

Notes on Electrodynamics

December 31, 2023

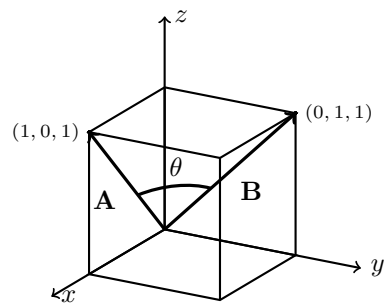
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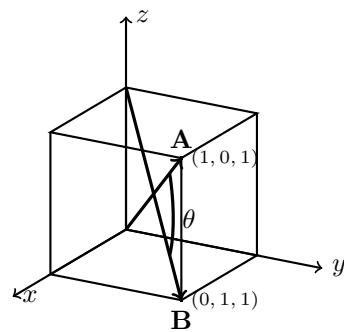
1 Vector Analysis

1.1 Vector Algebra

Face diagonals



Body diagonals



2 Potentials

2.1 Multipole Expansion

$$r = |\mathbf{r} - \mathbf{r}'|$$

Where r denotes the vector from a source point \mathbf{r}' to a field point \mathbf{r} .

3 Introduction to Electrostatics

3.1 Gauss's Law

3.1.1 Solid Angle

The following is all wikipedia.

Whereas an angle in radians, projected onto a circle, gives a length of a circular arc on the circumference, a solid angle in steradians, projected onto a sphere, gives the area of a spherical cap on the surface.

The point from which the object is viewed is called the apex of the solid angle, and the object is said to subtend its solid angle at that point.

In SI units, a solid angle is expressed in a dimensionless unit called a **steradian**.

One steradian corresponds to one unit of area on the unit sphere surrounding the apex, so an object that blocks all rays from the apex would cover a number of steradians equal to the total surface area of the unit sphere, 4π .

Just like a planar angle in radians is the ratio of the length of an arc to its radius, $\theta = s/r$, a solid angle in steradians is the ratio of the area covered on a sphere by an object to the area given by the square of the radius of said sphere. The formula is

$$\Omega = \frac{A}{r^2}$$

where A is the spherical surface area and r is the radius of the considered sphere.

Any area on a sphere which is equal in area to the square of its radius, when observed from its center, subtends precisely one steradian.

The solid angle of a sphere measured from any point in its interior is 4π sr (steradians), and the solid angle subtended at the center of a cube by one of its faces is one-sixth of that, or $2\pi/3$ sr.

In spherical coordinates there is a formula for the differential,

$$d\Omega = \sin\theta d\theta d\varphi$$

where θ is the colatitude (angle from the North Pole - z-axis range $[0, \pi]$) and φ is the longitude (range $[0, 2\pi]$).

The solid angle for an arbitrary oriented surface S subtended at a point P is equal to the solid angle of the projection of the surface S to the unit sphere with center P, which can be calculated as the surface integral:

$$\Omega = \iint_S \frac{\hat{r} \cdot \hat{n}}{r^2} dS = \iint_S \sin\theta d\theta d\varphi$$

where $\hat{r} = \vec{r}/r$ is the unit vector corresponding to \vec{r} , the position vector of an infinitesimal area of surface dS with respect to point P, and where \hat{n} represents the unit normal vector to dS . Even if the projection on the unit sphere to the surface S is not isomorphic, the multiple folds are correctly considered according to the surface orientation described by the sign of the scalar product $\hat{r} \cdot \hat{n}$.

Thus one can approximate the solid angle subtended by a small facet having flat surface area dS , orientation \hat{n} , and distance r from the viewer as:

$$d\Omega = 4\pi \left(\frac{dS}{A} \right) (\hat{r} \cdot \hat{n})$$

where the surface area of a sphere is $A = 4\pi r^2$.

3.1.2 Surface elements da

The following logic comes from Griffith's electrodynamics chapter 1.4.

In curvilinear coordinates, an infinitesimal displacement in the \hat{r} direction is simply dr .

$$dl_r = dr$$

An infinitesimal element of length in the $\hat{\theta}$ is

$$dl_\theta = r d\theta$$

And an infinitesimal element of length along the $\hat{\phi}$ direction is

$$dl_\phi = r \sin \theta d\phi$$

And thus, the general infinitesimal displacement becomes

$$d\mathbf{l} = dr\hat{r} + r d\theta\hat{\theta} + r \sin \theta d\phi\hat{\phi}$$

From there one can see how the infinitesimal volume element $d\tau$ is then

$$d\tau = dl_r dl_\theta dl_\phi = r^2 \sin \theta dr d\theta d\phi$$

However, the infinitesimal area element depends on the orientation of the surface!

For example, if we integrate over the surface of a sphere, where r stays constant but θ and ϕ vary,

$$d\mathbf{a} = dl_\theta dl_\phi \hat{r} = r^2 \sin \theta d\theta d\phi \hat{r}$$

Where as is the surface lies in the x-y plane, so that θ is constant, and let's say $\theta = \phi/2$, while r and ϕ vary,

$$d\mathbf{a} = dl_r dl_\phi \hat{\theta} = r \sin \theta dr d\phi \hat{\theta} = r dr d\phi \hat{\theta}$$

The area element on the sphere can be calculated from the cross products of other two elements,

3.1.3 Explanation of Jackson

If the electric field makes an angle θ with the unit normal, then the projection of the infinitesimal area element along the normal of the surface is $\cos \theta da$. Using the solid angle formula, we then get $d\Omega = \cos \theta da / r^2$ or $r^2 d\Omega = \cos \theta da$.

3.2 Exercises

1.2

The Dirac delta function in three dimensions can be taken as the improper limit as $\alpha \rightarrow 0$ of the Gaussian function

$$D(\alpha; x, y, z) = \frac{1}{(\alpha\sqrt{2\pi})^3} \exp \left[-\frac{1}{2\alpha^2} (x^2 + y^2 + z^2) \right]$$

Consider a general orthogonal coordinate system specified by the surfaces $u = \text{constant}$, $v = \text{constant}$, $w = \text{constant}$, with length elements du/U , dv/V , dw/W in the three perpendicular directions. Show that

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(u - u') \delta(v - v') \delta(w - w') \cdot UVW$$

by considering the limit of the Gaussian above. Note that as $\alpha \rightarrow 0$ only the infinitesimal length element need be used for the distance between the points in the exponent.

$$\begin{aligned} \delta(x) &= \lim_{b \rightarrow 0} \frac{1}{|b|\sqrt{\pi}} \exp \left[-\frac{1}{b^2} x^2 \right] \\ &= \lim_{b \rightarrow 0^+} \frac{1}{2\sqrt{\pi}b} \exp \left[-\frac{1}{4b} x^2 \right] \\ &= \lim_{b \rightarrow 0} \frac{1}{b\sqrt{2\pi}} \exp \left[-\frac{1}{2b^2} x^2 \right] \end{aligned}$$