Notes on Real Analysis

December 28, 2023

Contents

1 The real numbers

1.1 Background

More formally stated, a field is any set where addition and multiplication are well-defined operations that are commutative, associative, and obey the familiar distributive property $a(b + c) = ab + ac$. There must be an additive identity, and every element must have an additive inverse. Finally, there must be a multiplicative identity, and multiplicative inverses must exist for all nonzero elements of the field.

1.2 Preliminaries

1.2.1 $x_{n+1} = \frac{1}{2}x_n + 1$

The series is non-decreasing

$$x_n \leq x_{n+1}$$

And it just so happens that $x_n = 2 - \frac{1}{n^{n-1}} = \frac{2^{n+1}-1}{2}$ Write it all out to see the pattern.

$$\lim_{n\to\infty} \frac{2^{n+1} - 1}{2} = 2$$

$\frac{2^{n+1}-1}{2}$ grows as $\frac{2^{n+1}}{2}$.

1.2.2 Triangle Inequality

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \tag{1.1}$$

$$|ab| = |a||c| \tag{1.2}$$

$$|a + b| \leq |a| + |c| \tag{1.3}$$

If $a, b \geq 0$, then $|a + b| = a + b = |a| + |b|$. If $a, b < 0$, then $|a + b| = |a| + |b|$ (i.e., $| - 1 + -2| = | - 1| + | - 2| = 3|$). If $a < 0, b > 0$, then $| - 1 + 2| = |1| = 1$ and $| - 1| + |2| = 3$, so $|a + b| < |a| + |b|$. Similarly if $a > 0$ and $b < 0$.

$$|a - b| = |b - a| \tag{1.4}$$

think about it as the distance between two points.

$|a-b| = |(a-c)+(c-b)| <= |a-c|+|c-b|$ using the triangle inequality. So, $|a - b| \leq |a - c| + |c - b|$ for any $c \in \mathbb{R}$. So the above expression says something like "distance from a-to-b is equal to or less than the distance from a-to-c plus the distance from c-to-b".

Proofs for triangle and reverse triangle inequalities are in `https://en.wikipedia.org/wiki/Triangle_inequality`. Some takeaways are that:

$|b| \leq a$ can also be expressed as $-a \leq b \leq a$.
$-|x| \leq x \leq |x|$ so $-(|x| + |y|) \leq x + y \leq |x| + |y|$.
$|x| = |(x - y) + y| \leq |x - y| + |y|$ or $|x| - |y| \leq |x - y|$.
$-|x - y| \leq |x| - |y| \leq |x - y|$ also means $||x| - |y|| <= |x - y|$.
Additionally $|x| - |y| \leq |x + y| \leq |x| + |y|$.

1.2.3 Exercises

1.1.1

Prove that $\sqrt{3}$, $\sqrt{6}$, $\sqrt{4}$ are irrational.

stackexchange: prove that the square root of 3 is irrational: A supposed equation $m^2 = 3n^2$ is a **direct contradiction to the Fundamental Theorem of Arithmetic, because when the left-hand side is expressed as the product of primes, there are evenly many 3s there, while there are oddly many on the right.**

1.2.7

$$A = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$$
$$B = \{x \in \mathbb{R} : 1 \leq x \leq 4\}$$

so $A \cap B = \{x \in R : 1 <= x <= 2\}$.

$$f(A) = \{x \in \mathbb{R} : 0 \leq x \leq 4\}$$
$$f(B) = \{x \in \mathbb{R} : 1 \leq x \leq 16\}$$

so

$$f(A) \cap f(B) = \{x \in \mathbb{R} : 1 \leq x \leq 4\}$$

and $f(A \cap B) = \{x \in \mathbb{R} : 1 \leq x \leq 4\}$.

So in this case $f(A) \cap f(B) = f(A \cap B)$.

$$A \cup B = \{x \in \mathbb{R} : 0 \leq x \leq 4\}$$

so

$$f(A \cup B) = \{x \in \mathbb{R} : 0 \leq x \leq 16\}$$
$$f(A) \cup f(B) = \{x \in \mathbb{R} : 0 \leq x \leq 16\}$$

So $f(A) \cup f(B) = f(A \cup B)$ as well.

Counterer example of $f(A \cap B) = f(A) \cap f(B)$ is if $A = \{x \in \mathbb{R} : -6 \leq x \leq 3\}$ And $B = \{x \in \mathbb{R} : 3 \leq x \leq 6\}$. There A and B is $\emptyset$, so $f(A \cup B)$ is $\emptyset$. While f(A), f(B), and $f(A) \cup (B)$ are $\{x \in \mathbb{R} : 9 \leq x \leq 36\}$.

To show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, the statement $g(A \cap B) \subseteq g(A) \cap g(B)$ holds for all sets $A, B \subseteq \mathbb{R}$, we need to prove that every element in $g(A \cap B)$ is also an element of both $g(A)$ and $g(B)$. Let's proceed with the proof:

Let x be an arbitrary element in $g(A \cap B)$. This means that there exists an element $y \in A \cap B$ such that $g(y) = x$. Since $y \in A \cap B$, y is both in set A and set B. Therefore, $g(y) \in g(A)$ and $g(y) \in g(B)$. Since $g(y) = x$, we can conclude that $x \in g(A)$ and $x \in g(B)$, which implies that $x \in g(A) \cap g(B)$. Since x was an arbitrary element in $g(A \cap B)$, we have shown that every element in $g(A \cap B)$ is also an element of $g(A) \cap g(B)$.

Thus, we have proved that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

Following a similar line of thinking for $g(A \cup B) and g(A) \cup g(B)$. Let x be some element in $g(A \cup B)$, then there must be a y in either A or B such that $g(y) = x$. Therefore, $g(y) \in g(A) \cup g(B)$.

1.2.10

$y_1 = 1$

$$y_{n+1} = \frac{3}{4}y_n + 1 = \frac{3y_n + 4}{4}$$

$n = 1 : y_1 < 4$

Now we want to show that if we have $y_n < 4$, then so will $y_n + 1 < 4$ We are starting from the hypothesis that $y_n < 4$, then

$$\frac{3}{4}y_n + 1 < \frac{3}{4}4 + 1 = 4$$

Which means $y_{n+1} < 4$.

$y_1 = 1$, $y_2 = \frac{7}{4}$, $y_3 = \frac{37}{16}$.

$n = 1 : y_1 < y_2$

Induction hypothesis $y_n < y_n + 1$

$$\frac{3}{4}y_n + 1 < \frac{3}{4}y_{n+1} + 1 \to y_{n+1} < y_{n+2}$$

1.2.11

If a set A contains n elements, prove that the number of different subsets of A is equal to 2n. (Keep in mind that the empty set is considered to be a subset of every set.)

Every element in A can be in or not, thus m multiplications of 2.

1.2.12

Trivial case is: $(A_1)^c = A_1^c$.

Base case: $(A_1 \cup A2_)^c = A_1^c \cap A_2^c$ by de morgan's theorem.

$(A_1 \cup A_2 \cup A_3)^c = (B_1 \cup A_3)^c = B_1^c \cap A_3^c = A_1^c \cap A_2^c \cap A3^c$. Distributive property plus $B := A_1 \cup A2$. without loss of generality $B = A_1 \cup A_2 \cup \ldots A_k$.

1.3 The axiom of completeness

1.3.1

Let $Z_5 = \{0, 1, 2, 3, 4\}$. Additiona and multiplication modulo 5 would then be as,

$$7 + 9 = 16 \bmod 5 = 1$$

And

$$7 \times 9 = 63 \bmod 5 = 3$$

Additive inverse: in order for $z + y = 0$ in Z_5, then the modulo addition of z and y must be a multiple of 5, in this case. One way to define the additive inverse of z is to have it be $m - z$, where m is the modulo we are using. That way $z + y = z + (m - z) \bmod m = m \ \bmod m = 0$.

Multiplicative inverse: if $z \neq 0$ in Z_5, $\exists x$ such that $zx = 1$. So whatever product we end up getting must be $nm + 1$ for some integer n (always have a remainder of 1). Yet another way of thinking about this is that $zx \equiv 1 \bmod m$.

There is an interesting pattern that shows up here... If $z = 1$, then $zx = x \bmod 5 = 1$ has multiple solutions but two of them are $x = 1$ or $x = 6$ ($x = z$ or $x = z + m$). Similarly, if $z = 2$, then 2 possible solutions would be $x = 3$ or $x = 8$ ($x = z + 1$ or $x = z + m + 1$). If $z = 3$, then 2 possible solutions are $x = 2$ or $x = 7$ ($x = z - 1$ or $x = z + m - 1$). If then $z = 4$, then 2 possible solutions are $x = 4$ or $x = 9$ ($x = z$ or $x = z + m$ - again!!!). Since we have found the multiplicative inverse for each non-zero element in Z_5, we can conclude that for any $z \neq 0$ in Z_5, there exists an element x such that $zx = 1$.

Multiplicative inverses exist only when z and m are relatively prime. For example if $z = 5$, and we are in Z_{10} then there is no number such that $zx \equiv 1 \bmod 10$.

Keeping in mind that a **congruence class** is an equivalence relation on an algebraic structure (e.g., a ring) that is compatible with the structure in the sense that algebraic operations done with equivalent elements will yield equivalent elements. More humanly put, a congruence class is the set of all integers that have the same remainder as a when divided by n. Now in a ring Z_n, the units (i.e. the elements which have a multiplicative inverse) are the congruence classes of the elements m which are coprime to z, because for such an element, we have a Bezout's relation, $um + vz = 1$ or $um = 1 - vz$, which means the class of u is the inverse of that of m. We obtained the previous argument from stackexchange: prove that there exists a multiplicative inverse.

1.3.2

A real number A real number l is the greatest lower bound for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

1. l is a lower bound of A,

2. if m is any lower bound for A, then $l \geq m$.

Lemma: assume $l \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then $l = \inf A$ if and only if, $\forall \epsilon > 0$, $\exists a \in A$ satisfying $l + \epsilon > a$.

Given that l is a lower bound, l is the greatest lower bound iff any number greater than l is not a lower bound.

In the forward direction we want to prove that: if l is the greatest upper bound, then $l + \epsilon > a$. In the forward direction we want to prove that: if l is a lower bound satisfying $l + \epsilon > a$, then l is also the greatest upper bound.

For the former case, because $l + \epsilon > l$, then by definition $l + \epsilon$ is not a lower bound. Thus there must $\exists a \in A$ for which $l + \epsilon > a$. Note that we used the defintion to prove a point, we are not questioning the definition (whether l is the greatest upper bound).

For the latter direction, assume l is a lower bound such that $\forall \epsilon > 0$, $l + \epsilon$ is no longer a lower bound for A. If this is so, a number slightly greater than l (for any degree of slightly) is no longer a lower bound, then by Definition of the greatest lower bound, l can be the greatest lower bound.

1.3.3

If A is bounded below, and we define $B = \{b \in \mathbb{R} : \text{b is a lower bound of A}\}$. Show that $\sup B = \inf A$.

By the axiom of completeness we can start by knowing that any non-empty set of real numbers that is bounded above has a least upper bound. So as long as A is not an empty set, then B will not be empty either. In the case of the set B, any upper bound wil be equal or greater than any $b \in B$. And the least upper bound will be equal to or smaller than any other bound we can find (smaller than or equal to any $a \in A$).

At the same time, since B contains lower bounds of A, we already know that there is a $supB$ that exist and is smaller than any $a \in A$ (other upper bounds of B) while also being greater than or equal to any other bounds of A (members of B). That is, $\sup B$ is a lower bound of A and is greather than or equal to any other bounds of A. Hence, by defintion of the greatest lower bound $\sup B = \inf A$.

This exercise points to the interesting case when if a is an upper bound for A, and if $a \in A$, then it must be that $a = \sup A$. This statement is actually exercise **1.3.7** in the book.

1.3.4

Assume that A and B are nonempty, bounded above, and satisfy $B \subseteq A$. Show $\sup B \leq \sup A$.

If B and A are equal then their least upper bound will be equal. On the other hand, if A has elements that B doesn't, then those elements not in B can be smaller or greater than those in B. If the extra elements are smaller than

$b \in B$, then the least upper bounds of the two sets will not change. But if A contains elements that are greater than B, then $\sup A > \sup B$.

1.3.8

If $\sup A < \sup B$, then show that $\exists b \in B$ that is an upper bound for A.

Since B has a least upper bound, then there must be $b \in B$ such that $b \leq \sup B$. Similarly in A, there must exist $a \in A$ such that $a \leq \sup A$. Since we know that $\sup A < \sup B$, then we have $a \leq \sup A < \sup B$ ($\sup B$ is an upper bound for A).

If b is an upper bound of A, then that would mean that $b \geq a$ for any $a \in A$. If b was not an upper bound, then $b < a$ for all b. But if this was the case, then $\sup B$, which is an upper bound of B ($\sup B \geq b$) would be smaller than some a, leading us to a contradiction as $\sup A$ would then be greater than $\sup B$.

We can also see this as a case mention in problem 1.3.4 above.

1.4 Consequences of completeness

1.4.1

Density of Q in R: For every $a, b \in \mathbb{R}$ where $a < b$ and $a < 0$, $\exists r \in \mathbb{Q}$ satisfying $a < r < b$.

Since we want to prove the case of $a < 0$, we want to see if there is a rational number r such that $a < r < 0$. There rest of the proof in theorem 1.4.3 then applies as is.

1.4.1 Existence of square roots

$$T = \{t \in \mathbb{R} : t^2 < 2\}$$

Then we work out the expression

$$\left(\alpha + \frac{1}{n}\right) < \alpha^2 + \frac{2\alpha + 1}{n}$$

That bit $(2\alpha + 1)/n$ is what we need to fit between α^2 and 2 while keeping $\alpha^2 < 2$. Since we want to fill that space, we come up with

$$\frac{2\alpha + 1}{n} < 2 - \alpha^2$$

Which then simplifies to what Abbott uses.

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2 - \alpha^2) = 2$$

contradicts the fact that α is an upper bound because $(\alpha + \frac{1}{n})^2$ is also a member of T, and is larger than α alone.

For the case of $\alpha^2 > 2$, where we want to prove that this contradicts the fact that α is the least upper bound, then it must be the case in which α is an upper bound for $\{t \in \mathbb{R} : t^2 < 2\}$ but it must not be the smallest upper bound. That is, $\alpha > b$, where b is some other bound of T ($b = 2$ for example).

To show that the above would lead us to the said contradiction we can now take

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}$$

If we chose an n_0 such that

$$\alpha^2 - \frac{2\alpha}{n_0} > 2$$

That is, we look for an n_0 that can give us a number smaller than α but that it is an upper bound. The above implies that $-\frac{2\alpha}{n_0} > 2 - \alpha^2$, and consecuently

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 + (2 - \alpha^2) = 2$$

So we see that $\alpha - \frac{1}{n_0}$ is an upper bound and it is smaller than α, so $\alpha^2 > 2$ also be the case.

1.4.2 Exercises

1.4.1 v1 (not present in v2)

For every two real numbers a and b with $a < b$, there exists a rational number satisfying $a < r < b$.

As requirements we have that $a < b$ and this time we want to look at the case were $a < 0$, so b can be either negative, zero, or positive, it just has to be greater than a. However the rest of the proof can proceed as normal if we don't require $m \in \mathbb{N}$ but instead generalize $m \in \mathbb{Z}$. See How does this proof of density of Q in R require $a \geq 0$?.

The original proof uses the Archimedean Property and the Archimedean Principle for positive numbers to find a positive integer n such that $\frac{1}{n} < b - a$. Then, it uses the Density Property of the set of natural numbers to find a natural number m such that $m > n$ and $an < m$.

1.4.1

If we have $a = \frac{m}{n}$ and $b = \frac{p}{q}$, then

$$a + b = \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}$$

Which is a rational number since $m, n, p, q \in \mathbb{Z}$.

Similarly,

$$ab = \frac{m}{n}\frac{p}{q} = \frac{mp}{nq}$$

However, if we multiply (or add), let's say a, by $i \in \mathbb{I}$, we want to show that this would result in an irrational number. To prove this we will use a proof by contradition.

In the case of a sum, let's begin by assuming that if we add a rational number and an irrational numbert that the result is a rational number too. Then,

$$\frac{m}{n} + i = \frac{x}{y}$$

This would mean that we could isolate i

$$i = \frac{x}{y} - \frac{m}{n}$$

Which as we showed above, would mean that i is a rational number.

In the case of a multiplication,

$$i\frac{m}{n} = \frac{x}{y}$$

i could be isolated,

$$i = \frac{x}{y}\frac{n}{m}$$

Which again, would lead to a contradiction since the product of two rational numbers is also a rational number and the above says that we magically converted i into a rational number.

One would then think that the irratinoal numbers are also closed under addition or multiplication, but this is not the case. Consider $1 - \sqrt{2} \in \mathbb{I}$ (as we just saw) and $\sqrt{2}$.

$$(1 - \sqrt{2}) + \sqrt{2} = 1 \in \mathbb{Z}$$

Similarly,

$$\sqrt{2}\sqrt{2} = 2$$

1.4.3

Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

The proof we used for the nested property theorem, theorem 1.4.1. Following a similar schema, we can see that the set of right-hand endpoints $\{1/n : n \in \mathbb{N}\}$ which we have already seen that its greatest lower bound is 0. We have also saw in exercise 1.3.3 that the supremum of the left-hand endpoints must equal the infinimum of the right-hand endpoints, which is zero. So the approach we followed in theorem 1.4.1 doesn't apply here since 0 is not even part of any of the intervals I_n.

1.4.4 v1 (not present in v2)

Use the Archimedean property of $\mathbb{R}$ prove that $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$.

By the Archimedean property we know that for any $y > 0 \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $y > \frac{1}{n}$.

We also know that for a number to be an infinimum then it has to be a lower bound and it has to be greater than any otther lower bounds.

In the case of the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, we can see that 0 is indeed a lower bound. But let's say that we have another lower bound b that is greater than 0 (assume the greatest lower bound is b).

Since $0 \notin A$ and b is a lower bound, then $b > 0$ but $b \leq a$, for any $a \in A$. However, the Archimedean property tells us that for any b we could have, there will be a $\frac{1}{n}$ that is smaller than it. Thus disproving the fact that a $b > 0$ could be a lower bound.

1.4.5 v1 (not present in v2)

Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$. To prove this, we can use the results from the previous exercise by applying the theorem where we talked about the density of the rational numbers in the real numbers to $a - \sqrt{2}$ and $b - \sqrt{2}$ (which are real and irrational numbers.)

This case is easy to see since we know that between any two real numbers there exists a rational number

$$a < \frac{m}{n} < b$$

And we also know that we can convert a rational number into an irrational by adding or multiplying by an irrational number. So, if we shift the case we used to prove the density of the rational numbers by subtracting an irrational number

$$a - \sqrt{2} < \frac{m}{n} - \sqrt{2} < b - \sqrt{2}$$

We can arrive to the result we wanted to prove.

1.5 Cardinality

1.5.1 $N \sim R$

We known that for some real number $x_{n_0} \notin I$, so that

$$x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n$$

Since we are assuming that $\mathbb{R} = \{x_1, x_2, ...\}$ contains all the real numbers, then n_0 can be any of the real numbers. So for any given n_0, I_{n_0} will not contain it, nor any I_{n_0+1}, or any of its subsets. Going in the other direction, x_{n_0} will be in I_{n_0-1}, but I_{n_0-1} will not contain x_{n_0-1}. Thus, there is no one number that is contained by all sets I_n, Hence

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

Which contradicts the nested interval property.

When proving the nested interval property we showed that at the very least the supremum of each interval I_n would be in every I_n, which lead us to the discovery that their intersection would be a non-empty set. However, if we don't have the supremum of each set present for each I_n, then by our experience with the axiom of completeness, we would think that there are gaps in our list of numbers.

The "logical" issue in the above argument is that the real numbers are "enumerable", which doesn't show up in the original proof of the nested interval theorem because hte axiom of completeness creates a sort of "continuity" among the numbers (the filling in between the gaps).

1.5.1

Theorem 1.5.7: if $A \subseteq B$ and B is countable, then either A is countable, finite, or empty.

Recall that a set B is countable if $\mathbb{N} \sim B$. Which means that the sets $\mathbb{N}$ and B have the same cardinality as there exist some $f : N \to B$ that is 1-to-1 and onto.

Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, set $g(1) = f(n_1)$.

We know that f is 1-to-1, so every natural number gets mappted to a different member of B, that is, $n_1 \neq n_2$, then $f(n_1) \neq f(n_2)$. We also know that for every B, and clearly for every A (since $A \subseteq B$), there exists an $n \in \mathbb{N}$ such that $a = f(n)$ for every $a \in A$. Thus, we only need to prove that g is a 1-to-1 function.

Now back to our definition of g, if we now look at $A - \{f(n_1)\}$, and we define $n_2 = \min\{n \in \mathbb{N} : f(n) \in A - \{f(n_1)\}\}$, or more generally, $n_n = \min\{n \in \mathbb{N} : f(n) \in A, n \notin \{n_1, n_2, ..., n_{k-1}\}\}$, then we have $g(k) = f(n_k)$, which we know is different from all other values of g since f is 1-to-1.

1.5.3

Use the following outline to supply proofs for the statements in Theorem 1.5.8:

(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.5.3 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?) Now exaplain how the more general statement in (i) follows.

(b) Explain why induction cannot be used to prove part (ii) of Theorem 1.5.8 from part (i).

(c) Show how arranging $\mathbb{N}$ into a two-dimensional array leads to proof of Theorem 1.5.8

If $A_1, A_2, \ldots, A_m$ are each coutable sets, then the union $A_1 \cup A_2 \cup \ldots \cup A_m$ is countable.
Replace A_2 with $B_2 = A_2 - A_1 = \{x \in A_2 : x \notin A_1\}$. $A_1 \cup B_2 = A_1 \cup A_2$ while keeping the two sets disjoint.

If we followed the Example 1.5.3, then we could see how "laying out" the members of $A_1 \cup A_2 = A_1 \cup B_2$ could be mapped to by the natural numbers, $\{a_1, a_2, \ldots, b_1, b_2, \ldots\}$. For the induction step, we can follow logic similar to when we prove the generalization of the de morgan laws in 1.2.12.

If A_n is a countable set for each $b \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.
The same induction logic we used above could get us into trouble here because induction shows something for all $n \in \mathbb{N}$, not for infinity - and we have already seen that there are various types of infinities. If we followed the same logic we could end up with a problem similar as to when we claimed that the reals are enumerable.

If we were to arrange natural numbers in a two-dimensional array we end up with disjoint sets B such that $\cup_{n=1}^{\infty} B_n = \mathbb{N}$. (Note how this can be a construction similar to the first instance in this problem where $B_1 = A_1$, $B_2 = A_2 - A_1 = A_2 - B_1$, $B_3 = A_3 - B_2$, and so on.)

