Notes on Complex Analysis

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1 Complex Numbers

1.1 Basic Algebraic Properties

A handy thing to keep written down

$$z^{-1} = \left(\frac{a}{z^2 + b^2}, \frac{-b}{a^2 + y^2} \right)$$

The way to think about it is as if you wanted to find c and d such that

$$\frac{1}{a + bi} = c + di$$

The trick here is as follows

$$\frac{1}{a + bi} \left(\frac{a - bi}{a - bi} \right) = \frac{a - bi}{a^2 + b^2}$$

Also,

$$|z|^2 = z\bar{z} = (a + ib)(a - ib) = a^2 + b^2$$

The generalization of $|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$ does hold!

Note that the product of two complex numbers is very diffeent from the scalar or vector products done in vector spaces over the reals. This notion of a **billinear form** is what is often used to distinguish between different algebras.

Also note that $z_1 < z_2$ has no meaning, so the order field properties we are used to from real numbers don't apply as such. However $|z_1| < |z_2|$ does make sense.

The distance between two points (x_1, y_1) and (x_2, y_2) is $|z_1 - z_2|$.

The complex numbers lying on a circle with center z_0 and radius R satisfy the equation

$$|z - z_0| = R$$

A wonderful example of this last interpretation is

$$|z - 3i| + |z + 3i| = |z - 3i| + |z - (-3i)| = 12$$

This equation represents the set of all points whose distance from the two set points, $F_1(0, 3)$ and $F_2(0, -3)$, is 12. This turns out to be the ellipse with foci $F_1(0, 3)$ and $f_2(0, -3)$. Kline has some great exercises to get you acquainted with Ellipses, parabolas, and hyperbolas.

1.1.1 Exercises

2.2

Some interesting properties

$$z + \bar{z} = (a + ib) + (a - ib) = 2a = 2\,\mathrm{Re}(z)$$

Similarly,

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2\,\mathrm{Im}(z)$$

Following the same mechanics,

$$\mathrm{Re}(iz) = \mathrm{Re}(i(a + ib)) = \mathrm{Re}(ai - b) = -\,\mathrm{Im}(z)$$

And

$$\mathrm{Im}(iz) = \mathrm{Im}(ai - b) = \mathrm{Re}(z)$$

1.2 Triangle Inequality

There is a briliant example in this section, go read it!

The heart of the example is in noticing that the triangle inequality gives us an upper and a lower bound for the sum of two numbers. The upper bound comes from

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and the lower bound from

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

1.2.1 polynomials

If n is a positive integer, and if $a_0, a_1, a_2, \ldots$ are complex constants, where $a_n \neq 0$,

$$P(z) = a_0 + a_1 z + a_2 z^+ \ldots a_n z_6$$

if a polynomial of degree n. For some positive number R, the reciprocal $1/P(z)$ satisfies the inequality

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n| R^n}$$

whenever $|z| > R$. Geometrically, this tells us that the modulus of the reciprocal $1/P(z)$ is bounded from above when z is exterior to the circle $|z| = R$.

The tricky bit of the argument for the abvove is given when the author's say "Now that a sufficiently large positive number R can be found such that each of the quotients on the right in inequality (9) is less than the number $|a_n|/(2n)$ when $|z| > R$..." What ci-dessus means in practice is that when we have

$$|w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + \frac{|a_{n-1}|}{|z|}$$

We are actually dividing $|w||z|^n$ by R^n since $R^n \sim |z|^n$, and so Churchill is saying that we can find some z such that when we raise it to a power n and divide $|w||z|^n$ by it, that we will have all n terms on the right side of the inequality be less than the n-th term divided by n (go far out ebough and there will be a z value that will make this truth).

The rest of the argument in the book is algebra.

1.2.2 Exercises

Ex 4
Verify $\sqrt{2}|z| \geq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$. Suggestion: reduce the above inequality to $(|x| - |y|)^2 \geq 0$.

This exercise is similar to one in Folland's advanced calculus, exercise 2 in section 1.1 Euclidean Spaces and Vectors (we do provide a solution for that example in the calculus doc).

