

# Measures on Topological Spaces

Krzysztof Szymański  
338068@uw.edu.pl

March 16, 2025

# Contents

<b>1</b>	<b>Measure Theory Bank of Lemmas</b>	<b>2</b>
<b>2</b>	<b>An introduction to geometric measure theory</b>	<b>4</b>
2.1	Corollaries and the Lebesgue Differentiation Theorem . . . . .	9
2.2	Generalization to metric spaces . . . . .	9
<b>3</b>	<b>One (Cantor) set to rule them all</b>	<b>10</b>
3.1	Ternary Cantor . . . . .	10
3.2	Abstract Cantor . . . . .	11
3.3	Topology of the Cantor set . . . . .	13
3.4	The group structure . . . . .	14
3.5	Measure . . . . .	14
3.6	Normal number theorem . . . . .	16
<b>4</b>	<b>Measures on separable, metrisable topological spaces</b>	<b>17</b>
4.1	Basic properties . . . . .	17
4.2	Polish spaces . . . . .	19
<b>5</b>	<b>Measures on Topological Spaces, Problemset 1</b>	<b>23</b>
<b>6</b>	<b>Measures on Topological Spaces, Problemset 2</b>	<b>26</b>
<b>7</b>	<b>Measures on Topological Spaces, Problemset 3</b>	<b>30</b>

# Chapter 1

## Measure Theory Bank of Lemmas

**Lemma 1.1** (Generating a  $\sigma$ -algebra). *Fix a space  $X$ . For any family of sets  $\mathcal{A}$ ,  $\sigma(\mathcal{A})$  can be generated by any of the following sets of operations:*

1. complements and countable unions.
2. complements and increasing countable unions.

**Lemma 1.2.** *Let  $\mathcal{B}$  be a basis for the topology of  $X$ . Then*

$$\sigma(\mathcal{B}) = \text{Bor } X.$$

**Lemma 1.3.** *The set operations and the measure taking operations are continuous with respect to the symmetric difference pseudometric.*

**Lemma 1.4.** *A bounded measurable function  $f$  on a measure space  $(X, \mu)$  can be uniformly approximated by simple functions.*

*Proof.* Let

$$|f| \leq [-M, M].$$

We will construct the approximation by considering *bins* of the values of  $f$ , i.e. the sets

$$A_k := f^{-1} \left[ [k\varepsilon, (k+1)\varepsilon) \right]$$

for  $k \in \mathbb{Z}$ . In such a *bin*, all the values are within an  $\varepsilon$  of each other. Since  $f$  is bounded, all but finitely many of the bins are empty  $X$ , so the function

$$\tilde{f}_\varepsilon := \sum_{A_k \neq \emptyset} (k\varepsilon) \cdot \chi_{A_k}$$

□

**Remark.** This works equally well for almost everywhere bounded functions, giving almost everywhere uniform convergence.

**Lemma 1.5.** *Let  $A, A_1, \dots, A_k$  be measurable sets such that*

$$\forall k. \mu(A_i \cap A) \geq (1 - \delta_i) \mu(A).$$

*Then*

$$\mu(A \cap A_1 \cap A_2 \cap \dots \cap A_k) \geq \left(1 - \sum \delta_i\right) \mu(A).$$

*Proof.* Union bound on the sets

$$A \cap A_i^c.$$

□

**Remark.** This also works for an infinite sequence of sets  $A_k$ ; we obtain.

$$\mu \left( A \cap \bigcap_k A_k \right) \geq \left( 1 - \sum_k \delta_k \right) \mu(A).$$

## Chapter 2

# An introduction to geometric measure theory

In this chapter, we study the links between the topology and geometry of  $\mathbb{R}$  and the Lebesgue measure. We first give two examples of how the two structures agree, and one example of how they don't.

**Isometries.** Consider the group  $\text{Isom } \mathbb{R}$  of the isometries of  $\mathbb{R}$  with the euclidean metric. One easily shows that this group consists of functions of the form

$$x + a \text{ or } a - x$$

for  $a \in \mathbb{R}$ . The Lebesgue measure is invariant on transformations  $g \in \text{Isom } \mathbb{R}$ , i.e.

$$\lambda(gA) = \lambda(A)$$

for all measurable  $A \subseteq \mathbb{R}$ . A corollary of this is that the Lebesgue measure is invariant w.r.t. the addition operation on  $\mathbb{R}$ , which gives the reals the structure of a topological group.

**Affine transformations.** Similarly to the above, the Lebesgue measure work well with the action of the affine transformation group  $\text{Aff } \mathbb{R}$ . Directly from the definition, the group of affine transformations consists of the functions

$$g_{a,b}(x) := ax + b$$

for  $r \neq 0$ , and the interaction with measure is given by

$$\lambda(g_{a,b}A) = |a| \cdot \lambda(A).$$

**Topology.** There is a disconnect between the topological (nonempty interior) and measure-theoretic (positive measure) notions of *large* or *non-negligible* – the topological notion is strictly stronger! Indeed, a set with nonempty interior has positive measure, but if we enumerate the rationals as

$$\mathbb{Q} = \{q_1, q_2, \dots\}$$

the set

$$\mathbb{R} \setminus \bigcup_{n=1}^{\infty} \left( q_n - \frac{\varepsilon}{2^{n+1}}, q_n + \frac{\varepsilon}{2^{n+1}} \right)$$

has comeasure  $\varepsilon$ , but is nowhere dense.

However, there does exist a link between the two notions. It is a bit more subtle.

**Definition 2.1.** Fix a measurable set  $A \subseteq \mathbb{R}$ . A point  $x \in \mathbb{R}$  is called a **density point** iff

$$\lim_{\delta \rightarrow 0^+} \frac{\lambda(A \cap B(x, \delta))}{2\delta} = 1.$$

The  $2\delta$  in the numerator is of course  $\lambda(B(x, \delta))$ .

**Definition 2.2.** The set of density points of  $A$  will be denoted  $\phi(A)$ .

Note that a density point is by necessity an accumulation point. The promised link between geometry, measure and topology is provided by the theorem below.

**Theorem 2.1** (Lebesgue Density Theorem). Let  $A \subseteq \mathbb{R}$  be a measurable set. Then almost all points of  $A$  are density points of  $A$  in the sense that

$$\lambda^*(A \setminus \phi(A)) = 0.$$

**Remark.** Note that the theorem follows trivially for null sets. Also, for a given  $A$ , we may as well apply the theorem to  $A^c$  to get that almost all points outside of  $A$  have density 0.

For the proof of the **Lebesgue Density Theorem**, we will need a tool, which we introduce now and prove later.

**Definition 2.3.** A family  $\mathcal{J}$  of nontrivial closed intervals is called a **Vitali cover** of a set  $A$  (not necessarily measurable) if for any given  $\varepsilon > 0$  and  $x \in A$  there is an interval  $J \in \mathcal{J}$  such that

$$\text{diam } J < \varepsilon \wedge x \in J.$$

In particular

$$A \subseteq \bigcup \mathcal{J}.$$

**Theorem 2.2** (Vitali Covering Theorem). If  $\mathcal{J}$  is a Vitali cover of  $A$ , there exists a sequence of pairwise disjoint segments  $J_n \in \mathcal{J}$  such that

$$\lambda \left( A \setminus \bigcup_n J_n \right) = 0.$$

**Corollary.** If a set  $A$  has a Vitali Cover, then  $A$  is measurable.

**Why is this theorem useful?** Vitali's theorem may not sound very smart on first glance. Its strength lies in the *disjointness* of the cover. If we go about choosing the cover  $J_n$  without any guarantees, we can for example choose

$$\bigcup_{q_n \in \mathbb{Q}} \left( q_n - \frac{\varepsilon}{2^{n+1}}, q_n + \frac{\varepsilon}{2^{n+1}} \right)$$

and get stuck! We have only covered a subset of the reals of size  $\varepsilon$ , but we cannot use any other segment by density of  $\mathbb{Q}$ .