We also now, can see, that every B has an f_n that is 1-to-1 and onto (from the argument in part (a)).

So in order to prove that there exists a mapping that maps the natural numbers into our two-dimensional array of disjoint sets, it suffices to see that our way of ordering the sets is itself a mapping such that $f : \mathbb{N} \to \cup_{n=1}^{\infty} B_n$. The mapping could be expresed as $f(x_{m,n} \in \text{2D array}) = (m, n) \in \cup B_n$.

And since we organized the two-dimensional array in the way we have, then the we have $\mathbb{N} \subseteq \mathbb{N} \times \mathbb{N}$ and the mapping is bijective ($f(x_{m,n} \in \mathbb{N})$ is a specific mapping that creates order pairs that such that all order pairs are 1-to-1 and onto).

See Simplification in Proof of Countable Union of Countable Sets is Countable for a similar presentation.

1.5.4

Is $(a, b) \sim \mathbb{R}$, for any open interval (a, b)?

Let's take example 1.5.4 as inpiration. This example says that $f = x/(x^2-1)$ takes the interval $(-1, 1)$ onto $\mathbb{R}$, showing that $(-1, 1) \sim \mathbb{R}$.

The books mentions some calculus so let's see what we can see. First we have,

$$\lim_{x \to -1^+} \frac{x}{x^2 - 1} \to \frac{-1}{\text{some very small positive number}} \to -\infty$$

Similarly,

$$\lim_{x \to 1^-} \frac{x}{x^2 - 1} \to \frac{1}{\text{some very small positive number}} \to \infty$$

Its derivative is

$$f' = \frac{1}{(x^2-1)^2}((x2-1) - x(2x)) = \frac{-x^2-1}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

For the derivative, no matter whether x is negative or positive, its value will always be zero. Which is a no-graphing way of triple checking that the original function won't "fluctuate" and we can indeed map all of the domain of f to unique values in $\mathbb{R}$.

Now in order to see whether f is onto, we could see if we can build an inverse of f and see if it has any "holes". But that's rather complicated in our case, so let's try something else. We could use the intermediate value theorem but we don't yet know what "continuous" or any of the other requirements mean. However, we have seen that within $(-1, 1)$ the function goes from $-\infty$ to ∞, so we can imagine that any value within the image will be mapped to by somewhere in the domain.

To shift the function we have been using into the interval (a, b), let's define the midpoint between b and a as $m = (b-a)/2$ and let's define the transformed function f as

$$g(x) = f\left(\frac{(x-m)}{m}\right)$$

With the above transformation we are "stretching" the function by dividing x my the mid-way length between b and x and we are also shifting it by the same distance. Since the changes are all constant factors, all other properties we have been relying on remain the same.

1.5.5

Is $A \sim A$ for every set A?

If so, it'd mean that there is a function $f : A \to A$ that is 1-to-1 and onto. Which means that $f(a_1) = f(a_2)$ iff $a_1 = a_2$. However, $f(a_1) = a_1$ in our case. Given this, it is easy to see that such a mapping will also be onto.

If $A \sim B$, does that mean that $B \sim A$?

Well, for starters, we now know that there is an $f : A \to B$ that is 1-to-1 and onto. So $b_1 = b_2 = f(a_1) = f(a_2)$ iff $a_1 = a_2$. Also, for every $b_i = f(a_i)$ there is a unique a_i. Putting these two properties together we know that for every element in B there is a uniqie element that can be mapped to it from A. If we take this in the opposite direction, $B \to A$, we only need to prove that every every element we are mapping to is unique, which is given to us by the mapping being 1-to-1.

If $A \sim B$ and $B \sim C$, does that mean that $A \sim C$?

There exists an $f : A \to B$ that is 1-to-1 and onto. There also exists a $g : B \to C$ that is 1-to-1 and onto.

So if we start with $f(a_i) = b_i$, then $g \circ f(a_i) = g(b_i) = c_i$. Since $g(b_1) = g(b_2)$ iff $b_1 = b_2$, and $b_1 = f(a_1)$ and $b_2 = f(b_2)$, and $f(a_1) = f(a_2)$ iff $a_1 = a_2$, then by extension, $g(b_1) = g(b_2)$ iff $a_1 = a_2$.

Similarly, since for every c there exist a unique b, and for every b, there is a unique a. Then we can map every c to a unique a.

1.5.9

A real number $x \in \mathbb{R}$ is algebraic if there exists integers $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0 \tag{1.5}$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called transcendental numbers.

$\sqrt{2}$ is an algebraic number since $2x^2 - x^4 = 0$ can be a way to express $x^2 = 2$, $x = \sqrt{2}$.

Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

For example, $A_2 = \{x \in \mathbb{R} : x \text{ is a root of } a_0 + a_1 x + a_2 x^2 = 0\}$. Here problem 1.5.3 (theorem 1.5.8) will come in handy. Essentially, if every polynomial of degree n has a finite set of roots, then a piecewise application of $f : \mathbb{N} \to A_n$ is doable. If we also organize all A_n to have only non-zero coefficients, then we have a collection of countable and disjointsets. By the application of theorem 1.5.8, the union of all A_n is also countable.

Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

As we saw in theorem 1.5.6 $\mathbb{R}$ is not countable. And we just argued that the set of algebraic numbers is countable. Roughly speaking, this could have us belief that the set of transcendental numbers is uncountable.

1.5.11

Schröder-Bernstein theorem Assume there exists a 1-to-1 function $f : X \to Y$ and another 1-to-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-to-1, onto function $h : X \to Y$ and hence $X \sim Y$.

The strategy is to partition X and Y into components

$$X = A \cup A'$$

and

$$Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

If we assume that X and Y don't have the same cardinality, then it would be possible to map elements of Y that are not reachable from X, and vice versa. So a way that the partitioning shceme could help us show that there is a 1-1 and onto mapping h would be if it helped us see that all of the partitions of X and Y are reachable from some element in either of the partitions on the other side - we can get to any element in B or B' from A or A'.

The 1-1 and onto function $h(x)$ will probably look something like:

$$h(x) = \begin{cases} f(x) & x \in A \\ g^{-1}(x) & x \in A' \end{cases} \tag{1.6}$$

Some good links to dive into similar arguments:

1. Schröder-Bernstein Theorem proof help
2. Intuition behind Cantor-Bernstein-Schröder
3. Does this proof of the Schröder–Bernstein theorem work?
4. How to prove this version of the Cantor-Schroder-Bernstein theorem?
5. Understanding a proof of Schröder-Bernstein theorem

(b) Set $A_1 = X - g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y.

$A_1 = X - g(Y)$ can be seen as "all of X, except the parts that can be reached from Y with g". We then have $A_1 \subseteq X$. If $A_1 = \emptyset$, then all of $g(Y)$ can be mapped into some X, so g is 1-1 and onto, and we have found our answer. But let's carry out assuming that $A_1 \neq \emptyset$.

$A_1 \subseteq X$. So $f(A_1) \subseteq Y$. And consequently, $A_2 = g(f(A_1)) \subseteq X$ as well. Since A_1 contained all $x \in \mathbb{R}$ such that $x \notin g(Y)$, and A_2 was obtained by applying g on $f(A_1)$, then we have $A_1 \cap A_2 = \emptyset$.

We now need to look at how A_n and A_{n+1} relate to one another. $A_n = g(f(A_{n-1}))$, and $A_{n+1} = g(f(A_n)) = g(f(g(f(A_{n-1}))))$. If we assume that $A_n \cap A_{n+1} \neq \emptyset$, then it means that there is some $x \in X$ such that $g(f(A_n)) = x = g(f(A_{n+1}))$. And since both f and g are 1-1 mappings, this would mean that $A_n = A_{n+1}$. Thus, for $A_n \neq A_{n+1}$, we need that $A_n \cap A_{n+1} = \emptyset$, for all $n \in \mathbb{N}$.

(c) Let $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} f(A_n)$. Show that $f : A \to B$.

As we just saw, we have that all A_n are pairwise disjoint subsets of X, and by similar logic we can see that $\{f(A_n) : n \in \mathbb{N}\}$ are also pairwise disjoints subsets of Y. If we revisit our partitioning scheme, we could define

$$A = \cup_{n=1}^{\infty} A_n$$

and

$$B = \cup_{n=1}^{\infty} f(A_n)$$

If we were now to apply f on our partitions we would have something like the following,

$$f(X) = f\left(A \cup A'\right)$$

But since we know that $A \cap A' = \emptyset$, then we have

$$f(X) = f\left(A \cup A'\right) = f(A) \cup f(A')$$

If we look only at what happens with A we then have,

$$f(A) = f\left(\cup_{n=1}^{\infty} A_n\right) = \cup_{n=1}^{\infty} f(A_n) = f(B)$$

The second step because we saw in the previous step that the series of A_n is pairwise disjoint, the last step because that was our definition of B.

(d) Let $A' = X - A$ and $B' = Y - B$. Show $g : B' \to A'$.

If we follow a similar line of argument as we did above, we could make an argument that $g : B \to A$. So now, all we have to figure out is how A' and B' behave under our mappings.

We know that $f : X \to Y = A \cup A' \to B \cup B'$ and $g : Y \to X = B \cup B' \to A \cup A'$. Since the sets A and A' are disjoint, as well as B and B', combining that with our previous answers, we can see that g also maps $B' \to A'$.

In part b we covered A by looking at the infinite number of disjoint subsets that could make it up and we saw how they all get mapped to B, and viceversa for $B \to A$. Now that we have defined $A' = X - A$ and $B' = X - B$, we have covered all potential cases, so now we can make a better argument for the existence of a 1-1 and onto mapping that shows $X \sim Y$.

1.6 Cantor's Theorem

When we pick a, $f(a) = \{\{\emptyset\}, \{b\}, \{c\}, \{b, c\}\}$. If we then look at $f(a)$, $f(b)$, and $f(c)$ (look at the mappings for all elements of A), then we see that the set $B = \{a, b, c\}$ is never a possibility for any of them - B is not in the range of the function used. (A counter example to f being onto, already.)

Since we are looking for a mapping that is onto, we would assume that $B = f(a')$ for some $a' \in A$.

If we say that $a' \in B$, then $f(a') = B$ is not possible with the current mapping we are using, $f(a)$ does not contain a.

If we look at the case were $a' \notin B$, then $f(a')$ would contain a', which is again inconsistent with our mapping.

1.6.1 Exercises

1.6.4

This result is used in Chapter 3 when Abbott makes an argument for Cantor sets being uncountable.

2 Rearrangements of infinite series

2.1 Limit of a sequence

2.1.1 Exercises

2.2.2

$$\lim \frac{2n+1}{5n+4} = \frac{2}{5}$$

We want to show that,

$$\left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| < \epsilon$$

For any $\epsilon > 0 \in \mathbb{R}$ as long as $n \geq N \in \mathbb{N}$. To find N,

$$\left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| = \left| \frac{5(2n+1) - 2(5n+4)}{5(5n+4)} \right| = \left| \frac{-3}{5(5n+4)} \right| = \frac{3}{25n+20} < \epsilon$$

We can do the last simplification because N is some positive number. Simplifying the above we get,

$$3 < \epsilon(25N + 20)$$

or

$$\frac{3}{25\epsilon} - \frac{20}{25} = \frac{3}{25\epsilon} - \frac{4}{5} < N$$

Note that we could obtain a simpler expression if we note that

$$\frac{3}{5(5n+4)} < \frac{1}{n} < \epsilon$$

So $N > 1/\epsilon$, which follows the same pattern as our previous answer. The first answer gives us a more accurate convergence rate.

$$\lim \frac{2n^2}{n^3+3} = 0$$

Following similar logic, we are looking at

$$\left| \frac{2n^2}{n^3+3} \right| < \epsilon$$

Plain algebra doesn't really help us isolate n, so again look for patterns.

$$\frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon$$

Now for,

$$\lim \frac{\sin(n^2)}{n^{1/3}} = 0$$

sin will never be greater than one, so we can proceed as follows.

$$\frac{\sin(n^2)}{n^{1/3}} < \frac{1}{n^{1/3}} < \epsilon$$

So $N > 1/\epsilon^3$.

2.2.6

Uniqueness of limites: if the limit of a sequence that converges were not unique, then we should be able to find any other number that meets our requirements - trying to prove that there are multiple limits for a generic sequence is hard, same as trying to then generalize this result to any sequence. Intead our approach will be as follows.

If the sequence a_n converges to a, then let's assume that there it also converges to b. If we see that $a = b$, then we have our proof, else, we have a great publication. So we know that

$$|a_n - a| < \epsilon$$

and

$$|a_n - b| < \epsilon$$

For some arbitrary epsilon. Given the definition of converge, the above requirements must hold for some $N \in \mathbb{N}$, but that N does not need to be the same for the two expressions above. So let's follow along using some $N = \max(N_1, N_2)$, where N_1 and N_2 are the numbers that would make the above expressions hold for any ϵ.

We then have,

$$|a_N - a| < \epsilon$$

and

$$|a_N - b| < \epsilon$$

Now, we are assuming that $a \neq b$, meaning that $|a - b| > 0$. If we use the triangle inequality,

$$|a - b| = |a - a_N + a_N - b| = |(a - a_N) + (a_N - b)|$$

$$\leq |a - a_N| + |a_N - b|$$

Since $|x - y| = |y - x|$, we then have

$$|a - b| \leq |a - a_N| + |a_N - b| = |a_N - a| + |a_N - b|$$

$$< \epsilon + \epsilon = \epsilon'$$

Interestingly enough, we already saw a theorem of this sort back in theorem 1.2.6, which stated that two real numbers a and b are equal iff $\forall \epsilon > 0$ it follows that $|a-b| < \epsilon$. This is exactly what we are trying to get at. But for the sake of argument, we could also see that we reach a contradiction if we continue. Let's assume $\epsilon = 1/10$, then,

$$|a - b| < 2\epsilon = \frac{2}{10}|a - b|$$

And a number cannot be less than (a fraction of) itself.

2.2 The algebraic and order limit theorems

why does $|b_n - b| < |b|/2$ implies $|b_n| > |b|/2$?

We have been seeing that for a lot of convergence proofs we end up using the triangle inequality. And this is a good tool to try out here! Let's try this out firsrt,

$$|b_n| = |b_n - b + b| \leq |b_n - b| + |b| < \frac{|b|}{2} + |b| = \frac{3|b|}{2}$$

So we get a sense that we have to multiply $|b|$ by a number greater than 1 in order to go above $|b_n|$. But its a weak argument. Let's try it the other way now!

$$|b| = |b - b_n + b_n| \leq |b - b_n| + |b_n| = |b_n - b| + |b_n| < \frac{|b|}{2} + |b_n|$$

Which can be simplified to

$$\frac{|b|}{2} < |b_n|$$

Which is the bit we did need.

In the **order limit theorem** proof we also see the statement: $|a_N - a| < |a|$ implies that $a_N < 0$. One way to see this is by noting that

$$|a_N - a| < |a| \rightarrow -|a| < a_N - a < |a|$$

If $a < 0$, then $-|a| = a < 0$ and $|a| = -a > 0$, so

$$-|a| < a_N - a < |a| \rightarrow a < a_N - a < -a \rightarrow 2a < a_N < 0$$

The above tells us that in this case we would land with $a_N < 0$.

2.2.1 Exercises

2.3.1

If $x_n \to 0$, then we know that $|x_n| < \epsilon$ when $n \geq N$. Now let's look at $|\sqrt{x_n}| = \sqrt{x_n}$.

A useful thing to note here is that if $x_n \geq 0$, then the order limit theorem says that $x \geq 0$.

Coming back to $|\sqrt{x}_n| = \sqrt{x_n}$ we can also see that $|x_n| = x_n < \epsilon$ in this case, so $= \sqrt{x_n} = \sqrt{|x_n|}$. Note that we can now make the value under the square root as small as we want

$$|\sqrt{x}_n| < \sqrt{|x_n|} < \epsilon$$

If $x_n \to x$, then

$$|\sqrt{x}_n - \sqrt{x}| = |\sqrt{x}_n - \sqrt{x}| \left| \frac{\sqrt{x}_n + \sqrt{x}}{\sqrt{x}_n + \sqrt{x}} \right|$$

$$= \frac{|x_n - x|}{\sqrt{x}_n + \sqrt{x}}$$

Since $\sqrt{x}_n + \sqrt{x} \geq \sqrt{x}$, then we could further simplify the above expression to

$$\frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}$$

Because a smaller denominator, results in a larger overall quantity. We now know that the numerator matches the expression that we know convrges, so we can now chose a $\epsilon' = \epsilon\sqrt{x}$, so that

$$\frac{|x_n - x|}{\sqrt{x}} < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon'$$

2.3.2

We know that $x_n \to 2$, so if we look at $\frac{2x_n - 1}{3}$, we can apply the same operations to $|x_n \to 2|$,

$$\left| \frac{2x_n - 1}{3} - \frac{2(2) - 1}{3} \right| = \left| \frac{2x_n - 1}{3} - \frac{3}{3} \right|$$

$$= \frac{1}{3}|(2x_n - 1) - 3| = \frac{1}{3}|2x_n - 4| = \frac{2}{3}|x_n - 2| < \frac{2}{3}\epsilon$$

Let's now look at $1/x_n$.

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{2 - x_n}{2x_n}\right| = \frac{|x_n - 2|}{|2x_n|} = \frac{|x_n - 2|}{2|x_n|}$$

If we look at the denominator, we'll see a $|x_n|$ which we know is bounded, $|x_n| < M$, because x_n is a convergent series. However, an upper bound in the denominator, gives us a lower bound on the overall expression, which is not what we want. We instead need to look for an inequality of the form $x_n \geq \delta > 0$.

Let's look at,

$$|2| = |2 - x_n + n_x| \leq |x_n - 2| + |x_n| < \epsilon + |x_n|$$

Which means $|x_n| > |2| - \epsilon$. Similarly,

$$|x_n| = |x_n - 2 + 2| \leq |x_n - 2| + |2| < \epsilon + |2|$$

Which means that $|x_n| < \epsilon + |2|$. Putting these two expressions together we get $|2| - \epsilon < |x_n| < |2| + \epsilon$. The left side of it gives us a lower bound which can be greater than 0 if we pick a good value for ϵ.

Back to our original problem,

$$\frac{|x_n - 2|}{2|x_n|} < \frac{|x_n - 2|}{2(|2| - \epsilon)} = \frac{|x_n - 2|}{2}$$

If we chose $\epsilon = 1$.

2.3.3

Squeeze theorem: if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Let's assume that $\lim x_n = x$, $\lim y_n = y$, and $\lim z_n = z$ for now. By the order limit theorem we have know that $x \leq y \leq z$. Since $x = z = l$, then $y = l$.

2.3.5

If x_n converges, then it means that $|z_n - l| < \epsilon\ \forall \epsilon > 0 \in \mathbb{R}$ when $n \geq N \in \mathbb{N}$. So when $n \geq N$, all terms in x_n must be "ϵ close to whatever happens to be l."

This implies that all terms after $n > N$ must be "close to l"

$$|x_n - l| < \epsilon$$

and

$$|y_n - l| < \epsilon$$

Thus, if for some reason x_n and y_n diverged or converged to different values, then the above conditions would not be met, and thus z_n would not converge.

2.3.6
Consider
$$b_n = n - \sqrt{n^2+2n}$$

Find $\lim b_n$ given $1/n \to 0$; the fact that when $x \geq 0$ if $x_n \to x$, then $\sqrt{x}_n \to \sqrt{x}$; and the algebraic limit theorem.

$$n - \sqrt{n^2+2n} = n - \sqrt{n^2+2n}\left(\frac{n+\sqrt{n^2+2n}}{n+\sqrt{n^2+2n}}\right) = \frac{n^2-n^2-2n}{n+\sqrt{n^2+2n}}$$

$$= \frac{-2n}{n+\sqrt{n^2+2n}} = \frac{-2}{1+\sqrt{1+2/n}}$$

Now we can use the info given clearly,

$$\lim b_n = \lim\left(\frac{-2}{1+\sqrt{1+2/n}}\right) = \frac{-2}{1+\sqrt{1+\lim 2/n}} = \frac{-2}{1+\sqrt{1}} = -1$$

2.3.11
Cesaro mean: if $x_n \to x$, then does

$$y_n = \frac{x_1+x_2+\ldots+x_n}{n} \to x$$

?

If we think about ϵ-neighborhoods, then we can envision that at some point all members of the series hover extremely close around a given value, so n elements of the same value divided by n would be "the value". In a more math way,

$$|y_n - x| = \left|\frac{x_1+x_2+\ldots+x_n}{n} - x\right| = \left|\frac{|x_1-x|+|x_2-x|+\ldots+|x_n-x|}{n}\right|$$

Since x_n converges, we know that for any $n \geq N$, $|x_n - x| < \epsilon$. So

$$\left|\frac{|x_1-x|+|x_2-x|+\ldots+|x_n-x|}{n}\right| < \left|\frac{|x_1-x|+|x_2-x|+\ldots+\epsilon+\ldots+\epsilon+\ldots}{n}\right|$$

If we define $M = \max|x_1-x|, |x_2-x|, \ldots, |x_{n-1}-x|$, then

$$\left|\frac{|x_1-x|+|x_2-x|+\ldots+|x_n-x|}{n}\right| < \left|\frac{M+M+\ldots+\epsilon+\ldots+\epsilon+\ldots}{n}\right|$$

If we chose an n that's far out, we the above expression grows as M/n, which will tend to zero ($\lim \frac{1}{n} = 0$).

2.3.13

Iterated limits: given a doubly indexed array a_{mn} where $m, n \in \mathbb{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

Let $a_{mn} = m/(m+n)$ and compute the iterated limits

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$$

and

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right)$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$.

2.3 The monotone convergence theorem and a first look at infinite series

2.3.1 Exercises

2.4.1 For thiese sorts of problems it helps to always plugin a couple numbers to see the patterns. The sequence we are working with is

$$x_{n+1} = \frac{1}{4 - x_n}$$

where $x_1 = 3$.

Let's start by finding the first couple terms in the sequence,

$$x_1 = 3$$

$$x_2 = \frac{1}{4-3} = \frac{1}{1} = 1$$

$$x_3 = \frac{1}{4-1} = \frac{1}{3} = 1/3$$

$$x_4 = \frac{1}{4-1/3} = \frac{1}{\frac{12}{3} - \frac{1}{3}} = \frac{1}{11/3} = 3/11$$

$$x_5 = \frac{1}{4-3/11} = \frac{1}{\frac{44}{11} - \frac{3}{11}} = \frac{1}{41/11} = 11/41$$

$$x_6 = \frac{1}{4-11/41} = \frac{1}{\frac{164}{41} - \frac{11}{41}} = \frac{1}{153/41} = 41/153$$

We could even obtain the rest of the numbers in this sequence, given our initial conditions with

```
def new_term(numerator, denominator):
    new_denominator = 4*denominator - numerator
    return denominator, new_denominator
```

So it seems like we have a case of the $x_n > x_{n+1}$, or a decreasing function. Induction is demanded here and now since we already have the base case of $x_1 > x_2$. For the rest of the numbers, let's go step by step...

