

# Notes on Complex Analysis

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# 1 Complex Numbers

## 1.1 Basic Algebraic Properties

A handy thing to keep written down

$$z^{-1} = \left( \frac{a}{z^2 + b^2}, \frac{-b}{a^2 + y^2} \right)$$

The way to think about it is as if you wanted to find  $c$  and  $d$  such that

$$\frac{1}{a + bi} = c + di$$

The trick here is as follows

$$\frac{1}{a + bi} \left( \frac{a - bi}{a - bi} \right) = \frac{a - bi}{a^2 + b^2}$$

Also,

$$|z|^2 = z\bar{z} = (a + ib)(a - ib) = a^2 + b^2$$

The generalization of  $|z|^2 = (\Re(z))^2 + (\Im(z))^2$  does hold!

Note that the product of two complex numbers is very different from the scalar or vector products done in vector spaces over the reals. This notion of a **bilinear form** is what is often used to distinguish between different algebras.

Also note that  $z_1 < z_2$  has no meaning, so the order field properties we are used to from real numbers don't apply as such. However  $|z_1| < |z_2|$  does make sense.

The distance between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $|z_1 - z_2|$ .

The complex numbers lying on a circle with center  $z_0$  and radius  $R$  satisfy the equation

$$|z - z_0| = R$$

A wonderful example of this last interpretation is

$$|z - 3i| + |z + 3i| = |z - 3i| + |z - (-3i)| = 12$$

This equation represents the set of all points whose distance from the two set points,  $F_1(0, 3)$  and  $F_2(0, -3)$ , is 12. This turns out to be the ellipse with foci  $F_1(0, 3)$  and  $F_2(0, -3)$ . Kline has some great exercises to get you acquainted with Ellipses, parabolas, and hyperbolas.

### 1.1.1 Exercises

## 2.2

Some interesting properties

$$z + \bar{z} = (a + ib) + (a - ib) = 2a = 2\Re(z)$$

Similarly,

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2\Im(z)$$

Following the same mechanics,

$$\Re(iz) = \Re(i(a + ib)) = \Re(ai - b) = -\Im(z)$$

And

$$\Im(iz) = \Im(ai - b) = \Re(z)$$

## 1.2 Triangle Inequality

There is a brilliant example in this section, go read it!

The heart of the example is in noticing that the triangle inequality gives us an upper and a lower bound for the sum of two numbers. The upper bound comes from

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and the lower bound from

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

### 1.2.1 polynomials

If  $n$  is a positive integer, and if  $a_0, a_1, a_2, \dots$  are complex constants, where  $a_n \neq 0$ ,

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$$

if a polynomial of degree  $n$ . For some positive number  $R$ , the reciprocal  $1/P(z)$  satisfies the inequality

$$\left| \frac{1}{P(z)} \right| < \frac{2}{|a_n|R^n}$$

whenever  $|z| > R$ . Geometrically, this tells us that the modulus of the reciprocal  $1/P(z)$  is bounded from above when  $z$  is exterior to the circle  $|z| = R$ .

The tricky bit of the argument for the above is given when the author says "Now that a sufficiently large positive number  $R$  can be found such that each of the quotients on the right in inequality (9) is less than the number  $|a_n|/(2n)$  when  $|z| > R$ ..." What this means in practice is that when we have

$$|w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \dots + \frac{|a_{n-1}|}{|z|}$$

We are actually dividing  $|w||z|^n$  by  $R^n$  since  $R^n \sim |z|^n$ , and so Churchill is saying that we can find some  $z$  such that when we raise it to a power  $n$  and divide  $|w||z|^n$  by it, that we will have all  $n$  terms on the right side of the inequality be less than the  $n$ -th term divided by  $n$  (go far out enough and there will be a  $z$  value that will make this truth).

The rest of the argument in the book is algebra.

### 1.2.2 Exercises

#### Ex 4

Verify  $\sqrt{2}|z| \geq |\Re(z)| + |\Im(z)|$ . Suggestion: reduce the above inequality to  $(|x| - |y|)^2 \geq 0$ .

This exercise is similar to one in Folland's advanced calculus, exercise 2 in section 1.1 Euclidean Spaces and Vectors (we do provide a solution for that example in the calculus doc).

