HATCHER SOLUTIONS

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ABSTRACT. We present the solutions to the excercises found in Allen Hatcher's „Algebraic Topology" [Hat01] with our own perspective and commentary. We also present proofs of selected theorems (with what we hope is a more detailed explanation).

Dedicated to Jan Dymara, Światosław Gal & Jacek Świątkowski

1. Introduction

Blabla że co w ogóle robimy...

We denote a k-sphere by $\mathcal{S}^k$, k-disk by $\mathcal{D}^k$ and a $[0,1]$ segment by $\mathcal{I}$. By $\simeq$ we mean homotopy or, when placed between spaces, homotopy equivalence when placed between spaces. We denote homeomorphisms by $\approx$.

2. The Fundamental Group

2.1. Basic Constructions.

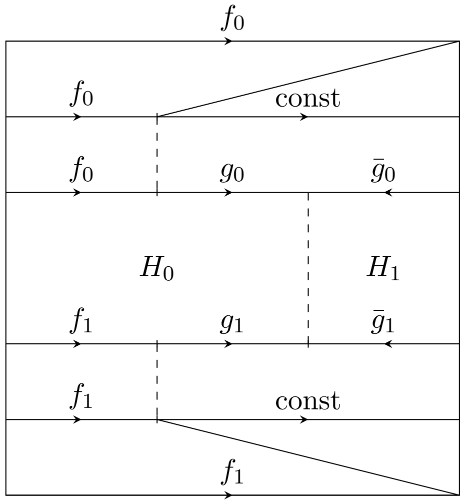
This section is about basic constructions.

Należałoby poprawić numerację problemów, żeby była tak jak w hatcherze, no i też żeby sekcje w Additional Topics nie były 2.4 itp tylko 2.A, 2.B itd

Problem 2.1. *Show that composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.*

Proof. The homotopy assumptions imply that g_i have the same endpoints, and so do the $f_i g_i$, which means the f_i have the same endpoints as well.

Since $g_0 \simeq g_1$, it is clear that $\bar{g_0} \simeq \bar{g_1}$. We proceed as shown on the homotpy diagram:



Algebraically, this corresponds to the computations

$$f_0 \simeq f_0 \operatorname{const}_{x_0} \simeq f_0(g_0\overline{g_0}) \simeq (f_0g_0)\overline{g_0} \simeq (f_1g_1)\overline{g_1} \simeq f_1(g_1\overline{g_1}) \simeq f_1 \operatorname{const}_{x_0} \simeq f_1,$$

where x_0 denotes the enpoint of the paths f_i. ☐

Problem 2.2. *Show that the change-of-basepoint homomorphism β_h depends only on the homotopy class of h.*

Proof. It suffices to show that for every two homotopic loops h and h' we have $\beta_h = \beta_{h'}$. Let H denote the homotopy $h \simeq_H h'$. Take any loop g. We have that $\beta_h(g) = hg\overline{h} \simeq h'g\overline{h'} = \beta_{h'}(g)$.
obrazek ☐

Problem 2.3. *For a path-connected space X, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphism β_h depend only on the endpoints of the path h.*

Proof. ($\Rightarrow$) Consider any two paths h, h' from points $y \in X$ to $x \in X$. We want to show that $\beta_h = \beta_{h'}$. Consider any loop $g \in \pi_1(X, x)$ and a loop $\bar{h}h'$ which also is in $\pi_1(X, x)$. Since $\pi_1(X)$ is abelian then $\pi_1(X, x)$ is also abelian. And so it follows that:

$$g \cdot \bar{h}h' \simeq \bar{h}h' \cdot g$$

Now multiply it by h on left hand side, and by $\bar{h}'$ on right hand side to get:

$$hg\bar{h} \simeq h'g\bar{h}'$$

That is exactly equivalent to $\beta_h = \beta_{h'}$.

