

## HATCHER SOLUTIONS

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ABSTRACT. We present the solutions to the excercises found in Allen Hatcher's „Algebraic Topology” [Hat01] with our own perspective and commentary. We also present proofs of selected theorems (with what we hope is a more detailed explanation).



*Dedicated to Jan Dymara, Świątosław Gal & Jacek Świątkowski*

## 1. INTRODUCTION

**Blabla że co w ogóle robimy...**

We denote a  $k$ -sphere by  $S^k$ ,  $k$ -disk by  $D^k$  and a  $[0, 1]$  segment by  $\mathcal{I}$ . By  $\simeq$  we mean homotopy or, when placed between spaces, homotopy equivalence when placed between spaces. We denote homeomorphisms by  $\approx$ .

## 2. THE FUNDAMENTAL GROUP

**2.1. Basic Constructions.** This section is about basic constructions.

**Należałooby poprawić numerację problemów, żeby była tak jak w hatcherze, no i też żeby sekcje w Additional Topics nie były 2.4 itp tylko 2.A, 2.B itd**

**Problem 2.1.** *Show that composition of paths satisfies the following cancellation property: If  $f_0 \cdot g_0 \simeq f_1 \cdot g_1$  and  $g_0 \simeq g_1$  then  $f_0 \simeq f_1$ .*

*Proof.* The homotopy assumptions imply that  $g_i$  have the same endpoints, and so do the  $f_i g_i$ , which means the  $f_i$  have the same endpoints as well.

Since  $g_0 \simeq g_1$ , it is clear that  $\bar{g}_0 \simeq \bar{g}_1$ . We proceed as shown on the homotpy diagram:

**obrazek**

Algebraically, this corresponds to the computations

$$f_0 \simeq f_0 \text{ const}_{x_0} \simeq f_0(g_0 \bar{g}_0) \simeq (f_0 g_0) \bar{g}_0 \simeq (f_1 g_1) \bar{g}_1 \simeq f_1(g_1 \bar{g}_1) \simeq f_1 \text{ const}_{x_0} \simeq f_1,$$

where  $x_0$  denotes the enpoint of the paths  $f_i$ .  $\square$

**Problem 2.2.** *Show that the change-of-basepoint homomorphism  $\beta_h$  depends only on the homotopy class of  $h$ .*

*Proof.* It suffices to show that for every two homotopic loops  $h$  and  $h'$  we have  $\beta_h = \beta_{h'}$ . Let  $H$  denote the homotopy  $h \simeq_H h'$ . Take any loop  $g$ . We have that  $\beta_h(g) = hg\bar{h} \simeq h'g\bar{h}' = \beta_{h'}(g)$ .  $\square$

**Problem 2.3.** *For a path-connected space  $X$ , show that  $\pi_1(X)$  is abelian iff all basepoint-change homomorphism  $\beta_h$  depend only on the endpoints of the path  $h$ .*

*Proof.* ( $\Rightarrow$ ) Consider any two paths  $h, h'$  from points  $y \in X$  to  $x \in X$ . We want to show that  $\beta_h = \beta_{h'}$ . Consider any loop  $g \in \pi_1(X, x)$  and a loop  $\bar{h}h'$  which also is in  $\pi_1(X, x)$ . Since  $\pi_1(X)$  is abelian then  $\pi_1(X, x)$  is also abelian. And so it follows that:

$$g \cdot \bar{h}h' \simeq \bar{h}h' \cdot g$$

Now multiply it by  $h$  on left hand side, and by  $\bar{h}'$  on right hand side to get:

$$hg\bar{h} \simeq h'g\bar{h}'$$

That is exactly equivalent to  $\beta_h = \beta_{h'}$ .