If we assume that $x_n \geq x_{n+1}$, then $4-x_n \leq 4-x_{n+1}$, since we are subtracting a bigger number on the left side. However, smaller denominators make for greater rational numbers, so $\frac{1}{4-x_n} \geq \frac{1}{4-x_{n+1}}$. This expression holds up becuause of our initial conditions (what if $x_1 = 4.5$?).

Intuitvely, we can also follow the pattern we saw. After x_2, the next couple terms in the sequence were numbers smaller than 1. If the new terms get smaller, then their subscessors will get close to $1/4$, or so we think!

But the monotone convergence theorem can help us out here. Since the sequence is decreasing and 3 can serve as our bound, then this sequence should converge.

This is cool so put attention... based on everything we have seen thus far $\lim x_n = \lim x_{n+1}$. So if we take the limit of each side of the recurisve equation to explicitly compute $\lim x_n$, then we get,

$$\lim x_n = x = \lim x_{n+1}$$

So

$$\lim x_n = x$$

and

$$\lim x_n = \frac{1}{4 - \lim x_n} = \frac{1}{4-x}$$

Which leads us to

$$x = \frac{1}{4-x} \rightarrow 4x - x^2 = 1$$

Or $x^2 - 4x + 1 = 0$. We can readily use the quadratic equation now to find our solution,

$$x = \frac{-(-4) \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

If $x = 2+\sqrt{3}$ then $x > 3$ and this contradicts everything that we have seen thus far about the sequence being decreasing. So $x = 2 - \sqrt{3}$ must be it (and this matches the answer we get from usign the code above)!

2.4.2 This time let's work with the recurisevely defined sequence

$$y_{n+1} = 3 - y_n$$

When $y_1 = 1$.

Plugin in a couple numbers,

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= 3 - y_1 = 3 - 1 = 2 \\
y_3 &= 3 - y_2 = 3 - 2 = 1 \\
y_4 &= 3 - y_3 = 3 - 1 = 2 \\
y_5 &= 3 - y_4 = 3 - 2 = 1 \\
y_6 &= 3 - y_5 = 3 - 1 = 2 \\
y_7 &= 3 - y_6 = 3 - 2 = 1
\end{aligned}$$

So this time the series is not monotone but alternating. Hence, we have no reason to believe that his converges, if anything it looks like it diverges (without doing any more work). Because of this, it would not be sensible to follow the same procedure as in **2.4.1** and apply the limit on both sides of the recurisve definition of this sequence, because the limits do not exist.

Now, if instead we were working with

$$y_{n+1} = 3 - 1/y_n$$

and $y_1 = 1$, then

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= 3 - 1/1 = 2 \\
y_3 &= 3 - 1/2 = 5/2 \\
y_4 &= 3 - 2/5 = \frac{15 - 2}{5} = 13/5 \\
y_5 &= 3 - 5/13 = \frac{39 - 5}{13} = 34/13 \\
y_6 &= 3 - 13/34 = \frac{102 - 13}{34} = 89/34 \approx 2.6176471
\end{aligned}$$

We can see that $y_1 \leq y_2$, so maybe $y_n \leq y_{n+1}$?

As in problem **2.4.1**, we have the hypothesis that $y_n \leq y_{n+1}$, That implies that $1/y_n \geq 1/y_{n+1}$, since 1 divided by a large denominator results in a small number. Which in turns implies that $3 - 1/y_n \leq 3 - 1/y_{n+1}$, since 3 minus $1/y_{n+1}$ is 3 minus a smaller number. Thus $y_{n+1} \leq y_{n+2}$. We see that the series is monotonically increasing, but is it bounded (and thus convergent)?

What we do know is that $y_n > 0$, then $y_{n+1} = 3 - 1/y_n \leq 3$. So 3 could be an upper bound as long as we are sure thant y_n stays positive.

The way for $y_n < 0$ to happen would be for $3 - 1/y_n < 0$ or $3 < 1/y_n$, or $y_n < 1/3$. Since we know that the sequence is increasing, as long as start with a number above $1/3$ we should be okay and we should be bounded by 3.

Now we actually have some justification for attempting to calculate the limits of our recursive formula. But the algebra is not simple, so we wont.

2.4.3 does

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converge?

Let's try following some of the similar steps as in the previous problems.

We can define the sequence as $x_1 = \sqrt{2}$ and

$$x_{n+1} = \sqrt{2+x_n}$$

We can see that $x_1 = \sqrt{2} \leq x_2 = \sqrt{2+\sqrt{2}}$, so it seems like the sequence would be increasing as n increases. If $x_{n+1} \geq x_n$, then

$$2 + x_{n+1} \geq 2 + x_n \to \sqrt{2+x_{n+1}} \geq \sqrt{2+x_n} \to x_{n+2} \geq x_{n+1}$$

(Remember problem 2.3.1? It comes in handy a lot!) So the sequence does seem to be an increasing one. Now let's see if we can argue that it is bounded.

Right now we know that $0 < x_1 = \sqrt{2} < 2$. So what if we said that $x_n < 2$? Well we could aso say that $2 + x_n < 4$, or that $\sqrt{2+x_n} = x_{n+1} < 2$. Which makes us believe that 2 is a bound for this sequence.

Given these two findings, we can assume that $\lim x_n$ exists and that $\lim x_n = \lim x_{n+1}$. So if we apply the limit to both sides of our re-defined sequence we get

$$\lim x_{n+1} = x = \lim \sqrt{2+x_n} = \sqrt{2+\lim x_n} = \sqrt{2+x}$$
$$\to x = \sqrt{2+x}$$

A bit of algebra leads us to

$$x = \sqrt{2+x} \to x^2 = 2 + x \to x^2 - x - 2 = 0$$

And a little bit of the quadratic equation would land us in

$$\frac{-(-1) \pm \sqrt{1-4(-2)}}{2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

x is either 2 or -1. For our case, $x = 2$ must be the one.

Now, what about the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

Let's go through this as we have been practicing... $x_1 = \sqrt{2}$, and

$$x_{n+1} = \sqrt{2x_n}$$

Since $\sqrt{2} > 1$, $x1 \leq x_2$, so again we are increasing. Now let's say $x_{n+1} \geq x_n$, in which case

$$x_{n+1} \geq x_n \to \sqrt{2x_{n+1}} \geq \sqrt{2x_n} \to x_{n+2} \geq x_{n+1}$$

Similarly, $x_1 = \sqrt{2} < 2$, so if $x_n < 2$, then $\sqrt{2x_n} = x_{n+1} < \sqrt{4} = 2$, so again x_n is bounded by 2.

Now that we got our excuse to apply the limits, we have

$$x = \sqrt{2x} \to x^2 = 2x$$

or $x = \pm 2$, and again we chose the positive value.

2.4.4

Show that the monotone convergence theorem can also be used to prove the archimedean property without making use of the axiom of completeness.

The archiemedean property states that: given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$. And that given any real number $y > 0$, there exists an n satisfying $1/n < y$.

The trick to seeing this is to find two sequences that are convergent, because if they are convergent, then they must have an upper bound, and thus the archiemedean property flows.

One way is to find some sequence we know and see how it the archimedean principle makes itself visible. For example, we know that $\lim 1/n = 0$, so $|1/n - 0| = 1/n < \epsilon$, and if we set ϵ to some real number y, then we have second statement of the archiemedean principle.

Another way, is to follow the exact same line of reasoning used in the book when proving it. In the book we went with a proof by contradaction - we assumed that the natural numbers were bounded. Since the natural numbers are monotonically increasing and we are assuming that they are bounded, then the sequence of natural numbers must converge to some number. Which is hard to argue.

Mathematically, we are saying that $\lim n = N$, so $\lim n+1 = N+1$, meaning that $N = N + 1$.

There is a similar argument made here TFAE: Completeness Axiom and Monotone Convergence Theorem.

Use the monotone convergence theorem to supply a proof for the nested interval property.

The nested interval property tells us that for each $n \in \mathbb{N}$, if we are given a close interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$, and if $I_{n+1} \subseteq I_n$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

For this part, we can also follow the logic of the book. The sequence of a_n is increasing and bounded by any b_n, so it must converge to some value. Similar argument can be made for the sequence b_n but this one being bounded from below and decreasing. Since $\lim a_n = a$ and $\lim b_n = b$ and $a_n \leq b_n$ for all n, then the order limit theorem tells us that $a \leq b$. From here can follow the argument in the book by looking at $a_n \leq a$ for a particular I_n. In this same instance, $a < b_n$, so $a \in I_n$ for all n.

2.4.5 Let $x_1 = 2$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

To show that $\forall n$, $x_n^2 \geq 2$, we can use induction. $x_1^2 = 4 \geq 2$. To see the rest of the argument,

$$x_{n+1}^2 = \frac{1}{4}x_n^2 + \frac{1}{x_n^2} + 1$$

Hence, if $x_n^2 \geq 2$, then $x_{n+1}^2 \geq 2$.

Now that we have an argument made for $x_n^2 \geq 2$ for all n, let's see if we can prove that $x_n - x_{n+1} \geq 0$.

One way of seeing this is by exploring whether the sequence is decreasing. We know that $x_1 = 2 > x_2 = 3/2$. Now let's poke around and see what we can find about x_n in general.

$$\frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{1}{2}\left(\frac{x_n^2 + 2}{x_n^2}\right) \leq \frac{1}{2}(x_n^2 + 2) \geq 1$$

So its a tad odd to carry on with an induction argument given the expression we just made.

So let's try another way,

$$x_n - x_{n+1} = x_n - \frac{1}{2}x_n - \frac{1}{x_n} = \frac{1}{2}x_n - \frac{1}{x_n}$$

If $x_n - x_{n+1} \geq 0$, then

$$\frac{1}{2}x_n - \frac{1}{x_n} \geq 0 \rightarrow \frac{1}{2}x_n \geq \frac{1}{x_n} \rightarrow x_n^2 \geq 2$$

Which matches what we had above.

If instead we had assumed that $x_n - x_{n+1} < 0$, then we would have ended up with $x_n^2 < 2$.

Since we have showned that the sequence is decreasing and bounded, then we know that it converges, and we can apply the limit to both sides of our definition. This will land us with $\lim x_n = \pm 2$, but only $+2$ is a valid limit for us.

2.4.6

Explain why the geometric mean is always less than the arithmetic mean: why is $\sqrt{xy} \leq (x+y)/2$, for any two positive real numbers x and y.

To see why, let's square both sides

$$\sqrt{xy} \leq \frac{x+y}{2} \to xy \leq \frac{1}{4}(x^2 + y^2 + 2xy) \to 4xy \leq x^2 + y^2 + 2xy$$

The last expression can be re-arranged to

$$0 \leq x^2 + y^2 - 2xy = (x-y)^2$$

Which helps us see that only in the case in which $x = y$, will the geometric and the arithmetic means be equal, otherwise, the arithmetic mean will always be greater.

If we have $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n}$$

and

$$y_{n+1} = \frac{x_n + y_n}{2}$$

and we want to see whether $\lim x_n$ and $\lim y_n$ esit, then the simplest thing is to investigate whether the sequences are monotone and bounded.

We already saw that the geometric mean is always less than the arithmetic mean, so $x_n \leq y_n$ for all $n \in \mathbb{N}$. Thus we can carry our initial condition as a general case and assume that $x_n \leq y_n$ throughout.

If we now look at x_{n+1} we see that if it was that if $x_{n+1} = \sqrt{x_n x_n}$. However, y_n is greater than or equal to x_n so $x_{n+1} \geq x_n$.

As for y_n, it helps if we rewrite it as $y_{n+1} = \frac{1}{2}x_n + \frac{1}{2}y_n$. Again, $y_{n+1} = y_n$ if $x_n = y_n$, but since x_n is equal to or smaller, then $y_{n+1} \leq y_n$.

So we see that x_n is increasing, while y_n is decreasing. We also see that if $x_n = y_n$ we have some sort of equilibrium where $x_{n+1} = x_n$ and $y_{n+1} = y_n$. Which shows us that $\lim x_n = \lim y_n$ when that is the case.

2.4.7

Limit superior let (a_n) be a bounded sequence.

Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

Let's see if the monotone convergence theorem can help us out here too. The sequence y_n essentially looks at all the elements of the sequence a_n that come after a given k. So for a given y_n we have a corresponding least-upper bound from the subsequence $\{a_k : k \geq n\}$. y_1 will then be the supremum of the entire sequence a_n. y_2 will be the supremum if a_n did not have its first element a_1, and so on and so forth. Since y_1 is the supremum for the entire sequence, all other spremums from sub-sequences will be less than or equal to it. Similar argument can be made for y_2, any of the following supremums will be smaller than or equal to it. So we can already see that the sequence is decreasing and bounded by $y_1 = \sup a_n = \alpha$.

The **limit superior** of a_n is defined as follows:

$$\limsup a_n = \lim y_n$$

Where y_n is the sequence we defined above, and we also made an argument for its existance.

Now what about $\liminf a_n$? Given our definition of y_n, we could define $z_n = \inf\{a_k : k \geq n\}$. In such an instance, z_1 would be the infinimum of the entire sequence. Similarly, z_2 would be the infinimum of a_n without its first element, so it could be the same (if the sequence is decreasing), it could be of greater value (if the sequence is increasing), or it could be equal (if the first two values are similar).

Since our definition starts with the entire sequence and then gets rid of the elements at the begining, we will not be finding any infinimum value that is smaller than any of the z_n that have been computed before. Thus we can see that z_n is increasing and bounded from below by $z_1 == \inf a_n = \beta$.

However, a_n is itself bounded, so this definition, although it results in an increasing sequence and the infinimum is a lower bound, it also converges and thus $\liminf a_n$ should exist.

From experience, definition, whatever you want to call it, $\inf a_n \leq \sup a_n$. And it would make sense to think that $\liminf a_n \leq \limsup a_n$. And thus the oreder limit theorem makes this make sense.

Now, if if $\liminf a_n = \limsup a_n$, thenthe squeeze theorem can get us $\lim a_n$ as it would be the same value as the limit superior and the limit inferior. And if we know that $\lim a_n$ exists, then Intuitvely we can imagine that after some point, all elements of a_n never leave some ϵ-neighborhood (all values get arbitrarily close to the value they converge to). In such an intansce, the supremum and infinimum would be equal to one another and the order limit theorem would again come in handy to see that all three limits must equal each other.

2.4.8

Find an explicit formula for the sequence of partial sums and determine if the following series converge:

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

Writing out the terms, we get

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1/2 + 1/4 + 1/8 + \dots$$

This is a geometric series where the common term is 1/2. The explicit formula for geometric series is

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

But since we are starting at $n = 1$, we need to subtract the first term. Thus

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = -1 + \sum_{n=1}^{\infty} r^n = -1 + \frac{1}{1-\frac{1}{2}} = 1$$

In the case of

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

This is the wikipedia example of a telescoping series.

For this series,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1$$

And finally,

$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$$

Is a telescoping series of sorts since $\log \frac{n+1}{n} = \log(n+1) - \log n$. So we have

$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) = (\log 2 - \log 1) + (log 3 - \log 2) + (\log 4 - \log 3) + (\log 5 - \log 4) + (\log 6 - \log 5) + \dots$$

$$= -\log 1 + \lim \log N$$

Which doesn't seem like it converges, since $\log x$ is an increasing function.

2.4.9

Show that if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

Abbott is trully a great teacher. The beauty of this proof starts with the fact that we are proving the contrapositive of the cauchy condensation test. The following argument is completely taken from Proof of Cauchy Condensation Test using contrapositive. (I thought about it, and thought about it, and kept on thinking about it and never got anywhere.)

Since the series b_n is decreasing, we were able to define

$$s_{2^{k+1}-1} = b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \ldots + (b_{2^k} + \ldots + b_{2^{k+1}-1})$$

$$\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \ldots + (b_{2^k} + \ldots + b_{2^k})$$

$$= b_1 + 2(b_2) + 2^2(b_4) + \ldots + 2^k(b_{2^k}) = t_k$$

One of the tricks used above is that $b_n \geq b_{n+1}$. If we went the other route, if replaced some of the terms in our sequence with the smallest terms within their bracketed groups then we could obtain a sequence that is smaller.

$$s_{2^{k+1}-1} = b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \ldots + (b_{2^k} + \ldots + b_{2^{k+1}-1})$$

$$\geq b_1 + (b_3 + b_3) + (b_7 + b_7 + b_7 + b_7) + \ldots + (b_{2^{k+1}-1} + \ldots + b_{2^{k+1}-1})$$

And we really recommend that you write out the terms and see the patterns but take a look at the end. $s_{2^{k+1}-1}$ ends in $(b_{2^k} + \ldots + b_{2^{k+1}-1})$, it goes all the way to the index term in s and the bracketing starts at the previous 2^k.

Then, we can see how

$$s_{2^k} = b_1 + b_2 + b_3 + \ldots = b_1 + (b_2 + b_3) + \ldots + (b_{2^{k-1}+1} + \ldots + b_{2^k})$$

We should also note that the factor infront of the last bracketing is the 2^n that happens to be closest. Hence,

$$s_{2^k} = b_1 + (b_2 + b_3) + \ldots + (b_{2^{k-1}+1} + \ldots + b_{2^k})$$

$$\geq b_1 + 2(b_3) + 2^2(b_7) + \ldots + 2^{k-1}(b_{2^k})$$

and since sequences are infinite, we could change the offset as such

$$s_{2^k} = b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + \ldots + (b_{2^{k-1}+1} + \ldots + b_{2^k})$$

$$\geq b_1 + b_2 + 2(b_4) + 2^2(b_8) + \ldots + 2^{k-1}(b_{2^k})$$

$$= b_1 + t_{k-1}$$

And so we get our proof.

However, a peek at wikipedia reveals another very great trick. We know that

$$t_k = b_1 + 2b_2 + 4b_4 + \ldots$$

This could be bracketed also as

$$t_k = (b_1 + b_2) + (b_2 + b_4 + b_4 + b_4) + (b_4 + b_8 + \ldots) + \ldots$$

And since the series is decreasing,

$$\begin{aligned} t_k &= (b_1 + b_2) + (b_2 + b_4 + b_4 + b_4) + (b_4 + b_8 + \dots) + \dots \\ &\leq (b_1 + b_1) + (b_2 + b_2 + b_3 + b_3) + (b_4 + b_4 + \dots) + \dots \\ &= 2\sum_{n=1}^{\infty} b_n \end{aligned}$$

Hence the cauchy condensation test can be expanded to

$$\sum_{n=1}^{\infty} b_n \leq \sum_{n=0}^{\infty} 2^n b_n \leq 2\sum_{n=1}^{\infty} b_n$$

2.4.10

The infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \dots$$

can be understood in terms of its sequence of partial products (like infinite series and partial sums)

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 \dots b_m$$

We will focus here on the special class of infinite products that looks as such

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \dots$$

Where $a_n \geq 0$.

If $a_n = 1/n$ we get this interesting looking partial product

$$\begin{aligned} p_m &= (1+1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\left(1 + \frac{1}{2}\right)\dots\left(1 + \frac{1}{m}\right) \\ &= \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\dots\left(\frac{m+1}{m}\right) \\ &= \frac{m+1}{1} = m + 1 \end{aligned}$$

However, since $m \to \infty$ in the infinite product, we could think that this case diverges.

Now, when $a_n = 1/n^2$, we get

$$p_m = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{4^2}\right)\left(1+\frac{1}{5^2}\right)\dots\left(1+\frac{1}{m^2}\right)$$
$$= \left(\frac{2}{1}\right)\left(\frac{2^2+1}{2^2}\right)\left(\frac{3^2+1}{3^2}\right)\left(\frac{4^2+1}{4^2}\right)\left(\frac{5^2+1}{5^2}\right)\dots\left(\frac{m^2+1}{m^2}\right)$$

Here the first term is 2 but as m grows, the terms begin to get closer and closer to 1.

Can we show that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges?

Abbott gave us the trick to this, if we use the identity $1 + x \leq e^x$, then we can rewrite the partial product as follows

$$p_m = (1+a_1)(1+a_2) + \dots (1+a_m) \leq e^{a_1}e^{a_1}\dots e^{a_m} = e^{s_m}$$

Hence, if $\sum_{n=1}^{\infty} a_n \to a$, then e^{s_m} is a number, and thus we get our proof - the series is increasing and bounded.

To prove the other route (if the infinite product converges then the series converges), it sufices to show that $p_m \geq s_m$. (Expand the product and you will see terms correspodning to the partial sum plus other additional terms.)

2.4 subsequences and the Bolzano-Weierstrass theorem

2.4.1 Exercises

2.5.3

Assume $a_1 + a_2 + 1_3 + \dots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any **regrouping** of the terms

$$(a_1 + a_2 + \dots + a_{n_1}) + (a_{n_1+1} + a_{n_1+2} + \dots + a_{n_2}) + (a_{n_2+1} + a_{n_2+2} + \dots + a_{n_3}) + \dots$$

leads to a series that converges to L.

A sequence of partial partial sums $s_n \to L$. A regrouping of terms ends up creating a subsequence of s_n. And since s_n converges, the any of its subsequences converge as well.

2.5.4

Assume the nested property is true and use it to provide a proof for the axiom of completeness.

Some brief reminders:

- Axiom of completeness: every non-empty set of real numbers that is bounded above has a least upper bound.
- nexted property: assume we are given a close interval I_n and that $I_{n+1} \subseteq I_n$, then $\cap I_n \neq \emptyset$.

When proving the nested interval property, we laid out all $I_n = [a_n, b_n]$ on the real number line. Since all intervals were nested, then we had an order for the set of $A = \{a_n :\in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$, where a_1 and b_1 were at the values fartest apart. Then we saw that all $b \in B$ served as upper bounds, and thus a supremum x must exist such that $a_n \leq x \leq b_n$. And since this inequality held for all intervals (as they were all nested and equal to or smaller than their predecessor) then we had a proof that there was an element that was part of all the intervals.

When proving the Bolzano-Weierstrass theorem, we also built nested intervals with the same properties as the ones we saw in the nested interval theorem. So if we assume that the nested interval theorem is true, then right away we can see that from the bisecting of our Bolzano-Weierstrass prove that there must be at least one element that is part of all the intervals.

If we start with the nested interval property then we also get an upper bound for our sequence (b_n).

