Notes on Electrodynamics

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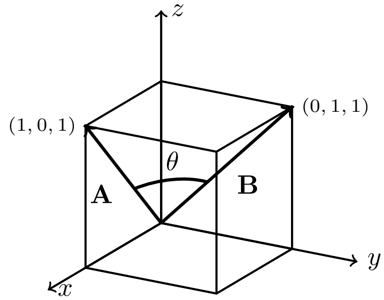
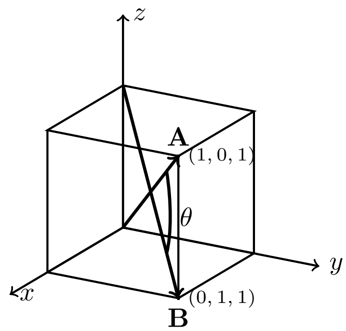
Face diagonals



Body diagonals



0.2 Potentials

0.2.1 Multipole Expansion

$$\boldsymbol{\mathscr{r}} = |\mathbf{r} - \mathbf{r}'|$$

Where $\boldsymbol{\mathscr{r}}$ denotes the vector from a source point $\mathbf{r}'$ to a field point $\mathbf{r}$.

Chapter 1

Introduction to Electrostatics

1.1 Charge distributions

1.1.1 Surface elements da

The following logic comes from Griffith's electrodynamics chapter 1.4.

In curvilinear coordinates, an infinitesimal displacement in the $\hat{\mathbf{r}}$ direction is simply dr.

$$dl_r = dr$$

An infinitesimal element of length in the $\hat{\boldsymbol{\theta}}$ is

$$dl_\theta = r d\theta$$

And an infinitesimal element of length along the $\hat{\boldsymbol{\phi}}$ direction is

$$dl_\phi = r \sin\theta d\phi$$

θ is the polar angle, the angle from the z axis, and its range is $[0, \pi]$. ϕ is the azimuthal angle, which is the angle around from the x axis and its range is $[0, 2\pi]$.

The general infinitesimal displacement becomes

$$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$$

From there one can see how the infinitesimal volume element $d\tau$ is then

$$d\tau = dl_r dl_\theta dl_\phi = r^2 \sin\theta dr d\theta d\phi$$

However, the inifinitesimal area element depends on the orientation of the surface!

For example, if we integrate over the surface of a sphere, where r stays contant but θ and ϕ vary,

$$d\mathbf{a} = dl_\theta dl_\phi \hat{\mathbf{r}} = r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}$$

Where as is the surface lies in the x-y plane, so that θ is contant, and let's say $\theta = \phi/2$, while r and ϕ vary,

$$d\mathbf{a} = dl_r dl_\phi \hat{\boldsymbol{\theta}} = r\sin\theta dr d\phi \hat{\boldsymbol{\theta}} = r dr d\phi \hat{\boldsymbol{\theta}}$$

The area element on the sphere can be calculated from the cross products of other two elements,

You should also, definetely, see Integration and differentiation in spherical coordinates.

Example 1

One good example to get practice is Griffith's Example 1.13: find the volume of a sphere of radius R.

As we saw above, the optimal way to an answer is to integrate the volume element $d\tau$ in spherical coordinates.

$$\begin{aligned} V = \int d\tau &= \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} dr\, d\theta\, d\phi r^2 \sin\theta \\ &= \left(\int_{r=0}^{R} dr\, r^2\right)\left(\int_{\theta=0}^{\pi} d\theta\, \sin\theta\right)\left(\int_{\phi=0}^{2\pi} d\phi\right) \\ &= \left(\frac{1}{3}R^3\right)\left(-\cos\theta\Big|_0^\pi\right)(2\pi) \\ &= \frac{4}{3}\pi R^3 \end{aligned}$$

1.1.2 Fundamental Theorems

This presentation is a quick summary of Griffith's 1.3.2 to 1.3.5.

The fundamental theorem of calculus states

$$\int_a^b dx \left(\frac{df}{dx}\right) = f(b) - f(a)$$

The fundamental theorem for gradients is

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$

Here T is a scalar-values function, $T : \mathbb{R}^3 \to \mathbb{R}$, and the application of the ∇ operator ($\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$) makes ∇T a vector-valued function, $\nabla T : \mathbb{R}^3 \to \mathbb{R}^3$.

The gradient points in the direction of maximum increase of the function T and it expreses the rate of change of T along a given direction.

The interesting bit of our fundamental theorem for gradients is that $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is path independent. And by consequence, $\oint (\nabla T) \cdot d\mathbf{l} = 0$ (if you go up the stairs, measure how many meters above sea-level you are, then go back down, and come back up, then your change is 0).

The fundamental theorem for divergences (Gauss's of Green's theorem) states that

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) \; d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$$

For the two previous fundamental theorems we saw that the boundary of a line was two points, and now we see that the boundary of a volume $\mathcal{V}$ is the closed region $\mathcal{S}$. Accordingly, if we sum how much quantity $\mathbf{v}$ spreads out over an entire volume, this is equal to how much is $\mathbf{v}$ is crossing through the closed enclosing surface.

The fundamental theorem for curls (Stoke's theorem) states that

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

Now the boundary of a region $\mathcal{S}$ is the perimeter (a line) $\mathcal{P}$. Here we see that the circulation of $\mathbf{v}$, $\oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$, over a perimeter, is equal to the sum of the flux of the "twisting" of $\mathbf{v}$ over a surface.

1.1.3 Work done by a particle

Taken from Kline, Chapter 7, Section 7, problem 3

AM is a straight vertical line. MO is a straight horizontal line. O is to the right of M.

A particle moving along AM is attracted to a fixed point O with a force that varies inversely as the square of the distance from O. Find the work done on the particle as it moves from A to a distance B (between A and M).

Suggestion: let x be the distance from A to any point P on AB and let r be the variable distance from P to O. The force F attracting P to O is k/r^2. But only the component of this force along AB serves to move the particle along AB. This component is $(k/r^2) \cos OPM = (k/r^2)(c-x)/r$. Then $dW/dx = k(c-x)/r^3$ and $r = \sqrt{p^2 + (c-x)^2}$.

$$
\begin{aligned}
W &= \int_A^B \frac{dW}{dx} dx \\
&= \int_A^B \frac{k(c-x)}{r^3} \\
&= \int_A^B \frac{k(c-x)}{\left(p^2+(c-x)^2\right)^{3/2}} dx
\end{aligned}
$$

The trick here is to note that if we chose $u = p^2 + (c-x)^2$, then $du = -2(c-x)dx$, so that

$$
\frac{k(c-x)}{\left(p^2+(c-x)^2\right)^{3/2}} dx \to -\frac{k}{2}\frac{1}{u^{3/2}} du
$$

Since $\int u^{-3/2} du = -2u^{-1/2} + C$. So putting everything together,

$$
\begin{aligned}
W &= \int_A^B \frac{k(c-x)}{\left(p^2+(c-x)^2\right)^{3/2}} dx \\
&= -\frac{k}{2}(-2)\frac{1}{u^{1/2}}\Big|_A^B \\
&= k\left(\frac{1}{\sqrt{p^2+(c-B)^2}} - \frac{1}{\sqrt{p^2+(c-A)^2}}\right) \\
&= \frac{k}{r_B} - \frac{k}{r_A}
\end{aligned}
$$

Note: the two interesting bits to note from this exercise are first, when we were looking for the component of motion that contributed to the work being done, the vertix of the angle used was the point we were "on". Second, note how our the u-substitution we used went, doing it the normal way may confuse you and try to get you for somewhere where to use $u = p^2 + (c-x)^2$, which is not needed.

