Measures on Topological Spaces

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Contents

Chapter 1

Measure Theory Bank of Lemmas

Lemma 1.1 (Generating a σ-algebra)**.** *Fix a space X. For any family of sets $\mathcal{A}$, $\sigma(\mathcal{A})$ can be generated by any of the following sets of operations:*

1. *the empty set, complements and countable unions.*
2. *the empty set, complements, finite unions and increasing countable unions.*

Lemma 1.2. *Let $\mathcal{B}$ be a basis for the topology of X. Then*

$$\sigma(\mathcal{B}) = \operatorname{Bor} X.$$

Lemma 1.3. *The set operations and the measure taking operations are continuous with respect to the symmetric difference pseudometric.*

Lemma 1.4. *A bounded measurable function f on a measure space (X, μ) can be uniformly approximated by simple functions.*

Proof. Let

$$|f| \leqslant [-M, M].$$

We will construct the approximation by considering *bins* of the values of f, i.e. the sets

$$A_k := f^{-1}\Big[[k\varepsilon, (k+1)\varepsilon)\Big]$$

for $k \in \mathbb{Z}$. In such a *bin*, all the values are within an ε of each other. Since f is bounded, all but finitely many of the bins are empty X, so the function

$$\widetilde{f}_\varepsilon := \sum_{A_k \neq \varnothing} (k\varepsilon) \cdot \chi_{A_k}$$

□

Remark. This works equally well for almost everywhere bounded functions, giving almost everywhere uniform convergence.

Lemma 1.5. *Let A, A_1, …, A_k be measurable sets such that*

$$\forall k.\, \mu(A_i \cap A) \geq (1 - \delta_i)\mu(A).$$

Then

$$\mu(A \cap A_1 \cap A_2 \cap \ldots \cap A_k) \geq \Big(1 - \sum \delta_i\Big)\, \mu(A).$$

Proof. Union bound on the sets

$$A \cap A_i^c.$$

□

Remark. This also works for an infinite sequence of sets A_k; we obtain.

$$\mu\left(A \cap \bigcap_k A_k\right) \geqslant \left(1 - \sum_k \delta_k\right) \mu(A).$$

Chapter 2

An introduction to geometric measure theory

In this chapter, we study the links between the topology and geometry of $\mathbb{R}$ and the Lebesgue measure. We first give two examples of how the two structures agree, and one example of how they don't.

Isometries. Consider the group $\operatorname{Isom}\mathbb{R}$ of the isometries of $\mathbb{R}$ with the euclidean metric. One easily shows that this group consists of functions of the form

$$x + a \text{ or } a - x$$

for $a \in \mathbb{R}$. The Lebesgue measure is invariant on transformations $g \in \operatorname{Isom}\mathbb{R}$, i.e.

$$\lambda(gA) = \lambda(A)$$

for all measurable $A \subseteq \mathbb{R}$. A corollary of this is that the Lebesgue measure is invariant w.r.t. the addition operation on $\mathbb{R}$, which gives the reals the structure of a topological group.

Affine transformations. Similarly to the above, the Lebesque measure work well with the action of the affine transformation group $\operatorname{Aff}\mathbb{R}$. Directly from the definition, the group of affine transformations consists of the functions

$$g_{a,b}(x) := ax + b$$

for $r \neq 0$, and the interaction with measure is given by

$$\lambda(g_{a,b}A) = |a| \cdot \lambda(A).$$

Topology. There is a disconnect between the topological (nonempty interior) and measure-theoretic (positive measure) notions of *large* or *non-negligable* – the topological notion is strictly stronger! Indeed, a set with nonempty interior has positive measure, but if we enumerate the rationals as

$$\mathbb{Q} = \{q_1, q_2, \ldots\}$$

the set

$$\mathbb{R} \setminus \bigcup_{n=1}^{\infty} (q_n - \frac{\varepsilon}{2^{n+1}}, q_n + \frac{\varepsilon}{2^{n+1}})$$

has comeasure ε, but is nowhere dense.

However, there does exist a link between the two notions. It is a bit more subtle.

Definition 2.1. *Fix a measurable set $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a **density point** iff*

$$\lim_{\delta \to 0^+} \frac{\lambda(A \cap B(x,\delta))}{2\delta} = 1.$$

The 2δ in the numerator is of course $\lambda(B(x,\delta))$.

Definition 2.2. *The set of density points of A will be denoted $\phi(A)$.*

Note that a density point is by neccesity an accumulation point. The promised link between geometry, measure and topology is provided by the theorem below.

Theorem 2.1 (Lebesgue Density Theorem)**.** *Let $A \subseteq \mathbb{R}$ be a measurable set. Then almost all points of A are density points of A in the sense that*

Ankified

$$\lambda^*(A \setminus \phi(A)) = 0.$$

Remark. Note that the theorem follows trivially for null sets. Also, for a given A, we may as well apply the theorem to A^c to get that almost all points outside of A have density 0.

For the proof of the **Lebesgue Density Theorem**, we will need a tool, which we introduce now and prove later.

Definition 2.3. *A family $\mathcal{J}$ of nontrivial closed intervals is called a **Vitali cover** of a set A (not necessarily measurable) if for any given $\varepsilon > 0$ and $x \in A$ there is an interval $J \in \mathcal{J}$ such that*

$$\operatorname{diam} J < \varepsilon \wedge x \in J.$$

In particular

$$A \subseteq \bigcup \mathcal{J}.$$

Theorem 2.2 (Vitali Covering Theorem)**.** *If $\mathcal{J}$ is a Vitali cover of A, there exists a (possibly finite) sequence of pariwaise disjoint segments $J_n \in \mathcal{J}$ such that*

Ankified

$$\lambda^* \left(A \setminus \bigcup_n J_n \right) = 0.$$

Why is this theorem useful? Vitali's theorem may not sound very smart on first glance. Its strength lies in the *disjointness* of the cover. If we go about choosing the cover J_n without any guarantees, we can for example choose

$$\bigcup_{q_n \in \mathbb{Q}} \left(q_n - \frac{\varepsilon}{2^{n+1}}, q_n + \frac{\varepsilon}{2^{n+1}} \right)$$

and get stuck! We have only covered a subset of the reals of size ε, but we cannot use any other segment by density of $\mathbb{Q}$.

Corllary. If a set A has a Vitali cover consisting of its subsets (up to measure zero), then A is measurable.

Proof of the Lebesgue Density Theorem. We represent

$$A \setminus \phi(A) = \bigcup_k A_k$$

for

$$A_k = \left\{ a \in A : \liminf_{\delta \to 0^+} \frac{\lambda(A \cap B(a,\delta))}{2\delta} < 1 - \frac{1}{k} \right\}.$$

It suffices to show

$$\lambda^*(A_k) = 0$$

for all k to finish the proof. Since we may represent A as

$$A = \bigcup_{z \in \mathbb{Z}} A \cap [z-1, z+1]$$

and being a density point of A is the same as being a density point of one of the *cutouts* in the sum above, we may assume without loss of generality that $A \subseteq [0,1]$.

By definition of outer measure, we can approximate A_k from above by an open set U such that

$$\lambda^*(A_k) \leqslant \lambda(U) \leqslant \lambda^*(A_k) + \varepsilon.$$

Construct a covering

$$\mathcal{J} = \left\{ [a,b] : [a,b] \subseteq U,\ \lambda\Big(A \cap [a,b]\Big) \leqslant \left(1 - \frac{1}{k}\right) \lambda[a,b] \right\}.$$

It is a Vitali cover of A_k. By Vitali's Theorem we can pick a pairwise disjoint sequence of intervals $J_i \in \mathcal{J}$ for which

$$\lambda^* \left(A_k \setminus \bigcup_i J_i \right) = 0.$$

This gives

$$\begin{aligned}
\lambda^*(A_k) &= \lambda^* \left(A_k \cap \bigcup_i J_i \right) \\
&\leqslant \sum_i \lambda^*(A_k \cap J_i) \\
&\leqslant \sum_i \lambda^*(A \cap J_i) \\
&\leqslant \left(1 - \frac{1}{k}\right) \sum_i \lambda(J_i) \\
&\leqslant \left(1 - \frac{1}{k}\right) \lambda(U) \\
&\leqslant \left(1 - \frac{1}{k}\right) (\lambda^*(A_k) + \varepsilon).
\end{aligned}$$

The passage from line 2 to 3 may seem trivial, but is in fact crucial. This is the place where we use $A_k \subseteq A$! Otherwise the theorem is quite absurd, even for simple examples like $[0,1]$. Since $\lambda^*(A_k) \leqslant \lambda(A) < \infty$, we can rearrange this to obtain

$$\lambda^*(A_k) \leqslant (k-1)\varepsilon.$$

Since ε can be picked arbitrarily close to 0, we get

$$\lambda^*(A_k) = 0.$$

□

Proof of the Vitali Covering Theorem. The key to avoiding the *trap* we wrote about after stating the VCT is to choose the segments to be as large as possible – or at least not embarassingly small.

