The Golden Section, the Pentagon and the Dodecahedron

C. Godsalve
email:seagods@hotmail.com

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1 Introduction

First of all, the author should mention that there is much more to the study of platonic solids (as they are called) than is mentioned here. No mention of group theory shall be made, and the constructions mentioned here are just one of many. There are plenty of other (and better) Internet resources on the subject, notably at wolfram.mathworld and wikipedia. This is just a case of the author typing up some old rough notes and sharing them. Note that the coordinates of the dodecahedron given here are not what are known as the canonical coordinates. These canonical coordinates come from a construction based on a cube in Euclid's Elements. This does not have a facet lying in the (x, y) plane, where the construction given here does.

In this article, it is assumed that the reader understands Cartesian coordinates, basic vectors, trigonometry, and a bit of algebra. The dodecahedron is one of the so called Platonic Solids. These are three dimensional geometric constructions that consist of a number of plane facets. These facets can be equilateral triangles, squares, or regular pentagons. The simplest is the tetrahedron with four equilateral triangles. Then there is the cube consisting of six squares. The octahedron consists of eight equilateral triangles; the dodecahedron which consists of twelve pentagons. Finally, the icosahedron consists of twenty equilateral triangles. It is known that this completes the list. There are no other solids which can be built up of just one type of regular polyhedra. That is just a property of Euclidean space. Other constructions such as Archimedean solids consist of different types of regular polyhedra.

2 The Golden Ratio

There is an important number involved in the construction of pentagons and dodecahedra, it is called the Golden Section, and it is also known as the Divine Proportion. There is a nice introduction in David Wells' "The Penguin Dictionary of Curious and Interesting Numbers". He is also the author of the "Penguin Dictionary of Curious and Interesting Geometry" which is wonderful resource also. The Golden Section (or Golden Ratio) is related to much in mathematics. For instance it relates to spirals, and Fibonacci series. It is also related to flowers, and the proportions of the Human body.

Suppose we have a straight line AB, and we wish to divide it into two. That is, we want to cut it at some point C. We could cut it at any point, but suppose we choose the point so that the ratios

$$\Phi = \frac{AB}{CB} = \frac{CB}{AC}. \tag{1}$$

Now,

$$AB = AC + CB$$

so

$$\frac{AB}{CB} = \frac{AC + CB}{CB}$$

or

$$\Phi = 1 + \frac{1}{\Phi}. \tag{2}$$

The value of Φ is a root of the equation $\Phi^2 - \Phi - 1 = 0$, indeed

$$\Phi = \frac{\sqrt{5}+1}{2}$$

also

$$\frac{1}{\Phi} = \phi = \frac{2}{1+\sqrt{5}} = \frac{2}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$\phi = \frac{\sqrt{5}-1}{2}. \tag{3}$$

Also, $\Phi - \phi = 1$.

It is easy to construct a right angle triangle with a hypotenuse of $\sqrt{5}/2$. The base is of unit length, and the height is one half the unit length or $AC = AB/2$. That is we consider AB to be our unit length, but to relate the to the points in the figure we continue to use AB rather than unity. Consider the construction in Fig.1. We draw an arc at centre C through B to intersect the hypotenuse at D. Then we construct a second arc at centre A, through D to intersect the base at E. Clearly $AE = AD = AC - CD = AC - 1/2AB$. So

$$AE/AB = \frac{\sqrt{5}-1}{2} = \phi \tag{4}$$

Incidentally, there is another and similar method of constructing the Golden section. This

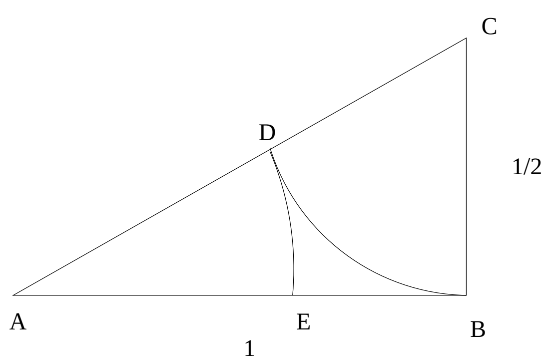

Figure 1: The line AB is divided into Golden Section ratio by drawing two arcs in a right angle triangle of hypotenuse $AB \times \sqrt{5}/2$.

