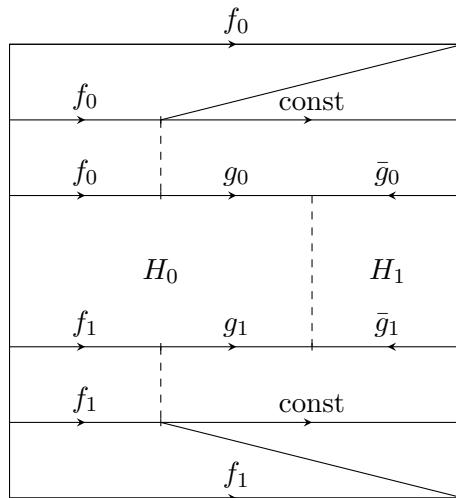
2.1. Basic Constructions. This section is about basic constructions.

Należały poprawić numerację problemów, żeby była tak jak w hatcherze, no i też żeby sekcje w Additional Topics nie były 2.4 itp tylko 2.A, 2.B itd

Problem 2.1 (1.1.1). *Show that composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.*

Proof. The homotopy assumptions imply that g_i have the same endpoints, and so do the $f_i g_i$, which means the f_i have the same endpoints as well.

Since $g_0 \simeq g_1$, it is clear that $\bar{g}_0 \simeq \bar{g}_1$. We proceed as shown on the homotopy diagram:



Algebraically, this corresponds to the computations

$$f_0 \simeq f_0 \text{ const}_{x_0} \simeq f_0(g_0 \bar{g}_0) \simeq (f_0 g_0) \bar{g}_0 \simeq (f_1 g_1) \bar{g}_1 \simeq f_1(g_1 \bar{g}_1) \simeq f_1 \text{ const}_{x_0} \simeq f_1,$$

where x_0 denotes the endpoint of the paths f_i . \square

Problem 2.2 (1.1.2). *Show that the change-of-basepoint homomorphism β_h depends only on the homotopy class of h .*

Proof. It suffices to show that for every two homotopic loops h and h' we have $\beta_h = \beta_{h'}$. Let H denote the homotopy $h \simeq_H h'$. Take any loop g . We have that $\beta_h(g) = hg\bar{h} \simeq h'g\bar{h}' = \beta_{h'}(g)$. obrazek \square

Problem 2.3 (1.1.3). *For a path-connected space X , show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphism β_h depend only on the endpoints of the path h .*

Proof. (\Rightarrow) Consider any two paths h, h' from points $y \in X$ to $x \in X$. We want to show that $\beta_h = \beta_{h'}$. Consider any loop $g \in \pi_1(X, x)$ and a loop $\bar{h}h'$ which also is in $\pi_1(X, x)$. Since $\pi_1(X)$ is abelian then $\pi_1(X, x)$ is also abelian. And so it follows that:

$$g \cdot \bar{h}h' \simeq \bar{h}h' \cdot g$$

Now multiply it by h on left hand side, and by \bar{h}' on right hand side to get:

$$hg\bar{h} \simeq h'g\bar{h}'$$

That is exactly equivalent to $\beta_h = \beta_{h'}$.

(\Leftarrow) Suppose that for any two paths from x to y they induce the same maps $\beta_h, \beta_{h'}$. In particular given any two loops g, h in $\pi_1(X, x_0)$ we have $\beta_g = \beta_h$. And so it follows that:

$$hg\bar{h} = \beta_h(g) = \beta_g(g) = g$$

Thus $gh = hg$ which means that $\pi_1(X, x)$ is abelian, and so of course $\pi_1(X)$ is abelian too. \square

Problem 2.4 (1.1.4). *A subspace $X \subseteq \mathbb{R}^n$ is said to be star-shaped if there is a point $x_0 \in X$ such that for each $x \in X$ the line segment from x_0 to x lies in X . Show that if a subspace $X \subseteq \mathbb{R}^n$ is locally star-shaped, in the sense that every point of X has a star-shaped neighbourhood in X , then every path in X is a piecewise linear path, that is, a path consisting of a finite number of straight lines traversed at constant speed. Show that this applies in particular when X is open or when X is a union of finitely many closed convex subsets.*

Proof. In the case that X is star-shaped as a whole, it is contractible, so the thesis definitely holds. Denote the path in question by γ . In general, we can use the compactness of the image of γ to cover it with finitely many star-shaped sets. However, in each of these we cannot just contract the curve, as we must keep the endpoints intact.

