Notes on linear algebra

September 17, 2023

Contents

1 Vector Spaces

1.1 Exercises

1.A.1

Find c and d such that

$$\frac{1}{a+bi} = c + di$$

The trick here is as follows

$$\frac{1}{a+bi}\left(\frac{a-bi}{a-bi}\right) = \frac{a-bi}{a^2+b^2}$$

So $c = a/|z|^2$ and $d = -b/|z|^2$, where $z = a + bi$ and $|z| = \sqrt{a^2+b^2}$.

1.A.2

Show that

$$\frac{-1\sqrt{3}i}{2}$$

is a cube root of 1.

$$(-1\sqrt{3}i)(-1\sqrt{3}i) = -2 - 2\sqrt{3}i$$
$$(-2 - 2\sqrt{3}i)(-1\sqrt{3}i) = 2 + 6 = 8$$

Ad since $\frac{1}{2^3} = 1/8$, then we get our proof.

1.A.3

To find 2 distinct square roots of i it helps to look at Euler's identity first

$$e^{i\pi} + 1 = 0$$

or

$$e^{i\pi} = -1$$

And since $i^2 = -1$, then

$$e^{i\pi} = -1 = i^2$$

Hence

$$e^{i\pi/2} = i$$

So our firsrt square root is $e^{i\pi/4} = \sqrt{i}$. If we go around once, then we get another square root

$$e^{(i\pi/2+2\pi i)/2} = e^{i5\pi/4}$$

We can still simply this further.

$$e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

and

$$e^{i5\pi/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{-1}{\sqrt{2}} + i\frac{-1}{\sqrt{2}} = -\frac{1+i}{\sqrt{2}}$$

Another interesting read is: Ed Scheinerman (2023) A Third Real Solution to $x = x^{-1}$, Not Really, The American Mathematical Monthly, 130:6, 514-514, DOI: 10.1080/00029890.2023.2184161

1.A.4
Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Let's define $\alpha = a + bi$ and $\beta = c + di$.

$$\alpha + \beta = (a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$\beta + \alpha = (c + di) + (a + bi) = (c + a) + (d + b)i$$

And since summation is commutative for the reals, QED.

1.A.5
Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ where $\alpha, \beta, \lambda \in \mathbb{C}$.

Same mechanics,

$$(\alpha + \beta) = (a + c) + (b + d)i$$

If we define $\lambda = x + yi$ then

$$(\alpha + \beta) + \lambda = [(a + c) + (b + d)i] + (x + yi) = (a + b + x) + (b + d + y)i$$

And hopefully you can see the rest of the argument and how we are save by the commutative property of the reals.

1.A.8
Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exist a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

The result from exercise 1.A.1 comes in handy here! this is because we get an expression for a an inverse complex number in terms that make it easy to carry out calculations the way we are used to.

In 1.A.1 we had $\beta = c + di = (c, d)$ equal to

$$(a, b)\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) = (1, 0) = 1$$

1.A.10
Find $x \in \mathbb{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8)$$

Let's try component by component,

- $4 + 2x_0 = 5$ results in $x_0 = 1/2$.
- $-3 + 2x_1 = 9$ results in $x_1 = 12/2 = 6$.
- $1 + 2x_2 = -6$ results in $x_2 = -7/2$.
- $7 + 2x_3 = 8$ results in $x_3 = 1/2$.

So $x = (1/2, 6, -7/2, 1/2)$.

1.A.11
Explain why there does not exist a $\lambda \in \mathbb{C}$ such that

$$\lambda\,(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i)$$

This one is funky but try note that $\lambda(2-3i) = (12-5i)$, and $\lambda(5+4i) = (7+22i)$. Then multipy $\lambda(2-3i)(7+22i)$ and compare that against $(12-5i)(5+4i)$, and so on.

1.B.1
Prove that $-(-v) = v$ for every $v \in V$.
Using the additive inverse property

$$(-v) + -(-v) = 0 \to (-v) + v = 0$$

And by the Uniqueness of the additive inverse, $-(-v) = v$.

1.B.3
Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that

$$v + 3x = w$$

There are two parts to this question: existance and Uniqueness. The existance part can be seen by using

$$x = \frac{1}{3}(w - v)$$

in our original expression. That is,

$$v + 3x = v + 3\frac{1}{3}(w - v) = v + w - v = w$$

Uniqueness can be seen by noting that if we had

$$v + 3x' = w$$

Then

$$x' = \frac{1}{3}(w - v)$$

And so

$$x - x' = 0$$

1.B.4

The empty set is not a vector space because it fails to satisfy the additive identity - can't have the existence of an element $0 \in V$ if the set is empty.

1.B.5

Show that the additive inverse condition - for every $v \in V$, $\exists w \in V$ such that $v + w = 0$ - can be replaced by the condition

$$0v = 0$$

for all $v \in V$. Where the 0 on the left is $0 \in \mathbb{F}$ and the 0 on the right is the additive identity of v.

Normally, we would think of the additive inverse for $v + w = 0$ as $w = -v$, so

$$0 = v + w = v + (-v) = 1v + (-1v) = 0v = 0$$

1.B.6

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Now, say we defien addition and scalar multiplication in $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as we normally would.

Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$?

No. For example, additive inverses and additive identities would not be unique - nor any of the other conditions required of a vector space.

1.2 Subspaces