time we extend the line AB to $2AB$, and draw two circles (or semi-circles) centred at B. One is of radius $AB/2 = BC$, and the other is of radius AB. Both are centred at B. Clearly the one of radius AC bisects AB at M say. If the perpendicular BC is extended, there is some point T where it intersects the semi-circle of radius AB. If we draw a line from M to T and draw an arc of radius MT (centred at M), it shall intersect the extended line of length $2AB$ at some point G. If the reader wishes to fill in the details, the reader shall find that $AG = \Phi AB$ and of course $AB = \phi AG$.

3 The Pentagon

The construction can be related to the pentagon in the following manner. We draw an arc, centred at A passing counterclockwise through B. We then find the midpoint, M, of the segment EB. Next we construct a vertical line passing through M which intersects the new arc at F. The construction is shown in Fig.2.

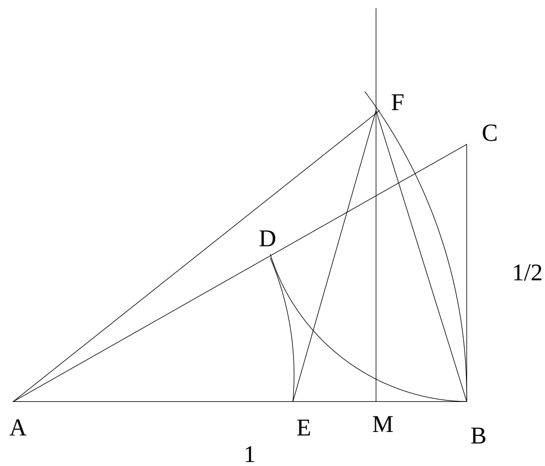

Figure 2: We construct an arc centred at A of length AB, and find the intersection of the vertical through the midpoint of EB.

It is clear that ABF is an isosceles triangle, also we have another isosceles triangle in EFB. What about the triangle AEF? It looks as though it might be an isosceles.

Now

$$AF^2 = AB^2 = AM^2 + MF^2,\ MF^2 = AB^2(1 - \frac{1}{4}(1+\phi)^2), \tag{5}$$

also

$$MF^2 + EM^2 = EF^2,$$

so

$$EF^2 = MF^2 + AB^2(\frac{1}{4}(1-\phi)^2). \tag{6}$$

Using eqns.5 and 6 to eliminate MF,

$$EF^2 = AB^2(1 + \frac{1}{4}((1-\phi)^2 - (1+\phi)^2)) = AB^2(1-\phi). \tag{7}$$

The reader can confirm that $1 - \phi = \phi^2$, and so AEF is indeed an isosceles triangle. We shall need this result to establish quantities such as $\sin 18°$.

Now, we have three isosceles triangles, EBF, AEF and ABF. We examine the angles in Fig.3. The angles of the outer isosceles triangle, which contains two more isosceles triangle add up to $2\alpha + 2\alpha + \alpha$. So $5\alpha = 180°$. The angles are $36°$ and $72°$, or $\pi/5$ and $2\pi/5$ in radians.

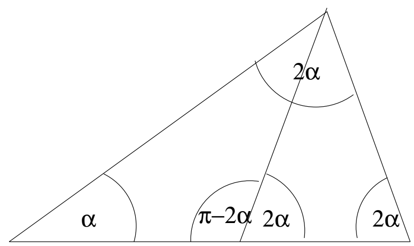

Figure 3: Adding up the angles in the outer triangle reveals them to be important for the construction of a pentagon.