Instead we show the following: if X is star-shaped with *star-point* x_0 , we can deform any path γ in X to a linear path going from $\gamma(0)$ to x_0 and then from x_0 to $\gamma(1)$. Without loss of generality $x_0 = 0$. The homotopy given by **Obrazek**

$$H_1(s, t) = [(1-t) + t|1-2s|] \gamma(s)$$

achieves this partially, as it deforms $\gamma(s)$ to $|1-2s|\gamma(s)$, which is equal to $\gamma(0), x_0, \gamma(1)$ at $s = 0, \frac{1}{2}, 1$. Now we turn to making the segments linear.

If δ is the first half of γ , the homotopy H_1 deforms it to the path $(1-s)\delta(s)$. We will deform it to $s\delta(0)$, which is certainly constant-speed. This is achieved through the homotopy

$$H_2(s, t) = (1-s)\delta((1-t)s).$$

Finally, applying the same procedure to the second half of γ , we will have indeed deformed γ to a piecewise-linear path consisting of two segments. \square

Open sets. In an open set, every point has a neighbourhood which is a Euclidean ball, so the space X is locally star-shaped. \square

Finite unions of convex sets. Call a subspace $X \subseteq \mathbb{R}^n$ *strictly locally star-shaped* iff every $x \in X$ has a star-shaped neighbourhood *for which it is the star-point*. Every convex set satisfies this property. We will show that if A and B are closed and strictly locally star-shaped, then so is their union.

Pick a point $x \in A \cup B$. Without loss of generality, $x \in A$. Then either $x \in B$ or x is separated from B . In the first case, for some small r we have a neighbourhood

$$B_r(x) \cap X = (B_r(x) \cap A) \cup (B_r(x) \cap B),$$

which is the union of star-shaped neighbourhood with the same star-point, so it is star shaped. In the second case, for some small r we have

$$B_r(x) \cap X = B_r(x) \cap A,$$

so x has a star-shaped neighbourhood. \square

Problem 2.5 (1.1.5). *Show that for a space X , the following three conditions are equivalent:*

- (a) Every map $\mathcal{S}^1 \rightarrow X$ is homotopic to a constant map, with image a point.
- (b) Every map $\mathcal{S}^1 \rightarrow X$ extends to a map $\mathcal{D}^2 \rightarrow X$.
- (c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected all maps $\mathcal{S}^1 \rightarrow X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

Proof. ($a \Rightarrow b$) Take any $f : \mathcal{S}^1 \rightarrow X$. We have that $f \simeq_H \text{const}_{x_0}$ for some $x_0 \in X$, with a homotopy $H : \mathcal{S}^1 \times \mathcal{I} \rightarrow X$. Since $H(x, 0) = f(x)$ and $H(x, 1) = x_0$ we can consider the quotient of $\mathcal{S}^1 \times \mathcal{I}$ by the relation identifying $\mathcal{S}^1 \times \{1\}$ with a single point. Such a cone-like space is clearly homeomorphic with \mathcal{D}^2 . Let $j : \mathcal{S}^1 \times \mathcal{I} \rightarrow \mathcal{D}^2$ be the quotient map. We have the induced map H' as shown in the diagram:

$$\begin{array}{ccccc} \mathcal{S}^1 & \xhookrightarrow{i} & \mathcal{S}^1 \times \mathcal{I} & \xrightarrow{j} & \mathcal{D}^2 \\ & & \downarrow H & & \\ & f \searrow & X & \swarrow H' & \end{array}$$

Now observe that $H' \circ j \circ i$ extends f to a disk.

($b \Rightarrow c$) **obrazek** Take any $g \in \pi_1(X, x_0)$. It can be represented as a map $g : \mathcal{S}^1 \rightarrow X$ and by assumption we get $\bar{g} : \mathcal{D}^2 \rightarrow X$ which extends g . Suppose that $s \in \mathcal{D}^2$ corresponds to the base point x_0 by g . Since disks are contractible we get homotopy H that in 0 is the boundary sphere \mathcal{S}^1 and in 1 it is a point in the middle of \mathcal{D}^2 and call it m . Now consider a path h from point s to m as in the diagram. The homotopy H contracts path h to a constant path in m . We will now show that $\beta_h(g) = 0 \in \pi_1(X, \bar{g}(m))$. Since $\beta_h(g) = hgh$ we conclude that H contracts hgh to a point $\bar{g}(m)$, which is exactly representation of 0 in $\pi_1(X, \bar{g}(m))$. Because β_h is isomorphism we see that g must be in the same class as trivial map in $\pi_1(X, x_0)$.

