The interplay between stochastic volatility and correlations in equity autocallables

Alvise De Col and Patrick Kuppinger investigate typical equity worst-of autocallable structures within industry-standard multidimensional stochastic local volatility models. Introducing the corresponding effective local volatility models, they show how correlations between the stochastic variances play a central role in autocallable prices and risk management. One key result is that the pricing behaviour cannot be explained by the common belief that stochastic volatility only reduces the spot-spot correlation

At odds with the push towards simple derivatives structures in the aftermath of the credit crisis, exotic equity autocallable structures remain as popular as ever. Investors with a sideways or mildly positive view on equity prices are attracted by the possibility of financing above-market coupons via shorting downside protection in the form of an embedded kick-in put. In autocallable structures, a series of discrete upper barriers may trigger an early termination of the structure and the repayment of the notional. To finance attractive coupons, autocallables are often written on the laggard component of a basket of equity underlyings.

It is a well-known fact that the local volatility (LV) model, a simplistic but still popular model choice in equity trading desks, is not suitable to price and risk manage these structures. Similar to barrier options, autocallable structures have strong path-dependent features. For example, the embedded kick-in put only materialises if no autocall event has taken place and the protection barrier has been breached. Therefore, the valuation crucially depends on a correct estimation of the corresponding conditional probabilities (of hitting the protection barrier, given that no autocall event has occurred) in line with the market: a property the LV model cannot guarantee. A typical choice is to augment the LV model with a stochastic volatility component, leading to a stochastic local volatility (SLV) model (Austing 2014; De Col & Kuppinger 2014; Guyon & Henry-Labordère 2012).

While the impact of stochastic volatility is well understood in the case of an autocallable with a single equity underlying, the same is not true when the autocallable is written on the laggard of a portfolio. Besides the effect on transitional probabilities, stochastic volatility also modifies the realised correlation between the different equities. SLV models preserve all one-dimensional spot marginals, but the joint density (and the correlation structure) is directly affected by the stochastic volatility and is therefore model dependent. A typical observation described in the literature is the realised spot-spot correlation is reduced due to the presence of stochastic volatility (Bergomi 2016; Brockhaus 2016).

From a practical perspective, one is naturally confronted with the question of how to compare the SLV and LV model valuations, and, ultimately, if it is even possible to split the two effects mentioned above to assess their impacts separately. Despite its limitations, LV is often used as a reference point in the pricing process. Moreover, it may not be possible to run live risk management reports on SLV due to the additional performance burden. Then, SLV is used on a regular basis to compute a valuation adjustment to correct the profit and loss (P&L) calculated off the LV model. Shedding light on the components that cause the difference between SLV and LV becomes a critical task in understanding the P&L drivers of a structured equity desk.

In this article, we build an effective LV model that agrees with the original SLV model on the valuation of any multidimensional terminal payout. In this way, we can reduce the impact of stochastic volatility on the transitional probabilities and concentrate on the coupling effect of the individual one-dimensional marginals in the SLV model (ie, the spot-spot correlation structure). We prove decorrelation is not the only effect caused by stochastic volatility and explain why the sensitivity of the price towards stochastic volatility might actually have the opposite sign to what one would expect to come from decorrelation alone. The key drivers of this rich pricing behaviour, observed, for example, in autocallables, are higher-order model parameters, the most important of which is the correlation between the stochastic variance drivers.

Model setup

We consider two equity underlyings, S_1 and S_2, each of which is described by a typical SLV model, coupled by a constant correlation. For simplicity, we shall assume zero rates, no dividends and the same initial values of the processes for the two underlyings. The two underlyings then follow the process:

$$\begin{aligned} \mathrm{d}S_i &= S_i A_i(S_i, t)\sqrt{v_i}\,\mathrm{d}W_{S_i} \\ \mathrm{d}v_i &= \alpha(v_i, t)\,\mathrm{d}t + \beta(v_i, t)\,\mathrm{d}W_{v_i} \end{aligned} \tag{1}$$

The leverage surfaces $A_i(S_i, t)$ are determined from Gyöngy's lemma in order to ensure the fit to vanilla prices:

$$A_i^2(S_i, t)\mathbb{E}[v_i(t) \mid S_i(t) = S_i] = \sigma_i^2(S_i, t)$$

where $\sigma_i(S_i, t)$ denotes the usual Dupire local volatility function of S_i.

