COMP3240 - 002

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(1-50pts) Logic. (a - 3pts) Exercise 1.3.2: The inverse, converse, and contrapositive of conditional sentences in English. Give the inverse, converse and contrapositive for the following statement:

If it snowed last night, then school will be cancelled.

Inverse : If it did not snow last night, then school won't be cancelled.

Converse : If school will be cancelled, then it snowed last night.

Contrapositive : If school won't be cancelled, then it was not snowed last night.

(b - 3pts) Exercise 1.4.4: Proving whether two logical expressions are equivalent. Determine whether the following pairs of expressions are logically equivalent. Prove your answer. If the pair is logically equivalent, then use a truth table to prove your answer.

p ∧ (p → q) and p ∧ q

p	**q**	**p ∧ (p → q)**	**p ∧ q**
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

They are logically equivalent.

(c - 3pts) Exercise 1.4.5. Define the following propositions: j: Sally got the job. l: Sally was late for her interview r: Sally updated her resume. Express the following pair of sentences using a logical expression. Then prove whether the two expressions are logically equivalent.

If Sally did not get the job, then she was late for interview or did not update her resume.

$$\neg j \rightarrow (l \vee \neg \mathrm{r})$$

If Sally updated her resume and did not get the job, then she was late for her interview.

$$(r \wedge \neg j) \rightarrow l$$

J	l	r	$\neg j \rightarrow \neg l \vee \neg r$	$r \wedge \neg j \rightarrow l$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	F
F	F	F	T	T

They are logically equivalent.

(d - 3pts) Exercise 1.4.6: Applying De Morgan's laws. Translate the English sentences into logical expressions using the propositional variables defined below. Then negate the entire logical expression. Apply De Morgan's law to the resulting expression and translate the final expression back into English. p: the applicant has written permission from his parents; e: the applicant is at least 18 years old; s: the applicant is at least 16 years old

The applicant has written permission from his parents and is at least 16 years old.

$$p \wedge e$$

Negation (Demorgan's law)

- ⇨ $\neg(p \wedge e)$
- ⇨ $\neg p \vee \neg e$
- ⇨ The applicant has not written permission from his parents or is less than 16 years old.

(e - 6pts) Exercise 1.5.2/1.5.4: Using the laws of logic to prove logical equivalence. Use the laws of propositional logic to prove the following:

• (p → q) ∧ (p → r) ≡ p → (q ∧ r)

$$(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\boldsymbol{p} \rightarrow \boldsymbol{r}) \equiv \boldsymbol{p} \rightarrow (\boldsymbol{q} \wedge \boldsymbol{r})$$

- ⇨ $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\boldsymbol{p} \rightarrow \boldsymbol{r})$
- ⇨ $(\neg \boldsymbol{p} \vee \boldsymbol{q}) \wedge (\neg \boldsymbol{p} \vee \boldsymbol{r})$ **/conditional**
- ⇨ $\neg \boldsymbol{p} \vee (\mathbf{q} \wedge \boldsymbol{r})$ **/distributive**

⇨ $p \rightarrow (q \wedge r)$ **/conditional**

• A conditional statement is logically equivalent to its contrapositive.

⇨ $p \rightarrow q \equiv \neg q \rightarrow \neg p$

⇨ $\neg q \rightarrow \neg p \equiv q \vee \neg p$ ***/conditional***

⇨ $q \vee \neg p \equiv \neg p \vee q$

⇨ $\neg p \vee q \equiv p \rightarrow q$ **/conditional**

⇨ $They\ are\ logically\ equivalent$

(f - 3pts) Exercise 1.6.3: Translating mathematical statements in English into logical expressions. Consider the following statements in English. Write a logical expression with the same meaning. The domain of discourse is the set of all real numbers.

There is a number whose cube is equal to 2

$$\exists x\ (x^3 = 2)$$

It is True.

(g - 3pts) Exercise 1.7.3: Translating quantified statements in English into logic. In the following question, the domain of discourse is the set of employees at a company. Define the following predicates: T(x): x is a member of the executive team B(x): x received a large bonus Translate the following English statement into a logical expression with the same meaning.

