# Measures on Topological Spaces

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# Contents

# Chapter 1

# Measure Theory Bank of Lemmas

**Lemma 1.1** (Generating a $\sigma$-algebra)**.** *Fix a space $X$. For any family of sets $\mathcal{A}$, $\sigma(\mathcal{A})$ can be generated by any of the following sets of operations:*

1. *complements and countable unions.*

**Lemma 1.2.** *Let $\mathcal{B}$ be a basis for the topology of $X$. Then*

$$\sigma(\mathcal{B}) = \mathrm{Bor}\, X.$$

**Lemma 1.3.** *The set operations and the measure taking operations are continuous with respect to the symmetric difference pseudometric.*

**Lemma 1.4.** *A bounded measurable function $f$ on a measure space $(X, \mu)$ can be uniformly approximated by simple functions.*

*Proof.* Let

$$|f| \leqslant [-M, M].$$

We will construct the approximation by considering *bins* of the values of $f$, i.e. the sets

$$A_k := f^{-1}\Big[[k\varepsilon, (k+1)\varepsilon)\Big]$$

for $k \in \mathbb{Z}$. In such a *bin*, all the values are within an $\varepsilon$ of each other. Since $f$ is bounded, all but finitely many of the bins are empty $X$, so the function

$$\widetilde{f}_\varepsilon := \sum_{A_k \neq \varnothing} (k\varepsilon) \cdot \chi_{A_k}$$

□

**Remark.** This works equally well for almost everywhere bounded functions, giving almost everywhere uniform convergence.

**Lemma 1.5.** *Let $A$, $A_1$, ..., $A_k$ be measurable sets such that*

$$\forall k.\, \mu(A_i \cap A) \geq (1 - \delta_i)\mu(A).$$

*Then*

$$\mu(A \cap A_1 \cap A_2 \cap \ldots \cap A_k) \geq \left(1 - \sum \delta_i\right) \mu(A).$$

*Proof.* Union bound on the sets
$$A \cap A_i^c.$$
□

**Remark.** This also works for an infinite sequence of sets $A_k$; we obtain.
$$\mu\left(A \cap \bigcap_k A_k\right) \geqslant \left(1 - \sum_k \delta_k\right) \mu(A).$$

# Chapter 2

# An introduction to geometric measure theory

In this chapter, we study the links between the topology and geometry of $\mathbb{R}$ and the Lebesgue measure. We first give two examples of how the two structures agree, and one example of how they don't.

**Isometries.** Consider the group $\operatorname{Isom}\mathbb{R}$ of the isometries of $\mathbb{R}$ with the euclidean metric. One easily shows that this group consists of functions of the form

$$x + a \text{ or } a - x$$

for $a \in \mathbb{R}$. The Lebesgue measure is invariant on transformations $g \in \operatorname{Isom}\mathbb{R}$, i.e.

$$\lambda(gA) = \lambda(A)$$

for all measurable $A \subseteq \mathbb{R}$. A corollary of this is that the Lebesgue measure is invariant w.r.t. the addition operation on $\mathbb{R}$, which gives the reals the structure of a topological group.

**Affine transformations.** Similarly to the above, the Lebesque measure work well with the action of the affine transformation group $\operatorname{Aff}\mathbb{R}$. Directly from the definition, the group of affine transformations consists of the functions

$$g_{a,b}(x) := ax + b$$

for $r \neq 0$, and the interaction with measure is given by

$$\lambda(g_{a,b}A) = |a| \cdot \lambda(A).$$

**Topology.** There is a disconnect between the topological (nonempty interior) and measure-theoretic (positive measure) notions of *large* or *non-negligable* – the topological notion is strictly stronger! Indeed, a set with nonempty interior has positive measure, but if we enumerate the rationals as

$$\mathbb{Q} = \{q_1, q_2, \ldots\}$$

the set

$$\mathbb{R} \setminus \bigcup_{n=1}^{\infty} (q_n - \frac{\varepsilon}{2^{n+1}}, q_n + \frac{\varepsilon}{2^{n+1}})$$

has comeasure $\varepsilon$, but is nowhere dense.

However, there does exist a link between the two notions. It is a bit more subtle.

**Definition 2.1.** *Fix a measurable set $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a **density point** iff*

$$\lim_{\delta \to 0^+} \frac{\lambda(A \cap B(x, \delta))}{2\delta} = 1.$$

*The $2\delta$ in the numerator is of course $\lambda(B(x, \delta))$.*

**Definition 2.2.** *The set of density points of $A$ will be denoted $\phi(A)$.*

Note that a density point is by neccesity an accumulation point. The promised link between geometry, measure and topology is provided by the theorem below.

**Theorem 2.1** (Lebesgue Density Theorem)**.** *Let $A \subseteq \mathbb{R}$ be a measurable set. Then almost all*
Ankified *points of $A$ are density points of $A$ in the sense that*

$$\lambda^*(A \setminus \phi(A)) = 0.$$

**Remark.** Note that the theorem follows trivially for null sets. Also, for a given $A$, we may as well apply the theorem to $A^c$ to get that almost all points outside of $A$ have density 0.

For the proof of the **Lebesgue Density Theorem**, we will need a tool, which we introduce now and prove later.