Additional assumptions are that the two variance processes have the same initial value as well as the same functional form for the drift $\alpha(\cdot, t)$ and the volatility of variance $\beta(\cdot, t)$. Standard choices for these include the Heston model (Heston 1993) ($\alpha(v, t) = -\lambda(v - \bar{v})$ and $\beta(v, t) = \eta\sqrt{v}$) and the Scott model (Scott 1987) ($\alpha(v, t) = -\lambda v(\ln v - m) + \eta^2 v/2$ and $\beta(v, t) = \eta v$). Moreover, for any particular choice of functional form, we assume the two underlyings' variance processes share the numerical values of those variance-specific model parameters, ie, the mean reversion speed λ, the volatility of variance (vov) η and the long-term means $\bar{v}$ or m. These assumptions are made to simplify the exposition and maintain analytical tractability to some degree, but they can be lifted without fundamentally changing the description of the effects of SLV on worst-of autocallable pricing beyond decorrelation. We further define:

$$\mathrm{d}W_{S_1}\,\mathrm{d}W_{S_2} = \rho_S\,\mathrm{d}t$$

Moreover, the correlations between the individual spot and corresponding variance process drivers are denoted by:

$$\mathrm{d}W_{S_i}\,\mathrm{d}W_{v_i} = \rho_{S_i v_i}\,\mathrm{d}t$$

where we assume $\rho_{S_1 v_1} = \rho_{S_2 v_2} = \rho_{Sv}$. Not allowing for idiosyncratic correlation between S_1 and v_2 or S_2 and v_1 yields:

$$\mathrm{d}W_{S_1}\,\mathrm{d}W_{v_2} = \mathrm{d}W_{S_2}\,\mathrm{d}W_{v_1} = \rho_S \rho_{Sv}\,\mathrm{d}t$$

We shall, however, assume an idiosyncratic correlation ρ_{vv} between v_1 and v_2, such that the resulting correlation can be written as (Kahl & Günther 2005):

$$\mathrm{d}W_{v_1}\,\mathrm{d}W_{v_2} = (\rho_S \rho_{Sv}^2 + \rho_{vv}(1-\rho_{Sv}^2))\,\mathrm{d}t$$

Common choices for the idiosyncratic correlation between the variance processes are either $\rho_{vv} = 1$ (in which case one has the numerical advantage of requiring only three Brownian drivers to simulate the two underlyings and their variances) or $\rho_{vv} = 0$. These two cases are referred to as the common driver case ($\rho_{vv} = 1$) and the independent drivers case ($\rho_{vv} = 0$), respectively. Note that, here, independent refers to the fact that the idiosyncratic variance drivers are uncorrelated.

Effective LV model. Let us denote by $\mathcal{O}(S_1, S_2)$ any terminal payout at time T depending on the two underlyings S_1 and S_2. Its expected value at inception is:

$$\mathbb{E}[\mathcal{O}(S_1, S_2)] = \int_0^\infty \int_0^\infty \mathcal{O}(S_1, S_2)\, p_T(S_1, S_2)\,\mathrm{d}S_1\,\mathrm{d}S_2$$

where $p_T(S_1, S_2)$ denotes the time-T joint terminal probability density. In the SLV framework (1), however, the density depends on four state variables, ie, $P_T(S_1, S_2, v_1, v_2)$. To proceed with the valuation of any terminal trade, the natural step is to reduce the dimensionality of the SLV system from four to two, by looking at the (S_1, S_2)-marginal. The two-dimensional density is defined as follows:

$$p_t(S_1, S_2) = \int_0^\infty \int_0^\infty P_t(S_1, S_2, v_1, v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2 \tag{2}$$

The four-state probability density is described by the Fokker-Planck equation:

$$\begin{aligned}
&\partial_t P_t \\
&\quad = \tfrac{1}{2}\partial^2_{S_1 S_1}(S_1^2 A_1^2 v_1 P_t) + \tfrac{1}{2}\partial^2_{S_2 S_2}(S_2^2 A_2^2 v_2 P_t) \\
&\qquad + \rho_S \partial^2_{S_1 S_2}(S_1 S_2 A_1 A_2 \sqrt{v_1 v_2} P_t) \\
&\qquad + (\mathbb{L}_{v_1} + \mathbb{L}_{v_2} + \mathbb{L}_{v_1 v_2} + \mathbb{L}_{v_1 S_1} + \mathbb{L}_{v_2 S_2} + \mathbb{L}_{v_1 S_2} + \mathbb{L}_{v_2 S_1}) P_t
\end{aligned}$$

where $P_t = P_t(S_1, S_2, v_1, v_2)$ and all operators $\mathbb{L}$ depend on the specific stochastic volatility process (via α and β). Each of them contains at least one partial derivative with respect to a variance parameter. The effective two-dimensional theory is obtained by integrating out both v_1 and v_2 according to (2), yielding:

$$\begin{aligned}
\partial_t p_t = \tfrac{1}{2}&\partial^2_{S_1 S_1}(S_1^2 \Sigma_{11}(S_1, S_2, t) p_t) \\
&+ \tfrac{1}{2}\partial^2_{S_2 S_2}(S_2^2 \Sigma_{22}(S_1, S_2, t) p_t) \\
&+ \partial^2_{S_1 S_2}(S_1 S_2 \Sigma_{12}(S_1, S_2, t) p_t)
\end{aligned} \tag{3}$$

where we have used the fact that all the $\mathbb{L}$ terms vanish when integrating out the variances. Note that in (3) we now have the two-dimensional state probability $p_t = p_t(S_1, S_2)$. Moreover, we have defined the local covariance functions:

$$\begin{aligned}
\Sigma_{ii}(S_1, S_2, t) &= A_i^2(S_i, t)\mathbb{E}[v_i(t) \mid S_1(t) = S_1, S_2(t) = S_2] \\
\Sigma_{12}(S_1, S_2, t) &= A_1(S_1, t) A_2(S_2, t)\rho_S \\
&\quad \times \mathbb{E}\Big[\sqrt{v_1(t) v_2(t)} \;\Big|\; S_1(t) = S_1, S_2(t) = S_2\Big]
\end{aligned} \tag{4}$$