Working backwards,

$$(\sqrt{2}|z|)^2 = 2|z|^2 = 2x^2 + 2y^2$$

Following the tip of the book, a thing to try is something like

$$(|\operatorname{Re}(z)| + |\operatorname{Im}(z)|)^2 = |x|^2 + |y|^2 + 2|x||y| = x^2 + y^2 + 2|x||y|$$

Then

$$\begin{aligned}(\sqrt{2}|z|)^2 - (|\operatorname{Re}(z)| + |\operatorname{Im}(z)|)^2 &= x^2 + y^2 - 2|x||y| \\ &= |x|^2 + |y|^2 - 2|x||y| \\ &= (|x| - |y|)^2 \geq 0\end{aligned}$$

Ex 6
Using the fact that $|z_1 - z_2|$ is the distance between the points z_1 and z_2, give a geometric argument that $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

This one is a very interesting case. $|z - 1|$ would correspond to the distance from a point z to the point 1 (the x-axis), $|z + i|$ would correspond to the distance from a point z to the point $-i$ (the y-axis). Since both are a unit from the origin, in their respective axis. The expression above then equates these two distances giving us a straightline that passes through the origin with a -1 slope.

Ex 7

Show that for R sufficiently large, the polynomial $P(z)$ satisfies

$$|P_n(z)| \leq 2|a_n||z|^n$$

whenever $|z| > R$. Suggestion: observe that there is a positive number R such that the modulus of each quotient in $|w| \leq |a_0|/|z|^n + |a_1|/|z|^{n-1} + \ldots + |a_{n-1}|/|z|$ is less than $|a_n|/n$ when $|z| > R$.

In the argument in which the original inuequality was made, there was a step where we use the following $|a_n + w| \geq ||a_n| - |w||$ to come up with a lower bound. If we instead looked for an upper bound, we could look at

$$|a_n + w| \leq |a_n| + |w| < |a_n| + \frac{1}{2}|a_n| < 2|a_n|$$

The rest of the argument flows when we plug that into expression (10),

$$|P_n(z)| = |a_n + w||z|^n < 2|a_n||z|^n$$

Ex 8

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Use simple algebra to show that

$$|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

then point out how the identity $|z_1 z_2| = |z_1||z_2|$ follows.

The trick to the first part is to make use of $|z| = \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}$
First of,

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

From there, we can see that

$$\begin{aligned} \mathrm{Re}(z_1 z_2)^2 &= (x_1 x_2 - y_1 y_2)(x_1 x_2 - y_1 y_2) \\ &= (x_1 x_2)^2 + (y_1 y_2)^2 - 2(x_1 x_2)(y_1 y_2) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Im}(z_1 z_2)^2 &= (x_1 y_2 + x_2 y_1)(x_1 y_2 + x_2 y_1) \\ &= (x_1 y_2)^2 + (x_2 y_1)^2 + 2(x_1 x_2)(y_1 y_2) \end{aligned}$$

It then follows that

$$\begin{aligned} |z_1 z_2| &= \sqrt{(x_1 x_2)^2 + (y_1 y_2)^2 - 2(x_1 x_2)(y_1 y_2) + (x_1 y_2)^2 + (x_2 y_1)^2 + 2(x_1 x_2)(y_1 y_2)} \\ &= \sqrt{(x_1 x_2)^2 + (y_1 y_2)^2 + (x_1 y_2)^2 + (x_2 y_1)^2} \\ &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \end{aligned}$$

Since $|z| = \sqrt{x^2 + y^2}$, we can see how the above reordering is equivalent to

$$\begin{aligned} |z_1 z_2| &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ &= \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \\ &= |z_1||z_2| \end{aligned}$$

Ex 9

If we use the result from the previous exercise and assume have $z = z_1 = z_2$, we have

$$|z^2| = |z||z| = |z|^2$$

We could use this as the base case for an induction argument ($n = 2$).