1.2 Gauss's Law

1.2.1 Solid Angle

The following is all wikipedia.

Whereas an angle in radians, projected onto a circle, gives a length of a circular arc on the circumference, a solid angle in steradians, projected onto a sphere, gives the area of a spherical cap on the surface.

The point from which the object is viewed is called the apex of the solid angle, and the object is said to subtend its solid angle at that point.

In SI units, a solid angle is expressed in a dimensionless unit called a **steradian**.

One steradian corresponds to one unit of area on the unit sphere surounding the apex, so an object that blocks all rays from the apex would cover a number of steradians equal to the total surface area of the unit sphere, 4π.

Just like a planar angle in radians is the ratio of the length of an arc to its radius, $\theta = s/r$, a solid angle in steradians is the ratio of the area covered on a sphere by an object to the area given by the square of the radius of said sphere. The formula is

$$\Omega = \frac{A}{r^2}$$

where A is the spherical surface area and r is the radius of the considered sphere.

Any area on a sphere which is equal in area to the square of its radius, when observed from its center, subtends precisely one steradian.

The solid angle of a sphere measured from any point in its interior is 4π sr (steridians), and the solid angle subtended at the center of a cube by one of its faces is one-sixth of that, or $2\pi/3$ sr.

In spherical coordinates there is a formula for the differential,

$$d\Omega = \sin\theta d\theta d\varphi$$

where θ is the colatitude (angle from the North Pole - z-axis range $[0, \pi]$) and φ is the longitude (range $[0, 2\pi]$).

The solid angle for an arbitrary oriented surface S subtended at a point P is equal to the solid angle of the projection of the surface S to the unit sphere with center P, which can be calculated as the surface integral:

$$\Omega = \iint_S \frac{\hat{r}\cdot\hat{n}}{r^2}\, dS = \iint_S \sin\theta\, d\theta\, d\varphi$$

where $\hat{r} = \vec{r}/r$ is the unit vector corresponding to $\vec{r}$, the position vector of an infinitesimal area of surface dS with respect to point P, and where $\hat{n}$ represents the unit normal vector to dS. Even if the projection on the unit sphere to the surface S is not isomorphic, the multiple folds are correctly considered according to the surface orientation described by the sign of the scalar product $\hat{r}\cdot\hat{n}$.

Thus one can approximate the solid angle subtended by a small facet having flat surface area dS, orientation $\hat{n}$, and distance r from the viewer as:

$$d\Omega = 4\pi\left(\frac{dS}{A}\right)(\hat{r}\cdot\hat{n})$$

where the surface area of a sphere is $A = 4\pi r^2$.

1.2.2 Explanation of Jackson

If the electric field makes an angle θ with the unit normal, then the projection of the infinitesimal area element along the normal of the surface is $\cos\theta da$. Using the solid angle formula, we then get $d\Omega = \cos\theta da/r^2$ or $r^2 d\Omega = \cos\theta da$.

1.3 Problems

1.3.1 Problem 1.2

The Dirac delta function in three dimensions can be taken as the improper limit as $\alpha \to 0$ of the Gaussian function

$$D(\alpha; x, y, z) = \frac{1}{\left(\alpha\sqrt{2\pi}\right)^3} \exp\left[-\frac{1}{2\alpha^2}\left(x^2 + y^2 + z^2\right)\right]$$

Consider a general orthogonal coordinate system specified by the surfaces $u =$ constant, $v =$ constant, $w =$ constant, with length elements du/U, dv/V, dw/W in the three perpendicular directions. Show that

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(u - u')\,\delta(v - v')\,\delta(w - w') \cdot UVW$$

by considering the limit of the Gaussian above. Note that as $\alpha \to 0$ only the infinitesimal length element need be used for the distance between the points in the exponent.

Browsing through the web, one will see the following defintions of the delta function,

$$\begin{aligned}
\delta(x) &= \lim_{b \to 0} \frac{1}{|b|\sqrt{\pi}} \exp\left[-\frac{1}{b^2}x^2\right] \\
&= \lim_{b \to 0^+} \frac{1}{2\sqrt{\pi b}} \exp\left[-\frac{1}{4b}x^2\right] \\
&= \lim_{b \to 0} \frac{1}{b\sqrt{2\pi}} \exp\left[-\frac{1}{2b^2}x^2\right]
\end{aligned}$$

The thing to note here is how the arguments of the exponent make it into the constant that's infront of the exponent. And the reason to even look at this now is because we found this amazingly interesting approach to this problem in West Texas CS Baird's site:

We start with

$$D(\alpha; x, y, z) = \frac{1}{\left(\alpha\sqrt{2\pi}\right)^3} \exp\left[-\frac{1}{2\alpha^2}\left(x^2 + y^2 + z^2\right)\right]$$

and then we make a change of variables, a displacement if you will, by changing $x \to x - x'$, etc. This gives us

$$D(\alpha; x - x', y - y', z - z') = \frac{1}{\left(\alpha\sqrt{2\pi}\right)^3} \exp\left[-\frac{1}{2\alpha^2}\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)\right]$$

Now comes the limit taking. First off is whether the limit is some actual number orr just infinity (we are skipping over the question of existence). Since an exponent grows faster than a ever-increasing number to a power, $\lim_{\alpha\to\infty}\alpha^3 \cdot e^{-\alpha} = 0$, then we would expect our function D to be bounded. But, anyway, if we take the limit as $\alpha \to 0$, and if we look for a case where $D \not\to 0$, then we would want $x - x'$ to "balance out" α just enough so that we have some non-zero bounded value.