*Proof of the Lebesgue Density Theorem.* We represent

$$A \setminus \phi(A) = \bigcup_k A_k$$

for

$$A_k = \left\{ a \in A : \liminf_{\delta \rightarrow 0^+} \frac{\lambda(A \cap B(a, \delta))}{2\delta} < 1 - \frac{1}{k} \right\}.$$

It suffices to show

$$\lambda^*(A_k) = 0$$

for all  $k$  to finish the proof. Since we may represent  $A$  as

$$A = \bigcup_{z \in \mathbb{Z}} A \cap [z - 1, z + 1]$$

and being a density point of  $A$  is the same as being a density point of one of the *cutouts* in the sum above, we may assume without loss of generality that  $A \subseteq [0, 1]$ .

By definition of outer measure, we can approximate  $A_k$  from above by an open set  $U$  such that

$$\lambda^*(A_k) \leq \lambda(U) \leq \lambda^*(A_k) + \varepsilon.$$

Construct a covering

$$\mathcal{J} = \left\{ [a, b] : [a, b] \subseteq U, \lambda(A \cap [a, b]) \leq \left(1 - \frac{1}{k}\right) \lambda[a, b] \right\}.$$

It is a Vitali cover of  $A_k$ . By **Vitali's Theorem** we can pick a pairwise disjoint sequence of intervals  $J_i \in \mathcal{J}$  for which

$$\lambda^* \left( A_k \setminus \bigcup_i J_i \right) = 0.$$

This gives

$$\begin{aligned} \lambda^*(A_k) &= \lambda^* \left( A_k \cap \bigcup_i J_i \right) \\ &\leq \sum_i \lambda^*(A_k \cap J_i) \\ &\leq \sum_i \lambda^*(A \cap J_i) \\ &\leq \left(1 - \frac{1}{k}\right) \sum_i \lambda(J_i) \\ &\leq \left(1 - \frac{1}{k}\right) \lambda(U) \\ &\leq \left(1 - \frac{1}{k}\right) (\lambda^*(A_k) + \varepsilon). \end{aligned}$$

The passage from line 2 to 3 may seem trivial, but is in fact crucial. This is the place where we use  $A_k \subseteq A$ ! Otherwise the theorem is quite absurd, even for simple examples like  $[0, 1]$ . Since  $\lambda^*(A_k) \leq \lambda(A) < \infty$ , we can rearrange this to obtain

$$\lambda^*(A_k) \leq (k - 1)\varepsilon.$$

Since  $\varepsilon$  can be picked arbitrarily close to 0, we get

$$\lambda^*(A_k) = 0.$$

□

*Proof of the Vitali Covering Theorem.* The key to avoiding the *trap* we wrote about after stating the **VCT** is to choose the segments to be as large as possible – or at least not embarassingly small.

Without loss of generality,  $A$  is bounded since we can sum the coverings of  $A \cap (n, n+1)$ . The sequence of segments we pick is denoted  $J_n$ . In that case we may also assume  $\bigcup \mathcal{J}$  is bounded. Its prefixes are

$$P_n := \bigcup_{i < n} J_i$$

$$\mathcal{J}_n := \{J \in \mathcal{J} : J \cap P_n = \emptyset\}$$

and the *width* of what we can choose is

$$\gamma_n := \sup_{J \in \mathcal{J}_n} \text{diam } J.$$

Note that in particular

$$P_1 = \emptyset,$$

$$\mathcal{J}_1 = \mathcal{J}$$

$$\gamma_1 \leq \text{diam } A < \infty.$$

At each step, we choose  $J_n$  so that

$$\text{diam } J_n \geq \frac{1}{2} \gamma_n,$$

or we stop if  $\gamma_n = 0$  at some point.

**Claim 1.** The sequence  $\gamma_n$  converges monotonically to 0.

*Proof of Claim 1.* Being the supremum of ever decreasing sets,  $\gamma_n$  is decreasing. It is also non-negative, so the sequence converges and  $\lim_n \gamma_n \geq 0$ . Suppose that  $\lim_n \gamma_n = c > 0$ . Then in the construction, we would almost always choose disjoint intervals of diameter at least  $c/2$ . This is impossible, since  $\bigcup \mathcal{J}$  was assumed to be bounded, so it has finite measure! □

The key to proving that the choice procedure is correct will be the **blowup**, which we define for  $J = [x - r, x + r]$  as

$$\tilde{J} := [x - 5r, x + 5r]$$

**Claim 2.** At all steps of the construction

$$A \subseteq \mathcal{J}_n \cup \bigcup_{i \geq n} \tilde{J}_i.$$



*Proof of Claim 2.* The set  $\mathcal{J}_n$  is closed as a union of closed intervals. Therefore, if  $a \in A \setminus \mathcal{J}_n$ , there is a nondegenerate interval  $I \ni a$ . Since  $\gamma_n \rightarrow 0$  by **Claim 1**,  $I$  is not considered in the construction of the sequence  $J_n$  for almost all  $n$ . Let  $n_0$  be the last step where it is considered. Then we must have  $I \cap J_{n_0+1} \neq \emptyset$ , because that is the step at which  $I$  is no longer considered.

We will show that this implies  $a \in \tilde{J}_{n_0+1}$ . Let  $J_{n_0+1} = [x - r, x + r]$  and  $y \in I \cap J_{n_0+1}$ . Then we have

$$\begin{aligned} d(a, x) &\leq d(a, y) + d(y, x) \\ &\leq \text{diam } I + r \\ &\leq (2 \cdot \text{diam } J_{n_0+1}) + r \\ &= 2 \cdot 2r + r \\ &= 5r, \end{aligned}$$

where the diameter bound comes from the definition of  $\gamma_{n_0}$  and the fact that  $I$  is still available at step  $n_0$  of the construction.  $\square$

To finish the proof of Vitali's Covering Theorem, we compute that for all  $n$

$$\begin{aligned} \lambda^*(A \setminus P_n) &\leq \lambda^*\left(A \cap \bigcup_{i \geq n} \tilde{J}_i\right) \\ &\leq \lambda\left(\bigcup_{i \geq n} \tilde{J}_i\right) \\ &\leq \sum_{i \geq n} \lambda(\tilde{J}_i) \\ &= 5 \sum_{i \geq n} \lambda(J_i). \end{aligned}$$

These are the tails of the convergent series

$$\sum_{i=1}^{\infty} \lambda(J_i) = \lambda\left(\bigcup_i J_i\right) < \infty,$$

so we get

$$\lambda^*\left(A \setminus \bigcup_i J_i\right) \leq \lambda^*(A \setminus P_n) \rightarrow 0.$$

$\square$

**Remark.** Retracing the argument behind **Claim 2.**, we might prove that for any  $\alpha < 1$ , if we define  $\gamma_n$  with a coefficient of  $\alpha$  instead of  $\frac{1}{2}$ , the constant used for blowing up intervals can be brought down to

$$1 + \frac{2}{\alpha}.$$

In particular, we can get arbitrarily close to 3.

## §2.1 Corollaries and the Lebesgue Differentiation Theorem

**Theorem 2.3** (Lebesgue Differentiation Theorem). *Let  $f \in L^1(\mathbb{R})$ . Then, for almost all  $x$ ,*

$$\lim_{\delta \rightarrow 0^+} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(s) \, d\lambda(s) = f(x).$$

*Proof.* For characteristic functions, this is just a restatement of the [Lebesgue Density Theorem](#).  $\square$

## §2.2 Generalization to metric spaces

The argument in the proof of the [VCT](#) was written so that it is easily generalizable to any metric space with a measure on its Borel sets.

To be more precise, what we need to lift the argument is that

$$\mu(B(x, 5r)) \leq C\mu(B(x, r))$$

for some constant  $C$ . We can also substitute any constant larger than 3 instead of 5.