($\Leftarrow$) Suppose that for any two paths from x to y they induce the same maps $\beta_h, \beta_{h'}$. In particular given any two loops g, h in $\pi_1(X, x_0)$ we have $\beta_g = \beta_h$. And so it follows that:

$$hg\bar{h} = \beta_h(g) = \beta_g(g) = g$$

Thus $gh = hg$ which means that $\pi_1(X, x)$ is abelian, and so of course $\pi_1(X)$ is abelian too. □

Problem 2.4. *A subspace $X \subseteq \mathbb{R}^n$ is said to be* star-shaped *if there is a point $x_0 \in X$ such that for each $x \in X$ the line segment from x_0 to x lies in X. Show that if a subspace $X \subseteq \mathbb{R}^n$ is locally star-shaped, in the sense that every point of X has a star-shaped neighbourhood in X, then every path in X is a piecewise linear path, that is, a path consisting of a finite number of straight lines traversed at constant speed. Show that this applies in particular when X is open or when X is a union of finitely many closed convex subsets.*

Proof. In the case that X is star-shaped as a whole, it is contractible, so the thesis definitely holds. Denote the path in question by γ. In general, we can use the compactness of the image of γ to cover it with finitely many star-shaped sets. However, in each of these we cannot just contract the curve, as we must keep the endpoints intact.

Instead we show the following: if X is star-shaped with *star-point* x_0, we can deform any path γ in X to a linear path going from $\gamma(0)$ to x_0 and then from x_0 to $\gamma(1)$. Without loss of generality $x_0 = 0$. The homotopy given by **Obrazek**

$$H_1(s, t) = \Big[(1 - t) + t|1 - 2s|\Big]\gamma(s)$$

achieves this partially, as it deforms $\gamma(s)$ to $|1 - 2s|\gamma(s)$, which is equal to $\gamma(0), x_0, \gamma(1)$ at $s = 0, \frac{1}{2}, 1$. Now we turn to making the segments linear.

If δ is the first half of γ, the homotopy H_1 deforms it to the path $(1 - s)\delta(s)$. We will deform it to $s\delta(0)$, which is certainly constant-speed. This is achieved through the homotopy

$$H_2(s, t) = (1 - s)\delta\big((1 - t)s\big).$$

Finally, applying the same procedure to the second half of γ, we will have indeed deformed γ to a piecewise-linear path consisting of two segments. □

Open sets. In an open set, every point has a neighbourhood which is a Euclidean ball, so the space X is locally star-shaped. □

Finite unions of convex sets. Call a subspace $X \subseteq \mathbb{R}^n$ *strictly locally star-shaped* iff every $x \in X$ has a star-shaped neighbourhood *for which it is the star-point.* Every convex set satisfies this property. We will show that if A and B are closed and strictly locally star-shaped, then so is their union.

Pick a point $x \in A \cup B$. Without loss of generality, $x \in A$. Then either $x \in B$ or x is separated from B. In the first case, for some small r we have a neighbourhood

$$B_r(x) \cap X = (B_r(x) \cap A) \cup (B_r(x) \cap B),$$

which is the union of star-shaped neighbourhood with the same star-point, so it is star shaped. In the second case, for some small r we have

$$B_r(x) \cap X = B_r(x) \cap A,$$

so x has a star-shaped neighbourhood. □

Problem 2.5. *Show that for a space X, the following three conditions are equivalent:*

(a) Every map $\mathcal{S}^1 \longrightarrow X$ is homotopic to a constant map, with image a point.
(b) Every map $\mathcal{S}^1 \longrightarrow X$ extends to a map $\mathcal{D}^2 \longrightarrow X$.
(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.
Deduce that a space X is simply-connected all maps $\mathcal{S}^1 \longrightarrow X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