( $\Leftarrow$ ) Suppose that for any two paths from  $x$  to  $y$  they induce the same maps  $\beta_h, \beta_{h'}$ . In particular given any two loops  $g, h$  in  $\pi_1(X, x_0)$  we have  $\beta_g = \beta_h$ . And so it follows that:

$$hg\bar{h} = \beta_h(g) = \beta_g(g) = g$$

Thus  $gh = hg$  which means that  $\pi_1(X, x)$  is abelian, and so of course  $\pi_1(X)$  is abelian too.  $\square$

**Problem 2.4.** *A subspace  $X \subseteq \mathbb{R}^n$  is said to be star-shaped if there is a point  $x_0 \in X$  such that for each  $x \in X$  the line segment from  $x_0$  to  $x$  lies in  $X$ . Show that if a subspace  $X \subseteq \mathbb{R}^n$  is locally star-shaped, in the sense that every point of  $X$  has a star-shaped neighbourhood in  $X$ , then every path in  $X$  is a piecewise linear path, that is, a path consisting of a finite number of straight lines traversed at constant speed. Show that this applies in particular when  $X$  is open or when  $X$  is a union of finitely many closed convex subsets.*

*Proof.* In the case that  $X$  is star-shaped as a whole, it is contractible, so the thesis definitely holds. Denote the path in question by  $\gamma$ . In general, we can use the compactness of the image of  $\gamma$  to cover it with finitely many star-shaped sets. However, in each of these we cannot just contract the curve, as we must keep the endpoints intact.

Instead we show the following: if  $X$  is star-shaped with *star-point*  $x_0$ , we can deform any path  $\gamma$  in  $X$  to a linear path going from  $\gamma(0)$  to  $x_0$  and then from  $x_0$  to  $\gamma(1)$ . Without loss of generality  $x_0 = 0$ . The homotopy given by **Obrazek**

$$H_1(s, t) = \left[ (1-t) + t|1-2s| \right] \gamma(s)$$

achieves this partially, as it deforms  $\gamma(s)$  to  $|1-2s|\gamma(s)$ , which is equal to  $\gamma(0), x_0, \gamma(1)$  at  $s = 0, \frac{1}{2}, 1$ . Now we turn to making the segments linear.

If  $\delta$  is the first half of  $\gamma$ , the homotopy  $H_1$  deforms it to the path  $(1-s)\delta(s)$ . We will deform it to  $s\delta(0)$ , which is certainly constant-speed. This is achieved through the homotopy

$$H_2(s, t) = (1-s)\delta((1-t)s).$$

Finally, applying the same procedure to the second half of  $\gamma$ , we will have indeed deformed  $\gamma$  to a piecewise-linear path consisting of two segments.  $\square$

*Open sets.* In an open set, every point has a neighbourhood which is a Euclidean ball, so the space  $X$  is locally star-shaped.  $\square$

*Finite unions of convex sets.* Call a subspace  $X \subseteq \mathbb{R}^n$  *strictly locally star-shaped* iff every  $x \in X$  has a star-shaped neighbourhood *for which it is the star-point*. Every convex set satisfies this property. We will show that if  $A$  and  $B$  are closed and strictly locally star-shaped, then so is their union.

Pick a point  $x \in A \cup B$ . Without loss of generality,  $x \in A$ . Then either  $x \in B$  or  $x$  is separated from  $B$ . In the first case, for some small  $r$  we have a neighbourhood

$$B_r(x) \cap X = (B_r(x) \cap A) \cup (B_r(x) \cap B),$$

which is the union of star-shaped neighbourhood with the same star-point, so it is star shaped. In the second case, for some small  $r$  we have

$$B_r(x) \cap X = B_r(x) \cap A,$$

so  $x$  has a star-shaped neighbourhood.  $\square$

**Problem 2.5.** Show that for a space  $X$ , the following three conditions are equivalent:

- (a) Every map  $S^1 \rightarrow X$  is homotopic to a constant map, with image a point.
- (b) Every map  $S^1 \rightarrow X$  extends to a map  $D^2 \rightarrow X$ .
- (c)  $\pi_1(X, x_0) = 0$  for all  $x_0 \in X$ .

Deduce that a space  $X$  is simply-connected all maps  $S^1 \rightarrow X$  are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

*Proof.* ( $a \Rightarrow b$ ) Take any  $f : S^1 \rightarrow X$ . We have that  $f \simeq_H \text{const}_{x_0}$  for some  $x_0 \in X$ , with a homotopy  $H : S^1 \times I \rightarrow X$ . Since  $H(x, 0) = f(x)$  and  $H(x, 1) = x_0$  we can consider the quotient of  $S^1 \times I$  by the relation identifying  $S^1 \times \{1\}$  with a single point. Such a cone-like space is clearly homeomorphic with  $D^2$ . Let  $j : S^1 \times I \rightarrow D^2$  be the quotient map. We have the induced map  $H'$  as shown in the diagram:

$$\begin{array}{ccccc} S^1 & \xhookrightarrow{i} & S^1 \times I & \xrightarrow{j} & D^2 \\ & \searrow f & \downarrow H & \swarrow & \\ & & X & & \end{array}$$

Now observe that  $H' \circ j \circ i$  extends  $f$  to a disk.

