Notes on differential equations

November 19, 2023

Contents

1 First Order Differential Equations

1.1 Linear Equations

$$\frac{dy}{dt} = f(y,t)$$

If f is a linear function on y, the we have a first order linear differential equation.

The simplest type first order linear equation is one in which the coefficients are constants. For example,

$$\frac{dy}{dt} = -ay + b$$

The above can be generalized into

$$\frac{dy}{dt} + p(t)y = g(t)$$

Where the coefficients are now functions of the independent variable. Furhtermore, the above can also be generalized as

$$p(t)\frac{dy}{dx} + q(t)y = g(t)$$

1.1.1 Method of Integrating Factors

Multiply the equation my the integrating factor and the equation is converted into one that can be integrated using the product rule for derivates.

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + y\frac{d\mu(t)}{dt} \sim p(t)\frac{dy}{dx} + q(t)y$$

A common presentation for equations that can readily be solved by the method of integrating factors,

$$\frac{dy}{dt} + cy = f(t)$$

Where c is a constant.

Also, make sure to remember to do the comparsion of $y\frac{d\mu(t)}{dt}$ properly. For example, using the last version we wrote, the integrating factor would come from the comparison of

$$y\frac{d\mu(t)}{dt} \sim yc\mu(t) \to \frac{d\mu(t)}{dt} \sim c\mu(t)$$

This integrating factor also looks like an exponential after differentiation.

1.1.2 Separable Equations

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Can be written in **differential form** as

$$M(x)dx + N(y)dy = 0$$

1.1.3 Notes

Sometimes equations of the form

$$\frac{dy}{dx} = f(x, y)$$

have a constant solution $y = y_0$.

For example,

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$$

Has a constant solution $y = 3$.

1.2 Modeling with First Order Equations

1.2.1 Example 1: Mixing

$$\frac{dQ}{dt} + \frac{r}{100}Q = \frac{r}{4}$$

Using the method of Integrating factors, we have

$$\frac{d}{dt}\left[\mu(t)Q(t)\right] = \mu\frac{dQ}{dt} + Q\frac{d\mu}{dt} = \mu\frac{dQ}{dt} + \mu\frac{r}{100}Q = \mu\frac{r}{4}$$

Comparing

$$Q\frac{d\mu}{dt} \sim \mu\frac{r}{100}Q$$

We have that

$$\frac{d\mu}{dt} = \frac{r}{100}\mu$$

So the integrating factor must be

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = ln|\mu| = \int \frac{r}{100} = \frac{r}{100}t + C_0$$

And so

$$\mu(t) = e^{\frac{r}{100}t + C_0} = C_1 e^{\frac{rt}{100}}$$

Our original equation becomes

$$\frac{d}{dt}\left[C_1 e^{\frac{rt}{100}}Q\right] = C_1 e^{\frac{rt}{100}}\frac{dQ}{dt} + C_1 e^{\frac{rt}{100}}\frac{r}{100}Q = C_1 e^{\frac{rt}{100}}\frac{r}{4}$$

Now, we can finally integrate both sides,

$$\int \frac{d}{dt}\left[C_1 e^{\frac{rt}{100}} Q\right] dt = C_1 e^{\frac{rt}{100}} Q$$
$$= \int C_1 e^{\frac{rt}{100}} \frac{r}{4} dt$$
$$= \frac{r}{4}\frac{100}{r} C_1 e^{\frac{rt}{100}} + C_2$$
$$= 25 C_1 e^{\frac{rt}{100}} + C_2$$

So our general solution is

$$C_1 e^{\frac{rt}{100}} Q = 25 C_1 e^{\frac{rt}{100}} + C_2$$

or

$$Q = 25 + C e^{\frac{-rt}{100}}$$

Since $Q(t=0) = Q_0$

$$Q_0 = 25 + C \rightarrow C = Q_0 - 25$$

And

$$Q(t) = 25 + (Q_0 - 25) e^{\frac{-rt}{100}}$$
$$= 25(1 - e^{\frac{-rt}{100}}) + Q_0 e^{\frac{-rt}{100}}$$