Proof. $(a \Rightarrow b)$ Take any $f : \mathcal{S}^1 \to X$. We have that $f \simeq_H \mathrm{const}_{x_0}$ for some $x_0 \in X$, with a homotopy $H : \mathcal{S}^1 \times \mathcal{I} \to X$. Since $H(x, 0) = f(x)$ and $H(x, 1) = x_0$ we can consider the quotient of $\mathcal{S}^1 \times \mathcal{I}$ by the relation identifying $\mathcal{S}^1 \times \{1\}$ with a single point. Such a cone-like space is clearly homeomorphic with $\mathcal{D}^2$. Let $j : \mathcal{S}^1 \times \mathcal{I} \to \mathcal{D}^2$ be the quotient map. We have the induced map H' as shown in the diagram:

$$\begin{tikzcd}
\mathcal{S}^1 \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathcal{S}^1 \times \mathcal{I} \arrow[r, two heads, "j"] \arrow[d, "H"] & \mathcal{D}^2 \arrow[dl, "H'"] \\
& X &
\end{tikzcd}$$

Now observe that $H' \circ j \circ i$ extends f to a disk.

$(b \Rightarrow c)$ **obrazek** Take any $g \in \pi_1(X, x_0)$. It can be represented as a map $g : \mathcal{S}^1 \to X$ and by assumption we get $\bar{g} : \mathcal{D}^2 \to X$ which extends g. Suppose that $s \in \mathcal{D}^2$ corresponds to the base point x_0 by g. Since disks are contractible we get homotopy H that in 0 is the boundary sphere $\mathcal{S}^1$ and in 1 it is a point in the middle of D^2 and call it m. Now consider a path h from point s to m as in the diagram. The homotopy H contracts path h to a constant path in m. We will now show that $\beta_h(g) = 0 \in \pi_1(X, \bar{g}(m))$. Since $\beta_h(g) = hg\bar{h}$ we conclude that H contracts $hg\bar{h}$ to a point $\bar{g}(m)$, which is exactly representation of 0 in $\pi_1(X, \bar{g}(m))$. Because β_h is isomorphism we see that g must be in the same class as trivial map in $\pi_1(X, x_0)$.

$(c \Rightarrow a)$
Since $\pi_1(X, x_0)$ is trivial, every loop is homotopic to a constant loop. Lets call this homotopy H. Since every loop can be represented as a map $\mathcal{S}^1 \to X$, by simply taking the homotopy H we get the thesis. □

Problem 2.6. *We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(\mathcal{S}^1, s_0) \to (X, x_0)$. Let $[\mathcal{S}^1, X]$ be the set of homotopy classes of maps $\mathcal{S}^1 \to X$ with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [\mathcal{S}^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[\mathcal{S}^1, X]$ and the set of conjugacy classes in $\pi_1(X)$ when X is path-connected.*

Problem 2.7. *Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the **Borsuk–Ulam theorem** to show that there is one plane $P \subseteq \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.*

Let us introduce some notation. Let λ denote the measure we are interested in. We can think of the set of all planes in $\mathbb{R}^3$, as $\mathcal{S}^2 \times \mathbb{R}$, thus for a plane P, let $P_{n,t}$ denote the plane with a normal vector $n \in \mathcal{S}^2$, moved along it by $t \in \mathbb{R}$. We denote by $P^+_{n,t}, P^-_{n,t}$ two halves of $\mathbb{R}^3$, respectively above and below the $P_{n,t}$ (in the sense of direction of n). We will need the following lemma:

Lemma 2.8. *Let $A \subseteq \mathbb{R}^3$ be a compact subset, there exists a continous, **even** map $h : \mathcal{S}^2 \to \mathbb{R}$, such that:*

$$\lambda(P^+_{n,h(n)} \cap A) = \lambda(P^-_{n,h(n)} \cap A)$$

The idea behind the lemma is to find a continous family of planes, each equally dividing given set. **Zauważyliśmy, że potrzeba żeby ta mapa była parzysta.**