Then for our hypothesis, we assume that $|z^m| = |z|^m$, when $n = m$, so it must also hold for $n = m + 1$,

$$|z^{m+1}| = |z^m z| = |z' z| = |z'||z| = |z^m||z| = |z|^m|z| = |z|^{m+1}$$

1.3 6 Complex Conjugates

1.3.1 Exercises

Ex 4

Show that,

$$\overline{z^2} = \bar{z}^2$$

Note that

$$\begin{aligned} \overline{z^2} &= \overline{zz} \\ &= \bar{z}\bar{z} \\ &= \bar{z}^2 \end{aligned}$$

Ex 9

By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using the inequality $|z_1 + z_2| \geq ||z_1| - |z_2||$, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}$$

To start with the factoring of our polynomial into two quadratic factors. The three constants we have are $A = 1$, $B = -4$, and $C = 3$. So we need some combination that when multiplied will equal $AC = 3$ and when summed $B = -4$. If we choose 1 and 3 we can get that combination, and since $A = 1$, then dividing both of those numbers by 1 results in no change. Thus we have,

$$\left|z^4 - 4z^2 + 3\right| = |(z-1)(z-3)| = |(z-1)|\,|(z-3)|$$

If we now use the inequality that was recommended to us, we get

$$
\begin{aligned}
\left|z^4 - 4z^2 + 3\right| &= \left|\left(z^2 - 1\right)\right| \left|\left(z^2 - 3\right)\right| \\
&\geq \left||z|^2 - |-1|\right| \left||z|^2 - |-3|\right| \\
&= \left||z|^2 - 1\right| \left||z|^2 - 3\right| \\
&= |4 - 1| \, |4 - 3| \\
&= 3
\end{aligned}
$$

The final result comes from inverting the two quantities.

Ex 10

Show that,

1. z is real if and only if $z = \bar{z}$
2. z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$

To see the why for the first one, note that if $z = x + iy = x - iy = \bar{z}$, then $(x, y) = (x, -y)$. Saying that $x = x$ leads us to a tautology, so no issue there, but $y = -y$ leads us to say that z must be real.

Using a similar argument,

$$\bar{z}^2 = \overline{z^2} = \overline{x^2 - y^2 + 2ixy} = x^2 - y^2 + 2ixy$$

The last step has $xy = -xy$, so this must be zero. And this happens when z is either real or pure imaginary.

Ex 13

Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radisu R can be written as

$$|z|^2 - 2\,\mathrm{Re}(zz_0) + |z_0|^2 = R$$

To see why, note that

$$
\begin{aligned}
|z - z_0| &= (z - z_0)^*(z - z_0) \\
&= (\bar{z} - \bar{z_0})(z - z_0) \\
&= \bar{z}z - \bar{z}z_0 - z\bar{z_0} + \bar{z_0}z_0 \\
&= |z|^2 - (z\bar{z_0} + \bar{z}z_0) + |z_0|^2 \\
&= |z|^2 - 2\,\mathrm{Re}(z\bar{z_0}) + |z_0|^2
\end{aligned}
$$

Ex 14

Show that the hyperbola $x^2 - y^2 = 1$ can be rewritten as $z^2 - \bar{z}^2 = 2$.

Well, $z^2 = (z+iy)(z+iy) = x^2 + 2ixy - y^2$. And we already saw that $\overline{z}^2 = \overline{z^2}$, so $\overline{z}^2 = x^2 - 2ixy - y^2$. From there the answer follows.

Just kidding, here are the rest of the steps. But keep in mind that,

1. $(z+\overline{z}) = z^2 + \overline{z}^2 + z\overline{z} + \overline{z}z$, and
2. $(z-\overline{z}) = z^2 + \overline{z}^2 - z\overline{z} - \overline{z}z$

So if, $x^2 - y^2 = 1$, then,

$$\begin{aligned} x^2 - y^2 = \mathrm{Re}^2(z) - \mathrm{Im}^2(z) &= \left(\frac{z+\overline{z}}{2}\right)^2 - \left(\frac{z-\overline{z}}{2i}\right)^2 \\ &= \frac{1}{4}\left(z+\overline{z}\right)^2 + \frac{1}{4}\left(z-\overline{z}\right)^2 \\ &= \frac{1}{2}\left(z^2+\overline{z}^2\right) \end{aligned}$$

Which is equivalent to saying that $2\left(x^2 - y^2\right) = z^2 + \overline{z}^2 = 2$, hence our answer.

