

HATCHER SOLUTIONS

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ABSTRACT. We present the solutions to the excercises found in Allen Hatcher's „Algebraic Topology” [Hat01] with our own perspective and commentary. We also present proofs of selected theorems (with what we hope is a more detailed explanation).



Dedicated to Jan Dymara, Świątosław Gal & Jacek Świątkowski

1. INTRODUCTION

Blabla że co w ogóle robimy...

We denote a k -sphere by S^k , k -disk by D^k and a $[0, 1]$ segment by \mathcal{I} . By \simeq we mean homotopy or, when placed between spaces, homotopy equivalence when placed between spaces. We denote homeomorphisms by \approx .

2. THE FUNDAMENTAL GROUP

2.1. Basic Constructions. This section is about basic constructions.

Należałooby poprawić numerację problemów, żeby była tak jak w hatcherze, no i też żeby sekcje w Additional Topics nie były 2.4 itp tylko 2.A, 2.B itd

Problem 2.1. *Show that composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.*

Proof. The homotopy assumptions imply that g_i have the same endpoints, and so do the $f_i g_i$, which means the f_i have the same endpoints as well.

Since $g_0 \simeq g_1$, it is clear that $\bar{g}_0 \simeq \bar{g}_1$. We proceed as shown on the homotpy diagram:

obrazek

Algebraically, this corresponds to the computations

$$f_0 \simeq f_0 \text{ const}_{x_0} \simeq f_0(g_0 \bar{g}_0) \simeq (f_0 g_0) \bar{g}_0 \simeq (f_1 g_1) \bar{g}_1 \simeq f_1(g_1 \bar{g}_1) \simeq f_1 \text{ const}_{x_0} \simeq f_1,$$

where x_0 denotes the enpoint of the paths f_i . \square

Problem 2.2. *Show that the change-of-basepoint homomorphism β_h depends only on the homotopy class of h .*

Proof. It suffices to show that for every two homotopic loops h and h' we have $\beta_h = \beta_{h'}$. Let H denote the homotopy $h \simeq_H h'$. Take any loop g . We have that $\beta_h(g) = hg\bar{h} \simeq h'g\bar{h}' = \beta_{h'}(g)$. \square

Problem 2.3. *For a path-connected space X , show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphism β_h depend only on the endpoints of the path h .*

Proof. (\Rightarrow) Consider any two paths h, h' from points $y \in X$ to $x \in X$. We want to show that $\beta_h = \beta_{h'}$. Consider any loop $g \in \pi_1(X, x)$ and a loop $\bar{h}h'$ which also is in $\pi_1(X, x)$. Since $\pi_1(X)$ is abelian then $\pi_1(X, x)$ is also abelian. And so it follows that:

$$g \cdot \bar{h}h' \simeq \bar{h}h' \cdot g$$

Now multiply it by h on left hand side, and by \bar{h}' on right hand side to get:

$$hg\bar{h} \simeq h'g\bar{h}'$$

That is exactly equivalent to $\beta_h = \beta_{h'}$.

(\Leftarrow) Suppose that for any two paths from x to y they induce the same maps $\beta_h, \beta_{h'}$. In particular given any two loops g, h in $\pi_1(X, x_0)$ we have $\beta_g = \beta_h$. And so it follows that:

$$hg\bar{h} = \beta_h(g) = \beta_g(g) = g$$

Thus $gh = hg$ which means that $\pi_1(X, x)$ is abelian, and so of course $\pi_1(X)$ is abelian too. \square

Problem 2.4. *A subspace $X \subseteq \mathbb{R}^n$ is said to be star-shaped if there is a point $x_0 \in X$ such that for each $x \in X$ the line segment from x_0 to x lies in X . Show that if a subspace $X \subseteq \mathbb{R}^n$ is locally star-shaped, in the sense that every point of X has a star-shaped neighbourhood in X , then every path in X is a piecewise linear path, that is, a path consisting of a finite number of straight lines traversed at constant speed. Show that this applies in particular when X is open or when X is a union of finitely many closed convex subsets.*

Proof. In the case that X is star-shaped as a whole, it is contractible, so the thesis definitely holds. Denote the path in question by γ . In general, we can use the compactness of the image of γ to cover it with finitely many star-shaped sets. However, in each of these we cannot just contract the curve, as we must keep the endpoints intact.

