

# Notes on Complex Analysis

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# 1 Complex Numbers

## 1.1 Basic Algebraic Properties

A handy thing to keep written down

$$z^{-1} = \left( \frac{a}{z^2 + b^2}, \frac{-b}{a^2 + y^2} \right)$$

The way to think about it is as if you wanted to find  $c$  and  $d$  such that

$$\frac{1}{a + bi} = c + di$$

The trick here is as follows

$$\frac{1}{a + bi} \left( \frac{a - bi}{a - bi} \right) = \frac{a - bi}{a^2 + b^2}$$

Also,

$$|z|^2 = z\bar{z} = (a + ib)(a - ib) = a^2 + b^2$$

The generalization of  $|z|^2 = (\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2$  does hold!

Note that the product of two complex numbers is very different from the scalar or vector products done in vector spaces over the reals. This notion of a **bilinear form** is what is often used to distinguish between different algebras.

Also note that  $z_1 < z_2$  has no meaning, so the order field properties we are used to from real numbers don't apply as such. However  $|z_1| < |z_2|$  does make sense.

The distance between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $|z_1 - z_2|$ .

The complex numbers lying on a circle with center  $z_0$  and radius  $R$  satisfy the equation

$$|z - z_0| = R$$

A wonderful example of this last interpretation is

$$|z - 3i| + |z + 3i| = |z - 3i| + |z - (-3i)| = 12$$

This equation represents the set of all points whose distance from the two set points,  $F_1(0, 3)$  and  $F_2(0, -3)$ , is 12. This turns out to be the ellipse with foci  $F_1(0, 3)$  and  $F_2(0, -3)$ . Kline has some great exercises to get you acquainted with Ellipses, parabolas, and hyperbolas.

### 1.1.1 Exercises

## 2.2

Some interesting properties

$$z + \bar{z} = (a + ib) + (a - ib) = 2a = 2\operatorname{Re}(z)$$

Similarly,

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2\operatorname{Im}(z)$$

Following the same mechanics,

$$\operatorname{Re}(iz) = \operatorname{Re}(i(a + ib)) = \operatorname{Re}(ai - b) = -\operatorname{Im}(z)$$

And

$$\operatorname{Im}(iz) = \operatorname{Im}(ai - b) = \operatorname{Re}(z)$$

## 1.2 Triangle Inequality

There is a brilliant example in this section, go read it!

The heart of the example is in noticing that the triangle inequality gives us an upper and a lower bound for the sum of two numbers. The upper bound comes from

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and the lower bound from

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

### 1.2.1 polynomials

If  $n$  is a positive integer, and if  $a_0, a_1, a_2, \dots$  are complex constants, where  $a_n \neq 0$ ,

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$$

if a polynomial of degree  $n$ . For some positive number  $R$ , the reciprocal  $1/P(z)$  satisfies the inequality

$$\left| \frac{1}{P(z)} \right| < \frac{2}{|a_n|R^n}$$

whenever  $|z| > R$ . Geometrically, this tells us that the modulus of the reciprocal  $1/P(z)$  is bounded from above when  $z$  is exterior to the circle  $|z| = R$ .

The tricky bit of the argument for the above is given when the author says "Now that a sufficiently large positive number  $R$  can be found such that each of the quotients on the right in inequality (9) is less than the number  $|a_n|/(2n)$  when  $|z| > R$ ..." What this means in practice is that when we have

$$|w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \dots + \frac{|a_{n-1}|}{|z|}$$

We are actually dividing  $|w||z|^n$  by  $R^n$  since  $R^n \sim |z|^n$ , and so Churchill is saying that we can find some  $z$  such that when we raise it to a power  $n$  and divide  $|w||z|^n$  by it, that we will have all  $n$  terms on the right side of the inequality be less than the  $n$ -th term divided by  $n$  (go far out enough and there will be a  $z$  value that will make this truth).

The rest of the argument in the book is algebra.