**Definition 2.3.** *A family $\mathcal{J}$ of nontrivial closed intervals is called a **Vitali cover** of a set $A$ (not necessarily measurable) if for any given $\varepsilon > 0$ and $x \in A$ there is an interval $J \in \mathcal{J}$ such that*

$$\operatorname{diam} J < \varepsilon \wedge x \in J.$$

*In particular*

$$A \subseteq \bigcup \mathcal{J}.$$

**Theorem 2.2** (Vitali Covering Theorem)**.** *If $\mathcal{J}$ is a Vitali cover of $A$, there exists a sequence of*
Ankified *pariwaise disjoint segments $J_n \in \mathcal{J}$ such that*

$$\lambda\left(A \setminus \bigcup_n J_n\right) = 0.$$

**Why is this theorem useful?** Vitali's theorem may not sound very smart on first glance. Its strength lies in the *disjointness* of the cover. If we go about choosing the cover $J_n$ without any guarantees, we can for example choose

$$\bigcup_{q_n \in \mathbb{Q}} \left(q_n - \frac{\varepsilon}{2^{n+1}}, q_n + \frac{\varepsilon}{2^{n+1}}\right)$$

and get stuck! We have only covered a subset of the reals of size $\varepsilon$, but we cannot use any other segment by density of $\mathbb{Q}$.

*Proof of the Lebesgue Density Theorem.* We represent

$$A \setminus \phi(A) = \bigcup_k A_k$$

for

$$A_k = \left\{ a \in A : \liminf_{\delta \to 0^+} \frac{\lambda(A \cap B(a,\delta))}{2\delta} < 1 - \frac{1}{k} \right\}.$$

It suffices to show

$$\lambda^*(A_k) = 0$$

for all $k$ to finish the proof. Since we may represent $A$ as

$$A = \bigcup_{z \in \mathbb{Z}} A \cap [z-1, z+1]$$

and being a density point of $A$ is the same as being a density point of one of the *cutouts* in the sum above, we may assume without loss of generality that $A \subseteq [0,1]$.

By definition of outer measure, we can approximate $A_k$ from above by an open set $U$ such that

$$\lambda^*(A_k) \leqslant \lambda(U) \leqslant \lambda^*(A_k) + \varepsilon.$$

Construct a covering

$$\mathcal{J} = \left\{ [a,b] : [a,b] \subseteq U,\ \lambda\Big(A \cap [a,b]\Big) \leqslant \left(1 - \frac{1}{k}\right) \lambda[a,b] \right\}.$$

It is a Vitali cover of $A_k$. By Vitali's Theorem we can pick a pairwise disjoint sequence of intervals $J_i \in \mathcal{J}$ for which

$$\lambda^* \left( A_k \setminus \bigcup_i J_i \right) = 0.$$

This gives

$$\begin{aligned}
\lambda^*(A_k) &= \lambda^* \left( A_k \cap \bigcup_i J_i \right) \\
&\leqslant \sum_i \lambda^*(A_k \cap J_i) \\
&\leqslant \sum_i \lambda^*(A \cap J_i) \\
&\leqslant \left(1 - \frac{1}{k}\right) \sum_i \lambda(J_i) \\
&\leqslant \left(1 - \frac{1}{k}\right) \lambda(U) \\
&\leqslant \left(1 - \frac{1}{k}\right) \left(\lambda^*(A_k) + \varepsilon\right).
\end{aligned}$$

The passage from line 2 to 3 may seem trivial, but is in fact crucial. This is the place where we use $A_k \subseteq A$! Otherwise the theorem is quite absurd, even for simple examples like $[0,1]$. Since $\lambda^*(A_k) \leqslant \lambda(A) < \infty$, we can rearrange this to obtain

$$\lambda^*(A_k) \leqslant (k-1)\varepsilon.$$

Since $\varepsilon$ can be picked arbitrarily close to 0, we get

$$\lambda^*(A_k) = 0.$$

□

*Proof of the Vitali Covering Theorem.* The key to avoiding the *trap* we wrote about after stating the VCT is to choose the segments to be as large as possible – or at least not embarassingly small.

Without loss of generality, $A$ is bounded since we can sum the coverings of $A \cap (n, n+1)$. The sequence of segments we pick is denoted $J_n$. In that case we may also assume $\bigcup \mathcal{J}$ is bounded. Its prefixes are

$$\begin{aligned} P_n &:= \bigcup_{i<n} J_i \\ \mathcal{J}_n &:= \{J \in \mathcal{J} : J \cap P_n = \varnothing\} \end{aligned}$$

and the *width* of what we can choose is

$$\gamma_n := \sup_{J \in \mathcal{J}_n} \operatorname{diam} J.$$

Note that in particular

$$\begin{aligned} P_1 &= \varnothing, \\ \mathcal{J}_1 &= \mathcal{J} \\ \gamma_1 &\leqslant \operatorname{diam} A < \infty. \end{aligned}$$

At each step, we choose $J_n$ so that

$$\operatorname{diam} J_n \geqslant \frac{1}{2}\gamma_n,$$

or we stop if $\gamma_n = 0$ at some point.