Without loss of generality, A is bounded since we can sum the coverings of $A \cap (n, n+1)$. The sequence of segments we pick is denoted J_n. In that case we may also assume $\bigcup \mathcal{J}$ is bounded. Its prefixes are

$$\begin{aligned} P_n &:= \bigcup_{i<n} J_i \\ \mathcal{J}_n &:= \{J \in \mathcal{J} : J \cap P_n = \varnothing\} \end{aligned}$$

and the *width* of what we can choose is

$$\gamma_n := \sup_{J \in \mathcal{J}_n} \operatorname{diam} J.$$

Note that in particular

$$\begin{aligned} P_1 &= \varnothing, \\ \mathcal{J}_1 &= \mathcal{J} \\ \gamma_1 &\leqslant \operatorname{diam} A < \infty. \end{aligned}$$

At each step, we choose J_n so that

$$\operatorname{diam} J_n \geqslant \frac{1}{2}\gamma_n,$$

or we stop if $\gamma_n = 0$ at some point. If indeed $\gamma_n = 0$, then by the definition of a Vitali cover, each $a \in A \setminus \mathcal{J}_{n-1}$ is an accumulation point of $\mathcal{J}_{n-1}$, but as a finite sum of closed intervals $\mathcal{J}_{n-1}$ is actually closed, so $a \in \mathcal{J}_{n-1}$.

Claim 1. The sequence γ_n converges monotonically to 0.

Proof of Claim 1. Being the supremum of ever decreasing sets, γ_n is decreasing. It is also positive, so the sequence converges and $\lim_n \gamma_n \geqslant 0$. Suppose that $\lim_n \gamma_n = c > 0$. Then in the construction, we would almost always choose disjoint intervals of diameter at least $c/2$. This is impossible, since $\bigcup \mathcal{J}$ was assumed to be bounded, so it has finite measure! □

If finitely many steps of the choice procedure are enough, there is nothing left to The key to proving that the choice procedure is correct will be the **blowup**, which we define for $J = [x-r, x+r]$ as

$$\widetilde{J} := [x - 5r,\, x + 5r]$$

Claim 2. At all steps of the construction

$$A \subseteq \mathcal{J}_n \cup \bigcup_{i \geqslant n} \widetilde{J}_i.$$

Proof of Claim 2. The set $\mathcal{J}_n$ is closed as a union of closed intervals. Therefore, if $a \in A \setminus \mathcal{J}_n$, there is a nondegenerate interval $I \ni a$. Since $\gamma_n \to 0$ by **Claim 1**, I is not considered in the construction of the sequence J_n for almost all n. Let n_0 be the last step where it is considered. Then we must have $I \cap J_{n_0+1} \neq 0$, because that is the step at which I is no longer considered.

We will show that this implies $a \in \widetilde{J}_{n_0+1}$. Let $J_{n_0+1} = [x - r, x + r]$ and $y \in I \cap J_{n_0+1}$. Then we have

$$\begin{aligned} d(a, x) &\leqslant d(a, y) + d(y, z) \\ &\leqslant \operatorname{diam} I + r \\ &\leqslant (2 \cdot \operatorname{diam} J_{n_0+1}) + r \\ &= 2 \cdot 2r + r \\ &= 5r, \end{aligned}$$

where the diameter bound comes from the definition of γ_{n_0} and the fact that I is still available at step n_0 of the construction. □

To finish the proof of Vitali's Covering Theorem, we compute that for all n

$$\begin{aligned} \lambda^* (A \setminus P_n) &\leqslant \lambda^* \left(A \cap \bigcup_{i \geqslant n} \widetilde{J}_i \right) \\ &\leqslant \lambda \left(\bigcup_{i \geqslant n} \widetilde{J}_i \right) \\ &\leqslant \sum_{i \geqslant n} \lambda(\widetilde{J}_i) \\ &= 5 \sum_{i \geqslant n} \lambda(J_i). \end{aligned}$$

These are the tails of the convergent series

$$\sum_{i=1}^{\infty} \lambda(J_i) = \lambda \left(\bigcup_i J_i \right) < \infty,$$

so we get

$$\lambda^* \left(A \setminus \bigcup_i J_i \right) \leqslant \lambda^* (A \setminus P_n) \to 0.$$

□

Remark. Retracing the argument behind **Claim 2.**, we might prove that for any $\alpha < 1$, if we define γ_n with a coefficient of α instead of $\frac{1}{2}$, the constant used for blowing up intervals can be brought down to

$$1 + \frac{2}{\alpha}.$$

In particular, we can get arbitrarily close to 3.

§2.1 Corollaries and the Lebesgue Differentiation Theorem

Theorem 2.3 (Lebesgue Differentiation Theorem)**.** *Let $f \in L^1(\mathbb{R})$. Then, for almost all x,*

$$\lim_{\delta \to 0^+} \frac{1}{2\delta} \int\limits_{x-\delta}^{x+\delta} f(s) \, \mathrm{d}\lambda(s) = f(x).$$

Proof. For characteristic functions, this is just a restatement of the Lebesgue Density Theorem. □

§2.2 Generalization to metric spaces

The argument in the proof of the VCT was written so that it is easliy generalizable to any metric space with a measure on its Borel sets.

To be more precise, what we need to lift the argument is that

$$\mu\left(B(x, 5r)\right) \leqslant C\mu\left(B(x, r)\right)$$

for some constant C. We can also substitute any constant larger that 3 instead of 5.

Chapter 3

One (Cantor) set to rule them all

§3.1 Ternary Cantor

Let us begin by making a construction. Take the closed interval $C_0 := [0,1]$ and remove the middle one third of it in such a way that the remaining two interval are closed. The result of this is

$$C_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Now, repeat the operation of cutting out the middle third and call the result C_2. We can repeat this *ad inifinitum* and obtain a decresing sequence of sets

$$[0,1] = C_0 \supset C_1 \supset C_2 \supset \dots.$$

Perhaps surprisingly, there are numbers which are not removed at any step, i.e. the intersection

$$\mathcal{C}_3 := \bigcap_{k=0}^{\infty} C_k$$

is nonempty! It contains 0 and 1. In fact, any number which can be written in base 3 using only 0's and 2's is an element of this intersection. These are in fact all such numbers. We introduce a tool to prove that.

Lemma 3.1. *Let $b \geqslant 2$ be a positional system base and x_0 be a number with k digits after the positional point. Then, the numbers formed by adjoining (perhaps infinitely many) digits to the base b representation of x_0 are all the numbers in the interval*

$$\left[x_0,\, x_0 + b^{-k}\right].$$

If we disallow the infinite extension by the digit $(b-1)$, we get the interval

$$\left[x_0,\, x_0 + b^{-k}\right).$$

Finally, if we allow only finite extensions, we get the b-ary numbers in the second interval.

Lemma 3.2. *A real number $x \in [0,1]$ is an element of $\mathcal{C}_3$ iff x can be written in base using only the digits 0 and 2.*

Proof (if). We proceed by induction with the induction thesis: *x belongs to $\mathcal{C}_3$ iff x can be written in base 3 so that its first k digits are 0 or 2.* It should be clear that this thesis is equivalent to the lemma statement. For each k, the statement is true by the previous lemma. □

§3.2 Abstract Cantor

The ternary Cantor set has many interesting properties. However, to study it, we will move to a more convenient representation. We can think of the Cantor set as the set of leaves of an infinite binary tree – starting from the root, at each level we choose whether to go right or left, or whether to insert 0 or 2 as the next digit in the base-3 representation of an $x \in \mathcal{C}_3$.