The only thing left to get the axiom of completeness from the use of the nested interval property on the Bolzano-Weierstrass theorem is to see that as we continue forming subintervals, each of of length $M(1/2)^{k-1}$, then as $k \to \infty$, the length will tend to zero (example 2.5.3). And since the length of the intervals tends to zero then the element x that we found above, can indeed be the least upper bound, since we can make the lenght of the interval as small as we want (ϵ) such that $x - \epsilon$ is no longer an upper bound, $x - \epsilon \leq a_n$.

The reason we are to assume that $1/2^{k-1}$ is a convergent sequence is because the lenght of the subinterval I_k is supposed to converge to 0, and using the normal ϵ convention we want to prove that something like $1/n < \epsilon$ is true for any $\epsilon > 0$. ϵ is a real number and so we are implicitly making use of the Archimedean property, which we originally proved using the Axion of Completeness, which we are trying to prove here.

2.5.6

Show that $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit.

The series $b^{1/n}$ is decreasing, since $\frac{1}{n} > \frac{1}{b+1}$. Since it is monotonically decreasing and abounded by b, then it must converge. Go again, if $\lim b^{1/n} = l$, then $0 \leq l \leq b$.

Since $b^{1/n} \to l$, then so must $b^{1/(n+1)} \to$. If we take the limit of both and equate then, we get

$$b^{1/n} = b^{1/(n+1)} \to b = b^{\frac{n}{n+1}} = b^{1+1/n} = bb^{1/n}$$

Hence

$$b^{1/n} = 1$$

If $b = 0$, $\lim b^{1/n} = 0$. Recall that one problem where we proved that $\lim a_n = 0$, then $\lim \sqrt{a_n} = 0$.

2.5.7

When $|b| < 1$, then b^n is decreasing and abounded by 1, hence convergent. We follow similar steps as in the previous problem, then $b^n = b^{n+1} = l$. And the only numbers that stay the same no matter how many powers you raise them to are 0 and 1. But since $|b| < 1$, the $|b^n| < 1$, so $\lim b^n = 0$.

2.5.8

Another way we could prove the Bolzano-Weierstrass theorem is by finding monotone subsequences in a bounded sequence. That way the monotone convergence theorem can be used to prove convergence.

If we can find a series of peak terms within our sequence, then we can build a monotone subsequence from the sequence of peak terms. Note that the peak points would define a decreasing subsequence.

However, if there is a finite number (or zero) peak points, then instead of peak points, we can look for "nadir points" and instead define a monotonically increasing subsequence.

2.5.9

Direct proof of Bolzano-Weierstrass theorem using the Axiom of Completeness

Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbb{R} : x < a_n \text{ for inifinetely many terms in } a_n\}$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$.

Since (a_n) is bounded, then S will also be bounded. Thus there must exist $s = \sup S$. If we chose a subsequence (a_{n_k}) that is monotonically increasing, whose values go up to the bound of (a_n), the by the monotone convergence theorem, this subsequence will converge. This subsequence should be such that $|a_{n_k} - s| < \epsilon$ for any $\epsilon > 0$ and any $k > N$. Which is possible because the parent sequence converges and we are restricting ourselves to a mnonotone increasing sequence.

2.5 The Cauchy Criterion

In **lemma 2.6.3, cauchy sequences are bounded**, there is a bit of logic that was used back in **theorem 2.3.2 every convergent sequence is bounded**, and also when de discussed the triangle inequality.

When we say that $|x_n - x| < \epsilon$, this can also be seen as $\epsilon < x_n - x < \epsilon$. Thus, $|x_n - x_m| < 1$, could be seen as $-1 < x_n - x_m < 1$ or $-(x_m + 1) < x_n < x_m + 1$.

2.5.1 Exercises

2.6.1

Every convergent sequence is a Cauchy sequence

If we have a sequence (x_n) such that $\lim x_n = x$, then

$$|x_n - x| < \epsilon$$

Whenever $n \geq N \in \mathbb{N}$.

Thus, if we look at a cauchy sequence, with $n, m \geq N$,

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| = |x_n - x| + |x_m - x| < \epsilon_1 + \epsilon_2$$

We could have gone the other way around and started with,

$$|x_n - x| \leq |x_n - x_m| + |x_m - x| < |x_n - x_m| + \epsilon$$

And from there we would be back to the first use of the triangle inequality.

Note: it could be easy to make an argument about making $x_m = x$ and then throwing in another epsilon. However, we would be fooling ourselves with the fake simplicity (why didn't we use that same argument instead of using the triangle inequality above?). To see why the fake simplicity is not correct, see the rest of contents in Abbott section 2.6. Where Abbott instead went and used the Bolzano-Weierstrass theorem.

Also, it is possible that the limit is not actually a memeber of the sequence, think of $\lim 1/n$.

2.6.3

If (x_n) and (y_n) are cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is to use the cauchy criterion. The cauchy criterion states that (x_n) and (y_n) must be convergent, and the algebraic limit theorem then implies $(x_n + y_n)$ is convergent and hence cauchy.

Remember that the cauchy criterion says that a sequence converges if and only if it is a cauchy sequence.

Now try giving a direct argument that $(x_n + y_n)$ is a cauchy sequence that does not use the cauchy criterion or the algebraic limit theorem.

We know that

$$|x_n - x| < \epsilon \Longleftrightarrow |x_n - x_m| < \epsilon$$

for all $\epsilon > 0$ when $n \geq N_1$. We also know that

$$|y_n - x| < \epsilon \Longleftrightarrow |y_n - y_m| < \epsilon$$

for all $\epsilon > 0$ when $n \geq N_2$.

To make a direct argument, we need to prove that

$$|(x_n + y_n) - (x_m + y_m)| < \epsilon$$

for all $\epsilon > 0$ when $n \geq \max N_1, N_2$.

To do so, note that

$$\begin{aligned} |(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\ &\leq |x_n - x_m| + |y_n - y_m| < \epsilon + \epsilon \end{aligned}$$

Give a direct argument for $(x_n y_n)$ now.

We now want to show that

$$|(x_n y_n) - (x_m y_m)| < \epsilon$$

Let's use a the same trick Abbott employed back when we were proving the algebraic limit theorem.

$$\begin{aligned} |(x_n y_n) - (x_m y_m)| &= |x_n y_n - x_m y_n + x_m y_n - x_m y_n| \\ &\leq |x_n y_n - x_m y_n| + |x_m y_n - x_m y_m| \\ &= |y_n||x_n - x_m| + |x_m||y_n - y_m| \\ &< M_2 \frac{\epsilon}{M_2} + M_1 \frac{\epsilon}{M_1} \end{aligned}$$

In the last step we used the fact that convergent sequences are bounded and defined M_1 as the upper bound for x_n and M_2 as the upper bound for y_n.

2.6.7

Exercises 2.3.1 (use MCT to prove AoC) and 2.4.1 (use NIP, assume AP, use BW methodology to prove AoC) establish the equivalence of the axiom of completeness and the monotone convergence theorem. They also show the nested interval property is equivalent to these two when the archimedean property is pressumed to be true.

Before we carry on, we want to note Rudin's statement of the **Archimedean property (and the density of the rational numbers within the reals)**: If $x, y \in \mathbb{R}$ and $x > 0$, then there is a $n > 0 \in \mathbb{N}$ such that

$$nx > y$$

Abbott states this as $n > x$ for any $x \in \mathbb{R}$.

If $x, y \in \mathbb{R}$ and $x < y$, then there is a $p \in \mathbb{Q}$ such that

$$x < p < y$$

Abbot states this as $1/n < y$ for any $y > 0 \in \mathbb{R}$ and $n \in \mathbb{N}$.

Assume the Bolzano-Weierstrass theorem to be true and use it to construct a proof of the monotone convergence theorem without making any appeal to the archimedean property. This shows that BW, AoC, and MCT are equivalent.

By the BW theorem, we know that a bounded sequence has a convergent subsequence. The subsequence generated by BW meets the condition that the elements we select must be so $n_k > n_{k-1} > \ldots > n_1$ so that $a_{n_k} \in I_k$. Which means that we are not going around picking elements as we want, we pick elements in the order that they show up in the original sequence.

Ultimately, we know that if BW applies then (a_n) is bounded and that we can build a subsequence $(a_{n_k}) \to x$.

For the MCT, the sequence must also be bounded and all elements must be monotonically increasing or decreasing. Let's pick the case in which the elements are monotonically increasing and our sequence is bounded above.

So

$$|a_{n_k} - x| < \epsilon$$

When $k \geq N$, and $a_{n_{k+1}} \geq a_{n_k}$.

Thus,

$$-(x + \epsilon) < a_{n_k} < x + \epsilon$$

or

$$-(x + \epsilon) < a_{n_k} \leq a_{n_{k+1}} < x + \epsilon$$

In this last expression, the added term works because the sequence is monotonically increasing, but we know that for any $k \geq N$, $|a_{n_k} - x| < \epsilon$ must hold. This last expression can also be rewritten as

$$|a_{n_{k+1}} - x| < \epsilon$$

And thus our prove.

Use the Cauchy criterion to prove BW, and find the point in the argument where the Archimedean property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness.

The argument for the BW theorem that Abbott shows us is rather generic. One thing we can do is to note when it makes use of the NIP. Originally this was to find what the limit for the subsequence could be. If we instead assumed the Cauchy criterion, then we'd know that the subsequence is bounded and that after some point, the elements of the sequence get close to each other.

Then the implicit use of thw Archimedean property comes from the assumption that $M(1/2)^{k-1}$ converges. ($M(1/2)^{k-1}$ is a rational number that we want to make smaller tthan some arbitrary real ϵ.)

How do we know it is impossible to prove the AoC starting from the Archimedean property?

The Archimedean property tells us that we can always find an $n \in \mathbb{N}$ that is greater than some real number. However, it tells us about upper bounds, it doesn't tell us about the least-upper bounds (or greatest upper bounds). So the Archimedean property leaves us with possible holes in our number line.

Also, the first part of the Archimedean property, $n > x$, is true if $x \in \mathbb{Q}$. So it doesn't give us any clues about us needing the reals.

2.6 Properties of infinite Series

Cauchy Criterion for Series

Since $n > m \geq N$,

$$s_m = a_1 + a_2 + \ldots + a_N + \ldots + a_m$$

and

$$s_n = a_1 + a_2 + \ldots + a_N + \ldots + a_m + \ldots + a_n$$

So

$$|s_n - s_m| = |(\cancel{a_1} + \cancel{a_2} + \ldots + \cancel{a_N} + \ldots + \cancel{a_m} + \ldots + a_n) - (\cancel{a_1} + \cancel{a_2} + \ldots + \cancel{a_N} + \ldots + \cancel{a_m})|$$

$$= |a_{m+1} + a_{m+2} + \ldots + a_n|$$

2.6.1 Exercises

2.7.1

The **alternating series tests** goes as follows: let (a_n) be a sequence satisfying

1. $a_1 \geq a_2 \geq \ldots a_n \geq a_{n+1} \geq \ldots$
2. $(a_n) \to 0$

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proving the alternating series tests ammounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - a_4 + \ldots \pm a_n$$

converges.

Different characterizations of completeness lead to different proofs.

Prove it by showing that (s_n) is a Cauchy sequence.

We have

$$s_n = a_1 - a_2 + a_3 - a_4 + \ldots \pm a_n$$

So if we chose an $n > m \geq N$

$$|s_n - s_m| = |(a_1 - a_2 + a_3 - a_4 + \ldots \pm a_n) - (a_1 - a_2 + a_3 - a_4 + \ldots \pm a_m)|$$

$$= | \pm a_{m+1} \mp a_{m+2} \pm \ldots \pm a_n|$$

In the special case in which $n = m + 1$, we get

$$|s_n - s_m| = |a_n| < \epsilon$$

The appended inequality is possible because $(a_n) \to 0$, so we can meet any ϵ as long as we move N farther out.

From there, we can see how we could chose the terms so that

$$|s_n - s_m| = |a_{m+1} - a_{m+2}| + \ldots + |a_{n-1} - a_n|$$

Prove it using the NIP

Let's play around with the sample alternating sequence

$$10 - 9 + 8 - 7 + 6 - 5 + 4 - 3 + 2 - 1 + 0$$

s_1	10
s_2	10-9 = 1
s_3	1+8 = 9
s_4	9-7 = 2
s_5	2+6 = 8
s_6	8-5 = 3
s_7	3+4 = 7
s_8	7-3 = 4
s_9	4+2 = 6
s_{10}	6-1 = 5

Using this as example, we can see how we could build intervals I_n such that $I_{n+1} \subseteq I_n$ by chosing $I_n = [s_{2n}, s_{2n+1}]$. That way we can guarantee that there will be an $x \in \cap I_n$. Also, since the length of I_n will become smaller and smaller, we can then have $|s_{2n} - x| < \epsilon$ or $|s_{2n+1} - x| < \epsilon$.

Consider the sequences (s_{2n}) and (s_{2n+1}), and show that the MCT leads to a third proof.

Another good way to llok at the alternating series is as follows:

$$s_n = (a_1 - a_2) + (a_3 - a_4) + \ldots \pm a_n$$

Each group consists of terms that are great than or equal to zero, give our preconditions. Each grouping resulting in the sum of some non-negative number that get's smaller and smaller.

We can also read it as:

$$s_n = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots$$

In this case, s_1 is an upper bound and we continuously subtract smaller and smaller numbers.

Comparing these two findings, we can see that the sequence is monotonically increasing and that its upper bound is s_1, so MCT says it must converge.

Now, let's turn to s_{2n}, which can equal s_2, s_4, s_6 etc., and s_{2n+1}, which can equal s_3, s_5, s_7, etc.

If you write out a couple terms you will see how Abbott was trying to help us see what we describe above and it offers us a slightly different argument.

s_{2n+1}, when we count from $n = 0$, and if we assume that $a_n \geq 0$ for all n, then we have a bounded and monotonically decreasing sequence - what we mentioned above. We can also see that $s_{2n+1} = s_{2n} + a_{2n+1}$, or $s_{2n} = s_{2n+1} - a_{2n+1}$. So we can do this,

$$|s_{2n} - l| \leq |s_{2n} - s_{2n+1}| + |s_{2n+1} - l| = |-a_{2n+1}| + |s_{2n+1} - l|$$

The first term on the right we can make it smaller by choosing greater by going farther to the right, the second term is just a restatement that s_{2n+1} converges.

2.7.2

Does $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converge?

This series look a tad like

$$\begin{aligned}\sum_{n=1} \frac{1}{2^n+n} &= \frac{1}{2+1} + \frac{1}{2^2+2} + \frac{1}{2^3+3} + \frac{1}{2^4+4} + \frac{1}{2^5+5} \cdots \\ &= \frac{1}{3} + \frac{1}{6} + \frac{1}{11} + \frac{1}{20} + \frac{1}{37} + \ldots\end{aligned}$$

In this case, $s_1 = 1/3$, $s_2 = 1/2$, and $s_3 = 13/22$. So at least from the first couple partial sums, the partial sums seem to be monotonically increasing.

Now we have to look for clues. The series $\sum \frac{1}{2^n}$ is a geometric series where $\frac{a}{1-r} = \frac{1}{1-1/2} = 2$.

Furhtermore, $\frac{1}{2^n} > \frac{1}{2^n+n}$ so by the comparsion tests, our series here should converge.

Does $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converge?

By a similar route, we saw in section 2.4 that $\sum 1/n^p$ when $p > 1$ converges. And since $\sin(n)$ oscillates between 1 and -1, then we can use the comparison test again to see that the series converges.

Does $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \ldots$ converge?

This series seems to go as $\sum(-1)^{n+1}\frac{n+1}{2n}$. Which made me think of the alternating series. But we need to see if the sequence $\frac{n+1}{2n}$ actually converges to 0. Luckily, some algebra can help us right away

$$\frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$$

No matter how big n gets, $(a_n) \to 1/2$. It seems like we cannot say that it converges. And since the series alternates between positive and negative values, then the series diverges.

Does $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \ldots$ converge?

The trick here is to group terms, we always see two positive terms followed by a negative one. And they always follow the pattern of the harmonic series but as

$$\frac{1}{n} + \frac{1}{n+1} - \frac{1}{n+2} \geq \frac{1}{n}$$

The inequality comes because $\frac{1}{n+1} > \frac{1}{n+2}$, so $\frac{1}{n+1} - \frac{1}{n+2}$ is a non-negative number that gets added to $1/n$. And by using the comparison test, we can tell this series diverges.

Does $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \ldots$ converge?

Interestingly, if we separate the terms for this series, it seems like we are adding $\sum \frac{1}{2n-1}$ and $\sum -\frac{1}{(2n)^2} = \sum -\frac{1}{4}\frac{1}{n^2}$.

The latter series could be argued it converges by using the algebraic limit theorem for series since we have $\sum 1/n^2$. However, the former seems like the harmonic series, but only taking odd numbers.

2.7.3

An alternate way to prove the comparison tests for infinite series theorem using the monotone converge theorem goes as follows.

If $0 \geq a_k \geq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges, then we know that the sequence of partial sums also converges. And since the sequence of partial sums converges then it must be bounded.

The limit of the sequence of partial sums of $\sum b_k = \lim s_m$ can also be seen as an upper bound for the sequence of partial sums of $\sum a_k$, since $0 \geq a_k \geq b_k$ for all $k \in \mathbb{N}$.

Also, since $0 \geq a_k$, then each successive term in the sequence of partial sums must be equal to or greater than the previous one, so the sequence of partial sums is a monotone sequence and thus converges as per the MCT.

2.7.5

Prove that the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

Abbott provides us with a wonderful hint: we can use the geometric series. In example 2.7.5, we saw that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

if and only if $|r| < 1$.

So let's say that $a = 1$, we then have

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

The above converges if and only if $|r| < 1$, so it means that r could be written as $1/n$ where we require that $|n| > 1$. We can then write the geometric series as

$$\sum_{k=0}^{\infty} \frac{1}{n^k} = \frac{1}{1-1/n} = \frac{n}{n-1}$$

We could do a comparison test now: we have a series with $1/n^k$, where k is the index, and the other as $1/n^p$, where n is now the index. If we used n as an index and p as some fixed number on both, we'd be comparing $1/p^n$ (geometric series with $|p| > 1$) with $1/n^p$. After a finite ammount of terms, the exponential p^n will be greater than n^p, so $\frac{1}{p^n} \leq \frac{1}{n^p}$. This last inequality tells us that a plan ol' comparison test will get us nowhere since the geometric series is a lower bound to a p-series.

The next logica trial would be to try and bound a p-series by something that looks like a geometric series. For example,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} + \dots \\
&\leq 1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{8^p} + \dots \\
&= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \dots \\
&= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \dots \\
&= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^2)^{p-1}} + \frac{1}{(2^3)^{p-1}} + \dots \\
&= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \frac{1}{2^{3(p-1)}} + \dots \\
&= \sum_{k=0} \left(\frac{1}{2^{p-1}}\right)^k = \frac{1}{1 - 2^{p-1}} \\
&= \frac{2^{p-1}}{2^{p-1} - 1}
\end{aligned}$$

2.7.7

Show that if $a_n > 0$ and $\lim na_n = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

These problems feel like it may be a place where we may be able to invoke theorem 2.7.3: if the series $\sum a_n$ converges, then $(a_n) \to 0$. But let's think about the problem a bit more first.

If $\lim na_n = l \neq 0$ then it means that $(na_n) \to l$. We also know that we are supposed to be looking for info that tells us whether $\sum a_n$ converges or diverges. To show that a series converges, we usually evaluate whether their partial sums have a limit, $\lim s_m$ exists, in this case the partial sums would look like $s_m = a_1 + 2a_2 + +3a_3 + \ldots + ma_m$.

Since $a_n > 0, \forall n$ and since $\lim na_n = l \neq 0$, then it means eventually, when $n \geq n$, the terms will be greater than zero as well. We could envision then that in order for the ns to balance with the a_ns in order to arive at a limit l, that the a_ns must grow like $\mathcal{O}(1/n)$ because if they grew like $\mathcal{O}(1)$ then the n would overpower the sequence. Similarly, if the a_n grew like $\mathcal{O}(1/n^2)$, then $\lim na_n = 0$.

And since $a_n \sim 1/n$, then $\sum a_n$ diverges because $\sum \frac{1}{n}$ diverges.

Assume $a_n > 0$ and $\lim n^2 a_n$ exists. Show that $\sum a_n$ converges.

Similar train of thought as above but this time, since we must balance out with n^2, then a_n must grow as any sort of p-series with $p > 1$, and since all of these converge, then $\sum a_n$ converges.

2.7.9

Ratio Test: given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the ratio test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

Then the series converges absolutely.

Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

can be seen as,

$$\left| \frac{a_{n+1}}{a_n} - r \right| < \epsilon$$

or

$$\frac{a_{n+1}}{a_n} \in (-(r+\epsilon), r+\epsilon)$$

So if we only want a one-sided Inequality, we can do

$$\left| \frac{a_{n+1}}{a_n} - r \right| < r + \epsilon = r'$$

Which can be rewritten as

$$|a_{n+1}| \leq |a_n| r'$$

Why does $|a_N| \sum (r')^n$ converge?

From above, we have $|a_{n+1}| \leq |a_n| r'$, so $|a_{N+1}| \leq |a_N| r'$, and $|a_{N+2}| \leq |a_{N+1}| r'$. The latter expression implying that $|a_{N+2}| \leq |a_{N+1}| r' \leq |a_N| (r')^2$. Hence $|a_{N+n}| \leq |a_N| (r')^n$.

Now, remember that for a series to converge we must have

$$|a_{m+1} + a_{m+2} + \ldots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \ldots + |a_n| < \epsilon$$

If we try to put what we know into a similar form we have

$$|a_N|+|a_{N+1}|+|a_{N+2}|+\ldots+|a_{N+n}| \leq |a_N|+|a_N|(r')+|a_N|(r')^2+\ldots++|a_N|(r')^n$$

Which leads us to

$$\sum |a_n| \leq \sum |a_N|(r')^n = |a_N| \sum (r')^n$$

Luckily for us, $\sum (r')^n$ is a geometric series which converges to $1/(1-r')$ since $0 < r' < 1$. Then, by the comparison test so does $\sum |a_n|$.

Since $\sum |a_n|$ converges, then, by the absolute convergence test, so does $\sum a_n$.

2.7.10
infinite Products: Review exercise 2.3.1.
Does

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$$

converge?

The reason Abbott tolds us to look at problem 2.3.1 is because the above can be rewritten as

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\left(1+\frac{1}{16}\right)\cdots$$

which itself can be rewritten as

$$\left(1+\frac{1}{2^0}\right)\left(1+\frac{1}{2^1}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{2^4}\right)\cdots$$

If we recall the formula $1+x \leq e^x$ then

$$\begin{aligned}
&\left(1+\frac{1}{2^0}\right)\left(1+\frac{1}{2^1}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{2^4}\right)\cdots \\
&\leq e^{1/2^0} e^{1/2^1} e^{1/2^2} e^{1/2^3} e^{1/2^4} \cdots \\
&= \exp \sum_{n=0} \frac{1}{2^n}
\end{aligned}$$

The latter expression converges to

$$\exp \frac{1}{1-1/2} = \exp 2$$

So our original product does converge and is bounded by e^2.

