Notes on differential equations

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1 Introduction to Asymptotic Approximations

We start with,

$$\frac{d^2x(t)}{dt^2} = -\frac{gR^2}{(R+x)^2}, \quad t \geq 0$$

But we want to study the error that arises from assuming that $x \ll R$, along with the behaviour that would be Introduced from the nonlineraity we'll scale the variables in our problem. We'll define the characteristic time as $\tau = t/t_c$ and the characteteristic value for the solution as $y(\tau) = x(t)/x_c$.

We will also chose the values $t_c = v_0/g$ and $x_c = v_0^2/g$. These values are within a constant the values we get when we solve for the maximum height an object would travel upward if launched with an initial velocity. Since $x \ll R$, we will also define an $\epsilon = v_0^2/Rg$ ($\epsilon \ll 1$).

In order to proceed with our transformation we will also need to figure out how to transform the right hand-side of our problem along with the time derivations. Thus,

$$-\frac{gR^2}{(R+x)^2} \to -\frac{g}{(1+x_c y/R)^2} \to -\frac{g}{(1+\epsilon y)^2}$$

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}\left(x_c y(\tau)\right) \\
&= x_c \frac{dy(\tau)}{d\tau}\frac{d\tau}{dt} \\
&= \frac{x_c}{t_c}\frac{dy(\tau)}{d\tau}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{x_c}{t_c}\frac{dy(\tau)}{d\tau}\right) \\
&= \frac{x_c}{t_c}\frac{d}{dt}\frac{dy(\tau)}{d\tau} \\
&= \frac{x_c}{t_c}\frac{d}{d(t_c\tau)}\frac{dy(\tau)}{d\tau} \\
&= \frac{x_c}{t_c^2}\frac{d^2y(\tau)}{d\tau^2}
\end{aligned}$$

Putting it all together,

$$\frac{d^2x}{dt^2} = \frac{x_c}{t_c^2}\frac{d^2y(\tau)}{d\tau^2} = \frac{\cancel{v_0^2}}{\cancel{g}}\frac{g^{\cancel{2}}}{\cancel{v_0^2}}\frac{d^2y(\tau)}{d\tau^2} = -\frac{g}{(1+\epsilon y)^2}$$

Hence, we get

$$\frac{d^2y(\tau)}{d\tau^2} = -\frac{1}{(1+\epsilon y)^2}, \quad \tau \geq 0$$

2 First Order Differential Equations

2.1 Linear Equations

$$\frac{dy}{dt} = f(y,t)$$

If f is a linear function on y, the we have a first order linear differential equation.

The simplest type first order linear equation is one in which the coefficients are constants. For example,

$$\frac{dy}{dt} = -ay + b$$

The above can be generalized into

$$\frac{dy}{dt} + p(t)y = g(t)$$

Where the coefficients are now functions of the independent variable. Furhter-more, the above can also be generalized as

$$p(t)\frac{dy}{dx} + q(t)y = g(t)$$

2.2 Method of Integrating Factors

Multiply the equation my the integrating factor and the equation is converted into one that can be integrated using the product rule for derivates.

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + y\frac{d\mu(t)}{dt} \sim p(t)\frac{dy}{dx} + q(t)y$$

A common presentation for equations that can readily be solved by the method of integrating factors,

$$\frac{dy}{dt} + cy = f(t)$$

Where c is a constant.

Also, make sure to remember to do the comparsion of $y\frac{d\mu(t)}{dt}$ properly. For example, using the last version we wrote, the integrating factor would come from the comparison of

$$y\frac{d\mu(t)}{dt} \sim yc\mu(t) \rightarrow \frac{d\mu(t)}{dt} \sim c\mu(t)$$

This integrating factor also looks like an exponential after differentiation.

Note: remember that you can forgoe the integration constant from the integration factor because it we will get it back once we do the first integration.

2.2.1 Exercises and Problems

Example 3

The integration by parts step that is skiped is done as follows.
Recall the rule for integration by parts

$$\int udv = uv - \int vdu$$

$$\begin{aligned} \int te^{-2t}dt &= \begin{bmatrix} u = t & v = -\frac{1}{2}e^{-2t} \\ du = dt & dv = e^{-2t}dt \end{bmatrix} \\ &= -\frac{1}{2}te^{-2t} + \int \frac{1}{2}e^{-2t}dt \\ &= -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + c \end{aligned}$$

Example 5
Here we are presented with the integral

$$\int e^{t^2/4}dt$$

2.3 Separable Equations

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Can be written in **differential form** as

$$M(x)dx + N(y)dy = 0$$

2.4 Notes

Sometimes equations of the form

$$\frac{dy}{dx} = f(x,y)$$

have a constant solution $y = y_0$.
For example,

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$$

Has a constant solution $y = 3$.