## Chapter 3

# One (Cantor) set to rule them all

### §3.1 Ternary Cantor

Let us begin by making a construction. Take the closed interval  $C_0 := [0, 1]$  and remove the middle one third of it in such a way that the remaining two interval are closed. The result of this is

$$C_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Now, repeat the operation of cutting out the middle third and call the result  $C_2$ . We can repeat this *ad infinitum* and obtain a decreasing sequence of sets

$$[0, 1] = C_0 \supset C_1 \supset C_2 \supset \dots$$

Perhaps surprisingly, there are numbers which are not removed at any step, i.e. the intersection

$$\mathcal{C}_3 := \bigcap_{k=0}^{\infty} C_k$$

is nonempty! It contains 0 and 1. In fact, any number which can be written in base 3 using only 0's and 2's is an element of this intersection. These are in fact all such numbers. We introduce a tool to prove that.

**Lemma 3.1.** *Let  $b \geq 2$  be a positional system base and  $x_0$  be a number with  $k$  digits after the positional point. Then, the numbers formed by adjoining (perhaps infinitely many) digits to the base  $b$  representation of  $x_0$  are all the numbers in the interval*

$$[x_0, x_0 + b^{-k}].$$

*If we allow only finite extensions, we get the  $b$ -ary rational numbers in that interval, and if we disallow the infinite extension by the digit  $(b - 1)$ , we get the interval*

$$[x_0, x_0 + b^{-k}).$$

**Lemma 3.2.** *A real number  $x \in [0, 1]$  is an element of  $\mathcal{C}_3$  iff  $x$  can be written in base using only the digits 0 and 2.*

*Proof (if).* We proceed by induction with the induction thesis:  $x$  belongs to  $\mathcal{C}_3$  iff  $x$  can be written in base 3 so that its first  $k$  digits are 0 or 2. It should be clear that this thesis is equivalent to the lemma statement. For each  $k$ , the statement is true by [the previous lemma](#).  $\square$

### §3.2 Abstract Cantor

The ternary Cantor set has many interesting properties. However, to study it, we will move to a more convenient representation. We can think of the Cantor set as the set of leaves of an infinite binary tree – starting from the root, at each level we choose whether to go right or left, or whether to insert 0 or 2 as the next digit in the base-3 representation of an  $x \in \mathcal{C}_3$ .

In this way, we can represent the ternary Cantor as

$$\mathcal{C} := \{0, 1\}^{\mathbb{N}}.$$

That the map we just described is a bijection follows from

**Lemma 3.3.** *Let  $d_k, \tilde{d}_k$  be two sequences of base  $b$  digits. Then the corresponding real numbers are equal iff  $d_k$  and  $\tilde{d}_k$  agree on some prefix and afterwards one of them is 0 and the other is  $(b-1)$ .*

*Proof.* The condition implies equality of numbers by the sum of a geometric series. The other direction follows by looking at the first moment the expansions differ at then bounding the series sum.  $\square$

What is missing from this description is the topology. We topologize  $\mathcal{C}$  by the metric

$$d(x, y) = \begin{cases} 0 & \text{for } x = y \\ \frac{1}{n} & \text{for } x \neq y, \end{cases}$$

which can also be written succinctly as

$$d(x, y) = \frac{1}{n_0(x, y)}$$

with the notation

$$n_0(x, y) := \inf \{n : x_n \neq y_n\}$$

for the first index at which  $x$  and  $y$  differ. The function  $d$  may not look like a metric at first sight, but in fact it has an even better property.

**Lemma 3.4.** *For the metric  $d$  described above we have for all  $x, y, z \in \mathcal{C}$*

$$d(x, z) \leq \max \{d(x, y), d(y, z)\}.$$

*In particular,  $d$  is an **ultrametric**.*

*Proof.* Recall that  $n_0(x, z)$  is the first position at which  $x$  and  $z$  differ. Then any  $y$  has to differ with at least one of  $y$  and  $z$  at  $n_0$ , but might even earlier. This gives

$$n_0(x, z) \geq \min(n_0(x, y), n_0(y, z)).$$

Since the function  $x \mapsto 1/x$  is decreasing, the thesis follows.  $\square$

We have established that  $(\mathcal{C}, d)$  is a metric space. It is, in fact, homeomorphic with the subspace topology of  $\mathcal{C}_3$  inherited from  $[0, 1]$ .

**Lemma 3.5.** *The function*

$$h_3 : \mathcal{C} \rightarrow \mathcal{C}_3$$

*defined by*

$$h_3(x) := \sum_{k=1}^{\infty} \frac{2x_k}{3^k}$$

*is a homeomorphism.*

*Proof.* Bijectivity follows from the number-system lemma 3.3 and 3.2. For continuity, put down  $n_0 := n_0(x, y)$  and compute

$$\begin{aligned} |h_3(x) - h_3(y)| &= \sum_{k=1}^{\infty} \frac{2|x_k - y_k|}{3^k} \\ &\leq \sum_{k=n_0}^{\infty} \frac{2}{3^k} \\ &= \frac{2}{3^{n_0}} \cdot \frac{3}{2} \\ &= \frac{1}{3^{n_0-1}}. \end{aligned}$$

The continuity of the inverse follows from the bound

$$|h_3(x) - h_3(y)| \geq \frac{2}{3^{n_0}}.$$

□

The function  $h_3$  in 3.5 can be understood as a base 3 expansion operator. When we consider a base 2 expansion instead, we lose bijectivity, but we can cover the whole interval.

**Lemma 3.6.** *The function*

$$h_2 : \mathcal{C} \rightarrow [0, 1]$$

*given by*

$$h_2(x) := \sum_{k=1}^{\infty} \frac{x_k}{2^k}$$

*is a continuous surjection.*

*Proof.* Surjectivity follows from number system properties, and continuity is essentially the same calculation as in the proof of 3.5. □

**Theorem 3.1** (The Universal Property of the Cantor Set). *Every metrizable compact space  $K$  is a continuous image of  $\mathcal{C}$ .*

*Proof.* Considering an element of  $\mathcal{C}$  as a binary expansion, we have by 3.6 a surjection

$$h_2 : \{0, 1\}^{\mathbb{N}} \twoheadrightarrow [0, 1].$$

The space  $K$  can be embedded into the Hilbert cube by the **Urysohn Metrization Theorem** ???. By compactness of  $K$ , the image of the embedding is a compact and thus a closed subset. We also have a surjection

$$h : \{0, 1\}^{\mathbb{N}} \rightarrow [0, 1]^{\mathbb{N}}$$

by using the previous surjection and *unweaving* the Cantor set into the product of countably many Cantor sets, i.e. using

$$\mathbb{N} \cong \mathbb{N} \times \mathbb{N} \implies \mathcal{C} = \{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N} \times \mathbb{N}} \cong \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}} = \mathcal{C}^{\mathbb{N}}.$$

The last step is using the fact that any closed set of  $\mathcal{C}$  is a retract of  $\mathcal{C}$ , which is ??.

□

**A warning against generalization.** If  $K$  is a compact set, it embeds into a *Tichonov Cube*

$$K \rightarrow [0,1]^{\Gamma}$$

and we can surject the Tichonov cube with a generalized Cantor set

$$\{0,1\}^{\Gamma},$$

but the universality theorem fails!