Now, if we examine the isosceles EBG we can find some useful results. Bearing in mind that

$$\phi^2 = 1 - \phi, \;\; \Phi^2 = 1 + \Phi, \\ \frac{1}{\phi} = \Phi, \;\; \frac{1}{\Phi} = \phi, \tag{8}$$

we go back to Fig.2 and see that $EM = AB(1-\phi)/2$. Clearly the acute angle in the triangle EMF is $18°$, so eqn.7 leads us to

$$\sin\, 18° = \frac{\phi}{2}, \;\cos\, 18° = \frac{\sqrt{2+\Phi}}{2}. \tag{9}$$

Then using the identities

$$\cos\,(A+B) = \cos\, A\, \cos\, B - \sin\, A\, \sin\, B, \\ \sin\,(A+B) = \sin\, A\, \cos\, B + \sin\, B\, \cos\, A, \tag{10}$$

(or otherwise) we can find the following

$$\sin\, 36° = \frac{\sqrt{2-\phi}}{2}, \;\cos\, 36° = \frac{\Phi}{2},$$

$$\sin\ 54^\circ = \frac{\Phi}{2},\ \cos\ 54^\circ = \frac{\sqrt{2-\phi}}{2},$$
$$\sin\ 72^\circ = \frac{\sqrt{2+\Phi}}{2},\ \cos\ 72^\circ = \frac{\phi}{2}. \tag{11}$$

So, can we draw a perfect pentagon? First we construct the triangle of Fig.1, and find the point E. Then we construct the second triangle (EBF) as in Fig.2. We find the midpoint M, construct a vertical, and then we mark out the intersection of this point with the arc of unit length (using $AB = 1$) as mentioned earlier. Now we have the isosceles with base angle 72 degrees. Next, we draw lines parallel to the opposite side of the isosceles triangle through the base These lines are drawn through A and B and shall end up as BC and AE in Fig.4. (This can be done by sliding one set square over another.) Finally, all we have to do is to mark off the base length along these new lines with a compass. You can always use a protractor, but the author quite likes doing the straight edge and compass stuff.

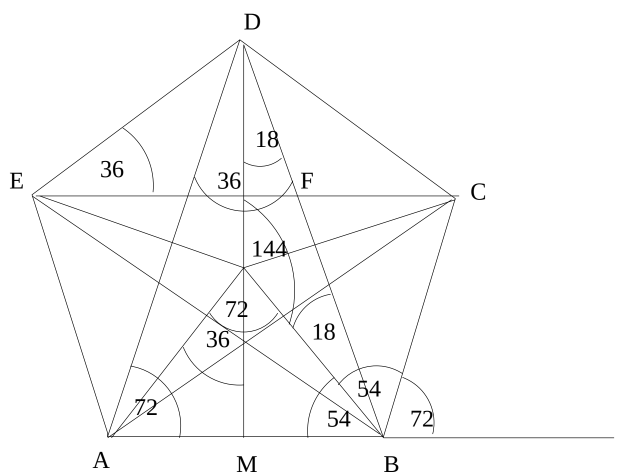

Figure 4: Examine the angles in the pentagon, the isosceles ABD is similar to the triangle EBF of Fig.2.

Now EC is parallel to AB, in the construction so far. We introduce a new point F (which is nothing to do with the F of Fig.2). This point is where EC intersects BD. Clearly that $AEFB$ is a rhombus, and that the length $EF = 1$. Examining the triangle ECD, it is clear that the length $EC = 2\ \cos\ 36^\circ$, and from the above we have $EC = \Phi$. The line BD intersects the line EC at F such that EC is divided according to the golden section ratio.

Further examination shows that (apart from a rotation and translation) ABD, BCE, CDA, DEB, and EAC are identical. Clearly, there is rotational symmetry here, and, for instance the bisector of $\angle ADB$ passes through the centre of rotation. Clearly the bisectors of the $36°$ angles in all the triangles just mentioned also pass through the centre of rotation, and they are marked in Fig.4. (All this hand waving falls far from any sort of rigorous proof, but it will do for as far as the purpose of this article is concerned, that is we want the coordinates of the vertices, the centre, and the radii of the inscribed and circumscribed circles.)