In this way, we can represent the ternary Cantor as

$$\mathcal{C} := \{0,1\}^{\mathbb{N}}.$$

That the map we just described is a bijection follows from

Lemma 3.3. *Let $d_k, \widetilde{d}_k$ be two sequences of base b digits. Then the corresponding real numbers are equal iff d_k and $\widetilde{d}$ agree on some prefix and afterwards one of them is 0 and the other is $(b-1)$.*

Proof. The condition implies equality of numbers by the sum of a geometric series. The other direction follows by looking at the first moment the expansions differ at then bounding the series sum. $\square$

What is missing from this description is the topology. We topologize $\mathcal{C}$ by the metric

$$d(x,y) = \begin{cases} 0 & \text{for } x = y \\ \frac{1}{n} & \text{for } x \neq y, \end{cases}$$

which can also be written succintly as

$$d(x,y) = \frac{1}{n_0(x,y)}$$

with the notation

$$n_0(x,y) := \inf\{n : x_n \neq y_n\}$$

for the first index at which x and y differ. The function d may not look like a metric at first sight, but in fact it has an even better property.

Lemma 3.4. *For the metric d described above we have for all $x, y, z \in \mathcal{C}$*

$$d(x,z) \leqslant \max\{d(x,y), d(y,z)\}.$$

In particular, d is an ***ultrametric****.*

Proof. Recall that $n_0(x,z)$ is the first position at which x and z differ. Then any y has to differ with at least one of y and z at n_0, but might even earlier. This gives

$$n_0(x,z) \geqslant \min\big(n_0(x,y), n_0(y,z)\big).$$

Since the function $x \mapsto 1/x$ is decreasing, the thesis follows. $\square$

We have established that $(\mathcal{C}, d)$ is a metric space. It is, in fact, homeomorphic with the subspace topology of $\mathcal{C}_3$ inherited from $[0,1]$.

Lemma 3.5. *The function*

$$h_3 : \mathcal{C} \to \mathcal{C}_3$$

defined by

$$h_3(x) := \sum_{k=1}^{\infty} \frac{2x_k}{3^k}$$

is a homeomorphism.

Proof. Bijectivity follows from the number-system lemma 3.3 and 3.2. For continuity, put down $n_0 := n_0(x, y)$ and compute

$$\begin{aligned} |h_3(x) - h_3(y)| &= \sum_{k=1}^{\infty} \frac{2|x_k - y_k|}{3^k} \\ &\leqslant \sum_{k=n_0}^{\infty} \frac{2}{3^k} \\ &= \frac{2}{3^{n_0}} \cdot \frac{3}{2} \\ &= \frac{1}{3^{n_0-1}}. \end{aligned}$$

The continuity of the inverse follows from the bound

$$|h_3(x) - h_3(y)| \geqslant \frac{2}{3^{n_0}}.$$

□

The function h_3 in 3.5 can be understood as a base 3 expansion operator. When we consider a base 2 expansion instead, we lose bijectivity, but we can cover the whole interval.

Lemma 3.6. *The function*

$$h_2 : \mathcal{C} \to [0, 1]$$

given by

$$h_2(x) := \sum_{k=1}^{\infty} \frac{x_k}{2^k}$$

is a continuous surjection.

Proof. Surjectivity follows from number system properties, and continuity is essentialy the same calculation as in the proof of 3.5. □

Theorem 3.1 (The Universal Property of the Cantor Set)**.** *Every metrizable compact space K is a continuous image of $\mathcal{C}$.*

Proof. Considering an element of $\mathcal{C}$ as a binary expansion, we have by 3.6 a surjection

$$h_2 : \{0, 1\}^{\mathbb{N}} \twoheadrightarrow [0, 1].$$

The space K can be embedded into the Hilbert cube by the **Urysohn Metrization Theorem ??**. By compactness of K, the image of the embedding is a compact and thus a closed subset. We also have a surjection

$$h : \{0, 1\}^{\mathbb{N}} \to [0, 1]^{\mathbb{N}}$$

by using the previous surjection and *unweaving* the Cantor set into the product of countably many Cantor sets, i.e. using

$$\mathbb{N} \cong \mathbb{N} \times \mathbb{N} \implies \mathcal{C} = \{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}\times\mathbb{N}} \cong \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} = \mathcal{C}^{\mathbb{N}}.$$

The last step is using the fact that any closed set of $\mathcal{C}$ is a retract of $\mathcal{C}$, which is **??**. □

A warning against generalization. If K is a compact set, it embeds into a *Tichonov Cube*

$$K \to [0,1]^{\Gamma}$$

and we can surject the Tichonov cube with a generalized Cantor set

$$\{0,1\}^{\Gamma},$$

but the universality theorem fails!

§3.3 Topology of the Cantor set

Definition 3.1 (Cantor Cylinder)**.** *Let*

$$\varphi : \mathbb{N} \multimap \{0,1\}$$

be a partial function with finite domain. Then we define the ***cylinder set*** *with base φ as*

$$[\varphi] := \{x \in \{0,1\} : x|_I = \varphi\}.$$

Lemma 3.7. *The sets $[\varphi]$ form a base of the topology of $\{0,1\}^{\mathbb{N}}$.*

Definition 3.2. *A set $A \subseteq \mathcal{C}$ is* ***determined*** *by $I \subseteq \mathbb{N}$, which we donte by $A \sim I$ if for all $x \in A$, $y \in \mathcal{C}$ we have*

$$x|_I = y|_I \implies y \in A.$$

Equivalently,

$$\pi_I^{-1}\pi_I[A] = A.$$

Lemma 3.8 (Clopen sets in the Cantor set)**.** *A set $A \subseteq \mathcal{C}$ is clopen iff $A \sim I$ for some finite $I \subseteq \mathbb{N}$. In particular, clopen sets can be written as a finite union of disjoint basis clopens $[\varphi_i]$ for φ_i with finite domain.*

(Direction one). If A is clopen, then

$$A = \bigcup_i [\varphi_i]$$

for some finitely many (by compactness) φ_i with finite domain I_i. Then

$$A \sim \bigcup_i I_i.$$

□

(Other direction). if $A \sim I$, blabla □

Immediately, a lemma follows.

Lemma 3.9 (Cantor set is zerodimensional)**.** *The Cantor set $\mathcal{C}$ is zerodimensional, i.e. it has a base of clopen sets.*

Theorem 3.2 (Topological characterisation of the Cantor set)**.** *If a topological space K is compact, metrizable, zerodimensional with no isolated points, then*

$$K \cong C.$$

§3.4 The group structure

The Cantor set has a natural abelian group structure given by its product structure. We can phrase it even more efficiently when we think of $\mathcal{C}$ as $\mathcal{P}(\mathbb{N})$ – the symmetric difference (or xor for the informatically inclined).

$$A \oplus B := A \Delta B$$

Every element has order two!

Fact. Together with the operation $\oplus$, the Cantor set $\mathcal{C}$ is a compact topological group, i.e. the function

$$(x, y) \mapsto x \oplus y$$

is continuous (in general the second element is inversed, but here every element is its own inverse anyway).

§3.5 Measure

We can define the measure on the Cantor set as a countable product of probability measures:

$$\nu = \bigotimes_{n=1}^{\infty} (\frac{1}{2}(\delta_0 + \delta_1)).$$

But we will do it by hand.

Definition 3.3. *Let $A \subseteq \mathcal{C}$ be clopen. Then*

$$A \sim \{1, 2, \ldots, n\}$$

for some n. Let

$$A' := \pi_{\{1,2,\ldots,n\}}[A]$$

We define its measure to be

$$\nu(A) := \frac{\#A'}{2^n}.$$

This makes sense with the probabilistic definition.

Theorem 3.3 (Well-definedness of the premeasure)**.** *The function*

$$\nu : \operatorname{Clop} \mathcal{C} \to \mathbb{R}$$

is a well-defined, additive function on the set algebra $\operatorname{Clop} \mathcal{C}$*.*

Proof. Since the Cantor set is compact, ν is automatically downward continuous on the empty set. By Caratheodory's Theorem, ν extends uniquely to a probabilistic measure on

$$\operatorname{Bor}\mathcal{C} = \sigma\left(\operatorname{Clop}\mathcal{C}\right).$$

What now??

$$\mathcal{A} := \{B \in \operatorname{Bor}\mathcal{C} : \forall\varepsilon > 0.\exists A \in \operatorname{Clop} C.\nu(A\Delta B) < \varepsilon\}.$$

We prove that this is a σ-algebra.

There is a nice formula for cylinders.