2.5 Modeling with First Order Equations

2.5.1 Example 1: Mixing

$$\frac{dQ}{dt} + \frac{r}{100}Q = \frac{r}{4}$$

Using the method of Integrating factors, we have

$$\frac{d}{dt}\left[\mu(t)Q(t)\right] = \mu\frac{dQ}{dt} + Q\frac{d\mu}{dt} = \mu\frac{dQ}{dt} + \mu\frac{r}{100}Q = \mu\frac{r}{4}$$

Comparing

$$Q\frac{d\mu}{dt} \sim \mu\frac{r}{100}Q$$

We have that

$$\frac{d\mu}{dt} = \frac{r}{100}\mu$$

So the integrating factor must be

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = ln|\mu| = \int \frac{r}{100} = \frac{r}{100}t + C_0$$

And so

$$\mu(t) = e^{\frac{r}{100}t+C_0} = C_1e^{\frac{rt}{100}}$$

Our original equation becomes

$$\frac{d}{dt}\left[C_1e^{\frac{rt}{100}}Q\right] = C_1e^{\frac{rt}{100}}\frac{dQ}{dt} + C_1e^{\frac{rt}{100}}\frac{r}{100}Q = C_1e^{\frac{rt}{100}}\frac{r}{4}$$

Now, we can finally integrate both sides,

$$\begin{aligned}
\int \frac{d}{dt}\left[C_1e^{\frac{rt}{100}}Q\right]dt &= C_1e^{\frac{rt}{100}}Q \\
&= \int C_1e^{\frac{rt}{100}}\frac{r}{4}dt \\
&= \frac{r}{4}\frac{100}{r}C_1e^{\frac{rt}{100}} + C_2 \\
&= 25C_1e^{\frac{rt}{100}} + C_2
\end{aligned}$$

So our general solution is

$$C_1e^{\frac{rt}{100}}Q = 25C_1e^{\frac{rt}{100}} + C_2$$

or

$$Q = 25 + Ce^{\frac{-rt}{100}}$$

Since $Q(t=0) = Q_0$

$$Q_0 = 25 + C \to C = Q_0 - 25$$

And

$$\begin{aligned} Q(t) &= 25 + (Q_0 - 25)e^{\frac{-rt}{100}} \\ &= 25(1 - e^{\frac{-rt}{100}}) + Q_0 e^{\frac{-rt}{100}} \end{aligned}$$

When we want to solve for the time T after which the salt level is within 2% of Q_L (the limiting ammount), we do it as follows:

$$\begin{aligned} 25.5 = 25 + 25e^{-rT/100} &\to \frac{1}{2} = 25e^{-rT/100} \\ = \frac{1}{50} = e^{-rT/100} &\to \ln(1/50) = \frac{-rT}{100} \\ = -\frac{100}{r}\ln(1/50) &= \frac{100}{r}\ln 50 \end{aligned}$$

2.5.2 Example 3: Chemicals in a pond

We will pick up from

$$\frac{dt}{dt} + \frac{1}{2}q = 10 + 5\sin(2t)$$

And we can see that we have a nice, simple, first order, linear equation, so we will proceed with the method of integrating factors.

$$\begin{aligned} \frac{d}{dt}\left[\mu(t)q(t)\right] &= \mu\frac{dq}{dt} + q\frac{d\mu}{dt} \\ &= \mu\frac{dt}{dt} + \frac{1}{2}\mu q = 10\mu + 5\mu\sin(2t) \end{aligned}$$

Means that the integrating factor will be

$$q\frac{d\mu}{dt} \sim \frac{1}{2}\mu q \to \frac{1}{\mu}\frac{d\mu}{dt} \sim \frac{1}{2}$$

Or

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = \int \frac{1}{2}$$

Which leads to $\mu(t) = e^{t/2}$.

So our equation becomes

$$\frac{d}{dt}\left[e^{t/2}q(t)\right] = e^{t/2}\frac{dt}{dt} + \frac{1}{2}e^{t/2}q = 10e^{t/2} + 5e^{t/2}\sin(2t)$$

Hence,

$$\begin{aligned} e^{t/2}q(t) &= \int 10e^{t/2}dt + \int 5e^{t/2}\sin(2t)dt \\ &= 20e^{t/2} + \int 5e^{t/2}\sin(2t)dt \end{aligned}$$

Here we have an interesting integral so let's break it down.