There is enough in the above for the reader to confirm that the radius ρ_1 of the circumcircle is $1/(2\ \sin 36°)$ That is

$$\rho_1 = \frac{1}{\sqrt{2-\phi}}. \tag{12}$$

The height of the centre of rotation O is also the radius ρ_2 of the inscribed circle, so

$$\cos\ 36° = \frac{\rho_2}{\rho_1} = \frac{\Phi}{2},$$

$$\rho_2 = \rho_1 \cos\ 36 = \frac{\Phi}{2\sqrt{2-\phi}}. \tag{13}$$

We shall mention a little more about the pentagon at the outset of the next section.

4 The Dodecahedron

To construct a dodecahedron out of cardboard, you need to mark out the diagram shown in Fig.5, in fact you need two such shapes.

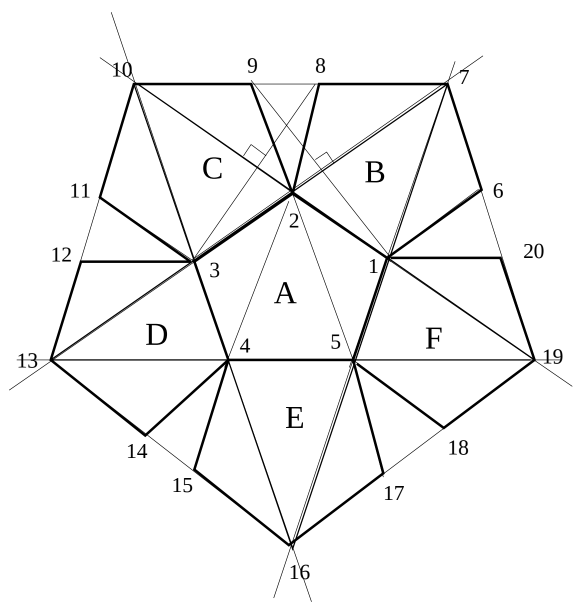

Figure 5: The shape needed is that of a ring of pentagons placed round an initial pentagon. (The initial 'pentagon' shown is just a rough sketch, so if this is printed, and the shape is cut out, the result won't be very good.) The numbering system of the vectors in the discussion below is given here.

There isn't much work to do once the initial pentagon is constructed. First, extend the edges of the pentagon far enough that, when all the edges are extended both ways, the lines intersect. What you have now is a five pointed star. A triangle is sitting on each edge of the initial pentagon. It is easy to see that these triangles are exact copies of the isosceles that the original pentagon was constructed from.

Next, draw lines between the points of the stars to form an outer pentagon. Then set the compass radius to the side length of the original pentagon, and mark out this length from the points of the star along the edges of the outer pentagon. You now have all the points you need for the ring of five pentagons.

Now, the we can write the vertices of the original pentagon as

$$\frac{\mathbf{r}_1}{\rho_1} = \frac{\sqrt{2+\Phi}}{2}\mathbf{i} + \frac{\phi}{2}\mathbf{j}$$
$$\frac{\mathbf{r}_2}{\rho_1} = \mathbf{j}$$
$$\frac{\mathbf{r}_3}{\rho_1} = -\frac{\sqrt{2+\Phi}}{2}\mathbf{i} + \frac{\phi}{2}\mathbf{j}$$
$$\frac{\mathbf{r}_4}{\rho_1} = -\frac{\sqrt{2-\phi}}{2}\mathbf{i} - \frac{\Phi}{2}\mathbf{j}$$
$$\frac{\mathbf{r}_5}{\rho_1} = \frac{\sqrt{2-\phi}}{2}\mathbf{i} - \frac{\Phi}{2}\mathbf{j} \tag{14}$$

These are the coordinates of pentagon A in Fig.5, and the directions are sketched in Fig.6.

Now, consider pentagons B and C. They are inverted versions of the original pentagon.

