# Notes on differential equations

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# 1 Introduction to Asymptotic Approximations

We start with,

$$\frac{d^2x(t)}{dt^2} = -\frac{gR^2}{(R+x)^2}, \quad t \geq 0$$

But we want to study the error that arises from assuming that $x \ll R$, along with the behaviour that would be Introduced from the nonlineraity we'll scale the variables in our problem. We'll define the characteristic time as $\tau = t/t_c$ and the characteteristic value for the solution as $y(\tau) = x(t)/x_c$.

We will also chose the values $t_c = v_0/g$ and $x_c = v_0^2/g$. These values are within a constant the values we get when we solve for the maximum height an object would travel upward if launched with an initial velocity. Since $x \ll R$, we will also define an $\epsilon = v_0^2/Rg$ ($\epsilon \ll 1$).

In order to proceed with our transformation we will also need to figure out how to transform the right hand-side of our problem along with the time derivations. Thus,

$$-\frac{gR^2}{(R+x)^2} \to -\frac{g}{(1+x_c y/R)^2} \to -\frac{g}{(1+\epsilon y)^2}$$

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}\left(x_c y(\tau)\right) \\
&= x_c \frac{dy(\tau)}{d\tau}\frac{d\tau}{dt} \\
&= \frac{x_c}{t_c}\frac{dy(\tau)}{d\tau}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{x_c}{t_c}\frac{dy(\tau)}{d\tau}\right) \\
&= \frac{x_c}{t_c}\frac{d}{dt}\frac{dy(\tau)}{d\tau} \\
&= \frac{x_c}{t_c}\frac{d}{d(t_c\tau)}\frac{dy(\tau)}{d\tau} \\
&= \frac{x_c}{t_c^2}\frac{d^2y(\tau)}{d\tau^2}
\end{aligned}$$

Putting it all together,

$$\frac{d^2x}{dt^2} = \frac{x_c}{t_c^2}\frac{d^2y(\tau)}{d\tau^2} = \frac{\cancel{v_0^2}}{\cancel{g}}\frac{\cancel{g^2}}{\cancel{v_0^2}}\frac{d^2y(\tau)}{d\tau^2} = -\frac{g}{(1+\epsilon y)^2}$$

Hence, we get

$$\frac{d^2y(\tau)}{d\tau^2} = -\frac{1}{(1+\epsilon y)^2}, \quad \tau \geq 0$$

## 2 First Order Differential Equations

### 2.1 Linear Equations

$$\frac{dy}{dt} = f(y, t)$$

If $f$ is a linear function on $y$, the we have a first order linear differential equation.

The simplest type first order linear equation is one in which the coefficients are constants. For example,

$$\frac{dy}{dt} = -ay + b$$

The above can be generalized into

$$\frac{dy}{dt} + p(t)y = g(t)$$

Where the coefficients are now functions of the independent variable. Furhtermore, the above can also be generalized as

$$p(t)\frac{dy}{dx} + q(t)y = g(t)$$

### 2.2 Method of Integrating Factors

Multiply the equation my the integrating factor and the equation is converted into one that can be integrated using the product rule for derivates.

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + y\frac{d\mu(t)}{dt} \sim p(t)\frac{dy}{dx} + q(t)y$$

A common presentation for equations that can readily be solved by the method of integrating factors,

$$\frac{dy}{dt} + cy = f(t)$$

Where $c$ is a constant.

Also, make sure to remember to do the comparsion of $y\frac{d\mu(t)}{dt}$ properly. For example, using the last version we wrote, the integrating factor would come from the comparison of

$$y\frac{d\mu(t)}{dt} \sim yc\mu(t) \rightarrow \frac{d\mu(t)}{dt} \sim c\mu(t)$$

This integrating factor also looks like an exponential after differentiation.

**Note**: remember that you can forgoe the integration constant from the integration factor because it we will get it back once we do the first integration.

### 2.2.1 Exercises and Problems

**Example 3**

The integration by parts step that is skiped is done as follows.
Recall the rule for integration by parts

$$\int udv = uv - \int vdu$$

$$\begin{aligned}
\int te^{-2t}dt &= \begin{bmatrix} u = t & v = -\frac{1}{2}e^{-2t} \\ du = dt & dv = e^{-2t}dt \end{bmatrix} \\
&= -\frac{1}{2}te^{-2t} + \int \frac{1}{2}e^{-2t}dt \\
&= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + c
\end{aligned}$$

## 2.3 Separable Equations

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Can be written in **differential form** as

$$M(x)dx + N(y)dy = 0$$

## 2.4 Notes

Sometimes equations of the form

$$\frac{dy}{dx} = f(x, y)$$

have a constant solution $y = y_0$.
For example,

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$$

Has a constant solution $y = 3$.

## 2.5 Modeling with First Order Equations

### 2.5.1 Example 1: Mixing

$$\frac{dQ}{dt} + \frac{r}{100}Q = \frac{r}{4}$$

Using the method of Integrating factors, we have

$$\frac{d}{dt}\left[\mu(t)Q(t)\right] = \mu\frac{dQ}{dt} + Q\frac{d\mu}{dt} = \mu\frac{dQ}{dt} + \mu\frac{r}{100}Q = \mu\frac{r}{4}$$

Comparing

$$Q\frac{d\mu}{dt} \sim \mu\frac{r}{100}Q$$

We have that

$$\frac{d\mu}{dt} = \frac{r}{100}\mu$$

So the integrating factor must be

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = ln|\mu| = \int \frac{r}{100} = \frac{r}{100}t + C_0$$