**Claim 1.** The sequence $\gamma_n$ converges monotonically to 0.

*Proof of Claim 1.* Being the supremum of ever decreasing sets, $\gamma_n$ is decreasing. It is also non-negative, so the sequence converges and $\lim_n \gamma_n \geqslant 0$. Suppose that $\lim_n \gamma_n = c > 0$. Then in the construction, we would almost always choose disjoint intervals of diameter at least $c/2$. This is impossible, since $\bigcup \mathcal{J}$ was assumed to be bounded, so it has finite measure! □

The key to proving that the choice procedure is correct will be the **blowup**, which we define for $J = [x - r, x + r]$ as

$$\widetilde{J} := [x - 5r,\, x + 5r]$$

**Claim 2.** At all steps of the construction

$$A \subseteq \mathcal{J}_n \cup \bigcup_{i \geqslant n} \widetilde{J}_i.$$

*Proof of Claim 2.* The set $\mathcal{J}_n$ is closed as a union of closed intervals. Therefore, if $a \in A \setminus \mathcal{J}_n$, there is a nondegenerate interval $I \ni a$. Since $\gamma_n \to 0$ by **Claim 1**, $I$ is not considered in the construction of the sequence $J_n$ for almost all $n$. Let $n_0$ be the last step where it is considered. Then we must have $I \cap J_{n_0+1} \neq 0$, because that is the step at which $I$ is no longer considered.

We will show that this implies $a \in \widetilde{J}_{n_0+1}$. Let $J_{n_0+1} = [x-r,\, x+r]$ and $y \in I \cap J_{n_0+1}$. Then we have

$$\begin{aligned} d(a,x) &\leqslant d(a,y) + d(y,z) \\ &\leqslant \operatorname{diam} I + r \\ &\leqslant (2 \cdot \operatorname{diam} J_{n_0+1}) + r \\ &= 2 \cdot 2r + r \\ &= 5r, \end{aligned}$$

where the diameter bound comes from the definition of $\gamma_{n_0}$ and the fact that $I$ is still available at step $n_0$ of the construction. □

To finish the proof of Vitali's Covering Theorem, we compute that for all $n$

$$\begin{aligned} \lambda^*(A \setminus P_n) &\leqslant \lambda^* \left( A \cap \bigcup_{i \geqslant n} \widetilde{J}_i \right) \\ &\leqslant \lambda \left( \bigcup_{i \geqslant n} \widetilde{J}_i \right) \\ &\leqslant \sum_{i \geqslant n} \lambda(\widetilde{J}_i) \\ &= 5 \sum_{i \geqslant n} \lambda(J_i). \end{aligned}$$

These are the tails of the convergent series

$$\sum_{i=1}^{\infty} \lambda(J_i) = \lambda \left( \bigcup_i J_i \right) < \infty,$$

so we get

$$\lambda^* \left( A \setminus \bigcup_i J_i \right) \leqslant \lambda^*(A \setminus P_n) \to 0.$$

□

**Remark.** Retracing the argument behind **Claim 2.**, we might prove that for any $\alpha < 1$, if we define $\gamma_n$ with a coefficient of $\alpha$ instead of $\frac{1}{2}$, the constant used for blowing up intervals can be brought down to

$$1 + \frac{2}{\alpha}.$$

In particular, we can get arbitrarily close to 3.

## §2.1 Corollaries and the Lebesgue Differentiation Theorem

**Theorem 2.3** (Lebesgue Differentiation Theorem). *Let $f \in L^1(\mathbb{R})$. Then, for almost all $x$,*

$$\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(s) \, \mathrm{d}\lambda(s) = f(x).$$

*Proof.* For characteristic functions, this is just a restatement of the Lebesgue Density Theorem. □

## §2.2 Generalization to metric spaces

The argument in the proof of the VCT was written so that it is easliy generalizable to any metric space with a measure on its Borel sets.

To be more precise, what we need to lift the argument is that

$$\mu\left(B(x, 5r)\right) \leqslant C\mu\left(B(x, r)\right)$$

for some constant $C$. We can also substitute any constant larger that 3 instead of 5.

# Chapter 3

# One (Cantor) set to rule them all

## §3.1 Ternary Cantor

Let us begin by making a construction. Take the closed interval $C_0 := [0,1]$ and remove the middle one third of it in such a way that the remaining two interval are closed. The result of this is

$$C_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Now, repeat the operation of cutting out the middle third and call the result $C_2$. We can repeat this *ad inifinitum* and obtain a decresing sequence of sets

$$[0,1] = C_0 \supset C_1 \supset C_2 \supset \ldots.$$

Perhaps surprisingly, there are numbers which are not removed at any step, i.e. the intersection

$$\mathcal{C}_3 := \bigcap_{k=0}^{\infty} C_k$$

is nonempty! It contains 0 and 1. In fact, any number which can be written in base 3 using only 0's and 2's is an element of this intersection. These are in fact all such numbers. We introduce a tool to prove that.