We have obtained the Markovian projection (Brunick & Shreve 2013), ie, an effective LV model, for the original two-underlyings SLV system in (1):

$$\mathrm{d}S_i = S_i \sqrt{\Sigma_{ii}(S_1, S_2, t)}\,\mathrm{d}W_i \tag{5}$$

for $i \in \{1, 2\}$, where the correlation between the stochastic drivers is locally defined as:

$$\mathrm{d}W_1\,\mathrm{d}W_2 = \frac{\Sigma_{12}(S_1, S_2, t)}{\sqrt{\Sigma_{11}(S_1, S_2, t)\Sigma_{22}(S_1, S_2, t)}}\,\mathrm{d}t$$

By construction, the SLV model (1) and the effective LV model (5) yield the same joint terminal spot density and thus agree on the valuation of any terminal payoff. Note that the effective LV model differs from the standard extension of the local volatility framework (Austing 2014) to two underlyings in two ways:

- the local volatilities are not described by the standard Dupire local volatilities $\sigma_i^2(S_i, t)$ but rather by functions $\Sigma_{11}(S_1, S_2, t)$ and $\Sigma_{22}(S_1, S_2, t)$, each depending on both underlying spot values; and
- the covariance between the underlyings is described by the local function $\Sigma_{12}(S_1, S_2, t)$, which is generally different from $\sigma_1(S_1, t)\sigma_2(S_2, t)\rho_S$.

However, by virtue of Gyöngy's lemma, the generalised local volatility functions Σ_{ii} and the standard Dupire local volatilities are connected as follows:

$$\mathbb{E}[\Sigma_{ii}(S_1, S_2, t) \mid S_i(t) = S_i] = \sigma_i^2(S_i, t)$$

Importantly, the effective LV model allows us to retain the stochastic volatility impact on joint terminal densities produced without additional stochastic drivers. In order to further investigate the joint terminal distribution, we consider the instantaneous correlation $\rho_{\mathrm{ELV}}(S_1, S_2, t)$ between the returns in the effective LV model:

$$\begin{aligned}
\rho_{\mathrm{ELV}}(S_1, S_2, t) &= \frac{\mathrm{d}S_1\,\mathrm{d}S_2}{\sqrt{\mathrm{d}S_1^2\,\mathrm{d}S_2^2}} \\
&= \frac{\Sigma_{12}(S_1, S_2, t)}{\sqrt{\Sigma_{11}(S_1, S_2, t)\Sigma_{22}(S_1, S_2, t)}} \\
&\triangleq \gamma(S_1, S_2, t)\rho_S
\end{aligned} \tag{6}$$

where, using (4), we have defined the local decorrelation function γ as:

$$\gamma(S_1, S_2, t) = \frac{\mathbb{E}[\sqrt{v_1(t) v_2(t)} \mid S_1(t) = S_1, S_2(t) = S_2]}{\sqrt{D_1 D_2}} \tag{7}$$

where $D_i = \mathbb{E}[v_i(t) \mid S_1(t) = S_1, S_2(t) = S_2]$ for $i = 1, 2$.

As a consequence of Hölder's inequality, we have $\gamma(S_1, S_2, t) \leqslant 1$. The upper bound is attained by setting the vov to zero, resulting in $\Sigma_{ii} = \sigma_i^2$ and $\Sigma_{12} = \sigma_1 \sigma_2 \rho_S$. However, by increasing the vov, the instantaneous spot-spot correlation $\rho_{\mathrm{ELV}}(S_1, S_2, t)$ in (6) decreases (Bergomi 2016). As mentioned above, this effect is often referred to as the decorrelation effect in SLV models.

Effect of stochastic volatility on worst-of autocallables

Switching from the standard LV to the SLV model, two separate effects are expected on worst-of autocallables, as the conditional probabilities affecting the barrier hitting events and the terminal joint densities of the underlying returns are modified. By virtue of the effective LV model, we are now able to separate the effect on the terminal density from any remaining impact on transitional probabilities due to stochastic volatility.

