Notes on Calculus

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Contents

1 Vectors

1.1 Parametric Equations in General

1.1.1 Cycloid

The parametric equation for $\overrightarrow{AP}$ looks the way it does because the starting point if at $\frac{3\pi}{2}$.

1.1.2 Involute of a Circle

The lenght of $\overrightarrow{BP}$ is $a\theta$ which is the length of the arc! ($s = \theta r$.)

1.2 The Dot Product

1.3 Vector Projections

Example 4

The magnitude of the projection of $\mathbf{F}$ onto $\mathbf{a}$ is equivalent to $\mathbf{F}\sin(30)$. That is,

$$|\text{proj}_{\mathbf{a}}\mathbf{F}| = |\mathbf{F}|\sin 30$$

Remember that to convert degrees to radians you must multiply degrees by $\pi/180$. This works out so because gravity is only acting along a single direction.

2 Setting the Stage

2.1 Euclidean Spaces and Vectors

2.1.1 Exercises

1.1.2

Given $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\begin{aligned} |\vec{x}+\vec{y}|^2 &= (\vec{x}+\vec{y})(\vec{x}+\vec{y}) \\ &= \vec{x}\cdot\vec{x}+\vec{y}\cdot\vec{y}+2\vec{x}\cdot\vec{y} \\ &= |\vec{x}|^2+|\vec{y}|^2+2\vec{x}\cdot\vec{y} \end{aligned}$$

Similarly,

$$\begin{aligned} |\vec{x}-\vec{y}|^2 &= (\vec{x}-\vec{y})(\vec{x}-\vec{y}) \\ &= \vec{x}\cdot\vec{x}+\vec{y}\cdot\vec{y}-2\vec{x}\cdot\vec{y} \\ &= |\vec{x}|^2+|\vec{y}|^2-2\vec{x}\cdot\vec{y} \end{aligned}$$

Hence

$$|\vec{x}+\vec{y}|^2+|\vec{x}-\vec{y}|^2 = 2\left(|\vec{x}|^2+|\vec{y}|^2\right)$$

1.1.7

Suppose $\vec{a}, \vec{b} \in \mathbb{R}^3$

Show that if $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{c}$ for some non-zero $\vec{c} \in \mathbb{R}^3$, then $\vec{a} = \vec{b}$.

We could try to simply stare at

$$\vec{a}\cdot\vec{c} = |\vec{a}||\vec{c}|\cos\theta_1 = |\vec{b}||\vec{c}|\cos\theta_2 = \vec{b}\cdot\vec{c}$$

Which tells us

$$|\vec{a}|\cos\theta_1 = |\vec{b}|\cos\theta_2$$

Let's try something else,

$$|a\times c|^2 = |a||c| - (a\cdot c)^2 = |b||c| - (b\cdot c)^2 = |b\times c|^2$$

We now have

$$|a||c| - (a\cdot c)^2 = |b||c| - (b\cdot c)^2$$

or

$$|a||c| = |b||c| \to |a| = |b|$$

So we can go back to our first attempt and see that

$$|a|\cos\theta_1 = |b|\cos\theta_2 \to \cos\theta_1 = \cos\theta_2$$

1.1.8

To see that $a \cdot (b \times c)$ is the determinant of the three vectors, simply write out the determinant for $b \times c$ and note that the explicit version of it is a "normal" vector. Since the dot product is defined as $x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_ny_n$, when $x, y \in \mathbb{R}^n$.

Putting these two facts together we can see how $a \cdot (b \times c)$ can be computed via a single determinant operation.

2.2 Subsets of Euclidean Space

Proposition 1.4

Remember that to be a boundary point of $S \subset \mathbb{R}^n$ **every** ball centered at $\vec{x}$ must contain points in S and in S^c.

If $\vec{x}$ is an interior point, then for some $\vec{x} \in S$, there is a ball $B(r, \vec{x}) \subset S$; but also, there is an $B(r, \vec{x}) \not\subset S^c$ for some $r > 0$. This is why you ought to be an interior point in S, in S^c, or be a boundary point - because to be a boundary point every single $B(r, \vec{x})$ must be in S AND in S^c.

This last statement is why it is also the case that if S is closed then S^c must be open: because any points must be interior points to either one or be a boundary point. So if S has all of the boundary points, then its compliment must be left with none and thus only have interior points and be an open set.

Example 1

The computation is broken into three scenarios:

1. $|x| < \rho$: $r = \rho - |x| > 0$
2. $|x| > \rho$
3. $|x| = \rho$

The interesting bit of the argument presented for the first case is that we are using $|y| \leq |y - x| + |x| < \rho$, so we make an argument for $B(\rho, x) \subset S = B(\rho, 0)$ by looking for an x and for a correspodning y and seeing that these can be built in such a way that $S \neq \emptyset$ and that any x that meets these conditions will be an interior point, $x \in S^{int}$.

For the second case, if $r = \rho - |x|$, then $r < 0$. So in this case $|y - x| < r$ is not possible since we are in euclidean space.

For the last argument, we note that if $0 < c < 1$, then we have interior points, whereas if we have $c \geq 1$ then $x \in S^c$. The interesting point is that since balls have a positive radius, which makes them expand in all available directions, then any small distance away from $|x| = \rho$ will place part of the ball in both S and S^c.

A noteworthy thing to mention is that this example proves that balls are open.

2.2.1 Exercises

1

$$S = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 4\}$$

x and y cannot be zero at the same time, so $(0, 0) \notin S$. So the set is open there, but it is closed outside of it since the disk is contained in S.

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 - x \leq y \leq 0\}$$

Here, $x^2 - x \leq 0$ which implies $x \in [0, 1]$ since $1^2 - 1 \leq 0$ but any number smaller than 0 will not meet the same contrain. And since $x \leq y \leq 0$, then $y \in [0, 1]$, as well. So S is closed.

2 Show that for any $S \subset \mathbb{R}^n$, S^{int} is open and ∂S and $\bar{S}$ are both closed. Hint: use the fact that balls are open.