Let's assume the above is possible and rewirte D as

$$D\left(\alpha; x-x', y-y', z-z'\right) = \frac{1}{\left(\alpha\sqrt{2\pi}\right)^3}\exp\left[-\frac{1}{2\alpha^2}\left((dx)^2+(dy)^2+(dz)^2\right)\right]$$

The infinitesimal displacements in the argument to the exponent look like an arc lenght, so let's rewrite it now as

$$D\left(\alpha; x-x', y-y', z-z'\right) = \frac{1}{\left(\alpha\sqrt{2\pi}\right)^3}\exp\left[-\frac{1}{2\alpha^2}(ds)^2\right]$$

Since an arc length is a concept independent of the coordinate system, we can use our given general orthogonal coordinate system, and rewrite our equation once more,

$$D\left(\alpha; u-u', v-v', w-w'\right) = \frac{1}{\left(\alpha\sqrt{2\pi}\right)^3}\exp\left[-\frac{1}{2\alpha^2}\left(\left(\frac{du}{U}\right)^2+\left(\frac{dv}{V}\right)^2+\left(\frac{dw}{W}\right)^2\right)\right]$$

And if we reverse the process, by writing the differentials as differences,

$$D\left(\alpha; u-u', v-v', w-w'\right) = \frac{1}{\left(\alpha\sqrt{2\pi}\right)^3}\exp\left[-\frac{1}{2\alpha^2}\left(\left(\frac{u-u'}{U}\right)^2+\left(\frac{v-v'}{V}\right)^2+\left(\frac{w-w'}{W}\right)^2\right)\right]$$

We can now look at the first couple definition of the delta function and make a final change of variahles. For example, if we let $b_1 = \alpha U$, $b_2 = \alpha V$, and $b_3 = \alpha W$,

$$D = \left[\frac{\exp\left[-\frac{\left(u-u'\right)^2}{2b_1^2}\right]}{b_1\sqrt{2\pi}}U\right]\left[\frac{\exp\left[-\frac{\left(v-v'\right)^2}{2b_2^2}\right]}{b_2\sqrt{2\pi}}V\right]\left[\frac{\exp\left[-\frac{\left(w-w'\right)^2}{2b_3^2}\right]}{b_3\sqrt{2\pi}}W\right]$$

If we now take the limit of the b_is, then we get our desired result.

Yet another way to look at this problem is as follows. The general property of the delta function is that

$$\int\int\int \delta(x)\delta(y)\delta(z)dxdydz = 1$$

If we use our limit,

$$\int\int\int \lim_{\alpha\to 0} D(\alpha; x, y, z)dxdydz = 1$$

In our general orthogonal coordinate system the above would look as

$$\int\int\int \lim_{\alpha\to 0} D(\alpha; x, y, z \to u, v, w)\frac{du}{U}\frac{dv}{V}\frac{dw}{W} = 1$$

Note that we do not know how to transform D, so we could wrap up our ignorance by defining an intermediate variable as such

$$F(u, v, w) = \lim_{\alpha\to 0} D(\alpha; x, y, z \to u, v, w)\frac{1}{UVW}$$

In which case we have

$$\int\int\int F(u, v, w)dudvdw = 1$$

If we conveniently generalize the above by taking into account the possibility of a translation - because we are integrating over the entire universe anyway - then we have,

$$\int\int\int F(u - u', v - v', w - w')dudvdw = 1$$

Which has the exact same form as the expression that we started with, except we are looking at some other set of integration variables. So we could make the claim that

$$\begin{aligned} F(u - u', v - v', w - w') &= \delta(u - u')\delta(v - v')\delta(w - w') \\ &= \lim_{\alpha\to 0} D(\alpha; x, y, z \to u, v, w)\frac{1}{UVW} \end{aligned}$$

And the most interesting part of this problem is the application!

Le'ts think again about spherical coordinates! The lenght elements along the three orthogonal coordinates are dr, $rd\theta$, and $r\sin\theta d\phi$. So $U = 1$, $V = 1/r$, and $W = 1/r\sin\theta$. So a three-dimensional delta function is spherical coordinates becomes

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')\frac{1}{r^2\sin\theta}$$

One interesting way of writing the above is by using $\delta(\cos\theta)$. Since we have the composition of a function as an argument to our delta function, we can use the following formula

$$\delta(f(x)) = \sum_i \frac{1}{\left|\frac{df}{dx}(x_i)\right|}\delta(x - x_i)$$

where the sum extends over all simple roots.

So,

$$\delta(\cos\theta - \cos\theta') = \sum \frac{1}{|-\sin\theta'|}\delta(\theta - \theta') = \frac{\delta(\theta - \theta')}{\sin\theta'}$$

Thus,

$$\delta\left(\mathbf{x}-\mathbf{x}'\right)=\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\phi-\phi')\frac{1}{r^2}$$

Now, in cylindrical coordinates, the lenght elements are dr, $rd\theta$, and dz, so $U=1$, $V=1/r$, and $W=1$. And

$$\delta\left(\mathbf{x}-\mathbf{x}'\right)=\delta(r-r')\delta(\theta-\theta')\delta(z-z')\frac{1}{r}$$

1.3.2 Problem 1.3

Using Dirac Delta functions in the appropiate coordinate, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

(d) The same as part (c), but using spherical coordinates.

Chapter 2

Intro to QFT

2.1 Inivitation

2.1.1 Polarization Vectors

In the invitation, we are presented with the polarization vector $\epsilon^\mu = (0, 1, i, 0)$.

To understand this notation let's go through some examples taken from Lecture 14: Polarization.

We say that a plane wave is **linearly polarized** if there is no phase difference between E_x and E_y:

$$\mathbf{E}_0 = (E_x, E_y, 0)$$

Then a linearly polarized plane wave in the x direction looks like $\mathbf{E} = E_0 e^{i(kz-wt)}(1, 0, 0)$. And a linearly polarized plane wave in the y direction looks like $\mathbf{E} = E_0 e^{i(kz-wt)}(0, 1, 0)$.

The thing to remember is that we are implicitly only looking at the real part, so a linearly polarized wave in the x direction is actually $(E_0 \cos(kz - wt), 0, 0)$.

Circular polarization is when the electric field components are one-quarter out of phase $(\pi/2)$. Then the field can be written as,

$$\begin{aligned}
\mathbf{E}_0 &= \left(E_0, E_0 e^{i\pi/2}, 0\right) \\
&= (E_0, iE_0, 0) \\
&= E_0 \left(e^{i(kz-wt)}, ie^{i(kz-wt)}, 0\right)
\end{aligned}$$

And since we only care about the real parts,

$$\mathbf{E}_0 = (cos(kz - wt), -sin(kz - wt), 0)$$

This is interesting because if you jump to the wikipedia page for "List of trigonometric identities" and look for the "Shift by one quarter period" table, you'll see that

$$\sin(\theta + \frac{\pi}{2}) = \cos(\theta)$$

and

$$\cos(\theta + \frac{\pi}{2}) = -\cos(\theta)$$

(A shit by a quarter wavelength is essentially differentiation!)

But here is the catch and the connection with P&S: we can also get the same result if we have

$$\mathbf{E} = E_0 \left(\cos(wt - kz)\hat{x} + \sin(wt - kz)\hat{y} \right)$$

We can also write is as,

$$E_0 e^{i(wt-kz)}(0, 1, i, 0)$$

and remembering that you only care about the real part. If you do so, you'll end up with terms such as $\cos \hat{x} - \sin \hat{y}$ which just so happen to again be the a sin and a cos (our original plane wave components) shifted by $\pi/2$.