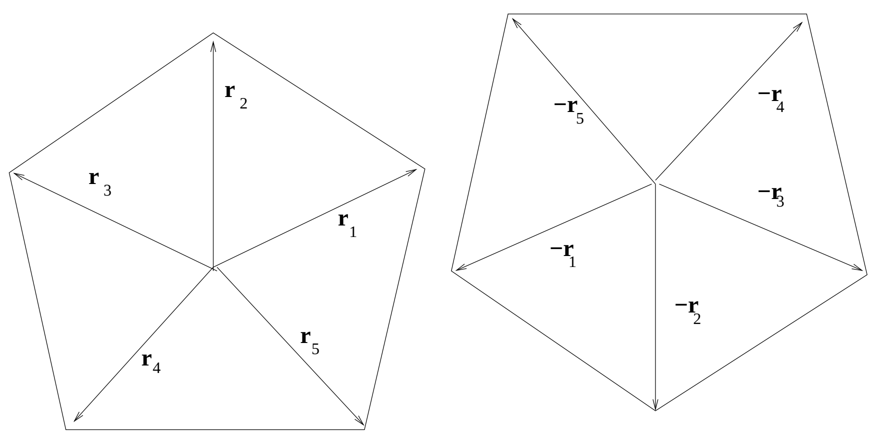

Figure 6: A sketch of the directions of the vectors in eqn.14, we also show the vertices of an inverted pentagon in terms of the vectors of the original.

Shifting the origin to the centre of either would give us the coordinates of the inverted pentagons to be $-\mathbf{r}_1$, $-\mathbf{r}_2$, $-\mathbf{r}_3$, $-\mathbf{r}_4$, and $-\mathbf{r}_1$. The origin of the pentagon B is at $\mathbf{r}_2 + \mathbf{r}_1$, and the origin of the pentagon C is at $\mathbf{r}_2 + \mathbf{r}_3$.

Next, look at the two edges, ($\mathbf{r}_2$, -$\mathbf{r}_2$+$\mathbf{r}_1$), and $\mathbf{r}_2$, -$\mathbf{r}_2$+$\mathbf{r}_3$). Suppose we make folds along the corresponding edges of the original pentagon (A) in Fig.5. If we fold each of the outer pentagons (B and C in Fig.5) "up" by the same angle, at what angle shall the pentagons B and C share an edge?

We can answer this question more easily if we look at Fig.7. Here we extend the fold lines, and draw lines between vertices 3 and 8, and 1 and 9. By symmetry,the line between vertices 3 and 8 is perpendicular to the fold line between vertices 1 and 2.

To see this, look at the triangle between points $\mathbf{r}_1$, $\mathbf{r}_3$ and $\mathbf{r}_8$ in Fig.7. Extending the edge between $\mathbf{r}_1$ and $\mathbf{r}_2$ to cut the line between $\mathbf{r}_3$ and $\mathbf{r}_8$ gives us a new point. The triangle is symmetric with respect to a reflection along the edge between $\mathbf{r}_1$ and $\mathbf{r}_2$. That is to say, the intersection point described is the midpoint between the points at $\mathbf{r}_3$ and $\mathbf{r}_8$. We shall denote this midpoint $\mathbf{r}_{M38}$. That is the position vector midway between points 3 and 8. Because of the reflection symmetry lines between points 3 and 8, and 1 and 2 are perpendicular. The same goes for the fold line between vertices 2 and 3, and the line connecting vertices 1 and 9.

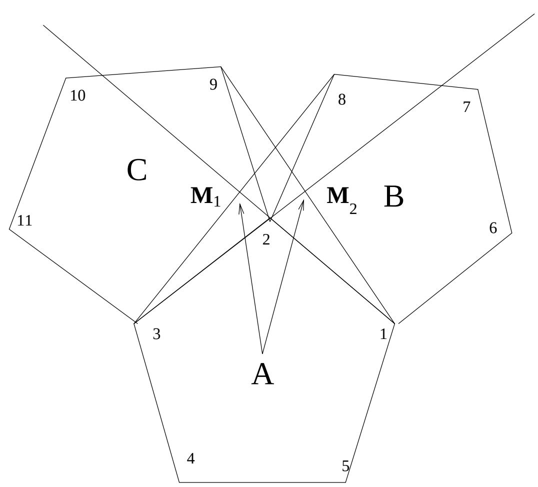

Figure 7: Folds are made between vertices 1 and 2, and vertices 2 and 3. Then $\mathbf{M_1}$ and $\mathbf{M}_2$ are the mid points of vertices 1 and 9, and vertices 3 and 8. (This is a very rough sketch.)