Every executive team member got a large bonus.

$$\forall x\ (T(x) \rightarrow B(x))$$

(h - 5pts) Exercise 1.8.4: Using De Morgan's law for quantified statements to prove logical equivalence. Use De Morgan's law for quantified statements and the laws of propositional logic to show the following equivalence:

¬∀x(¬P(x) → Q(x)) ≡ ∃x(¬P(x) ∧ ¬Q(x))

⇨ $\neg\forall x(\neg P(x) \rightarrow Q(x))$

⇨ $\neg\forall x(P(x) \vee Q(x))$ **/conditional**

⇨ $\exists x\ \neg(P(x) \vee Q(x))$ **=> Demorgan's law**

- ⇨ $\exists \boldsymbol{x}\, (\neg \boldsymbol{P}(\boldsymbol{x}) \land \neg \boldsymbol{Q}(\boldsymbol{x}))$ **/Demorgan's law again**

(i - 5pts) Exercise 1.9.5: Applying De Morgan's law with nested quantifiers. The domain for variables x and y is a group of people. The predicate F(x, y) is true if and only if x is a friend of y. For the purposes of this problem, assume that for any person x and person y, either x is a friend of y or x is an enemy of y. Therefore, ¬F(x, y) means that x is an enemy of y. Translate the following statement into a logical expression. Then negate the expression by applying De Morgan's law and then translate the logical expression back into English.

Everyone is a friend of someone.

$$\forall x \exists y F(x, y)$$

Negation it (De Morgan's law)

$$\neg \forall x \exists y F(x, y)$$

- ⇨ $\exists x\, \forall y \neg F(x, y)$
- ⇨ Someone is an enemy of everyone.

(j - 3pts) Exercise 1.10.6: Statements with nested quantifiers: English to logic. The domain for the variables x and y are the set of musicians in an orchestra. The predicates S, B, and P are defined as: S(x): x plays a string instrument

B(x): x plays a brass instrument

P(x, y): x practices more than y

Give a quantified expression that is equivalent to the following English statement: ***There is a brass player who practices more than all the string players.***

There is a brass player => $\exists x B(x)$

Who practices more than all the string player => $\forall y (S(y) \rightarrow \mathrm{P}(\mathrm{x}, \mathrm{y}))$

Therefore, $\exists x (B(x)$ ^ $\forall y (S(y) \rightarrow \mathrm{P}(\mathrm{x}, \mathrm{y}))\,)$

(k - 6pts) Exercise 1.11.1/1.11.2: Valid and invalid arguments expressed in logical notation. Indicate whether the argument is valid or invalid. Prove using a truth table.

• ***¬(p → q)***

q → p

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∴ ¬q

p	q	¬(p → q)	q → p	¬q
T	T	F	T	F
T	F	T	T	T
F	T	F	F	F
F	F	F	T	T

It is valid.

• ***p → q***

¬p

——-

∴ ¬q

p	q	p → q	¬p	¬q
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

It is invalid.

(l - 3pts) Exercise 1.12.5: Proving arguments in English are valid or invalid. Give the form of the argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

If I get a job then I will buy a new car and a new house.

I won't buy a new house.

∴ I will not get a job.

j => get a job

h => buy a house

c => buy a car

If I get a job then I will buy a new car and a new bouse => $j \rightarrow (c \wedge \mathrm{h})$

I won't buy a new house => $\neg h$

I will not get a job => $\neg j$

- ⇨ $j \rightarrow (c \wedge \mathrm{h})$ /hypothesis
- ⇨ $\neg j \vee (c \wedge \mathrm{h})$ /conditional 1
- ⇨ $(\neg j \vee c) \wedge (\neg j \vee \mathrm{h})$ /distribution 2
- ⇨ $(\neg j \vee \mathrm{h})$ /simplification
- ⇨ $(\neg \mathrm{h})$ /hypothesis
- ⇨ ¬j /disjunctive syllogism
- ⇨ $It\ is\ valid$

(m - 4pts) Exercise 1.13.1: Proving the validity of arguments with quantified statements. Prove that the given argument is valid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. Then use the rules of inference to prove that the form is valid. The domain is the set of students at an elementary school.

Every student who has a permission slip can go on the field trip.

Every student has a permission slip.

——————————————-

∴ Every student can go on the field trip.