( $b \Rightarrow c$ ) **obrazek** Take any  $g \in \pi_1(X, x_0)$ . It can be represented as a map  $g : S^1 \rightarrow X$  and by assumption we get  $\bar{g} : D^2 \rightarrow X$  which extends  $g$ . Suppose that  $s \in D^2$  corresponds to the base point  $x_0$  by  $g$ . Since disks are contractible we get homotopy  $H$  that in 0 is the boundary sphere  $S^1$  and in 1 it is a point in the middle of  $D^2$  and call it  $m$ . Now consider a path  $h$  from point  $s$  to  $m$  as in the diagram. The homotopy  $H$  contracts path  $h$  to a constant path in  $m$ . We will now show that  $\beta_h(g) = 0 \in \pi_1(X, \bar{g}(m))$ . Since  $\beta_h(g) = h\bar{g}\bar{h}$  we conclude that  $H$  contracts  $h\bar{g}\bar{h}$  to a point  $\bar{g}(m)$ , which is exactly representation of 0 in  $\pi_1(X, \bar{g}(m))$ . Because  $\beta_h$  is isomorphism we see that  $g$  must be in the same class as trivial map in  $\pi_1(X, x_0)$ .

( $c \Rightarrow a$ )

Since  $\pi_1(X, x_0)$  is trivial, every loop is homotopic to a constant loop. Lets call this homotopy  $H$ . Since every loop can be represented as a map  $S^1 \rightarrow X$ , by simply taking the homotopy  $H$  we get the thesis.  $\square$

**Problem 2.6.** *We can regard  $\pi_1(X, x_0)$  as the set of basepoint-preserving homotopy classes of maps  $(S^1, s_0) \rightarrow (X, x_0)$ . Let  $[S^1, X]$  be the set of homotopy classes of maps  $S^1 \rightarrow X$  with no conditions on basepoints. Thus there is a natural map  $\Phi : \pi_1(X, x_0) \rightarrow [S^1, X]$  obtained by ignoring basepoints. Show that  $\Phi$  is onto if  $X$  is path-connected, and that  $\Phi([f]) = \Phi([g])$  iff  $[f]$  and  $[g]$  are conjugate in  $\pi_1(X, x_0)$ . Hence  $\Phi$  induces a one-to-one correspondence between  $[S^1, X]$  and the set of conjugacy classes in  $\pi_1(X)$  when  $X$  is path-connected.*

**Problem 2.7.** *Let  $A_1, A_2, A_3$  be compact sets in  $\mathbb{R}^3$ . Use the **Borsuk–Ulam theorem** to show that there is one plane  $P \subseteq \mathbb{R}^3$  that simultaneously divides each  $A_i$  into two pieces of equal measure.*

Let us introduce some notation. Let  $\lambda$  denote the measure we are interested in. We can think of the set of all planes in  $\mathbb{R}^3$ , as  $S^2 \times \mathbb{R}$ , thus for a plane  $P$ , let  $P_{n,t}$  denote the plane with a normal vector  $n \in S^2$ , moved along it by  $t \in \mathbb{R}$ . We denote by  $P_{n,t}^+, P_{n,t}^-$  two halves of  $\mathbb{R}^3$ , respectively above and below the  $P_{n,t}$  (in the sense of direction of  $n$ ). We will need the following lemma:

**Lemma 2.8.** *Let  $A \subseteq \mathbb{R}^3$  be a compact subset, there exists a continuous, even map  $h : S^2 \rightarrow \mathbb{R}$ , such that:*

$$\lambda(P_{n,h(n)}^+ \cap A) = \lambda(P_{n,h(n)}^- \cap A)$$

The idea behind the lemma is to find a continous family of planes, each equally dividing given set. **Zauważliśmy, że potrzeba żeby ta mapa była parzysta.**