In our tests, we have used an SLV model of Scott type with $\rho_{Sv} = -85\%$, $\kappa = 180\%$ and $\rho_S = 50\%$ if not specified otherwise. The mean reversion level is chosen such that, for each time t, $\mathbb{E}[v_i(t)] = 1$, following Bergomi's forward variance model approach (Bergomi 2016). Purely concentrating on the effect of stochastic volatility, and in line with our theoretical discussion below, we have chosen simplified identical market data for both underlyings, with zero rates and dividends but a realistic volatility surface with time-dependent (and not flat) smiles. All results were obtained via Monte Carlo simulation, where we used the particle method (Guyon & Henry-Labordère 2012) to determine the effective local volatility. More specifically, we approximate numerically the conditional expectations needed in the effective LV model for a discrete set of underlying values $S_{1,k}$, $S_{2,l}$ and times t_j according to:

$$\mathbb{E}[v_i(t_j) \mid S_1(t_j) = S_{1,k}, S_2(t_j) = S_{2,l}] \approx \frac{\sum_{n=1}^{N} v_i^n(t_j) \mathbf{1}_{S_{1,k},\Delta S_1}(S_1^n(t_j)) \mathbf{1}_{S_{2,l},\Delta S_2}(S_2^n(t_j))}{\sum_{n=1}^{N} \mathbf{1}_{S_{1,k},\Delta S_1}(S_1^n(t_j)) \mathbf{1}_{S_{2,l},\Delta S_2}(S_2^n(t_j))}$$

where n stands for the Monte Carlo path index and $\mathbf{1}_{S_{1,k},\Delta S_1}(\cdot)$ denotes the box function of width ΔS_1 centred around $S_{1,k}$. In our analysis, we used $N = 100{,}000$ paths for the estimation of the local covariance and a regular rectangular grid with 151×151 cells. Clearly, if performance of the effective LV model becomes critical, more refined methods with enhanced kernels should be considered.

In the tests, we consider a typical worst-of autocallable structure. In particular, we assume a one-year maximum maturity and quarterly coupons of 3%, with coupons and autocall barriers at 102% of the inception level; moreover, if the trade is not terminated early, the client is short a kick-in put with strike at 100% and a European (ie, terminal) protection kick-in barrier at 85%. It is worth noting, though, that the results of our analysis are to a large extent independent of the specific autocallable structure considered and, more importantly, exemplify well the properties of typical large autocallable portfolios.

In terms of the decorrelation effect described in the previous section, we would expect a negative pricing impact of the effective LV compared with valuation under the standard LV model. This is a consequence of the fact that worst-of autocallables are usually long the correlation between the different equity components. In our prototypical autocallable above, the client has a positive exposure to the laggard component in the basket. The spot-spot correlation affects directly its future expectation value, the so-called worst-of forward (WoF):

$$\mathrm{WoF}(T) = \int_0^\infty \int_0^\infty \min(S_1, S_2) p_T(S_1, S_2)\, \mathrm{d}S_1\, \mathrm{d}S_2 \tag{8}$$

where T is the maturity of the WoF. The lower the spot-spot correlation, the lower the value of the WoF. By virtue of the autocallable positive exposure to the laggard, the spot-spot correlation sensitivity is also expected to be positive.

In figure 1, we show the pricing results under LV, common and independent SLV drivers, and the corresponding effective LV models for different values of the vov. As expected, these results show the price add-on coming from the impact of stochastic volatility on transitional densities (the difference between the SLV and effective LV results in figure 1) is comparable in common and independent drivers of SLV; in both cases, the effect is around 40 basis points for the maximum value of the vov considered. Surprisingly, however, the price add-on coming from the impact of stochastic volatility on terminal densities (the difference between effective LV and LV results) has opposite signs in the two versions of SLV. In particular, the increase in price in the effective LV model in the common driver case cannot be understood by decorrelation alone.

This unexpected behaviour is entirely due to modifications of the joint densities and can be traced back directly to the WoF component (for terminal products like the WoF, as expected, SLV and corresponding effective LV models are equivalent up to numerical effects, see figure 2). In the remainder of this article, and without loss of generality, we concentrate on the WoF to shed light on this unexpected pricing behaviour.

■ **A Dupire-like equation for WoFs.** We will derive an equation that describes the evolution of the WoF value in terms of the local volatility and the joint terminal distribution. The same formalism applied to European call options in one dimension is used to derive the Dupire equation. Taking the partial derivative with respect to the maturity T in (8), we obtain:

$$\begin{aligned}
\partial_T\, \mathrm{WoF}(T) &= \int_0^\infty \int_0^\infty \min(S_1, S_2) \partial_T p_T(S_1, S_2)\, \mathrm{d}S_1\, \mathrm{d}S_2 \\
&= \frac{1}{2} \int_0^\infty \int_0^\infty ((S_1^2 \Sigma_{11} p_T) \partial^2_{S_1 S_1} + (S_2^2 \Sigma_{22} p_T) \partial^2_{S_2 S_2} \\
&\qquad + 2(S_1 S_2 \Sigma_{12} p_T) \partial^2_{S_1 S_2}) \\
&\qquad \times \min(S_1, S_2)\, \mathrm{d}S_1\, \mathrm{d}S_2
\end{aligned} \tag{9}$$

where we have used (3) for the partial derivative of the joint terminal density in the effective LV model and integrated by parts twice in order to arrive at (9). Noting that:

$$\begin{aligned}
\partial^2_{S_1 S_1} \min(S_1, S_2) &= -\delta(S_1 - S_2) \\
\partial^2_{S_2 S_2} \min(S_1, S_2) &= -\delta(S_1 - S_2) \\
\partial^2_{S_1 S_2} \min(S_1, S_2) &= \delta(S_1 - S_2)
\end{aligned}$$

we obtain:

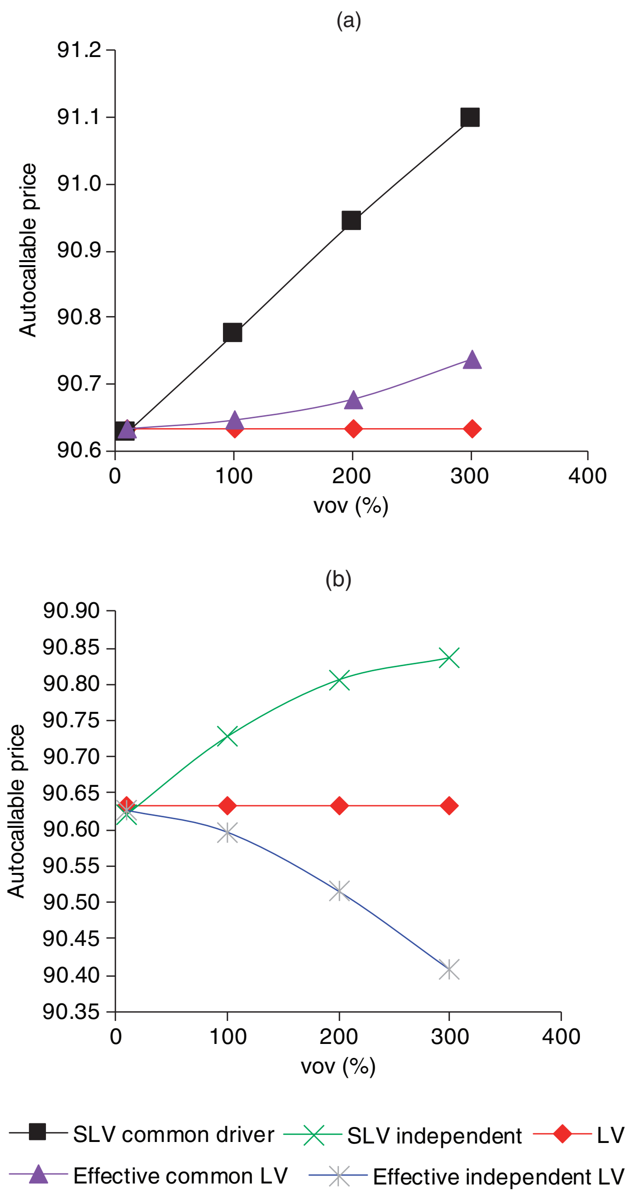
$$\begin{aligned}
\partial_T\, \mathrm{WoF}(T) &= -\frac{1}{2} \int_0^\infty S^2 (\Sigma_{11}(S, S, T) + \Sigma_{22}(S, S, T) - 2\Sigma_{12}(S, S, T)) \\
&\qquad \times p_T(S, S)\, \mathrm{d}S \\
&= -\frac{1}{2} \int_0^\infty S^2 \sigma^2_{S_1 - S_2}(S, S, T) p_T(S, S)\, \mathrm{d}S
\end{aligned} \tag{10}$$

where we have defined:

$$\sigma^2_{S_1 - S_2}(S, S, T) = \Sigma_{11}(S, S, T) + \Sigma_{22}(S, S, T) - 2\Sigma_{12}(S, S, T)$$

In (10), one recognises the known fact that a WoF can be written in terms of a spread option.

1 Price of a standard worst-of autocallable as a function of vov



(a) LV versus SLV with common variance driver and corresponding effective LV model. (b) LV versus SLV with independent variance drivers and corresponding effective LV model

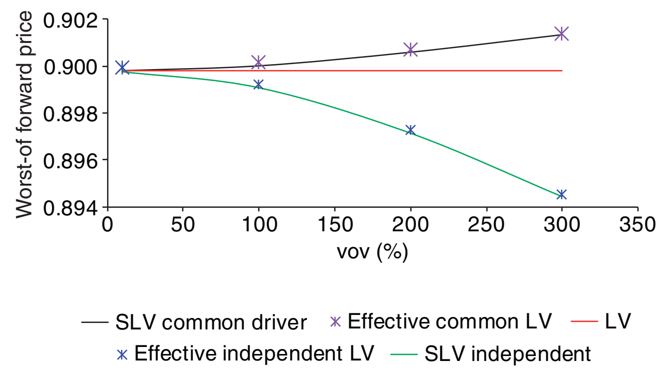
Effect of stochastic volatility on WoFs. To assess the impact of vov on the pricing of the WoF, we consider the effect of stochastic volatility as a perturbation to the standard LV model for which vov is equal to zero. More specifically, we look at the difference:

$$\Delta\,\mathrm{WoF}(T) = \mathrm{WoF}_{\mathrm{SLV}}(T) - \mathrm{WoF}_{\mathrm{LV}}(T)$$