### 1.2.2 Exercises

#### Ex 4

Verify  $\sqrt{2}|z| \geq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$ . Suggestion: reduce the above inequality to  $(|x| - |y|)^2 \geq 0$ .

This exercise is similar to one in Folland's advanced calculus, exercise 2 in section 1.1 Euclidean Spaces and Vectors (we do provide a solution for that example in the calculus doc).

Working backwards,

$$(\sqrt{2}|z|)^2 = 2|z|^2 = 2x^2 + 2y^2$$

Following the tip of the book, a thing to try is something like

$$(|\operatorname{Re}(z)| + |\operatorname{Im}(z)|)^2 = |x|^2 + |y|^2 + 2|x||y| = x^2 + y^2 + 2|x||y|$$

Then

$$\begin{aligned} (\sqrt{2}|z|)^2 - (|\operatorname{Re}(z)| + |\operatorname{Im}(z)|)^2 &= x^2 + y^2 - 2|x||y| \\ &= |x|^2 + |y|^2 - 2|x||y| \\ &= (|x| - |y|)^2 \geq 0 \end{aligned}$$

#### Ex 6

Using the fact that  $|z_1 - z_2|$  is the distance between the points  $z_1$  and  $z_2$ , give a geometric argument that  $|z - 1| = |z + i|$  represents the line through the origin whose slope is  $-1$ .

This one is a very interesting case.  $|z - 1|$  would correspond to the distance from a point  $z$  to the point 1 (the x-axis),  $|z + i|$  would correspond to the distance from a point  $z$  to the point  $-i$  (the y-axis). Since both are a unit from the origin, in their respective axis. The expression above then equates these two distances giving us a straightline that passes through the origin with a  $-1$  slope.

**Ex 7**

Show that for  $R$  sufficiently large, the polynomial  $P(z)$  satisfies

$$|P_n(z)| \leq 2|a_n||z|^n$$

whenever  $|z| > R$ . Suggestion: observe that there is a positive number  $R$  such that the modulus of each quotient in  $|w| \leq |a_0|/|z|^n + |a_1|/|z|^{n-1} + \dots + |a_{n-1}|/|z|$  is less than  $|a_n|/n$  when  $|z| > R$ .

In the argument in which the original inequality was made, there was a step where we use the following  $|a_n + w| \geq ||a_n| - |w||$  to come up with a lower bound. If we instead looked for an upper bound, we could look at

$$|a_n + w| \leq |a_n| + |w| < |a_n| + \frac{1}{2}|a_n| < 2|a_n|$$

The rest of the argument flows when we plug that into expression (10),

$$|P_n(z)| = |a_n + w||z|^n < 2|a_n||z|^n$$

**Ex 8**

Let  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ . Use simple algebra to show that

$$|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

then point out how the identity  $|z_1 z_2| = |z_1||z_2|$  follows.

The trick to the first part is to make use of  $|z| = \sqrt{(\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2}$   
First of,

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

From there, we can see that

$$\begin{aligned} \operatorname{Re}(z_1 z_2)^2 &= (x_1 x_2 - y_1 y_2)(x_1 x_2 - y_1 y_2) \\ &= (x_1 x_2)^2 + (y_1 y_2)^2 - 2(x_1 x_2)(y_1 y_2) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im}(z_1 z_2)^2 &= (x_1 y_2 + x_2 y_1)(x_1 y_2 + x_2 y_1) \\ &= (x_1 y_2)^2 + (x_2 y_1)^2 + 2(x_1 x_2)(y_1 y_2) \end{aligned}$$

It then follows that

$$\begin{aligned} |z_1 z_2| &= \sqrt{(x_1 x_2)^2 + (y_1 y_2)^2 - 2(x_1 x_2)(y_1 y_2) + (x_1 y_2)^2 + (x_2 y_1)^2 + 2(x_1 x_2)(y_1 y_2)} \\ &= \sqrt{(x_1 x_2)^2 + (y_1 y_2)^2 + (x_1 y_2)^2 + (x_2 y_1)^2} \\ &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \end{aligned}$$