The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \ldots$ certainly converges. Why? Does it converge to zero?

No simple pattern seems evident at first to let's try coming up with formulas for it. The following seem to work

$$\prod_{n=0} \frac{2n+1}{2n+2} = \prod_{n=1} \frac{2n-1}{2n} = \prod_{n=1} \left(1 - \frac{1}{2n}\right)$$

The last one, once again, has the form $1 + x$, where $x = -1/2n$. So let's try looking at

$$\prod_{n=1} \left(1 - \frac{1}{2n}\right) \leq \prod_{n=1} \exp\left(-\frac{1}{2n}\right)$$
$$\exp\left(-\frac{1}{2}\sum_{n=1} \frac{1}{n}\right)$$

The $\sum 1/n$ of course is a harmonic series and it diverges. However, notice that the divergent term is within a e^{-x} term, so it actually grows smaller and smaller as n grows. So the $n = 1$ is an upper bound. Hence, we can see how the product converges and how it convegres to zero - $\lim_{x\to\infty} e^{-x} = 0$.

In 1655, John Wallis famously derived the formula

$$\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\left(\frac{8\cdot 8}{7\cdot 9}\right)\ldots = \frac{\pi}{2}$$

Show that the left side converges to something.

While we think about proving the convergence we came up with this way of rewriting the product

$$\begin{aligned}
\prod_{n=1} \frac{2n\cdot 2n}{(2n-1)(2n+1)} &= \prod_{n=1} \frac{(2n)^2}{(2n)^2-1} \\
&= \prod_{n=1} \left(1 + \frac{(2n)^2-(2n)^2+1}{(2n)^2-1}\right) \\
&= \prod_{n=1} \left(1 + \frac{1}{(2n)^2-1}\right) \\
&\leq \exp\left(\sum_{n=1} \frac{1}{(2n)^2-1}\right)
\end{aligned}$$

2.7.12

Summation by parts: Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \ldots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

Let's fish for useful information. One of the terms in the formula is

$$\begin{aligned} s_n - s_{m-1} &= (x_1 + \ldots + x_{m-1} + x_m + x_{m+1} \ldots + x_n) - (x_1 + \ldots x_{m-1}) \\ &= x_m + x_{m+1} + \ldots + x_n \end{aligned}$$

Our next idea was to actualy write out the terms of the sum and see what patterns emerged.

First,

$$x_m y_m = (s_m - s_{m-1}) y_m = s_m y_m - s_{m-1} y_m$$

The next term, $j = m + 1$,

$$x_{m+1} y_{m+1} = (s_{m+1} - s_m) y_{m+1} = s_{m+1} y_{m+1} - s_m y_{m+1}$$

so thus far,

$$\begin{aligned} \sum_{j=m}^{m+1} x_j y_j &= -s_{m-1} y_m + s_m y_m - s_m y_{m+1} + s_{m+1} y_{m+1} \\ &= s_{m+1} y_{m+1} - s_{m-1} y_m + s_m (y_m - y_{m+1}) \end{aligned}$$

Let's add another term and this time let's say that $n = (m + 1) + 1$,

$$x_n y_n = (s_n - s_{m+1}) y_n = s_n y_n - s_{m+1} y_n$$

Our sum then becomes

$$\begin{aligned} \sum_{j=m}^{n=(m+1)+1} x_j y_j &= s_{m+1} y_{m+1} - s_{m-1} y_m + s_m (y_m - y_{m+1}) + s_n y_n - s_{m+1} y_n \\ &= s_n y_n - s_{m-1} y_m + s_m (y_m - y_{m+1}) + s_{m+1} (y_{m+1} - y_n) \end{aligned}$$

Note, that if we go this way, the last term was $s_{m+1}(y_{m+1} - y_n)$. So our sum does not contain a s_n term, we are 1 step away from the sort of expression we want. To get the s_n term we would need something like $s_n(y_n - y_{n+1}) = s_n y_n - s_n y_{n+1}$. And it just so happens that we do already have the $s_n y_n$ bit at the begining of the previous expression, so all we need to do is to add a $s_n y_{n+1}$ to cancel out the last term we are looking to introduce.

Let's try working backwards now. If we expand out the series we have

$$\begin{aligned}
\sum_{j=m}^{n} x_j y_j = & \quad s_n y_{n+1} - s_{m-1} y_m + s_m(y_m - y_{m+1}) + s_{m+1}(y_{m+1} - y_{m+2}) \\
& + s_{m+2}(y_{m+2} - y_{m+3}) + \ldots + s_n(y_n - y_{n+1}) \\
= & \quad s_n y_{n+1} - s_{m-1} y_m + s_m y_m - s_m y_{m+1} + s_{m+1} y_{m+1} - s_{m+1} y_{m+2} \\
& + s_{m+2} y_{m+2} - s_{m+2} y_{m+3} + \ldots + s_n y_n - s_n y_{n+1} \\
= & \quad s_n y_{n+1} + (s_m - s_{m-1}) y_m + (s_{m+1} - s_m) y_{m+1} + (s_{m+2} - s_{m+1}) y_{m+2} \\
& + \ldots + s_n y_n + s_n y_{n+1} \\
= & \quad s_n y_{n+1} + x_m y_m + x_{m+1} y_{m+1} + x_{m+2} y_{m+2} + \ldots + x_n y_n + s_n y_{n+1}
\end{aligned}$$

2.7.13

Abel's Test: Abel's test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \ldots \geq 0$$

then the series $\sum_{k=1} x_k y_k$ converges.

In the previous exercise we saw that

$$\sum_{k=m}^{n} x_k y_k = s_n y_{n+1} - s_{m-1} y_m + \sum_{k=m}^{n} s_k(y_k - y_{k+1})$$

With $s_0 = 0$. So if $k = m = 1$, then the above simplies to

$$\sum_{k=m}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=m}^{n} s_k(y_k - y_{k+1})$$

Now, use the comparison test to argue that $\sum_{k=1} s_k(y_k - y_{k+1})$ converges absolutely, and show how this leads directly to a proof of Abel's test.

Let's look at the series in parts. Since (y_k) is monotonically decreasing and non-negative, then $y_k - y_{k+1}$ will always be equal to or greather than zero. As per s_k, since we know that (s_k) converges, we also know that the partial sums are bounded by some M. Thus

$$0 \leq \sum_{k=1} |s_k(y_k - y_{k+1})| \leq \sum M y_1 = M y_1 n$$

So our upper bound there doesn't tell us anything. Turns out that the trick is to look again at $\sum y_k - y_{k+1}$. If you write it out you'll see that it is a telescoping series (memorize the way it looks!). So we can refine our upper bound as such,

$$0 \leq \sum_{k=1} |s_k(y_k - y_{k+1})| \leq M(y_1 - y_n)$$

And since the upper bound is now a finite number, then by the comparison test, $\sum x_k y_k$ must converge as well.

2.7.14

Dirichlet's Test: Dirichlet's test for convergence states that if the partial sums of $\sum x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \ldots \geq 0$ with $\lim y_k = 0$, then the series $\sum x_k y_k$ converges.

The hypothesis for Dirichlet's test differs from Abel in the fact that we did not assume that $(y_k) \to 0$, instead we use the restructuring in terms of partial sums to create a telescoping series that gave us a finite upper bound. Dirichlet also doesn't require $\sum x_k$ to converge - which is how the alternating series test can be seen as a case of Dirichlet's test.

2.7 Double Summations and Products of infinite Series

2.7.1 Exercises

2.8.1

Compute $\lim s_{nn}$ for

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & -1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

This is the grid of real numbers $\{a_{ij} : i, j \in \mathbb{N}\}$, where $a_{ij} = 1/2^{j-i}$ if $j > i$, $a_{ij} = -1$ if $j = i$, and $a_{ij} = 0$ if $j < i$.

If we add row by row, then we can see that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

is the geometric series with $a = 1/2$ (since the geometric series is defined as $\sum_{k=0} ar^k$ and the first term must be $ar^0 = a$) and $r = 1/2$. That plus the starting term of -1 means that every row will add up to zero. So the entire sum is zero is we go this route.

On the other hand, if we add column by column, then we have the series

$$-1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \dots$$

Which is another geometric series, but this time with $a = -1$ and $r = 1/2$. So the double sum is now -2 (vs 0 in the previous case).

When we do the sum over squares, we end up doing the same sum as when we sum by columns.

2.8.2

Show that if the iterated series

$$\sum_{i=1} \sum_{j=1} |a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbb{N}$ the series $\sum_{j=1} |a_{ij}|$ converges to some real number b_i and the series $\sum_{i=1} b_i$ converges as well), then the iterated series

$$\sum_{i=1} \sum_{j=1} a_{ij}$$

converges.

Abbott once again gives us a great starting point. If we assume that $\sum_{j=1} |a_{ij}|$ converges then $\sum_{j=1} a_{ij}$ must also converge.

If we refer to $\sum_{j=1} a_{ij}$ as c_i, then $c_i \sim |s_{i,m+1} + \dots + s_{i,n}|$ which is less than $b_i \sim |s_{i,m+1}| + \dots + |s_{i,n}|$. Then we can assume that $\sum_{i=1} b_i$ converges, then so must $\sum_{i=1} c_i$ since $|s_{i,m+1} + \dots + s_{i,n}| \leq |s_{i,m+1}| + \dots + |s_{i,n}|$.

2.8.3

If we define

$$t_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|$$

and we assume that

$$t_{mn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges.

Then $\lim_{n\to\infty} t_{nn}$ exist since

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$$

And since t_{nn} converges, then we can use it as a Cauchy sequence.

By the same logic, and borrowing some of the notation from the previous exercise, since $c_i \leq b_i$, then $|t_{nn} - t_{mm}| = |b_{m+1} + \ldots + b_n|$ and in turn $|s_{nn} - s_{mm}| = |c_{m+1} + \ldots + c_n| \leq |b_{m+1} + \ldots + b_n| < \epsilon$.

2.8.4

Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$, when

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|$$

and

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}$$

The fact that $t_{mn} \leq B$ comes from the fact that B is the supremum. For the rest of the inequality it is useful to look back to **lemma 1.3.8** which stated the following: assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

Now show that there exists and N such that

$$|s_{mn} - S| < \epsilon$$

for all $m, n > N$.

Here we assume $S = \sup\{s_{mn} : m, n \in \mathbb{N}\}$.

There is an interesting consequence from the argument for t_{mn} in which if we find some $\epsilon > 0$, then

$$|t_{mn} - B| = \left|\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|\right| < \epsilon$$

From there,

$$
\begin{aligned}
|s_{mn} - S| &= \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} - \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \right| \\
&= \left| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij} \right| \\
&\leq \left| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} |a_{ij}| \right| \\
&= |t_{mn} - B| \\
&< \epsilon
\end{aligned}
$$

2.8.5

In the previous examples we have shown how that the absolute convergence theorem applies to double sums and that "rectangular" partial sums of t_{mn} and s_{mn} do exist. Along with showing the equivalence between the conventional notation for convergence and the Cauchy type convergence for double sums. Here, we aim to show that for double sums that absolutely converge, it doesn't matter if you sum by rows and then by columns, or vice versa, the limit for the double sums exist and any rearrangement gives you the same answer.

If we defined the sum of a row as

$$r_i = \sum_{j=1}^{n} a_{ij}$$

Show that for all $m \geq N$

$$|(r_1 + r_2 + \ldots + r_m) - S| < \epsilon$$

Another bit to keep in mind from our previous proofs is that $\lim r_i$ exists and we can make use of the algebraic limit theorem to perform the rest of the double sum and have some confidence that the sum of the limits for each row is the sum of the rows. Similarly, the order limit theorem will give us some confidence in stating that the sum of the rows - since each row is less than or equal to the simit of the double sums - will result in a number that is less than or equal to the double sum.

So we can make another argument for convergence by re-arranging the above

$$\begin{aligned} |(r_1 + r_2 + \ldots + r_m) - S| &= \left| \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \right| \\ &= \left| \sum_{i=1}^{m} \left(r_i - \sum_{j=1}^{n} a_{ij} \right) \right| \\ &\leq \sum_{i=1}^{m} \left| r_i - \sum_{j=1}^{n} a_{ij} \right| \end{aligned}$$

Here, in the last step, the term inside of the absolute value is another restatement of the limit of r_i, which since we know it converges can be restated as

$$|r_i - \sum_{j=1}^{n} a_{ij}| < \epsilon$$

So we can make it

$$|r_i - \sum_{j=1}^{n} a_{ij}| < \epsilon/m$$

And thus our previous set of inequalities ends as

$$\begin{aligned} |(r_1 + r_2 + \ldots + r_m) - S| &= \left| \sum_{i=1}^{m} \left(r_i - \sum_{j=1}^{n} a_{ij} \right) \right| \\ &\leq \sum_{i=1}^{m} \left| r_i - \sum_{j=1}^{n} a_{ij} \right| \\ &\leq m \frac{\epsilon}{m} = \epsilon \end{aligned}$$

The exact same argument can be made when we add by columns since our convergence argument for **2.8.3** still holds since we never specified any limitations on how the partial sums were to be made.

2.8.6

If we now perform the double summation along diagonals (think about diagonals in a matrix going left to right) where $i + j$ is equal a constant, for example

$$d_2 = a_{11} \quad d_3 = a_{12} + a_{21} \quad d_4 = a_{13} + a_{22} + a_{31}$$

and in general

$$d_k = a_{1,k-1} + a_{2,k-2} + \ldots a_{k-1,1}$$

Then $\sum_{k=2}^{\infty} d_k$ represents another way of performing the double summation.

Assuming everything up-to-this point is correct, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

What we can do here to show that the sum along diagonals converges absolutely is to note that for $n \geq 2$

$$\sum_{k=2}^{n} |d_k| \leq t_{nn}$$

Since t_{nn} contains all the terms in $\sum_{k=2}^{n} |d_k|$ and more. And since the $\sum_{k=2}^{n} |d_k|$ is monotonically increasing and bounded, then it converges.

Now, let's try and imitate our proof for theorem 2.8.1 to show that $\sum_{k=2} d_k$ converges to $\lim_{n\to\infty} s_{nn} = S$.

This one gets tricky. Our first attempt was to go this route

$$\begin{aligned}
|\sum_{k=2} d_k - S| &= |\sum_{k=2} d_k - s_{mn} + s_{mn} - S| \\
&\leq |\sum_{k=2} d_k - s_{mn}| + |s_{mn} - S| \\
&< |\sum_{k=2} d_k - s_{mn}| + \epsilon \\
&= |\sum_{k=2} d_k - \sum_{i}^{m} \sum_{j}^{n} a_{ij}| + \epsilon
\end{aligned}$$

Since we know that $\lim s_{mn} = S$, we could arange the above as the argument for a Cauchy series, i.e., something like $|s_{mn} - spq| < \epsilon$ since the diagonal sums do contain a decent deal of whats in s_{mn}, we just need to look far enough and make the diagonal as close as possible to s_{mn}. For example, the sum up to $d_{k=3}$ will contain all but 1 elements of s_{22}. Similarly, the sum up to $d_{k=4}$ will contain all but 3 elements of s_{33}.

Following that logic we can see how $|\sum_{k=2} d_k - \sum_{i}^{m} \sum_{j}^{n} a_{ij}|$ becomes arbitrarily small as $n \to \infty$ when $m = n$ since $\lim s_{nn}$ converges.

2.8.7

Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

Show that the iterated iterated sum $\sum_i \sum_j |a_i b_j|$ converges so that we may apply theorem 2.8.1.

$$
\begin{aligned}
\sum_i \sum_j |a_i b_j| &= \sum_i \sum_j |a_i||a_j| \\
&= \sum_i \left(|a_i| \sum_j |b_j| \right) \\
&= \sum_i |a_i| \sum_j |b_j| \\
&= A \sum_j |b_j| \quad \text{algebraic limit theorem for series } \sum_k ca_k = cA \\
&= AB
\end{aligned}
$$

Let $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_i \sum_j a_i b_j = \sum_j \sum_i a_i b_j = \sum_k d_k = AB$$

$$
\begin{aligned}
\lim s_{nn} &= \sum_i \sum_j a_i b_j \\
&= \lim \left(\sum_i a_i \right) \left(\sum_j b_j \right) \\
&= \lim s_n^a s_n^b \\
&= AB
\end{aligned}
$$

We already know that $\sum \sum |a_i b_j|$ converges to AB and we know that

$$\left(\sum_i a_i \right) \left(\sum_j a_j \right) = \sum_k d_k$$

and we know that $\lim_{n\to\infty} s_{nn} = AB$. Thus we can call upon theorem 2.8.1 to beleive that

$$\lim_{n\to\infty} s_{nn} = \sum_k d_k = \sum_i \sum_j a_i b_j = \sum_j \sum_i a_i b_j = AB$$

3 Basic Topology of R

3.1 Open and Closed Sets

The empy set $\emptyset$ is considered an open subset of the real line, becuase if it wans't it would imply that there is something in the emptyset that is "closed" (negation of "for all $a \in O$ there exist a $V_\epsilon \subseteq O$").

There is a theorem mentioned in topology that goes something along the lines of: the union of two non-disjoint intervals is an interval. The idea behind it being that in order for something to be an interval, it must must not have gaps between its endpoints. For example, the union of $[0,1] \cup [2,3]$ is not an internal since it doesn't include $[1,2]$.

Open balls or ϵ-neighborhoods make sense right away until you ought to give a proper argument. The way to read the definitions for these is to think that you always want a "ball" around some point, and you want to see if balls of any radius will be able to fit within another interval. For example, in Abbott example 3.2.2(ii), we take ϵ to be some positive value by looking at the difference between the point $x \in (c,d)$ and its left and right endpoints. ($x - c > 0$ and $d - x > 0$). (It must be a strict inequality because otherwise $|x - a| < 0$ would cause us trouble.) And that definition of ϵ gives you a simple way to look for open balls in (c,d).

To see the above, draw out a line and see how no matter what radius you end up using, as long as it is less than ϵ, all the open balls you draw will fit within (c,d).

Theorem 3.2.3

Note the interesting bit of logic being used here: since $O_{\lambda'}$ is open then $V_\epsilon(a) \subseteq O_{\lambda'}$. Since $O_{\lambda'}$ is some arbitrary member of $O = \cup_{\lambda \in \Lambda} O_\lambda$, then $O_{\lambda'} \subseteq O$. Putting these together $V_\epsilon(a) \subseteq O_{\lambda'} \subseteq O$.

Definition 3.2.4 By now, we've seen that ϵ-neighborhoods are equivalent to open balls,

$$V_\epsilon(a) \sim B(r = \epsilon, a) \sim \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}| < r(= \epsilon)\}$$

The neighborhood part comes from $x \in S^{int} = \{\mathbf{x} \in S : B(r, \mathbf{x}) \in S \text{ for some } r > 0\}$. Which is a math way of saying that a neighborhood consist of all points close to our center who are still members of the set in question.

Folland gives a conventional view of a closed set by first defining **boundary points** of a set as points where every ball centered on them contains points both in S and S^c, These points may then belong to either the set in question or its compliment ($x \in S \cup S^v$). And the set of all boundary points creates a boundary for S

$$\partial S = \{\mathbf{x} \in \mathbb{R}^n : B(r, \mathbf{x}) \cap S \neq \emptyset \text{ and } B(r, \mathbf{x}) \cap S^c \neq \emptyset\}$$

However, instead of following this route, Abbott goes onto to construct closed intervals with limit points which instead of relying on geometry, relies on our knowledge of series.

The definition of a limit point is literally the defintion of the limit of a sequence in terms of ϵ-neighborhoods as $\lim(a_n) = x \to |a_n - x| < \epsilon$.

Theorem 3.2.5

Notice how in the first part of the proof we are being told about an $\epsilon = 1/n$. This is because when we prove that a limit exist, we must always provide a relationship for an $\epsilon = \epsilon(n)$ so that $|a_n - x| < \epsilon$.

Example 3.2.9

The first example is $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, when $\epsilon = 1/n - 1/(n+1)$. Abbott helps us out here because otherwise we would have to figure out how to prove that 1.) the limit point is zero (but that is not in A), 2.) every other $x \in \mathbb{R}$ is either $1/n$ or it never intersects with other member of A (think about some $(1/n - \epsilon, 1/n + \epsilon)$).

In our case it helps to see it this way,

$$\begin{aligned}
v_\epsilon(1/n) = \{x \in \mathbb{R} : \left|x - \frac{1}{n}\right| < \epsilon\} &= \left(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon\right) \\
&= \left(\frac{1}{n} - \frac{1}{n} + \frac{1}{n+1}, \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}\right) \\
&= \left(\frac{1}{n+1}, \frac{2}{n} - \frac{1}{n+1}\right) \\
&= \left(\frac{1}{n+1}, \frac{2n+2-n}{n(n+1)}\right) \\
&= \left(\frac{1}{n+1}, \frac{n+2}{n(n+1)}\right) \\
&= \left(\frac{n}{n(n+1)}, \frac{n+2}{n(n+1)}\right)
\end{aligned}$$

The only number we can put in between that interval, following our requirements, is $\frac{n+1}{n(n+1)} = \frac{1}{n}$.

Theorem 3.2.13 In the first part of the proof, we want to prove that O^c contains all of its limit points. Containing all of its limits points means that every ϵ-neighborhood of any limit point would contain other points in O^c. That is why if $x \in O$ would lead to a contradiction: if a single $x \notin O^c$, then at least one of its $V_\epsilon(x) \subseteq O$ (O and O^c have no members in commmon, so if an ϵ-neighborhood is contained in one then it cannot contain points in the other.)

When reading the converse statement, again, keep in mind that the concept of limit points is to look for neighborhoods that are fully contained within a set, every $o \in v_\epsilon$ is also $x \in O$ if $v_\epsilon \in O$. So looking at a limit point that is not part of O^c, means that $v_\epsilon(x) \in O$ by definition.

3.1.1 Exercises

3.2.1

(a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be finite gets used?

Theorem 3.2.3 states that: the intersection of a finite collection of open sets is open. The "finite-ness" is used when we look for the smallest ϵ-neighborhood contained in every O_k. If we assume that we are looking at the intersection of an infinite ammount of open sets, then the nested interval property doesn't hold and we cannot guarantee that there is an element contained within the inifity of neighborhoods (it may or mat not exist). See 1.4.2.

(b) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \ldots\}$ whose intersection $\cup_{n=1}^{\infty} O_n$ is closed, not empty and not all of $\mathbb{R}$.
TODO

3.2.2
Let

$$A = \left\{(-1)^n + \frac{2}{n} : n \in \mathbb{N}\right\},$$

$$B = \{x \in \mathbb{Q} : 0 < x < 1\},$$

and

$$C = \left\{\frac{(-1)^n n}{n+1} : n \in \mathbb{N}\right\}$$

(a) what are the limit points?

