# My Continued Fractions at GitHub

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# 1 Introduction

Hwere, we shall simply state what continued fractions are and list whatever theorems and results the code here uses. (I shall put in some decent references later.) I started off by writing a class for simple fractions, that is fractions of the form

$$F = a + \frac{b}{c} = \frac{B}{c}. \tag{1.1}$$

OK, we have overloaded *,/,+, and - operators and so on. The integers $a$, $b$, $c$, and $B$ are going to be positive in every case. The class has a bool member to give $F$ a sign. The thing to note here is the integers are Gnu Multiple Precision numbers.

That is to say, these numbers can be arbitrarily large: the are not constrained by 32 or 64 bits to represent them. We have the option to GetFloat — get the floating point value. we can set a limit to the number of bytes we can use, and JumpToFloat if the integers go over that limit. Once that is done, the fraction only has a floating point value. The GMPfrac class has a bool member called float which is false until JumpToFloat is called. It also has a member to represent the floating point value.

So, the integers are members of Gnu Multiple Precision's mpz_class. (These are signed integers). The rationals $\frac{a}{b}$ and $\frac{B}{c}$ are handled using the GMP mpq_class, and the floating point numbers are handled using the GMP mpf_class. The thing about the GMP rationals, is that the results of all arithmetic operations are automatically reduced using successive applications of the Euclidean Algorithm (which computes the gratest common factor of two integers). It might turn out that we don't really need the fractions class that I wrote, but I have included it in MyLib (https://github.com/seagods/MyLib.git). Anyhow, what do we mean by a continued fraction?

## 2 What is a Continued Fraction

A famous example of an *infinitely* continued fraction is

$$\sqrt{2} = CF_{\text{inf}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}, \tag{2.1}$$

In practice, we can make this a finite fraction by just stopping at some point. We can define the following as *convergents* of the continued fraction, namely the finite continued fractions

$$CF_0 = 1, \;\; CF_1 = 1 + \frac{1}{2},$$

$$CF_2 = 1 + \cfrac{1}{2 + \cfrac{1}{2}},$$

$$CF_3 = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}, \tag{2.2}$$

and so on. Clearly

$$CF_3 = 1 + \cfrac{1}{2 + \cfrac{2}{5}} = 1 + \frac{5}{12} = \frac{17}{12}. \tag{2.3}$$

Any finite continued fraction is a rational number. The next thing to note is that we can represent the same continued fraction in infinitely many ways! Suppose we have some positive or negative intager $p$. Looking at $CF_3$, we see that

$$CF_3 = 1 + \cfrac{(p/p)}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = 1 + \cfrac{p}{2p + \cfrac{p}{2 + \cfrac{1}{2}}}. \tag{2.4}$$

Similarly, we can introduce another non zero integer $q$, and

$$CF_3 = 1 + \cfrac{p}{2p + \cfrac{p(q/q)}{2 + \cfrac{1}{2}}} 1 + \cfrac{p}{2p + \cfrac{pq}{2q + \cfrac{q}{2}}}. \tag{2.5}$$

So given any rational, we may express it as a finite continued fraction in an infinity of ways.

## 2.1 Notation for a General Continued Fraction

Obviously, this way of writing things down get's out of hand as we get to $CF_{50}$, and a more compact notation is necessary. We write our general contined fraction using integers $a$ and $b$, and put

$$CF_{\text{inf}} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{b_4}{c_4 + \dots}}}}. \tag{2.6}$$

Two common notations are to write

$$CF_\infty = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \ldots = b_0 + \mathop{\mathrm{K}}_{i=1}^{\infty} \frac{a_i}{b_i}. \tag{2.7}$$

the $K$ in the second representation comes from the German "Ketternbruch" which translates as "chain fraction". We can represent our $CF_3$ for $\sqrt{2}$ as

$$CF_3 = 1 + \frac{1|}{|2} + \frac{1|}{|2} + \frac{1|}{|2} = 1 + \mathop{\mathrm{K}}_{i=1}^{3} \frac{1}{2}. \tag{2.8}$$

# 3 How do we Compute Continued Fractions

We start with the finite fraction

$$CF_3 = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3}}}. \tag{3.1}$$

One we we might proceed is to use our GMPfrac class to compute

$$F_1 = b_2 + \frac{a_3}{b_3},$$

then

$$F_2 = a_2/F_1,$$

and

$$F_3 = b_1 + F_2, \;\; F_4 = a_1/F_3, \;\; \text{and } \; CF_3 = b_0 + F_4. \tag{3.2}$$