*Proof. (of the Lemma 2.8) TODO*  $\square$

*Proof. utworzyć begin-solution i tutaj zmienić* Construct  $h : S^2 \rightarrow \mathbb{R}$  as in Lemma 2.8 for  $A_1$ . Let  $g : S^2 \rightarrow \mathbb{R}^2$  be defined as follows:

$$\begin{array}{ccc} S^2 & \xrightarrow{g} & \mathbb{R}^2 \\ \Downarrow & & \Downarrow \\ n & \longmapsto & \left( \begin{array}{l} \lambda(P_{n,h(n)}^+ \cap A_2) \\ \lambda(P_{n,h(n)}^+ \cap A_3) \end{array} \right) \end{array}$$

**Ten diagram to tylko fleks z użycia quivera, można to napisać normalnie i pewnie będzie ładniej.**

Since  $h$  is continuous,  $g$  is continuous as well. By the **Borsuk-Ulam theorem**, there exists some  $n \in S^2$ , such that:

$$\begin{pmatrix} \lambda(P_{n,h(n)}^+ \cap A_2) \\ \lambda(P_{n,h(n)}^+ \cap A_3) \end{pmatrix} = \begin{pmatrix} \lambda(P_{-n,h(-n)}^+ \cap A_2) \\ \lambda(P_{-n,h(-n)}^+ \cap A_3) \end{pmatrix} = \begin{pmatrix} \lambda(P_{n,h(n)}^- \cap A_2) \\ \lambda(P_{n,h(n)}^- \cap A_3) \end{pmatrix}$$

The last equality follows from evenness of  $h$ , and the fact that  $P_{-n,t}^+ = P_{n,t}^-$ . **Zamiast tych pmatrixów można pisać  $A_i$**   $\square$

**Problem 2.9.** Show that the isomorphism  $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$  in [Hat01, Proposition 1.12] is given by  $[f] \mapsto (p_{1*}([f]), p_{2*}([f]))$  where  $p_1$  and  $p_2$  are the projections of  $X \times Y$  onto its two factors.

**Problem 2.10.** Given a map  $f : X \rightarrow Y$  and a path  $h : I \rightarrow X$  from  $x_0$  to  $x_1$ , show that  $f_*\beta_h = \beta_{fh}f_*$ , namely that the diagram below commutes.

$$\begin{array}{ccc} \pi_1(X, x_1) & \xrightarrow{\beta_h} & \pi_1(X, x_0) \\ f_* \downarrow & & \downarrow f_* \\ \pi_1(Y, f(x_1)) & \xrightarrow{\beta_{fh}} & \pi_1(Y, f(x_0)) \end{array}$$

*Proof.* Take any loop  $\gamma \in \pi_1(X, x_1)$ .

$$f_*\beta_h([\gamma]) = f_*([h\gamma\bar{h}]) = [f \circ h\gamma\bar{h}] = [(f \circ h)(f \circ \gamma)\overline{(f \circ h)}] = \beta_{fh}([f \circ \gamma]) = \beta_{fh}f_*([\gamma])$$

$\square$

**Problem 2.11.** Show that there are no retractions  $r : X \rightarrow A$  in the following cases:

- (a)  $X = \mathbb{R}^3$  with  $A$  any subspace homeomorphic to  $S^1$ .
- (b)  $X = S^1 \times D^2$  with  $A$  its boundary torus  $S^1 \times S^1$ .
- (c)  $X = S^1 \times D^2$  and  $A$  the circle shown in the figure

**rysunek**

- (d)  $X = D^2 \vee D^2$  with  $A$  its boundary  $S^1 \vee S^1$ .
- (e)  $X$  a disk with two points on its boundary identified and  $A$  its boundary  $S^1 \vee S^1$ .
- (f)  $X$  the Möbius band and  $A$  its boundary circle.

