Notes on Fourier Analysis

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1 Fourier Series

1.1 It all Adds Up

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi n t) + b_n \sin(2\pi n t) \right)$$

Given that

$$e^{2\pi i n t} = \cos(2\pi n t) + i \sin(2\pi n t)$$

and

$$e^{2\pi i n t} = \cos(2\pi n t) - i \sin(2\pi n t)$$

Since $\cos(-x) = \cos(x)$ while $\sin(-x) = -\sin(x)$.

The above also mean that

$$\cos(2\pi n t) = \frac{1}{2} \left(e^{2\pi i n t} + e^{-2\pi i n t} \right)$$

and

$$\sin(2\pi n t) = \frac{1}{2i} \left(e^{2\pi i n t} - e^{-2\pi i n t} \right) = -\frac{i}{2} \left(e^{2\pi i n t} - e^{-2\pi i n t} \right)$$

In the last step we used the fact that $i^{-1} = -i$. $(1 = (-i)i = i \cdot i^{-1})$.

Using these expressions to rewrite our series of sines and cosines we get

$$\begin{aligned}
\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi n t) + b_n \sin(2\pi n t) \right) &= \frac{a_0}{2} + \sum_{n=1}^{N} \left[\frac{a_n}{2} \left(e^{2\pi i n t} + e^{-2\pi i n t} \right) + \frac{b_n}{2i} \left(e^{2\pi i n t} - e^{-2\pi i n t} \right) \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} \left[\frac{a_n}{2} \left(e^{2\pi i n t} + e^{-2\pi i n t} \right) - \frac{i b_n}{2} \left(e^{2\pi i n t} - e^{-2\pi i n t} \right) \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} \left[\frac{a_n}{2} e^{2\pi i n t} + \frac{a_n}{2} e^{-2\pi i n t} - \frac{i b_n}{2} e^{2\pi i n t} + \frac{i b_n}{2} e^{-2\pi i n t} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} \left[\frac{1}{2} \left(a_n - i b_n \right) e^{2\pi i n t} + \frac{1}{2} \left(a_n + i b_n \right) e^{-2\pi i n t} \right]
\end{aligned}$$

If we look at the terms within the square braces we see that we have a complex number times and exponent and its complex conjugate,

$$c_n := \frac{1}{2} \left(a_n - i b_n \right)$$

and

$$\bar{c_n} = \frac{1}{2} \left(a_n + i b_n \right)$$

So

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi n t) + b_n \sin(2\pi n t) \right) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[c_n e^{2\pi i n t} + \bar{c_n} \overline{e^{2\pi i n t}} \right]$$

Here is where the additional requirement comes in. If we have $\bar{c_n} = c_{-n}$, then $a_{-n} = a_n$ and $b_{-n} = -b_n$. We can then reindex our series,

$$\begin{aligned}
\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(2\pi nt) + b_n \sin(2\pi nt)) &= \frac{a_0}{2} + \sum_{n=1}^{N} \left[c_n e^{2\pi int} + \bar{c_n} \overline{e^{2\pi int}} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} c_n e^{2\pi int} + \sum_{n=-1}^{-N} c_{-n} e^{-2\pi int}
\end{aligned}$$

and if we have $c_0 = \frac{1}{2}(a_0 - ib_0) = \frac{a_0}{2}$, $(b_0 = 0)$, then we can see where

$$\sum_{n=-N}^{N} c_n e^{2\pi int}$$

comes from.