Let's plugin a couple numbers! In A, when $n = 1$ we got $-1 + 2 = 1$; when $n = 2$, $1 + 1 = 2$; $n = 3$, $-1 + 2/3 = -1/3$; $n = 4$, $1 + 1/2 = 3/2$; so $A = \{1, 2, -\frac{1}{3}, \frac{3}{2}, -\frac{3}{5}, \ldots\}$. Holistically, the first term oscillates between -1 and 1, while the second one is monotonically decreasing starting at 2 and then decreasing to 0 as n grows unbounded. Since we are adding or subtracting monotonically decreasing terms to -1 and 1, and since the first two terms are 1 and 2, then all other values will oscillate between the two, getting closer and closer to both of those values. So our limit points here are 1 and -1. (they are literally the limit of a sequence contained in A.)

Now, B. Back in **example 3.2.9**, we saw that a property of $\mathbb{Q}$ is that its set of limit points is **ALL** of $\mathbb{R}$. So the set of limit points for B is not just 0 and 1 but the entire interval $[0, 1]$.

Finally C. Again, plugin in some numbers, $n = 1$; $-1/2$; $n = 2$, $2/3$, etc. So, we get $C = \{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots, \frac{10}{11}, -\frac{11}{12}, \ldots\}$. We again see that the elements of C approach -1 and 1 as n grows without bounds. So the limit points are -1 and 1.

Let's do a bit more work for C, just because. We can see that a limit point for C is 1 because the even terms grow as $(\frac{n}{n+1})$. The difference between consecutive even terms is $\left|\frac{n}{n+1} - \frac{n+2}{n+3}\right| = \left|\frac{3(n+3)-(n+1(n+2))}{(n+1)(n+3)}\right| = \left|\frac{-2}{(n+1)(n+3)}\right| = \frac{2}{(n+1)(n+3)}$. We want to use this data to verify that every ϵ-neighborhood of a limit point does indeed intercept our set C at some point other than the limit point we are looking at. Which means that we want to see $V_\epsilon(1) \cap C \neq \emptyset$ and $V_\epsilon(1) \cap C \neq 1$. We can see $V_\epsilon(1) \cap C \neq \emptyset$ right away since $\frac{n}{n+1} \neq 1$ ever. Leaving us with the task of verifying that $\{x \in \mathbb{R} : |x - 1| < \epsilon\} \cap C \neq \emptyset$ for all $\epsilon > 0$. The previous expression can in turn be simplified to $\{c_n \in C : |c_n - 1| < \epsilon\}$ **which just magically became a restatement of the criteria for the convergence of a series!** - think about it, we want to find values in our neighborhood that intercept with C, so instead of searching through all of $\mathbb{R}$ we could search through C. This onw reduces to $\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \epsilon$. Since $\frac{1}{n+1} > \frac{2}{(n+1)(n+3)}$, the distance between two terms in the series, we should be able to find elements of C within our ϵ-neighborhoods.

Note that in all of these cases we could readily evaluate the limits to get the limit points because the limit points are not actually memebers of the sets.

(b) Are the sets open or closed?

A is not closed because it doesn't contain all of its limit points, $-1 \notin A$. But A is also not open since every $V_\epsilon(a), a \in A$ will have some irrational number that is not part of A, so there is not an $V_\epsilon(a) \subseteq A$ (tere isn't a neighborhood that is completely within A).

Similarly, B is not closed, because it doesn't not contain its limit points, and also not open because every $V_\epsilon(b), b \in B$ is not completely contained within B (continuous ϵ-neighborhoods are not a subset of B).

Same story for C.

(c) Do the set contain isolated points?

Every point in A except 1 and -1 are isolated points.

In the case of B, B is dense in $[0, 1]$ - every $b \in B \subseteq [0, 1]$ - so $B \setminus [0, 1] = \emptyset$, so B has no isolated points.

Finally, all elements in C are isolated points.

(d) Find the closure of the sets.

Our closures are, $\bar{A} = A \cup \{-1, 1\}$, $\bar{B} = [0, 1]$, and $\bar{C} = C \cup \{-1, 1\}$.

3.2.3

Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbb{Q}$

Abbott already shared with us that all the reals are the limit points for $\mathbb{Q}$. So $\mathbb{Q}$ is not closed. But we also already noted in the previous exercise that continuous ϵ-neighborhoods are not contained in $\mathbb{Q}$ - for all elements in $\mathbb{Q}$, it is not the case that there is an ϵ-neighborhood that is contained within $\mathbb{Q}$, $V_\epsilon(q) \not\subseteq \mathbb{Q}$.

Another way of putting it: $\mathbb{Q}$ is not is not open because irrationals in the ϵ-neighborhoods are not part of $\mathbb{Q}$. And $\mathbb{Q}$ is not closed because it contains irrational limit points.

(b) $\mathbb{N}$

It is not open by the same logic as above. $\mathbb{N}$ has no limit points, but it is considered closed.

Think about the two definitions for limit points that we have: there is no sequence contained within $\mathbb{N}$ with a limit and also, it is not the case that every ϵ-neighborhood intercepts $\mathbb{N}$.

(c) $\{x \in \mathbb{R} : x \neq 0\}$

$\mathbb{R} \setminus \{0\}$ is open because every element has an ϵ-neighborhood that is contained within $\mathbb{R} \setminus \{0\}$. It is not closed because 0 is a limit point but it is not included - for example, $1/n \to 0$.

(d) $\left\{\sum \frac{1}{n^2} : n \in \mathbb{N}\right\}$

The limit of the sum is $\pi^2/6$, its elements are $\{1, 5/4, \ldots\}$. So similar sitation as above, it is not closed because it doesn't contain its limit. It is also not open since it only contains rationals.

(e) $\left\{\sum \frac{1}{n} : n \in \mathbb{N}\right\}$

Similar logic to the previous answer but this time the set is closed since $\sum 1/n$ doesn't converge, so there is no limit point. It is not open for the same reasons as above.

3.2.4

Let A be nonempty and bounded above so that $s = \sup A$ exists.
(a) Show that $s \in \bar{A}$

Interestingly, this exercise is a great connection to 1.3 where we find out that if a set is closed then the supremum of the set is a member of the set.

In general, $\bar{A} = A \cup L$, where L is the set of limit points. So one way to try and prove this is to see whether $s \in L$, or in general $s \in A$. If you remember **lemma 1.3.8**, it states that if $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$, then $s = \sup A$ if and only if, **for every** choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

Now, if you remember the definition of a limit point, it states that a limit point is a limit point if every ϵ-neighborhood of a limit point intercepts the set A at some point other than itself. Using lemma 1.38, we can see then that every $V_\epsilon(s)$ will contain some a such that $s - \epsilon < a$. Hence, **we can think of the supremum as a limit point**.

(b) Can an open set contain its supremum?

This is a great exercise because it gives us the opportunity to go from hypothesis to proof. Based on what we know about supremums, it would be reasonable to doubt that an open set will contain its supremum. So let's try and prove that directly.

If A is an open set, then it means that every $a \in A$ has a $V_\epsilon(a)$ such that $V_\epsilon(a) \subseteq A$. That is, there is at least one ϵ-neighborhood for every element of A that is equal to or a subset of A. Which means that at least one ϵ-neighborhood contains to elements external to A. This means that if we wanted to show that $s \notin A$ then we would need to show that every $V_\epsilon(s)$ contains elements that are outside of A. To see this, remember that s is the least upper bound, so any $s + \epsilon$ will be an upper bound but it will be outside the range of A. If $s + \epsilon \in A$, then it would mean that $s \neq \sup A$.

3.2.5

Theorem 3.2.8 A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also contained in F.

Quick reminder, a Cauchy sequence (a_n) is one which for every $\epsilon > 0$ $\exists N \in \mathbb{N}$ such that whenever $m, n \geq N$ $|a_n - a_m| < \epsilon$. And we also know that every convergent sequence is also a Cauchy sequence.

Let's assume that every Cauchy sequence we know is contained in F and their limits are too contained in F. Since Cauchy sequences are convergent sequences, then it means that all the limit points we can find will be contained in

F, thus meeting the defintion of a closed set.

The above argument was the proof for the backwards direction, now let's go forward. Assume that F is closed. Since F is closed, then it must contain all of its limit points. Since every limit point, l, is the limit of some sequence contained in F, then it means that we have $\lim(f_n) = l$. And since every convergent sequence is a cauchy seuqnece, then we can state our theorem.

3.2.6

TODO: it asks whether the Cantor set is closed, which is an interesting exercise.

3.2.12

Let A be an uncountable set and let B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both $\{x : x \in A, x < s\}$ and $\{x : x \in A, x > s\}$ are uncountable. Show that B is nonempty and open.

We are asked for a couple things right from the begining. First, $A \not\sim \mathbb{N}$ and $B \subseteq \mathbb{R}$. Furthermore, $B \neq \emptyset$ and is open. Remainder, B being open means that for every $s \in B$, $B \subseteq \mathbb{R}$ (we got that) and $V_\epsilon(s) \subseteq B$ for some $\epsilon > 0$. So for B to be open it must be a subset of the reals and every single one of its members must have an ϵ-neighborhood completely contained within B. Yet another way of saying this is that $\forall s \in B$, $(s - \epsilon, s + \epsilon) \in B$, for some $\epsilon > 0$. (Finding a single $s \in B$ whose $V_\epsilon(s)$ was not contained in B would also serve our purposes in someway.)

Before we go and dive into what we think are solutions, there is another very important thing to mention. Notice how the problem satets "let B be the set of real numbers that divides A into two uncountable sets". In a way this means that B is sort of like a function - we don't chose it, instead we feed it some uncountable A and out comes a B that divides the A into two uncountable sets.

One argument that can eb made is as follows: let's call $C_1 = \{x : x \in A, x < s\}$ and $C_2 = \{x : x \in A, s < x\}$. To show that $B \neq \emptyset$, consider $x \in C_1$ and $y \in C_2$. Since C_1 and C_2 are uncountable, then there are nonempty and thus $(x + 2)/2 \in B$.

Now to show that B is open consider $(-\infty, s)$ and (s, ∞) when $s \in B$. We know there must exist an $x \in (-\infty, s)$ and a $y \in (s, \infty)$. So we can form an open interval (x, y). Since A, C_1, and C_2 are uncountable, we can have an inifity of such intervals. If we look at the midpoints between any x and any y, then we will be able to obtain some $s \in B$. From these we can form our open interval as $(s-\epsilon, s+\epsilon) \in B$. To make the last connection revisit example 3.2.2 (ii).

There are also some great answers in the internet that merit discussion. Let's take a look at Show a set is nonempty and open. There is a particular argument made there that pretty much goes as follows: if B is not open, then $\forall \epsilon > 0$, there would exists an $x_\epsilon \in (x - \epsilon, x + \epsilon)$ such that either $(-\infty, x_\epsilon) \cap A$ or $(x_\epsilon, \infty) \cap A$ is countable.

The thing that makes this worth mentioning is that it serves as remmainder that is not necessarily the case that if a set is not open then it is close. As some of the previous exercsies showed us, sets can be neither open nor close when they contain "holes".

As for the rest of the argument made in the above matheschange link, along with A is uncountable and B divides A in two uncountable sets. Show that B is nonempty and open and Show that an uncountable set can be separated into two disjoint uncountable subsets relie on the following principles.

Consider, $B_1 = \{x \in \mathbb{R} : (-\infty, x) \cup A \text{ is uncountable}\}$. The way to read this is that B_1 is the set of xs such that an interval $(-\infty, x)$ overlaps with A resulting in an uncountable set of numbers within their overlap. For example, if $A = [0, 1]$, then x could be 1, but it could also be 2, 3, etc., $(-\infty, 1), (-\infty, 2), (-\infty, 3)$ all result in an uncoutable set when we look at their interception with A. Also, because it is the case that for any y that is $y > x$ for any $x \in B_1$, then B_1 is **upward closed** ($(-\infty, x)$ is the upper set).

Following the same line of arguments, $B_2 = \{x \in \mathbb{R} : (x, \infty) \cup A \text{ is uncountable}\}$ corresponds to a **downward closed set** where for any $x \in B_2$ if $x > y$ for some $y \in \mathbb{R}$, then $y \in B_2$.

The rest of the argument relies on something we proved back in 1.5.1, which is that the union of countable sets is a countable set. Which should point us to a proof by contradiction since we have been talking about uncountable sets. The thing to note is that since B_1 is upward close, then for any $x \in B_1$, $x + \epsilon \in B_1$, so we only need to make an argument for $x - \epsilon \in B_1$. And the proof by contradiciton goes as follows: if it is not the case that $x - \epsilon \in B_1$ for some $\epsilon > 0$, then $(-\infty, x - 1/n) \cup A$ must be countable for all $n \in \mathbb{N}$ (if not open, then there is a whole, and the whole is because we have some countable amount of elements between $x - 1/n$ and x).

This in turn means that

$$\bigcup_{n \in \mathbb{N}} \left(-\infty, x - \frac{1}{n}\right) \cup A = (-\infty, x) \cup A$$

is countable, and we land in a contradiciton.

Last things to mention are that $B_1 = (-\infty, b_1)$, where b_1 is either $\inf B_1$ or $-\infty$. Similarly, B_2 is (b_2, ∞), with b_2 either $\sup B_2$ or ∞. Therefore $B_1 \cup B_2 = \mathbb{R}$, $b_1 > b_2$, and $B = B_1 \cap B_2 \neq \emptyset$.

3.2.13

Prove that the only sets that are both open and closed are $\mathbb{R}$ and $\emptyset$.

The way to do these sorts of proves is to look for a contradiction, but before we tak that path, let's verify our conditions. Starting out simple, $\emptyset$. The $\emptyset$ is both open and close by matter of convention, notation, etc.

Now $\mathbb{R}$. $\mathbb{R}$ is open because every element of the reals has an ϵ-neighborhood that contains only reals. The reals are also closed because any limit point that it can have will be a real itself, so all limit points of $\mathbb{R}$ are members of $\mathbb{R}$.

Now, let's fish for contraditions. One way is to use what we found in 3.1.1, that is: open sets cannot contain their supremum (or infinimum). So let's say there is some A that is both open and closed but it is not $\mathbb{R}$ or $\emptyset$. If this is the case, then A must be unbounded (a closed set would contain its supremum, but an open set would not, thus one would think there is no supremum, hence the set would be unbounded). And if A has continuous [ϵ-neighborhoods] elements, then it will be open. But since A is closed, then it must also contain all of its limit points, So A is a set that is unbounded, contains all of its limit points, and has continuous ϵ-neighborhoods. The only set having that property are the reals.

Another route of argument is to note that if A is open and closed, then A^c must also be open and closed. Furthermore, if we want $A^c \neq \emptyset$, and again we run into trouble because A^c must also be unbounded but contain all of its limit points while not being equal to the reals. And also worth mentioning that the limit points don't have ϵ-neighborhoods contained within A or A^c for that matter.

3.2.14

A dual notion to the closure of a set is the interior of a set. The interior of E is deonted E^o and defined as

$$E^o = \{x \in E : \exists V_\epsilon(x) \subseteq E\}$$

Results about closures and interiors posses a useful symmetry.

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^o = E$.

We know that $\overline{E} = E \cup L$, where L is the set of limit points of E. If we have $\overline{E} = E \cup L = E$, then $L \subseteq E$ (E must contain all of its limit points). Thus we agreement with out previous defintion of closed sets.

Similarly, if $E^o = E$, then it means that every $x \in E$ has an ϵ-neighborhood that is contained in E, again matching our definition.

(b) Show that $\overline{E}^c = (E^c)^o$, and similarly that $(E^o)^c = \overline{E^c}$.

First, let's shake it and see what comes out

$$\overline{E}^c = (\overline{E})^c = (E \cup L)^c = E^c \cap L^c$$

If E is closed, then $E = \overline{E}$, and $E^c = (E^c)^o$. So right away we can simplify $\overline{E}^c = (\overline{E})^c = E^c = (E^c)^o$.

Now, if E is open, then $E = E^o$, and $E^c = \overline{(E^c)}$. Doing the same sort of manipulation as above would actually give us the second equality we are looking to prove. So great, but let's try something else so we can prove this second case. If we look at $\overline{E}^c = (\overline{E})^c = (E \cup L)^c$, then we are looking into the universe of elements that do not belog to E and are not limit points of E. Since these elements are do not intercept E, then they are contained within E^c. However, being contained within E^c does not mean that there is an $V_\epsilon(x) \subseteq E^c$ for every $x \in E^c$. This is only the case if $x \in (E^c)^o$, otherwise, there will be areas of E where an ϵ-neighborhood will contain points outside of E^c.

3.2.15

A set A is called an F_σ set if it can be written as the countable union of closed sets. A set B is called a G_δ set if it can be written as the countable intersection of open sets.

This exercise is lovely, first, because we saw in **theorem 3.2.14 (i)** that the union of a finite collection of closed sets is closed. And then in **theorem 3.2.3 (ii)** that the intersection of a finite collection of open sets is open.

Since a countable amount of sets can be either finite or inifinite, it must mean that we can form sets F_σ and G_δ by looking at a countably inifite amount of sets in each case. Otherwise we would end with contradicitons.

Some great pages to read are wiki page on F_σ sets and wiki page on G_δ sets.

(a) Show that a closed interval $[a, b]$ is a G_δ set.

The interesting bit here is that we have to figure out how to use the intersection of a countable amount of open sets to form a closed set. The countable amount must be infitine, as otherwise a finitie intersection of open sets results in an open set.

With this countably infinite intersection of open sets, we must be able to create a set that contains all of its limit points.

Approaching this problem "geometrically" helps a tad: open sets contain neighborhood that are completely contained within themselves but do not contain any of their limits. Requiring at least one ϵ-neighborhood for each member of an open set to be contained within itself gives us a sense of "continuity" - a lack of "holes" - while a lack of limit points means that we can get close to them and as the intersection of open sets looks for an inifite amount of open sets, for the limit points should show uo themselves as the limit of an inifite number of open sets.

In short, something like

$$\bigcap_{n\in\mathbb{N}} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$$

Could just be the trick we need since each individual set is open but as $n \to \infty$, then $1/n \to 0$, so the endpoints of the open intervals get closer and closer to a and b and they always contain a and b.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

Similar logic as above, we could have the G_δ set

$$\bigcap_{n\in\mathbb{N}} \left(a, b + \frac{1}{n}\right)$$

or the F_σ set

$$\bigcup_{n\in\mathbb{N}} \left[a + \frac{1}{n}, b\right]$$

For the case of the F_σ set, note how we look at $a + \frac{1}{n}$, so in this case we are always using an endpoint to the right of a, so the lower bound never contains a but it approaches it as $n \to \infty$.

(c) Show that $\mathbb{Q}$ is an F_σ set, and the set of irrationals $\mathbb{I}$ forms a G_δ set. (We will see in section 3.5 that $\mathbb{Q}$ is not a G_δ set, nor $\mathbb{I}$ is an F_σ set. See 3.4)

Since $\mathbb{Q}$ is countable, then $\cup_{q\in\mathbb{Q}}\{q\} = \mathbb{Q}$.

To see that the irrationals form a G_δ set, let's take the compliment of the above,

$$\mathbb{I} = \mathbb{Q}^c = (\cup_{q\in\mathbb{Q}}\{q\}) = \cap_{q\in\mathbb{Q}}\{q\}^c$$

To finish the argument quickly, note that since $\{q\}$ is closed, then $\{q\}^c$ must be open.

Now, to carry the conversation into a future topic, see

1. are singletons always closed?
2. Are Singleton sets in R both closed and open?

The general argument for why a singleton is closed in the topology of $\mathbb{R}$ is quite interesting. To show that $\{x\}$ is closed, you want to argue that $\mathbb{R} \setminus \{x\}$, its complement, is open. Again, in an open set, every member has an ϵ-neighborhood contained within their set. So we want to see that for any $a \in \mathbb{R} \setminus \{x\}$, $V_\epsilon(a) \subseteq \mathbb{R} \setminus \{x\}$.

If we look at the $V_\epsilon(a)$ where $\epsilon = |a - x|$, then we are looking at the open interval $(a - \epsilon, a + \epsilon)$. Since ϵ is the distance from a to x, then $a \pm \epsilon$ puts us right on-top of x, but since an ϵ-neighborhood is an open interval, then $x \notin (a - \epsilon, a + \epsilon)$, meaning that $V_\epsilon(a) \subseteq \mathbb{R} \setminus \{x\}$. Hence $\mathbb{R} \setminus \{x\}$ is open.

The logic to the above argument is similar to Example 3.2.2 (ii).

3.2 Compact Sets

3.2.1 Exercises

3.3.1

Show that if K is compact and nonempty then $\sup K$ and $\inf K$ both exist and are elements of K.

Being compact means that K is closed and bounded, and a bounded set of real numbers will have a supremum and infinitum (axiom of completeness).

Furthermore, since the supremum and infinimum are limit points and are thus contained within K. Lemma 1.3.8 says that for every $\epsilon > 0$, there exists an element $a \in K$ such that $s - \epsilon < a$. So we can get arbitrarily close to the supremum s and form a sequence such that $|a - s| < \epsilon$.

3.3.2

Decide which of the following sets are compact. For those that are not compact, give an example of a sequence contained in the given set that does not posses a subsequence converging to a limit in the set.

Note that this problem is similar to 3.1.1.

(a) $\mathbb{N}$

The set of natural numbers is not compact. The natural numbers themselves are a seuqnece that does not converge. A simple series to show this, $n, n \in \mathbb{N}$.

(b) $\mathbb{Q} \cap [0, 1]$

Remember that the reals are the limit points of the rational numbers. However, the intersection of the rationals and the closed interval $[0, 1]$ is the set of rationals in the interval $[0, 1]$ (including 0 and 1). Since the intersection does leave us with the endpoints then we do have some of the limit points but not all of them (since that would require the interval to have all of the reals between 0 and 1 as well).

A simple series to make our point, $(x_n) \to 1/\sqrt{2}$.

(c) The cantor set

This one is a bit more surprising, one could think that maybe the cantor set is not compact becuase maybe the reals are the limit points for it, as the case for the rational numbers, but this is not the case!

First, the cantor set is bounded since $C \subset [0,1]$. Furthermore, the cantor set, C, is made up of an infitine intersection of closed sets (Abbott gives us that hint). And as per theorem 3.2.14 (ii), the intersection of an arbitrary collection of closed sets is open. Hence, the cantor set is bounded and close and thus compact.