**Problem 2.12.** Construct infinitely many nonhomotopic retractions  $S^1 \vee S^1 \rightarrow S^1$ .

*Proof.* Let  $r_n$  be the retraction that fixes the left circle and winds the right circle in the wedge sum  $n$  times around the right. Then, if  $\gamma$  is the path that winds around the right circle once

$$(r_n)_*[\gamma] = [\delta]^n,$$

so the retractions  $r_n$  induce different homomorphisms. Suppose now that for some  $m \neq n$  we have  $r_m \simeq r_n$ . Then by Lemma 1.19. **dodać odnośnik** we have

$$(r_n)_* = \beta_h(r_m)_*,$$

where  $h$  is the path traced by basepoint under the homotopy. Since both  $r_n$  and  $r_m$  fix the basepoint and  $\pi_1 S^1$  is abelian, by **dodać odnośnik** Problem 1.1.3.  $\beta_h$  does nothing, to  $(r_n)_* = (r_m)_*$ .  $\square$

**Problem 2.13.** Using [Hat01, Lemma 1.15], show that if a space  $X$  obtained from a path-connected subspace  $A$  by attaching a cell  $e^n$  with  $n \geq 2$ , then the inclusion  $A \hookrightarrow X$  induces a surjection on  $\pi_1$ . Apply this to show:

- (a) The wedge sum  $S^1 \vee S^2$  has a fundamental group  $\mathbb{Z}$ .
- (b) For a path-connected CW-complex  $X$  the inclusion map  $X^{(1)} \hookrightarrow X$  of its 1-skeleton induces a surjection  $\pi_1(X^{(1)}) \rightarrow \pi_1(X)$ .

## 2.2. Van Kampen's Theorem.

## 2.3. Covering Spaces. **Covering Spacey – mem z kevinem nad okresem**

**Problem 2.14.** Show that if a path-connected, locally path-connected path-connected space  $X$  has  $\pi_1(X)$  finite, then every map  $X \rightarrow S^1$  is nullhomotopic.

*Proof.* Denote the map by  $f : X \rightarrow S^1$ . We will use the covering space  $p : \mathbb{R} \rightarrow S^1$ . Since  $\pi_1(X)$  is finite, every element  $g \in \pi_1(X)$  has finite order, so does  $f_*(g)$ , so it is the identity element. Therefore the image  $f_*(\pi_1(X))$  is the trivial subgroup, so  $f$  has a lift  $\tilde{f} : X \rightarrow \mathbb{R}$ . As  $\mathbb{R}$  is contractible,  $\tilde{f}$  is nullhomotopic, hence so is  $f = p \circ \tilde{f}$ .  $\square$

## 2.4. Additional Topics.

### 2.4.1. Graphs and Free Groups.

### 2.4.2. $\mathcal{K}(G, 1)$ -Spaces and Graphs of Groups.

## 3. HOMOLOGY

### 3.1. Simplicial and Singular Homology.

### 3.2. Computations and Applications.

### 3.3. The Formal Viewpoint.

### 3.4. Additional Topics.

#### 3.4.1. Homology and Fundamental Group.

#### 3.4.2. Classical Applications.

### Proposition 3.1.

- (1) For  $D^k \simeq \Delta \subseteq S^n$ , for all  $i$ :  $\tilde{H}_i(S^n - \Delta) = 0$
- (2) For  $S^k \simeq \Sigma \subseteq S^n$ ,  $\tilde{H}_i(S^n - \Sigma) = \mathbb{Z}$  for  $i = n - k - 1$ , and 0 otherwise.

**Problem 3.2.** Compute  $H_i(S^n - X)$  when  $X$  is a subspace of  $S^n$  homeomorphic to  $S^k \vee S^l$  or to  $S^k \sqcup S^l$ .

*Proof.* In both cases we will use Prop. 3.1 and **Mayer-Vietoris sequence** (**tu sformułować jakoś ładnie MV i dać odnośnik - to w sam raz na tekst nasz w rozdziale 3.2, no bo w Hatcherze to nie jest ujęte w żadne proposition**)

label=(a) Let  $X \simeq S^k \vee S^l$ . In order to use the **Mayer-Vietoris** we need to define:

$$A := S^n - S^k, \quad B := S^n - S^l$$

Observe, that

$$A \cap B = S^n - X, \quad A \cup B = S^n - \{\star\} \simeq \mathbb{R}^n$$

By **odnośnik do Mayera**, we get:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \tilde{H}_{i+1}(A \cup B) & \longrightarrow & \tilde{H}_i(A \cap B) & \longrightarrow & \tilde{H}_i(A \cup B) \longrightarrow \dots \\ & & \parallel & & \parallel & & \parallel \\ & & & & \tilde{H}_i(S^n - S^k) & & \\ \dots & \longrightarrow & \tilde{H}_{i+1}(\mathbb{R}^n) & \longrightarrow & \tilde{H}_i(S^n - X) & \longrightarrow & \tilde{H}_i(\mathbb{R}^n) \longrightarrow \dots \\ & & & & \oplus & & \\ & & & & \tilde{H}_i(S^n - S^l) & & \end{array}$$