### §3.3 Topology of the Cantor set

**Definition 3.1** (Cantor Cylinder). *Let*

$$\varphi : \mathbb{N} \multimap \{0,1\}$$

*be a partial function with finite domain. Then we define the **cylinder set** with base  $\varphi$  as*

$$[\varphi] := \{x \in \{0,1\}^{\mathbb{N}} : x|_I = \varphi\}.$$

**Lemma 3.7.** *The sets  $[\varphi]$  form a base of the topology of  $\{0,1\}^{\mathbb{N}}$ .*

**Definition 3.2.** *A set  $A \subseteq \mathcal{C}$  is **determined** by  $I \subseteq \mathbb{N}$ , which we denote by  $A \sim I$  if for all  $x \in A$ ,  $y \in \mathcal{C}$  we have*

$$x|_I = y|_I \implies y \in A.$$

*Equivalently,*

$$\pi_I^{-1}\pi_I[A] = A.$$

**Lemma 3.8** (Clopen sets in the Cantor set). *A set  $A \subseteq \mathcal{C}$  is clopen iff  $A \sim I$  for some finite  $I \subseteq \mathbb{N}$ . In particular, clopen sets can be written as a finite union of disjoint basis clopens  $[\varphi_i]$  for  $\varphi_i$  with finite domain.*

*(Direction one).* If  $A$  is clopen, then

$$A = \bigcup_i [\varphi_i]$$

for some finitely many (by compactness)  $\varphi_i$  with finite domain  $I_i$ . Then

$$A \sim \bigcup_i I_i.$$

□

*(Other direction).* if  $A \sim I$ , blabla

□

Immediately, a lemma follows.

**Lemma 3.9** (Cantor set is zerodimensional). *The Cantor set  $\mathcal{C}$  is zerodimensional, i.e. it has a base of clopen sets.*

**Theorem 3.2** (Topological characterisation of the Cantor set). *If a topological space  $K$  is compact, metrizable, zerodimensional with no isolated points, then*

$$K \cong \mathcal{C}.$$

### §3.4 The group structure

The Cantor set has a natural abelian group structure given by its product structure. We can phrase it even more efficiently when we think of  $\mathcal{C}$  as  $\mathcal{P}(\mathbb{N})$  – the symmetric difference (or xor for the informatically inclined).

$$A \oplus B := A \Delta B$$

Every element has order two!

**Fact.** Together with the operation  $\oplus$ , the Cantor set  $\mathcal{C}$  is a compact topological group, i.e. the function

$$(x, y) \mapsto x \oplus y$$

is continuous (in general the second element is inversed, but here every element is its own inverse anyway).

### §3.5 Measure

We can define the measure on the Cantor set as a countable product of probability measures:

$$\nu = \bigotimes_{n=1}^{\infty} \left( \frac{1}{2}(\delta_0 + \delta_1) \right).$$

But we will do it by hand.

**Definition 3.3.** *Let  $A \subseteq \mathcal{C}$  be clopen. Then*

$$A \sim \{1, 2, \dots, n\}$$

*for some  $n$ . Let*

$$A' := \pi_{\{1, 2, \dots, n\}}[A]$$

*We define its measure to be*

$$\nu(A) := \frac{\#A'}{2^n}.$$

This makes sense with the probabilistic definition.

**Theorem 3.3** (Well-definedness of the premeasure). *The function*

$$\nu : \text{Clop } \mathcal{C} \rightarrow \mathbb{R}$$

*is a well-defined, additive function on the set algebra  $\text{Clop } \mathcal{C}$ .*

**Proof.** Since the Cantor set is compact,  $\nu$  is automatically downward continuous on the empty set. By Caratheodory's Theorem,  $\nu$  extends uniquely to a probabilistic measure on

$$\text{Bor } \mathcal{C} = \sigma(\text{Clop } \mathcal{C}).$$

What now??

$$\mathcal{A} := \{B \in \text{Bor } \mathcal{C} : \forall \varepsilon > 0. \exists A \in \text{Clop } \mathcal{C}. \nu(A \Delta B) < \varepsilon\}.$$

We prove that this is a  $\sigma$ -algebra.

There is a nice formula for cylinders.

**Lemma 3.10** (Measure of a cylinder). *For a partial function*

$$\varphi : \mathbb{N} \multimap \{0, 1\},$$

*its cylinder has measure*

$$\nu[\phi] = 2^{-|\text{dom } \varphi|}.$$

*The result holds even if  $\text{dom } \varphi$  is infinite, in which case the measure is 0.*

*Proof.* For finite-domain partial functions  $\phi$ , take

$$\text{dom } \varphi =: I \subseteq \{1, 2, \dots, n\} =: [n]$$

for some  $n$ . Then

$$|\pi_{[n]}[\varphi]| = \frac{2^{n-|I|}}{2^n} = 2^{-|I|}.$$

For infinite-domain functions  $\phi$ , taking a decreasing intersection

$$[\phi] = \bigcap_n [\phi|_{[n]}]$$

shows that the measure of the intersection is 0. □

**Theorem 3.4.** *The measure  $\nu$  is the Haar measure on  $\mathcal{C}$ , that is, the unique probability measure invariant under group actions*

$$\nu(x \oplus B) = \nu(B)$$

*for all  $x \in \mathcal{C}$ ,  $B \subseteq \mathcal{C}$ .*

*Proof.* Let us first consider  $B = [\varphi]$ , and  $I = \text{dom } \varphi$ . Then

$$\nu(x \oplus [\varphi]) = \nu([x \oplus \varphi]) = \nu([\varphi]).$$

A clopen is a disjoint sum of  $[\varphi_i]$  for finitely many  $\varphi_i$ , so additivity on clopens follows. Now, take a superficially different measure

$$\nu_x(B) := \nu(x \oplus B).$$

Since  $\nu$  and  $\nu_x$  agree on clopens, by uniqueness in Caratheodory's Theorem they agree on all sets. □

Note the isomorphism

$$(C, \oplus) \cong (\mathcal{P}(\mathbb{N}), \Delta)$$

of (topological) groups.



### §3.6 Normal number theorem

**Definition 3.4.** Let  $A \subseteq \mathcal{C}$ . We call  $A$  a **tail set** if

$$A \sim \{k : k \geq n\}$$

for all  $n$ . Equivalently, if  $x \in A$  and  $x(n) = y(n)$  for almost all  $n$ , then  $y \in A$ .

**Example.** A naturally occurring example of a tail set is

$$A_\beta := \left\{ x \in \mathcal{C} : \lim_n \frac{x(1) + \dots + x(n)}{n} = \beta \right\}.$$

**Theorem 3.5** (Kolmogorov zero-one law for the Cantor set). *A borel tail set  $A \subseteq \mathcal{C}$  has measure 0 or 1.*

*Proof.* Take a basis set  $[\varphi]$ . We have

$$\nu([\varphi] \cap A) = \nu([\varphi]) \cdot \nu(A).$$

From this immediately follows that this work for any  $B \in \text{Clop } \mathcal{C}$ . Now approximate  $A$  by a clopen  $B$  so that

$$\nu(A \Delta B) < \varepsilon.$$

To finish the proof, compute

$$\nu(A) \cdot \nu(B) = \nu(A \cap B) \geq \nu(A) - \varepsilon \nu(A).$$

□

Returning to the example we have  $\nu(A_\beta) \in \{0, 1\}$ . We have

$$\nu(A_\beta) = \nu(A_{1-\beta}).$$

**Theorem 3.6** (Borel's normal number theorem).

$$\nu\left(A_{\frac{1}{2}}\right) = 1.$$

**Remark.** According to Billingsley, this theorem was the founding work of modern probability theorem, which is founded on limit theorems.

*Proof.* Denote for  $\alpha < \frac{1}{2}$

$$B_n^\alpha = \left\{ x \in \mathcal{C} : \frac{x_1 + x_2 + \dots + x_n}{n} \leq \alpha \right\}.$$

We claim that there exists a  $\theta$  such that

$$\nu(B_n^\alpha) \leq \theta^n.$$

Then

$$\nu(B_n^\alpha) = \frac{c_n}{2^n},$$

where

$$c_n = \sum_{k=1}^{\lfloor \alpha n \rfloor} \binom{n}{k}.$$

□

## Chapter 4

# Measures on separable, metrisable topological spaces

### §4.1 Basic properties

For brevity, we will denote the class of separable and metrisable topological spaces by  $\mathcal{SM}$ . A lot of the time, it is easier to work with such spaces in a *common box*, i.e. use a universal space in which all of these spaces can be embedded. Luckily, we have such a space – the Hilbert Cube.