Lemma 3.10 (Measure of a cylinder)**.** *For a partial function*

$$\varphi : \mathbb{N} \multimap \{0, 1\},$$

its cylinder has measure

$$\nu\left[\phi\right] = 2^{-|\operatorname{dom}\varphi|}.$$

The result holds even if $\operatorname{dom}\varphi$ *is infinite, in which case the measure is* 0*.*

Proof. For finite-domain partial functions ϕ, take

$$\operatorname{dom}\varphi =: I \subseteq \{1, 2, \ldots, n\} =: [n]$$

for some n. Then

$$\left|\pi_{[n]}[\varphi]\right| = \frac{2^{n-|I|}}{2^n} = 2^{-|I|}.$$

For infinite-domain functions ϕ, taking a decreasing intersection

$$[\phi] = \bigcap_n \left[\phi|_{[n]}\right]$$

shows that the measure of the intersection is 0. □

Theorem 3.4. *The measure* ν *is the Haar measure on* $\mathcal{C}$*, that is, the unique probability measure invariant under group actions*

$$\nu\left(x \oplus B\right) = \nu(B)$$

for all $x \in \mathcal{C}$*,* $B \subseteq \mathcal{C}$*.*

Proof. Let us first consider $B = [\varphi]$, and $I = \operatorname{dom}\varphi$. Then

$$\nu\left(x \oplus [\varphi]\right) = \nu\left([x \oplus \varphi]\right) = \nu\left([\varphi]\right).$$

A clopen is a disjoint sum of $[\varphi_i]$ for finitely many φ_i, so additivity on clopens follows. Now, take a superficially different measure

$$\nu_x(B) := \nu\left(x \oplus B\right).$$

Since ν and ν_x agree on clopens, by uniqueness in Caratheodory's Theorem they agree on all sets. □

Note the isomorphism

$$(C, \oplus) \cong (\mathcal{P}(\mathbb{N}), \Delta)$$

of (topological) groups.

§3.6 Normal number theorem

Definition 3.4. *Let $A \subseteq \mathcal{C}$. We call A a **tail** set if*

$$A \sim \{k : k \geqslant n\}$$

for all n. Equivalently, if $x \in A$ and $x(n) = y(n)$ for almost all n, then $y \in A$.

Example. A naturally occuring example of a tail set is

$$A_\beta := \left\{ x \in \mathcal{C} : \lim_n \frac{x(1) + \ldots x(n)}{n} = \beta \right\}.$$

Theorem 3.5 (Kolmogorov zero-one law for the Cantor set)**.** *A borel tail set $A \subseteq \mathcal{C}$ has measure 0 or 1.*

Proof. Take a basis set $[\varphi]$. We have

$$\nu\left([\varphi] \cap A\right) = \nu\left([\varphi]\right) \cdot \nu(A).$$

From this immediately follows that this work for any $B \in \operatorname{Clop} \mathcal{C}$. Now approximate A by a clopen B so that

$$\nu\left(A \Delta \mathcal{C}\right).$$

To finish the proof, compute

$$\nu(A) \cdot \nu(B) = \nu(A \cap B) \geqslant \nu(A) - \varepsilon \nu(A).$$

□

Returning to the example we have $\nu(A_\beta) \in \{0, 1\}$. We have

$$\nu(A_\beta) = \nu(A_{1-\beta}).$$

Theorem 3.6 (Borel's normal number theorem)**.**

$$\nu\left(A_{\frac{1}{2}}\right) = 1.$$

Remark. According to Billingsley, this theorem was the founding work of modern probability theorem, which is founded on limit theorems.

Proof. Denote for $\alpha < \frac{1}{2}$

$$B_n^\alpha = \left\{ x \in \mathcal{C} : \frac{x_1 + x_2 + \ldots + x_n}{n} \leqslant \alpha \right\}.$$

We claim that there exists a θ such that

$$\nu\left(B_n^\alpha\right) \leqslant \theta^n.$$

Then

$$\nu\left(B_n^\alpha\right) = \frac{c_n}{2^n},$$

where

$$c_n = \sum_{k=1}^{\lfloor \alpha n \rfloor}.$$

□

Chapter 4

Measures on separable, metrisable topological spaces

§4.1 Basic properties

For brevity, we will denote the class of separable and metrisable topological spaces by $\mathcal{SM}$. A lot of the time, it is easier to work with such spaces in a *common box*, i.e. use a universal space in which all of these spaces can be embedded. Luckily, we have such a space – the Hilbert Cube.

Ankified

Theorem 4.1. *Every $\mathcal{SM}$ topological space embeds in the Hilbert Cube.*

Proof. Fix a metric $d \leqslant 1$ and a countable dense subset $x_n \in X$. We define the embedding as

$$f_n(x) := d(x, x_n).$$

This is a product of continuous functions, so it continuous. It is injective, as if $f(x) = f(y)$, then a subsequence of x_n convergent to x is also convergent to y, so $x = y$.

The most difficult fact is that this is open. To see this, take $x \in U \subseteq X$ with U open. We will show that $f[U]$ is open. The neighbourhood U contains some ball $B(x, r)$. We can find an element x_k of the countable dense set such that $d(x, x_k) < r/4$. Then

$$x \in B(x_k, r/2) \subseteq B(x, r) \subseteq U,$$

which implies

$$f(x) \in f\left[B(x_k, r/2)\right] \subseteq f[U].$$

But, by definition of f,

$$f\left[B(x_k, r/2)\right] = f[X] \cap \pi_k^{-1}(-\infty, r/2).$$

Since x was an arbitrary element of U, we have that $f[U]$ is open, so f is a homeomorphism onto its image! $\square$

You may wonder how this relates to the fact that all compact metrisable embed in the Hilbert Cube (as closed sets!). In turns out that compact metrisable spaces are $\mathcal{SM}$. We only need separability, and compactness together with a covering by balls gives us a countable dense subset rather easily.

Lemma 4.1. *If a metrisable topological K is compact, then it is separable.*

Proof. For each n, finitely many balls of radius $1/n$ cover K by compactness. Taking the centers of all such balls over all n yields a countable dense subset. □

The proof above can be trivially extended to totally bounded spaces and Lindelöf spaces. In the second case, we have countably many centers of balls at each step.

Lemma 4.2. *Let K be a metrisable topological space with is either*

1. *compact,*
2. *Lindelöf,*
3. *σ-compact,*
4. *or totally bounded.*

Then K is separable.

We will now investigate for a moment how properties of $\mathcal{SM}$ spaces are reflected in functions on such spaces. Since we care about topology, we restrict our attention to continuous functions. Unfortunately, even continuous functions on an arbitrary $\mathcal{SM}$ space can have an untame structure. Therefore we restrict our attention to bounded functions.

Definition 4.1. *The space of bounded, continuous functions from a topological space X to $\mathbb{R}$ is denoted by*

$$C_b(X).$$

If we want bounded functions into $\mathbb{C}$, we use the notation

$$C_b(X;\mathbb{C}).$$

This function space has the obvious structure of a linear space, and even an algebra with pointwise addition, scaling and multiplication. This space also has its own topology induced by the supremum norm.

Lemma 4.3. *Let X be an arbitrary.*

$$C_b(X)$$

is a Banach algebra under pointwise operations and the supremum norm.

Proof for compact spaces. Take a Cauchy sequence f_n. For each $x \in X$, $f_n(x)$ is Cauchy, so it converges. Therefore, the sequence of functions coverges pointwise to a limit function f. Suppose the convergence is not uniform. Then for some $\varepsilon > 0$ we can take a sequence x_n such that

$$f_n(x_n) - f(x_n) \geqslant \varepsilon.$$

By the $\varepsilon/3$ trick, f is continous. By compactness of X, x_n has a subsequence convergent to x_0. Since $f_n(x_0) \to f(x_0)$ and $f(x_n) \to f(x_0)$. TODO! □

Proof. The only thing one need to check is that Cauchy sequences actually converge. Let f_n be a Cauchy sequence. For any $x \in X$, $f_n(x)$ is a Cauchy sequence of real numbers, so it converges. Therefore, f_n converges pointwise to a function f. Note that so far we don't know if the convergence is uniform, or even if the function is continuous.

Let N_ε be the point after which the sequence f_n is ε-close. Then for $n, m > N_\varepsilon$ we have

$$|f_n - f_m| \leqslant \varepsilon$$

uniformly on X. Keeping n constant and passing with m to the limit we have

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leqslant \varepsilon.$$

Therefore, f_n converges to f uniformly. In particular, f is continuous and bounded. $\square$

We recall here a useful theorem, whose proof can be found in the literature.