**Lemma 3.1.** *Let $b \geqslant 2$ be a positional system base and $x_0$ be a number with $k$ digits after the positional point. Then, the numbers formed by adjoining (perhaps infinitely many) digits to the base $b$ representation of $x_0$ are all the numbers in the interval*

$$\left[x_0,\, x_0 + b^{-k}\right].$$

*If we allow only finite extensions, we get the $b$-ary rational numbers in that interval, and if we disallow the infinite extension by the digit $(b-1)$, we get the interval*

$$\left[x_0,\, x_0 + b^{-k}\right).$$

**Lemma 3.2.** *A real number $x \in [0,1]$ is an element of $\mathcal{C}_3$ iff $x$ can be written in base using only the digits 0 and 2.*

*Proof (if).* We proceed by induction with the induction thesis: *$x$ belongs to $\mathcal{C}_3$ iff $x$ can be written in base 3 so that its first $k$ digits are 0 or 2.* It should be clear that this thesis is equivalent to the lemma statement. For each $k$, the statement is true by the previous lemma. □

## §3.2 Abstract Cantor

The ternary Cantor set has many interesting properties. However, to study it, we will move to a more convenient representation. We can think of the Cantor set as the set of leaves of an infinite binary tree – starting from the root, at each level we choose whether to go right or left, or whether to insert 0 or 2 as the next digit in the base-3 representation of an $x \in \mathcal{C}_3$.

In this way, we can represent the ternary Cantor as

$$\mathcal{C} := \{0,1\}^{\mathbb{N}}.$$

That the map we just described is a bijection follows from

**Lemma 3.3.** *Let $d_k, \widetilde{d}_k$ be two sequences of base $b$ digits. Then the corresponding real numbers are equal iff $d_k$ and $\widetilde{d}$ agree on some prefix and afterwards one of them is $0$ and the other is $(b-1)$.*

*Proof.* The condition implies equality of numbers by the sum of a geometric series. The other direction follows by looking at the first moment the expansions differ at then bounding the series sum. □

What is missing from this description is the topology. We topologize $\mathcal{C}$ by the metric

$$d(x,y) = \begin{cases} 0 & \text{for } x = y \\ \frac{1}{n} & \text{for } x \neq y, \end{cases}$$

which can also be written succintly as

$$d(x,y) = \frac{1}{n_0(x,y)}$$

with the notation

$$n_0(x,y) := \inf\{n : x_n \neq y_n\}$$

for the first index at which $x$ and $y$ differ. The function $d$ may not look like a metric at first sight, but in fact it has an even better property.

**Lemma 3.4.** *For the metric $d$ described above we have for all $x, y, z \in \mathcal{C}$*

$$d(x,z) \leqslant \max\{d(x,y), d(y,z)\}.$$

*In particular, $d$ is an **ultrametric**.*

*Proof.* Recall that $n_0(x,z)$ is the first position at which $x$ and $z$ differ. Then any $y$ has to differ with at least one of $y$ and $z$ at $n_0$, but might even earlier. This gives

$$n_0(x,z) \geqslant \min\bigl(n_0(x,y), n_0(y,z)\bigr).$$

Since the function $x \mapsto 1/x$ is decreasing, the thesis follows. □

We have established that $(\mathcal{C}, d)$ is a metric space. It is, in fact, homeomorphic with the subspace topology of $\mathcal{C}_3$ inherited from $[0,1]$.

**Lemma 3.5.** *The function*

$$h_3 : \mathcal{C} \to \mathcal{C}_3$$

*defined by*

$$h_3(x) := \sum_{k=1}^{\infty} \frac{2x_k}{3^k}$$

*is a homeomorphism.*

*Proof.* Bijectivity follows from the number-system lemma 3.3 and 3.2. For continuity, put down $n_0 := n_0(x, y)$ and compute

$$\begin{aligned} |h_3(x) - h_3(y)| &= \sum_{k=1}^{\infty} \frac{2|x_k - y_k|}{3^k} \\ &\leqslant \sum_{k=n_0}^{\infty} \frac{2}{3^k} \\ &= \frac{2}{3^{n_0}} \cdot \frac{3}{2} \\ &= \frac{1}{3^{n_0-1}}. \end{aligned}$$

The continuity of the inverse follows from the bound

$$|h_3(x) - h_3(y)| \geqslant \frac{2}{3^{n_0}}.$$

□

The function $h_3$ in 3.5 can be understood as a base 3 expansion operator. When we consider a base 2 expansion instead, we lose bijectivity, but we can cover the whole interval.

**Lemma 3.6.** *The function*

$$h_2 : \mathcal{C} \to [0, 1]$$

*given by*

$$h_2(x) := \sum_{k=1}^{\infty} \frac{x_k}{2^k}$$

*is a continuous surjection.*

*Proof.* Surjectivity follows from number system properties, and continuity is essentialy the same calculation as in the proof of 3.5. □

**Theorem 3.1** (The Universal Property of the Cantor Set)**.** *Every metrizable compact space $K$ is a continuous image of $\mathcal{C}$.*

*Proof.* Considering an element of $\mathcal{C}$ as a binary expansion, we have by 3.6 a surjection

$$h_2 : \{0, 1\}^{\mathbb{N}} \twoheadrightarrow [0, 1].$$

The space $K$ can be embedded into the Hilbert cube by the **Urysohn Metrization Theorem ??**. By compactness of $K$, the image of the embedding is a compact and thus a closed subset. We also have a surjection

$$h : \{0, 1\}^{\mathbb{N}} \to [0, 1]^{\mathbb{N}}$$

by using the previous surjection and *unweaving* the Cantor set into the product of countably many Cantor sets, i.e. using

$$\mathbb{N} \cong \mathbb{N} \times \mathbb{N} \implies \mathcal{C} = \{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}\times\mathbb{N}} \cong \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} = \mathcal{C}^{\mathbb{N}}.$$

The last step is using the fact that any closed set of $\mathcal{C}$ is a retract of $\mathcal{C}$, which is **??**. □

**A warning against generalization.** If $K$ is a compact set, it embeds into a *Tichonov Cube*

$$K \to [0,1]^{\Gamma}$$

and we can surject the Tichonov cube with a generalized Cantor set

$$\{0,1\}^{\Gamma},$$

but the universality theorem fails!

