Notes on differential equations

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Contents

1 First Order Differential Equations

1.1 Linear Equations

1.1.1 Method of Integrating Factors

$$p(t)\frac{dy}{dx} + q(t)y = g(t)$$

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + y\frac{d\mu(t)}{dt} \sim p(t)\frac{dy}{dx} + q(t)y$$

1.1.2 Separable Equations

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Can be written in **differential form** as

$$M(x)dx + N(y)dy = 0$$

1.1.3 Notes

Sometimes equations of the form

$$\frac{dy}{dx} = f(x, y)$$

have a constant solution $y = y_0$.

For example,

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$$

Has a constant solution $y = 3$.

1.2 Modeling with First Order Equations

1.2.1 Example 1: Mixing

$$\frac{dQ}{dt} + \frac{r}{100}Q = \frac{r}{4}$$

Using the method of Integrating factors, we have

$$\frac{d}{dt}\left[\mu(t)Q(t)\right] = \mu\frac{dQ}{dt} + Q\frac{d\mu}{dt} = \mu\frac{dQ}{dt} + \mu\frac{r}{100}Q = \mu\frac{r}{4}$$

Comparing

$$Q\frac{d\mu}{dt} \sim \mu\frac{r}{100}Q$$

We have that

$$\frac{d\mu}{dt} = \frac{r}{100}\mu$$

So the integrating factor must be

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = ln|\mu| = \int \frac{r}{100} = \frac{r}{100}t + C_0$$

And so

$$\mu(t) = e^{\frac{r}{100}t + C_0} = C_1 e^{\frac{rt}{100}}$$

Our original equation becomes

$$\frac{d}{dt}\left[C_1 e^{\frac{rt}{100}} Q\right] = C_1 e^{\frac{rt}{100}} \frac{dQ}{dt} + C_1 e^{\frac{rt}{100}} \frac{r}{100} Q = C_1 e^{\frac{rt}{100}} \frac{r}{4}$$

Now, we can finally integrate both sides,

$$\begin{aligned}\int \frac{d}{dt}\left[C_1 e^{\frac{rt}{100}} Q\right] dt = C_1 e^{\frac{rt}{100}} Q \\ &= \int C_1 e^{\frac{rt}{100}} \frac{r}{4} dt \\ &= \frac{r}{4}\frac{100}{r} C_1 e^{\frac{rt}{100}} + C_2 \\ &= 25 C_1 e^{\frac{rt}{100}} + C_2\end{aligned}$$

So our general solution is

$$C_1 e^{\frac{rt}{100}} Q = 25 C_1 e^{\frac{rt}{100}} + C_2$$

or

$$Q = 25 + C e^{\frac{-rt}{100}}$$

Since $Q(t=0) = Q_0$

$$Q_0 = 25 + C \to C = Q_0 - 25$$

And

$$\begin{aligned}Q(t) &= 25 + (Q_0 - 25) e^{\frac{-rt}{100}} \\ &= 25(1 - e^{\frac{-rt}{100}}) + Q_0 e^{\frac{-rt}{100}}\end{aligned}$$

When we want to solve for the time T after which the salt level is within 2% of Q_L (the limiting ammount), we do it as follows:

$$25.5 = 25 + 25e^{-rT/100} \to \frac{1}{2} = 25e^{-rT/100}$$

$$= \frac{1}{50} = e^{-rT/100} \to \ln(1/50) = \frac{-rT}{100}$$

$$= -\frac{100}{r}\ln(1/50) = \frac{100}{r}\ln 50$$