And so

$$\mu(t) = e^{\frac{r}{100}t + C_0} = C_1 e^{\frac{rt}{100}}$$

Our original equation becomes

$$\frac{d}{dt}\left[C_1 e^{\frac{rt}{100}}Q\right] = C_1 e^{\frac{rt}{100}}\frac{dQ}{dt} + C_1 e^{\frac{rt}{100}}\frac{r}{100}Q = C_1 e^{\frac{rt}{100}}\frac{r}{4}$$

Now, we can finally integrate both sides,

$$\begin{aligned}\int \frac{d}{dt}\left[C_1 e^{\frac{rt}{100}}Q\right]dt = C_1 e^{\frac{rt}{100}}Q &\\ &= \int C_1 e^{\frac{rt}{100}}\frac{r}{4}dt \\ &= \frac{r}{4}\frac{100}{r}C_1 e^{\frac{rt}{100}} + C_2 \\ &= 25C_1 e^{\frac{rt}{100}} + C_2\end{aligned}$$

So our general solution is

$$C_1 e^{\frac{rt}{100}}Q = 25C_1 e^{\frac{rt}{100}} + C_2$$

or

$$Q = 25 + Ce^{\frac{-rt}{100}}$$

Since $Q(t=0) = Q_0$

$$Q_0 = 25 + C \to C = Q_0 - 25$$

And

$$\begin{aligned}Q(t) &= 25 + (Q_0 - 25)e^{\frac{-rt}{100}} \\ &= 25(1 - e^{\frac{-rt}{100}}) + Q_0 e^{\frac{-rt}{100}}\end{aligned}$$

When we want to solve for the time $T$ after which the salt level is within 2% of $Q_L$ (the limiting ammount), we do it as follows:

$$25.5 = 25 + 25e^{-rT/100} \rightarrow \frac{1}{2} = 25e^{-rT/100}$$

$$= \frac{1}{50} = e^{-rT/100} \rightarrow \ln(1/50) = \frac{-rT}{100}$$
$$= -\frac{100}{r}\ln(1/50) = \frac{100}{r}\ln 50$$

### 2.5.2 Example 3: Chemicals in a pond

We will pick up from

$$\frac{dt}{dt} + \frac{1}{2}q = 10 + 5\sin(2t)$$

And we can see that we have a nice, simple, first order, linear equation, so we will proceed with the method of integrating factors.

$$\frac{d}{dt}\left[\mu(t)q(t)\right] = \mu\frac{dq}{dt} + q\frac{d\mu}{dt}$$
$$= \mu\frac{dt}{dt} + \frac{1}{2}\mu q = 10\mu + 5\mu\sin(2t)$$

Means that the integrating factor will be

$$q\frac{d\mu}{dt} \sim \frac{1}{2}\mu q \rightarrow \frac{1}{\mu}\frac{d\mu}{dt} \sim \frac{1}{2}$$

Or

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = \int \frac{1}{2}$$

Which leads to $\mu(t) = e^{t/2}$.

So our equation becomes

$$\frac{d}{dt}\left[e^{t/2}q(t)\right] = e^{t/2}\frac{dt}{dt} + \frac{1}{2}e^{t/2}q = 10e^{t/2} + 5e^{t/2}\sin(2t)$$

Hence,

$$e^{t/2}q(t) = \int 10e^{t/2}dt + \int 5e^{t/2}\sin(2t)dt$$
$$= 20e^{t/2} + \int 5e^{t/2}\sin(2t)dt$$

Here we have an interesting integral so let's break it down.

#### 2.5.2.1 An interesting integral

In the previous expression we ended up with

$$\int 5e^{t/2}\sin(2t)dt$$

The tip here is a chain of integrations by parts and $u$-substitutions. First, let's recall the rule for integration by parts

$$\int udv = uv - \int vdu$$

Now, let's get to it.

$$\begin{aligned}
\int e^{t/2}\sin(2t)dt &= \begin{bmatrix} u = e^{t/2} & v = -\frac{1}{2}\cos(2t) \\ du = \frac{1}{2}e^{t/2}dt & dv = \sin(2t)dt \end{bmatrix} \\
&= -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{4}\left[\frac{1}{2}e^{t/2}\sin(2t) - \frac{1}{4}\int e^{t/2}\sin(2t)dt\right] \\
&= -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{2^3}e^{t/2}\sin(2t) - \frac{1}{2^4}\int e^{t/2}\sin(2t)dt
\end{aligned}$$

Notice that we got our initial integral back, so now some algebra will lead us to

$$\left(\int e^{t/2}\sin(2t)dt\right)\left(1 + \frac{1}{2^4}\right) = -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{2^3}e^{t/2}\sin(2t)$$

Which can be simplified to

$$\begin{aligned}
\int e^{t/2}\sin(2t)dt &= -\frac{2^4}{2}\frac{1}{2^4+1}e^{t/2}\cos(2t) + \frac{2^4}{2^3}\frac{1}{2^4+1}e^{t/2}\sin(2t) \\
&= -\frac{2^3}{2^4+1}e^{t/2}\cos(2t) + \frac{2}{2^4+1}e^{t/2}\sin(2t)
\end{aligned}$$

Now, we can put everything together!

$$\begin{aligned}
e^{t/2}q(t) &= 20e^{t/2} + \int 5e^{t/2}\sin(2t)dt \\
&= 20e^{t/2} + 5\left[-\frac{2^3}{2^4+1}e^{t/2}\cos(2t) + \frac{2}{2^4+1}e^{t/2}\sin(2t)\right] \\
&= 20e^{t/2} - \frac{40}{17}e^{t/2}\cos(2t) + \frac{10}{17}e^{t/2}\sin(2t) + C
\end{aligned}$$

Notice that we trhew in an integration coefficient at the end. And our final answer is now

$$q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) + Ce^{-t/2}$$