Theorem 4.2 (Stone-Weierstrass)**.** *Let K be a compact, Hausdorff space and let $W \subseteq C_b(K)$ be a subalgebra. If the subalgebra distinguishes points, we have*

$$\overline{W} = C_b(K).$$

Proof. A classic proof due to Lebesgue can be found in Engelking (or prof. Szwarc's functional analysis notes). $\square$

The first and so far only property we investigate for $C_b(X)$ is separability. It turns out that this space is rarely separable, and an exact characterisation can be given in terms of X.

Ankified

Theorem 4.3. *Let X be an $\mathcal{SM}$ topological space. Then $C_b(X)$ is separable iff X is compact.*

Proof (⇐). We know by 4.1 that X can be regarded as a subspace of $[0, 1]^{\mathbb{N}}$. Since X is compact, it is a closed subset of the cube. By the Stone-Weierstrass Theorem 4.2, the algebra generated by coordinate projections, which consists of finite linear combinations of finite products of coordinate projections, is dense in $C_b(X)$. This subalgebra is in general not countable, however. One small fix is required to find a countable dense subset of $C_b(X)$ – only take rational coefficients in linear combinations. $\square$

Proof (⇒). We'll mirror the proof of the nonseparability of $C_b(\mathbb{R})$ – we will find $\mathfrak{c}$ many balls of radius $\frac{1}{2}$.

Since X is noncompact and metrisable, we have a discrete sequence of elements of X. Call it a_n and let $A = \{x_n : n \in \mathbb{N}\}$. For a subset I of the natural numbers, we define the function $f_I : A \to [0, 1]$ by

$$f_I(x_i) := \begin{cases} 1 & i \in I \\ 0 & i \notin I. \end{cases}$$

We can extend all these to function $\widetilde{f}_I : A \to [0, 1]$ with the Tietze Extension Theorem. Now, for $I \neq J$, if we look at an element $x_0 \in I \Delta J$ we get

$$\left\| \widetilde{f}_I - \widetilde{f}_J \right\|_\infty \geqslant \left| \widetilde{f}_I(x_0) - \widetilde{f}_J(x_0) \right| = 1.$$

$\square$

§4.2 Polish spaces

We now turn to a subclass of $\mathcal{SM}$ spaces which is particularly useful and important.

Definition 4.2. *A **Polish space** is an $\mathcal{SM}$ space X, which is completely metrisable.*

Please note that this depends on the topology and not on any given metric for the space, as the example below shows.

Example. The space $(0,1)$ is Polish, since it is homeomorphic to $(0, \infty)$. However, it is definitely not complete with regards to its standard metric! An explicit complete metric can be given by

$$d(x,y) := \left|\operatorname{tg}\frac{x\pi}{2} - \operatorname{tg}\frac{y\pi}{2}\right|,$$

which is the pullback of the complete metric from $(0, \infty)$ by a homeomorphism.

Example. An even weirder example is

$$\mathbb{R} \setminus \mathbb{Q} \cong \mathbb{N}^{\mathbb{N}}.$$

That this is completely metrisable can be seen from the following theorem.

Theorem 4.4 (Alexandroff)**.** *A subspace Y of a Polish space X is itself Polish iff Y is a G_δ subset of X.*

Proof. Let ρ' be a new metric on Y given by

$$\rho'(y_1,y_2) = \rho(y_1,y_2) + \sum_n \min\left(\frac{1}{2^n}, \left|\frac{1}{\rho(y_1,V_n^c)} - \frac{1}{\rho(y_2,V_n^c)}\right|\right)$$

The rest of the details can be found in Kerchis' classical book on Descriptive Set Theory. □

Definition 4.3. *In a topological space X, the* ***Borel subsets*** *of X are precisely the elements of* $\operatorname{Bor} X := \sigma(\tau_X)$.

Definition 4.4. *For a topological, but especially $\mathcal{SM}$ or Polish space X, we denote the set of* ***probability measures*** *on* $\operatorname{Bor} X$ *by* $\mathbb{P}(X)$.

We need a tool before stating proving the next theorem.

Lemma 4.4. *A closed set in a metric space is G_δ. Conversely, any open set is F_σ.*

Proof. We will use the ε-neighbourhoods of F, i.e.

$$F_\varepsilon := \{x \in X : d(x,F) < \varepsilon\},$$

which are open. Since F is closed, we have

$$F = \bigcap_{n=1}^{\infty} F_{1/n}.$$

□

Theorem 4.5 (First Regularity Theorem)**.** *For any $\mathcal{SM}$ (in fact, any metrisable) space X and $\mu \in \mathbb{P}(X)$, the measure μ is* ***regular****, that is for any $B \in \operatorname{Bor} X$ and $\varepsilon > 0$ there are two sets*
Ankified *$F \subseteq B \subseteq V$, respectively closed and open, such that*

$$\mu(V \setminus F) < \varepsilon.$$

Proof. Let $\mathcal{A}$ be the family of all sets with the given propety. We will prove that it is a σ-algebra and that it contains closed sets. By 1.1, we only have to check for complements, finite sums and ascending countable sums.

Closed sets are in $\mathcal{A}$ because they are G_δ, see Lemma 4.4, and because a probability measure is downward continuous.

Closure under complements is inherent in the definition. If $F \subseteq B \subseteq V$, then

$$V^c \subseteq B^c \subseteq F^c,$$

F^c is open, V^c is closed and

$$F^c \setminus V^c = V \setminus F,$$

so the approximation still works.

Finite sums are easy. If $F_i \subseteq B_i \subseteq V_i$ for $1 \leqslant i \leqslant n$ with

$$\nu\left(V_i \setminus F_i\right) \leqslant \frac{\varepsilon}{n},$$

then

$$\bigcup_{i=1}^n F_i \subseteq \bigcup_{i=1}^n B_i \bigcup_{i=1}^n V_i$$

and

$$\bigcup_{i=1}^n V_i \setminus \bigcup_{i=1}^n F_i \subseteq \bigcup_{i=1}^n (V_i \setminus F_i).$$

Passing to measure

$$\nu\left(\bigcup_{i=1}^n V_i \setminus \bigcup_{i=1}^n F_i\right) \leqslant n \cdot \frac{\varepsilon}{n} = \varepsilon.$$

This works, because we have implicitly used that finite sums of closed sets are closed, and the same for open sets. However, this fails for countable sums in the *closed* part. How do we repair this? Let's do the setup first. For an increasing sequence B_n take approximations $F_n \subseteq B_n \subseteq V_n$ such that

$$\mu\left(V_n \setminus F_n\right) < \frac{\varepsilon}{2^{n+1}}$$

and let $B = \bigcup_n B_n$. By summing prefixes of F_n, we may assume that F_n is an increasing sequence. Then

$$\bigcup_{n=1}^\infty F_n \subseteq \bigcup_{n=1}^\infty B_n \subseteq \bigcup_{n=1}^\infty V_n$$

and

$$\bigcup_{n=1}^\infty V_n \setminus \bigcup_{n=1}^\infty F_n \subseteq \bigcup_{n=1}^\infty (V_n \setminus F_n)$$

(this is a general identity on sets). We are now facing the closed set problem again. Fortunately, all measures are upward continuous, so we can take a really good approximation by a prefix! Namely,

$$\mu\left(\bigcup_{n=1}^\infty F_n \setminus \bigcup_{n=1}^N F_n\right) < \frac{\varepsilon}{2},$$

and obtain an approximation

$$\bigcup_{n=1}^N F_n \subseteq B \subseteq \bigcup_{n=1}^\infty V_n$$

with
$$\mu\left(\bigcup_{n=1}^{\infty} V_n \setminus \bigcup_{n=1}^{N} F_n\right) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2} = \varepsilon.$$
□

Remark. This proof does not use separability at all, and only uses metrisability to obtain the supplementary lemma 4.4. Similarly, we only need that μ is a probability measure to get downward continuity. Therefore, the proof lifts immediately to σ-finite measures on spaces where closed sets are G_δ.

Remark. This implies that we don't care much about Descriptive Set Theory. For $X \in \mathcal{SM}$ and $\mu \in \mathbb{P}(X)$, we only care about F_σ and G_δ sets. More formally, for any $B \in \operatorname{Bor} X$, any set is up to a set of measure 0 an F_σ from below and G_δ from above.

Remark. An analytical set is an image of a Polish space.