($c \Rightarrow a$)

Since $\pi_1(X, x_0)$ is trivial, every loop is homotopic to a constant loop. Lets call this homotopy H . Since every loop can be represented as a map $\mathcal{S}^1 \rightarrow X$, by simply taking the homotopy H we get the thesis. \square

Problem 2.6 (1.1.6). We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(\mathcal{S}^1, s_0) \rightarrow (X, x_0)$. Let $[\mathcal{S}^1, X]$ be the set of homotopy classes of maps $\mathcal{S}^1 \rightarrow X$ with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \rightarrow [\mathcal{S}^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[\mathcal{S}^1, X]$ and the set of conjugacy classes in $\pi_1(X)$ when X is path-connected.

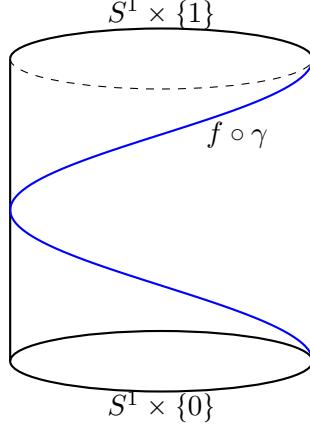
Problem 2.7 (1.1.7). Define $f : \mathcal{S}^1 \times \mathcal{I} \rightarrow \mathcal{S}^1 \times \mathcal{I}$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $\mathcal{S}^1 \times \mathcal{I}$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles.

Proof. For the first part definie homotpy $f_t : \mathcal{S}^1 \times \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{S}^2 \times \mathcal{I}$ as follows:

$$(1) \quad f_t(\theta, s) = (\theta + 2\pi s(1-t), s)$$

This ofcourse is stationary on $\mathcal{S} \times \{0\}$ because the term $2\pi s(1-t)$ vanishes for $s = 0$ for all t , but we see that is not stationary on the boundary circle $\mathcal{S}^1 \times \{1\}$

For the second part consider a loop $\gamma(t) = (\theta_0, t)$, where θ_0 is a point on S^1 . The image of γ under map f will be a path connecting points $(\theta_0, 0)$ and $(\theta_0, 1)$ and going around the cylinder once (As pictured below).



Now suppose for contradiction that there is homotopy f_t that is stationary on both boundary. Then we would be able to construct homotopy between paths $f \circ \gamma$ and γ . Now consider loop based in θ_0 $(f \circ \gamma) \cdot \bar{\gamma}$. This loops is nontrivial generator of group $\pi(S^1 \times I, \theta_0)$. But we get that:

$$(2) \quad (f \circ \gamma) \cdot \bar{\gamma} \simeq \gamma \cdot \bar{\gamma} \simeq \text{const}_{\theta_0}$$

That is contradiction. \square

Problem 2.8 (1.1.8). *Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \rightarrow \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?*

Proof. Consider torus embeded in \mathbb{R}^3 in the usual way and projection π on to plane $z = 0$. Ofcourse if $\pi(a, b) = \pi(c, d)$ it must be that points $b = d$. So that automatically contradicts the thesis of Borsuk Ulam theorem on Torus. \square

Problem 2.9 (1.1.9). *Let A_1, A_2, A_3 be compact sets in \mathbb{R}^3 . Use the **Borsuk–Ulam theorem** to show that there is one plane $P \subseteq \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.*

Let us introduce some notation. Let λ denote the measure we are interested in. We can think of the set of all planes in \mathbb{R}^3 , as $S^2 \times \mathbb{R}$, thus for a plane P , let $P_{n,t}$ denote the plane with a normal vector $n \in S^2$, moved along it by $t \in \mathbb{R}$. We denote by $P_{n,t}^+, P_{n,t}^-$ two halves of \mathbb{R}^3 , respectively above and below the $P_{n,t}$ (in the sense of direction of n). We will need the following lemma:

Lemma 2.10. *Let $A \subseteq \mathbb{R}^3$ be a compact subset, there exists a continuous, even map $h : S^2 \rightarrow \mathbb{R}$, such that:*

$$\lambda(P_{n,h(n)}^+ \cap A) = \lambda(P_{n,h(n)}^- \cap A)$$

The idea behind the lemma is to find a continous family of planes, each equally dividing given set. **Zauważliśmy, że potrzeba żeby ta mapa była parzysta.**