Now, take $\mathbf{M}_1$

$$\mathbf{M}_1 = \mathbf{r}_3 + \frac{1}{2}(-\mathbf{r}_3 + \mathbf{r}_2 + \mathbf{r}_1 - \mathbf{r}_5) = \frac{1}{2}(\mathbf{r}_3 + \mathbf{r}_2 + \mathbf{r}_1 - \mathbf{r}_5)$$

$$\mathbf{M}_2 = \mathbf{r}_1 + \frac{1}{2}(-\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 - \mathbf{r}_4) = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_4 - \mathbf{r}_3). \tag{15}$$

Then

$$\mathbf{r}_8 = \mathbf{M_1} + \frac{1}{2}(-\mathbf{r}_3 + \mathbf{r}_2 - \mathbf{r}_5 + \mathbf{r}_1)$$

$$\mathbf{r}_9 = \mathbf{M_2} + \frac{1}{2}(-\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_4 - \mathbf{r}_4). \tag{16}$$

Now we can make the folds. If the fold angle is Θ then

$$\mathbf{r}_8 \to \mathbf{M_1} + \frac{1}{2}(-\mathbf{r}_3 + \mathbf{r}_2 - \mathbf{r}_5 + \mathbf{r}_1)\cos\Theta + \frac{1}{2}|\mathbf{r}_3 + \mathbf{r}_2 - \mathbf{r}_5 + \mathbf{r}_1|\sin\Theta\mathbf{k}$$

$$\mathbf{r}_9 \to \mathbf{M_2} + \frac{1}{2}(-\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_4 - \mathbf{r}_4)\cos\Theta + \frac{1}{2}|-\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_4 - \mathbf{r}_4|\sin\Theta\mathbf{k}. \tag{17}$$

Now, if Θ is such that the two vertices 8 and 9 meet, then we can equate the projections onto the (x,y) plane, that is

$$\mathbf{M_1} + \frac{1}{2}(\mathbf{r}_3 + \mathbf{r}_2 - \mathbf{r}_5 + \mathbf{r}_1)\cos\Theta = \mathbf{M_2} + \frac{1}{2}(-\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_4 - \mathbf{r}_4)\cos\Theta. \tag{18}$$

On using eqn.15, we arrive at

$$(\mathbf{r}_4 - \mathbf{r}_5) = (2(\mathbf{r}_3 - \mathbf{r}_1) - (\mathbf{r}_4 - \mathbf{r}_5)) \tag{19}$$

. Now, $\mathbf{r}_4 - \mathbf{r}_5$ is a vector from the vertex 5 to vertex 4, it is of unit length. Also $(\mathbf{r}_3 - \mathbf{r}_1)$ is a vector from vertex 1 to vertex 3. This is parallel to $\mathbf{r}_4 - \mathbf{r}_5$, and of length Φ. So we have

$$\cos\Theta = \frac{1}{2\Phi - 1} = \frac{1}{\sqrt{5}}. \tag{20}$$

Of course, if the side of the pentagon is unity, then the width (the distance from vertex 3 to 1) is Φ.

We examine the $\mathbf{i}$ components of where the corners meet, we get (when scaled by ρ_1)

$$\textit{component in } \mathbf{i} = \frac{1}{\sqrt{5}}\sqrt{2+\Phi} - \left(1 + \frac{1}{\sqrt{5}}\right)\frac{\sqrt{2-\phi}}{2} \tag{21}$$

Now, the square of the second term on the r.h.s of eqn.21 is

$$\left(1 + \frac{1}{\sqrt{5}}\right)^2 \left(\frac{2-\phi}{2}\right)^2$$

$$= \frac{1}{40}(6 + 2\sqrt{5})(5 - \sqrt{5}) = \frac{1}{20}(10 + 2\sqrt{5}). \tag{22}$$