S(x) => has a permission slip

F(x) => go on the field trip

Every student who has a permission slip can go on the field trip => $\forall x(S(x) \rightarrow F(x))$

Every student has a permission slip => $\forall x S(x)$

Every student can go on the field trip => $\forall x F(x)$

- ⇨ $\forall x(S(x) \rightarrow F(x))$ /hypothesis
- ⇨ $\forall x(\neg S(x) \vee F(x))$ / conditional1
- ⇨ $\neg S(x) \vee F(x)$
- ⇨ $\forall x S(x)$ /hypothesis
- ⇨ $S(x)$
- ⇨ $F(x)$ /Disjunctive syllogism
- ⇨ $\forall x F(x)$
- ⇨ It is valid

(a - 10pts) Exercise 2.4.1: Proving statements about integers with direct proofs. The statement below involves odd and even integers. An odd integer is an integer that can be expressed as 2k+ 1 where k is an integer. An even integer is an integer that can be expressed as 2k where k is an integer. Prove the following statements using a direct proof.

The square of an odd integer is an odd integer

I can show it just giving an example. The square of 3 is 9.

or

Let's suppose that there is an integer n = 2k + 1

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

- ⇨ $4k^2 + 4k$ is definitely even and add 1 here

⇨ Therefore $4k^2 + 4k + 1$ is odd

(b - 10pts) Exercise 2.4.4: Direct proof or counterexample. Prove whether the statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample.

If x + y is an even integer, then x and y are both even integers.

If x = 1, y = 3, then x + y = 4 which is even but x and y are both odd integers.

So the statement is false.

(c - 10pts) Exercise 2.5.4: Proving conditional statements by contrapositive. Prove the following statement by contrapositive.

For every pair of real numbers x and y, if x 3 + xy2 ≤ x 2y + y 3 , then x ≤ y

Its contrapositive is **For some pair of real numbers x and y, if x > y, then x^3 + xy^2 > x^2y + y^3**

⇨ If x – y > 0 , then $x^3 + xy^2 - x^2y - \ y^3 \ > 0$

⇨ $x^3 + xy^2 - x^2y - \ y^3 \ > 0$

⇨ $x(x^2 + y^2) - y(x^2 + \ y^2) \ > 0$

⇨ $(x - y)(x^2 + y^2) \ > 0$

⇨ $(x - y) \quad > 0$ and $((x^2 + y^2)$ is definitely a positive number.

⇨ Therefore, If x – y > 0 , then $x^3 + xy^2 - x^2y - \ y^3 \ > 0$

⇨ Of course ***if x 3 + xy2 ≤ x 2y + y 3 , then x ≤ y***

⇨ This is true.

(d - 10pts) Exercise 2.6.6: Proofs by contradiction. Give a proof for the following statement.

If a group of 9 kids have won a total of 100 trophies, then at least one of the 9 kids has won at least 12 trophies.

Let's contradict it. Set x_i as the number of trophies won by each students.

⇨ A group of 9 kids have won a total of 100 trophies and no kids has won at least 12 trophies.

⇨ $x_1 + \ x_2 + \ x_3 + \ x_4 + \ x_5 + \ x_6 + \ x_7 + \ x_8 + \ x_9 = 100$

⇨ Assume that all students won 11 trophies which is max, 11 * 9 = 99 which is less than 100.

- ⇨ Therefore the statement is inconsistent.

(e – 10pts) Exercise 2.7.2: Proofs by cases. Prove the following statement.

If x is an integer, then x 2 + 5x − 1 is odd.

Case 1, x is even

For some integer k, x = 2k.

- ⇨ Plug the 2k into the $x^2 + 5x - 1$
- ⇨ $(2k)^2 + 5(2k) - 1$
- ⇨ $4k^2 + 10k - 1$
- ⇨ $4k^2$ is a definitely even number and 10k is also even number too.
- ⇨ $4k^2 + 10k$ is even number but we minus 1 from that
- ⇨ Therefore, $4k^2 + 10k - 1$ is odd.

Case 2, x is odd

For some integer k, x = 2k – 1

- ⇨ Plug the 2k - 1 into the $x^2 + 5x - 1$
- ⇨ $(2k - 1)^2 + 5(2k - 1) - 1$
- ⇨ $4k^2 - 4k + 1 + 10k - 5 - 1$
- ⇨ $4k^2 + 6k - 5$
- ⇨ $4k^2 + 6k$ is definitely even number and minus 5 from it which is odd number
- ⇨ Therefore, $4k^2 + 6k - 5$ is odd number.

x 2 + 5x − 1 is odd in any cases.