Proof. *(of the Lemma 2.8)* **TODO** □

Proof. **utworzyć begin-solution i tutaj zmienić)** Construct $h : \mathcal{S}^2 \to \mathbb{R}$ as in Lemma 2.8 for A_1. Let $g : \mathcal{S}^2 \to \mathbb{R}^2$ be defined as follows:

$$\begin{array}{ccc} \mathcal{S}^2 & \xrightarrow{\quad g \quad} & \mathbb{R}^2 \\ \Cup & & \Cup \\ n & \longmapsto & \begin{pmatrix} \lambda(P^+_{n,h(n)} \cap A_2) \\ \lambda(P^+_{n,h(n)} \cap A_3) \end{pmatrix} \end{array}$$

Ten diagram to tylko fleks z użycia quivera, można to napisać normalnie i pewnie będzie ładniej.
Since h is continous, g is continous as well. By the **Borsuk-Ulam theorem**, there exists some $n \in \mathcal{S}^2$, such that:

$$\begin{pmatrix} \lambda(P^+_{n,h(n)} \cap A_2) \\ \lambda(P^+_{n,h(n)} \cap A_3) \end{pmatrix} = \begin{pmatrix} \lambda(P^+_{-n,h(-n)} \cap A_2) \\ \lambda(P^+_{-n,h(-n)} \cap A_3) \end{pmatrix} = \begin{pmatrix} \lambda(P^-_{n,h(n)} \cap A_2) \\ \lambda(P^-_{n,h(n)} \cap A_3) \end{pmatrix}$$

The last equality follows from evenness of h, and the fact that $P^+_{-n,t} = P^-_{n,t}$. **Zamiast tych pmatrixów można pisać A_i** □

Problem 2.9. *Show that the isomorphism $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ in [Hat01, Proposition 1.12] is given by $[f] \mapsto (p_{1*}([f]), p_{2*}([f]))$ where p_1 and p_2 are the projections of $X \times Y$ onto its two factors.*

Problem 2.10. *Given a map $f : X \to Y$ and a path $h : \mathcal{I} \to X$ from x_0 to x_1, show that $f_*\beta_h = \beta_{fh} f_*$, namely that the diagram below commutes.*

$$\begin{tikzcd} \pi_1(X, x_1) \arrow[r, "\beta_h"] \arrow[d, "f_*"'] & \pi_1(X, x_0) \arrow[d, "f_*"] \\ \pi_1(Y, f(x_1)) \arrow[r, "\beta_{fh}"'] & \pi_1(Y, f(x_0)) \end{tikzcd}$$

Proof. Take any loop $\gamma \in \pi_1(X, x_1)$.

$$f_*\beta_h([\gamma]) = f_*([h\gamma\bar{h}]) = [f \circ h\gamma\bar{h}] = [(f \circ h)(f \circ \gamma)\overline{(f \circ h)}] = \beta_{fh}([f \circ \gamma]) = \beta_{fh} f_*([\gamma])$$

□

Problem 2.11. *Show that there are no retractions $r : X \to A$ in the following cases:*
(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to $\mathcal{S}^1$.
(b) $X = \mathcal{S}^1 \times \mathcal{D}^2$ with A its boundry torus $\mathcal{S}^1 \times \mathcal{S}^1$.
(c) $X = \mathcal{S}^1 \times \mathcal{D}^2$ and A the circle shown in the figure
rysunek
(d) $X = \mathcal{D}^2 \vee \mathcal{D}^2$ with A its boundry $\mathcal{S}^1 \vee \mathcal{S}^1$.
(e) X a disk with two points on its boundry identified and A its boundry $\mathcal{S}^1 \vee \mathcal{S}^1$.
(f) X the Möbius band and A its boundry circle.

Problem 2.12. *Construct infinitely many nonhomotopic retractions $\mathcal{S}^1 \vee \mathcal{S}^1 \to \mathcal{S}^1$.*

Proof. Let r_n be the retraction that fixes the left circle and winds the right circle in the wedge sum n times around the right. Then, if γ is the path that winds around the right circle once

$$(r_n)_*[\gamma] = [\delta]^n,$$

so the retractions r_n induce different homomorphisms. Suppose now that for some $m \neq n$ we have $r_m \simeq r_n$. Then by Lemma 1.19. **dodać odnośnik** we have

$$(r_n)_\star = \beta_h (r_m)_\star,$$

where h is the path traced by basepoint under the homotopy. Since both r_n and r_m fix the basepoint and $\pi_1 \mathcal{S}^1$ is abelian, by **dodać odnośnik** Problem 1.1.3. β_h does nothing, to $(r_n)_\star = (r_m)_\star$.