Ankified **Theorem 4.6.** *If X is a Polish space and $\mu \in \mathbb{P}(X)$, then μ is **tight**, i.e. for every $\varepsilon > 0$ there is a compact K such that*
$$\mu(K) > 1 - \varepsilon.$$

Proof. Let $d(-,-)$ be a complete metric on X and let x_n be a countable dense set.
$$X = \bigcup_{k=1}^{\infty} B\left(x_k, \frac{1}{n}\right).$$
By upward continuity of measure, we can take a k_n such that the first k_n balls are large, i.e. larger than
$$1 - \frac{\varepsilon}{2^n}.$$
Now denote
$$K_n := \overline{\bigcup_{k=1}^{k_n} B\left(x_k, \frac{1}{n}\right)},\ K = \bigcap_{n=1}^{\infty} K_n.$$
To see that K is large in μ, use Lemma 1.5. Note that K is closed.

To prove compactness, we will find for a sequence x_k in K a convergent subsequence. We mirror the proof that $[0, 1]$ is sequentially compact. Infinitely many elements of x_k will belong to one of the (finitely many!) balls that make up K_1, and of those inifitely many will land in one of the balls that make up K_2 etc. Therefore, x_n has a Cauchy subsequence. Since X is Polish, this subsequence is convergent in X, but as K is closed, the limit is actually in K. □

Example. Let $X = \mathbb{N}^{\mathbb{N}}$.

Theorem 4.7 (Lusin)**.** *If $X \in \mathcal{SM}$ and $f : X \to \mathbb{R}$ is a borel function, then for any $\mu \in \mathbb{P}(X)$ and $\varepsilon > 0$ there exists a closed set F of large measure such that f is continuous on F.*

Lemma 4.5. *Let (X, d) be a complete metric space and $F_n \subseteq X$ be a descending sequence of nonempty closed sets. If*
$$\operatorname{diam} F_n \to 0$$
then the intersection $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

Chapter 5

Measures on Topological Spaces, Problemset 1

Problem 4

Extension 1

We show that the set can be the graph of a function! Let Z be a borel set of positive measure and define

$$T_Z = \{x : \lambda(Z_x) > 0\}.$$

Then T_Z is a measurable set by Fubini's Theorem. We can pick a compact subset T'_Z. A compact set of positive measure has at least $\mathfrak{c}$ elements, and there are as many borel sets. Then, enumerate borel sets of $\mathbb{R}^2$.

Problem 5

Set of undefined density at 0

TODO

Set of density t at 0

Presented in class by **Michał Baran**. Fix $t \in (0, 1)$.

The set we will construct will be symmetric around 0. We will find a sequence b_n such that with

$$A_n = \left(\frac{1}{n} - b_n, \frac{1}{n}\right)$$

we will have for all n

$$\frac{t}{n} = \lambda\left(\bigcup_{k=n}^{\infty} A_n\right) = \sum_{k=n}^{\infty} b_k,$$

so

$$b_n = \sum_{k=n}^{\infty} b_k - \sum_{k=n+1}^{\infty} b_k = \frac{t}{n(n+1)}.$$

Consider

$$A := \bigcup_{k=1}^{\infty} A_k \cup -A_k.$$

We will bound the fraction

$$\frac{\lambda\left(A \cap (-\delta, \delta)\right)}{2\delta} = \frac{\lambda\left(A \cap (0, \delta)\right)}{\delta}$$

from above and below. For $\delta \in (1/(n+1), 1/n]$ we have

$$\bigcup_{k=n+1} A_k \subseteq A \cap (0, \delta) \subseteq \bigcup_{k=n} A_k,$$

passing to measure

$$\frac{t}{n+1} \leqslant \lambda\left(A \cap (0, \delta)\right) \leqslant \frac{t}{n}.$$

When divided by δ, we get the result by the squeeze theorem.

Remark. The solution would work equally well if instead of $a_n = 1/n$ we used a sequence that converges to 0 monotonically and satisfies

$$\frac{a_n - a_{n+1}}{a_n} \to 0.$$

Problem 9

Presented in class by **dr Arturo Martinez Celiz**.

Wlog, everything happens within (0, 1). Following the hint, take a countable sequence A_i such that the set $B := \bigcup_i A_i$ has maximal measure.

By this choice, for any $C \in \mathcal{A}$, we have

$$\lambda\Big((C \cup B)\, \Delta B\Big) = 0,$$

so that

$$\phi(C \cup B) = \phi(B)$$

and

$$C \subseteq \phi(C \cup B) = \phi(B).$$

Since C was arbitrary

$$\bigcup \mathcal{A} \subseteq \phi(B) \implies B \subseteq \bigcup \mathcal{A} \phi(B).$$

Since $\lambda(B) = \lambda\left(\phi(B)\right)$ we know that the sum of $\mathcal{A}$ is measurable.

Problem 10

Presented in class by **Szymon Smolarek**.

We take a cover of **regular sets**, i.e. a family for which there exists a constant C such that

$$\operatorname{diam}^2 A \leqslant C\lambda_2(A).$$

It can be proven that if such a family is a Vitali cover of a set $A \subseteq \mathbb{R}^2$, an analogue of the VCT holds.

The family of all triangles does not satisfy the regularity condition – think of keeping one segment constant and bringing the third verted evert closer to the segment. To deal with this, we subdivide the family $\mathcal{T}$ into subfamilies

$$\mathcal{T}_n := \left\{T \in \mathcal{T} : \operatorname{diam}^2 T \leqslant n\lambda_2(A)\right\}.$$

Reducing to a given subfamily, we can cover each triangle T by arbitrarily small traingles similar to T contained within T. This gives us a regular Vitali cover $\widetilde{T}_n$ of $\bigcup \mathcal{T}_n$.

Problem 11

Stated in class by **Szymon Smolarek**.

Theorem 5.1 (Steinhaus theorem for the Cantor Set)**.** *For any measurable set A, the set*

$$A \oplus A$$

contains an open neighbourhood of 0.

Theorem 5.2 (Vitali Covering Theorem for the Cantor Set)**.** *If a family of clopens $\mathcal{J} \subseteq \operatorname{Clop}\mathcal{C}$ is a Vitali cover of A, then there is a sequence $J_n \in \mathcal{J}$ such that*

$$\nu^*\left(A \setminus \bigcup_n J_n\right) = 0.$$

Theorem 5.3 (Lebesgue Density Theorem for the Cantor Set)**.** *Let $A \subseteq \mathcal{C}$. An element $a \in A$ is a **density point** of A if*

$$\lim_{n\to\infty} \frac{\nu\left(A \cap [a|_{[n]}]\right)}{2^{-n}} = 1.$$

If A is measurable, then almost all points of A are density points of A.

The proofs are quite the same, as $\mathcal{C}$ is a topological group and the measure ν is its Haar measure.

Problem 12

Hint. Use Baire's theorem.

Chapter 6

Measures on Topological Spaces, Problemset 2

Problem 1

Such an a exists by compactness of A and continuity of metric. If we have two a_1, a_2 such that

$$\rho(x, a_1) = \rho(x, a_2)$$

then a_1, a_2 must agree and disagree with x at all places, so in fact $a_1 = a_2$, thus r_A is well-defined. For any $a \in A$, $d(a, A) = 0 = d(a, a)$, so r_A is a retraction. What remains to be shown is continuity.

Let x, y agree up to $n_0(x, y)$. Then $r_A(x)$ and $r_A(y)$ also agree up to $n_0(x, y)$ – if they differed earlier, we could use $r_A(x)$ instead of $r_A(y)$ and get a closer point a in the definition! So we have

$$n_0(x, y) \leqslant n_0\left(r_A(x), r_A(y)\right)$$

and

$$d(x, y) \geqslant d\left(r_A(x), r_A(y)\right).$$

Remark. The metric $d(x, y) = 1/n_0(x, y)$, i.e. the first moment where x and y differ, won't work, because it can't tell apart points from which x differs at the same position!

Problem 2

Problem 3

Any $A, B \in \operatorname{Clop} \mathcal{C}$ can be written as disjoint sums of the basis cylinders $[\varphi]$ by 3.8. Since the condition distributes over disjoint sums, we will prove the statement for $A = [\varphi]$ and $B = [\psi]$ with

$$\left|\operatorname{dom} \varphi\right|, \left|\operatorname{dom} \psi\right| < \infty.$$

Let $I = \operatorname{dom} \varphi$, $J = \operatorname{dom} \psi$ be the disjoint(!) domains of φ, ψ. There is a function τ on $I \cup J$ such that

$$\tau|_I = \phi,\ \tau|_J = \psi.$$

For such a function,

$$[\varphi] \cap [\psi] = [\tau] \, .$$

Now take an n such that $I \cup J \subseteq \{1, 2, \ldots, n\}$ and denote the last set as $[n]$. By 3.10 we compute

$$\begin{aligned} \nu\,[\varphi] &= 2^{-|I|} \\ \nu\,[\psi] &= 2^{-|J|} \\ \nu\,[\tau] &= 2^{-|I \cup J|}, \end{aligned}$$

and $|I \cup J| = |I| + |J|$ finishes the proof.

