Notes on linear algebra

September 30, 2023

Contents

1 Vector Spaces

1.1 Exercises

1.A.1

Find c and d such that

$$\frac{1}{a+bi} = c + di$$

The trick here is as follows

$$\frac{1}{a+bi}\left(\frac{a-bi}{a-bi}\right) = \frac{a-bi}{a^2+b^2}$$

So $c = a/|z|^2$ and $d = -b/|z|^2$, where $z = a + bi$ and $|z| = \sqrt{a^2+b^2}$.

1.A.2

Show that

$$\frac{-1\sqrt{3}i}{2}$$

is a cube root of 1.

$$(-1\sqrt{3}i)(-1\sqrt{3}i) = -2 - 2\sqrt{3}i$$
$$(-2 - 2\sqrt{3}i)(-1\sqrt{3}i) = 2 + 6 = 8$$

Ad since $\frac{1}{2^3} = 1/8$, then we get our proof.

1.A.3

To find 2 distinct square roots of i it helps to look at Euler's identity first

$$e^{i\pi} + 1 = 0$$

or

$$e^{i\pi} = -1$$

And since $i^2 = -1$, then

$$e^{i\pi} = -1 = i^2$$

Hence

$$e^{i\pi/2} = i$$

So our firsrt square root is $e^{i\pi/4} = \sqrt{i}$. If we go around once, then we get another square root

$$e^{(i\pi/2+2\pi i)/2} = e^{i5\pi/4}$$

We can still simply this further.

$$e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

and

$$e^{i5\pi/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{-1}{\sqrt{2}} + i\frac{-1}{\sqrt{2}} = -\frac{1+i}{\sqrt{2}}$$

Another interesting read is: Ed Scheinerman (2023) A Third Real Solution to $x = x^{-1}$, Not Really, The American Mathematical Monthly, 130:6, 514-514, DOI: 10.1080/00029890.2023.2184161

1.A.4
Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Let's define $\alpha = a + bi$ and $\beta = c + di$.

$$\alpha + \beta = (a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$\beta + \alpha = (c + di) + (a + bi) = (c + a) + (d + b)i$$

And since summation is commutative for the reals, QED.

1.A.5
Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ where $\alpha, \beta, \lambda \in \mathbb{C}$.

Same mechanics,

$$(\alpha + \beta) = (a + c) + (b + d)i$$

If we define $\lambda = x + yi$ then

$$(\alpha + \beta) + \lambda = [(a + c) + (b + d)i] + (x + yi) = (a + b + x) + (b + d + y)i$$

And hopefully you can see the rest of the argument and how we are save by the commutative property of the reals.

1.A.8
Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exist a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

The result from exercise 1.A.1 comes in handy here! this is because we get an expression for a an inverse complex number in terms that make it easy to carry out calculations the way we are used to.

In 1.A.1 we had $\beta = c + di = (c, d)$ equal to

$$(a, b)\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right) = (1, 0) = 1$$

1.A.10
Find $x \in \mathbb{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8)$$

Let's try component by component,

- $4 + 2x_0 = 5$ results in $x_0 = 1/2$.
- $-3 + 2x_1 = 9$ results in $x_1 = 12/2 = 6$.
- $1 + 2x_2 = -6$ results in $x_2 = -7/2$.
- $7 + 2x_3 = 8$ results in $x_3 = 1/2$.

So $x = (1/2, 6, -7/2, 1/2)$.

1.A.11
Explain why there does not exist a $\lambda \in \mathbb{C}$ such that

$$\lambda\,(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i)$$

This one is funky but try note that $\lambda(2-3i) = (12-5i)$, and $\lambda(5+4i) = (7+22i)$. Then multipy $\lambda(2-3i)(7+22i)$ and compare that against $(12-5i)(5+4i)$, and so on.

1.B.1
Prove that $-(-v) = v$ for every $v \in V$.
Using the additive inverse property

$$(-v) + -(-v) = 0 \rightarrow (-v) + v = 0$$

And by the Uniqueness of the additive inverse, $-(-v) = v$.

1.B.3
Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that

$$v + 3x = w$$

There are two parts to this question: existance and Uniqueness. The existance part can be seen by using

$$x = \frac{1}{3}(w - v)$$

in our original expression. That is,

$$v + 3x = v + 3\frac{1}{3}(w - v) = v + w - v = w$$

Uniqueness can be seen by noting that if we had

$$v + 3x' = w$$

Then

$$x' = \frac{1}{3}(w - v)$$

And so

$$x - x' = 0$$

1.B.4

The empty set is not a vector space because it fails to satisfy the additive identity - can't have the existence of an element $0 \in V$ if the set is empty.

1.B.5

Show that the additive inverse condition - for every $v \in V$, $\exists w \in V$ such that $v + w = 0$ - can be replaced by the condition

$$0v = 0$$

for all $v \in V$. Where the 0 on the left is $0 \in \mathbb{F}$ and the 0 on the right is the additive identity of v.

Normally, we would think of the additive inverse for $v + w = 0$ as $w = -v$, so

$$0 = v + w = v + (-v) = 1v + (-1v) = 0v = 0$$

1.B.6

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Now, say we defien addition and scalar multiplication in $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as we normally would.

Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$?

No. For example, additive inverses and additive identities would not be unique - nor any of the other conditions required of a vector space.

1.2 Subspaces

One topic covered in the book is proving that sets are subspaces. Let's see a couple worked out cases before attempting example 1.35.