Since  $|z| = \sqrt{x^2 + y^2}$ , we can see how the above reordering is equivalent to

$$\begin{aligned} |z_1 z_2| &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ &= \sqrt{x_1^2 + y_1^2} \sqrt{x_2^2 + y_2^2} \\ &= |z_1| |z_2| \end{aligned}$$

### Ex 9

If we use the result from the previous exercise and assume have  $z = z_1 = z_2$ , we have

$$|z^2| = |z| |z| = |z|^2$$

We could use this as the base case for an induction argument ( $n = 2$ ).

Then for our hypothesis, we assume that  $|z^m| = |z|^m$ , when  $n = m$ , so it must also hold for  $n = m + 1$ ,

$$|z^{m+1}| = |z^m z| = |z'| |z| = |z'| |z| = |z^m| |z| = |z|^m |z| = |z|^{m+1}$$

## 1.3 6 Complex Conjugates

### 1.3.1 Exercises

#### Ex 4

Show that,

$$\overline{z^2} = \bar{z}^2$$

Note that

$$\begin{aligned} \overline{z^2} &= \overline{z \bar{z}} \\ &= \bar{z} \bar{\bar{z}} \\ &= \bar{z}^2 \end{aligned}$$

#### Ex 9

By factoring  $z^4 - 4z^2 + 3$  into two quadratic factors and using the inequality  $|z_1 + z_2| \geq ||z_1| - |z_2||$ , show that if  $z$  lies on the circle  $|z| = 2$ , then

$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| \leq \frac{1}{3}$$

To start with the factoring of our polynomial into two quadratic factors. The three constants we have are  $A = 1$ ,  $B = -4$ , and  $C = 3$ . So we need some combination that when multiplied will equal  $AC = 3$  and when summed  $B = -4$ . If we choose 1 and 3 we can get that combination, and since  $A = 1$ , then dividing both of those numbers by 1 results in no change. Thus we have,

$$|z^4 - 4z^2 + 3| = |(z - 1)(z - 3)| = |(z - 1)| |(z - 3)|$$

If we now use the inequality that was recommended to us, we get

$$\begin{aligned}
 |z^4 - 4z^2 + 3| &= |(z^2 - 1)| |(z^2 - 3)| \\
 &\geq ||z|^2 - 1| ||z|^2 - 3| \\
 &= ||z|^2 - 1| ||z|^2 - 3| \\
 &= |4 - 1| |4 - 3| \\
 &= 3
 \end{aligned}$$

The final result comes from inverting the two quantities.

### Ex 10

Show that,

1.  $z$  is real if and only if  $z = \bar{z}$
2.  $z$  is either real or pure imaginary if and only if  $\bar{z}^2 = z^2$

To see the why for the first one, note that if  $z = x + iy = x - iy = \bar{z}$ , then  $(x, y) = (x, -y)$ . Saying that  $x = x$  leads us to a tautology, so no issue there, but  $y = -y$  leads us to say that  $z$  must be real.

Using a similar argument,

$$\bar{z}^2 = \overline{z^2} = \overline{x^2 - y^2 + 2ixy} = x^2 - y^2 + 2ixy$$

The last step has  $xy = -xy$ , so this must be zero. And this happens when  $z$  is either real or pure imaginary.

### Ex 13

Show that the equation  $|z - z_0| = R$  of a circle, centered at  $z_0$  with radius  $R$  can be written as

$$|z|^2 - 2\operatorname{Re}(z\bar{z}_0) + |z_0|^2 = R^2$$

To see why, note that

$$\begin{aligned}
 |z - z_0| &= (z - z_0)^*(z - z_0) \\
 &= (\bar{z} - \bar{z}_0)(z - z_0) \\
 &= \bar{z}z - \bar{z}z_0 - z\bar{z}_0 + \bar{z}_0z_0 \\
 &= |z|^2 - (z\bar{z}_0 + \bar{z}z_0) + |z_0|^2 \\
 &= |z|^2 - 2\operatorname{Re}(z\bar{z}_0) + |z_0|^2
 \end{aligned}$$

### Ex 14

Show that the hyperbola  $x^2 - y^2 = 1$  can be rewritten as  $z^2 - \bar{z}^2 = 2$ .