**Theorem 4.1.** *Every  $\mathcal{SM}$  topological space embeds in the Hilbert Cube.*

*Proof.* Fix a metric  $d \leq 1$  and a countable dense subset  $x_n \in X$ . We define the embedding as

$$f_n(x) := d(x, x_n).$$

This is a product of continuous functions, so it continuous. It is injective, as if  $f(x) = f(y)$ , then a subsequence of  $x_n$  convergent to  $x$  is also convergent to  $y$ , so  $x = y$ .

The most difficult fact is that this is open. To see this, take  $x \in U \subseteq X$  with  $U$  open. We will show that  $f[U]$  is open. The neighbourhood  $U$  contains some ball  $B(x, r)$ . We can find an element  $x_k$  of the countable dense set such that  $d(x, x_k) < r/4$ . Then

$$x \in B(x_k, r/2) \subseteq B(x, r) \subseteq U,$$

which implies

$$f(x) \in f[B(x_k, r/2)] \subseteq f[U].$$

But, by definition of  $f$ ,

$$f[B(x_k, r/2)] = f[X] \cap \pi_k^{-1}(-\infty, r/2).$$

Since  $x$  was an arbitrary element of  $U$ , we have that  $f[U]$  is open, so  $f$  is a homeomorphism onto its image!  $\square$

You may wonder how this relates to the fact that all compact metrisable embed in the Hilbert Cube (as closed sets!). It turns out that compact metrisable spaces are  $\mathcal{SM}$ . We only need separability, and compactness together with a covering by balls gives us a countable dense subset rather easily.

**Lemma 4.1.** *If a metrisable topological  $K$  is compact, then it is separable.*

*Proof.* For each  $n$ , finitely many balls of radius  $1/n$  cover  $K$  by compactness. Taking the centers of all such balls over all  $n$  yields a countable dense subset.  $\square$

The proof above can be trivially extended to totally bounded spaces and Lindelöf spaces. In the second case, we have countably many centers of balls at each step.

**Lemma 4.2.** *Let  $K$  be a metrisable topological space with is either*

1. *compact,*
2. *Lindelöf,*
3.  *$\sigma$ -compact,*
4. *or totally bounded.*

*Then  $K$  is separable.*

We will now investigate for a moment how properties of  $\mathcal{SM}$  spaces are reflected in functions on such spaces. Since we care about topology, we restrict our attention to continuous functions. Unfortunately, even continuous functions on an arbitrary  $\mathcal{SM}$  space can have an untame structure. Therefore we restrict our attention to bounded functions.

**Definition 4.1.** *The space of bounded, continuous functions from a topological space  $X$  to  $\mathbb{R}$  is denoted by*

$$C_b(X).$$

*If we want bounded functions into  $\mathbb{C}$ , we use the notation*

$$C_b(X; \mathbb{C}).$$

This function space has the obvious structure of a linear space, and even an algebra with pointwise addition, scaling and multiplication. This space also has its own topology induced by the supremum norm.

**Lemma 4.3.** *Let  $X$  be an arbitrary.*

$$C_b(X)$$

*is a Banach algebra under pointwise operations and the supremum norm.*

*Proof for compact spaces.* Take a Cauchy sequence  $f_n$ . For each  $x \in X$ ,  $f_n(x)$  is Cauchy, so it converges. Therefore, the sequence of functions converges pointwise to a limit function  $f$ . Suppose the convergence is not uniform. Then for some  $\varepsilon > 0$  we can take a sequence  $x_n$  such that

$$f_n(x_n) - f(x_n) \geq \varepsilon.$$

By the  $\varepsilon/3$  trick,  $f$  is continuous. By compactness of  $X$ ,  $x_n$  has a subsequence convergent to  $x_0$ . Since  $f_n(x_0) \rightarrow f(x_0)$  and  $f(x_n) \rightarrow f(x_0)$ . TODO!  $\square$

*Proof.* The only thing one need to check is that Cauchy sequences actually converge. Let  $f_n$  be a Cauchy sequence. For any  $x \in X$ ,  $f_n(x)$  is a Cauchy sequence of real numbers, so it converges. Therefore,  $f_n$  converges pointwise to a function  $f$ . Note that so far we don't know if the convergence is uniform, or even if the function is continuous.

Let  $N_\varepsilon$  be the point after which the sequence  $f_n$  is  $\varepsilon$ -close. Then for  $n, m > N_\varepsilon$  we have

$$|f_n - f_m| \leq \varepsilon$$

uniformly on  $X$ . Keeping  $n$  constant and passing with  $m$  to the limit we have

$$|f_n(x) - f(x)| = \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \leq \varepsilon.$$

Therefore,  $f_n$  converges to  $f$  uniformly. In particular,  $f$  is continuous and bounded.  $\square$

We recall here a useful theorem, whose proof can be found in the literature.

**Theorem 4.2** (Stone-Weierstrass). *Let  $K$  be a compact, Hausdorff space and let  $W \subseteq C_b(K)$  be a subalgebra. If the subalgebra distinguishes points, we have*

$$\overline{W} = C_b(K).$$

*Proof.* A classic proof due to Lebesgue can be found in Engelking (or prof. Szwarc's functional analysis notes).  $\square$

The first and so far only property we investigate for  $C_b(X)$  is separability. It turns out that this space is rarely separable, and an exact characterisation can be given in terms of  $X$ .

**Theorem 4.3.** *Let  $X$  be an  $\mathcal{SM}$  topological space. Then  $C_b(X)$  is separable iff  $X$  is compact.*

*Proof* ( $\Leftarrow$ ). We know by 4.1 that  $X$  can be regarded as a subspace of  $[0, 1]^{\mathbb{N}}$ . Since  $X$  is compact, it is a closed subset of the cube. By the Stone-Weierstrass Theorem 4.2, the algebra generated by coordinate projections, which consists of finite linear combinations of finite products of coordinate projections, is dense in  $C_b(X)$ . This subalgebra is in general not countable, however. One small fix is required to find a countable dense subset of  $C_b(X)$  – only take rational coefficients in linear combinations.  $\square$

*Proof* ( $\Rightarrow$ ). We'll mirror the proof of the nonseparability of  $C_b(\mathbb{R})$  – we will find  $\mathfrak{c}$  many balls of radius  $\frac{1}{2}$ .

Since  $X$  is noncompact and metrisable, we have a discrete sequence of elements of  $X$ . Call it  $a_n$  and let  $A = \{x_n : n \in \mathbb{N}\}$ . For a subset  $I$  of the natural numbers, we define the function  $f_I : A \rightarrow [0, 1]$  by

$$f_I(x_i) := \begin{cases} 1 & i \in I \\ 0 & i \notin I. \end{cases}$$

We can extend all these to function  $\tilde{f}_I : A \rightarrow [0, 1]$  with the Tietze Extension Theorem. Now, for  $I \neq J$ , if we look at an element  $x_0 \in I \Delta J$  we get

$$\|\tilde{f}_I - \tilde{f}_J\|_{\infty} \geq |\tilde{f}_I(x_0) - \tilde{f}_J(x_0)| = 1.$$

$\square$

## §4.2 Polish spaces

We now turn to a subclass of  $\mathcal{SM}$  spaces which is particularly useful and important.

**Definition 4.2.** A **Polish space** is an  $\mathcal{SM}$  space  $X$ , which is completely metrisable.

Please note that this depends on the topology and not on any given metric for the space, as the example below shows.

Ankified

**Example.** The space  $(0, 1)$  is Polish, since it is homeomorphic to  $(0, \infty)$ . However, it is definitely not complete with regards to its standard metric! An explicit complete metric can be given by

$$d(x, y) := \left| \operatorname{tg} \frac{x\pi}{2} - \operatorname{tg} \frac{y\pi}{2} \right|,$$

which is the pullback of the complete metric from  $(0, \infty)$  by a homeomorphism.