Given (10), this difference evolves to linear order as follows:

$$\partial_T \Delta\,\mathrm{WoF}(T) \approx -\frac{1}{2}\int_0^\infty S^2(\Delta\sigma^2_{S_1-S_2}(S,S,T)p_T(S,S) + \sigma^2_{S_1-S_2}(S,S,T)\Delta p_T(S,S))\,\mathrm{d}S \tag{11}$$

2 Price of a WoF with one-year maturity as a function of vov



Due to the terminal nature of the payoff, the effective local volatility models agree with the corresponding SLV models (up to numerical error)

To understand the qualitative behaviour of the pricing impact due to stochastic volatility, we consider a short maturity approximation. For a short time-to-maturity T, changes in the density are induced by modifications to the local volatility, which acts as a perturbation source for the density. This can be understood via linear response theory (Risken 1989), as one can express $\Delta p_T(S,S)$ as a response to $\Delta\sigma$. The second term is (at least) an order in T larger than the first one. As a consequence, we may approximate (11) by using the dominating first term. We verified the dominance of this term in our numerical tests for $T = 6$ months, which showed the contribution of the first term is around three times larger (at least) in the common driver case and 10 times larger in the independent drivers case compared with the second term. Note that until this point the simplifying assumptions regarding identical market data and SLV parameterisation have not explicitly entered the theoretical discussion. However, to obtain further insight into the mechanics of the SLV pricing, we shall now make use of the fact that these assumptions imply $\Sigma_{11} = \Sigma_{22}$. Then, integrating over time from 0 to T, we obtain:

$$\begin{aligned}
\Delta\,\mathrm{WoF}(T) &\approx -\frac{1}{2}\int_0^T\int_0^\infty S^2\Delta\sigma^2_{S_1-S_2}(S,S,t)p_t(S,S)\,\mathrm{d}S\,\mathrm{d}t \\
&= -\int_0^T\int_0^\infty S^2(\Delta\Sigma_{11}(S,S,t) - \rho_S\Delta(\gamma(S,S,t)\Sigma_{11}(S,S,t))) \\
&\qquad \times p_t(S,S)\,\mathrm{d}S\,\mathrm{d}t \\
&= \int_0^T\int_0^\infty S^2(\rho_S\Sigma_{11}(S,S,t)\Delta\gamma(S,S,t) - (1-\rho_S)\Delta\Sigma_{11}(S,S,t)) \\
&\qquad \times p_t(S,S)\,\mathrm{d}S\,\mathrm{d}t
\end{aligned} \tag{12}$$

where we have used that $\Sigma_{12} = \rho_S\gamma\Sigma_{11}$. Furthermore, we note that $\Delta\Sigma_{11} = \Sigma_{11} - \sigma_1^2$ and $\Delta\gamma = \gamma - 1$. We recognise that the stochastic volatility might impact the value of the WoF in two qualitatively different ways, described by the two terms in (12): the first is the local decorrelation expressed via $\Delta\gamma$; the second is the impact of the fact that Σ_{11} depends on both S_1 and S_2, whereas its standard local volatility counterpart depends only on S_1 or S_2, respectively. We will refer to this double-dependency of the resulting Σ_{ii} as bilocality.

In general, lifting the above-made assumptions on market data and SLV parameterisation, different scaling factors will appear in front of the bilocality and decorrelation terms in (12) that will render the analysis more intricate but would not fundamentally alter the conclusions, especially when the local volatilities do not differ much from each other. Going forward, we will investigate the impact of decorrelation and bilocality on the pricing of WoF payouts under the two extreme variance correlation scenarios of the SLV model, namely SLV with a common driver and SLV with independent (idiosyncratic variance) drivers.

■ **Decorrelation.** As we have seen in (7), decorrelation can be measured via $\gamma(S_1, S_2, t)$. For the WoF, we need to investigate the decorrelation factor on the main diagonal (ie, the line $S_1 = S_2$). Let us first look at the common driver case $\rho_{vv} = 1$. In order to understand the term $\mathbb{E}[\sqrt{v_1 v_2} \mid S_1(t) = S, S_2(t) = S]$ for a small time t, we look at the drivers of the variance processes for $t' < t$. We expect the spot paths not to deviate much from the most likely path $S_1(t') = S_2(t')$. In particular, when stepping from t' to $t' + \Delta t$, we only consider equal spot increments $\Delta W_{S_1}(t') = \Delta W_{S_2}(t')$:

$$\begin{aligned}\mathbb{E}[\Delta W_{v_1}(t')\Delta W_{v_2}(t') \mid S, S] &\approx \mathbb{E}[\Delta W_{v_1}(t')\Delta W_{v_2}(t') \mid \Delta W_{S_1}(t') = \Delta W_{S_2}(t')] \\ &\approx \rho_{Sv}^2 \mathbb{E}[\Delta W_{S_1}(t')\Delta W_{S_2}(t') \mid \Delta W_{S_1}(t') = \Delta W_{S_2}(t')] \\ &\quad + (1 - \rho_{Sv}^2)\Delta t \approx \Delta t \end{aligned} \tag{13}$$