Using the result that $2\sqrt{5} = 4\Phi - 2$,

$$\left(1 + \frac{1}{\sqrt{5}}\right)\left(\frac{\sqrt{2-\phi}}{2}\right) = \frac{1}{\sqrt{5}}\sqrt{2+\Phi}. \tag{23}$$

The two terms on the r.h.s of eqn.21 cancel, and in the coordinate basis we have chosen, there is no $\mathbf{i}$ component. This is entirely as expected by symmetry. Now consider the $\mathbf{j}$ component of the point where vertices 8 and 9 meet after the fold, that is the $\mathbf{j}$ component of the point

$$\mathbf{r} = -\mathbf{r}_3 + \mathbf{r}_2 + \mathbf{r}_1 - \mathbf{r}_5 + \frac{1}{\sqrt{5}}(-\mathbf{r}_3 + \mathbf{r}_2 + \mathbf{r}_1 - \mathbf{r}_5),$$

when scaled to ρ_1, this is

$$= \frac{\phi}{2} + 1 + \frac{\phi}{2} + \frac{\Phi}{2} + \frac{1}{\sqrt{5}}\left(-\frac{\phi}{2} + 1 + \frac{\phi}{2} + \frac{\Phi}{2}\right)$$

$$= \frac{3}{2}\Phi + \frac{1}{\sqrt{5}}\left(1 + \frac{\Phi}{2}\right) = \frac{1}{\sqrt{5}}\left(1 + \frac{3\sqrt{5}\Phi}{2} + \frac{\Phi}{2}\right)$$
$$= \frac{1}{\sqrt{5}}\frac{4 + 15 + 3\sqrt{5} + \sqrt{5} + 1}{4} = \frac{5 + \sqrt{5}}{\sqrt{5}} = 2\Phi. \tag{24}$$

Remember that this is actually twice the vector we wanted and scaled to ρ_1. (The first terms unscaled by $1/\sqrt{5}$ give the vector from vertex 3 to vertex 8.) The point where vertices 8 and 9 meet upon folding has (x, y) coordinates $(0, \rho_1\Phi)$, so when all the outer pentagons are folded, the end points where the edges meet lie on a pentagon of radius $\rho_1\Phi$. Where do the vertices like vertex 7 of Fig.7 end up?

Clearly they lie on a different pentagon, and we can say that the (x, y) coordinates of this pentagon lie on a radius

$$\rho = \rho_2 + \frac{1}{\sqrt{5}}(\rho_2 + \rho_1) = \rho_1\left(\frac{\Phi}{2} + \frac{1}{\sqrt{5}}\left(\frac{\Phi}{2} + 1\right)\right) = \rho_1\Phi. \tag{25}$$

So the (x, y) coordinates are all on a decagon of radius $\rho_1\Phi$ as depicted in Fig.8.

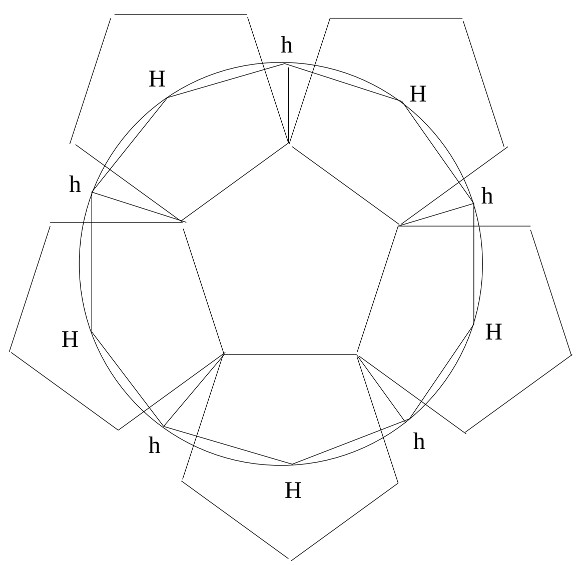

Figure 8: When folded, the x, y coordinates all lie on a decagon, and the z coordinates alternate between H and h, where $h < H$.

