Notes Mathematical Methods

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1 Tips and Tricks

If working with a differential equation that has too many independent variables try something like this

$$\frac{dv(t)}{dt} = f(x, v, t)$$

To get rid of t, use the chain rule to instead look at

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dt}$$

2 determinants and Matrices

2.1 determinants

2.1.1 Homogeneous Linear Equations

$$a \cdot (b \times c) = (a \times b) \cdot c$$

An $n \times n$ homogeneous system of linear equations has a unique non-trivial solution if and only if its determinant is non-zero. If this determinant is zero, then the system has either no nontrivial solutions or an infinite number of solutions.

If the determinant is zero, then one of the vectors lies in the plane spaned by one of the other two vectors - it is not independent which is equivalent to saying that is linear combination of the other(s).

3 Vector Spaces

3.1 Vectors in Function Spaces

Scwarz Inequality

$$
\begin{aligned}
I &= \langle f - \lambda g | f - \lambda g \rangle \\
&= \left(\langle f| - \lambda^* \langle g| \right) \left(|f\rangle - \lambda |g\rangle \right) \\
&= \langle f|f \rangle - \lambda \langle f|g \rangle - \lambda^* \langle g|f \rangle + \lambda^* \lambda \langle g|g \rangle \\
&= \langle f|f \rangle + |\lambda|^2 \langle g|g \rangle - \left(\langle \lambda g|f \rangle + \langle f|\lambda g \rangle \right) \\
&= \langle f|f \rangle + |\lambda|^2 \langle g|g \rangle - \left(\langle f|\lambda g \rangle^* + \langle f|\lambda g \rangle \right) \\
&= |f|^2 + |\lambda|^2 |g|^2 - \left(\langle f|\lambda g \rangle^* + \langle f|\lambda g \rangle \right) \\
&\geq 0
\end{aligned}
$$

If we were looking at complex numbers, the term in the braces would equal $2\Re(z)$.

$$
\begin{aligned}
\partial_{\lambda^*} I &= \partial_{\lambda^*} \left(\left(\langle f| - \lambda^* \langle g| \right) \left(|f - \lambda g\rangle \right) \right) \\
&= - \langle g|f \rangle + \lambda \langle g|g \rangle = 0
\end{aligned}
$$

The above implies that

$$\lambda = \frac{\langle g|f \rangle}{\langle g|g \rangle}$$

If we plug that in, we have,

$$
\begin{aligned}
I &= \langle f|f \rangle + |\lambda|^2 \langle g|g \rangle - \langle \lambda g|f \rangle - \langle f|\lambda g \rangle \\
&= \langle f|f \rangle + \frac{\langle f|g \rangle \langle g|f \rangle}{\langle g|g \rangle^2} \langle g|g \rangle - \frac{\langle f|g \rangle}{\langle g|g \rangle} \langle g|f \rangle - \frac{\langle g|f \rangle}{\langle g|g \rangle} \langle f|g \rangle \\
&= \langle f|f \rangle + \frac{\langle f|g \rangle \langle g|f \rangle}{\langle g|g \rangle} - \frac{\langle f|g \rangle}{\langle g|g \rangle} \langle g|f \rangle - \frac{\langle g|f \rangle}{\langle g|g \rangle} \langle f|g \rangle \\
&= \langle f|f \rangle - \frac{\langle g|f \rangle}{\langle g|g \rangle} \langle f|g \rangle \\
&\geq 0
\end{aligned}
$$

So

$$\langle f|f \rangle \langle g|g \rangle \geq \langle g|f \rangle \langle f|g \rangle = |\langle f|g \rangle|^2$$