Notes on differential equations

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Contents

1 First Order Differential Equations

1.1 Linear Equations

1.1.1 Method of Integrating Factors

$$p(t)\frac{dy}{dx} + q(t)y = g(t)$$

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + y\frac{d\mu(t)}{dt} \sim p(t)\frac{dy}{dx} + q(t)y$$

Another way of writing the above could be

$$\frac{dy}{dt} + cy = f(t)$$

If c is a coefficient, then great, but this is another common presentation for equations that canr eadily be solved by the method of integrating factors.

Also, make sure to remember to do the comparsion of $y\frac{d\mu(t)}{dt}$ properly. For example, using the last version we wrote, the integrating facotr would come from the comparison of

$$y\frac{d\mu(t)}{dt} \sim cy\mu(t) \rightarrow \frac{d\mu(t)}{dt} \sim c\mu(t)$$

That's why we get so many exponentials.

1.1.2 Separable Equations

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Can be written in **differential form** as

$$M(x)dx + N(y)dy = 0$$

1.1.3 Notes

Sometimes equations of the form

$$\frac{dy}{dx} = f(x, y)$$

have a constant solution $y = y_0$.

For example,

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1+2y^2}$$

Has a constant solution $y = 3$.

1.2 Modeling with First Order Equations

1.2.1 Example 1: Mixing

$$\frac{dQ}{dt} + \frac{r}{100}Q = \frac{r}{4}$$

Using the method of Integrating factors, we have

$$\frac{d}{dt}\left[\mu(t)Q(t)\right] = \mu\frac{dQ}{dt} + Q\frac{d\mu}{dt} = \mu\frac{dQ}{dt} + \mu\frac{r}{100}Q = \mu\frac{r}{4}$$

Comparing

$$Q\frac{d\mu}{dt} \sim \mu\frac{r}{100}Q$$

We have that

$$\frac{d\mu}{dt} = \frac{r}{100}\mu$$

So the integrating factor must be

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = ln|\mu| = \int \frac{r}{100} = \frac{r}{100}t + C_0$$

And so

$$\mu(t) = e^{\frac{r}{100}t + C_0} = C_1 e^{\frac{rt}{100}}$$

Our original equation becomes

$$\frac{d}{dt}\left[C_1 e^{\frac{rt}{100}}Q\right] = C_1 e^{\frac{rt}{100}}\frac{dQ}{dt} + C_1 e^{\frac{rt}{100}}\frac{r}{100}Q = C_1 e^{\frac{rt}{100}}\frac{r}{4}$$

Now, we can finally integrate both sides,

$$\begin{aligned}\int \frac{d}{dt}\left[C_1 e^{\frac{rt}{100}}Q\right]dt &= C_1 e^{\frac{rt}{100}}Q \\ &= \int C_1 e^{\frac{rt}{100}}\frac{r}{4}dt \\ &= \frac{r}{4}\frac{100}{r}C_1 e^{\frac{rt}{100}} + C_2 \\ &= 25C_1 e^{\frac{rt}{100}} + C_2\end{aligned}$$

So our general solution is

$$C_1 e^{\frac{rt}{100}}Q = 25C_1 e^{\frac{rt}{100}} + C_2$$

or

$$Q = 25 + Ce^{\frac{-rt}{100}}$$

Since $Q(t=0) = Q_0$

$$Q_0 = 25 + C \to C = Q_0 - 25$$

And

$$\begin{aligned} Q(t) &= 25 + (Q_0 - 25)e^{\frac{-rt}{100}} \\ &= 25(1 - e^{\frac{-rt}{100}}) + Q_0 e^{\frac{-rt}{100}} \end{aligned}$$

When we want to solve for the time T after which the salt level is within 2% of Q_L (the limiting ammount), we do it as follows:

$$\begin{aligned} 25.5 = 25 + 25e^{-rT/100} &\to \frac{1}{2} = 25e^{-rT/100} \\ = \frac{1}{50} = e^{-rT/100} &\to \ln(1/50) = \frac{-rT}{100} \\ = -\frac{100}{r}\ln(1/50) &= \frac{100}{r}\ln 50 \end{aligned}$$

