Notes on Calculus

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1.1 Euclidean Spaces and Vectors

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1.1.2

Given $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\begin{aligned} |\vec{x}+\vec{y}|^2 &= (\vec{x}+\vec{y})(\vec{x}+\vec{y}) \\ &= \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} + 2\vec{x}\cdot\vec{y} \\ &= |\vec{x}|^2 + |\vec{y}|^2 + 2\vec{x}\cdot\vec{y} \end{aligned}$$

Similarly,

$$\begin{aligned} |\vec{x}-\vec{y}|^2 &= (\vec{x}-\vec{y})(\vec{x}-\vec{y}) \\ &= \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} - 2\vec{x}\cdot\vec{y} \\ &= |\vec{x}|^2 + |\vec{y}|^2 - 2\vec{x}\cdot\vec{y} \end{aligned}$$

Hence

$$|\vec{x}+\vec{y}|^2 + |\vec{x}-\vec{y}|^2 = 2\left(|\vec{x}|^2 + |\vec{y}|^2\right)$$

1.1.7

Suppose $\vec{a}, \vec{b} \in \mathbb{R}^3$

Show that if $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{c}$ for some non-zero $\vec{c} \in \mathbb{R}^3$, then $\vec{a} = \vec{b}$.

We could try to simply stare at

$$\vec{a}\cdot\vec{c} = |\vec{a}||\vec{c}|\cos\theta_1 = |\vec{b}||\vec{c}|\cos\theta_2 = \vec{b}\cdot\vec{c}$$

Which tells us

$$|\vec{a}|\cos\theta_1 = |\vec{b}|\cos\theta_2$$

Let's try something else,

$$|a\times c|^2 = |a||c| - (a\cdot c)^2 = |b||c| - (b\cdot c)^2 = |b\times c|^2$$

We now have

$$|a||c| - (a\cdot c)^2 = |b||c| - (b\cdot c)^2$$

or

$$|a||c| = |b||c| \to |a| = |b|$$

So we can go back to our first attempt and see that

$$|a|\cos\theta_1 = |b|\cos\theta_2 \to \cos\theta_1 = \cos\theta_2$$

1.1.8

To see that $a \cdot (b \times c)$ is the determinant of the three vectors, simply write out the determinant for $b \times c$ and note that the explicit version of it is a "normal" vector. Since the dot product is defined as $x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$, when $x, y \in \mathbb{R}^n$.

Putting these two facts together we can see how $a \cdot (b \times c)$ can be computed via a single determinant operation.

1.2 Subsets of Euclidean Space

Proposition 1.4

Remember that to be a boundary point of $S \subset \mathbb{R}^n$ **every** ball centered at $\vec{x}$ must contain points in S and in S^c.

If $\vec{x}$ is an interior point, then for some $\vec{x} \in S$, there is a ball $B(r, \vec{x}) \subset S$; but also, there is an $B(r, \vec{x}) \not\subset S^c$ for some $r > 0$. This is why you ought to be an interior point in S, in S^c, or be a boundary point - because to be a boundary point every single $B(r, \vec{x})$ must be in S AND in S^c.

This last statement is why it is also the case that if S is closed then S^c must be open: because any points must be interior points to either one or be a boundary point. So if S has all of the boundary points, then its compliment must be left with none and thus only have interior points and be an open set.

1.2.1 Exercises