Instead we show the following: if X is star-shaped with *star-point* x_0 , we can deform any path γ in X to a linear path going from $\gamma(0)$ to x_0 and then from x_0 to $\gamma(1)$. Without loss of generality $x_0 = 0$. The homotopy given by **Obrazek**

$$H_1(s, t) = \left[(1-t) + t|1-2s| \right] \gamma(s)$$

achieves this partially, as it deforms $\gamma(s)$ to $|1-2s|\gamma(s)$, which is equal to $\gamma(0), x_0, \gamma(1)$ at $s = 0, \frac{1}{2}, 1$. Now we turn to making the segments linear.

If δ is the first half of γ , the homotopy H_1 deforms it to the path $(1-s)\delta(s)$. We will deform it to $s\delta(0)$, which is certainly constant-speed. This is achieved through the homotopy

$$H_2(s, t) = (1-s)\delta((1-t)s).$$

Finally, applying the same procedure to the second half of γ , we will have indeed deformed γ to a piecewise-linear path consisting of two segments. \square

Open sets. In an open set, every point has a neighbourhood which is a Euclidean ball, so the space X is locally star-shaped. \square

Finite unions of convex sets. Call a subspace $X \subseteq \mathbb{R}^n$ *strictly locally star-shaped* iff every $x \in X$ has a star-shaped neighbourhood *for which it is the star-point*. Every convex set satisfies this property. We will show that if A and B are closed and strictly locally star-shaped, then so is their union.

Pick a point $x \in A \cup B$. Without loss of generality, $x \in A$. Then either $x \in B$ or x is separated from B . In the first case, for some small r we have a neighbourhood

$$B_r(x) \cap X = (B_r(x) \cap A) \cup (B_r(x) \cap B),$$

which is the union of star-shaped neighbourhood with the same star-point, so it is star shaped. In the second case, for some small r we have

$$B_r(x) \cap X = B_r(x) \cap A,$$

so x has a star-shaped neighbourhood. \square

Problem 2.5. Show that for a space X , the following three conditions are equivalent:

- (a) Every map $S^1 \rightarrow X$ is homotopic to a constant map, with image a point.
- (b) Every map $S^1 \rightarrow X$ extends to a map $D^2 \rightarrow X$.
- (c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected all maps $S^1 \rightarrow X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

Proof. ($a \Rightarrow b$) Take any $f : S^1 \rightarrow X$. We have that $f \simeq_H \text{const}_{x_0}$ for some $x_0 \in X$, with a homotopy $H : S^1 \times I \rightarrow X$. Since $H(x, 0) = f(x)$ and $H(x, 1) = x_0$ we can consider the quotient of $S^1 \times I$ by the relation identifying $S^1 \times \{1\}$ with a single point. Such a cone-like space is clearly homeomorphic with D^2 . Let $j : S^1 \times I \rightarrow D^2$ be the quotient map. We have the induced map H' as shown in the diagram:

$$\begin{array}{ccccc} S^1 & \xhookrightarrow{i} & S^1 \times I & \xrightarrow{j} & D^2 \\ & \searrow f & \downarrow H & \swarrow & \\ & & X & & \end{array}$$

Now observe that $H' \circ j \circ i$ extends f to a disk.

($b \Rightarrow c$) **obrazek** Take any $g \in \pi_1(X, x_0)$. It can be represented as a map $g : S^1 \rightarrow X$ and by assumption we get $\bar{g} : D^2 \rightarrow X$ which extends g . Suppose that $s \in D^2$ corresponds to the base point x_0 by g . Since disks are contractible we get homotopy H that in 0 is the boundary sphere S^1 and in 1 it is a point in the middle of D^2 and call it m . Now consider a path h from point s to m as in the diagram. The homotopy H contracts path h to a constant path in m . We will now show that $\beta_h(g) = 0 \in \pi_1(X, \bar{g}(m))$. Since $\beta_h(g) = h\bar{g}\bar{h}$ we conclude that H contracts $h\bar{g}\bar{h}$ to a point $\bar{g}(m)$, which is exactly representation of 0 in $\pi_1(X, \bar{g}(m))$. Because β_h is isomorphism we see that g must be in the same class as trivial map in $\pi_1(X, x_0)$.

