Notes on Fourier Analysis

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Contents

1 The Laplace Transform

$$
\begin{aligned}
L[1] &= \int_0^\infty e^{-st} dt \\
&= -\frac{1}{s} e^{-st} \Big|_0^\infty \\
&= \frac{1}{s}
\end{aligned}
$$

$$
\begin{aligned}
L[t] &= \int_0^\infty t e^{-st} dt \rightarrow \begin{bmatrix} u = t & v = -\frac{1}{s} e^{-st} \\ du = dt & dv = e^{-st} dt \end{bmatrix} \\
&= -\frac{t}{s} e^{-st} \Big|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st} dt \\
&= \frac{1}{s^2}
\end{aligned}
$$

$$
\begin{aligned}
L[t^2] &= \int_0^\infty t^2 e^{-st} dt \rightarrow \begin{bmatrix} u = t^2 & v = -\frac{1}{s} e^{-st} \\ du = 2t dt & dv = e^{-st} dt \end{bmatrix} \\
&= -\frac{t^2}{s} e^{-st} \Big|_0^\infty + \int_0^\infty \frac{2}{s} t e^{-st} dt \\
&= \frac{2}{s} \int_0^\infty t e^{-st} dt \\
&= \frac{2}{s^3}
\end{aligned}
$$

There is an interesting pattern showing here, let's try a more generic case now.

$$
\begin{aligned}
L[t^n] &= \int_0^\infty t^n e^{-st} dt \rightarrow \begin{bmatrix} u = t^n & v = -\frac{1}{s} e^{-st} \\ du = n t^{n-1} dt & dv = e^{-st} dt \end{bmatrix} \\
&= \cancelto{0}{-\frac{t^n}{s} e^{-st} \Big|_0^\infty} + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt
\end{aligned}
$$

Note that we can tell that the "boundary term" will be zero for any n as an exponential grows faster than t^n.

However, we get an interesting recurrence relation:

$$
L[t^n] = \int_0^\infty t^n e^{-st} dt = \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt
$$

By doing a couple terms we can even see how

$$L[t^n] = \frac{n!}{s^{n+1}}$$

Now, let's look into some other functions.

$$\begin{aligned}
L[e^{kt}] &= \int_0^\infty e^{kt}e^{-st}dt \\
&= \int_0^\infty e^{-(s-k)t}dt
\end{aligned}$$

Note that if $s < k$, the the integral blows up. So let's assume $s > k$,

$$\begin{aligned}
L[e^{kt}] &= \int_0^\infty e^{-(s-k)t}dt \\
&= \frac{1}{s-k}
\end{aligned}$$

$$\begin{aligned}
L[\sin(kt)] &= \int_0^\infty \sin(kt)e^{-st}dt \rightarrow \begin{bmatrix} u = \sin kt & v = -\frac{1}{s}e^{-st} \\ du = k\cos(kt)dt & dv = e^{-st}dt \end{bmatrix} \\
&= \cancelto{0}{-\frac{1}{s}\sin(kt)e^{-st}\Big|_0^\infty} + \frac{k}{s}\int_0^\infty \cos(kt)e^{-st}
\end{aligned}$$

We now need to calculate $L[\cos(kt)]$,

$$\begin{aligned}
L[\cos(kt)] &= \int_0^\infty \cos(kt)e^{-st}dt \rightarrow \begin{bmatrix} u = \cos kt & v = -\frac{1}{s}e^{-st} \\ du = -k\sin(kt)dt & dv = e^{-st}dt \end{bmatrix} \\
&= -\frac{1}{s}\cos(kt)e^{-st}\Big|_0^\infty - \frac{k}{s}\int_0^\infty \sin(kt)e^{-st} \\
&= \frac{1}{s} - \frac{k}{s}\int_0^\infty \sin(kt)e^{-st}
\end{aligned}$$

We get two interesting results here together. First,

$$\begin{aligned}
L[\sin(kt)] &= \int_0^\infty \sin(kt)e^{-st}dt \\
&= \frac{k}{s}\int_0^\infty \cos(kt)e^{-st} \\
&= \frac{k}{s}\left(\frac{1}{s} - \frac{k}{s}\int_0^\infty \sin(kt)e^{-st}\right) \\
&= \frac{k}{s^2} - \frac{k^2}{s^2}\int_0^\infty \sin(kt)e^{-st}
\end{aligned}$$