Because  $\mathbb{R}^n$  is contractible, its homology groups are trivial, thus we get an isomorphism for every  $i$ :

$$\tilde{H}_i(\mathcal{S}^n - X) \simeq \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^k) \oplus \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^l)$$

Assuming  $k \neq l$ , for  $i = n - k - 1$  and  $i = n - l - 1$ , we get that  $\tilde{H}_i(\mathcal{S}^n - X) \simeq \mathbb{Z}$  for other  $i$  it is 0. Otherwise, that is when  $k = l$ , for  $i = n - k - 1 = n - l - 1$ , we get  $\tilde{H}_i(\mathcal{S}^n - X) \simeq \mathbb{Z}^2$ , and for other  $i$  it is again 0.

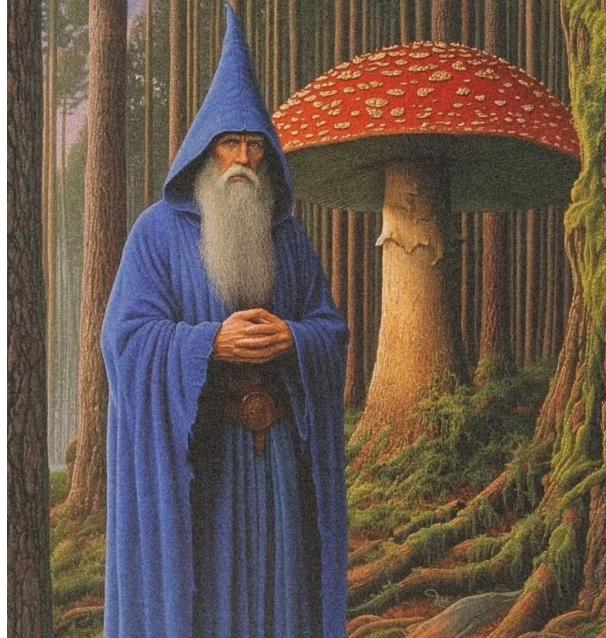
label=(b) **Dokończyć**

□

### 3.4.3. Simplicial Approximation.

**Theorem 2C.3.** If  $X$  is a finite simplicial complex, or more generally a retract of a finite simplicial complex, and  $f: X \rightarrow X$  is a map with  $\tau(f) \neq 0$ , then  $f$  has a fixed point.

$$\tau(f) := \sum_{n=0}^{\infty} (-1)^n \text{tr}(f_* : H_n X \rightarrow H_n X)$$



## 4. COHOMOLOGY

### 4.1. Cohomology Groups.

### 4.2. Cup Product.

**Theorem 3.2.** If a chain complex  $C$  of free abelian groups has homology groups  $H_n(C)$ , then the cohomology groups  $H^n(C; G)$  of the cochain complex  $\text{Hom}(C_n, G)$  are determined by split exact sequences

$$0 \rightarrow \text{Ext}(H_{n-1}(C), G) \rightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n(C), G) \rightarrow 0$$

□



### 4.3. Poincare Duality.

### 4.4. Additional Topics.

#### 4.4.1. Universal Coefficients for Homology.

#### 4.4.2. The General Künneth Formula.

#### 4.4.3. $H$ -Spaces and Hopf Algebras.

#### 4.4.4. The Cohomology of $\text{SO}(n)$ .

#### 4.4.5. Bockstein Homomorphisms.

#### 4.4.6. Limits and Ext.

#### 4.4.7. Transfer Homomorphisms.

#### 4.4.8. Local Coefficients.

### REFERENCES

- [Hat01] Allen Hatcher. *Algebraic Topology*. en. Cambridge, England: Cambridge University Press, Dec. 2001 (cit. on pp. 1, 5).