The general case

Now take arbitrary $A, B \in \operatorname{Bor} \mathcal{C}$ such that $A \sim I$, $B \sim J$. Approximate A, B by clopens A', B' to within an ε, i.e. so that

$$\nu(A \Delta A'),\, \nu(B \Delta B') < \varepsilon.$$

We cannot use the clopen statement we just proved since a priori A' and B' could be determined by sets with nonempty intersection. We can, however, improve the approximation with

$$\widetilde{A} := \pi_I^{-1} \pi_I A'.$$

The set $\widetilde{A}$ is still a clopen – since A' was determined by a finite set K, $\widetilde{A}$ is determined by $K \cap I$. Additionally we have

$$\widetilde{A} \Delta A \subseteq A' \Delta A,$$

so we have improved the approximation! Now, do the same for B' and use the statement for clopens to finish up the solution.

Warning! The reasoning below does not work! (For tail sets, for example)

We can approximate A, B by decreasing sequences of clopens by putting down

$$A_n := \pi_{[n]}^{-1} \pi_{[n]} A$$

and the same for B_n. We also approximate their intersection by decreasing clopens in the same way, i.e.

$$C_n := \pi_{[n]}^{-1} \pi_{[n]} (A \cap B).$$

For these approximations

$$C_n = A_n \cap B_n,$$

so by the first subproblem

$$\nu(C_n) = \nu(A_n \cap B_n) = \nu(A_n) \cdot \nu(B_n).$$

Since the measure ν is probabilistic, and hence continuous, by passing to the limit $n \to \infty$ we get what we need.

Problem 4

Presented in class by **Szymon Smolarek**.

We estimate the complement of this set, which we denote by

$$B(\varepsilon) = A(\varepsilon)^c.$$

Consider a sequence of partial functions

$$\varphi_k : [kn,\, (k+1)n) \multimap\!\!\to \{0,1\}$$

given by $\varphi_k(j) := \varepsilon(j - kn)$, thinking of ε as a partial function $\varepsilon : [0,n) \multimap\!\!\to \{0,1\}$. Then

$$B(\varepsilon) \subseteq \bigcap_{k=1}^{\infty} [\varphi_k]^c,$$

so by the previous problem and 3.10 we have

$$\nu(B(\varepsilon)) \leqslant \nu\left(\bigcap_{k=1}^{\infty} [\varphi]^c\right) \leqslant \prod_{k=1}^{\infty} \nu\left([\varphi_k]^c\right) = 0,$$

since $\nu\left([\varphi_k]^c\right) = 1 - 1/2^n$.

Problem 6

Any clopen $C \in \operatorname{Clop}\mathcal{C}$ is a disjoint sum of basis cylinders by 3.8. Since $\oplus$ is a group operation, the function

$$l_x(y) = x \oplus y$$

is bijective, so on the level of sets l_x distributes over disjoint sums. We check the property for a cylinder $[\varphi]$. This is easy, since

$$\nu\Big(x \oplus [\varphi]\Big) = \nu[x \oplus \varphi] = 2^{-|\operatorname{dom}\varphi|} = \nu[\varphi]$$

by 3.10. Now consider the family of sets

$$\mathcal{A} := \Big\{A : \forall x \in \mathcal{C}.\, \nu(A) = \nu(x \oplus A)\Big\}.$$

We will show that this is a σ-algebra. Since we have already shown that it contains all the clopens, which form a basis of the topology on $\mathcal{C}$, it will automatically be equal to $\operatorname{Bor}\mathcal{C}$ by 1.2.

A σ-algebra can be generated by complements and countable sums (see 1.1). As mentioned before, l_x respects these operations, so

$$\nu(x \oplus A^c) = \nu\Big((x \oplus A)^c\Big) = 1 - \nu(x \oplus A) = 1 - \nu(A) = \nu(A^c)$$

and

$$\nu\left(x \oplus \bigcup_i A_i\right) = \nu\left(\bigcup_i x \oplus A_i\right) = \sum_i \nu(x \oplus A_i) = \sum_i \nu(A_i) = \nu\left(\bigcup_i A_i\right).$$

Problem 7

The identification is

$$A \mapsto \chi_A,\, x \mapsto \{n : x_n = 1\}\,.$$

One easily checks that these two are mutually inverse. Addition modulo 2 comes out to 1 iff exactly one of the summands is 1, and this corresponds exactly to belonging to the symmetric difference.

Problem 8

A filter cannot contain both A and A^c, since then it would contain $A \cap A^c = \varnothing$. Thus, a filter containing for all A either A or A^c is maximal.

For the other direction, suppose neither A nor A^c is in a filter $\mathcal{F}$. We define its *extension* by A as

$$\mathcal{F}_A = \{A' \cap F : A \subseteq A', F \in \mathcal{F}\}\,.$$

We check that this is a filter.

1. If $\varnothing \in \mathcal{F}_A$, $\mathcal{F}$ contains a set disjoint with A, so by the superset property it contains A^c.
2. Let $A_1 \cap F_1, A_2 \cap F_2 \in \mathcal{F}_A$. Then
$$A \subseteq A_1 \cap A_2, F_1 \cap F_2 \in \mathcal{F},$$
so $(A_1 \cap F_1) \cap (A_2 \cap F_2) = (A_1 \cap A_2) \cap (F_1 \cap F_2) \in \mathcal{F}_A$.
3. Let $B \supseteq A' \cap F$. Then
$$B = B \cup (A' \cap F) = (B \cup A') \cap (B \cup F)$$
and $A \subseteq A' \cup B$, $F \subseteq B \cup F$, so $B \in \mathcal{F}_A$.

Of course, $A \in \mathcal{F}_A \setminus \mathcal{F}$, so $\mathcal{F}$ was not maximal in the first place.

Remark. One can check that $\mathcal{F}_A$ is the minimal filter containing $\mathcal{F}$ and A.

Remark. A filter is free iff it contains the Frechet filter.

Problem 9

Non-measurability

Take an $F \subseteq \mathcal{C}$ corresponding to $\mathcal{F}$ is the sense of problem 7. Assume F is measurable. By the Zero-One Law, either $\nu(F) = 1$ or $\nu(F) = 0$.

Take the set of complements of F, i.e. $F \oplus \mathbb{1}$. We have

$$F \cap (F \oplus \mathbb{1}) = \varnothing,\, F \cup (F \oplus \mathbb{1}) = \mathcal{C}.$$

We now have $\nu(F) = 1/2$, which is a contradiction.

Remark. The only principal ultrafilters are generated by singletons, so they are definitely measurable.

Outer/inner measure

Approximate the filter F by a tail Borel set.

Problem 10

By the $\pi - \lambda$ Lemma, it is enough to check this for dyadic intervals, i.e. the intervals

$$\left[\frac{k}{2^l}, \frac{k+1}{2^l}\right].$$

By 3.1, modulo a point this corresponds to the cylinder specifying k in the binary number system. This has size $1/2^l$ by 3.10, which agrees with the Lebesgue measure.

Problem 11

Just do what's in the hint :). For measurability of f, it is enough to check that $f^-1[C]$ is measurable for a basis cylinder $C = [\varphi]$. But

$$f^{-1}[C] = A_\varphi,$$

which is measurable.

Chapter 7

Measures on Topological Spaces, Problemset 3

Problem 1

By transfinite induction, each $B \in \operatorname{Bor} Y$ is of the form $\widetilde{B} \cap Y$ for some $\widetilde{B} \in \operatorname{Bor} X$.

The axiom $\nu(\varnothing) = 0$ is immediate from $\varnothing = \varnothing \cap Y$.