Well,  $z^2 = (z + iy)(z + iy) = x^2 + 2ixy - y^2$ . And we already saw that  $\bar{z}^2 = \overline{z^2}$ , so  $\bar{z}^2 = x^2 - 2ixy - y^2$ . From there the answer follows.

Just kidding, here are the rest of the steps. But keep in mind that,

1.  $(z + \bar{z}) = z^2 + \bar{z}^2 + z\bar{z} + \bar{z}z$ , and

2.  $(z - \bar{z}) = z^2 + \bar{z}^2 - z\bar{z} - \bar{z}z$

So if,  $x^2 - y^2 = 1$ , then,

$$\begin{aligned} x^2 - y^2 &= \operatorname{Re}^2(z) - \operatorname{Im}^2(z) = \left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 \\ &= \frac{1}{4}(z + \bar{z})^2 + \frac{1}{4}(z - \bar{z})^2 \\ &= \frac{1}{2}(z^2 + \bar{z}^2) \end{aligned}$$

Which is equivalent to saying that  $2(x^2 - y^2) = z^2 + \bar{z}^2 = 2$ , hence our answer.

### Ex 15

Follow the steps below to give an algebraic derivation of the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

(a) Show that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = z_1\bar{z}_1 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + z_2\bar{z}_2$$

This first step is just plain algebra, though keep in mind that  $z\bar{z} = |z|^2$  and that  $\overline{z_1\bar{z}_2} = \bar{z}_1z_2$ .

(b) Point out why  $z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2\operatorname{Re}(z_1\bar{z}_2) \leq 2|z_1||\bar{z}_2|$

Now we are using the identity  $z + \bar{z} = 2\operatorname{Re}(z)$ , where  $z = z_1\bar{z}_2$  along with the inequality  $\operatorname{Re}(z) \leq |\operatorname{Re}(z)| \leq |z|$ .

(c) Use the results in parts (a) and (b) to obtain the inequality

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2,$$

and note how the triangle inequality follows.

This last inequality is obtained by noting that

$$\begin{aligned} |z_1 + z_2|^2 &= z_1\bar{z}_1 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + z_2\bar{z}_2 \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2| \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

and since both, left and right, quantities are positive, then we get  $|z_1 + z_2| \leq |z_1| + |z_2|$ .



## 1.4 8 Product's and Powers in Exponential Form

### 1.4.1 De Moivre's Theorem

$$(a + ib)^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

De Moivre's Theorem

## 1.5 9 Arguments of products and Quotients

### 1.5.1 Roots of Complex Numbers

Nth roots: for any positive integer  $n$ , the  $n$ th distinct roots of  $(a + ib)^n = r^n(\cos nx + i \sin nx)$  are

$$r^{\frac{1}{n}} \left[ \cos \frac{x + 2\pi k}{n} + i \sin \frac{x + 2\pi k}{n} \right]$$

for  $k = 0, 1, \dots, n - 1$ .

### 1.5.2 Exercises

#### 9.1

Find the principal argument  $\text{Arg } z$  when

1.  $z = \frac{-2}{1 + \sqrt{3}i}$

$$\text{Arg}(-2) = \pi$$

$$\text{Arg}(1 + \sqrt{3}i) = \frac{\pi}{3}$$

To get the above answer, we drew it out and searched for special triangles. It just so happens that we can use the 30-60-90 triangle whose  $\tan \sqrt{3} = \pi/3$ .

If we subtract the two, we get  $2\pi/3$  which falls within the valid range for a principal argument,  $-\pi < \Theta \leq \pi$ .

