Notes on Complex Analysis

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1 Complex Numbers

1.1 Basic Algebraic Properties

A handy thing to keep written down

$$z^{-1} = \left(\frac{a}{z^2 + b^2}, \frac{-b}{a^2 + y^2} \right)$$

also,

$$|z|^2 = |z\bar{z}| = (a + ib)(a - ib) = a^2 + b^2$$

The generalization of $|z|^2 = (\Re(z))^2 + (\Im(z))^2$ does hold!

Note that the product of two complex numbers is very diffeent from the scalar or vector products done in vector spaces over the reals. This notion of a **billinear form** is what is often used to distinguish between different algebras.

Also note that $z_1 < z_2$ has no meaning, so the order field properties we are used to from real numbers don't apply as such. However $|z_1| < |z_2|$ does make sense.

The distance between two points (x_1, y_1) and (x_2, y_2) is $|z_1 - z_2|$.

The complex numbers lying on a circle with center z_0 and radius R satisfy the equation

$$|z - z_0| = R$$

A wonderful example of this last interpretation is

$$|z - 3i| + |z + 3i| = |z - 3i| + |z - (-3i)| = 12$$

This equation represents the set of all points whose distance from the two set points, $F_1(0, 3)$ and $F_2(0, -3)$, is 12. This turns out to be the ellipse with foci $F_1(0, 3)$ and $f_2(0, -3)$. Kline has some great exercises to get you acquainted with Ellipses, parabolas, and hyperbolas.

1.1.1 Exercises

2.2

Some interesting properties

$$z + \bar{z} = (a + ib) + (a - ib) = 2a = 2\Re(z)$$

Similarly,

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2\Im(z)$$

Following the same mechanics,

$$\Re(iz) = \Re(i(a + ib)) = \Re(ai - b) = -\Im(z)$$

And

$$\Im(iz) = \Im(ai - b) = \Re(z)$$

1.2 Triangle Inequality

There is a briliant example in this section, go read it!

The heart of the example is in noticing that the triangle inequality gives us an upper and a lower bound for the sum of two numbers. The upper bound comes from

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and the lower bound from

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

1.2.1 Exercises

Ex 8

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Use simple algebra to show that

$$|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

then point out how the identity $|z_1 z_2| = |z_1||z_2|$ follows.

The trick to the first part is to make use of $|z| = \sqrt{(\Re(z))^2 + (\Im(z))^2}$
First of,

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

From there, we can see that

$$\begin{aligned}
\Re(z_1z_2)^2 &= (x_1x_2 - y_1y_2)(x_1x_2 - y_1y_2) \\
&= (x_1x_2)^2 + (y_1y_2)^2 - 2(x_1x_2)(y_1y_2)
\end{aligned}$$

and

$$\begin{aligned}
\Im(z_1z_2)^2 &= (x_1y_2 + x_2y_1)(x_1y_2 + x_2y_1) \\
&= (x_1y_2)^2 + (x_2y_1)^2 + 2(x_1x_2)(y_1y_2)
\end{aligned}$$

It then follows that

$$\begin{aligned}
|z_1z_2| &= \sqrt{(x_1x_2)^2 + (y_1y_2)^2 - 2(x_1x_2)(y_1y_2) + (x_1y_2)^2 + (x_2y_1)^2 + 2(x_1x_2)(y_1y_2)} \\
&= \sqrt{(x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (x_2y_1)^2} \\
&= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}
\end{aligned}$$

Since $|z| = \sqrt{x^2 + y^2}$, we can see how the above reordering is equivalent to

$$\begin{aligned}
|z_1z_2| &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\
&= \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \\
&= |z_1||z_2|
\end{aligned}$$

Ex 9
If we use the result from the previous exercise and assume have $z = z_1 = z_2$, we have

$$|z^2| = |z||z| = |z|^2$$

We could use this as the base case for an induction argument ($n = 2$).

Then for our hypothesis, we assume that $|z^m| = |z|^m$, when $n = m$, so it must also hold for $n = m + 1$,

$$|z^{m+1}| = |z^m z| = |z'z| = |z'||z| = |z^m||z| = |z|^m|z| = |z|^{m+1}$$

1.3 De Moivre's Theorem

$$(a + ib)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$r = |z| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-} 1\frac{y}{x}$$

De Moiver's Theorem

1.4 Roots of Complex Numbers

Nth roots: for any positive integer n, the nth distinct roots of $(a + ib)^n = r^n(\cos nx + i\sin nx)$ are

$$r^{\frac{1}{n}}\left[\cos\frac{x + 2\pi k}{n} + i\sin\frac{x + 2\pi k}{n}\right]$$

for $k = 0, 1, \ldots, n - 1$.