**Example.** An even weirder example is

$$\mathbb{R} \setminus \mathbb{Q} \cong \mathbb{N}^{\mathbb{N}}.$$

That this is completely metrisable can be seen from the following theorem.

**Theorem 4.4** (Alexandroff). *A subspace  $Y$  of a Polish space  $X$  is itself Polish iff  $Y$  is a  $G_\delta$  subset of  $X$ .*

*Proof.* Let  $\rho'$  be a new metric on  $Y$  given by

$$\rho'(y_1, y_2) = \rho(y_1, y_2) + \sum_n \min \left( \frac{1}{2^n}, \left| \frac{1}{\rho(y_1, V_n^c)} - \frac{1}{\rho(y_2, V_n^c)} \right| \right)$$

The rest of the details can be found in Kerchis' classical book on Descriptive Set Theory.  $\square$

**Definition 4.3.** *In a topological space  $X$ , the **Borel subsets** of  $X$  are precisely the elements of  $\operatorname{Bor} X := \sigma(\tau_X)$ .*

**Definition 4.4.** *For a topological, but especially  $\mathcal{SM}$  or Polish space  $X$ , we denote the set of **probability measures** on  $\operatorname{Bor} X$  by  $\mathbb{P}(X)$ .*

We need a tool before stating proving the next theorem.

**Lemma 4.4.** *A closed set in a metric space is  $G_\delta$ . Conversely, any open set is  $F_\sigma$ .*

*Proof.* We will use the  $\varepsilon$ -neighbourhoods of  $F$ , i.e.

$$F_\varepsilon := \{x \in X : d(x, F) < \varepsilon\},$$

which are open. Since  $F$  is closed, we have

$$F = \bigcap_{n=1}^{\infty} F_{1/n}.$$

$\square$

**Theorem 4.5** (First Regularity Theorem). *For any  $\mathcal{SM}$  (in fact, any metrisable) space  $X$  and  $\mu \in \mathbb{P}(X)$ , the measure  $\mu$  is **regular**, that is for any  $B \in \operatorname{Bor} X$  and  $\varepsilon > 0$  there are two sets  $F, V$ , respectively closed and open, such that*

$$\mu(V \setminus F) < \varepsilon.$$

*Proof.* Let  $\mathcal{A}$  be the family of all sets with the given property. We will prove that it is a  $\sigma$ -algebra and that it contains closed sets. By 1.1, we only have to check for complements and countable sums.

Closed sets are in  $\mathcal{A}$  because they are  $G_\delta$ , see Lemma 4.4, and because a probability measure is downward continuous.

Closure under complements is inherent in the definition. If  $F \subseteq B \subseteq V$ , then

$$V^c \subseteq B^c \subseteq F^c,$$

$F^c$  is open,  $V^c$  is closed and

$$F^c \setminus V^c = V \setminus F,$$

so the approximation still works.

What about increasing countable sums? For an increasing sequence  $B_n$  take approximations  $F_n \subseteq B_n \subseteq V_n$  such that

$$\mu(V_n \setminus F_n) < \frac{\varepsilon}{2^n}$$

and let  $B = \bigcup_n B_n$ . By summing prefixes of  $F_n$ , we may assume that  $F_n$  is an increasing sequence. Then

$$\bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} V_n$$

and

$$\bigcup_{n=1}^{\infty} V_n \setminus \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} (V_n \setminus F_n)$$

(this is a general identity on sets). There is a slight problem – an ascending sum of closed sets might not be closed. Fortunately, we work with a probability measure, so we can take a finite prefix of length  $N$  with

$$\mu\left(\bigcup_{n=1}^{\infty} F_n \setminus \bigcup_{n=1}^N F_n\right) < \delta$$

and obtain an approximation

$$\bigcup_{n=1}^N F_n \subseteq B \subseteq \bigcup_{n=1}^{\infty} V_n$$

with

$$\mu\left(\bigcup_{n=1}^{\infty} V_n \setminus \bigcup_{n=1}^N F_n\right) < \varepsilon + \delta.$$

□

**Remark.** This implies that we don't care much about Descriptive Set Theory. For  $X \in \mathcal{SM}$  and  $\mu \in \mathbb{P}(X)$ , we only care about  $F_\sigma$  and  $G_\delta$  sets. More formally, for any  $B \in \text{Bor } X$ , any set is up to a set of measure 0 an  $F_\sigma$  from below and  $G_\delta$  from above.

**Remark.** An analytical set is an image of a Polish space.

**Theorem 4.6.** *If  $X$  is a Polish space and  $\mu \in \mathbb{P}(X)$ , then  $\mu$  is **tight**, i.e. for every  $\varepsilon > 0$  there is a compact  $K$  such that*

$$\mu(K) > 1 - \varepsilon.$$

*Proof.* Let  $d(-, -)$  be a complete metric on  $X$  and let  $x_n$  be a countable dense set.

$$X = \bigcup_{k=1}^{\infty} B\left(x_k, \frac{1}{n}\right).$$

By upward continuity of measure, we can take a  $k_n$  such that the first  $k_n$  balls are large, i.e. larger than

$$1 - \frac{\varepsilon}{2^n}.$$

Now denote

$$K_n := \bigcup_{k=1}^{k_n} \overline{B\left(x_k, \frac{1}{n}\right)}, \quad K = \bigcap_{n=1}^{\infty} K_n.$$

To see that  $K$  is large in  $\mu$ , use Lemma 1.5. Note that  $K$  is closed.

To prove compactness, we will find for a sequence  $x_k$  in  $K$  a convergent subsequence. We mirror the proof that  $[0, 1]$  is sequentially compact. Infinitely many elements of  $x_k$  will belong to one of the (finitely many!) balls that make up  $K_1$ , and of those infinitely many will land in one of the balls that make up  $K_2$  etc. Therefore,  $x_n$  has a Cauchy subsequence. Since  $X$  is Polish, this subsequence is convergent in  $X$ , but as  $K$  is closed, the limit is actually in  $K$ .  $\square$

**Example.** Let  $X = \mathbb{N}^{\mathbb{N}}$ .

**Theorem 4.7** (Lusin). *If  $X \in \mathcal{SM}$  and  $f : X \rightarrow \mathbb{R}$  is a borel function, then for any  $\mu \in \mathbb{P}(X)$  and  $\varepsilon > 0$  there exists a closed set  $F$  of large measure such that  $f$  is continuous on  $F$ .*

## Chapter 5

# Measures on Topological Spaces, Problemset 1

### Problem 4

#### Extension 1

We show that the set can be the graph of a function! Let  $Z$  be a borel set of positive measure and define

$$T_Z = \{x : \lambda(Z_x) > 0\}.$$

Then  $T_Z$  is a measurable set by Fubini's Theorem. We can pick a compact subset  $T'_Z$ . A compact set of positive measure has at least  $\mathfrak{c}$  elements, and there are as many borel sets. Then, enumerate borel sets of  $\mathbb{R}^2$ .

### Problem 5

#### Set of undefined density at 0

TODO

#### Set of density $t$ at 0

Presented in class by **Michał Baran**. Fix  $t \in (0, 1)$ .

The set we will construct will be symmetric around 0. We will find a sequence  $b_n$  such that with

$$A_n = \left(\frac{1}{n} - b_n, \frac{1}{n}\right)$$

we will have for all  $n$

$$\frac{t}{n} = \lambda\left(\bigcup_{k=n}^{\infty} A_k\right) = \sum_{k=n}^{\infty} b_k,$$

so

$$b_n = \sum_{k=n}^{\infty} b_k - \sum_{k=n+1}^{\infty} b_k = \frac{t}{n(n+1)}.$$



Consider

$$A := \bigcup_{k=1}^{\infty} A_k \cup -A_k.$$

We will bound the fraction

$$\frac{\lambda(A \cap (-\delta, \delta))}{2\delta} = \frac{\lambda(A \cap (0, \delta))}{\delta}$$

from above and below. For  $\delta \in (1/(n+1), 1/n]$  we have

$$\bigcup_{k=n+1} A_k \subseteq A \cap (0, \delta) \subseteq \bigcup_{k=n} A_k,$$

passing to measure

$$\frac{t}{n+1} \leq \lambda(A \cap (0, \delta)) \leq \frac{t}{n}.$$

When divided by  $\delta$ , we get the result by the squeeze theorem.