Proof. (of the Lemma 2.10) TODO

\square

Proof. **utworzyć begin-solution i tutaj zmienić** Construct $h : \mathcal{S}^2 \rightarrow \mathbb{R}$ as in Lemma 2.10 for A_1 . Let $g : \mathcal{S}^2 \rightarrow \mathbb{R}^2$ be defined as follows:

$$\begin{array}{ccc} \mathcal{S}^2 & \xrightarrow{g} & \mathbb{R}^2 \\ \Downarrow & & \Downarrow \\ n & \longmapsto & \begin{pmatrix} \lambda(P_{n,h(n)}^+ \cap A_2) \\ \lambda(P_{n,h(n)}^+ \cap A_3) \end{pmatrix} \end{array}$$

Ten diagram to tylko fleks z użycia quivera, można to napisać normalnie i pewnie będzie ładniej.

Since h is continuous, g is continuous as well. By the **Borsuk-Ulam theorem**, there exists some $n \in \mathcal{S}^2$, such that:

$$\begin{pmatrix} \lambda(P_{n,h(n)}^+ \cap A_2) \\ \lambda(P_{n,h(n)}^+ \cap A_3) \end{pmatrix} = \begin{pmatrix} \lambda(P_{-n,h(-n)}^+ \cap A_2) \\ \lambda(P_{-n,h(-n)}^+ \cap A_3) \end{pmatrix} = \begin{pmatrix} \lambda(P_{n,h(n)}^- \cap A_2) \\ \lambda(P_{n,h(n)}^- \cap A_3) \end{pmatrix}$$

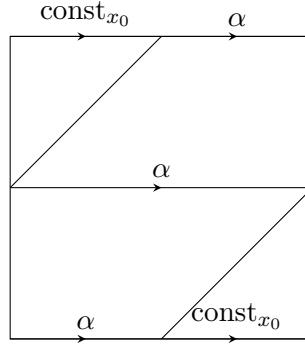
The last equality follows from evenness of h , and the fact that $P_{-n,t}^+ = P_{n,t}^-$. **Zamiast tych pmatrixów można pisać A_i** \square

Problem 2.11 (1.1.10). *From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this*

Proof. Consider two loops $\alpha \in X \times \{y_0\}$ and $\beta \in \{x_0\} \times Y$. They represent classes $[(\alpha, \text{const}_{y_0})]$ and $[(\text{const}_{x_0}, \beta)]$ (We use a little abuse of notation here) The goal is to proof that

$$[(\alpha, \text{const}_{y_0})][(\text{const}_{x_0}, \beta)] = [(\text{const}_{x_0}, \beta)][(\alpha, \text{const}_{y_0})]$$

The diagram below show easy homotopy of loops: $\alpha \cdot \text{const}_{x_0} \simeq \text{const}_{x_0} \cdot \alpha$



We use this homotopy to construct the needed one to show the thesis.

Nie wiem czy tego nie dokładniej napisać jakoś? \square

Problem 2.12 (1.1.11). *If X_0 is the path-component of a space X containing the basepoint x_0 show the inclusion $X_0 \hookrightarrow X$ induces an isomorphism $\pi_1(X_0, x_0) \rightarrow \pi_1(X, x_0)$*

Proof. Let $\iota : X_0 \rightarrow X$ be the inclusion in question. The goal is to show that ι_* is an isomorphism. First we will show that ι_* is 1–1. Take any two loops $[\alpha], [\beta] \in \pi_1(X_0, x_0)$ and suppose that $\iota_*([\alpha]) = \iota_*([\beta])$. That imply that $\iota \circ \alpha \simeq \iota \circ \beta$. Which gives us a homotopy $H : \mathcal{I} \times \mathcal{I} \rightarrow X$ of this two loops. And since X_0 is path component containing x_0 and each point in $Im(H)$ is path connected to x_0 we get that H is also an homotopy of paths $\alpha \simeq \beta$ which lives in X_0 .

Similarly we proof that the ι_* is onto. Let α be an arbitrary loop in $\pi_1(X, x_0)$. Its easy to see that $\alpha(t) \in X_0$ for all $t \in \mathcal{I}$. And so α represents a loop in $\pi_1(X_0, x_0)$ aswell. \square

Problem 2.13 (1.1.12). Show that every homomorphism $\pi_1(\mathcal{S}^1) \rightarrow \pi_1(\mathcal{S}^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : \mathcal{S}^1 \rightarrow \mathcal{S}^1$.