But before we move on, there is another lovely argument to be made, The cantor set is defined as

$$C = [0,1] \setminus \left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \ldots \right]$$

Its complement is

$$C^c = (-\infty, 0) \cup (1, \infty) \cup \left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \ldots \right]$$

This time we get to use theorem 3.23 (i) which says that the union of of an arbitrary collection of oepn sets is open. And since C^c is open, then C must be closed.

(d) $\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$

This is one of our favourite p-series. Since $p > 1$, then it converges, and this one in particular converges to $\pi^2/6$. Since the series converges, then its bounded. Since its bounded, then by the Bolzano-Weierstrass theorem there is a subsequence that converges. The only thing we need if for the limit to be within the set.

One could make the argument that based on the criteria for convergence, if we find an n that is sufficiently large then the limit point "is in". However this argument falls short. The sum approaches its limit, but the sum of rationals is not exactly equal to the irrational number.

(e) $\left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

This sequence converges to 1, and since 1 is in the set then it is closed. It is also bounded. Hence it is compact.

3.3.3

Prove the converse of theorem 3.3.4 by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

For K to be compact, every sequence in K must have a subsequence that converges to a limit that is also in K.

We know that K is bounded, so there exists an $M > 0$such that $|a| < M$ for all $a \in K$. Because every $|a| < M$, then every sequence in K is bounded. Then, the Bolzano-Weierstrass theorem then tells us from any of such sequences that we can form, there will be a subsequence that will converge to some limit.

Furthermore, because a closed set contains all of its limit points, then every limit from any of the convergent subsequences that can be made will be contained within K. This last statement is the defintion of compactness.

3.2.4

Assume K is compact and F is closed. Decide if the following sets are compact, closed, both, or neither.

(a) $K \cap F$

As per theorem 3.2.14 (i), the union of a finite collection collection of closed sets is closed. So this union is closed. Since K is bounded, then the intersection of the two sets will also be bounded. Thus the intersection is compact.

(b) $\overline{F^c \cup K^c}$

F^c is open, and K^c is open and unbounded ($[0,1]^c = (-\infty, 0) \cup (1, \infty)$). The union of an arbitrary collection of open sets is open. Here comes an interesting bit, the complement of a bounded set is an unbounded set but the complement of an unbounded set is not necessarily a bounded set (i.e., $(0, \infty)^c = (-\infty, 0]$). So $F^c \cup K^c$ is open and could be bounded or unbounded.

The closure then is the union of $F^c \cup K^c$ and its set of limit points. So the set is closed but not necessarily bounded.

(c) $K \setminus F = \{x \in K : x \notin F\}$

Since $K \setminus F = K \cap F^c$, we know right away that the resulting set is bounded. However, we have the intersection of a closed set and an open set so we need more information to give a defintie answer. Some examples, if $K = F = [0,1]$, then $K \setminus F$ is open. - we are taking away the limit points. If $K = [0,1]$ and $F^c = (2,3)$, then $K \setminus F$ is empty, so it is open and closed.

(d) $\overline{K \cap F^c}$

Borrowing from our previous answer, we have the intersection of a closed and bounded set with an open set. So the resulting set is bounded. We got this far last time, but this time we are looking at the closure of the same set, and since the closure of a set always results in a closed set, then we end up with something that is both bounded and closed, thus compact.

3.3.7

As some more evidence of the surprising nature of Cantor set, follow these steps to show that the sum $C + C = \{x + y : x, y \in C\}$ is equal to the closed interval $[0, 2]$. (Keep in mind that C has zero length and contains no intervals.)

Because $C \subseteq [0, 1]$, $C + C \subseteq [0, 2]$, so we only need to prove the reverse inclusion $[0, 2] \subseteq C + C$. Thus, given $s \in [0, 2]$, we must find two elements $x, y \in C$ satisfying $x + y = s$.

(a) Show that there exists $x_1, y_1 \in C_1$ for which $x_1 + y_1 = s$. Show in general that, for an arbitrary $n \in \mathbb{N}$, we can always find $x_n, y_n \in C_n$ for which $x_n + y_n = s$.

Finding an $x, y \in C_0$ such that $x + y = s$ is the trivial case. Now, to find an $x_1, y_1 \in C_1$.

$C_1 = [0, 1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. And with $C_1 + C_1 = \left(\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]\right) + \left(\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]\right)$ we can cover the entire range of $[0, 2]$ including the middle third we took out because any $x_1 \in \left[0, \frac{1}{3}\right]$ added to any $y_1 \in \left[0, \frac{1}{3}\right]$ will cover the range of $\left[0, \frac{2}{3}\right]$.

Similarly, any $x_1 \in \left[0, \frac{1}{3}\right]$ added to any $y_1 \in \left[\frac{2}{3}, 1\right]$ will cover the range of $\left[\frac{2}{3}, 1\frac{1}{3}\right]$.

Lastly, any $x_1 \in \left[\frac{1}{3}, \frac{2}{3}\right]$ added to any $y_1 \in \left[\frac{2}{3}, 1\right]$ will cover the range of $\left[1\frac{1}{3}, 2\right]$.

so, from our possible combinations we end up with $x_1 + y_1 \in \left[0, \frac{2}{3}\right] \cup \left[\frac{2}{3}, 1\frac{1}{3}\right] \cup \left[1\frac{1}{3}, 2\right] = [0, 2]$. symbolically, $C_1 + C_1 = [0, 2]$. This forms the base case for induction. Now we need to formulate the inductive step.

To help us take the inductive step, there is an interesting pattern to notice. For example, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ and $C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right)$. If you compute $3 \cdot C_2$ you'll get $3C_2 = \left(\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]\right) \cup \left(\left[2, 2\frac{1}{3}\right] \cup \left[2\frac{2}{3}, 3\right]\right)$. Which we can rewrite as $3C_2 = C_1 \cup (2 + C_1)$. So if we symbolically repeat the steps we performed in the base step, we get

$$\begin{aligned}
3C_2 + 3C_2 &= (C_1 \cup (2 + C_1)) + (C_1 \cup (2 + C_1)) \\
&= (C_1 + C_1) \cup (C_1 + (2 + C_1)) \cup (4 + C_1 + C_1) \\
&= [0, 2] \cup [2, 4] \cup [4, 6] \\
&= [0, 6]
\end{aligned}$$

Note in particular the middle term that resulted in the interval [2.4]. If we were doing a simple $(x+y)+(x+y)$ then we just get $2x+2y$. But in the case of $(x \cup y)+(x \cup y)$ it equals $(x+x) \cup (x+y) \cup (x+y) \cup (y+y) = (x+x) \cup (x+y) \cup (y+y)$. We made the previous reasoning by thinking about $a + (x \cup y)$ being equal to $a + x \cup a + y$ and generalizing from there.

To complete the inductive step, generalize the above by changing c_1 to C_n and C_2 to C_{n+1}.

(b) Keeping in mind that the sequences (x_n) and (y_n) do not necessarily converge, show how they can nevertheless be used to produce the desired $x, y \in C$ satisfying $x + y = s$.

(x_n) and (y_n) may not necessarily converge since the xs and ys are being picked from non-continuous intervals.

Back in problem 3.2.1, we showed that the cantor set is compact, So every sequence in C has a subsequence that converges to a limit that is also in C. So we can chose some (x_{n_k}) and some (y_{n_k}) such that they converge to x and y respectively. Since x and y are contained in $C \subseteq C$, then $x + y = s \in [0, 2]$.

A similar line or argument can be made by noting that since C is bounded, then (x_n) and (y_n) are bounded, and the Bolzano-Weierstrass theorem tells us that we can find a subsequence that does converge. From there, the same reasoning can be reused.

3.3.8

Let K and L be nonempty compact sets, and define

$$d = \inf\{|x - y| : x \in K, y \in L\}$$

This turns out to be a reasonable definition for the distance between K and L.

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

We'll combine the answer to this with the answer to the next question.

(b) Show that it's possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.

These two questions are super interesting. Conventional experience would have us think that as long as two sets are disjoint, then $d > 0$ since the two sets do not share any elements so $x \neq y$. However the second question gives us a hint: when the two sets are closed then somehow someway $d = 0$. Which makes us think that being able to have a zero "distance" is somehow possible due to the difference between a pair of sets that are closed and bounded and other pair that is just closed.

Because we are comparing a pair that is closed and bounded with a pair of sets that is just closed, we may be thinking of the same boring old set of sets in which to be closed you must be bounded. However, if we think back to limit points, we can have a set that is closed if the set doesn't have any limit points (it doesn't have to contain its limit points if these don't exist).

For example, if $K = n : n \in \mathbb{N}$ and $L = \{n + \frac{1}{n} : n \in \mathbb{N}\}$, then neither set has limit points. And if there are no limit points then they can be considered as closed.

However, if K and L are compact, then any sequence contained in them has a subsequence that converges to some limit that is also contained within them. From there, you can define $d = \lim_{n\to\infty} |x_n - y_n| = |x_0 - y_0|$.

3.3.9

Follow these steps to prove the final implication of theorem 3.3.8.

Assume K satisfies i) K is compact and ii) K is closed and bounded, and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiciton, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

One way to do this is to follow the sort of thing we saw in Example 3.3.7, since we saw that by dividing an interval we could obtain subsets that had no open cover. So we can define $|I_n| = |I_0|/2^{n-1}$ (we do $n-1$ so that $I_{n=1} = I_0$).

Again, following the sort of reasoning we saw in example 3.3.7, we can see that these $I_n \cap K$ cannot be finitely covered and that $\lim I_n = \lim |I_0|/2^{n-1} = 0$.

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

The nested interval property tells us that there msut exist some $x \in I_n$ for all n. So we need to show that that x is also in K. This additonal detail comes from looking at our previous answer: since K is compact, then $I_n \cap K$ will also be compact, and now we can use the nested compact set property which tells us that there is an $x \in I_n \cap K$ for all n, so $x \in K$ and $x \in I_n$.

(c) Because $x \in K$, there must exists an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiciton.

We began this whole journey assuming that there was an open cover for K, so as there must be an $x \in I_n \cap K$, so too must there be an $x \in O_{\lambda_0}$. Now, since $\lim |I_n| = 0$, then we should be able to find some $I_{n_0} \subseteq O_{\lambda_0}$, that is an I_{n_0} that fits in O_{λ_0}. Which finally leads us to a contradiction because we assumed that $I_n \cap K$ could not be finitely covered.

3.3.10

Here is an alternate proof to the one from given in Exercise **3.3.9** for the final implication in the Heine-Borel theorem.

Consider the special case where K is a closed interval. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$.

We start with knowledge that $\{O_\lambda\}$ is an open cover for $[a, b]$. Meaning that $[a, b] \subseteq \bigcup_{\lambda \in \Lambda} Q_\lambda$. Remember that the union of an arbitrary number of open sets is itself an open set, so an open cover is itself open. Furthermore, every O_λ is open, meaning that for every $a \in O_\lambda$, there exists and $V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} \subseteq O_\lambda$.

Thus, since $x \in [a, b] \subseteq \bigcup_\lambda Q_\lambda$, then there exists $V_\epsilon(x) \subseteq \bigcup_\lambda Q_\lambda$ for all $x \in S$.

(a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exist.

S is nonempty since $x = b$ has the entire cover and $x = a$ must have some cover $\{O_\lambda\}$, as long as $a\ in\{O_\lambda\}$ for some $\lambda \in \Lambda$.

Furthermore, S is also bounded since our cover if for $[a, b]$ and $x \leq b$ for all $x \in S$.

(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

Making an allusion to the definiton of the supremum: let's say that $s < b$, then because $[a, b]$ does have an open cover, then these should be some $s < y$ such that $[a, y]$ also has a cover, so $\sup S$ wouldn'T equal this value that is less than y, our first constradiction.

And if we say that $s > b$, then we are demanding the existence of another neighborhood that will cover the difference between s and b, thus leading to another contradiciton.

Since our cover includes the endpoints of our interval, we can mimick of construction of a finite subcover from back in example 3.3.7, one big open set that goes $(a - \epsilon, b + \epsilon)$.

(c) Finally, prove the theorem for an arbitrary closed and bounded set K.

The process we just argued is generalizable to any a and b for simple intervals. But if there were gaps, and we were looking at sets, then we would need to look at the infinimum and supremum of the set to build a finite open subcover.

And if the set did not contain any limit points, that's where the last equivalence with the other definitions of compactness comes into play: we need the set to be bounded, otherwise we wouldn't be able to construct a finite subcover. Think back to Exercise 3.1.1 and how $\mathbb{N}$ are a closed interval. Our construct only works if the supremum and infinimum exist.

3.3 Perfect Sets and Connected Sets

Theorem 3.4.3

A closed set must contain all of its limit points - if there are any. If we now think about the set's isolated points, then we have three possible cases

1. there are no isolated points - since we are assuming that the set is nonempty, this means that only limit points exist in the set. This means that we can always construct a (x_n) that converges to x for any given x but it is different from x.
2. there is a finite number of isolated points.
3. there is an infinite number of isolated points.

A perfect set would be of the first kind.

Now, let's think about the case in which the number of limit points is finite or infinite. For the finite case, let's think of a set such as $\{1, 2, 3\}$, hard to make an argument for any $(x_n) \to x$, no?.

During the proof, note that the intervals I_n that we are building are compact because they are closed and bounded. And because all compact sets contain sequences that converge to a member that is contained within their set, that is why we can expect there to be some y_2 for an x_1 within the interior of I_1, for example.

Finally, note that the last bit of the argument, where we let $K_n = I_n \cap P$, that step is similar to what we did in exercise 3.2.1.

The last bit, the contradiction, is a contradiction because we began assuming that P was countable, so every single x_n should have been a member of P. But then the nested compact set property allowed us to produce an x_n that was not within our list.

3.3.1 Exercises

3.4.1

If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

If P is a perfect set, then it contains no isolated points - its all made up an uncountable number of limit points - and is closed. K on the other hand is closed, bounded, and contains all of its limit points.

Right away we can see that their intersection will be close and bounded, so their intersection will result in a compact set. However, the intersection is not necessarily a perfect set itself.

3.4.2

Does there exist a perfect set consisting of only rational numbers?

No. $\mathbb{Q}$ is a countable set.

3.4.3

Review the portion of the proof given in Example 3.4.2 and follow these steps to complete the argument.

A closed set contains all of its limit points, and limit points are xs such that $(x_n) \to x$ and all x_n are different from x.

(a) Because $x_1 \in C$, argue that there exists and $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq 1/3$.

Since the Cantor set C is not empty and its made of a finite intersection of C_n sets, then $C \cap C_1$ will not be empty and it will be closed. Furthermore, the Cantor set is itself bounded, so there will be limit points within $C \cap C_1$, meaning that there must exist some $(x_n) \to x$ where all x_n differ from x.
And since the limit points are contained within the interception and because we have a whole in the middle third of C_1, then any x_n and x will have to meet the criteria of $|x - x_n| \leq 1/3$.

(b) Finish the proof by showing that for each $n \in \mathbb{N}$, there exists $x_n \in C \cap C_n$, different from x, satisfying $|x - x_n| \leq 1/3^n$.

Given that for each C_n we continue removing the middle $1/3$ of each subinterval (all closed intervals have a length of $1/3^n$), then we will always have a $|x - x_n| \leq 1/3^n$ as all limit points are contained within their respective closed intervals.
In the argument given in Example 3.4.2 we already showed that the Cantor sets are notempty, so the above statement holds.

3.4.4

Repeat the Cantor construction from Section 3.1 starting with the interval $[0, 1]$. This time, however, remove the open middle fourth from each component.

Following the construction we used in Section 3.1, $C_0 = [0,1]$. Then on C_1, we remove $1/4$ of the interval. Leaving C_1 with a total lengh of $3/4$.

Then on C_2 we remove a another $1/4$ from the length of C_1 which is $3/4$, so the total lenght of the interval we take away is now $\frac{1}{4}\frac{3}{4}$. Living C_2 a length of $\frac{3}{4}\left(\frac{3}{4}\right) = \frac{3}{4}^2$.

Similarly, in C_3 we take away another $1/4$ from C_2, that is $\frac{1}{4}\frac{3}{4}^2$, leaving c_3 with a lenght of $\frac{3}{4}^3$. So one could infer that C_n we take away $\frac{1}{4}\frac{3}{4}^{n-1}$ of its length leaving C_n with a total lenght of $\frac{3}{4}^n$

For completion, if $C_0 = [0,1]$, then $C_1 = [0, 3/8] \cup [5/8, 1]$. To literally remove the middle fourth we need to do some trial and experiment. If we start by difiving $[0,1]$ into fourths, then we can't tell which one is the middle one but it makes it obvious that we have to "shift" one of the forths so that it "wraps" around the interval leaving us with an abovious middle. But since we have to perform a "shift", then it may help us to divide the interval into eights. When we performed the "shift", we have to leave half of one fourth off the ends (you should really be drawing this). This leaves us with one fourth from each end plus the one eighth we got from shifting.

Wikipedia seems to, again, have the perfect example, Smith–Volterra–Cantor set. This page contains the exact same example we are working on.

(a) Is the resulting set compact? Perfect?

To do the proof we can re-use everything from section 3.1 and Exercise 3.3.1. The only modification is that the argument for the limit points now needs to have $|x - x_n| \le (3/8)^n$.

(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

From our logic above, the lenght of what we take away from each C_n is

$$\frac{1}{4} + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{3}{4}^2\right) + \dots \frac{1}{4}\left(\frac{3}{4}\right)^{n-1} = \frac{1}{4}\sum_{n\in\mathbb{N}}\left(\frac{3}{4}\right)^{n-1} = \frac{1}{4}\sum_{n=0}\left(\frac{3}{4}\right)^{n}$$

The summation is a geometric series, and since $|r| = |3/4| < 1$, then it converges to $1/(1 - \frac{3}{4} = 4)$ which then cancels out with our $1/4$ term serving as a common multiplier leaving us with an answer of 1. So, apparently, we once again have ended up with a Cantor set of lenght 0.

Now, the proper way to get the dimension is by noting that $C_1 = [0, 3/8] \cup [5/8, 1]$. So if we magnify the cantor set by a factor of $8/3$, then we get two copies of C_0, so now we ought to solve the equation $2 = \frac{8}{3}^d$, where $d = \log(2)/\log(8/3) \approx 0.707$

3.4.6

A set $E \subseteq \mathbb{R}$ is connected if and only if, for all empty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Let's try a proof by contradiciton. In order for E to be connected, it needs to be able to be written as $A \cup B$ for any possible nonmepty pair of A and B and we need a series (x_n) that converges to x but the elements in the series and the limit need to be in different sets.

So let's assume without losing a sense of generality that both (x_n) and x are in A. Since the seires and the limit are also arbitrary this would make A a compact set. Which in turn would make B an open set. If A and B are disjoint, then it is not possible to use both of these to form any general type of connected set E - there would always be one endpoint missing (the one from the non-overlapping open set), and this would always make the other end part of the connected set.

($\Rightarrow$) However, a more constructive way to do the proof in the forward direction is by noting that if (x_n) is in either A or B but its limit point is **not** in the other set (this is where we assume the contrary), then there is no "shared" element between the two sets so they are separed - $\overline{A} \cap B = A\overline{B} = \emptyset$.

If A and B are separated, then $E = A \cup B$ would be disconnected. Which is a contradiction.

($\Leftarrow$) Now going backwards, we assume that (x_n) is in A so that x is a limit point of A but it is actually in B, that is $\overline{A} \cap B = x$. So A and B are not separated. Which in turn means that E cannot be disconnected, so it must be connected.

3.4.7

A set E is **totally disconnected** if, given any two distinct points $x, y \in E$, there exists separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.

(a) Show that $\mathbb{Q}$ is totally disconnected.

If we simply chose $A = (-\infty, x) \cap \mathbb{Q}$ and $B = \mathbb{Q} \cap (y, \infty)$ then we are leaving a gap in the middle. But if we remember that between any x and y there is an z such that $x < z < y$ and i is a real number, then we can chose $A = (-\infty, z) \cap \mathbb{Q}$ and $B = \mathbb{Q} \cap (z, \infty)$

(b) Is the set of irrational numbers totally disconnected?

The density argument also works the otherway around - interestingly Rudin presents it this way in theorem 1.20 (b). If $x \in \mathbb{I}$, $y \in \mathbb{I}$, and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$. Then we can construct $A = (-\infty, p) \cap \mathbb{I}$ and $B = \mathbb{I} \cap (p, \infty)$

3.4.8

Follow these steps to show that the Cantor set is totally disconnected in the sense described in Exercise 3.3.1. Let $C = \bigcap_{n=0}^{\infty} C_n$, as defined in Section 3.1.

(a) Given $x, y \in C$, with $x < y$, set $\epsilon = y - x$. For each $n = 0, 1, 2, \ldots$, the set C_n consists of a finite number of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval of C_N.

There are some useful bits in Exercise 3.3.1 and from the discussion of lenght and dimension of the Cantor set in Section 3.1. The bit that matters right now is that the lenght of subintervals is $1/3^n$ for C_n. Hence, for some N large enough, we will have $1/3^N < \epsilon = y - x$, meaning that the distance between x and y is greater than the lenght of any subinterval of length $1/3^N$ for some sufficiently large N no matter which x or y we pick.

(b) Show that C is totally disconnected.

Again, to show that C is totally disconnected we must prove that given any x and y there exist separated sets $x \in A$, $y \in B$, such that $C = A \cup B$.

And for a and B to be separated we must show that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

There are two ways to kind of see that C is totally disconnected. The first is by noting that every C_n is a closed set, so any A and B that we can form from a finite number of closed sets will themselves be closed, so $A = \overline{A}$ and $B = \overline{B}$. And since we can find an x and a y in different sets, then the intersection of A and B will be empty, showing that they are separated, and thus C is totally disconnected.

The other way follows the argument we used in Exercise 3.3.1. Since x and y are in different closed intervals of C_n, then between them there will be some z that should be where a gap is (one of the removed subintervals), so $z \notin C$ and $x < z < y$.

Then we can construct $A = (-\infty, z) \cap C$ and $B = C \cap (z, \infty)$.

3.4.9

Let $r = \{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rational numbers, and for each $n \in \mathbb{N}$ set $\epsilon_n = 1/2^n$. Define $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$, and let $F = O^c$.

(a) Argue that F is a closed, nonempty set consisting only of irrational numbers.

The union of an arbitrary collection of open sets is an open set. So O must be open, and F must be closed.

Furthermore, since the epsilon neighborhoods are around rational numbers, then $\mathbb{Q} \subset O$ and F will just be irrational numbers.

It remains to prove that F is nonempty. But to see that, intuitevely, one could argue that since the rationals are countably infinite and the reals are uncountable, that a countably inifite number of intervals of lengh $1/2^n$ won't be able to cover all of $\mathbb{R}$. A more convincing argument may be to note that $\sum_{n\in\mathbb{N}} \frac{1}{2^n} = \frac{1}{1/2} = 2$ (geometric series).