2.5.2.1 An interesting integral

In the previous expression we ended up with

$$\int 5e^{t/2}\sin(2t)dt$$

The tip here is a chain of integrations by parts and u-substitutions. First, let's recall the rule for integration by parts

$$\int udv = uv - \int vdu$$

Now, let's get to it.

$$\begin{aligned}
\int e^{t/2}\sin(2t)dt &= \begin{bmatrix} u = e^{t/2} & v = -\frac{1}{2}\cos(2t) \\ du = \frac{1}{2}e^{t/2}dt & dv = \sin(2t)dt \end{bmatrix} \\
&= -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{4}\left[\frac{1}{2}e^{t/2}\sin(2t) - \frac{1}{4}\int e^{t/2}\sin(2t)dt\right] \\
&= -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{2^3}e^{t/2}\sin(2t) - \frac{1}{2^4}\int e^{t/2}\sin(2t)dt
\end{aligned}$$

Notice that we got our initial integral back, so now some algebra will lead us to

$$\left(\int e^{t/2}\sin(2t)dt\right)\left(1 + \frac{1}{2^4}\right) = -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{2^3}e^{t/2}\sin(2t)$$

Which can be simplified to

$$\begin{aligned}
\int e^{t/2}\sin(2t)dt &= -\frac{2^4}{2}\frac{1}{2^4+1}e^{t/2}\cos(2t) + \frac{2^4}{2^3}\frac{1}{2^4+1}e^{t/2}\sin(2t) \\
&= -\frac{2^3}{2^4+1}e^{t/2}\cos(2t) + \frac{2}{2^4+1}e^{t/2}\sin(2t)
\end{aligned}$$

Now, we can put everything together!

$$\begin{aligned}
e^{t/2}q(t) &= 20e^{t/2} + \int 5e^{t/2}\sin(2t)dt \\
&= 20e^{t/2} + 5\left[-\frac{2^3}{2^4+1}e^{t/2}\cos(2t) + \frac{2}{2^4+1}e^{t/2}\sin(2t)\right] \\
&= 20e^{t/2} - \frac{40}{17}e^{t/2}\cos(2t) + \frac{10}{17}e^{t/2}\sin(2t) + C
\end{aligned}$$

Notice that we trhew in an integration coefficient at the end. And our final answer is now

$$q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) + Ce^{-t/2}$$

3 Numerical Analysis

3.1 Errors

As NR explains: Arithmetic among numbers in floating-point representation is not exact, even if the operands happen to be exactly represented. For example, two floating numbers are added by first right-shifting (dividing by two) the mantissa of the smaller (in magnitude) one and simultaneously increasing its exponent until the two operands have the same exponent. Low-order (least significant) bits of the smaller operand are lost by this shifting. If the two operands differ too greatly in magnitude, then the smaller operand is effectively re- placed by zero, since it is right-shifted to oblivion.

The smallest (in magnitude) floating-point number that, when added to the floating-point number 1.0, produces a floating-point result different from 1.0 is term- ed the **machine accuracy**, ϵ_m.

It is important to understand that ϵ_m is not the smallest floating-point number that can be represented on a machine. That number depends on how many bits there are in the exponent, while ϵ_m depends on how many bits there are in the mantissa.

Pretty much any arithmetic operation among floating numbers should be thought of as introducing an additional fractional error of at least ϵ_m. This type of error is called roundoff error.

Roundoff errors accumulate with increasing amounts of calculation. If, in the course of obtaining a calculated value, you perform N such arithmetic operations, you might be so lucky as to have a total roundoff error on the order of $\sqrt{N}\epsilon_m$, if the roundoff errors come in randomly up or down. (The square root comes from a random-walk.) However, this estimate can be very badly off the mark for two reasons:

(1) It very frequently happens that the regularities of your calculation, or the peculiarities of your computer, cause the roundoff errors to accumulate preferentially in one direction. In this case the total will be of order $N\epsilon_m$.

(2) Some especially unfavorable occurrences can vastly increase the roundoff error of single operations. Generally these can be traced to the subtraction of two very nearly equal numbers, giving a result whose only significant bits are those (few) low-order ones in which the operands differed.