## §3.3 Topology of the Cantor set

**Definition 3.1** (Cantor Cylinder)**.** *Let*

$$\varphi : \mathbb{N} \multimap \{0,1\}$$

*be a partial function with finite domain. Then we define the* ***cylinder set*** *with base $\varphi$ as*

$$[\varphi] := \{x \in \{0,1\} : x|_I = \varphi\}.$$

**Lemma 3.7.** *The sets $[\varphi]$ form a base of the topology of $\{0,1\}^{\mathbb{N}}$.*

**Definition 3.2.** *A set $A \subseteq \mathcal{C}$ is* ***determined*** *by $I \subseteq \mathbb{N}$, which we donte by $A \sim I$ if for all $x \in A$, $y \in \mathcal{C}$ we have*

$$x|_I = y|_I \implies y \in A.$$

*Equivalently,*

$$\pi_I^{-1}\pi_I[A] = A.$$

**Lemma 3.8** (Clopen sets in the Cantor set)**.** *A set $A \subseteq \mathcal{C}$ is clopen iff $A \sim I$ for some finite $I \subseteq \mathbb{N}$. In particular, clopen sets can be written as a finite union of disjoint basis clopens $[\varphi_i]$ for $\varphi_i$ with finite domain.*

*(Direction one).* If $A$ is clopen, then

$$A = \bigcup_i [\varphi_i]$$

for some finitely many (by compactness) $\varphi_i$ with finite domain $I_i$. Then

$$A \sim \bigcup_i I_i.$$

□

*(Other direction).* if $A \sim I$, blabla □

Immediately, a lemma follows.

**Lemma 3.9** (Cantor set is zerodimensional)**.** *The Cantor set $\mathcal{C}$ is zerodimensional, i.e. it has a base of clopen sets.*

**Theorem 3.2** (Topological characterisation of the Cantor set)**.** *If a topological space $K$ is compact, metrizable, zerodimensional with no isolated points, then*

$$K \cong C.$$

## §3.4 The group structure

The Cantor set has a natural abelian group structure given by its product structure. We can phrase it even more efficiently when we think of $\mathcal{C}$ as $\mathcal{P}(\mathbb{N})$ – the symmetric difference (or xor for the informatically inclined).

$$A \oplus B := A \Delta B$$

Every element has order two!

**Fact.** Together with the operation $\oplus$, the Cantor set $\mathcal{C}$ is a compact topological group, i.e. the function

$$(x, y) \mapsto x \oplus y$$

is continuous (in general the second element is inversed, but here every element is its own inverse anyway).

## §3.5 Measure

We can define the measure on the Cantor set as a countable product of probability measures:

$$\nu = \bigotimes_{n=1}^{\infty} (\frac{1}{2}(\delta_0 + \delta_1)).$$

But we will do it by hand.

**Definition 3.3.** *Let $A \subseteq \mathcal{C}$ be clopen. Then*

$$A \sim \{1, 2, \dots, n\}$$

*for some $n$. Let*

$$A' := \pi_{\{1,2,\dots,n\}}[A]$$

*We define its measure to be*

$$\nu(A) := \frac{\#A'}{2^n}.$$

This makes sense with the probabilistic definition.

**Theorem 3.3** (Well-definedness of the premeasure)**.** *The function*

$$\nu : \operatorname{Clop} \mathcal{C} \to \mathbb{R}$$

*is a well-defined, additive function on the set algebra* $\operatorname{Clop} \mathcal{C}$*.*

**Proof.** Since the Cantor set is compact, $\nu$ is automatically downward continuous on the empty set. By Caratheodory's Theorem, $\nu$ extends uniquely to a probabilistic measure on

$$\operatorname{Bor}\mathcal{C} = \sigma\left(\operatorname{Clop}\mathcal{C}\right).$$

What now??

$$\mathcal{A} := \{B \in \operatorname{Bor}\mathcal{C} : \forall\varepsilon > 0.\exists A \in \operatorname{Clop} C.\nu(A\Delta B) < \varepsilon\}.$$

We prove that this is a $\sigma$-algebra.

There is a nice formula for cylinders.