**Remark.** The solution would work equally well if instead of  $a_n = 1/n$  we used a sequence that converges to 0 monotonically and satisfies

$$\frac{a_n - a_{n+1}}{a_n} \rightarrow 0.$$

## Problem 9

Presented in class by **dr Arturo Martinez Celiz**.

Wlog, everything happens within  $(0, 1)$ . Following the hint, take a countable sequence  $A_i$  such that the set  $B := \bigcup_i A_i$  has maximal measure.

By this choice, for any  $C \in \mathcal{A}$ , we have

$$\lambda((C \cup B) \Delta B) = 0,$$

so that

$$\phi(C \cup B) = \phi(B)$$

and

$$C \subseteq \phi(C \cup B) = \phi(B).$$

Since  $C$  was arbitrary

$$\bigcup \mathcal{A} \subseteq \phi(B) \implies B \subseteq \bigcup \mathcal{A} \phi(B).$$

Since  $\lambda(B) = \lambda(\phi(B))$  we know that the sum of  $\mathcal{A}$  is measurable.

## Problem 10

Presented in class by **Szymon Smolarek**.

We take a cover of **regular sets**, i.e. a family for which there exists a constant  $C$  such that

$$\text{diam}^2 A \leq C \lambda_2(A).$$

It can be proven that if such a family is a Vitali cover of a set  $A \subseteq \mathbb{R}^2$ , an analogue of the **VCT** holds.

The family of all triangles does not satisfy the regularity condition – think of keeping one segment constant and bringing the third vertex ever closer to the segment. To deal with this, we subdivide the family  $\mathcal{T}$  into subfamilies

$$\mathcal{T}_n := \{T \in \mathcal{T} : \text{diam}^2 T \leq n\lambda_2(A)\}.$$

Reducing to a given subfamily, we can cover each triangle  $T$  by arbitrarily small triangles similar to  $T$  contained within  $T$ . This gives us a regular Vitali cover  $\tilde{\mathcal{T}}_n$  of  $\bigcup \mathcal{T}_n$ .

## Problem 11

Stated in class by **Szymon Smolarek**.

**Theorem 5.1** (Steinhaus theorem for the Cantor Set). *For any measurable set  $A$ , the set*

$$A \oplus A$$

*contains an open neighbourhood of 0.*

**Theorem 5.2** (Vitali Covering Theorem for the Cantor Set). *If a family of clopens  $\mathcal{J} \subseteq \text{Clop } \mathcal{C}$  is a Vitali cover of  $A$ , then there is a sequence  $J_n \in \mathcal{J}$  such that*

$$\nu^* \left( A \setminus \bigcup_n J_n \right) = 0.$$

**Theorem 5.3** (Lebesgue Density Theorem for the Cantor Set). *Let  $A \subseteq \mathcal{C}$ . An element  $a \in A$  is a **density point** of  $A$  if*

$$\lim_{n \rightarrow \infty} \frac{\nu(A \cap [a|_{[n]}])}{2^{-n}} = 1.$$

*If  $A$  is measurable, then almost all points of  $A$  are density points of  $A$ .*

The proofs are quite the same, as  $\mathcal{C}$  is a topological group and the measure  $\nu$  is its Haar measure.

## Problem 12

**Hint.** Use Baire's theorem.

## Chapter 6

# Measures on Topological Spaces, Problemset 2

### Problem 1

Such an  $a$  exists by compactness of  $A$  and continuity of metric. If we have two  $a_1, a_2$  such that

$$\rho(x, a_1) = \rho(x, a_2)$$

then  $a_1, a_2$  must agree and disagree with  $x$  at all places, so in fact  $a_1 = a_2$ , thus  $r_A$  is well-defined. For any  $a \in A$ ,  $d(a, A) = 0 = d(a, a)$ , so  $r_A$  is a retraction. What remains to be shown is continuity.

Let  $x, y$  agree up to  $n_0(x, y)$ . Then  $r_A(x)$  and  $r_A(y)$  also agree up to  $n_0(x, y)$  – if they differed earlier, we could use  $r_A(x)$  instead of  $r_A(y)$  and get a closer point  $a$  in the definition! So we have

$$n_0(x, y) \leq n_0(r_A(x), r_A(y))$$

and

$$d(x, y) \geq d(r_A(x), r_A(y)).$$

**Remark.** The metric  $d(x, y) = 1/n_0(x, y)$ , i.e. the first moment where  $x$  and  $y$  differ, won't work, because it can't tell apart points from which  $x$  differs at the same position!

### Problem 2

### Problem 3

Any  $A, B \in \text{Clop } \mathcal{C}$  can be written as disjoint sums of the basis sets  $[\varphi]$  by 3.8. Since the condition distributes over disjoint sums, we will prove the statement for  $A = [\varphi]$  and  $B = [\psi]$  with

$$|\text{dom } \varphi|, |\text{dom } \psi| < \infty.$$

Let  $I = \text{dom } \varphi$ ,  $J = \text{dom } \psi$  be the disjoint(!) domains of  $\varphi, \psi$ . There is a function  $\tau$  on  $I \cup J$  such that

$$\tau|_I = \varphi, \tau|_J = \psi.$$

For such a function,

$$[\varphi] \cap [\psi] = [\tau].$$

Now take an  $n$  such that  $I \cup J \subseteq \{1, 2, \dots, n\}$  and denote the last set as  $[n]$ . By 3.10 we compute

$$\begin{aligned}\nu[\varphi] &= 2^{-|I|} \\ \nu[\psi] &= 2^{-|J|} \\ \nu[\tau] &= 2^{-|I \cup J|},\end{aligned}$$

and  $|I \cup J| = |I| \cup |J|$  finishes the proof. Now take arbitrary  $A, B \in \text{Bor } \mathcal{C}$  such that  $A \sim I$ ,  $B \sim J$ . Approximate  $A, B$  by clopens  $A', B'$  to within an  $\varepsilon$ , i.e. so that

$$\nu(A \Delta A'), \nu(B \Delta B') < \varepsilon.$$

We cannot use the clopen statement we just proved since a priori  $A'$  and  $B'$  could be determined by sets with nonempty intersection. We can, however, improve the approximation with

$$\tilde{A} := \pi_I^{-1} \pi_I A'.$$

The set  $\tilde{A}$  is still a clopen – since  $A'$  was determined by a finite set  $K$ ,  $\tilde{A}$  is determined by  $K \cap I$ . Additionally we have

$$\tilde{A} \Delta A \subseteq A' \Delta A,$$

so we have improved the approximation! Now, do the same for  $B'$  and use the statement for clopens to finish up the solution.

**Warning!** The reasoning below does not work! (For tail sets, for example)

We can approximate  $A, B$  by decreasing sequences of clopens by putting down

$$A_n := \pi_{[n]}^{-1} \pi_{[n]} A$$

and the same for  $B_n$ . We also approximate their intersection by decreasing clopens in the same way, i.e.

$$C_n := \pi_{[n]}^{-1} \pi_{[n]} (A \cap B).$$

For these approximations

$$C_n = A_n \cap B_n,$$

so by the first subproblem

$$\nu(C_n) = \nu(A_n \cap B_n) = \nu(A_n) \cdot \nu(B_n).$$

Since the measure  $\nu$  is probabilistic, and hence continuous, by passing to the limit  $n \rightarrow \infty$  we get what we need.

