Notes on Calculus

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1 Vectors

1.1 Parametric Equations in General

1.1.1 Cycloid

The parametric equation for $\overrightarrow{AP}$ looks the way it does because the starting point if at $\frac{3\pi}{2}$.

1.1.2 Involute of a Circle

The lenght of $\overrightarrow{BP}$ is $a\theta$ which is the length of the arc! ($s = \theta r$.)

1.2 The Dot Product

1.3 Vector Projections

Example 4

The magnitude of the projection of $\mathbf{F}$ onto $\mathbf{a}$ is equivalent to $\mathbf{F}\sin(30)$. That is,

$$|\text{proj}_{\mathbf{a}}\mathbf{F}| = |\mathbf{F}|\sin 30$$

Remember that to convert degrees to radians you must multiply degrees by $\pi/180$. This works out so because gravity is only acting along a single direction.

1.4 The Cross Product

1.5 Rotation of a Rigid Body

$|\mathbf{r}(t)|$ and θ are constant, but $\mathbf{r}(t)$ changes direction as time changes. That change in direction corresponds to a change to the vector $\mathbf{r}(t)$ and thats why it corresponds to the arc swept btween t and $t + \Delta t$.

Proposition 4.3, its proof is similar to an exercise we did for Folland, see 2.1.1

1.6 Equations for Planes and Distance Problems

Proposition 2.1 (foudn in section 1.2) the vector parametric equation for the line through the point $P_0(b_1, b_2, b_3)$, whose position vector is $\overrightarrow{OP_0} = \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and parallel to $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is $\mathbf{r}(t) = \mathbf{b} + t\mathbf{a}$.

2 Setting the Stage

2.1 Euclidean Spaces and Vectors

2.1.1 Exercises

The following exercises begin on page 8.

1.1.2

Given $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\begin{aligned} |\vec{x}+\vec{y}|^2 &= (\vec{x}+\vec{y})(\vec{x}+\vec{y}) \\ &= \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} + 2\vec{x}\cdot\vec{y} \\ &= |\vec{x}|^2 + |\vec{y}|^2 + 2\vec{x}\cdot\vec{y} \end{aligned}$$

Similarly,

$$\begin{aligned} |\vec{x}-\vec{y}|^2 &= (\vec{x}-\vec{y})(\vec{x}-\vec{y}) \\ &= \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} - 2\vec{x}\cdot\vec{y} \\ &= |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x}\cdot\vec{y} \end{aligned}$$

Hence

$$|\vec{x}+\vec{y}|^2 + |\vec{x}-\vec{y}|^2 = 2\left(|\vec{x}|^2 + |\vec{y}|^2\right)$$

1.1.7

Suppose $\vec{a}, \vec{b} \in \mathbb{R}^3$

Show that if $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{c}$ for some non-zero $\vec{c} \in \mathbb{R}^3$, then $\vec{a} = \vec{b}$.

We could try to simply stare at

$$\vec{a}\cdot\vec{c} = |\vec{a}||\vec{c}|\cos\theta_1 = |\vec{b}||\vec{c}|\cos\theta_2 = \vec{b}\cdot\vec{c}$$

Which tells us

$$|\vec{a}|\cos\theta_1 = |\vec{b}|\cos\theta_2$$

Let's try something else,

$$|a\times c|^2 = |a||c| - (a\cdot c)^2 = |b||c| - (b\cdot c)^2 = |b\times c|^2$$

We now have

$$|a||c| - (a\cdot c)^2 = |b||c| - (b\cdot c)^2$$

or

$$|a||c| = |b||c| \to |a| = |b|$$

So we can go back to our first attempt and see that

$$|a|\cos\theta_1 = |b|\cos\theta_2 \to \cos\theta_1 = \cos\theta_2$$

1.1.8

To see that $a \cdot (b \times c)$ is the determinant of the three vectors, simply write out the determinant for $b \times c$ and note that the explicit version of it is a "normal" vector. Since the dot product is defined as $x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_ny_n$, when $x, y \in \mathbb{R}^n$.

Putting these two facts together we can see how $a \cdot (b \times c)$ can be computed via a single determinant operation.

2.2 Subsets of Euclidean Space

Proposition 1.4

Remember that to be a boundary point of $S \subset \mathbb{R}^n$ **every** ball centered at $\vec{x}$ must contain points in S and in S^c.

