Notes on Functional Analysis and PDEs

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1 Growth Functions

For a given function $g(n)$,

$\Theta\left(g(n)\right) = \{f(n) : \exists c_1, c_2 > 0 \text{ such that } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \forall n \geq n_0 \in \mathbb{N}\}$

In the above defintion c_1 and c_2 are constants. So $f(n) = \Theta\left(g(n)\right)$ means that $f(n)$ is bound by $g(n)$ to within a constant factor - bound above and below.

This also means that $0 \leq c_1 \leq \frac{f(n)}{g(n)} \leq c_2$ for sufficiently large n. We say that $g(n)$ **is an asymptotically tight bound for** $f(n)$.

Θ expresses an asymptotic bound from above and from below. For an **asymptotic upper bound** we have

$$O\left(g(n)\right) = \{f(n) : \exists c > 0 \text{ such that } 0 \leq f(n) \leq cg(n) \quad \forall n \geq n_0 \in \mathbb{N}\}$$

The above can also be seen as $0 \leq \frac{f(n)}{g(n)} \leq c$.

The **asymtptotic lower bound** is similarly defined as

$$\Omega = \{f(n) : \exists c > 0 \text{ such that } 0 \leq cg(n) \leq f(n) \quad \forall n \geq n_0 \in \mathbb{N}\}$$

or the set of functions that meet the following inequality $0 \leq c \leq \frac{f(n)}{g(n)}$.

Note that we can remove the "tightness" of the upper and lower bounds by converting the last inequality for both definitions into a strict inequality (swap the last "$\leq$" for a "$<$").

With this language we can define limits and define an order. For example,

- $f(n) = \Theta(g(n))$ is like $a = b$
- $f(n) = O(g(n))$ is like $a \leq b$
- $f(n) = \Omega(g(n))$ is like $a \geq b$
- $f(n) = o(g(n))$ is like $a < b$. This denotes an upper bound that is not asymptotically tight.

Another thing that comes in handy is to remember the rates fo growth of polynomials and exponentials

$$\lim_{n \to \infty} \frac{n^b}{a^n} = 0$$

This is equivalent to saying $n^b = o(a^n)$.

Titchmarsh uses the following notation: $f(x) = O\{\phi(x)\}$ means $|f(x)| < A\phi(x)$ if x is sufficiently close to some limit. In particular $O(1)$ means a bounded function (really think about it).

And $f(x) = o\{\phi(x)\}$ means $f(x)/\phi(x) \to 0$ as x tends to a gven limit. So in this way Titchmarsh notation matches the conventional mathematical notation.

2 The Exponential Function

There is a handy thing to note for the proof of part (a) of the first theorem.

If you look closely to

$$e^z = \sum_{k=0}^{\infty} \frac{z^n}{n!}$$

Then we ought to wonder why $e^0 = 1$ since the first term in the series would be 0^0.

Looking around you may think to use l'hopital (Bernoulli's) rule and do something like: if $y = x^x$

$$\lim_{x \to 0} y = \lim e^{\ln y} = e^{\lim \ln y} = e^{\lim \ln x^x} = e^{\lim x \ln x}$$

Remember that e^z is continuous, so we can just pass the limit through it. Then

$$\lim_{x \to 0} x \ln x = \lim \frac{\ln x}{1/x}$$

One application of Bernoulli's rule later, we have

$$\lim_{x \to 0} x \ln x = \lim \frac{\ln x}{1/x} = \lim \frac{1/x}{-1/x^2} = \lim -x = 0$$

So

$$\lim_{x \to 0} y = \lim e^{\ln y} = e^{\lim x \ln x} = e^0$$

So going that route leads us to a circular argument. Instead, it helps to unfold the series and see that

$$e^z = \sum_{n=0} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \dots$$

3 Introduction to Inner Product Spaces