2.1.2 Cross Sections

The next thing we want to document is how to solve the differential cross section (per unit solid angle). The expression given was

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{cm}^2}(1 + \cos^2\theta)$$

That when integrated gives the total cross section

$$\sigma_{total} = \frac{4\pi\alpha^2}{3E_{cm}^2}$$

The trick here is to identity $d\Omega = \sin\theta \, d\theta \, d\phi$. So essentially the problem is to integrate

$$\begin{aligned}
\int d\Omega \; (1 + \cos\theta) &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (1 + \cos\theta) \sin\theta \, d\theta \, d\phi \\
&= \left(\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\phi \right) + \left(\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \cos\theta \sin\theta \, d\theta \, d\phi \right) \\
&= (4\pi) + \left(-\int u^2 du \right) \\
&= (4\pi) + \left(-\frac{1}{3} \cos^3\theta \Big|_0^{\pi} du \right) \\
&= (4\pi) \left(2\pi \frac{8}{3} \right) \\
&= \frac{16\pi}{3}
\end{aligned}$$

Note that we did a u-substitution in the second integral: $u = \cos\theta$, so $du = -\sin\theta\, d\theta$.

2.2 The Klein-Gordon Field

2.2.1 Klein-Gordon Inconsistensies

The Schrodinger equation canr readily be abtain treating the energy and momentum as operators. In quantum mechanics $E = i\partial_t$ and $p = -i\nabla$. Using the realtionship $E = \frac{p^2}{2}$, we get

$$i\partial_t \left|\psi\right\rangle = -\frac{1}{2m}\nabla^2 \left|\psi\right\rangle$$

Note that we are still in natural units, otherwise there would be an $\hbar$.

The Klein-Gordon equation comes if we instead use $E^2 - p^2 = m^2$,

$$-\partial_t^2 \left|\psi\right\rangle + \nabla^2 \left|\psi\right\rangle = m^2 \left|\psi\right\rangle$$

And remember that $\partial^2 = \partial_t^2 - \nabla^2$, so there comes $(\partial^2 - m)\psi = 0$ equation.

2.2.1.1 Klein-Gordon Solution

One method to solve the equation is to use Fourier transforms. So we can begin by expressing the wave function $\phi(x,t)$ as a Fourier integral,

$$\psi(x,t) = \int \frac{d^4k}{(2\pi)^4}\tilde{\psi}(k,\omega)e^{-i(k\cdot x - \omega t)}$$

Here, $\tilde{\psi}(k,\omega)$ is the Fourier transform of $\psi(x,t)$. One thing that's often missed is the rgument of the exponent: if you look back to examples where we talk about plane waves (just go back to the previous section), a wave with positive momentum moving in positive direction x has the argument $wt - xk$. And the convention is to hae such a wave in the Fourier transform.

With this Fourier transofrm in place, we take the Fourier transform of our entire differential equation to reduce the solution into an algebraic problem.

For example, applying the Fourier transform to the time derivative goes as follows:

$$\begin{aligned}
\mathcal{F}\{\partial_t\psi\} &= \int \frac{d^4k}{(2\pi)^4}\partial_t\psi e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \to \begin{bmatrix} u = e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} & v = \psi \\ du = i\omega e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\, dt & dv = \partial_t\psi\, dt \end{bmatrix} \\
&= \cancelto{0}{\psi(\mathbf{x},t)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}}\Big|_{t=-\infty}^{t=\infty} - \int \frac{d^4k}{(2\pi)^4}\psi(\mathbf{x},t)\,(i\omega)\,e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
&= -i\omega\tilde{\psi}(\mathbf{k},\omega)
\end{aligned}$$

We made use of the integration by parts technique ($\int u\,dv = uv - \int v\,du$) to transfer the time derivative on ψ to the exponent term.

As per the boudnary term, we have a couple words to say: physically, the wave function (or even field) should vanish at infinity. Mathematically, in order for the Fourier transform to converge to a value the function that is being "transformed", ψ, must be a **rapidly decreasing function** (a function of Schwartz space).

Similarly then,

$$
\begin{aligned}
\mathcal{F}\{\partial_t^2\psi\} &= \int \frac{d^4k}{(2\pi)^4}\partial_t^2\psi e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \rightarrow \begin{bmatrix} u = e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} & v = \partial_t\psi \\ du = i\omega e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,dt & dv = \partial_t^2\psi\,dt \end{bmatrix} \\
&= \cancelto{0}{\partial_t\psi(\mathbf{x},t)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}}\Bigg|_{t=-\infty}^{t=\infty} - \int \frac{d^4k}{(2\pi)^4}\partial_t\psi(\mathbf{x},t)\,(i\omega)\,e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
&= -i\omega\mathcal{F}\{\partial_t\psi\} \\
&= -\omega^2\tilde{\psi}(\mathbf{k},\omega)
\end{aligned}
$$

The boundary term again goes to zero here. We will skip the physical argument and just mention that a rapidly decreasin function also requires all of its derivatives tend to zero.

For the spatial derivatives, each ∂_i will bring down a factor of $-ik_i$, resulting in a factor of $-k_i^2$ for each ∂_i^2. And this is something that always triped me up: one may be inclined to write $-k^2$ in the Fourier transform but that same one ought to remember that k is actually $\vec{k}$ (a vector) and that is why we must write $-|k|^2$, because we want the [L-2] norm.

But anyways, the Fourier transform of the Laplacian is,

$$
\begin{aligned}
\mathcal{F}\{\nabla\psi\} &= \int \frac{d^4k}{(2\pi)^4}\nabla\psi e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \rightarrow \begin{bmatrix} u = e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} & v = \psi \\ du = -i\mathbf{k}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,d\mathbf{x} & dv = \nabla\psi\,d\mathbf{x} \end{bmatrix} \\
&= \cancelto{0}{\psi(\mathbf{x},t)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}}\Bigg|_{\mathbf{x}=-\infty}^{\mathbf{x}=\infty} - \int \frac{d^4k}{(2\pi)^4}\psi(\mathbf{x},t)\,(-i\mathbf{k})\,e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
&= -i\mathbf{k}\tilde{\psi}(\mathbf{k},\omega)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}\{\nabla^2\psi\} &= \int \frac{d^4k}{(2\pi)^4}\nabla^2\psi e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \rightarrow \begin{bmatrix} u = e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} & v = \nabla\psi \\ du = -i\mathbf{k}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,d\mathbf{x} & dv = \nabla^2\psi\,d\mathbf{x} \end{bmatrix} \\
&= \cancelto{0}{\nabla\psi(\mathbf{x},t)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}}\Bigg|_{\mathbf{x}=-\infty}^{\mathbf{x}=\infty} - \int \frac{d^4k}{(2\pi)^4}\nabla\psi(\mathbf{x},t)\,(-i\mathbf{k})\,e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
&= -i\mathbf{k}\mathcal{F}\{\nabla\psi\} \\
&= -|\mathbf{k}|^2\tilde{\psi}(\mathbf{k},\omega)
\end{aligned}
$$

And since the above must hold throughout all of space, that's where

$$\left(-\omega^2 + |\mathbf{k}|^2 + m^2\right) \tilde{\psi}(\mathbf{k}, \omega) = 0$$

comes from!