Proof. We use the fact that $\pi_1(\mathcal{S}^1) \cong \mathbb{Z}$ and the generator can be represented by: $f(t) = e^{2\pi t}$. Any map $\mathbb{Z} \rightarrow \mathbb{Z}$ is only depended by where it maps the generator of \mathbb{Z} . So suppose we have map $\psi : \pi_1(\mathcal{S}^1) \rightarrow \pi_1(\mathcal{S}^1)$ of the form $\psi([f]) = [f]^n$. Then we can define map $\varphi : \mathcal{S}^1 \rightarrow \mathcal{S}^1$ as $\varphi(z) = z^n$. Then by what we know about fundamental group of circle we get:

$$\varphi_*([f]) = [\varphi \circ f] = [f]^n$$

The last equality was proven in section about fundamental group of circle in Hatcher: [Hat01] \square

Problem 2.14 (1.1.13). Given a space X and a path-connected subspace A containing the basepoint x_0 , show that the path map $\pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoints in A is homotopic to a path in A .

Proof. For sake of clarity we will denote homotopy classes of loops in subspace A as $[\cdot]_A$ and $[\cdot]_X$ for loops in X . **Lepiej zrobić te implikacje w nawiasie:** Also we denote the inclusion $A \hookrightarrow X$ as ι .

(\Leftarrow)

For this implication, we assume that every path in X with endpoints in A is homotopic to some path in A . This of course holds also for loops based in x_0 . The goal is to show that ι_* is surjective. Take any loop $[\alpha]_X \in \pi_1(X, x_0)$. We need to find a loop α' in A that is sent to α by ι_* . By the assumption there exist a loop α' in A which is homotopic to α . And so it holds that:

$$\iota_*([\alpha']_A) = [\iota \circ \alpha']_X = [\alpha']_X = [\alpha]_X$$

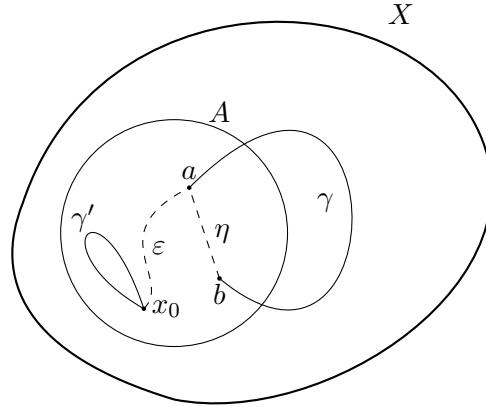
(\Rightarrow)

Now assume that induced map ι_* is surjective. The goal now is to for any path γ in X from arbitrary points a, b construct a new path from a to b that is in A .

From assumptions we know that A is path connected, so we get two paths: first from x_0 to a that we will call ϵ , and second from a to b that we call η . Consider an loop $\bar{\varepsilon} \cdot \gamma \cdot \bar{\eta} \cdot \bar{\varepsilon}$ based in x_0 . Since map ι_* is surjective we get that there exist an loop γ' in A that satisfies $\gamma' \simeq_X \bar{\varepsilon} \cdot \gamma \cdot \bar{\eta} \cdot \bar{\varepsilon}$. And by properties of homotopic equivalence we get that

$$\begin{aligned} \bar{\varepsilon} \cdot \gamma' \cdot \epsilon \cdot \eta &\simeq_X \bar{\varepsilon} \cdot \varepsilon \cdot \gamma \cdot \bar{\eta} \cdot \bar{\varepsilon} \cdot \epsilon \cdot \eta \\ \bar{\varepsilon} \cdot \gamma' \cdot \epsilon \cdot \eta &\simeq_X \gamma \end{aligned}$$

And that gives that desired path is $\bar{\varepsilon} \cdot \gamma' \cdot \epsilon \cdot \eta$



\square

Problem 2.15 (1.1.14). Show that the isomorphism $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ in [Hat01, Proposition 1.12] is given by $[f] \mapsto (p_{1*}([f]), p_{2*}([f]))$ where p_1 and p_2 are the projections of $X \times Y$ onto its two factors.

