

# Notes on Calculus

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# 1 Vectors

## 1.1 Parametric Equations in General

### 1.1.1 Cycloid

The parametric equation for  $\overrightarrow{AP}$  looks the way it does because the starting point is at  $\frac{3\pi}{2}$ .

### 1.1.2 Involute of a Circle

The length of  $\overrightarrow{BP}$  is  $a\theta$  which is the length of the arc! ( $s = \theta r$ .)

## 1.2 The Dot Product

## 1.3 Vector Projections

### Example 4

The magnitude of the projection of  $\mathbf{F}$  onto  $\mathbf{a}$  is equivalent to  $|\mathbf{F}| \sin(30)$ . That is,

$$|\text{proj}_{\mathbf{a}} \mathbf{F}| = |\mathbf{F}| \sin 30$$

Remember that to convert degrees to radians you must multiply degrees by  $\pi/180$ . This works out so because gravity is only acting along a single direction.

## 1.4 The Cross Product

## 1.5 Rotation of a Rigid Body

$|\mathbf{r}(t)|$  and  $\theta$  are constant, but  $\mathbf{r}(t)$  changes direction as time changes. That change in direction corresponds to a change to the vector  $\mathbf{r}(t)$  and that's why it corresponds to the arc swept between  $t$  and  $t + \Delta t$ .

**Proposition 4.3**, its proof is similar to an exercise we did for Folland, see 2.1.1

## 1.6 Equations for Planes and Distance Problems

**Proposition 2.1** (found in section 1.2) the vector parametric equation for the line through the point  $P_0(b_1, b_2, b_3)$ , whose position vector is  $\overrightarrow{OP_0} = \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ , and parallel to  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  is  $\mathbf{r}(t) = \mathbf{b} + t\mathbf{a}$ .

## 2 Setting the Stage

### 2.1 Euclidean Spaces and Vectors

#### 2.1.1 Exercises

The following exercises begin on page 8.

##### 1.1.2

Given  $\vec{x}, \vec{y} \in \mathbb{R}^n$ ,

$$\begin{aligned} |\vec{x} + \vec{y}|^2 &= (\vec{x} + \vec{y})(\vec{x} + \vec{y}) \\ &= \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} + 2\vec{x} \cdot \vec{y} \\ &= |\vec{x}|^2 + |\vec{y}|^2 + 2\vec{x} \cdot \vec{y} \end{aligned}$$

Similarly,

$$\begin{aligned} |\vec{x} - \vec{y}|^2 &= (\vec{x} - \vec{y})(\vec{x} - \vec{y}) \\ &= \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} - 2\vec{x} \cdot \vec{y} \\ &= |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x} \cdot \vec{y} \end{aligned}$$

Hence

$$|\vec{x} + \vec{y}|^2 + |\vec{x} - \vec{y}|^2 = 2(|\vec{x}|^2 + |\vec{y}|^2)$$

##### 1.1.7

Suppose  $\vec{a}, \vec{b} \in \mathbb{R}^3$

Show that if  $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$  and  $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$  for some non-zero  $\vec{c} \in \mathbb{R}^3$ , then  $\vec{a} = \vec{b}$ .

We could try to simply stare at

$$\vec{a} \cdot \vec{c} = |\vec{a}||\vec{c}| \cos \theta_1 = |\vec{b}||\vec{c}| \cos \theta_2 = \vec{b} \cdot \vec{c}$$

Which tells us

$$|\vec{a}| \cos \theta_1 = |\vec{b}| \cos \theta_2$$

Let's try something else,

$$|a \times c|^2 = |a||c| - (a \cdot c)^2 = |b||c| - (b \cdot c)^2 = |b \times c|^2$$

We now have

$$|a||c| - (a \cdot c)^2 = |b||c| - (b \cdot c)^2$$

or

$$|a||c| = |b||c| \rightarrow |a| = |b|$$

So we can go back to our first attempt and see that

$$|a| \cos \theta_1 = |b| \cos \theta_2 \rightarrow \cos \theta_1 = \cos \theta_2$$

### 1.1.8

To see that  $a \cdot (b \times c)$  is the determinant of the three vectors, simply write out the determinant for  $b \times c$  and note that the explicit version of it is a "normal" vector. Since the dot product is defined as  $x \cdot y = x_1y_1 + x_2y_2 + \dots + x_ny_n$ , when  $x, y \in \mathbb{R}^n$ .

Putting these two facts together we can see how  $a \cdot (b \times c)$  can be computed via a single determinant operation.

## 2.2 Subsets of Euclidean Space

### Proposition 1.4

Remember that to be a boundary point of  $S \subset \mathbb{R}^n$  **every** ball centered at  $\vec{x}$  must contain points in  $S$  and in  $S^c$ .

If  $\vec{x}$  is an interior point, then for some  $\vec{x} \in S$ , there is a ball  $B(r, \vec{x}) \subset S$ ; but also, there is an  $B(r, \vec{x}) \not\subset S^c$  for some  $r > 0$ . This is why you ought to be an interior point in  $S$ , in  $S^c$ , or be a boundary point - because to be a boundary point every single  $B(r, \vec{x})$  must be in  $S$  AND in  $S^c$ .