**Lemma 3.10** (Measure of a cylinder)**.** *For a partial function*

$$\varphi : \mathbb{N} \multimap \{0, 1\},$$

*its cylinder has measure*

$$\nu\left[\phi\right] = 2^{-|\operatorname{dom}\varphi|}.$$

*The result holds even if* $\operatorname{dom}\varphi$ *is infinite, in which case the measure is* $0$*.*

*Proof.* For finite-domain partial functions $\phi$, take

$$\operatorname{dom}\varphi =: I \subseteq \{1, 2, \ldots, n\} =: [n]$$

for some $n$. Then

$$\left|\pi_{[n]}[\varphi]\right| = \frac{2^{n-|I|}}{2^n} = 2^{-|I|}.$$

For infinite-domain functions $\phi$, taking a decreasing intersection

$$[\phi] = \bigcap_n \left[\phi|_{[n]}\right]$$

shows that the measure of the intersection is 0. □

**Theorem 3.4.** *The measure* $\nu$ *is the Haar measure on* $\mathcal{C}$*, that is, the unique probability measure invariant under group actions*

$$\nu\left(x \oplus B\right) = \nu(B)$$

*for all* $x \in \mathcal{C}$*,* $B \subseteq \mathcal{C}$*.*

*Proof.* Let us first consider $B = [\varphi]$, and $I = \operatorname{dom}\varphi$. Then

$$\nu\left(x \oplus [\varphi]\right) = \nu\left([x \oplus \varphi]\right) = \nu\left([\varphi]\right).$$

A clopen is a disjoint sum of $[\varphi_i]$ for finitely many $\varphi_i$, so additivity on clopens follows. Now, take a superficially different measure

$$\nu_x(B) := \nu\left(x \oplus B\right).$$

Since $\nu$ and $\nu_x$ agree on clopens, by uniqueness in Caratheodory's Theorem they agree on all sets. □

Note the isomorphism

$$(C, \oplus) \cong (\mathcal{P}(\mathbb{N}), \Delta)$$

of (topological) groups.

## §3.6 Normal number theorem

**Definition 3.4.** *Let $A \subseteq \mathcal{C}$. We call $A$ a **tail** set if*

$$A \sim \{k : k \geqslant n\}$$

*for all $n$. Equivalently, if $x \in A$ and $x(n) = y(n)$ for almost all $n$, then $y \in A$.*

**Example.** A naturally occuring example of a tail set is

$$A_\beta := \left\{ x \in \mathcal{C} : \lim_n \frac{x(1) + \dots x(n)}{n} = \beta \right\}.$$

**Theorem 3.5** (Kolmogorov zero-one law for the Cantor set)**.** *A borel tail set $A \subseteq \mathcal{C}$ has measure $0$ or $1$.*

*Proof.* Take a basis set $[\varphi]$. We have

$$\nu([\varphi] \cap A) = \nu([\varphi]) \cdot \nu(A).$$

From this immediately follows that this work for any $B \in \operatorname{Clop} \mathcal{C}$. Now approximate $A$ by a clopen $B$ so that

$$\nu(A \Delta \mathcal{C}).$$

To finish the proof, compute

$$\nu(A) \cdot \nu(B) = \nu(A \cap B) \geqslant \nu(A) - \varepsilon \nu(A).$$

□

Returning to the example we have $\nu(A_\beta) \in \{0, 1\}$. We have

$$\nu(A_\beta) = \nu(A_{1-\beta}).$$

**Theorem 3.6** (Borel's normal number theorem)**.**

$$\nu\left(A_{\frac{1}{2}}\right) = 1.$$

**Remark.** According to Billingsley, this theorem was the founding work of modern probability theorem, which is founded on limit theorems.

# Chapter 4

# Measures on Topological Spaces, Problemset 1

## Problem 4

### Extension 1

We show that the set can be the graph of a function! Let $Z$ be a borel set of positive measure and define

$$T_Z = \{x : \lambda(Z_x) > 0\}.$$

Then $T_Z$ is a measurable set by Fubini's Theorem. We can pick a compact subset $T'_Z$. A compact set of positive measure has at least $\mathfrak{c}$ elements, and there are as many borel sets. Then, enumerate borel sets of $\mathbb{R}^2$.

## Problem 5

### Set of undefined density at $0$

**TODO**

### Set of density $t$ at $0$

Presented in class by **Michał Baran**. Fix $t \in (0, 1)$.

The set we will construct will be symmetric around 0. We will find a sequence $b_n$ such that with

$$A_n = \left(\frac{1}{n} - b_n, \frac{1}{n}\right)$$

we will have for all $n$

$$\frac{t}{n} = \lambda\left(\bigcup_{k=n}^{\infty} A_n\right) = \sum_{k=n}^{\infty} b_k,$$

so

$$b_n = \sum_{k=n}^{\infty} b_k - \sum_{k=n+1}^{\infty} b_k = \frac{t}{n(n+1)}.$$

Consider

$$A := \bigcup_{k=1}^{\infty} A_k \cup -A_k.$$

We will bound the fraction

$$\frac{\lambda\left(A \cap (-\delta, \delta)\right)}{2\delta} = \frac{\lambda\left(A \cap (0, \delta)\right)}{\delta}$$

from above and below. For $\delta \in (1/(n+1), 1/n]$ we have

$$\bigcup_{k=n+1} A_k \subseteq A \cap (0, \delta) \subseteq \bigcup_{k=n} A_k,$$

passing to measure

$$\frac{t}{n+1} \leqslant \lambda\left(A \cap (0, \delta)\right) \leqslant \frac{t}{n}.$$

When divided by $\delta$, we get the result by the squeeze theorem.

**Remark.** The solution would work equally well if instead of $a_n = 1/n$ we used a sequence that converges to 0 monotonically and satisfies

$$\frac{a_n - a_{n+1}}{a_n} \to 0.$$

## Problem 9

Presented in class by **dr Arturo Martinez Celiz**.