where we have used the Cholesky decomposition of the variance drivers in terms of the spot drivers and noted that, approximately, the part orthogonal to the spot drivers is not affected by the conditioning. From (13), we conclude that the final variances are mainly constructed by fully correlated increments. Hence, $v_1(t) \approx v_2(t)$, which in turn yields $\mathbb{E}[\sqrt{v_1 v_2} \mid S, S] \approx \mathbb{E}[v_1 \mid S, S]$. As a consequence, one obtains that $\gamma(S, S, t) \approx 1$ and, therefore, $\Delta\gamma(S, S, t) \approx 0$ in (12). This means we expect only a small impact from decorrelation on WoF payoffs in the common driver case.

On the other hand, for the independent drivers case $\rho_{vv} = 0$ we numerically observe a significant amount of decorrelation, as depicted in figure 3. We can infer from this graph that the decorrelation effect is fundamentally different in the two variance driver correlation cases. While only a very small decorrelation (of less than 2%) is observed for the common driver case, a significant decorrelation (of more than 15%) can be seen for the independent drivers case. For the independent drivers case, we observe that $\gamma(S, S, t)$ is a decreasing function of S with a pronounced slope (see figure 3(a)). As a consequence, we obtain $\Delta\gamma(S, S, t) < 0$ in (12).

In conclusion, for the WoF we have found there is no decorrelation for SLV with a common variance driver but a significant amount of decorrelation for SLV with independent variance drivers.

■ **Bilocality.** We now consider the second effect that will affect the pricing of WoFs: the bilocality as discussed above in (12). Let us first consider the common driver case. From the findings in the previous section, we know there is no decorrelation effect on the WoF in this case. Hence, (12) simplifies to:

$$\Delta\,\mathrm{WoF}(T) \approx -\int_0^T \int_0^\infty S^2 \Delta\Sigma_{11}(S, S, t)(1 - \rho_S)p_t(S, S)\,\mathrm{d}S\,\mathrm{d}t$$

To assess the impact of bilocality on the pricing of the WoF in the common driver case, we need to understand what happens to Σ_{11} on the main diagonal. Consider first the case:

$$S_2 \gg \mathbb{E}[S_2 \mid S_1] \approx \rho_S S_1$$

In order to achieve a large S_2, typically its variance v_2 needs to be large as well. As v_1 and v_2 have an idiosyncratic correlation of $\rho_{vv} = 1$, we may conclude that v_1 is large in these scenarios as well. Specifically, we may write:

$$\mathbb{E}[v_1 \mid S_1, S_2 \gg \rho_S S_1] > \mathbb{E}[v_1 \mid S_1]$$

The same holds true for very small values of S_2. However, in order to attain the average value $\mathbb{E}[v_1 \mid S_1]$ when integrating out S_2, we must have it that:

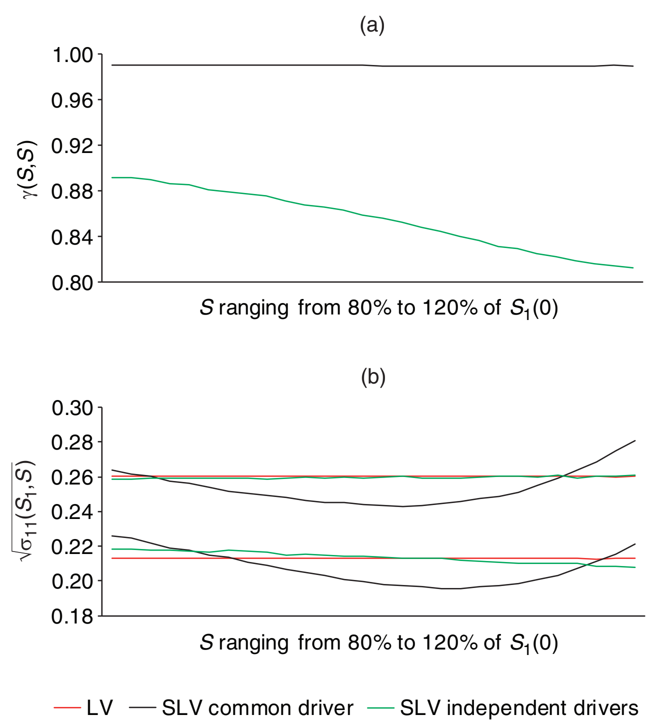
$$\mathbb{E}[v_1 \mid S_1, S_2 \approx S_1] < \mathbb{E}[v_1 \mid S_1]$$

As a consequence, we expect a negative bilocality effect on the main diagonal, ie, $\Delta\Sigma_{11}(S, S, t) < 0$. Test results shown in figure 3 (lower plot) confirm this qualitative argument. We conclude that in the common driver case we have $\Delta\,\mathrm{WoF}(T) > 0$.