Now, in order for us to not have a trivial solution, $\psi(\mathbf{x}, t) = 0$, it must be so that $\left(-\omega^2 + |\mathbf{k}|^2 + m^2\right) = 0$, and so our dispersion relation comes about

$$\omega^2 = |\mathbf{k}|^2 + m^2$$

So $\omega = \pm\sqrt{|\mathbf{k}|^2 + m^2}$. So essentially any function will do as long as the dispersion relation (momentum conservation) is respected.

A thing that people do, specially since we are talking about Fourier transforms is to define

$$\psi(\mathbf{k}, \omega) = A(\vec{k})\delta\left(\omega - \sqrt{|\mathbf{k}|^2 + m^2}\right) + B(\vec{k})\delta\left(\omega + \sqrt{|\mathbf{k}|^2 + m^2}\right)$$

Note that $A(\vec{k})$ and $B(\vec{k})$ are arbitrary functions that only depend on $\mathbf{k}$ since the Dirac delta functions specifies the value for ω.

With that, we can finally write a solution for the differential equation we started with,

$$\begin{aligned}\psi(\mathbf{x}, t) &= \int \frac{d^4k}{(2\pi)^4} \left[A(\vec{k})\delta\left(\omega - \sqrt{|\mathbf{k}|^2 + m^2}\right) e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + B(\vec{k})\delta\left(\omega + \sqrt{|\mathbf{k}|^2 + m^2}\right) e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)}\right] \\ &= \int \frac{d^4k}{(2\pi)^4} \left[A(\vec{k})e^{-i(\mathbf{k}\cdot\mathbf{x} - \sqrt{|\mathbf{k}|^2 + m^2}t)} + B(\vec{k})e^{-i(\mathbf{k}\cdot\mathbf{x} + \sqrt{|\mathbf{k}|^2 + m^2}t)}\right]\end{aligned}$$

Hopefully thiis result makes sense: we obtained a family of function as solution and only initial conditions or boundary conditions will result in a specific sort of function.

2.2.1.2 Klein-Gordon Negative Density

The expression to compute a probability current is derived by using the analogous continuity equation from fluid dynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

Using the Schrodinger equation and using the fact that $\rho = |\psi|^2 = \psi^*\psi$ one can arrive at the probability density

$$\rho = -\frac{i}{2m}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right)$$

For the Klein Gordon equation, the way to massage it and get an expression for $\partial_t\rho$ and for its probability current is to: take Klein-Gordon multiply it by

ψ^*, take the complex conjugate multiply it by ψ, subtract the two and rearange to get something like the continuity equation.

Following those steps we have,

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\psi = 0$$

and

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\psi^* = 0$$

Multiplying them with ψ^* and ψ respectively we get,

$$\psi^*\partial_t^2\psi - \psi^*\nabla^2\psi + \psi^* m^2\psi = 0$$

and

$$\psi\partial_t^2\psi^* - \psi\nabla^2\psi^* + \psi m^2\psi^* = 0$$

Subtracting the latter from the former,

$$\begin{aligned}
&\psi^*\partial_t^2\psi - \psi^*\nabla^2\psi + \psi^* m^2\psi - \psi\partial_t^2\psi^* + \psi\nabla^2\psi^* - \psi m^2\psi^* \\
&= \left(\psi^*\partial_t^2\psi - \psi\partial_t^2\psi^*\right) - \left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) + \left(\psi^* m^2\psi - \psi m^2\psi^*\right) \\
&= \left(\psi^*\partial_t^2\psi - \psi\partial_t^2\psi^*\right) - \left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) \\
&= \partial_t\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right) - \nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)
\end{aligned}$$

The time derivative is then equated to $\partial_t\rho$ and the spatial derivatives to the current. though one interesting tangent to take here is concerning the missing $\frac{i}{2m}$ factor that these equations have.

If you remember, the imaginery part of of a complex number can be isolated by taking the different of it with its complex conjugate: $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$. (Recall that if $z = a + ib$, then $\operatorname{Im} z = b$, not ib.) It just so happens that we are doing the same sort of operation here, so we can throw an $i/2$ factor into our equation to ensure we get a real quantity.

For the $1/m$ factor we have to do some dimensional analysis. To get the probability we have to integrate $\int d^3x\, |\psi|^2$, which is dimensionless. So the $|\psi|^2$ term needs to cancel out the integration over space. Hence $[|\psi|^2] = [L]^{-3} = [M]^3$. From there we can say that $[\psi] = [L]^{-3/2}$ and $[\psi^*\partial_t^2\psi] = [L]^{-3/2}[L]^{-2}[L]^{-3/2} = [L]^{-5} = [M]^5$.

This same term we just evaluated needs to match the dimensions of the time derivative, $[L]^{-1}$,a probability density, $[L]^{-3}$. It should be then that $[\partial_t\rho] = [L]^{-4} = [M^4]$. But hey, what a coincidence that if we were to divide $\psi^*\partial_t^2\psi$ by a mass that we would get just the right dimensions for our probability current! And so it goes that the expression we are used to seeing turns out to be excused as such in the Klein-Gordon case.

Suppose we had a $\psi = e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$ as solution.

Then,

$$\begin{aligned}
\rho &= -\frac{i}{2m}\left(e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\partial_t e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} - e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\partial_t e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)\\
&= -\frac{i}{2m}\left(e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\left(i\omega\right)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} - e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\left(-i\omega\right)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)\\
&= -\frac{i}{2m}\left(i\omega\right)(2)\\
&= \frac{\omega}{m}
\end{aligned}$$

And since $\omega = \pm\sqrt{|\mathbf{k}|^{+}m^2}$, then the density can be negative!

2.2.1.3 A Propagator

Describes the amplitude for a particle to propagate from one point to another.

With a source $j(\mathbf{x},t)$, the Klein-Gordon equation becomes

$$\left(\partial^2 + m^2\right)\psi(\mathbf{x},t) = j(\mathbf{x},t)$$

In momentum space the equation is

$$\left(-\omega^2 + |\mathbf{k}|^2 + m^2\right)\tilde{\psi}(\mathbf{k},\omega) = \tilde{j}(\mathbf{k},\boldsymbol{\omega})$$

$$\tilde{\psi}(\mathbf{k},\omega) = \frac{\tilde{j}(\mathbf{k},\boldsymbol{\omega})}{\omega^2 - |\mathbf{k}|^2 - m^2 + i\epsilon}$$

$i\epsilon$ Feynman's prescription.