□

Problem 2.13. *Using [Hat01, Lemma 1.15], show that if a space X obtained from a path-connected subspace A by attaching a cell e^n with $n \geqslant 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Apply this to show:*

(a) The wedge sum $\mathcal{S}^1 \vee \mathcal{S}^2$ has a fundamental group $\mathbb{Z}$.
(b) For a path-connected CW-complex X the inclusion map $X^{(1)} \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^{(1)}) \to \pi_1(X)$.

2.2. Van Kampen's Theorem.

2.3. Covering Spaces. Covering Spacey – mem z kevinem nad okręgiem

Problem 2.14. *Show that if a path-connected, locally path-connected path-connected space X has $\pi_1(X)$ finite, then every map $X \to \mathcal{S}^1$ is nullhomotopic.*

Proof. Denote the map by $f : X \to \mathcal{S}^1$. We will use the covering space $p : \mathbb{R} \to \mathcal{S}^1$. Since $\pi_1(X)$ is finite, every element $g \in \pi_1(X)$ has finite order, so does $f_\star(g)$, so it is the identity element. Therefore the image $f_\star(\pi_1(X))$ is the trivial subgroup, so f has a lift $\widetilde{f} : X \to \mathbb{R}$. As $\mathbb{R}$ is contractible, $\widetilde{f}$ is nullhomotopic, hence so is $f = p \circ \widetilde{f}$. □

2.4. Additional Topics.

2.4.1. *Graphs and Free Groups.*

2.4.2. *$\mathcal{K}(G, 1)$-Spaces and Graphs of Groups.*

3. Homology

3.1. Simplicial and Singular Homology.

3.2. Computations and Applications.

3.3. The Formal Viewpoint.

3.4. Additional Topics.

3.4.1. *Homology and Fundamental Group.*

3.4.2. *Classical Applications.*

Proposition 3.1.

(1) *For* $\mathcal{D}^k \simeq \Delta \subseteq \mathcal{S}^n$*, for all* i*:* $\tilde{H}_i(\mathcal{S}^n - \Delta) = 0$
(2) *For* $\mathcal{S}^k \simeq \Sigma \subseteq \mathcal{S}^n$*,* $\tilde{H}_i(\mathcal{S}^n - \Sigma) = \mathbb{Z}$ *for* $i = n - k - 1$*, and* 0 *otherwise.*

Problem 3.2. *Compute* $H_i(\mathcal{S}^n - X)$ *when* X *is a subspace of* $\mathcal{S}^n$ *homeomorphic to* $\mathcal{S}^k \vee \mathcal{S}^l$ *or to* $\mathcal{S}^k \sqcup \mathcal{S}^l$*.*

Proof. In both cases we will use Prop. 3.1 and **Mayer-Vietoris sequence** (**tu sformułować jakoś ładnie MV i dać odnośnik - to w sam raz na tekst nasz w rozdziale 3.2, no bo w Hatcherze to nie jest ujęte w żadne proposition**)

label=(a) Let $X \simeq \mathcal{S}^k \vee \mathcal{S}^l$. In order to use the **Mayer-Vietoris** we need to define:

$$A := \mathcal{S}^n - \mathcal{S}^k, \qquad B := \mathcal{S}^n - \mathcal{S}^l$$

Observe, that

$$A \cap B = \mathcal{S}^n - X, \qquad A \cup B = \mathcal{S}^n - \{\star\} \simeq \mathbb{R}^n$$