This last statement is why it is also the case that if  $S$  is closed then  $S^c$  must be open: because any points must be interior points to either one or be a boundary point. So if  $S$  has all of the boundary points, then its complement must be left with none and thus only have interior points and be an open set.

### Example 1

The computation is broken into three scenarios:

1.  $|x| < \rho$ :  $r = \rho - |x| > 0$
2.  $|x| > \rho$
3.  $|x| = \rho$

The interesting bit of the argument presented for the first case is that we are using  $|y| \leq |y - x| + |x| < \rho$ , so we make an argument for  $B(\rho, x) \subset S = B(\rho, 0)$  by looking for an  $x$  and for a corresponding  $y$  and seeing that these can be built in such a way that  $S \neq \emptyset$  and that any  $x$  that meets these conditions will be an interior point,  $x \in S^{int}$ .

For the second case, if  $r = \rho - |x|$ , then  $r < 0$ . So in this case  $|y - x| < r$  is not possible since we are in euclidean space.

For the last argument, we note that if  $0 < c < 1$ , then we have interior points, whereas if we have  $c \geq 1$  then  $x \in S^c$ . The interesting point is that since balls have a positive radius, which makes them expand in all available directions, then

any small distance away from  $|x| = \rho$  will place part of the ball in both  $S$  and  $S^c$ .

**A noteworthy thing to mention is that this example proves that balls are open.**

### 2.2.1 Exercises

1

$$S = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 4\}$$

$x$  and  $y$  cannot be zero at the same time, so  $(0, 0) \notin S$ . So the set is open there, but it is closed outside of it since the disk is contained in  $S$ .

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 - x \leq y \leq 0\}$$

Here,  $x^2 - x \leq 0$  which implies  $x \in [0, 1]$  since  $1^2 - 1 \leq 0$  but any number smaller than 0 will not meet the same constrain. And since  $x \leq y \leq 0$ , then  $y \in [0, 1]$ , as well. So  $S$  is closed.

**2** Show that for any  $S \subset \mathbb{R}^n$ ,  $S^{int}$  is open and  $\partial S$  and  $\bar{S}$  are both closed. Hint: use the fact that balls are open.

## 3 Euclidean Spaces

### 3.1 Smooth Functions on Euclidean Space

#### 3.1.1 $C^\infty$ Versus Analytic Functions

Starting out with the simplest notation that anyone may have seen, we can write the partial derivatives as

$$\frac{\partial^2 f}{\partial x \partial y}$$

There is a theorem called **Schwarz's theorem (or Clairaut's theorem on equality of mixed partials)** that states that for a function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  (note that  $U$  is an open set -  $U \subset \mathbb{R}^n$  not  $U \subseteq \mathbb{R}^n$ ) if  $p = (p^1, p^2, \dots, p^n) \in \mathbb{R}^n$  and  $p \in U$  (some neighborhood of  $p$  is contained in  $U$ ) and  $f$  has **continuous second partial derivatives on that neighborhood of  $p$** , then for all  $i$  and  $j$  in  $\{1, 2, \dots, n\}$ ,

$$\frac{\partial^2 f(p)}{\partial x^i \partial x^j} = \frac{\partial^2 f(p)}{\partial x^j \partial x^i}$$

**The partial derivatives of this function commute at that point.**

We wanted to mention Schwarz's theorem on equality of mixed partial derivatives to present some of the context we will be visiting over and over. Now, let's carry on with a little bit more notation. From our abstraction in the statement of Schwarz's theorem, we can see that mixed partial derivatives have the following form,

$$\frac{\partial^{i+j+k} f}{\partial x^i \partial x^j \partial x^k}$$

In Tu's notation the above then becomes

$$\frac{\partial^j f}{\partial x^{i_1} \dots \partial x^{i_j}}$$

where  $\partial x^{i_1}$  means that we are taking the partial derivative of degree  $i$  with respect to the coordinate  $x^1$ . There is also the implicit requirement that the  $j$  partial derivatives of degree  $i$  will sum to  $j$ , so essentially  $i = 1$ .

The "weirdness" in the notation comes because we assume that  $f$  is a vector-valued function so  $f$  can stand for the function or for one of its components, usually context will tell us.

The above notation could very well be reworded into

$$\frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \dots \partial x_j^{\alpha_j}}$$

where  $\alpha = \alpha_1 + \dots + \alpha_j$ , we also moved the superscript for the dimension to a subscript just to simplify the latex - but keep in mind that **Tu uses the second**

**superscript to note the dimension.**

Lastly, there is a very interesting result that is mentioned in this section: not all smooth functions are analytic. This may sound odd at first because in the example given  $f(x) = 0$  for  $x \leq 0$ . But keep in mind that in order to be analytic at a point  $p \in U$ , the function's Taylor series must equal the actual function in a neighborhood around  $p$ . So for some  $\delta + \epsilon$ , where  $\epsilon > 0$ , the function does not equal the non-zero part of the function.

The wikipedia page for Taylor series explains this quite well:  $[f(x)]$  is infinitely differentiable at  $x = 0$ , and has all derivatives zero there. Consequently, the Taylor series of  $f(x)$  about  $x = 0$  is identically zero. However,  $f(x)$  is not the zero function, so does not equal its Taylor series around the origin. Thus,  $f(x)$  is an example of a non-analytic smooth function.

### **3.1.2 Taylor's Theorem with Remainder**