First, let's say we have

$$U = \{(x_1, x_2, x_3) \in \mathbb{F}^4 : x_1 + 2x_2 = 0\}$$

and we claim this set is a subspace of $\mathbb{F}^3$.

We can see that the additive identity is part of the subspace, $0 \in U$, since $x_1 + 2x_2 = 0$.

Now let's see if this subspace is closed under addition. If we have $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$, then

$$u + v = (u_1, u_2, u_3) + (w_1, w_2, w_3) = (u_1 + w_1, u_2 + w_2, u_3 + w_3)$$

Given the defintion of our set, we should have the above $u + w = (u_1 + w_1, u_2 + w_2, u_3 + w_3)$ met the requirement that $(u_1 + w_1) + 2(u_2 + w_2) = 0$. To see this, start from that restraint on u and w:

$$u_1 + 2u_2 + w_1 + 2w_2 = 0 + 0 = (u_1 + w_1) + 2(u_2 + w_2)$$

Now, we just need to show that our subspace is closed under scalar multiplication. This process is similar. We want to see that $au = (au_1, au_2, au_3)$ stil matches the constrain given so that $au_1 + a2u_2 = 0$. Which we see does satisfy our constraint since $a(u_1 + 2u_2) = au_1 = a2u_2 = 0$. ■

Similarly, we can now take a look at example 1.35:

If $b \in \mathbb{F}$, then

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

From the constraint imposed on our subspace, we want to see

$$u + w = (u_1 + w_1, u_2 + w_2, u_3 + w_3, u_4 + w_4) \rightarrow (u_3 + w_3) = 5(u_4 + w_4) + 2b$$

If we add the individual constrants we see that we indeed match the above

$$u_3 + w_3 = 5u_4 + b + 5w_4 + b \rightarrow (u_3 + w_3) = 5(u_4 + w_4) + 2b$$

Similarly, we can see that this subspace is closed under scalar multiplication because $au = (au_1, au_2, au_3, au_4)$ would meet the constrain given when if $au_3 = a5u_4 + ab$. Which is indeed the case because $a(u_3) = a(5u_4 + b)$ matches the previous expression.

The only requirement we haven't meet is to show that this subspace meets the additive identity requirement. This is when $b = 0$ becomes a must. ■

The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

Note: in this example $\mathbb{R}$ actually stands for $\mathbb{R}^n$. What this means is that you need to keep in mind that the inputs and outputs to functions are tuples, not single numbers.

Now, let's revisit $\mathbb{R}^{[0,1]}$. People say $\mathbb{R}^{[0,1]}$ denotes the set of functions from the set $[0, 1]$ to $\mathbb{R}$ (set of real-valued functions on $[0, 1]$).

Let's start with noting the following:

$$\mathbb{R}^{[0,1]} = \{f | f : [0, 1] \to \mathbb{R}\}$$

Our first stop is checking the additive identity, $0 \in \mathbb{R}^{[0,1]}$? We could take the name literally but as Axler pointed out, the first requirement is a way to checking that the subspace is not empty and to show that $0 \in \mathbb{R}^{[0,1]}$.

To start, $f(x) = 0, \forall x \in [0, 1]$ is a continuous function. So $0 \in \mathbb{R}^{[0,1]}$.

Next, assume we have $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$. And we have $f(x) = a$ and $g(x) = b$, where $a, b \in \mathbb{R}$, then

$$(f + g)(x) = f(x) + g(x) = a + b \in \mathbb{R}$$

These cases are valid since the sum of continuous functions is itself a continuous function, and constant functions are constinuous.

Finally, if $\lambda \in [0, 1]$, then

$$(\lambda f)(x) = \lambda f(x) = \lambda a \in \mathbb{R} \quad \blacksquare$$

Note: this is also an entrypoint into an interesting argument for showing that $[0, 1]$ and $\mathbb{R}$ have the same cardinality.

The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^R$.

Note: in this example $\mathbb{R}$ actually stands for $\mathbb{R}^n$. What this means is that you need to keep in mind that the inputs and outputs to functions are tuples, not single numbers.

$\mathbb{R}^R$ denotes the set of functions from $\mathbb{R}$ to $\mathbb{R}$:

$$\mathbb{R}^R = \{f | f : \mathbb{R} \to \mathbb{R}\}$$

In this case, our [proposed] subspace is defined as follows:

$$U = \{f | f : \mathbb{R} \to \mathbb{R}, \text{ where } \forall c \in \mathbb{R}, \exists f'(c)\}$$

We resorted to the internet to gain some insights and What is R^R as a vector space?.

The accepted answer mentions this great point that we can think of $\mathbb{F}^n$ as the assignment of elements of the set $\{1, 2, \ldots, n\}$ to elements of $\mathbb{F}$ by $g \in \mathbb{F}^{\{1,2,\ldots,n\}}$ or as $g : \mathbb{N} \to \mathbb{F}$. The function g behaves such that the index of an element of the n-tuple corredsponds to a member of the n-tuple or, $g(1) = x_1$, $g(2) = x_2$, and so on correspond to $(x_1, x_2, \ldots, x_n)$.

This means that in our case, $f \in U$, these are functions that map a uncountably infinite number of elements to an uncountably infinite long tuple.

The math exchange answer also gives us $g : \mathbb{R} \to \{0\}$ as the additive identity. There, $\{0\}$ is an uncountably infinite tuple of all zeros, which is also differentiable everywhere within our domain. Then the rest of the argument is very similar to when we discussed $\mathbb{R}^{[0,1]}$.