By **odnośnik do Mayera**, we get:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \tilde{H}_{i+1}(A \cup B) & \longrightarrow & \tilde{H}_i(A \cap B) & \longrightarrow & \tilde{H}_i(A) \oplus \tilde{H}_i(B) & \longrightarrow & \tilde{H}_i(A \cup B) & \longrightarrow & \dots \\
 & & \| & & \| & & \| & & \| & & \\
\dots & \longrightarrow & \tilde{H}_{i+1}(\mathbb{R}^n) & \longrightarrow & \tilde{H}_i(\mathcal{S}^n - X) & \longrightarrow & \begin{array}{c}\tilde{H}_i(\mathcal{S}^n - \mathcal{S}^k) \\ \oplus \\ \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^l)\end{array} & \longrightarrow & \tilde{H}_i(\mathbb{R}^n) & \longrightarrow & \dots
\end{array}$$

Because $\mathbb{R}^n$ is contractible, its homology groups are trivial, thus we get an isomorphism for every i:

$$\tilde{H}_i(\mathcal{S}^n - X) \simeq \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^k) \oplus \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^l)$$

Assuming $k \neq l$, for $i = n - k - 1$ and $i = n - l - 1$, we get that $\tilde{H}_i(\mathcal{S}^n - X) \simeq \mathbb{Z}$ for other i it is 0. Otherwise, that is when $k = l$, for $i = n - k - 1 = n - l - 1$, we get $\tilde{H}_i(\mathcal{S}^n - X) \simeq \mathbb{Z}^2$, and for other i it is again 0.

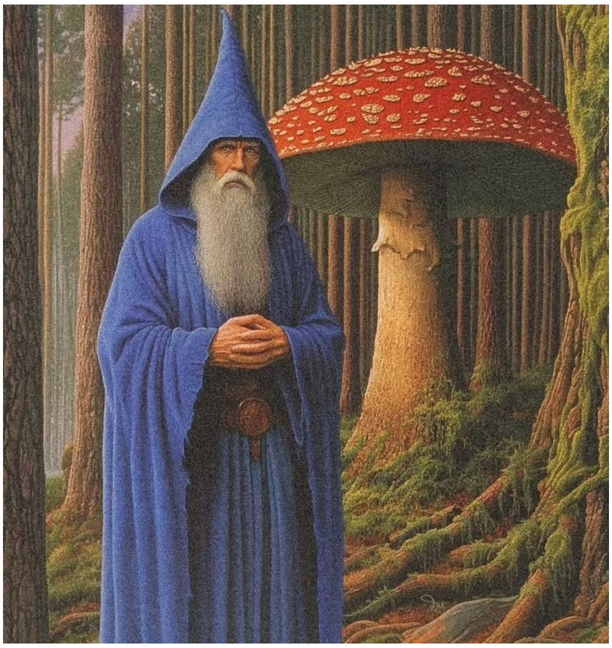
lbbel=(b) **Dokończyć**

□

3.4.3. *Simplicial Approximation.*

Theorem 2C.3. *If X is a finite simplicial complex, or more generally a retract of a finite simplicial complex, and $f: X \to X$ is a map with $\tau(f) \neq 0$, then f has a fixed point.*

$$\tau(f) := \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}(f_* : H_n X \to H_n X)$$



4. Cohomology

4.1. Cohomology Groups.

4.2. Cup Product.

Theorem 3.2. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined by split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \longrightarrow 0 \qquad \square$$

4.3. Poincare Duality.

4.4. Additional Topics.

4.4.1. *Universal Coefficients for Homology.*

4.4.2. *The General Künneth Formula.*

4.4.3. *H-Spaces and Hopf Algebras.*

4.4.4. *The Cohomology of* $\mathrm{SO}(n)$.

4.4.5. *Bockstein Homomorphisms.*

4.4.6. *Limits and* Ext.

4.4.7. *Transfer Homomorphisms.*

4.4.8. *Local Coefficients.*

References

[Hat01] Allen Hatcher. *Algebraic Topology*. en. Cambridge, England: Cambridge University Press, Dec. 2001 (cit. on pp. 1, 5, 6).