Proof. In the proposition 1.1.12, the isomorphism is given by $[f] \rightarrow ([g], [h])$, where map $f = (g, h)$. But for any f it holds that $p_1 f = g$ and $p_2 f = h$, and so it also holds for homotopy classes. And so the isomorphism is given by $[f] \rightarrow (p_{1*}([f]), p_{2*}([f]))$. \square

Problem 2.16 (1.1.15). Given a map $f : X \rightarrow Y$ and a path $h : I \rightarrow X$ from x_0 to x_1 , show that $f_* \beta_h = \beta_{fh} f_*$, namely that the diagram below commutes.

$$\begin{array}{ccc} \pi_1(X, x_1) & \xrightarrow{\beta_h} & \pi_1(X, x_0) \\ f_* \downarrow & & \downarrow f_* \\ \pi_1(Y, f(x_1)) & \xrightarrow{\beta_{fh}} & \pi_1(Y, f(x_0)) \end{array}$$

Proof. Take any loop $\gamma \in \pi_1(X, x_1)$.

$$f_* \beta_h([\gamma]) = f_*([h\gamma\bar{h}]) = [f \circ h\gamma\bar{h}] = [(f \circ h)(f \circ \gamma)\overline{(f \circ h)}] = \beta_{fh}([f \circ \gamma]) = \beta_{fh} f_*([\gamma])$$

\square

Problem 2.17 (1.1.16). Show that there are no retractions $r : X \rightarrow A$ in the following cases:

- (a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1 .
- (b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
- (c) $X = S^1 \times D^2$ and A the circle shown in the figure

rysunek

- (d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
- (e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.
- (f) X the Möbius band and A its boundary circle.

Proof. We will use proposition 1.1.17 from [Hat01]. It states that if X retracts onto a subspace A then inclusion map induces an injection map of fundamental groups.

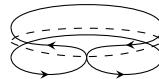
- (a) Suppose that there is such a retraction. Then by proposition 1.1.17 we would have injection $\pi_1(A, x_0) \hookrightarrow \pi_1(X, x_0)$. Since A is S^1 embedded in X which is R^3 we get injection $\mathbb{Z} \hookrightarrow \{e\}$. And that gives contradiction.
- (b) We can calculate the fundamental groups of X and A :

$$\pi_1(X) = \pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z}$$

$$\pi_1(A) = \pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$$

If there was retraction of X onto A then it would induce injection of $\mathbb{Z} \times \mathbb{Z} \hookrightarrow \mathbb{Z}$. Suppose there is injective homomorphism φ of these groups. Then let $a = \varphi((1, 0))$ and $b = \varphi((0, 1))$. Observe that both those elements are non zero since kernel of φ must be 0. Now consider $\varphi((b, -a)) = b\varphi((1, 0)) - a\varphi((0, 1)) = ba - ab = 0$. Which means that kernel is non zero and that is contradiction.

- (c) By the same calculations as previously we get that $\pi_1(X) \cong \mathbb{Z}$ and $\pi_1(A) \cong \mathbb{Z}$, so there is no contradiction in algebraic way. Suppose that there is a retraction. Then the inclusion $i : A \hookrightarrow X$ induces an injection i_* . This map sends loop that goes around A to the same loop just embedded in X . But this loop in X is nullhomotopic. To see that, firstly observe that the loop we consider is homotopic to loop in a figure below



Then we can deformation retract $S^1 \times D^2$ to the circle denoted by dashed line on a figure. Since deformation retractions induce isomorphism of fundamental groups we get that our loop is mapped to an loop that goes around circle clockwise and then counter clockwise, which is ofcourse nullhomotopic.

napisać to lepiej

And since we get nontrivial elent of $\pi_1(A)$ being mapped to 0 we see that ι_* is not injective and that is contradiction.

- (d) Observe that space X is simply connected. By lemma 1.1.15 in [Hat01] any loop $\gamma : I \rightarrow X$ is homotopic to a finite product of loops each contained in one of the disks. But disks are simply connected and so each of factors is homotopic to trivial loop. This implies that the whole loop is homotopic to a trivial one, and so X is simply connected.

Now observe that A has nontrivial fundamental group. There is retraction of $S^1 \wedge S^1$ to S^1 (Just colapse on of the cricle to a point) and it will induce an injection of \mathbb{Z} into fundamental group of this wedge, and this proofs that there is an nontrivial element.

Now suppose that there is retraction of X onto A . That will induce an injection of nontrivial group $\pi_1(S^1 \wedge S^1)$ to trivial group. But this is impossible so that gives a contradicion.

- (e) For this problem we need to show that there is no injection $\pi_1(S^1 \wedge S^1) \rightarrow \mathbb{Z}$. (Since X deformation retracts onto a circle) Take a loop ω_1 to be loop that goes around first circle in $S^1 \wedge S^1$ and ω_2 to be loop that goes around the second circle.