Ex 15

Follow the steps below to give an algebraic derivation of the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

(a) Show that

$$|z_1 + z_2|^2 = (z_1 + z_2)\left(\overline{z_1} + \overline{z_2}\right) = z_1\overline{z_1} + \left(z_1\overline{z_2} + \overline{z_1\overline{z_2}}\right) + z_2\overline{z_2}$$

This first step is just plain algebra, though keep in mind that $z\overline{z} = |z|^2$ and that $\overline{z_1\overline{z_2}} = \overline{z_1}z_2$.

(b) Point out why $z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\,\mathrm{Re}\left(z_1\overline{z_2}\right) \leq 2|z_1||\overline{z_2}|$

Now we are using the identity $z + \overline{z} = 2\,\mathrm{Re}(z)$, where $z = z_1\overline{z_2}$ along with the inequality $\mathrm{Re}(z) \leq |\mathrm{Re}(z)| \leq |z|$.

(c) Use the results in parts (a) and (b) to obtain the inequality

$$|z_1 + z_2|^2 \leq \left(|z_1| + |z_2|\right)^2,$$

and note how the triangle inequality follows.

This last inequality is obtained by boting that

$$\begin{aligned} |z_1 + z_2|^2 &= z_1\overline{z_1} + \left(z_1\overline{z_2} + \overline{z_1\overline{z_2}}\right) + z_2\overline{z_2} \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1||\overline{z_2}| \\ &= \left(|z_1| + |z_2|\right)^2 \end{aligned}$$

and since both, left and right, quantities are positive, then we get $|z_1 + z_2| \leq |z_1| + |z_2|$.

1.4 8 Product's and Powers in Exponential Form

1.4.1 De Moivre's Theorem

$$(a+ib)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$r = |z| = \sqrt{a^2+b^2}$$

$$\theta = \tan^-1\frac{y}{x}$$

De Moiver's Theorem

1.5 9 Arguments of products and Quotients

1.5.1 Roots of Complex Numbers

Nth roots: for any positive integer n, the nth distinct roots of $(a+ib)^n = r^n(\cos nx + i\sin nx)$ are

$$r^{\frac{1}{n}}\left[\cos\frac{x+2\pi k}{n} + i\sin\frac{x+2\pi k}{n}\right]$$

for $k = 0, 1, \dots, n-1$.

1.5.2 Exercises

9.1

Find the principal argument $\operatorname{Arg} z$ when

1. $z = \frac{-2}{1+\sqrt{3}i}$

$$\operatorname{Arg}(-2) = \pi$$

$$\operatorname{Arg}(1+\sqrt{3}i) = \frac{\pi}{3}$$

   To get the above answer, we drew it out and searched for special triangles. It just so happens that we can use the 30-60-90 triangle whose $\tan\sqrt{3} = \pi/3$.

   If we subtract the two, we get $2\pi/3$ which falls within the valid range for a principal argument, $-\pi < \Theta \leq \pi$.

2. $z = \left(\sqrt{3}-i\right)^6$

   If we graph where $\sqrt{3}-i$ is, we will find it in the foruth quadrant, so the principal argument must be negative. The angle in this case is $\tan 1/\sqrt{3}$, which is now the 30-degrees side of the 30-60-90 triangle and so $\Theta = -\pi/6$.

However, now we have to raise it to the 6-th power. $\left(e^{i\theta}\right)^6 = e^{-i\pi} = -1$. Also, $r = |z| = \sqrt{\overline{z}z} = \sqrt{\mathrm{Re}\,(z)^2 + \mathrm{Im}\,(z)^2} = 2$. So $z^6 = (2^6)(-1) = -64$. Since the value is all real and it lies in the negative side of the graph, the principal argument is $\Theta = \pi$.

9.4

Using the fact that the modulus $|e^{i\theta} - 1|$ is the distance between teh points $e^{i\theta}$ and 1 (See section 4), give a geometric argument to find a value of θ in the interval $0 \leq \theta < 2\pi$ that satisifes $|e^{i\theta} - 1| = 2$.