The propagator $D_F(x-y)$ is the inverse transform of $\frac{1}{\omega^2-|\mathbf{k}|^2-m^2+i\epsilon}$

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}}{\omega^2 - |\mathbf{k}|^2 - m^2 + i\epsilon}$$

2.2.2 Causality Arguments: The Non-relativistic Case

Consider the amplitude for a free particle to propagate from $\mathbf{x}_0$ to $\mathbf{x}$

$$U(t) = \langle\mathbf{x}|\, e^{-iHt}\,|\mathbf{x}_0\rangle$$

In nonrelativistic quantum mechanics we have $E = \frac{1}{2m}p^2$, so

$$\begin{aligned}
U(t) &= \langle\mathbf{x}|\, e^{-i(\mathbf{p}^2/2m)t}\,|\mathbf{x}_0\rangle && (2.2.1)\\
&= \int d^3p\ \langle\mathbf{x}|\, e^{-i(\mathbf{p}^2/2m)t}\,|\mathbf{p}\rangle\,\langle\mathbf{p}|\mathbf{x}_0\rangle && (2.2.2)\\
&= \int \frac{d^3p}{(2\pi)^3} e^{-i(\mathbf{p}^2/2m)t} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} && (2.2.3)\\
&= \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t} && (2.2.4)
\end{aligned}$$

Note: we think Peskin and Schroeder have a typo when doing the expansion in the momentum state. The integral starts with $\int \frac{d^3p}{(2\pi)^3}$ but the identity used to project the eigenstate into momentum state is just $\mathbb{I} = \int dp \, |p\rangle \langle p|$, see the section below. There is a $1/(2\pi)^3$ that comes from simplifying $\langle \mathbf{x}|\mathbf{p}\rangle \langle \mathbf{p}|\mathbf{x}_0\rangle$ though. More on this later.

To go from 2.2.1 to 2.2.2, we changed the basis to momentum eigenstates in order to apply the Hamiltonian operator and "extract" the exponential. If we have the operator $\hat{p}$ act on a pure momentum eigenstate $|p\rangle$, then we get $\hat{p}\,|p\rangle = p\,|p\rangle$. So changing basis helps us work with real quantities. See the next section on formalism for more details (its a brief summary of stuff from quantum mechanics).

When you see the integrals over states in quantum mechanics, especially when involving time evolution and position states, the calculation often utilizes the well-known expressions for the overlap of position and momentum eigenstates $\langle \mathbf{x}|\mathbf{p}\rangle$ and $\langle \mathbf{p}|\mathbf{x_0}\rangle$. These expressions are crucial in converting between position and momentum representations, which allows us to analyze the system's dynamics. In such cases, we have these useful identities - the next section will remind you where these come from:

$$\langle \mathbf{x}|\mathbf{p}\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{x}}$$

and

$$\langle \mathbf{p}|\mathbf{x_0}\rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{x_0}}$$

We used these identities to go from 2.2.2 to 2.3.1,

$$\begin{aligned} &\int \frac{d^3p}{(2\pi)^3} e^{-i(\mathbf{p}^2/2m)t} \langle \mathbf{x}|\mathbf{p}\rangle \langle \mathbf{p}|\mathbf{x}_0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-i(\mathbf{p}^2/2m)t} \frac{1}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0} \end{aligned}$$

If these details don't ring a bell, read the following section.

As per the solution, see 2.3.2. The trick is to complete the square for the arguments of the exponentials and extract a Gaussian integral.

2.2.2.1 A Bit of Formalism

The couple facts used in the above integral go as follows.

First, this comes from the section "outer products" in Wikipedia: Bra-Ket Notation, we have to keep in mind that $|\psi\rangle \langle\psi|$ defines an **outer product**. In a finite-dimensional vector space the outer product is defined as

$$|\phi\rangle\langle\psi| = \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_N\end{pmatrix}\begin{pmatrix}\psi_1^* & \psi_2^* & \dots & \psi_N^*\end{pmatrix} = \begin{pmatrix}\phi_1\psi_1^* & \phi_1\psi_2^* & \dots & \phi_1\psi_N^*\\ \phi_2\psi_1^* & \phi_2\psi_2^* & \dots & \phi_2\psi_N^*\\ \vdots & \vdots & \ddots & \vdots\\ \phi_N\psi_1^* & \phi_N\psi_2^* & \dots & \phi_N\psi_N^*\end{pmatrix}$$

One of the uses of the outer product is to construct **projection operators**. Given a ket $|\psi\rangle$ of norm 1, the orthogonal projection onto the subspace spanned by $|\psi\rangle$ is $|\psi\rangle\langle\psi|$.

The "Unit operator" section on Wikipedia: Bra-Ket Notation, also has this: if we have a complete orthonormal basis $\{e_i|i \in \mathbb{N}\}$, functional analysis tells us that any $|\psi\rangle$ can also be written as

$$|\psi\rangle = \sum_{i\in\mathbb{N}} \langle e_i|\psi\rangle |e_i\rangle \tag{2.2.5}$$

This is how we "project" ψ into a new basis.

It mentions that it can also be shown that

$$\mathbb{I} = \sum_{i\in\mathbb{N}} |e_i\rangle\langle\psi|$$

There is also this result called resolution of the identity in Borel functional calculus that allows us to generalized this result to the continuous case,

$$\mathbb{I} = \int dx\ |x\rangle\langle x| = \int dp\ |p\rangle\langle p|$$

The analogous to 2.2.5 in the continuous case is

$$|\psi\rangle = \int dx\, \psi(x)\, |x\rangle$$

where $\psi(x) = \langle x|\psi\rangle$ by analogy.

The next bit of muscle memory to train is to remember that

$$\delta^4(k) = \int \frac{d^4x}{(2\pi)^4} e^{ik\cdot x}$$

and that

$$\langle x|x'\rangle = \delta(x - x')$$

Along with the following bit of formalism comes from James Binney's book, section 2.3.2.

Position operator $\hat{x}$ acts in position space as $\hat{x}\psi = x\psi$. The momentum operator in the position representation is $\hat{p} = -i\nabla$.

A position eigenstate $|x\rangle$ satisfies $\hat{x}|x\rangle = x|x\rangle$. Similarly, momentum eigenstate $|p\rangle$ satisfies $\hat{p}|p\rangle = p|p\rangle$.