Let us next focus on the independent drivers case. Here, for the full density function we can write:

$$\begin{aligned} p_t(S_1, S_2, v_1, v_2) &= p_t(v_1, v_2 \mid S_1, S_2)p_t(S_1, S_2) \\ &\approx p_t(v_1 \mid S_1, S_2)p_t(v_2 \mid S_1, S_2)p_t(S_1, S_2) \\ &\approx p_t(v_1 \mid S_1)p_t(v_2 \mid S_2)p_t(S_1, S_2) \\ &= \frac{p_t(S_1, v_1)p_t(S_2, v_2)p_t(S_1, S_2)}{p_t(S_1)p_t(S_2)} \end{aligned} \tag{14}$$

3 Decorrelation and bilocality



(a) $\gamma(S, S, 6\text{ months})$ as a function of S for the two variance driver correlation cases $\rho_{vv} = 1$ (common driver) and $\rho_{vv} = 0$ (independent drivers). (b) Bilocality in terms of the behaviour of $\sqrt{\Sigma_{11}(S_1, S, 6\text{ months})}$ as a function of S for two fixed values of S_1 (upper $S_1(0)$, lower $1.05S_1(0)$)

A. Summary of decorrelation and bilocality effects relevant for pricing of a WoF under SLV with common and independent variance driver, respectively

SLV variance driver	Common, $\rho_{vv} = 1$	Independent, $\rho_{vv} = 0$
Decorrelation	Little, $\Delta\gamma(S,S,t) \approx 0$	$\Delta\gamma(S,S,t) < 0$
Bilocality	$\Delta\Sigma_{ii}(S,S,t) < 0$	Little, $\Delta\Sigma_{ii}(S,S,t) \approx 0$

where we have used that, approximately, conditional on S_1 and S_2, v_1 and v_2 are independent. Moreover, again approximately, conditional on S_1, v_1 does not depend on S_2 (equivalently for v_2). Note that while the above relations are exact for the drivers of the processes, they are only approximately true for the processes themselves. From (14), it follows that:

$$\begin{aligned}\mathbb{E}[v_1 \mid S_1, S_2] &= \frac{\int_0^\infty \int_0^\infty v_1\, p_t(S_1, S_2, v_1, v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2}{\int_0^\infty \int_0^\infty p_t(S_1, S_2, v_1, v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2} \\ &\approx \frac{\int_0^\infty v_1\, p_t(S_1, v_1)\,\mathrm{d}v_1}{\int_0^\infty p_t(S_1, v_1)\,\mathrm{d}v_1} \\ &= \mathbb{E}[v_1 \mid S_1]\end{aligned}$$

and, therefore:

$$\Sigma_{11}(S_1, S_2, t) \approx \sigma_1^2(S_1, t)$$

Hence, in the independent variance drivers case, we expect to see no bilocality effect, ie, $\Delta\Sigma_{11}(S,S,t) \approx 0$ in (12). This is confirmed by the tests shown in figure 3, which demonstrate that there is only a little dependency of Σ_{11} on S_2 and that it is rather close to the local variance.

■ **The total pricing impact on WoFs and autocallables.** Summarising our above findings with regard to decorrelation and bilocality in table A, we expect the following behaviour for the pricing of WoFs and autocallables (focusing on the effect on terminal densities), both of which exhibit long correlation and short volatility. For the common driver SLV case, the results predict an increase in the price compared with standard LV (due to little decorrelation and a negative bilocality impact); for the independent driver SLV case, the results predict a decrease in the price compared with standard LV (due to a negative decorrelation impact and little bilocality). Both of these predictions are in line with the test results presented in figure 1 and figure 2.

Final remarks

Our results demonstrate the valuation impact of SLV versus LV for worst-of autocallables, aside from the known impact on transitional densities relevant for path-dependent payoffs, cannot be understood by the generally accepted concept of decorrelation alone. In particular, we have shown that depending on the correlation structure among the stochastic variances and the product considered, bilocality can outweigh decorrelation and yield an additional positive pricing impact as opposed to a negative one, which would be expected from decorrelation.

The formalism we have used can be extended straightforwardly to a variety of relevant payoffs with different sensitivities towards correlation and volatility. In particular, if we were to consider the case of arithmetic basket options, both sensitivities would be positive and, hence, the overall SLV pricing impact would typically be negative.

Our results are of relevance for structurers/traders trying to compare LV and SLV results when quoting a price to clients. Moreover, from a risk management perspective it is critical to separate the two effects of stochastic volatility; while decorrelation changes the level of realised correlation, bilocality changes the level of realised volatility. In particular, a consequence of our findings is that one cannot correct a volatility effect via correlation adjustments in a product-agnostic way.

Eventually, it will be important that trading has a view on the crucial and often overlooked variance-variance correlations (see De Col & Kuppinger (2018) for a discussion on the importance of the same model parameter in a foreign exchange context) so as to ensure the appropriate pricing and risk management of autocallable (and similar) structures. ■

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