When we want to solve for the time T after which the salt level is within 2% of Q_L (the limiting ammount), we do it as follows:

$$25.5 = 25 + 25e^{-rT/100} \rightarrow \frac{1}{2} = 25e^{-rT/100}$$
$$= \frac{1}{50} = e^{-rT/100} \rightarrow \ln(1/50) = \frac{-rT}{100}$$
$$= -\frac{100}{r}\ln(1/50) = \frac{100}{r}\ln 50$$

1.2.2 Example 3: Chemicals in a pond

We will pick up from

$$\frac{dt}{dt} + \frac{1}{2}q = 10 + 5\sin(2t)$$

And we can see that we have a nice, simple, first order, linear equation, so we will proceed with the method of integrating factors.

$$\frac{d}{dt}\left[\mu(t)q(t)\right] = \mu\frac{dq}{dt} + q\frac{d\mu}{dt}$$
$$= \mu\frac{dt}{dt} + \frac{1}{2}\mu q = 10\mu + 5\mu\sin(2t)$$

Means that the integrating factor will be

$$q\frac{d\mu}{dt} \sim \frac{1}{2}\mu q \rightarrow \frac{1}{\mu}\frac{d\mu}{dt} \sim \frac{1}{2}$$

Or

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = \int \frac{1}{2}$$

Which leads to $\mu(t) = e^{t/2}$.

So our equation becomes

$$\frac{d}{dt}\left[e^{t/2}q(t)\right] = e^{t/2}\frac{dt}{dt} + \frac{1}{2}e^{t/2}q = 10e^{t/2} + 5e^{t/2}\sin(2t)$$

Hence,

$$\begin{aligned} e^{t/2}q(t) &= \int 10e^{t/2}dt + \int 5e^{t/2}\sin(2t)dt \\ &= 20e^{t/2} + \int 5e^{t/2}\sin(2t)dt \end{aligned}$$

Here we have an interesting integral so let's break it down.

1.2.2.1 An interesting integral

In the previous expression we ended up with

$$\int 5e^{t/2}\sin(2t)dt$$

The tip here is a chain of integrations by parts and u-substitutions. First, let's recall the rule for integration by parts

$$\int udv = uv - \int vdu$$

Now, let's get to it.

$$\begin{aligned} \int e^{t/2}\sin(2t)dt &= \begin{bmatrix} u = e^{t/2} & v = -\frac{1}{2}\cos(2t) \\ du = \frac{1}{2}e^{t/2}dt & dv = \sin(2t)dt \end{bmatrix} \\ &= -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{4}\left[\frac{1}{2}e^{t/2}\sin(2t) - \frac{1}{4}\int e^{t/2}\sin(2t)dt\right] \\ &= -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{2^3}e^{t/2}\sin(2t) - \frac{1}{2^4}\int e^{t/2}\sin(2t)dt \end{aligned}$$

Notice that we got our initial integral back, so now some algebra will lead us to

$$\left(\int e^{t/2}\sin(2t)dt\right)\left(1 + \frac{1}{2^4}\right) = -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{2^3}e^{t/2}\sin(2t)$$

Which can be simplified to

$$
\begin{aligned}
\int e^{t/2}\sin(2t)dt &= -\frac{2^4}{2}\frac{1}{2^4+1}e^{t/2}\cos(2t) + \frac{2^4}{2^3}\frac{1}{2^4+1}e^{t/2}\sin(2t) \\
&= -\frac{2^3}{2^4+1}e^{t/2}\cos(2t) + \frac{2}{2^4+1}e^{t/2}\sin(2t)
\end{aligned}
$$

Now, we can put everything together!

$$
\begin{aligned}
e^{t/2}q(t) &= 20e^{t/2} + \int 5e^{t/2}\sin(2t)dt \\
&= 20e^{t/2} + 5\left[-\frac{2^3}{2^4+1}e^{t/2}\cos(2t) + \frac{2}{2^4+1}e^{t/2}\sin(2t)\right] \\
&= 20e^{t/2} - \frac{40}{17}e^{t/2}\cos(2t) + \frac{10}{17}e^{t/2}\sin(2t) + C
\end{aligned}
$$

Notice that we trhew in an integration coefficient at the end. And our final answer is now

$$
q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) + Ce^{-t/2}
$$