

Notes on Calculus

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1 Setting the Stage

1.1 Euclidean Spaces and Vectors

1.1.1 Exercises

1.1.2

Given $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\begin{aligned} |\vec{x} + \vec{y}|^2 &= (\vec{x} + \vec{y})(\vec{x} + \vec{y}) \\ &= \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} + 2\vec{x} \cdot \vec{y} \\ &= |\vec{x}|^2 + |\vec{y}|^2 + 2\vec{x} \cdot \vec{y} \end{aligned}$$

Similarly,

$$\begin{aligned} |\vec{x} - \vec{y}|^2 &= (\vec{x} - \vec{y})(\vec{x} - \vec{y}) \\ &= \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} - 2\vec{x} \cdot \vec{y} \\ &= |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x} \cdot \vec{y} \end{aligned}$$

Hence

$$|\vec{x} + \vec{y}|^2 + |\vec{x} - \vec{y}|^2 = 2(|\vec{x}|^2 + |\vec{y}|^2)$$

1.1.7

Suppose $\vec{a}, \vec{b} \in \mathbb{R}^3$

Show that if $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$ for some non-zero $\vec{c} \in \mathbb{R}^3$, then $\vec{a} = \vec{b}$.

We could try to simply stare at

$$\vec{a} \cdot \vec{c} = |\vec{a}||\vec{c}| \cos \theta_1 = |\vec{b}||\vec{c}| \cos \theta_2 = \vec{b} \cdot \vec{c}$$

Which tells us

$$|\vec{a}| \cos \theta_1 = |\vec{b}| \cos \theta_2$$

Let's try something else,

$$|a \times c|^2 = |a||c| - (a \cdot c)^2 = |b||c| - (b \cdot c)^2 = |b \times c|^2$$

We now have

$$|a||c| - (a \cdot c)^2 = |b||c| - (b \cdot c)^2$$

or

$$|a||c| = |b||c| \rightarrow |a| = |b|$$

So we can go back to our first attempt and see that

$$|a| \cos \theta_1 = |b| \cos \theta_2 \rightarrow \cos \theta_1 = \cos \theta_2$$

1.1.8

To see that $a \cdot (b \times c)$ is the determinant of the three vectors, simply write out the determinant for $b \times c$ and note that the explicit version of it is a "normal" vector. Since the dot product is defined as $x \cdot y = x_1y_1 + x_2y_2 + \dots + x_ny_n$, when $x, y \in \mathbb{R}^n$.

Putting these two facts together we can see how $a \cdot (b \times c)$ can be computed via a single determinant operation.

1.2 Subsets of Euclidean Space

1.2.1 Exercises