(b) Does F contain any nonempty open intervals? Is F totally disconnected?

This may sound a bit like Exercise 3.1.1, because in order to make an argument about a set being open we need to show that all of the elements in a have an ϵ-neighborhood that is also contained within the set, but let's follow another path. Remember how when we looked at the density of the reals we saw that between any tow rational numbers there was an irrational - and how Rudin presents its converse: between any two irrational numbers there is a rational one?

Well, if we think about it, F has a ton of irrationals, but if you look to the left or right of them you should encounter a rational number before the next irrational. The thing is that that rational number will now be part of O, so no matter how small you go, you will always have wholes from the numbers that went into O.

To show that F is totally disconnected we must prove that given any x and y there exist separated sets $x \in A$, $y \in B$, such that $F = A \cup B$.

And for A and B to be separated we must show that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

And to do so, we can follow the exact same argument we used in Exercise 3.3.1 and in Exercise 3.3.1 since in between any two irrationals in F we will have a gap from elements that were taken by O.

(c) Is it possible to show whether F is perfect? If not, can we modify this construction to produce a nonempty perfect set of irrational numbers?

To have a perfect set we need all elements of F to be a limit point. If we use the conventional notion of convergence then it begs the question of what is it that our limit points should be, but if we look back at the Cauchy criterion then if we can find any pair of x_n and x_m such that for any $\epsilon > 0$, $|x_n - x_m| < \epsilon$

as long as n and m are grater than some N, then it kinda helps us see what we need to have in order to have a perfect set.

Note that the order in which we cover the reals is not specified, so we could use our $1/2^n$ convers to block out a continuous strip of $\mathbb{R}$, or something of the sort, and we would be left with a perfect set.

3.4 Baire's Theorem

We first saw countable unions of closed sets and countable intersections of open sets back in Exercise 3.1.1.

3.5.1
Argue that a set A is G_δ set if and only if its complement is an F_σ set.

A G_σ set is a countable intersection of open sets. Theorem 3.2.3 (ii) says that the intersection of a finite collection of open sets is open - nothing is said about countably infinite intersections of open sets. So if G_σ is a finite intersection of open sets, then it is open, and F_δ must be closed.

The way to make this argument comes from DeMorgan's law. For example, if $G_\sigma = \bigcap O_\lambda$, then $G_\sigma^c = \left(\bigcap O_\lambda\right)^c = \bigcup O_\lambda^c = \bigcup C_n = F_\delta$. Where O_λ is some open set and C_n is some closed set.

3.5.2

(a) The **countable** union of F_σ sets is an F_σ set.

The countable union of countable unions of closed sets is a countable union of closed sets. See Theorem 1.5.8 and Exercise 1.5.1.

(b) The **finite** intersection of F_σ sets is an F_σ set.

Distributive law.

(c) The **finite** union of G_δ sets is a G_δ set.

Distributive law.

(d) The **countable** intersection of G_δ sets is a G_δ set.

The countable intersection of countable intersections of open sets is a countable intersection of open sets. See Exercise 1.5.1.

3.5.3
This is Exercise 3.1.1

3.5.4
Starting with $n = 1$, inductively construct a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ satisfying $I_n \subseteq G_n$. Give special attention to the issue

of the endpoints of each I_n. Show how this leads to a proof of Theorem 3.5.2.

To start the construction requested of us, note that $I_n \subseteq G_n$ entails that I_n is a closed set that contains I_{n+1}. By Theorem 1.4.1, the Nested Interval Property, we know that $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$.

This construction has in a way been foreshadowed throughout. Back in Theorem 1.5.6 (ii), when we proved that the reals are uncountable, we saw that a contradiction arosed when we thought the reals to be countable because at the very least the supremum would be in I_n. Exercise 1.4.2 then also shows us that for the nested interval property to work that the intervals must not be open. And in Exercise 3.1.1 (b) we saw that an open set cannot contain its supremum.

This time, our construction must be such that a our dense, open set G_n contains a closed interval I_n.

Roughly speaking, if we bind I_1, then all subsequent I_n will be bounded as well as $I_n \supseteq I_{n+1}$. Since all I_n will be bounded and closed, then they will be compact sets. Since I_n is compact, then any open covering has a finite subcover (G_n). So our construction is completely doable.

Furhtermore, since all G_n are open and dense, then we can continue on defining smaller and smaller I_n intervals. Hence creating a countable collection of G_n sets is also feasible.

3.5.5

Show that it is impossible to write

$$\mathbb{R} = \bigcup_{n=1}^{\infty} F_n,$$

where for each $n \in \mathbb{N}$, F_n is a closed set containing no nonempty open intervals.

In words, the exercise asks us to show that the reals cannot be formed from a infinetely countable union of closed sets containing no nonempty open intervals. (Also remember that we proved back in Exercise 3.1.1 that $\mathbb{R}$ and $\emptyset$ are the only sets that are both open and closed.)

This could be thought of as an argument about creating an uncountable set from a countable collection of sets but a more readily made argument is by noting how much this looks like the complement of Exercise 3.4.

So by contradiction, if $\mathbb{R} = \bigcup_{n\in\mathbb{N}} F_n$, then

$$\mathbb{R}^c = \bigcap_{n\in\mathbb{N}} F_n^c = \bigcap_{n\in\mathbb{N}} G_n = \emptyset$$

Which contradicts what we argued back in Exercise 3.4.

Also note that the reason we are able to have $F_n^c = G_n$ is because we required all sets F_n to contain no nonempty open intervals. If we drop this requirement, then F_n^c would not necessarily be bounded and the construction we made in Exercise 3.4 would not be possible.

3.5.6

Show how the previous exercise implies that the set $\mathbb{I}$ of irrationals cannot be an F_σ set, and $\mathbb{Q}$ cannot be a G_δ set.

Let's put together a couple things we have discovered thus far. Back in Exercise 3.1.1, we argued that $\mathbb{Q}$ is an F_σ set and that $\mathbb{I}$ is a G_δ set.

Furhtermore, in Exercise 3.4 we argued that a set A is G_δ set if and only if its complement is an F_σ set.

Now we could say that like $\mathbb{R}$ and $\emptyset$ are both open and closed even in the face of Theorem 3.2.13 which states that a set O is open if and only if O^c is closed and that a set F is closed if and only if F^c is open, then maybe it is possible for some sets to also be F_σ and G_δ.

But if we assumed that $\mathbb{I}$ is a F_σ set, then as per Exercise 3.4, a countable union of F_σ sets is an F_σ set. So $\mathbb{Q} \cup \mathbb{I} = \mathbb{R}$ would be a F_σ, which we just argued is not possible.

Hence, we should be inclined to think that $\mathbb{I} = \mathbb{Q}^c$ is indeed $F_\sigma^c = G_\delta$ set and not possible for it to be both at the same time. consecuently, $\mathbb{Q}$ could not be a G_δ set.

3.5.7

Using Exercise 3.5.6 and versions of the satements in Exercise 3.5.2, construct a set that is neither in F_σ nor in G_δ.

This time we will go looking for sets that are countable intersections of F_σ sets (countable intersections of sets that are countable unions of closed sets) and countable unions of G_δ sets (countable unions of sets that are countable intersections of open sets).

In symbolic forms, we are fishing for sets such as $\bigcap_n \bigcup_{n'} F$ and $\bigcup_n \bigcap_{n'} O$.

But after some thinking and searching, we think Example of a Borel set that is neither F nor G said it best. Also, note how the mathexchange thread is about Borel sets, and the wiki page on Borel sets then talks about measure theory.

3.5.8

Show that a set E is nowhere-dense in $\mathbb{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbb{R}$.

That is, E is nowhere-dense in $\mathbb{R}$ if and only if $\left(\overline{E}\right)^c$ is dense in $\mathbb{R}$. Where

$$\left(\overline{E}\right)^c = (E \cup L)^c = (E^c \cap L^c)$$

The proof in the forward direction is: if E is nowhere-dense in $\mathbb{R}$ ($\overline{E}$ contains no nonempty open intervals), then $\left(\overline{E}\right)^c$ is dense in $\mathbb{R}$.

The proof in the backward direction is: if $\left(\overline{E}\right)^c$ is dense in $\mathbb{R}$, then E is nowhere-dense in $\mathbb{R}$ ($\overline{E}$ contains no nonempty open intervals).

($\Rightarrow$) If $\overline{E}$ contains no nonempty open intervals then for any a and b, $(a, b) \notin \overline{E}$. And so for a c such that $a < c < b$, $c \notin \overline{E}$, which means that $c \in \overline{E}^c$.

Since a, b, and c are arbitrary and $a < c < b$, then $\overline{E}^c$ is dense.

For the sake of notation, remember that $\overline{E}^c = \mathbb{R} \setminus \overline{E}$.

Trying to verbalize it: if $\overline{E}$ contains no nonempty open intervals, then all nonempty open intervals must be in $\overline{E}^c$. Since the closure of $\overline{E}^c$ is $\overline{E}^c$ (all open nonempty intervals) the limit points of all nonempty open intervals, then $\overline{\overline{E}^c}$ must equal the reals.

($\Leftarrow$) If $\left(\overline{E}\right)^c$ is dense in $\mathbb{R}$ then $\overline{E}^c \cup L_{\overline{E}^c} = \mathbb{R}$.

The urge to apply DeMorgan's law tells us that $\left(\overline{E}^c \cup L_{\overline{E}^c}\right)^c = \overline{E} \cap L^c_{\overline{E}^c} = \mathbb{R}^c = \emptyset$. Which read out loud says that there are no common elements between $\overline{E}$ and everything that is not a limit point of $\overline{E}^c$. Meaning that $\overline{E} \subseteq L_{\overline{E}^c}$ or $\overline{E} \supseteq L_{\overline{E}^c}$ (if $\overline{E}$ and L^c are disjoint, then $\overline{E}$ and L aren't).

From the above, we could argue that $\overline{E}$ must contain no nonempty open intervals as follows: consider some limit point of $\overline{E}^c$ that is also in $\overline{E}$, let's call it a. If there was a nonempty open interval in $\overline{E}$, then there would be some ϵ-neighborhood, $(a - \epsilon, a + \epsilon)$, around said point. But since $\overline{E}^c$ is dense in $\mathbb{R}$, then any number in $(a - \epsilon, a + \epsilon)$ that we could find it should be in $\overline{E}^c$, not in ϵE. It would thus be inconsistent to think that $\overline{E}$ can sustain a nonempty open interval.

3.5.9

Deside whether the following sets are dense in $\mathbb{R}$, nowhere-dense in $\mathbb{R}$, or somewhere in between.

(a) $A = \mathbb{Q} \cap [0, 5]$

This is the set of rationals in $[0, 5]$. Since every real in $[0, 5]$, that is not a rational, is a limit point of A, then A is dense in $[0, 5]$ but not dense in all of $\mathbb{R}$.

(b) $B = \{1/n : n \in \mathbb{N}\}$

B is nowhere dense, since $\overline{B} = B \cup \{0\}$ does not contain any nonempty open interval.

(c) The set of irrationals

If the previous exercise thought us anything is to start by looking at the closure of the set in question. Here $\overline{\mathbb{I}} = \mathbb{I} \cup \mathbb{Q} = \mathbb{R}$ so the irrationals are dense in $\mathbb{R}$.

(d) The Cantor set

Couple facts we know about the Cantor set: it is closed, $C = \overline{C}$, and it contains no intervals. Since $\overline{C}$ contain no nonempty open intervals, we'd say that the Cantor set is nowhere-dense.

One could possibly want to make a similar argument for $C^c = \bigcup C_n^c$, where C_n^c would be a bunch of open intervals, but in that case the closure of the complement does contain nonempty open sets (so it is not nowhere-dense) but every point in the reals is also not a limit point of it so it is also not dense in $\mathbb{R}$.

So we are sticking with saying that the Cantor set is nowhere-dense.

3.5.10

The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

For contradiciton, assume E_1, E_2, E_3, ... are each nowhere-dense sets and satisfy $\mathbb{R} = \bigcup_{n\in\mathbb{N}} E_n$.

Since each E_n is nowhere-dense, then $\overline{E}_n$ contains no nonempty open sets. From here we could do an qualitative argument but also note that we can use Exercise 3.4.

Since $E_n \subseteq \overline{E}_n$, then using Exercise 3.4 we can see that is is impossible to write $\mathbb{R} = \bigcup_{n\in\mathbb{N}} \overline{E}_n$. Consequently, $\mathbb{R} \neq \bigcup_{n\in\mathbb{N}} E_n$.

3.5 Open questions

- The section of perfect and connected sets would make a great reminder for the cut property that was mentioned as an exercise in the latest version of the book in chapter 1.
- We should make a clear mention of how we discovered that the Canotr set is closed and that it has no intervals. These facts are used a lot in exercises.

4 Functional Limits and continuity

4.1 Functional Limits

4.1.1 Exercises

4.2.1

(a) Supply the details for how Corollary 4.2.4 part (ii) follows from the Sequential Criterion for Functional Limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

For reference, the Algebraic Limit Theorem is Theorem 2.3.3.

The Sequential Criterion for Functional Limits here tells us that since we know that since $\lim_{x\to c} f(x_n) \to L$ and $\lim_{x\to c} g(x_n) \to M$, then we know that $(x_n) \to c$ ($x_n \neq c$ for all n) and that $f(x_n) \to L$ and $g(x_n) \to M$.

Since the later two expressions are sequences on their own, we can use now the Algebraic Limit Theorem, $\lim(a_n + b_n) = a + b$, implying that $\lim_{x\to c}(f(x_n) + f(x_n)) \to L + M$.

(b) Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the Sequential Criterion for Functional Limits in Theorem 4.2.3.

To do the same proof without the Sequential Criterion for Functional Limits we ought to go back to $\epsilon - \delta$ proofs.

Since we know that $\lim_{x\to c} f(x_n) \to L$ and $\lim_{x\to c} g(x_n) \to M$, then we can define a δ_f such that when $0 < |x - c| < \delta_f$ $|f(x) - L| < \epsilon_f$ (for any $\epsilon_f > 0$) and a δ_g sucht that when $0 < |x - c| < \delta_g$ $|g(x) - M| < \epsilon_g$ (for any $\epsilon_g > 0$).

Then if we look for a $\delta = \min\{\delta_f, \delta_g\}$ such that

$$|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \epsilon_f + \epsilon_g = \epsilon$$

If you notice, this is pretty much the same sort of logic we used to prove the Algebraic Limit Theorem, so we could have a guess for what the proof for the next bit could be.

(c) Repeat (a) and (b) for Corollary 4.2.4 part (iii).

The proof using the Sequential Criterion for Functional Limits is essentially the same as in part (a). (You should definetely write it out or read it out loud though!) So let's carry on with the proof without it.

Similar to what we did while proving the Algebraic Limit Theorem, let's work backwards from what we want.

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&\leq |f(x)g(x) - Lg(x)| + |Lg(x) - LM| \\
&= |g(x)|\,|f(x) - L| + |L|\,|g(x) - M|
\end{aligned}$$

Again, we want to find the "worst case scenario" for $g(x)$. And since $g(x_n)$ is a convergent sequence, then it will have some bound M. If we also recall our δ_f and δ_g from part (b), then we have

$$\begin{aligned}
|f(x)g(x) - LM| &\leq |g(x)|\,|f(x) - L| + |L|\,|g(x) - M| \\
&\leq M\,|f(x) - L| + |L|\,|g(x) - M| \\
&< |M|\epsilon_f + |L|\epsilon_g \\
&= |M|\left(\frac{\epsilon}{2|M|}\right) + |L|\left(\frac{\epsilon}{2|L|}\right) = \epsilon
\end{aligned}$$

At the end there, we went ahead and figured out how to define ϵ_f and ϵ_g in terms of some arbitrary ϵ that could be used for the entire thing.

4.2.2

For each stated limit, find the largest possible δ-neighborhood that is a proper response to the ϵ-challenge.

(a) $\lim_{x\to 3}(5x-6) = 9$, where $\epsilon = 1$.

(b) $\lim_{x\to 4}\sqrt{x} = 2$, where $\epsilon = 1$.

(c) $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = 1$. (The function $[[x]]$ returns the greatest integer less than or ewual to x.)

(d) $\lim_{x\to\pi}[[x]]$, where $\epsilon = 0.01$.

5 Introduction

5.1 Gaps in the rationals

$$A = \{p \in \mathbb{Q}^+ : p^2 < 2\}$$

$$B = \{p \in \mathbb{Q}^+ : p^2 > 2\}$$

Rudin shows that A contains no largest number - for every p in A, we can find a q in A such that $p < q$. It also shows that B coontains no smaller number - for ever p, we can find a q such that $p > q$.

The trick to obtain equation 4 is to subtract 2 from $\left(\frac{2p+2}{p+2}\right)^2$.

Then, to get the $q^2 < 2$ or $q^2 > 2$ inequalities, convert

$$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2} \rightarrow q^2 = 2 + \frac{2(p^2-2)}{(p+2)^2}$$

5.2 Consequences of completeness

There is a very important theorem presented in 1.11 and given as an exercise in Abbott as 1.3.

Abbott introuces it to elucidate the point to the interesting case when if a is an upper bound for A, and if $a \in A$, then it must be that $a = \sup A$.

The way to read Rudin's proof is as follows:

Suppose S is an ordered set with the least-upper-bound property (set is not-empty, bounded above, and supremum exist). $B \subset S$ is not empty, and B is bounded below. Let L be the set of all lower bounds of B, $L = \{y \in S : y \leq x, x \in B\}$. Then $\alpha = \sup L$ exists in ($\alpha \in S$) and

$$\alpha = \inf B \text{ and } \alpha \in S$$

- B is bounded below thus $L \neq \emptyset$.
- All $x \in B$ are upper bounds of L.

Since S has the least-upper-bound property, and since L is bounded above, then $\alpha = \sup L$ must exist in S - this is the definition of the least-upper-bound-property that we assumed applies for our "universe" S.

By our definition of the suppremum,

- α is an upper bound of L.
- If $\gamma < \alpha$ then γ must not be an upper bound of L ($\gamma \in L$ but $\gamma \notin B$).
- α is the smallest upper bound, so $\alpha \leq x$ for all $x \in B$.
- If $\alpha \leq x$ then it means that it is a lower bound of B and thus $\alpha \in L$.

The last two properties we listed mean that if there were some β such that $\alpha < \beta$, then $\beta \in B$ - β is an upper bound of L but not in L.

This restatement of the least upper bound (any upper boundes greater than the least upper bounds are greater than or equal to it) also happens to be a statement for the definition of the greatest lower bound. Look at it this way, we said that there exist an α that is a lower bound of B ($\alpha \leq x$) and any $\beta > \alpha$ is not a lower bound. In some other places that last phrase might have been written as $\alpha \geq y$, where y is any other lower bound of B (which again is the defintion of all elements of L).

5.3 Fields

We have that

$$y = 0 + y = -x + (x + y) = -x + (x + z)$$

If $z = -x$, then

$$y = -x + (x + y) = -x + (x + (-x)) = (-x + x) + (-x) = -x$$

So $x + y = 0 = x + (-x)$ when we use $x + y = x + z$.

To prove

$$-(-x) = x$$

We can start with $x + y = x + (-x) = 0$. Since to ever x corresponds an element $-x$ such that $x + (-x) = 0$, then $-(x) + -(y) = 0 = -(x) - (-x) = 0 = -x + x$.

We can use similar logic to prove proposition 1.15.

$$y = 1y = x\left(\frac{1}{x}\right)y = \left(\frac{1}{x}\right)(xy) = \left(\frac{1}{x}\right)(xz) = z$$

To prove (b) take $z = 1$. To prove (c) take $z = 1/x$.

If $x \neq 0$, then there exists an $1/x$ such that $1(1/x) = 1$. We know that $x(1/x) = 1$, then $1/1 = (1/x)(1/(1/x))$.

In the case of **proposition 1.16(c)** Since

$$(-x)y + xy = 0$$

and 1.14(c) says that if $x + y = 0$, then $y = -x$, the above expression can be seen as $(-x)y + z = 0$ so $z = -(-x)y = xy$. In the last step we used 1.14(d).

Theorem 1.13

For part c, where it states that

$$|zw| = |z||w|$$

The intermediate steps are as follows:

$$|zw|^2 = |(zw)(\bar{z}\bar{w})|^2 = [(a+bi)(c+di)]\,[(a-bi)(c-di)]$$

$$= [(ac-bd)+(ad+bc)i]\,[(ac-bd)-(ad+bc)i]$$

$$= (ac-bd)^2 + (ad+bc)^2 - i(ac-bd)(ad+cb) + i(ac-bd)(ad+cb)$$

$$(ac-bd)^2 + (ad+bc)^2 = |z|^2|w|^2 = (|z||w|)^2$$

5.4 constructing $\mathbb{R}$ from $\mathbb{Q}$

Property no. 2 of a cut says that when $p \in \alpha$ and $q \in \mathbb{Q}$

$$q < p \to q \in \alpha$$

or, equivalently,

$$q \notin \alpha \to p < q$$

The second implication that Rudin states is not so easy to follow. But a way to make sense of it is by noting that the condition we are examining, when $p \in \alpha$ and $q \in \mathbb{Q}$

$$q < p \to q \in \alpha$$

means that the cut α is "**closed downwards**", meaning that for all $p \in \alpha$ is $q < p$ then $p \in \alpha$ (yes, we just restated it because it helps when you read it out loud).

This also means that if you were an r and you were less than an s, but you knew that you are not in the cut α then the thing greater than you, s, must not be in the cut either, because all custs are closed downwards.

Again, α is closed downawards and contains no greatest element.

A cut being a propoer subset of another cut is being used, along with the property of $\mathbb{R}$ being an ordered field, to show that one is less than the other.

In step 3, when we say that $\gamma \subset \beta$ (since $\alpha \subset \beta, \forall \alpha \in A$), we are making an appeal to Dedekin cuts being closed downwards.

Also, since we showed that $\alpha_0 \subset \gamma \subset \beta$, then by $\mathbb{R}$ being an ordered field (every set is at most less, equal, or greater than other sets) and by the fact that cuts are closed downwards ($\gamma \subset \beta$), then γ is another cut and from property (I) $\gamma \neq \mathbb{Q}$.

In step 4 we will be refering to some rational number $-p$. Not that this does not necessarily imply that $p > 0$. For example, if we look for a p that is greater than all values of a cut, then its negative, $-p$ will get us into the real of 0^* and thus give us some numbers from which we can prove the last axiom addition for a fields.

In step 4, when we look for an $s \notin \alpha$ and for a $p = -s - 1$. Note that $p < s$. Also note that $-p - 1 = s$. This way of looking for a p is what allows us to find one that meets the criteria to be a member of β. Then because we were able to find another rational number $q \in \alpha$, $\alpha \neq \beta$, then we can say that $\beta \neq \mathbb{Q}$.

6 Basic Topology