Notes on Complex Analysis

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Contents

1 Complex Numbers

1.1 Basic Algebraic Properties

A handy thing to keep written down

$$z^{-1} = \left(\frac{a}{z^2 + b^2}, \frac{-b}{a^2 + y^2} \right)$$

also,

$$|z|^2 = |z\bar{z}| = (a + ib)(a - ib) = a^2 + b^2$$

The generalization of $|z|^2 = (\Re(z))^2 + (\Im(z))^2$ does hold!

Note that the product of two complex numbers is very diffeent from the scalar or vector products done in vector spaces over the reals. This notion of a **billinear form** is what is often used to distinguish between different algebras.

Also note that $z_1 < z_2$ has no meaning, so the order field properties we are used to from real numbers don't apply as such. However $|z_1| < |z_2|$ does make sense.

The distance between two points (x_1, y_1) and (x_2, y_2) is $|z_1 - z_2|$.

The complex numbers lying on a circle with center z_0 and radius R satisfy the equation

$$|z - z_0| = R$$

A wonderful example of this last interpretation is

$$|z - 3i| + |z + 3i| = |z - 3i| + |z - (-3i)| = 12$$

This equation represents the set of all points whose distance from the two set points, $F_1(0, 3)$ and $F_2(0, -3)$, is 12. This turns out to be the ellipse with foci $F_1(0, 3)$ and $f_2(0, -3)$. Kline has some great exercises to get you acquainted with Ellipses, parabolas, and hyperbolas.

1.1.1 Exercises

2.2

Some interesting properties

$$z + \bar{z} = (a + ib) + (a - ib) = 2a = 2\Re(z)$$

Similarly,

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2\Im(z)$$

Following the same mechanics,

$$\Re(iz) = \Re(i(a + ib)) = \Re(ai - b) = -\Im(z)$$

And

$$\Im(iz) = \Im(ai - b) = \Re(z)$$

1.2 Triangle Inequality

There is a briliant example in this section, go read it!

The heart of the example is in noticing that the triangle inequality gives us an upper and a lower bound for the sum of two numbers. The upper bound comes from

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and the lower bound from

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

1.2.1 Exercises

Ex 8

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Use simple algebra to show that

$$|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

then point out how the identity $|z_1 z_2| = |z_1||z_2|$ follows.

The trick to the first part is to make use of $|z| = \sqrt{(\Re(z))^2 + (\Im(z))^2}$
First of,

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

From there, we can see that

$$\begin{aligned} \Re(z_1 z_2)^2 &= (x_1 x_2 - y_1 y_2)(x_1 x_2 - y_1 y_2) \\ &= (x_1 x_2)^2 + (y_1 y_2)^2 - 2(x_1 x_2)(y_1 y_2) \end{aligned}$$

and

$$\begin{aligned} \Im(z_1 z_2)^2 &= (x_1 y_2 + x_2 y_1)(x_1 y_2 + x_2 y_1) \\ &= (x_1 y_2)^2 + (x_2 y_1)^2 + 2(x_1 x_2)(y_1 y_2) \end{aligned}$$

It then follows that

$$\begin{aligned} |z_1 z_2| &= \sqrt{(x_1 x_2)^2 + (y_1 y_2)^2 - 2(x_1 x_2)(y_1 y_2) + (x_1 y_2)^2 + (x_2 y_1)^2 + 2(x_1 x_2)(y_1 y_2)} \\ &= \sqrt{(x_1 x_2)^2 + (y_1 y_2)^2 + (x_1 y_2)^2 + (x_2 y_1)^2} \\ &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \end{aligned}$$

Since $|z| = \sqrt{x^2 + y^2}$, we can see how the above reordering is equivalent to

$$\begin{aligned} |z_1 z_2| &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ &= \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \\ &= |z_1||z_2| \end{aligned}$$

Ex 9

If we use the result from the previous exercise and assume have $z = z_1 = z_2$, we have

$$|z^2| = |z||z| = |z|^2$$

We could use this as the base case for an induction argument ($n = 2$).

Then for our hypothesis, we assume that $|z^m| = |z|^m$, when $n = m$, so it must also hold for $n = m + 1$,

$$|z^{m+1}| = |z^m z| = |z'z| = |z'||z| = |z^m||z| = |z|^m|z| = |z|^{m+1}$$

1.3 De Moivre's Theorem

$$(a + ib)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^- 1\frac{y}{x}$$

De Moiver's Theorem

1.4 Roots of Complex Numbers

Nth roots: for any positive integer n, the nth distinct roots of $(a + ib)^n = r^n(\cos nx + i\sin nx)$ are

$$r^{\frac{1}{n}}\left[\cos\frac{x + 2\pi k}{n} + i\sin\frac{x + 2\pi k}{n}\right]$$

for $k = 0, 1, \ldots, n - 1$.

2 Inifinite Series, Products, and Integrals

2.1 Uniform Convergence

Note: when we speak of uniform convergence, the interval can be closed or open. Titchmarsh just uses (a, b) to cover the general case.

The more general case for the first test of uniform Convergence we see is stated as follows:

The series $\sum u_n(x)$ is uniformly convergent ($\forall \epsilon > 0$, we can find $n_0 \geq N$ depending on ϵ but not on x, such that $|s(x) - s| < \epsilon$, for every $n \geq n_0$ for every value in (a, b)) if $|u_n(x)| \leq v_n(x)$, and $\sum v_n(x)$ is uniformly convergent.

If we try to make an argument by contradiciton and assume that $\sum u_n(x)$ is not uniformly convergent, then the series could still converge but it could be the case that as x approaches some point on the interval (a, b), n_0 may become infinetely large. Additionally, the series could just be a divergent series. Either way it means we are not able to find an n_0 such that $|s(x) - s| < \epsilon$ for any $n \geq n_0$, for any $\epsilon > 0$ and for any $x \in (a, b)$. This means that

$$|u_{n+1}(x) + u_{n+2}(x) + \dots|$$

keeps on changing as n or x change.

Since,

$$|u_{n+1}(x) + u_{n+2}(x) + \dots| \leq |u_{n+1}(x)| + |u_{n+2}(x)| + \dots$$

Then any $v_n(x)$ such that $v_{n+1}(x) \geq |u_{n+1}(x)|$ would also grow indefinetely and thus lead to a contradiciton.

Examples

A proof to see why $\sum_{n=0} x^n$ is uniformly convergent in $c \in [a, b]$ when $-1 < a < b < 1$ can be seen by comparing the exercise 2.5.3 from Abbott.

The trigonometric series $\sum_{n=1} \frac{\cos nx}{n^2}$ is convergent anywhere because $-1 \leq \cos x \leq 1$ so $\left|\frac{\cos nx}{n^2}\right|$ behaves like a convergent p-series.

Similarly the Dirichlet series $\sum_{n=1} n^{-s} = \sum_{n=1} \frac{1}{n^s}$ is uniformly convergent in $x \in [a, b]$ if $1 < a < b$ because its absolute value is eequal to a convergent p series. The sum is refered to as the Riemann zeta function $\zeta(s)$.