Wlog, everything happens within (0, 1). Following the hint, take a countable sequence $A_i$ such that the set $B := \bigcup_i A_i$ has maximal measure.

By this choice, for any $C \in \mathcal{A}$, we have

$$\lambda\Big((C \cup B)\, \Delta B\Big) = 0,$$

so that

$$\phi(C \cup B) = \phi(B)$$

and

$$C \subseteq \phi(C \cup B) = \phi(B).$$

Since $C$ was arbitrary

$$\bigcup \mathcal{A} \subseteq \phi(B) \implies B \subseteq \bigcup \mathcal{A} \phi(B).$$

Since $\lambda(B) = \lambda\left(\phi(B)\right)$ we know that the sum of $\mathcal{A}$ is measurable.

## Problem 10

Presented in class by **Szymon Smolarek**.

We take a cover of **regular sets**, i.e. a family for which there exists a constant $C$ such that

$$\operatorname{diam}^2 A \leqslant C\lambda_2(A).$$

It can be proven that if such a family is a Vitali cover of a set $A \subseteq \mathbb{R}^2$, an analogue of the VCT holds.

The family of all triangles does not satisfy the regularity condition – think of keeping one segment constant and bringing the third verted evert closer to the segment. To deal with this, we subdivide the family $\mathcal{T}$ into subfamilies

$$\mathcal{T}_n := \left\{T \in \mathcal{T} : \operatorname{diam}^2 T \leqslant n\lambda_2(A)\right\}.$$

Reducing to a given subfamily, we can cover each triangle $T$ by arbitrarily small traingles similar to $T$ contained within $T$. This gives us a regular Vitali cover $\widetilde{T}_n$ of $\bigcup \mathcal{T}_n$.

# Problem 11

Stated in class by **Szymon Smolarek**.

**Theorem 4.1** (Steinhaus theorem for the Cantor Set)**.** *For any measurable set $A$, the set*

$$A \oplus A$$

*contains an open neighbourhood of* $0$.

**Theorem 4.2** (Vitali Covering Theorem for the Cantor Set)**.** *If a family of clopens $\mathcal{J} \subseteq \operatorname{Clop}\mathcal{C}$ is a Vitali cover of $A$, then there is a sequence $J_n \in \mathcal{J}$ such that*

$$\nu^*\left(A \setminus \bigcup_n J_n\right) = 0.$$

**Theorem 4.3** (Lebesgue Density Theorem for the Cantor Set)**.** *Let $A \subseteq \mathcal{C}$. An element $a \in A$ is a **density point** of $A$ if*

$$\lim_{n\to\infty} \frac{\nu\left(A \cap [a|_{[n]}]\right)}{2^{-n}} = 1.$$

*If $A$ is measurable, then almost all points of $A$ are density points of $A$.*

The proofs are quite the same, as $\mathcal{C}$ is a topological group and the measure $\nu$ is its Haar measure.

# Problem 12

**Hint.** Use Baire's theorem.

# Chapter 5

# Measures on Topological Spaces, Problemset 2

## Problem 1

Such an $a$ exists by compactness of $A$ and continuity of metric. If we have two $a_1, a_2$ such that

$$\rho(x, a_1) = \rho(x, a_2)$$

then $a_1, a_2$ must agree and disagree with $x$ at all places, so in fact $a_1 = a_2$, thus $r_A$ is well-defined. For any $a \in A$, $d(a, A) = 0 = d(a, a)$, so $r_A$ is a retraction. What remains to be shown is continuity.

Let $x, y$ agree up to $n_0(x, y)$. Then $r_A(x)$ and $r_A(y)$ also agree up to $n_0(x, y)$ – if they differed earlier, we could use $r_A(x)$ instead of $r_A(y)$ and get a closer point $a$ in the definition! So we have

$$n_0(x, y) \leqslant n_0 \left(r_A(x), r_A(y)\right)$$

and

$$d(x, y) \geqslant d \left(r_A(x), r_A(y)\right).$$

**Remark.** The metric $d(x, y) = 1/n_0(x, y)$, i.e. the first moment where $x$ and $y$ differ, won't work, because it can't tell apart points from which $x$ differs at the same position!

## Problem 2

## Problem 3

Any $A, B \in \operatorname{Clop} \mathcal{C}$ can be written as disjoint sums of the basis sets $[\varphi]$ by 3.8. Since the condition distributes over disjoint sums, we will prove the statement for $A = [\varphi]$ and $B = [\psi]$ with

$$|\operatorname{dom} \varphi|, |\operatorname{dom} \psi| < \infty.$$

Let $I = \operatorname{dom} \varphi$, $J = \operatorname{dom} \psi$ be the disjoint(!) domains of $\varphi$, $\psi$. There is a function $\tau$ on $I \cup J$ such that

$$\tau|_I = \phi,\ \tau|_J = \psi.$$

For such a function,

$$[\varphi] \cap [\psi] = [\tau] \,.$$

Now take an $n$ such that $I \cup J \subseteq \{1, 2, \ldots, n\}$ and denote the last set as $[n]$. By 3.10 we compute

$$\nu\,[\varphi] = 2^{-|I|}$$
$$\nu\,[\psi] = 2^{-|J|}$$
$$\nu\,[\tau] = 2^{-|I \cup J|},$$