$(c \Rightarrow a)$

Since $\pi_1(X, x_0)$ is trivial, every loop is homotopic to a constant loop. Lets call this homotopy H . Since every loop can be represented as a map $S^1 \rightarrow X$, by simply taking the homotopy H we get the thesis. \square

Problem 2.6. *We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \rightarrow (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \rightarrow X$ with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \rightarrow [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$ when X is path-connected.*

Problem 2.7. *Let A_1, A_2, A_3 be compact sets in \mathbb{R}^3 . Use the **Borsuk–Ulam theorem** to show that there is one plane $P \subseteq \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.*

Let us introduce some notation. Let λ denote the measure we are interested in. We can think of the set of all planes in \mathbb{R}^3 , as $S^2 \times \mathbb{R}$, thus for a plane P , let $P_{n,t}$ denote the plane with a normal vector $n \in S^2$, moved along it by $t \in \mathbb{R}$. We denote by $P_{n,t}^+, P_{n,t}^-$ two halves of \mathbb{R}^3 , respectively above and below the $P_{n,t}$ (in the sense of direction of n). We will need the following lemma:

Lemma 2.8. *Let $A \subseteq \mathbb{R}^3$ be a compact subset, there exists a continuous, even map $h : S^2 \rightarrow \mathbb{R}$, such that:*

$$\lambda(P_{n,h(n)}^+ \cap A) = \lambda(P_{n,h(n)}^- \cap A)$$

The idea behind the lemma is to find a continous family of planes, each equally dividing given set. **Zauważliśmy, że potrzeba żeby ta mapa była parzysta.**

Proof. (of the Lemma 2.8) TODO \square

Proof. utworzyć begin-solution i tutaj zmienić Construct $h : S^2 \rightarrow \mathbb{R}$ as in Lemma 2.8 for A_1 . Let $g : S^2 \rightarrow \mathbb{R}^2$ be defined as follows:

$$\begin{array}{ccc} S^2 & \xrightarrow{g} & \mathbb{R}^2 \\ \Downarrow & & \Downarrow \\ n & \longmapsto & \left(\begin{array}{l} \lambda(P_{n,h(n)}^+ \cap A_2) \\ \lambda(P_{n,h(n)}^+ \cap A_3) \end{array} \right) \end{array}$$

Ten diagram to tylko fleks z użycia quivera, można to napisać normalnie i pewnie będzie ładniej.

Since h is continuous, g is continuous as well. By the **Borsuk-Ulam theorem**, there exists some $n \in S^2$, such that:

$$\begin{pmatrix} \lambda(P_{n,h(n)}^+ \cap A_2) \\ \lambda(P_{n,h(n)}^+ \cap A_3) \end{pmatrix} = \begin{pmatrix} \lambda(P_{-n,h(-n)}^+ \cap A_2) \\ \lambda(P_{-n,h(-n)}^+ \cap A_3) \end{pmatrix} = \begin{pmatrix} \lambda(P_{n,h(n)}^- \cap A_2) \\ \lambda(P_{n,h(n)}^- \cap A_3) \end{pmatrix}$$

The last equality follows from evenness of h , and the fact that $P_{-n,t}^+ = P_{n,t}^-$. **Zamiast tych pmatrixów można pisać A_i** \square

Problem 2.9. Show that the isomorphism $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ in [Hat01, Proposition 1.12] is given by $[f] \mapsto (p_{1*}([f]), p_{2*}([f]))$ where p_1 and p_2 are the projections of $X \times Y$ onto its two factors.