Working backwards,

$$(\sqrt{2}|z|)^2 = 2|z|^2 = 2x^2 + 2y^2$$

Following the tip of the book, a thing to try is something like

$$(|\Re(z)| + |\Im(z)|)^2 = |x|^2 + |y|^2 + 2|x||y| = x^2 + y^2 + 2|x||y|$$

Then

$$\begin{aligned} (\sqrt{2}|z|)^2 - (|\Re(z)| + |\Im(z)|)^2 &= x^2 + y^2 - 2|x||y| \\ &= |x|^2 + |y|^2 - 2|x||y| \\ &= (|x| - |y|)^2 \geq 0 \end{aligned}$$

#### Ex 6

Using the fact that  $|z_1 - z_2|$  is the distance between the points  $z_1$  and  $z_2$ , give a geometric argument that  $|z - 1| = |z + i|$  represents the line through the origin whose slope is  $-1$ .

This one is a very interesting case.  $|z - 1|$  would correspond to the distance from a point  $z$  to the point 1 (the x-axis),  $|z + i|$  would correspond to the distance from a point  $z$  to the point  $-i$  (the y-axis). Since both are a unit from the origin, in their respective axis. The expression above then equates these two distances giving us a straightline that passes through the origin with a  $-1$  slope.

**Ex 7**

Show that for  $R$  sufficiently large, the polynomial  $P(z)$  satisfies

$$|P_n(z)| \leq 2|a_n||z|^n$$

whenever  $|z| > R$ . Suggestion: observe that there is a positive number  $R$  such that the modulus of each quotient in  $|w| \leq |a_0|/|z|^n + |a_1|/|z|^{n-1} + \dots + |a_{n-1}|/|z|$  is less than  $|a_n|/n$  when  $|z| > R$ .

In the argument in which the original inuequality was made, there was a step where we use the following  $|a_n + w| \geq ||a_n| - |w||$  to come up with a lower bound. If we instead looked for an upper bound, we could look at

$$|a_n + w| \leq |a_n| + |w| < |a_n| + \frac{1}{2}|a_n| < 2|a_n|$$

The rest of the argument flows when we plug that into expression (10),

$$|P_n(z)| = |a_n + w||z|^n < 2|a_n||z|^n$$

**Ex 8**

Let  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ . Use simple algebra to show that

$$|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

then point out how the identity  $|z_1 z_2| = |z_1||z_2|$  follows.

The trick to the first part is to make use of  $|z| = \sqrt{(\Re(z))^2 + (\Im(z))^2}$   
First of,

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

From there, we can see that

$$\begin{aligned} \Re(z_1 z_2)^2 &= (x_1x_2 - y_1y_2)(x_1x_2 - y_1y_2) \\ &= (x_1x_2)^2 + (y_1y_2)^2 - 2(x_1x_2)(y_1y_2) \end{aligned}$$

and

$$\begin{aligned} \Im(z_1 z_2)^2 &= (x_1y_2 + x_2y_1)(x_1y_2 + x_2y_1) \\ &= (x_1y_2)^2 + (x_2y_1)^2 + 2(x_1x_2)(y_1y_2) \end{aligned}$$

It then follows that

$$\begin{aligned} |z_1 z_2| &= \sqrt{(x_1x_2)^2 + (y_1y_2)^2 - 2(x_1x_2)(y_1y_2) + (x_1y_2)^2 + (x_2y_1)^2 + 2(x_1x_2)(y_1y_2)} \\ &= \sqrt{(x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (x_2y_1)^2} \\ &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \end{aligned}$$

Since  $|z| = \sqrt{x^2 + y^2}$ , we can see how the above reordering is equivalent to

$$\begin{aligned} |z_1 z_2| &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ &= \sqrt{x_1^2 + y_1^2} \sqrt{x_2^2 + y_2^2} \\ &= |z_1| |z_2| \end{aligned}$$

### Ex 9

If we use the result from the previous exercise and assume have  $z = z_1 = z_2$ , we have

$$|z^2| = |z| |z| = |z|^2$$

We could use this as the base case for an induction argument ( $n = 2$ ).

Then for our hypothesis, we assume that  $|z^m| = |z|^m$ , when  $n = m$ , so it must also hold for  $n = m + 1$ ,

$$|z^{m+1}| = |z^m z| = |z^m| |z| = |z|^m |z| = |z|^{m+1}$$

## 1.3 Complex Conjugates

### 1.3.1 Exercises

#### Ex 4

Show that,

$$\overline{z^2} = \bar{z}^2$$

Note that

$$\begin{aligned} \overline{z^2} &= \overline{z z} \\ &= \bar{z} \bar{z} \\ &= \bar{z}^2 \end{aligned}$$

#### Ex 10

Show that,

1.  $z$  is real if and only if  $z = \bar{z}$
2.  $z$  is either real or pure imaginary if and only if  $\bar{z}^2 = z^2$

To see the why for the first one, note that if  $z = x + iy = x - iy = \bar{z}$ , then  $(x, y) = (x, -y)$ . Saying that  $x = x$  leads us to a tautology, so no issue there, but  $y = -y$  leads us to say that  $z$  must be real.