and $|I \cup J| = |I| \cup |J|$ finishes the proof. Now take arbitrary $A, B \in \operatorname{Bor} \mathcal{C}$ such that $A \sim I$, $B \sim J$. Approximate $A, B$ by clopens $A', B'$ to within an $\varepsilon$, i.e. so that

$$\nu(A \Delta A'),\, \nu(B \Delta B') < \varepsilon.$$

We cannot use the clopen statement we just proved since a priori $A'$ and $B'$ could be determined by sets with nonempty intersection. We can, however, improve the approximation with

$$\widetilde{A} := \pi_I^{-1} \pi_I A'.$$

The set $\widetilde{A}$ is still a clopen – since $A'$ was determined by a finite set $K$, $\widetilde{A}$ is determined by $K \cap I$. Additionally we have

$$\widetilde{A} \Delta A \subseteq A' \Delta A,$$

so we have improved the approximation! Now, do the same for $B'$ and use the statement for clopens to finish up the solution.

**Warning!** The reasoning below does not work! (For tail sets, for example)

We can approximate $A, B$ by decreasing sequences of clopens by putting down

$$A_n := \pi_{[n]}^{-1} \pi_{[n]} A$$

and the same for $B_n$. We also approximate their intersection by decreasing clopens in the same way, i.e.

$$C_n := \pi_{[n]}^{-1} \pi_{[n]} (A \cap B).$$

For these approximations

$$C_n = A_n \cap B_n,$$

so by the first subproblem

$$\nu(C_n) = \nu(A_n \cap B_n) = \nu(A_n) \cdot \nu(B_n).$$

Since the measure $\nu$ is probabilistic, and hence continuous, by passing to the limit $n \to \infty$ we get what we need.

## Problem 6

Any clopen $C \in \operatorname{Clop} \mathcal{C}$ is a disjoint sum of basis cylinders by 3.8. Since $\oplus$ is a group operation, the function

$$l_x(y) = x \oplus y$$

is bijective, so on the level of sets $l_x$ distributes over disjoint sums. We check the property for a cylinder $[\varphi]$. This is easy, since

$$\nu\Big(x \oplus [\varphi]\Big) = \nu[x \oplus \varphi] = 2^{-|\operatorname{dom}\varphi|} = \nu[\varphi]$$

by 3.10. Now consider the family of sets

$$\mathcal{A} := \Big\{A : \forall x \in \mathcal{C}.\, \nu(A) = \nu(x \oplus A)\Big\}.$$

We will show that this is a $\sigma$-algebra. Since we have already shown that it contains all the clopens, which form a basis of the topology on $\mathcal{C}$, it will automatically be equal to $\operatorname{Bor}\mathcal{C}$ by 1.2.

A $\sigma$-algebra can be generated by complements and countable sums (see 1.1). As mentioned before, $l_x$ respects these operations, so

$$\nu(x \oplus A^c) = \nu\Big((x \oplus A)^c\Big) = 1 - \nu(x \oplus A) = 1 - \nu(A) = \nu(A^c)$$

and

$$\nu\left(x \oplus \bigcup_i A_i\right) = \nu\left(\bigcup_i x \oplus A_i\right) = \sum_i \nu(x \oplus A_i) = \sum_i \nu(A_i) = \nu\left(\bigcup_i A_i\right).$$

## Problem 7

The identification is

$$A \mapsto \chi_A,\ x \mapsto \{n : x_n = 1\}.$$

One easily checks that these two are mutually inverse. Addition modulo 2 comes out to 1 iff exactly one of the summands is 1, and this corresponds exactly to belonging to the symmetric difference.

## Problem 8

A filter cannot contain both $A$ and $A^c$, since then it would contain $A \cap A^c = \varnothing$. Thus, a filter containing for all $A$ either $A$ or $A^c$ is maximal.

For the other direction, suppose neither $A$ nor $A^c$ is in a filter $\mathcal{F}$. We define its *extension* by $A$ as

$$\mathcal{F}_A = \{A' \cap F : A \subseteq A', F \in \mathcal{F}\}.$$

We check that this is a filter.

1. If $\varnothing \in \mathcal{F}_A$, $\mathcal{F}$ contains a set disjoint with $A$, so by the superset property it contains $A^c$.
2. Let $A_1 \cap F_1, A_2 \cap F_2 \in \mathcal{F}_A$. Then

   $$A \subseteq A_1 \cap A_2, F_1 \cap F_2 \in \mathcal{F},$$

   so $(A_1 \cap F_1) \cap (A_2 \cap F_2) = (A_1 \cap A_2) \cap (F_1 \cap F_2) \in \mathcal{F}_A$.
3. Let $B \supseteq A' \cap F$. Then

   $$B = B \cup (A' \cap F) = (B \cup A') \cap (B \cup F)$$

   and $A \subseteq A' \cup B$, $F \subseteq B \cup F$, so $B \in \mathcal{F}_A$.

Of course, $A \in \mathcal{F}_A \setminus \mathcal{F}$, so $\mathcal{F}$ was not maximal in the first place.

**Remark.** One can check that $\mathcal{F}_A$ is the minimal filter containing $\mathcal{F}$ and $A$.

## Problem 9

The only principal ultrafilters are generated by singletons, so they are definitely measurable.