Notes Mathematical Methods

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Contents

1 Useful Identities

1.1 Trigonometry

$$tan^{-1}(1) = \pi/4 \tag{1.1}$$

Or $-\pi/4$, if you are measuring backwards.

2 Tips and Tricks

If working with a differential equation that has too many independent variables try something like this

$$\frac{dv(t)}{dt} = f(x, v, t)$$

To get rid of t, use the chain rule to instead look at

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dt}$$

3 Preliminaries

3.1 Infinite Series

A necessary condition for convergence of an Infinite series to a limit S is that $\lim_{n\to\infty} u_n = 0$, where u_n is a term in the series. However, the above condition is not sufficient to guarantee convergence.

The argument for this is presented in Abbott in Theorem 2.7.2, the Cauchy criterion for series, and in Theorem 2.7.3.

D'Alembert (Cauchy) Ratio Test

While looking at the D'Alembert (Cauchy) ratio test, keep in mind that the geometric series defines $r = u_{n+1}/u_n$.

Also, in Example 1.1.4, we have the $n \geq 2$ condition because if $n = 1$, then we would have an undeterminate result.

Cauchy or Maclaurin Integral Test

The trick to making sense of the first set of inequalities goes as follows...

s_i, by the fact that $f(x)$ is monotonic decreasing, will overestimate the area under the curve. So the difference between $s_i \geq \int_1^{i+1} f(x)dx$ and $s_i \leq a_1 + \int_1^i f(x)dx$ is that there is a "shift" between the two (the graph as shown in the book).

Where as $\int_1^{i+1} f(x)dx$ starts from 1 and always goes to $\{2, 3, 4, \ldots\}$, $\int_1^i f(x)dx$ cancels out its first "term" since $\int_1^1 f(x)dx = 0$.

So when we look at the first two partial sums in the context of the second inequality we get

1. $s_1 = a_1 \leq a_1 + \int_1^1 f(x)dx = a_1$. The claim that $a_1 \leq a_1$ is true.
2. $s_2 = a_1 + a_2 \leq a_1 + \int_1^2 f(x)dx$. Here the claim is that $f(2) \leq \int_1^2 f(x)dx$.
3. $s_3 = a_1 + a_2 + a_3 \leq a_1 + \int_1^3 f(x)dx$. Here the claim is that $f(2) + f(3) \leq \int_1^3 f(x)dx$.

4 determinants and Matrices

4.1 determinants

4.1.1 Homogeneous Linear Equations

$$a \cdot (b \times c) = (a \times b) \cdot c$$

An $n \times n$ homogeneous system of linear equations has a unique non-trivial solution if and only if its determinant is non-zero. If this determinant is zero, then the system has either no nontrivial solutions or an infinite number of solutions.

If the determinant is zero, then one of the vectors lies in the plane spaned by one of the other two vectors - it is not independent which is equivalent to saying that is linear combination of the other(s).

5 Vector Spaces

5.1 Vectors in Function Spaces

Scwarz Inequality

$$
\begin{aligned}
I &= \langle f - \lambda g | f - \lambda g \rangle \\
&= \left(\langle f| - \lambda^* \langle g| \right) \left(|f\rangle - \lambda |g\rangle \right) \\
&= \langle f|f\rangle - \lambda \langle f|g\rangle - \lambda^* \langle g|f\rangle + \lambda^* \lambda \langle g|g\rangle \\
&= \langle f|f\rangle + |\lambda|^2 \langle g|g\rangle - \left(\langle \lambda g|f\rangle + \langle f|\lambda g\rangle \right) \\
&= \langle f|f\rangle + |\lambda|^2 \langle g|g\rangle - \left(\langle f|\lambda g\rangle^* + \langle f|\lambda g\rangle \right) \\
&= |f|^2 + |\lambda|^2 |g|^2 - \left(\langle f|\lambda g\rangle^* + \langle f|\lambda g\rangle \right) \\
&\geq 0
\end{aligned}
$$

If we were looking at complex numbers, the term in the braces would equal $2\Re(z)$.

$$
\begin{aligned}
\partial_{\lambda^*} I &= \partial_{\lambda^*} \left(\left(\langle f| - \lambda^* \langle g| \right) \left(|f - \lambda g\rangle \right) \right) \\
&= - \langle g|f\rangle + \lambda \langle g|g\rangle = 0
\end{aligned}
$$

The above implies that

$$
\lambda = \frac{\langle g|f\rangle}{\langle g|g\rangle}
$$

If we plug that in, we have,

$$
\begin{aligned}
I &= \langle f|f\rangle + |\lambda|^2 \langle g|g\rangle - \langle \lambda g|f\rangle - \langle f|\lambda g\rangle \\
&= \langle f|f\rangle + \frac{\langle f|g\rangle \langle g|f\rangle}{\langle g|g\rangle^2} \langle g|g\rangle - \frac{\langle f|g\rangle}{\langle g|g\rangle} \langle g|f\rangle - \frac{\langle g|f\rangle}{\langle g|g\rangle} \langle f|g\rangle \\
&= \langle f|f\rangle + \frac{\langle f|g\rangle \langle g|f\rangle}{\langle g|g\rangle} - \frac{\langle f|g\rangle}{\langle g|g\rangle} \langle g|f\rangle - \frac{\langle g|f\rangle}{\langle g|g\rangle} \langle f|g\rangle \\
&= \langle f|f\rangle - \frac{\langle g|f\rangle}{\langle g|g\rangle} \langle f|g\rangle \\
&\geq 0
\end{aligned}
$$

So

$$
\langle f|f\rangle \langle g|g\rangle \geq \langle g|f\rangle \langle f|g\rangle = |\langle f|g\rangle|^2
$$