Which can be re-arranged into,

$$\begin{aligned}\left(1+\frac{k^2}{s^2}\right)\int_0^\infty \sin(kt)e^{-st}dt &= \frac{s^2+k^2}{s^2}\int_0^\infty \sin(kt)e^{-st}dt \\ &= \frac{k}{s^2}\end{aligned}$$

Meaning that,

$$\begin{aligned}L[\sin(kt)] &= \int_0^\infty \sin(kt)e^{-st}dt \\ &= \frac{k}{s^2+k^2}\end{aligned}$$

And now that we have this result, we can go back to,

$$\begin{aligned}L[\cos(kt)] &= \int_0^\infty \cos(kt)e^{-st}dt \\ &= \frac{1}{s} - \frac{k}{s}\int_0^\infty \sin(kt)e^{-st} \\ &= \frac{1}{s} - \frac{k}{s}\frac{k}{s^2+k^2} \\ &= \frac{1}{s} - \frac{k^2}{s(s^2+k^2)} \\ &= \frac{s^2+k^2-k^2}{s(s^2+k^2)} \\ &= \frac{s}{s^2+k^2}\end{aligned}$$

2 Fourier Series

2.1 It all Adds Up

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right)$$

Given that

$$e^{2\pi int} = \cos(2\pi nt) + i\sin(2\pi nt)$$

and

$$e^{2\pi int} = \cos(2\pi nt) - i\sin(2\pi nt)$$

Since $\cos(-x) = \cos(x)$ while $\sin(-x) = -\sin(x)$.

The above also mean that

$$\cos(2\pi nt) = \frac{1}{2}\left(e^{2\pi int} + e^{-2\pi int}\right)$$

and

$$\sin(2\pi nt) = \frac{1}{2i}\left(e^{2\pi int} - e^{-2\pi int}\right) = -\frac{i}{2}\left(e^{2\pi int} - e^{-2\pi int}\right)$$

In the last step we used the fact that $i^{-1} = -i$. $(1 = (-i)i = i \cdot i^{-1})$.

Using these expressions to rewrite our series of sines and cosines we get

$$\begin{aligned}
\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right) &= \frac{a_0}{2} + \sum_{n=1}^{N}\left[\frac{a_n}{2}\left(e^{2\pi int} + e^{-2\pi int}\right) + \frac{b_n}{2i}\left(e^{2\pi int} - e^{-2\pi int}\right)\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N}\left[\frac{a_n}{2}\left(e^{2\pi int} + e^{-2\pi int}\right) - \frac{ib_n}{2}\left(e^{2\pi int} - e^{-2\pi int}\right)\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N}\left[\frac{a_n}{2}e^{2\pi int} + \frac{a_n}{2}e^{-2\pi int} - \frac{ib_n}{2}e^{2\pi int} + \frac{ib_n}{2}e^{-2\pi int}\right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N}\left[\frac{1}{2}\left(a_n - ib_n\right)e^{2\pi int} + \frac{1}{2}\left(a_n + ib_n\right)e^{-2\pi int}\right]
\end{aligned}$$

If we look at the terms within the square braces we see that we have a complex number times and exponent and its complex conjugate,

$$c_n := \frac{1}{2}\left(a_n - ib_n\right)$$

and

$$\bar{c_n} = \frac{1}{2}\left(a_n + ib_n\right)$$

So

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt) + b_n \sin(2\pi nt)\right) = \frac{a_0}{2} + \sum_{n=1}^{N}\left[c_n e^{2\pi int} + \bar{c_n}\overline{e^{2\pi int}}\right]$$

Here is where the additional requirement comes in. If we have $\bar{c_n} = c_{-n}$, then $a_{-n} = a_n$ and $b_{-n} = -b_n$. We can then reindex our series,

$$\begin{aligned}
\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(2\pi nt) + b_n \sin(2\pi nt)) &= \frac{a_0}{2} + \sum_{n=1}^{N} \left[c_n e^{2\pi int} + \bar{c_n} \overline{e^{2\pi int}} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{N} c_n e^{2\pi int} + \sum_{n=-1}^{-N} c_{-n} e^{-2\pi int}
\end{aligned}$$

and if we have $c_0 = \frac{1}{2}(a_0 - ib_0) = \frac{a_0}{2}$, $(b_0 = 0)$, then we can see where

$$\sum_{n=-N}^{N} c_n e^{2\pi int}$$

comes from.

3 Fourier Series