The state $|p\rangle$ in which a measurement of the momentum will yield the value p has to be an eigenstate of $\hat{p}$. To find the wave function $u_p(x) = \langle x|p\rangle$ of this important state, $|p\rangle$, we can use the following argument

$$\hat{p}\phi(x) = -i\nabla\phi(x) = p\phi$$

This results in a differential equation with a simple solution,

$$-i\nabla u_p(x) = pu_p(x) \rightarrow u_p(x) = Ae^{ipx}$$

Thus the wavefunction of a particle of well-defined momentum is a plane wave. To normalize the wave function, we need to find A which can be done as follows:

$$\begin{aligned}
\delta(p-p') &= \langle p|p'\rangle \\
&= \int dx\ \langle p|x\rangle \langle x|p'\rangle \\
&= |A|^2 \int dx\, u_p^*(x) u_{p'}(x) \\
&= |A|^2 \int dx\, e^{-ipx} e^{ip'x} \\
&= |A|^2 \int dx\, e^{i(p-p')x} \\
&= 2\pi |A|^2 \delta(p-p')
\end{aligned}$$

Here we used $\mathbb{I} = \int dx\ |x\rangle \langle x|$ again. Along with $\delta(x-x') = \int \frac{dk}{2\pi} e^{ik(x-x')}$.

2.2.3 Causality Arguments: The Relativistic Case

And now let's look at

$$\begin{aligned}
U(t) &= \langle \mathbf{x}|\, e^{-it\sqrt{\mathbf{p}^2+m^2}}\, |\mathbf{x}_0\rangle \\
&= \int d^3p\ \langle \mathbf{x}|\, e^{-it\sqrt{\mathbf{p}^2+m^2}}\, |\mathbf{p}\rangle \langle \mathbf{p}|\mathbf{x}_0\rangle \\
&= \int d^3p\, e^{-it\sqrt{\mathbf{p}^2+m^2}}\, \langle \mathbf{x}|\mathbf{p}\rangle \langle \mathbf{p}|\mathbf{x}_0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\, e^{-it\sqrt{\mathbf{p}^2+m^2}} e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}_0} \\
&= \int \frac{d^3p}{(2\pi)^3}\, e^{-it\sqrt{\mathbf{p}^2+m^2}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}
\end{aligned}$$

The result then happens to be

$$\frac{1}{2\pi^2|\mathbf{x}-\mathbf{x}_0|}\int_0^\infty dp\, p \sin(p|\mathbf{x}-\mathbf{x}_0|) e^{-it\sqrt{\mathbf{p}^2+m^2}}$$

The trick for this one is to use spherical variables to simplify it a bit.

$$\int \frac{d^3p}{(2\pi)^3}\, e^{-it\sqrt{\mathbf{p}^2+m^2}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}$$

becomes

$$\int \frac{dp}{(2\pi)^3} d\Omega\, p^2 e^{-it\sqrt{p^2+m^2}} e^{ip\mathbf{r}\cos\theta}$$

where $r = |\mathbf{x}-\mathbf{x}_0|$ and $p = |\mathbf{p}|$.

From here we can do the angular integrals first,

$$\begin{aligned}
\int d\Omega\, e^{ipr\cos\theta} &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta e^{ipr\cos\theta} \\
&= 2\pi \int_0^{\pi} d\theta\, \sin\theta e^{ipr\cos\theta} \\
&= -2\pi \int_1^{-1} du\, e^{ipru} \\
&= -2\pi \left(\frac{e^{ipru}}{ipr}\right)\Big|_{u=1}^{u=-1} \\
&= -\frac{2\pi}{ipr}\left(e^{-ipr} - e^{ipr}\right)
\end{aligned}$$

Next, we can make use of the formula $\operatorname{Im} z = \frac{z-\overline{z}}{2i}$ and apply it to $z = e^{ix}$. Or just do it manually: $e^{ix} = \cos + i\sin$, $e^{-ix} = \cos - i\sin$, so $e^{-ix} - e^{ix} = -i\sin - i\sin = -2i\sin$.

$$\begin{aligned}
\int d\Omega\, e^{ipr\cos\theta} &= -\frac{2\pi}{ipr}\left(e^{-ipr} - e^{ipr}\right) \\
&= -\frac{2\pi}{ipr}\left(-2i\sin(pr)\right) \\
&= \frac{4\pi}{pr}\sin(pr)
\end{aligned}$$

Pluging this back in, We got

$$\begin{aligned}
\int \frac{dp}{(2\pi)^3} d\Omega\, |p|^2 e^{-it\sqrt{p^2+m^2}} e^{ipr\cos\theta} &= \int \frac{dp}{(2\pi)^3} p \frac{4\pi}{r}\sin(pr) e^{-it\sqrt{p^2+m^2}} \\
&= \frac{1}{2\pi^2|\mathbf{x}-\mathbf{x}_0|}\int_0^{\infty} dp\, p \sin(p|\mathbf{x}-\mathbf{x}_0|) e^{-it\sqrt{p^2+m^2}}
\end{aligned}$$

2.3 Integrals in QFT

2.3.1 Kinetic operator and a Green Function

This problem is part of MontePython: Implementing Quantum Monte Carlo using Python.

There is a kinetic operator defined as

$$G_K\left(\mathbf{R}', \mathbf{R}, t\right) = \frac{1}{(2\pi)^{3N}} \int e^{-i\mathbf{k}\mathbf{R}'} e^{-Dtk^2} e^{-i\mathbf{k}\mathbf{R}} d\mathbf{k}$$

This integral then is integrated and the following Green function is obtained

$$G_K\left(\mathbf{R}', \mathbf{R}, t\right) = \frac{1}{(4\pi Dt)^{3N/2}} e^{-\left(\mathbf{R}-\mathbf{R}'\right)/4Dt}$$

And as we saw in Jackson problem 1.2, the Green function we just got is equivalent to $\delta\left(\mathbf{R}-\mathbf{R}'\right)$.

But coming back to the integral, the trick is to complete the square!

$$\begin{aligned} G_K\left(\mathbf{R}', \mathbf{R}, t\right) &= \frac{1}{(2\pi)^{3N}} \int e^{-i\mathbf{k}\mathbf{R}'} e^{-Dtk^2} e^{-i\mathbf{k}\mathbf{R}} d\mathbf{k} \\ &= \frac{1}{(2\pi)^{3N}} \int e^{-Dtk^2 + -i\mathbf{k}\cdot\left(\mathbf{R}-\mathbf{R}'\right)} d\mathbf{k} \end{aligned}$$

This last integral has the form $\int e^{ax^2+bx}dx$, which can be solved as follows,

$$\begin{aligned} \int e^{ax^2+bx}dx &= \int \exp\left\{a\left(x^2 + \frac{b}{a}x\right)\right\} dx \\ &= \int \exp\left\{a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right)\right\} dx \\ &= \int \exp\left\{a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}\right\} dx \\ &= e^{-b^2/4a} \int e^{a\left(x+\frac{b}{2a}\right)^2} dx \end{aligned}$$

Now, let's look at $\int e^{a(x+b/2a)^2}dx$. First, we need to make a transformation. Let's define $u = x + b/2a$, then $du = dx$, the measure remains invariant (so do the limits of integration), so $\int e^{a(x+b/2a)^2}dx = \int e^{au^2}du$. Now, if we define $a = -c$, then

$$\begin{aligned} \int e^{a(x+b/2a)^2}dx &= \int e^{au^2}du \\ &= \int e^{-cu^2}du \\ &= \sqrt{\frac{\pi}{c}} = \sqrt{\frac{\pi}{-a}} \end{aligned}$$

This whole thing works if $a < 0$ but it turns out that this is also valid if $\mathrm{Re}\,(a) \leq 0$ but $a \neq 0$.