By retraction from previous point we see that ω_1 and ω_2 are not homotopic. and also for any powers i, j the loops ω_1^i is not homotopic to ω_2^j .

Now suppose that $\iota_*(\omega_1) = a$ and $\iota_*(\omega_2) = b$, and consider:

$$\iota_*(\omega_1^b \cdot \omega_2^{-a}) = \iota_*(\omega_1^b) + \iota_*(\omega_2^{-a}) = ab - ba = 0$$

And as previously we showed that kernel is non zero. So we get contradiction.

- (f) The fundamental group of Mobius strip is \mathbb{Z} since there is an deformation retraction onto a circile goind around the strip in the middle. If there were retraction onto boundary circle it would induce an injection:

$$\pi_1(A) \hookrightarrow \pi_1(X) \rightarrow \mathbb{Z}$$

Now just observe that by the deformation retraction the generator of $\pi_1(A)$ will be send to a loop that goes around the middle cricle twice so in groups we will get map $\mathbb{Z} \rightarrow \mathbb{Z}$ given by $1 \rightarrow 2$, which is not injective. That gives a contradiction.

□

Problem 2.18 (1.1.17). *Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \rightarrow S^1$.*

Proof. Let r_n be the retraction that fixes the left circle and winds the right circle in the wedge sum n times around the right. Then, if γ is the path that winds around the right circle once

$$(r_n)_*[\gamma] = [\delta]^n,$$

so the retractions r_n induce different homomorphisms. Suppose now that for some $m \neq n$ we have $r_m \simeq r_n$. Then by Lemma 1.19. **dodać odnośnik** we have

$$(r_n)_* = \beta_h(r_m)_*,$$

where h is the path traced by basepoint under the homotopy. Since both r_n and r_m fix the basepoint and $\pi_1 S^1$ is abelian, by **dodać odnośnik** Problem 1.1.3. β_h does nothing, to $(r_n)_* = (r_m)_*$.

□

Problem 2.19 (1.1.18). *Using [Hat01, Lemma 1.15], show that if a space X obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1 . Apply this to show:*

- (a) *The wedge sum $S^1 \vee S^2$ has a fundamental group \mathbb{Z} .*
- (b) *For a path-connected CW-complex X the inclusion map $X^{(1)} \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^{(1)}) \rightarrow \pi_1(X)$.*

2.2. Van Kampen's Theorem.

2.3. Covering Spaces. **Covering Spacey – mem z kevinem nad okre¶iem**

Problem 2.20 (1.3.9). *Show that if a path-connected, locally path-connected path-connected space X has $\pi_1(X)$ finite, then every map $X \rightarrow S^1$ is nullhomotopic.*

Proof. Denote the map by $f : X \rightarrow S^1$. We will use the covering space $p : \mathbb{R} \rightarrow S^1$. Since $\pi_1(X)$ is finite, every element $g \in \pi_1(X)$ has finite order, so does $f_*(g)$, so it is the identity element. Therefore the image $f_*(\pi_1(X))$ is the trivial subgroup, so f has a lift $\tilde{f} : X \rightarrow \mathbb{R}$. As \mathbb{R} is contractible, \tilde{f} is nullhomotopic, hence so is $f = p \circ \tilde{f}$. \square

2.4. Additional Topics.

2.4.1. Graphs and Free Groups.

2.4.2. $\mathcal{K}(G, 1)$ -Spaces and Graphs of Groups.

3. HOMOLOGY

3.1. Simplicial and Singular Homology.

3.2. Computations and Applications.

3.3. The Formal Viewpoint.

3.4. Additional Topics.

3.4.1. Homology and Fundamental Group.

3.4.2. Classical Applications.

Proposition 3.1.

- (1) *For $D^k \simeq \Delta \subseteq S^n$, for all i : $\tilde{H}_i(S^n - \Delta) = 0$*
- (2) *For $S^k \simeq \Sigma \subseteq S^n$, $\tilde{H}_i(S^n - \Sigma) = \mathbb{Z}$ for $i = n - k - 1$, and 0 otherwise.*

Problem 3.2 (2.B.1). *Compute $H_i(S^n - X)$ when X is a subspace of S^n homeomorphic to $S^k \vee S^l$ or to $S^k \sqcup S^l$.*