Using a similar argument,

$$\bar{z}^2 = \overline{z^2} = \overline{x^2 - y^2 + 2ixy} = x^2 - y^2 + 2ixy$$

The last step has  $xy = -xy$ , so this must be zero. And this happens when  $z$  is either real or pure imaginary.

**Ex 13**

Show that the equation  $|z - z_0| = R$  of a circle, centered at  $z_0$  with radius  $R$  can be written as

$$|z|^2 - 2\Re(z\bar{z}_0) + |z_0|^2 = R^2$$

To see why, note that

$$\begin{aligned} |z - z_0|^2 &= (z - z_0)(\bar{z} - \bar{z}_0) \\ &= (z - z_0)(\bar{z} - \bar{z}_0) \\ &= z\bar{z} - z\bar{z}_0 - z_0\bar{z} + z_0\bar{z}_0 \\ &= |z|^2 - (z\bar{z}_0 + \bar{z}z_0) + |z_0|^2 \\ &= |z|^2 - 2\Re(z\bar{z}_0) + |z_0|^2 \end{aligned}$$

**Ex 14**

Show that the hyperbola  $x^2 - y^2 = 1$  can be rewritten as  $z^2 - \bar{z}^2 = 2$ .

Well,  $z^2 = (x + iy)(x + iy) = x^2 + 2ixy - y^2$ . And we already saw that  $\bar{z}^2 = \overline{z^2}$ , so  $\bar{z}^2 = x^2 - 2ixy - y^2$ . From there the answer follows.

## 1.4 De Moivre's Theorem

$$(a + ib)^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

De Moivre's Theorem

## 1.5 Roots of Complex Numbers

$n$ th roots: for any positive integer  $n$ , the  $n$ th distinct roots of  $(a + ib)^n = r^n(\cos nx + i \sin nx)$  are

$$r^{\frac{1}{n}} \left[ \cos \frac{x + 2\pi k}{n} + i \sin \frac{x + 2\pi k}{n} \right]$$

for  $k = 0, 1, \dots, n - 1$ .

## 2 Infinite Series, Products, and Integrals

### 2.1 Uniform Convergence

**Note:** when we speak of uniform convergence, the interval can be closed or open. Titchmarsh just uses  $(a, b)$  to cover the general case.

The more general case for the first test of uniform Convergence we see is stated as follows:

The series  $\sum u_n(x)$  is uniformly convergent ( $\forall \epsilon > 0$ , we can find  $n_0 \geq N$  depending on  $\epsilon$  but not on  $x$ , such that  $|s(x) - s| < \epsilon$ , for every  $n \geq n_0$  for every value in  $(a, b)$ ) if  $|u_n(x)| \leq v_n(x)$ , and  $\sum v_n(x)$  is uniformly convergent.

If we try to make an argument by contradicton and assume that  $\sum u_n(x)$  is not uniformly convergent, then the series could still converge but it could be the case that as  $x$  approaches some point on the interval  $(a, b)$ ,  $n_0$  may become infinitely large. Additionally, the series could just be a divergent series. Either way it means we are not able to find an  $n_0$  such that  $|s(x) - s| < \epsilon$  for any  $n \geq n_0$ , for any  $\epsilon > 0$  and for any  $x \in (a, b)$ . This means that

$$|u_{n+1}(x) + u_{n+2}(x) + \dots|$$

keeps on changing as  $n$  or  $x$  change.

Since,

$$|u_{n+1}(x) + u_{n+2}(x) + \dots| \leq |u_{n+1}(x)| + |u_{n+2}(x)| + \dots$$

Then any  $v_n(x)$  such that  $v_{n+1}(x) \geq |u_{n+1}(x)|$  would also grow indefinitely and thus lead to a contradicton.

#### Examples

A proof to see why  $\sum_{n=0} x^n$  is uniformly convergent in  $c \in [a, b]$  when  $-1 < a < b < 1$  can be seen by comparing the exercise 2.5.3 from Abbott.

The trigonometric series  $\sum_{n=1} \frac{\cos nx}{n^2}$  is convergent anywhere because  $-1 \leq \cos x \leq 1$  so  $|\frac{\cos nx}{n^2}|$  behaves like a convergent p-series.

Similarly the Dirichlet series  $\sum_{n=1} n^{-s} = \sum_{n=1} \frac{1}{n^s}$  is uniformly convergent in  $x \in [a, b]$  if  $1 < a < b$  because its absolute value is equal to a convergent p series. The sum is referred to as the Riemann zeta function  $\zeta(s)$ .