If $\vec{x}$ is an interior point, then for some $\vec{x} \in S$, there is a ball $B(r, \vec{x}) \subset S$; but also, there is an $B(r, \vec{x}) \not\subset S^c$ for some $r > 0$. This is why you ought to be an interior point in S, in S^c, or be a boundary point - because to be a boundary point every single $B(r, \vec{x})$ must be in S AND in S^c.

This last statement is why it is also the case that if S is closed then S^c must be open: because any points must be interior points to either one or be a boundary point. So if S has all of the boundary points, then its compliment must be left with none and thus only have interior points and be an open set.

Example 1

The computation is broken into three scenarios:

1. $|x| < \rho$: $r = \rho - |x| > 0$
2. $|x| > \rho$
3. $|x| = \rho$

The interesting bit of the argument presented for the first case is that we are using $|y| \leq |y - x| + |x| < \rho$, so we make an argument for $B(\rho, x) \subset S = B(\rho, 0)$ by looking for an x and for a correspodning y and seeing that these can be built in such a way that $S \neq \emptyset$ and that any x that meets these conditions will be an interior point, $x \in S^{int}$.

For the second case, if $r = \rho - |x|$, then $r < 0$. So in this case $|y - x| < r$ is not possible since we are in euclidean space.

For the last argument, we note that if $0 < c < 1$, then we have interior points, whereas if we have $c \geq 1$ then $x \in S^c$. The interesting point is that since balls have a positive radius, which makes them expand in all available directions, then any small distance away from $|x| = \rho$ will place part of the ball in both S and S^c.

A noteworthy thing to mention is that this example proves that balls are open.

2.2.1 Exercises

1

$$S = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 4\}$$

x and y cannot be zero at the same time, so $(0, 0) \notin S$. So the set is open there, but it is closed outside of it since the disk is contained in S.

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 - x \leq y \leq 0\}$$

Here, $x^2 - x \leq 0$ which implies $x \in [0, 1]$ since $1^2 - 1 \leq 0$ but any number smaller than 0 will not meet the same contrain. And since $x \leq y \leq 0$, then $y \in [0, 1]$, as well. So S is closed.

2 Show that for any $S \subset \mathbb{R}^n$, S^{int} is open and ∂S and $\bar{S}$ are both closed. Hint: use the fact that balls are open.

3 Euclidean Spaces

3.1 Smooth Functions on Euclidean Space

3.1.1 C^∞ Versus Analytic Functions

Starting out with the simplest notation that anyone may have seen, we can write the partial drivates as

$$\frac{\partial^2 f}{\partial x \partial y}$$

There is a theorem called **Schwarz's theorem (or Clairaut's theorem on equality of mixed partials)** that states that for a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ (note that U is an open set - $U \subset \mathbb{R}^n$ not $U \subseteq \mathbb{R}^n$) if $p = (p^1, p^2, \ldots, p^n) \in \mathbb{R}^n$ and $p \in U$ (some neighboorhod of p is contained in U) and f has **continuous second partial derivatives on that neighboorhod of** p, then for all i and j in $\{1, 2, \ldots, n\}$,

$$\frac{\partial^2 f(p)}{\partial x^i \partial x^j} = \frac{\partial^2 f(p)}{\partial x^j \partial x^i}$$

The partial derivatives of this function commute at that point.

We wanted to mention Schwarz's theorem on equality of mixed partial derivatives to present some of the context we will be visiting over and over. Now, let's carry on with a little bit more notation. From our abstraction in the statement of Schwarz's theorem, we can see that mixed partial derivates have the following form,

$$\frac{\partial^{i+j+k} f}{\partial x^i \partial x^j \partial x^k}$$

In Tu's notation the above then becomes

$$\frac{\partial^j f}{\partial x^{i_1} \ldots \partial x^{i_j}}$$

where ∂x^{i_1} means that we are taking the partial derivative of degree i with respect to the coordinate x^1. There is also the implicit requirement that the j partial derivatives of degree i will sum to j, so essentially $i = 1$.

The above notation cold very well be reworded into

$$\frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \ldots \partial x_j^{\alpha_j}}$$

where $\alpha = \alpha_1 + \ldots + \alpha_j$, we also moved the superscript for the dimension to a subscript just to simplify the latex - but keep in mind that **Tu uses the second superscript to note the dimension**.