Notes on Fourier Analysis

October 8, 2023

Contents

1 Fourier Series

1.1 It all Adds Up

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right)$$

Given that

$$e^{2\pi int} = \cos(2\pi nt) + i\sin(2\pi nt)$$

and

$$e^{2\pi int} = \cos(2\pi nt) - i\sin(2\pi nt)$$

Since $\cos(-x) = \cos(x)$ while $\sin(-x) = -\sin(x)$.

The above also mean that

$$\cos(2\pi nt) = \frac{1}{2}\left(e^{2\pi int} + e^{-2\pi int}\right)$$

and

$$\sin(2\pi nt) = \frac{1}{2i}\left(e^{2\pi int} - e^{-2\pi int}\right) = -\frac{i}{2}\left(e^{2\pi int} - e^{-2\pi int}\right)$$

In the last step we used the fact that $i^{-1} = -i$. $(1 = (-i)i = i \cdot i^{-1})$.

Using these expressions to rewrite our series of sines and cosines we get

$$\begin{aligned}
\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right) &= \frac{a_0}{2} + \sum_{n=1}^{N} \left[\frac{a_n}{2}\left(e^{2\pi int} + e^{-2\pi int}\right) + \frac{b_n}{2i}\left(e^{2\pi int} - e^{-2\pi int}\right)\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} \left[\frac{a_n}{2}\left(e^{2\pi int} + e^{-2\pi int}\right) - \frac{ib_n}{2}\left(e^{2\pi int} - e^{-2\pi int}\right)\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} \left[\frac{a_n}{2}e^{2\pi int} + \frac{a_n}{2}e^{-2\pi int} - \frac{ib_n}{2}e^{2\pi int} + \frac{ib_n}{2}e^{-2\pi int}\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} \left[\frac{1}{2}\left(a_n - ib_n\right)e^{2\pi int} + \frac{1}{2}\left(a_n + ib_n\right)e^{-2\pi int}\right]
\end{aligned}$$

If we look at the terms within the square braces we see that we have a complex number times and exponent and its complex conjugate,

$$c_n := \frac{1}{2}\left(a_n - ib_n\right)$$

and

$$\bar{c_n} = \frac{1}{2}\left(a_n + ib_n\right)$$

So

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[c_n e^{2\pi int} + \bar{c_n}\overline{e^{2\pi int}}\right]$$

Here is where the additional requirement comes in. If we have $\bar{c_n} = c_{-n}$, then $a_{-n} = a_n$ and $b_{-n} = -b_n$. We can then reindex our series,

$$\begin{aligned}
\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(2\pi nt) + b_n \sin(2\pi nt)) &= \frac{a_0}{2} + \sum_{n=1}^{N} \left[c_n e^{2\pi int} + \bar{c_n} \overline{e^{2\pi int}} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} c_n e^{2\pi int} + \sum_{n=-1}^{-N} c_{-n} e^{-2\pi int}
\end{aligned}$$

and if we have $c_0 = \frac{1}{2}(a_0 - ib_0) = \frac{a_0}{2}$, $(b_0 = 0)$, then we can see where

$$\sum_{n=-N}^{N} c_n e^{2\pi int}$$

comes from.