2.  $z = (\sqrt{3} - i)^6$

If we graph where  $\sqrt{3} - i$  is, we will find it in the fourth quadrant, so the principal argument must be negative. The angle in this case is  $\tan 1/\sqrt{3}$ , which is now the 30-degrees side of the 30-60-90 triangle and so  $\Theta = -\pi/6$ .

However, now we have to raise it to the 6-th power.  $(e^{i\theta})^6 = e^{-i\pi} = -1$ .

Also,  $r = |z| = \sqrt{\bar{z}z} = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2} = 2$ . So  $z^6 = (2^6)(-1) = -64$ . Since the value is all real and it lies in the negative side of the graph, the principal argument is  $\Theta = \pi$ .

#### 9.4

Using the fact that the modulus  $|e^{i\theta} - 1|$  is the distance between the points  $e^{i\theta}$  and 1 (See section 4), give a geometric argument to find a value of  $\theta$  in the interval  $0 \leq \theta < 2\pi$  that satisfies  $|e^{i\theta} - 1| = 2$ .

We saw in section 4 that  $|z - z_0| = R$  can be interpreted as the set of complex numbers  $z$  ( $e^{i\theta}$ ) centered at  $z_0$  (1) and with a radius of  $R$  (2). So what complex numbers, lying on the unit circle, also lie on the circle centered at 1 and whose radius is 2? The point that meets this criteria is  $\theta = \pi$ .

#### 9.6

Show that if  $\operatorname{Re} z_1 > 0$  and  $\operatorname{Re} z_2 > 0$ , then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where principal arguments are used.

If  $\operatorname{Re} z > 0$ , then that means that when we draw  $z$  in a polar graph, the point  $z$  will lie in either the first or the fourth quadrants. Meaning that  $\operatorname{Arg} z \in (\pi/2, -\pi/2)$ .

Because of this,  $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \in (\pi, -\pi)$ , which meets the constraints for the range of the principal argument, no need to add or subtract any multiples of  $2\pi$ .

#### 9.8

Prove that two nonzero complex numbers  $z_1$  and  $z_2$  have the same moduli if and only if there are complex numbers  $c_1$  and  $c_2$  such that  $z_1 = c_1 c_2$  and  $z_2 = c_1 \bar{c}_2$ .

Suggestion: Note that

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right) \exp\left(i\frac{\theta_1 - \theta_2}{2}\right) = \exp(i\theta_1)$$

Remember that  $\overline{e^{i\theta}} = e^{-i\theta}$ .

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right) \overline{\exp\left(i\frac{\theta_1 - \theta_2}{2}\right)} = \exp(i\theta_2)$$

$$|z_1|^2 = z_1 \overline{z_1} = c_1 c_2 \overline{c_1 c_2} = c_1 \overline{c_1} c_2 \overline{c_2} = |c_1|^2 |c_2|^2$$

$$|z_2|^2 = z_2 \overline{z_2} = c_1 \overline{c_2} c_1 \overline{\overline{c_2}} = c_1 \overline{c_2} \overline{c_1} c_2 = c_1 \overline{c_1} c_2 \overline{c_2} = |c_1|^2 |c_2|^2$$

Ignoring the modulus of all complex numbers we are thinking about, we can equate  $c_1 \sim e^{i\theta_1}$  and  $c_2 \sim e^{i\theta_2}$ . Thus  $z_1 = c_1 c_2 \sim \exp(i(\theta_1 + \theta_2))$  and  $z_1 = c_1 \overline{c_2} \sim \exp(i(\theta_1 - \theta_2))$ .