We saw in section 4 that $|z - z_0| = R$ can be interpreted as the set of complex numbers z ($e^{i\theta}$) centered at z_0 (1) and with a radius of R (2). So what complex numbers, lying on the unit circle, also lie on the circle centered at 1 and whose radisu is 2? The point that meets this criteria is $\theta = \pi$.

9.6

Show that if $\mathrm{Re}\, z_1 > 0$ and $\mathrm{Re}\, z_2 > 0$, then

$$\mathrm{Arg}\,(z_1 z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2,$$

where principal arguments are used.

If $\mathrm{Re}\, z > 0$, then that means that when we draw z in a polar graph, the point z will lie in either the first or the fourth quadrants. Meaning that $\mathrm{Arg}\, z \in (\pi/2, -\pi/2)$.

Because of this, $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 \in (\pi, -\pi)$, which meets the constraints for the range of the principal argument, no need to add or subtract any multiples of 2π.

9.8

Prove that two nonzero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1 \overline{c_2}$.

Suggestion: Note that

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right) \exp\left(i\frac{\theta_1 - \theta_2}{2}\right) = \exp\,(i\theta_1)$$

Remember that $\overline{e^{i\theta}} = e^{-i\theta}$.

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right) \overline{\exp\left(i\frac{\theta_1 - \theta_2}{2}\right)} = \exp\,(i\theta_2)$$

$$
\begin{aligned}
|z_1|^2 = z_1\overline{z_1} = c_1c_2\overline{c_1c_2} = c_1\overline{c_1}c_2\overline{c_2} = |c_1|^2|c_2|^2 \\
|z_2|^2 = z_2\overline{z_2} = c_1\overline{c_2}\overline{c_1\overline{c_2}} = c_1\overline{c_2}\overline{c_1}c_2 = c_1\overline{c_1}c_2\overline{c_2} = |c_1|^2|c_2|^2
\end{aligned}
$$

2 Inifinite Series, Products, and Integrals

2.1 Uniform Convergence

Note: when we speak of uniform convergence, the interval can be closed or open. Titchmarsh just uses (a, b) to cover the general case.

The more general case for the first test of uniform Convergence we see is stated as follows:

The series $\sum u_n(x)$ is uniformly convergent ($\forall \epsilon > 0$, we can find $n_0 \geq N$ depending on ϵ but not on x, such that $|s(x) - s| < \epsilon$, for every $n \geq n_0$ for every value in (a, b)) if $|u_n(x)| \leq v_n(x)$, and $\sum v_n(x)$ is uniformly convergent.

If we try to make an argument by contradiciton and assume that $\sum u_n(x)$ is not uniformly convergent, then the series could still converge but it could be the case that as x approaches some point on the interval (a, b), n_0 may become infinetely large. Additionally, the series could just be a divergent series. Either way it means we are not able to find an n_0 such that $|s(x) - s| < \epsilon$ for any $n \geq n_0$, for any $\epsilon > 0$ and for any $x \in (a, b)$. This means that

$$|u_{n+1}(x) + u_{n+2}(x) + \dots|$$

keeps on changing as n or x change.

Since,

$$|u_{n+1}(x) + u_{n+2}(x) + \dots| \leq |u_{n+1}(x)| + |u_{n+2}(x)| + \dots$$

Then any $v_n(x)$ such that $v_{n+1}(x) \geq |u_{n+1}(x)|$ would also grow indefinetely and thus lead to a contradiciton.

Examples

A proof to see why $\sum_{n=0} x^n$ is uniformly convergent in $c \in [a, b]$ when $-1 < a < b < 1$ can be seen by comparing the exercise 2.5.3 from Abbott.

The trigonometric series $\sum_{n=1} \frac{\cos nx}{n^2}$ is convergent anywhere because $-1 \leq \cos x \leq 1$ so $\left|\frac{\cos nx}{n^2}\right|$ behaves like a convergent p-series.

Similarly the Dirichlet series $\sum_{n=1} n^{-s} = \sum_{n=1} \frac{1}{n^s}$ is uniformly convergent in $x \in [a, b]$ if $1 < a < b$ because its absolute value is eequal to a convergent p series. The sum is refered to as the Riemann zeta function $\zeta(s)$.