Problem 2.10. Given a map $f : X \rightarrow Y$ and a path $h : I \rightarrow X$ from x_0 to x_1 , show that $f_*\beta_h = \beta_{fh}f_*$, namely that the diagram below commutes.

$$\begin{array}{ccc} \pi_1(X, x_1) & \xrightarrow{\beta_h} & \pi_1(X, x_0) \\ f_* \downarrow & & \downarrow f_* \\ \pi_1(Y, f(x_1)) & \xrightarrow{\beta_{fh}} & \pi_1(Y, f(x_0)) \end{array}$$

Proof. Take any loop $\gamma \in \pi_1(X, x_1)$.

$$f_*\beta_h([\gamma]) = f_*([h\gamma\bar{h}]) = [f \circ h\gamma\bar{h}] = [(f \circ h)(f \circ \gamma)\overline{(f \circ h)}] = \beta_{fh}([f \circ \gamma]) = \beta_{fh}f_*([\gamma])$$

\square

Problem 2.11. Show that there are no retractions $r : X \rightarrow A$ in the following cases:

- (a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1 .
- (b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
- (c) $X = S^1 \times D^2$ and A the circle shown in the figure
rysunek
- (d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
- (e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.
- (f) X the Möbius band and A its boundary circle.

Problem 2.12. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \rightarrow S^1$.

Problem 2.13. Using [Hat01, Lemma 1.15], show that if a space X obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1 . Apply this to show:

- (a) The wedge sum $S^1 \vee S^2$ has a fundamental group \mathbb{Z} .
- (b) For a path-connected CW-complex X the inclusion map $X^{(1)} \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^{(1)}) \rightarrow \pi_1(X)$.

2.2. Van Kampen's Theorem.

2.3. Covering Spaces. **Covering Spacey – mem z kevinem nad okregiem**

Problem 2.14. Show that if a path-connected, locally path-connected path-connected space X has $\pi_1(X)$ finite, then every map $X \rightarrow S^1$ is nullhomotopic.

Proof. Denote the map by $f : X \rightarrow S^1$. We will use the covering space $p : \mathbb{R} \rightarrow S^1$. Since $\pi_1(X)$ is finite, every element $g \in \pi_1(X)$ has finite order, so does $f_*(g)$, so it is the identity element. Therefore the image $f_*(\pi_1(X))$ is the trivial subgroup, so f has a lift $\tilde{f} : X \rightarrow \mathbb{R}$. As \mathbb{R} is contractible, \tilde{f} is nullhomotopic, hence so is $f = p \circ \tilde{f}$. \square

2.4. Additional Topics.

2.4.1. Graphs and Free Groups.

2.4.2. $\mathcal{K}(G, 1)$ -Spaces and Graphs of Groups.

3. HOMOLOGY

3.1. Simplicial and Singular Homology.

3.2. Computations and Applications.

3.3. The Formal Viewpoint.

3.4. Additional Topics.

3.4.1. Homology and Fundamental Group.

3.4.2. Classical Applications.

Proposition 3.1.

- (1) For $\mathcal{D}^k \simeq \Delta \subseteq \mathcal{S}^n$, for all i : $\tilde{H}_i(\mathcal{S}^n - \Delta) = 0$
- (2) For $\mathcal{S}^k \simeq \Sigma \subseteq \mathcal{S}^n$, $\tilde{H}_i(\mathcal{S}^n - \Sigma) = \mathbb{Z}$ for $i = n - k - 1$, and 0 otherwise.

Problem 3.2. Compute $H_i(\mathcal{S}^n - X)$ when X is a subspace of \mathcal{S}^n homeomorphic to $\mathcal{S}^k \vee \mathcal{S}^l$ or to $\mathcal{S}^k \sqcup \mathcal{S}^l$.

Proof. In both cases we will use Prop. 3.1 and Mayer-Vietoris sequence (**tu sformułować jakoś ładnie MV i dać odnośnik - to w sam raz na tekst nasz w rozdziale 3.2, no bo w Hatcherze to nie jest ujęte w żadne proposition**)

label=(a) Let $X \simeq \mathcal{S}^k \vee \mathcal{S}^l$. In order to use the Mayer-Vietoris we need to define:

$$A := \mathcal{S}^n - \mathcal{S}^k, \quad B := \mathcal{S}^n - \mathcal{S}^l$$

Observe, that

$$A \cap B = \mathcal{S}^n - X, \quad A \cup B = \mathcal{S}^n - \{\star\} \simeq \mathbb{R}^n$$