## Problem 6

Any clopen  $C \in \text{Clop } \mathcal{C}$  is a disjoint sum of basis cylinders by 3.8. Since  $\oplus$  is a group operation, the function

$$l_x(y) = x \oplus y$$

is bijective, so on the level of sets  $l_x$  distributes over disjoint sums. We check the property for a cylinder  $[\varphi]$ . This is easy, since

$$\nu(x \oplus [\varphi]) = \nu[x \oplus \varphi] = 2^{-|\text{dom } \varphi|} = \nu[\varphi]$$

by 3.10. Now consider the family of sets

$$\mathcal{A} := \left\{ A : \forall x \in \mathcal{C}. \nu(A) = \nu(x \oplus A) \right\}.$$

We will show that this is a  $\sigma$ -algebra. Since we have already shown that it contains all the clopens, which form a basis of the topology on  $\mathcal{C}$ , it will automatically be equal to  $\text{Bor } \mathcal{C}$  by 1.2.

A  $\sigma$ -algebra can be generated by complements and countable sums (see 1.1). As mentioned before,  $l_x$  respects these operations, so

$$\nu(x \oplus A^c) = \nu((x \oplus A)^c) = 1 - \nu(x \oplus A) = 1 - \nu(A) = \nu(A^c)$$

and

$$\nu\left(x \oplus \bigcup_i A_i\right) = \nu\left(\bigcup_i x \oplus A_i\right) = \sum_i \nu(x \oplus A_i) = \sum_i \nu(A_i) = \nu\left(\bigcup_i A_i\right).$$

## Problem 7

The identification is

$$A \mapsto \chi_A, x \mapsto \{n : x_n = 1\}.$$

One easily checks that these two are mutually inverse. Addition modulo 2 comes out to 1 iff exactly one of the summands is 1, and this corresponds exactly to belonging to the symmetric difference.

## Problem 8

A filter cannot contain both  $A$  and  $A^c$ , since then it would contain  $A \cap A^c = \emptyset$ . Thus, a filter containing for all  $A$  either  $A$  or  $A^c$  is maximal.

For the other direction, suppose neither  $A$  nor  $A^c$  is in a filter  $\mathcal{F}$ . We define its *extension* by  $A$  as

$$\mathcal{F}_A = \{A' \cap F : A \subseteq A', F \in \mathcal{F}\}.$$

We check that this is a filter.

1. If  $\emptyset \in \mathcal{F}_A$ ,  $\mathcal{F}$  contains a set disjoint with  $A$ , so by the superset property it contains  $A^c$ .
2. Let  $A_1 \cap F_1, A_2 \cap F_2 \in \mathcal{F}_A$ . Then

$$A \subseteq A_1 \cap A_2, F_1 \cap F_2 \in \mathcal{F},$$

$$\text{so } (A_1 \cap F_1) \cap (A_2 \cap F_2) = (A_1 \cap A_2) \cap (F_1 \cap F_2) \in \mathcal{F}_A.$$

3. Let  $B \supseteq A' \cap F$ . Then

$$B = B \cup (A' \cap F) = (B \cup A') \cap (B \cup F)$$

$$\text{and } A \subseteq A' \cup B, F \subseteq B \cup F, \text{ so } B \in \mathcal{F}_A.$$

Of course,  $A \in \mathcal{F}_A \setminus \mathcal{F}$ , so  $\mathcal{F}$  was not maximal in the first place.

**Remark.** One can check that  $\mathcal{F}_A$  is the minimal filter containing  $\mathcal{F}$  and  $A$ .

### Problem 9

The only principal ultrafilters are generated by singletons, so they are definitely measurable.

## Chapter 7

# Measures on Topological Spaces, Problemset 3

### Problem 1

By transfinite induction, each  $B \in \text{Bor } Y$  is of the form  $\tilde{B} \cap Y$  for some  $\tilde{B} \in \text{Bor } X$ .

The axiom  $\nu(\emptyset) = 0$  is immediate from  $\emptyset = \emptyset \cap Y$ .

For countable additivity, take a sequence of pairwise disjoint sets  $B_n \in \text{Bor } Y$ . By an earlier observation, we may represent them as  $Y \cap \tilde{B}_n$  for a sequence  $\tilde{B}_n \in \text{Bor } X$ . By Problem 2, we may in fact assume

$$\mu(\tilde{B}_n) = \mu^*(Y \cap \tilde{B}_n)$$

if we take  $\tilde{B}_n$  to be measurable hulls. If  $B_n, B_m$  give disjoint sets in  $Y$ , then we have

$$\tilde{B}_n \cap \tilde{B}_m \subseteq (\tilde{B}_n \setminus Y) \cup (\tilde{B}_m \setminus Y)$$

so when we pass to outer measure we see that  $\mu(\tilde{B}_n \cap \tilde{B}_m) = 0$ . Note that  $B_n \cap Y = \tilde{B}_n \cap Y$ . We have

$$0 \leq \mu^*\left(\bigcup_{n=1}^{\infty} \tilde{B}_n \setminus Y\right) \leq \sum_{n=1}^{\infty} \mu^*(\tilde{B}_n \setminus Y) = 0,$$

so

$$\begin{aligned} \mu^*\left(\bigcup_{n=1}^{\infty} \tilde{B}_n \cap Y\right) &= \mu^*\left(\bigcup_{n=1}^{\infty} \tilde{B}_n\right) \\ &= \mu\left(\bigcup_{n=1}^{\infty} \tilde{B}_n\right) \\ &= \sum_{n=1}^{\infty} \mu(\tilde{B}_n) \\ &= \sum_{n=1}^{\infty} \mu^*(\tilde{B}_n \cap Y). \end{aligned}$$

## Problem 2

By definition of outer measure (as an infimum), we can choose measurable sets  $H_n \supseteq Z$  such that

$$\mu^*(Z) \leq \mu(H_n) < \mu^*(Z) + \frac{1}{n}.$$

Take

$$H = \bigcap_{n=1}^{\infty} H_n.$$

This  $H$  contains  $Z$ , so we have

$$\mu(H) = \mu^*(Z)$$

by squeezing. By regularity of borel measures (see 4.5), we can take a  $G_\delta$  upper approximation of  $H$  with the same measure.

For the second part, take two measurable hulls  $H_1$  and  $H_2$ . Since  $Z \subseteq H_1 \cap H_2$ , we have

$$H_1 \Delta H_2 \subseteq (H_1 \setminus Z) \cup (H_2 \setminus Z),$$

and the RHS has (outer) measure 0, so the LHS does as well.

**Observation.** Hulls work well with set unions, i.e. for a countable union of  $Z_i$  with hulls  $H_i$ , the union of  $H_i$  is a hull for that union. Intersections and complements are more problematic.

## Problem 3

By a *base* I understand a family  $\mathcal{U}$  of subsets of  $X$  such that  $\sigma(\mathcal{U}) = \text{Bor } X$ . Consider the family  $\mathcal{A}$  of borel subsets of  $X$  on which  $\mu$  and  $\nu$  agree. We will show that it is a  $\sigma$ -algebra.

Because these are probability measures, the family  $\mathcal{A}$  is closed under complements. By upward continuity of measures,  $\mathcal{A}$  is closed under finite unions. Therefore it is a  $\sigma$ -algebra, so  $\mathcal{A} = \text{Bor } X$  and  $\mu = \nu$ .

## Cardinality

Since  $X$  has at least two distinct points  $x_1, x_2$ , we have at least  $\mathfrak{c}$  measures, as witnessed by

$$p\delta_{x_1} + (1-p)\delta_{x_2}.$$

On the other hand, we have

**Lemma 7.1.** *If  $X$  is  $\mathcal{SM}$ , then  $X$  is second countable.*

*Proof.* By 4.1,  $X \hookrightarrow [0, 1]^{\mathbb{N}}$  which is second countable, so  $X$  is second countable as well.  $\square$

Since the values of a probability measure  $\mu$  are determined by its values on a countable basis, we know that there are at most as many measures as functions in  $[0, 1]^{\mathbb{N}}$ . The cardinality of the Hilbert Cube is

$$\mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c},$$

which gives the upper bound.

**Question.** What is the finite union assumption needed for?