Proof. In both cases we will use Prop. 3.1 and Mayer-Vietoris sequence (**tu sformu³owaæ jako¶ ³adnie MV i daæ odno¶nik - to w sam raz na tekst nasz w rozdziale 3.2, no bo w Hatcherze to nie jest ujete w ¶adne proposition**)

label=(a) Let $X \simeq S^k \vee S^l$. In order to use the **Mayer-Vietoris** we need to define:

$$A := S^n - S^k, \quad B := S^n - S^l$$

Observe, that

$$A \cap B = S^n - X, \quad A \cup B = S^n - \{\star\} \simeq \mathbb{R}^n$$

By **odnośnik do Mayera**, we get:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \tilde{H}_{i+1}(A \cup B) & \longrightarrow & \tilde{H}_i(A \cap B) & \longrightarrow & \tilde{H}_i(A) \oplus \tilde{H}_i(B) \longrightarrow \tilde{H}_i(A \cup B) \longrightarrow \dots \\ & & \parallel & & \parallel & & \parallel \\ \dots & \longrightarrow & \tilde{H}_{i+1}(\mathbb{R}^n) & \longrightarrow & \tilde{H}_i(\mathcal{S}^n - X) & \longrightarrow & \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^k) \oplus \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^l) \longrightarrow \tilde{H}_i(\mathbb{R}^n) \longrightarrow \dots \end{array}$$

Because \mathbb{R}^n is contractible, its homology groups are trivial, thus we get an isomorphism for every i :

$$\tilde{H}_i(\mathcal{S}^n - X) \simeq \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^k) \oplus \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^l)$$

Assuming $k \neq l$, for $i = n - k - 1$ and $i = n - l - 1$, we get that $\tilde{H}_i(\mathcal{S}^n - X) \simeq \mathbb{Z}$ for other i it is 0. Otherwise, that is when $k = l$, for $i = n - k - 1 = n - l - 1$, we get $\tilde{H}_i(\mathcal{S}^n - X) \simeq \mathbb{Z}^2$, and for other i it is again 0.

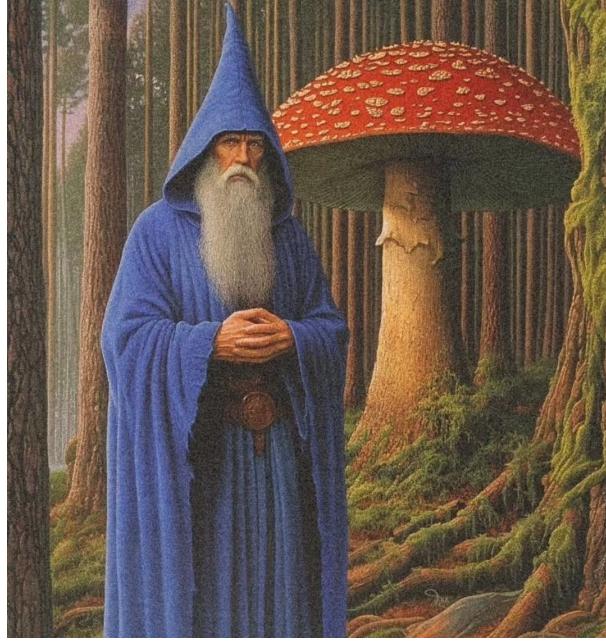
label=(b) **Dokończyć**

□

3.4.3. Simplicial Approximation.

Theorem 2C.3. If X is a finite simplicial complex, or more generally a retract of a finite simplicial complex, and $f: X \rightarrow X$ is a map with $\tau(f) \neq 0$, then f has a fixed point.

$$\tau(f) := \sum_{n=0}^{\infty} (-1)^n \text{tr}(f_* : H_n X \rightarrow H_n X)$$



4. COHOMOLOGY

4.1. Cohomology Groups.

4.2. Cup Product.

Theorem 3.2. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C; G)$ of the cochain complex $\text{Hom}(C_n, G)$ are determined by split exact sequences*

$$0 \rightarrow \text{Ext}(H_{n-1}(C), G) \rightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n(C), G) \rightarrow 0$$

□



4.3. Poincare Duality.

4.4. Additional Topics.

4.4.1. Universal Coefficients for Homology.

4.4.2. The General Künneth Formula.

4.4.3. H -Spaces and Hopf Algebras.

4.4.4. The Cohomology of $\text{SO}(n)$.

4.4.5. Bockstein Homomorphisms.

4.4.6. Limits and Ext.

4.4.7. Transfer Homomorphisms.

4.4.8. Local Coefficients.

REFERENCES

- [Hat01] Allen Hatcher. *Algebraic Topology*. en. Cambridge, England: Cambridge University Press, Dec. 2001 (cit. on pp. 1, 7–10).