1.2.2 Example 3: Chemicals in a pond

We will pick up from

$$\frac{dt}{dt} + \frac{1}{2}q = 10 + 5\sin(2t)$$

And we can see that we have a nice, simple, first order, linear equation, so we will proceed with the method of integrating factors.

$$\begin{aligned} \frac{d}{dt}\left[\mu(t)q(t)\right] &= \mu\frac{dq}{dt} + q\frac{d\mu}{dt} \\ &= \mu\frac{dt}{dt} + \frac{1}{2}\mu q = 10\mu + 5\mu\sin(2t) \end{aligned}$$

Means that the integrating factor will be

$$q\frac{d\mu}{dt} \sim \frac{1}{2}\mu q \to \frac{1}{\mu}\frac{d\mu}{dt} \sim \frac{1}{2}$$

Or

$$\int \frac{1}{\mu}\frac{d\mu}{dt}dt = \int \frac{1}{2}$$

Which leads to $\mu(t) = e^{t/2}$.

So our equation becomes

$$\frac{d}{dt}\left[e^{t/2}q(t)\right] = e^{t/2}\frac{dt}{dt} + \frac{1}{2}e^{t/2}q = 10e^{t/2} + 5e^{t/2}\sin(2t)$$

Hence,

$$\begin{aligned} e^{t/2}q(t) &= \int 10e^{t/2}dt + \int 5e^{t/2}\sin(2t)dt \\ &= 20e^{t/2} + \int 5e^{t/2}\sin(2t)dt \end{aligned}$$

Here we have an interesting integral so let's break it down.

1.2.2.1 An interesting integral

In the previous expression we ended up with

$$\int 5e^{t/2}\sin(2t)dt$$

The tip here is a chain of integrations by parts and u-substitutions. First, let's recall the rule for integration by parts

$$\int udv = uv - \int vdu$$

Now, let's get to it.

$$\begin{aligned}
\int e^{t/2}\sin(2t)dt &= \begin{bmatrix} u = e^{t/2} & v = -\frac{1}{2}\cos(2t) \\ du = \frac{1}{2}e^{t/2}dt & dv = \sin(2t)dt \end{bmatrix} \\
&= -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{4}\left[\frac{1}{2}e^{t/2}\sin(2t) - \frac{1}{4}\int e^{t/2}\sin(2t)dt\right] \\
&= -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{2^3}e^{t/2}\sin(2t) - \frac{1}{2^4}\int e^{t/2}\sin(2t)dt
\end{aligned}$$

Notice that we got our initial integral back, so now some algebra will lead us to

$$\left(\int e^{t/2}\sin(2t)dt\right)\left(1 + \frac{1}{2^4}\right) = -\frac{1}{2}e^{t/2}\cos(2t) + \frac{1}{2^3}e^{t/2}\sin(2t)$$

Which can be simplified to

$$\begin{aligned}
\int e^{t/2}\sin(2t)dt &= -\frac{2^4}{2}\frac{1}{2^4+1}e^{t/2}\cos(2t) + \frac{2^4}{2^3}\frac{1}{2^4+1}e^{t/2}\sin(2t) \\
&= -\frac{2^3}{2^4+1}e^{t/2}\cos(2t) + \frac{2}{2^4+1}e^{t/2}\sin(2t)
\end{aligned}$$

Now, we can put everything together!

$$\begin{aligned}
e^{t/2}q(t) &= 20e^{t/2} + \int 5e^{t/2}\sin(2t)dt \\
&= 20e^{t/2} + 5\left[-\frac{2^3}{2^4+1}e^{t/2}\cos(2t) + \frac{2}{2^4+1}e^{t/2}\sin(2t)\right] \\
&= 20e^{t/2} - \frac{40}{17}e^{t/2}\cos(2t) + \frac{10}{17}e^{t/2}\sin(2t) + C
\end{aligned}$$

Notice that we trhew in an integration coefficient at the end. And our final answer is now

$$q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) + Ce^{-t/2}$$