For countable additivity, take a sequence of pairwise disjoint sets $B_n \in \operatorname{Bor} Y$. By an earlier observation, we may represent them as $Y \cap \widetilde{B}_n$ for a sequence $\widetilde{B}_n \in \operatorname{Bor} X$. By Problem 2, we may in fact assume

$$\mu(\widetilde{B}_n) = \mu^*(Y \cap B_n)$$

if we take $\widetilde{B}_n$ to be measurable hulls. If B_n, B_m give disjoint sets in Y, then we have

$$\widetilde{B}_n \cap \widetilde{B}_m \subseteq (\widetilde{B}_n \setminus Y) \cup (\widetilde{B}_m \setminus Y)$$

so when we pass to outer measure we see that $\mu(\widetilde{B}_n \cap \widetilde{B}_m) = 0$. Note that $B_n \cap Y = \widetilde{B}_n \cap Y$. We have

$$0 \leqslant \mu^* \left(\bigcup_{n=1}^{\infty} \widetilde{B}_n \setminus Y \right) \leqslant \sum_{n=1}^{\infty} \mu^*(\widetilde{B}_n \setminus Y) = 0,$$

so

$$\begin{aligned} \mu^* \left(\bigcup_{n=1}^{\infty} \widetilde{B}_n \cap Y \right) &= \mu^* \left(\bigcup_{n=1}^{\infty} \widetilde{B}_n \right) \\ &= \mu \left(\bigcup_{n=1}^{\infty} \widetilde{B}_n \right) \\ &= \sum_{n=1}^{\infty} \mu \left(\widetilde{B}_n \right) \\ &= \sum_{n=1}^{\infty} \mu^* \left(\widetilde{B}_n \cap Y \right). \end{aligned}$$

Problem 2

By definition of outer measure (as an infimum), we can choose measurable sets $H_n \supseteq Z$ such that

$$\mu^*(Z) \leqslant \mu(H_n) < \mu^*(Z) + \frac{1}{n}.$$

Take

$$H = \bigcap_{n=1}^{\infty} H_n.$$

This H contains Z, so we have

$$\mu(H) = \mu^*(Z)$$

by squeezing. By regularity of borel measures (see 4.5), we can take a G_δ upper approximation of H with the same measure.

For the second part, take two measurable hulls H_1 and H_2. Since $Z \subseteq H_1 \cap H_2$, we have

$$H_1 \Delta H_2 \subseteq (H_1 \setminus Z) \cup (H_2 \setminus Z),$$

and the RHS has (outer) measure 0, so the LHS does as well.

Observation. Hulls work well with set unions, i.e. for a countable union of Z_i with hulls H_i, the union of H_i is a hull for that union. Intersections and complements are more problematic.

Problem 3

By a *base* I understand a basis for the topology, in particular $\sigma(\mathcal{U}) = \operatorname{Bor} X$. First, we will show that $\mu = \nu$ on open sets. Take an open set U. It can be represented as

$$U = \bigcup_{n=1}^{\infty} U_n = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} U_n$$

for some $U_n \in \mathcal{U}$. Since $\mathcal{U}$ is closed under finite sums, the inner sums are also in $\mathcal{U}$, so μ and ν agree on them. But since the outer sums on RHS are increasing, $\mu(U) = \nu(V)$. Applying the $\pi - \lambda$ Lemma shows that μ and ν agree on all Borel sets.

Alternatively, since these are probability measures, they also agree on closed sets (by complements), so the regularity property 4.5 does the job.

Cardinality

Since X has at least two distinct points x_1, x_2, we have at least $\mathfrak{c}$ measures, as witnessed by

$$p\delta_{x_1} + (1-p)\delta_{x_2}.$$

On the other hand, we have

Lemma 7.1. *If X is $\mathcal{SM}$, then X is second countable.*

Proof. By 4.1, $X \hookrightarrow [0, 1]^{\mathbb{N}}$ which is second countable, so X is second countable as well. □

Since the values of a probability measure μ are determined by its values on a countable basis, we know that there are at most as many measures as functions in $[0, 1]^{\mathbb{N}}$. The cardinality of the Hilbert Cube is

$$\mathfrak{c}^{\aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c},$$

which gives the upper bound.

Problem 4

Let $\mathcal{F}$ be the family of functions which have this property. It is of course closed under uniform limits, but because of the regularity property of Borel measures (see 4.5) and Egorov's Theorem it is also closed under almost uniform limits, so also under pointwise limits.

Problem 5

Think of ω^ω as an infinite product. There is a number n such that

$$A_1 := \mu\left(\pi_0^{-1}[k, n]\right) > 1 - \frac{\varepsilon}{2}.$$

The intervals denote finite subsets of ω. Analogously, define

$$A_k := \mu\left(\pi_k^{-1}[0, n_k]\right) > 1 - \frac{\varepsilon}{2^k}.$$

By upward continuity of μ, for each k such an n_k exists. Then

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) > 1 - \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = 1 - \varepsilon.$$

On the other hand,

$$K := \bigcap_{k=1}^{\infty} A_k = \prod_{k=1}^{\infty} [0, n_k],$$

which is a product of compact sets, so compact by Tychonoff.

Problem 6

Forward

We will first state and prove useful propeties of the Baire space ω^ω.

Lemma 7.2. *Every Polish space is a continuous image of the Baire Space ω^ω.*

An idea that may pop into your head is to pick a countable dense subset $x_n \in X$ and define

$$f : \omega^\omega \to X$$

via

$$f(s) = \lim_n x_{s_n}.$$

This is sort of the right idea, but runs into the problem that there are nonconvergent sequences, to its only a partial function. By analogy with the Cantor set, the Baire space retracts onto any of its closed subspaces. This does not save us, since $\operatorname{dom} f$ is not closed – the convergence of the sequence depends only on a tail set of indices, and the metric of ω^ω is defined via prefixes. There is a solution.

Proof. Define the set $D \subseteq \omega^\omega$ as

$$D := \left\{ s \in \omega^\omega : d(x_{s_n}, \lim_n x_{s_n}) < \frac{1}{n} \right\}.$$

This set is closed, so there is a retraction $r : \omega^\omega \to D$. The function is uniformly continuous on D, so $f \circ r$ surjects X. $\square$

We now proceed to try and surject the Souslin scheme result with the Baire Space. Assume the result is nonempty. For a sequence $\sigma \in \omega^{<\omega}$ denote by $[\sigma]$ the cylinder of sequences beginning with σ. These are clopen sets. Moreover, the sets $[n]$ for $n \in \omega$ form a partition of ω^ω into disjoint open sets. For each n, take a function

$$f_n : \omega^\omega \to F_n$$

and *merge* them using the disjoint sets S_n. That is, we put down $g_1 = f_n$ on S_n. Now, we can do this for prefixes of any arbitrary (finite) length k, giving us a function g_k. The key insight is that *surjects the Souslin scheme up to level* k. More concretely

$$g_k(\sigma) \in F_{\sigma|k}.$$

If we could take a pointwise limit $g = \lim_{k=1} g_k$, we would have $g(\sigma) \in$

$$g(\sigma) \in \bigcap_{k=1}^{\infty} F_{\sigma|k}.$$

We run into three problems

1. we may happen upon an empty set $F_{\sigma|k}$ somewhere in the scheme,
2. the limit might not exist,
3. the infinite intersection may be empty or contain more than one point, in which case g may not be a surjection.

Removing empty sets. If some set in the Souslin scheme is empty, we replace its subtree with on of its siblings. This does not change the result of the whole operation. If all siblings are empty, we travel up a level and treat the parent as though it was empty. Since the set we are trying to surject is nonempty, at some point we will be able to use a nonempty sibling. This fixes the first problem. Turns out the other two have a rather elegant solution, which I stole from Kechris.

Ensuring nonemptiness. If we have that $\operatorname{diam} F_\sigma < 1/l$, where l is the length of σ, then each intersection of $F_{\sigma|k}$ is a singleton. We can easily do this, since X can be covered by finitely many closed balls of radius $1/n$ for any n, since it is separable. Now insert a new level of the Souslin scheme tree in between two existing ones, where we take an F_σ and subdivide it into

$$F_\sigma \cap B\left(x_i, \frac{1}{|\sigma|}\right),$$

where $\{x_i\}$ is a countable dense set.

Mopping up. Take the initial Souslin scheme, insert the levels needed to have diameters tending to zero along every path and remove empty sets. Not that removing empty sets does not move any set downward in the tree, so the diameter bound is maintained. Then do the g_k construction. Because of the diameter bound, the convergence is now uniform, so we get a continuous function.

Formalisms. There are some issues with the constructions I used. They have to be done level-by-level to work, and an induction principle is needed!

Problem 9

The pushforward operator $f[-]$ is a covariant functor from the category of measurable spaces and Borel maps into an appropriate category (even the forgetful Set will suffice, though we could take sth like measure algebras). The function f has a Borel section s, so the operator $f[-]$ has a section $s[-]$, and in particular it must be surjective.