By **odnośnik do Mayera**, we get:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \tilde{H}_{i+1}(A \cup B) & \longrightarrow & \tilde{H}_i(A) \oplus \tilde{H}_i(B) & \longrightarrow & \tilde{H}_i(A \cup B) \longrightarrow \dots \\ & & \parallel & & \parallel & & \parallel \\ \dots & \longrightarrow & \tilde{H}_{i+1}(\mathbb{R}^n) & \longrightarrow & \tilde{H}_i(\mathcal{S}^n - X) & \longrightarrow & \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^k) \\ & & & & & & \oplus \\ & & & & & & \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^l) \\ & & & & & & \longrightarrow \tilde{H}_i(\mathbb{R}^n) \longrightarrow \dots \end{array}$$

Because \mathbb{R}^n is contractible, its homology groups are trivial, thus we get an isomorphism for every i :

$$\tilde{H}_i(\mathcal{S}^n - X) \simeq \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^k) \oplus \tilde{H}_i(\mathcal{S}^n - \mathcal{S}^l)$$

Assuming $k \neq l$, for $i = n - k - 1$ and $i = n - l - 1$, we get that $\tilde{H}_i(\mathcal{S}^n - X) \simeq \mathbb{Z}$ for other i it is 0. Otherwise, that is when $k = l$, for $i = n - k - 1 = n - l - 1$, we get $\tilde{H}_i(\mathcal{S}^n - X) \simeq \mathbb{Z}^2$, and for other i it is again 0.

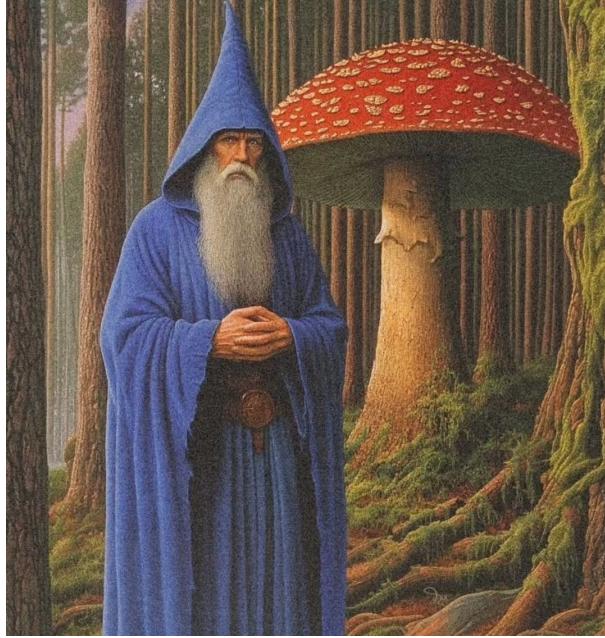
label=(b) **Dokończyć**

□

3.4.3. Simplicial Approximation.

Theorem 2C.3. If X is a finite simplicial complex, or more generally a retract of a finite simplicial complex, and $f : X \rightarrow X$ is a map with $\tau(f) \neq 0$, then f has a fixed point.

$$\tau(f) := \sum_{n=0}^{\infty} (-1)^n \text{tr}(f_* : H_n X \rightarrow H_n X)$$



4. COHOMOLOGY

4.1. Cohomology Groups.

4.2. Cup Product.

Theorem 3.2. If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C; G)$ of the cochain complex $\text{Hom}(C_n, G)$ are determined by split exact sequences

$$0 \rightarrow \text{Ext}(H_{n-1}(C), G) \rightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n(C), G) \rightarrow 0 \quad \square$$



4.3. Poincare Duality.

4.4. Additional Topics.

- 4.4.1. *Universal Coefficients for Homology.*
- 4.4.2. *The General Künneth Formula.*
- 4.4.3. *H-Spaces and Hopf Algebras.*
- 4.4.4. *The Cohomology of $\mathrm{SO}(n)$.*
- 4.4.5. *Bockstein Homomorphisms.*
- 4.4.6. *Limits and Ext.*
- 4.4.7. *Transfer Homomorphisms.*
- 4.4.8. *Local Coefficients.*

REFERENCES

- [Hat01] Allen Hatcher. *Algebraic Topology*. en. Cambridge, England: Cambridge University Press, Dec. 2001 (cit. on pp. 1, 5).