If we look at the suggested expressions, we'd have something like

$$\exp(i(\theta_1 + \theta_2)) \exp(i(\theta_1 - \theta_2)) \sim (c_1 c_2) (c_1 \overline{c_2}) \sim z_1 z_2$$

and

$$\exp(i(\theta_1 + \theta_2)) \overline{\exp(i(\theta_1 - \theta_2))} \sim (c_1 c_2) (\overline{c_1 \overline{c_2}}) \sim z_1 \overline{z_2}$$

This doesn't quite tell us much of anything, so let's try flipping the assignments and instead assume that  $c_1 \sim \exp(\frac{i}{2}(\theta_1 + \theta_2))$  and  $c_2 \sim \exp(\frac{i}{2}(\theta_1 - \theta_2))$ .

Then  $z_1 \sim e^{i\theta_1}$  and  $z_2 \sim e^{i\theta_2}$  would mean that  $c_1 c_2 \sim z_1$  and  $c_1 \overline{c_2} \sim z_2$ .

The big take away from such a manipulation is to note how the "angular" part of a complex number just works out in certain situations.

To see the prove the other way, you can write  $c_1 = a_1 + ib_1$  and  $c_2 = a_2 + ib_2$ , carry out the multiplication, then use the fact that  $|z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$ , and note that  $|z_1| = |z_2|$ .

## 9.9

Establish the identity

$$1 + z + z^2 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}, \quad (z \neq 1)$$

and then use it to derive **Lagrange's trigonometric identity**:

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)}, \quad (0 < \theta < 2\pi)$$

Suggestion: As for the first identity, write  $S = 1 + z + z^2 + \dots + z^n$  and consider the difference  $S - zS$ . To derive the second identity, write  $z = e^{i\theta}$  into the first one.

## 2 Infinite Series, Products, and Integrals

### 2.1 Uniform Convergence

**Note:** when we speak of uniform convergence, the interval can be closed or open. Titchmarsh just uses  $(a, b)$  to cover the general case.

The more general case for the first test of uniform Convergence we see is stated as follows:

The series  $\sum u_n(x)$  is uniformly convergent ( $\forall \epsilon > 0$ , we can find  $n_0 \geq N$  depending on  $\epsilon$  but not on  $x$ , such that  $|s(x) - s| < \epsilon$ , for every  $n \geq n_0$  for every value in  $(a, b)$ ) if  $|u_n(x)| \leq v_n(x)$ , and  $\sum v_n(x)$  is uniformly convergent.

If we try to make an argument by contradicton and assume that  $\sum u_n(x)$  is not uniformly convergent, then the series could still converge but it could be the case that as  $x$  approaches some point on the interval  $(a, b)$ ,  $n_0$  may become infinitely large. Additionally, the series could just be a divergent series. Either way it means we are not able to find an  $n_0$  such that  $|s(x) - s| < \epsilon$  for any  $n \geq n_0$ , for any  $\epsilon > 0$  and for any  $x \in (a, b)$ . This means that

$$|u_{n+1}(x) + u_{n+2}(x) + \dots|$$

keeps on changing as  $n$  or  $x$  change.

Since,

$$|u_{n+1}(x) + u_{n+2}(x) + \dots| \leq |u_{n+1}(x)| + |u_{n+2}(x)| + \dots$$

Then any  $v_n(x)$  such that  $v_{n+1}(x) \geq |u_{n+1}(x)|$  would also grow indefinitely and thus lead to a contradicton.

#### Examples

A proof to see why  $\sum_{n=0} x^n$  is uniformly convergent in  $c \in [a, b]$  when  $-1 < a < b < 1$  can be seen by comparing the exercise 2.5.3 from Abbott.

The trigonometric series  $\sum_{n=1} \frac{\cos nx}{n^2}$  is convergent anywhere because  $-1 \leq \cos x \leq 1$  so  $|\frac{\cos nx}{n^2}|$  behaves like a convergent p-series.

Similarly the Dirichlet series  $\sum_{n=1} n^{-s} = \sum_{n=1} \frac{1}{n^s}$  is uniformly convergent in  $x \in [a, b]$  if  $1 < a < b$  because its absolute value is equal to a convergent p series. The sum is referred to as the Riemann zeta function  $\zeta(s)$ .