Putting everything back together,

$$\int e^{ax^2+bx}dx = \sqrt{\frac{\pi}{-a}}e^{-b^2/4a}$$

Looking back at our original problem, we can chose $a = -Dt$ and $b = i\mathbf{r}$, where $\mathbf{r} = \mathbf{R} - \mathbf{R}'$. and so

$$\begin{aligned}
G_K\left(\mathbf{R}', \mathbf{R}, t\right) &= \frac{1}{(2\pi)^{3N}} \int e^{-Dtk^2+-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \\
&= \frac{1}{(2\pi)^{3N}} \int e^{-Dt\left(k^2+\frac{i\mathbf{k}}{Dt}\cdot\mathbf{r}\right)} d\mathbf{k} \\
&= \frac{1}{(2\pi)^{3N}} \int e^{-Dt\left(k^2+\frac{i\mathbf{r}}{Dt}\cdot\mathbf{k}+\left(\frac{i\mathbf{r}}{2Dt}\right)^2-\left(\frac{i\mathbf{r}}{2Dt}\right)^2\right)} d\mathbf{k} \\
&= \frac{1}{(2\pi)^{3N}} \int e^{-Dt\left(k+\frac{i\mathbf{r}}{2Dt}\right)^2-\frac{\mathbf{r}^2}{4Dt}} d\mathbf{k} \\
&= \frac{1}{(2\pi)^{3N}} e^{-\mathbf{r}^2/4Dt} \int e^{-Dt\left(k+\frac{i\mathbf{r}}{2Dt}\right)^2} d\mathbf{k}
\end{aligned}$$

Again, since $\int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$, then

$$\int e^{-Dt\left(k+\frac{i\mathbf{r}}{2Dt}\right)^2} dk = \sqrt{\frac{\pi}{Dt}}$$

Thus,

$$\begin{aligned}
G_K\left(\mathbf{R}', \mathbf{R}, t\right) &= \frac{1}{(2\pi)^{3N}} e^{-\mathbf{r}^2/4Dt} \int e^{-Dt\left(k+\frac{i\mathbf{r}}{2Dt}\right)^2} d\mathbf{k} \\
&= \frac{1}{(2\pi)^{3N}} e^{-\mathbf{r}^2/4Dt} \left(\sqrt{\frac{\pi}{Dt}}\right)^{3N} \\
&= \frac{1}{(4\pi^2)^{3N/2}} \frac{\pi^{3N/2}}{(Dt)^{3N/2}} e^{-\mathbf{r}^2/4Dt} \\
&= \frac{1}{(4\pi Dt)^{3N/2}} e^{-\left(\mathbf{R}-\mathbf{R}'\right)^2/4Dt}
\end{aligned}$$

2.3.2 Klein-gordon Non-relativistic Amplitude

The following integral comes from Peskin and schroeder Section 2.1

$$\int \frac{d^3p}{(2\pi)^3}\, e^{-i(\mathbf{p}^2/2m)t} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \tag{2.3.1}$$

$$= \left(\frac{m}{2\pi it}\right)^{3/2} e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t} \tag{2.3.2}$$

This integral follows the same procedure as 2.3.1.

Here we have $a = -\frac{it}{2m}$ and $b = i(\mathbf{x} - \mathbf{x_0})$. Since $\operatorname{Re} a = -t/2m \leq 0$ and $a \neq 0$ when $t \neq 0$ oru solution is valid for $t \neq 0$. Then

$$\sqrt{\frac{\pi}{-a}} = \sqrt{\frac{2\pi m}{it}}$$

and

$$\begin{aligned} -\frac{b^2}{4a} &= \frac{2m\left(\mathbf{x} - \mathbf{x}_0\right)^2}{-4it} \\ &= \frac{im\left(\mathbf{x} - \mathbf{x}_0\right)^2}{2t} \end{aligned}$$

So the solution is

$$\begin{aligned} &\frac{1}{(2\pi)^3}\left(\frac{2\pi m}{it}\right)^{3/2} e^{\frac{im(\mathbf{x}-\mathbf{x}_0)^2}{2t}} \\ &= \left(\frac{m}{2\pi it}\right)^{3/2} e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t} \end{aligned}$$

2.4 Tensors

Working with a spacetime with a metric signature $(+ - --)$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$x^\mu = (x^0, \mathbf{x})$ and $x_\mu = g_{\mu\nu}x^\nu = (x^0, -\mathbf{0})$. To see this explciitly, let's carry out the summations.

When $\mu = 0$, $x_0 = g_{00}x^0 + g_{01}x^1 + g_{02}x^2 + g_{03}x^3 = x^0 + 0 + 0 + 0 = x^0$. For $\mu = 1$, $x_1 = g_{10}x^0 + g_{11}x^1 + g_{12}x^2 + g_{13}x^3 = 0 + x^1 + 0 + 0 = -x^0$. Similarly, when $\mu = 2$, $x_2 = -x^2$, and when $\mu = 3$, $x_3 = -x^3$.

2.4.1 Inner Product

Why does $p \cdot x = g_{\mu\nu}p^\mu x^\nu = g^{\mu\nu}p_\mu x_\nu = p_\mu x^\nu = p^\mu x_\nu = p^0x^0 - \mathbf{p} \cdot \mathbf{x}$?

The metric tensor defines the geometry of spacetime, including the way distances and angles are measured. The implied summation, einstein summation, effectively 'weights' the components according to the geometry of spacetime.

$g_{\mu\nu}$ lowers indices, while $g^{\mu\nu}$ raises them. δ^μ_ν is a diagonal matrix with ones on the diagonal and zeros elsewhere. It selects the μ-th component when used in a summation, acting like an identity element.

$g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$

$$g^{\mu\alpha}g_{\alpha\nu} = \sum_{\alpha=0} g^{\mu\alpha}g_{\alpha\nu} = g^{\mu 0}g_{0\nu} + g^{\mu 1}g_{1\nu} + g^{\mu 2}g_{2\nu} + g^{\mu 3}g_{3\nu}$$

This is essentially a matrix multiplication.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Going component by component, take the second row and second column, $\mu = 1$ and $\nu = 1$. The element for that entry is given by

$$g^{1\alpha}g_{\alpha 1} = g^{10}g_{01} + g^{11}g_{11} + g^{12}g_{21} + g^{13}g_{31} = 0 + (-1)(-1) + 0 + 0 = 1$$

Chapter 3

Variational Calculus

Normed linear space - we want to define ”closeness” via a norm and continuity to study functionals that look like

$$J[y] = \int_a^b F(x, y, y')dx$$

where the appropiate function space is the set of all functions with two continuous derivatives.