Notes on Fluid Dynamics

December 29, 2023

Contents

1 Continuum mechanics

1.1 Exercises

1.1

(B) Consider an unsteady one-dimensional flow where the density and velocity depend on x and t. A Galilean transformation into a new set of variables x', t' is given by the equation $x = x' + Vt'$, $t = t'$, where V is a constant velocity. For the moment let $f = f(x,t)$ stand for a function that we wish to express in the $x' - t'$ coordinate system. By careful use of the chain rules of calculus, find the expressions for $\partial f/\partial t'$ and $\partial f/\partial x'$. Next, consider the substantial derivatives of ρ and v, which are

$$\frac{\partial \rho'}{\partial t'} + v'\frac{\partial \rho'}{\partial x'},$$

$$\frac{\partial v'}{\partial t'} + v'\frac{\partial v'}{\partial x'},$$

Show that the substantial derivatives above have exactly the same mathematical form when transformed into the $x - t$ coordinate system (note that $\rho' = \rho$ and $v' = v - V$).

Let's start with the chain rule to obtain the coordinate transformation for our partial derivatives. Since $f = f(x,t) = f\left(x(x',t'), t(t')\right)$, the new partial derivatives will be

$$\frac{\partial f}{\partial t'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial t'}$$

Since $t = t'$, $\partial t/\partial t' = 1$, and since $x = x' + Vt'$, $\partial x/\partial t' = \partial_{t'}\left(x' + Vt'\right) = V$. So

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x}$$

Similarly,

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\cancelto{1}{\frac{\partial x}{\partial x'}} + \frac{\partial f}{\partial t}\cancelto{0}{\frac{\partial t}{\partial x'}} = \frac{\partial f}{\partial x}$$

The first simplification comes about because $\partial_{x'}x = \partial_{x'}\left(x' + Vt'\right)$. The second because t does not depend on x'.

Now we can look at the substantial derivatives.

$$\begin{aligned}
\frac{\partial \rho'}{\partial t'} + v'\frac{\partial \rho'}{\partial x'} \rightarrow \frac{\partial \rho}{\partial t'} + (v - V)\frac{\partial \rho}{\partial x'} \\
= \frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial x} + (v - V)\frac{\partial \rho}{\partial x} \\
= \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}
\end{aligned}$$

And,

$$\begin{aligned}
\frac{\partial v'}{\partial t'} + v' \frac{\partial v'}{\partial x'} \to \frac{\partial (v - V)}{\partial t'} + (v - V) \frac{\partial (v - V)}{\partial x'} \\
&= \frac{\partial v}{\partial t} + V \frac{\partial v}{\partial x} + (v - V) \frac{\partial v}{\partial x} \\
&= \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}
\end{aligned}$$

So $\frac{D\rho}{Dt}$ and $\frac{Dv}{Dt}$ are both invariant under Galilean transformation.