Once the folding is completed all the way round, we have vertices of height h at the (x, y) points $\mathbf{r}_1\Phi$, $\mathbf{r}_2\Phi$, $\mathbf{r}_3\Phi$, $\mathbf{r}_4\Phi$, and $\mathbf{r}_5\Phi$. We also have have vertices at height H at the (x, y) coordinates $-\mathbf{r}_1\Phi$, $-\mathbf{r}_2\Phi$, $-\mathbf{r}_3\Phi$, - $\mathbf{r}_4\Phi$, and $\mathbf{r}_5\Phi$.

We must now work out the heights. When scaled to ρ_1,

$$|-\mathbf{r}_3 + \mathbf{r}_2 + \mathbf{r}_1 - \mathbf{r}_5|^2 = \left(\sqrt{2 + \Phi} - \frac{\sqrt{2 - \phi}}{2}\right)^2 + \left(1 + \frac{\Phi}{2}\right)^2$$

$$= \left(2+\Phi+\frac{2-(\Phi-1)}{4} - \sqrt{(2+\Phi)(2-\phi)}\right) + \left(1+\Phi+\frac{\Phi^2}{4}\right) \tag{26}$$

The term in the square root multiplies out as 5, so

$$|-\mathbf{r}_3+\mathbf{r}_2+\mathbf{r}_1-\mathbf{r}_5|^2 =$$

$$\left(2+\Phi+\frac{2-(\Phi-1}{4} - (2\Phi-1)\right) + \left(1+\Phi+\frac{1+\Phi}{4}\right) = 5. \tag{27}$$

So, before the fold, the perpendicular distance from the fold line to the vertex is $\rho_1\sqrt{5}/2$. and the height h is $\rho_1\sqrt{5}/2\sin\Theta$. Also,

$$\sin^2\Theta = (1-2\frac{1}{\sqrt{5}}+\frac{1}{5}) = \frac{4}{5}(1-\phi) = \frac{4\phi^2}{5}$$

$$\sin\Theta = \frac{2\phi}{2\Phi-1} \tag{28}$$

The distance from the fold line to the other vertices is $\rho = \rho_2+\rho_1 = \rho_1(1+\Phi/2)$. So we get the height H on multiplying by $\sin\Theta$.

Now, we have all the coordinates for our folded shape. What we need to do next is make another folded shape. This will start off with the initial pentagon rotated through 180^2, and this time the z coordinates are the same as in the original in magnitude but negative. Now all we have to do is shift the first folded shape down by $(H+h)/2$, and the second folded shape up by the same amount, and we have (at last) a dodecahedron.

Of course, if you are not making a cardboard model, the last five vertices have (x, y) coordinates of $-\mathbf{r_1}$, $-\mathbf{r_2}$, $-\mathbf{r_3}$, $-\mathbf{r_4}$, and $-\mathbf{r_5}$. I have written some graphics programs in openGL, and a dodecahedron is represented in Fig.9. I won't go into much detail on the following solids.

The icosahedron is made up of 20 equilateral triangles, and is the it platonic dual of the dodecahedron. If you take any platonic solid, place a vertex at the centre of each facet and connect the new vertices together you have its platonic dual. An icosahedron is represented in Fig.10.

Now if you take an icosahedron, and then mark off points one third, and two thirds of an edge distance from each vertex along all the edges you get a new set of points. These make up twelve pentagons and twenty hexagons. It's depicted in Fig.11. It is called a truncated icosahedron, and was famously used by the Architect Buckminster-Fuller in his geodesic domes. It is now also famous as a soccer ball design, and is also the structure of the molecule C60. This molecule was discovered in 1985, and the discoverers were awarded the 1996 Nobel prize. This was awarded to Robert Curl, Harry Kroto, and Richard Smalley. The molecule is also known as buckminsterfullerene, and sometimes as the bucky-ball.

5 A few more details

We shall finish with just a few details on the nodes of dodecahedron. A little repetition will do no harm. We shall just give a few numbers regarding a dodecahedron where each length of each edge is unity. It is not too hard to work out the following

$$\cos 18^\circ = \frac{1}{2}\sqrt{\sqrt{5}\Phi},\ \sin 18^\circ = \phi/2, \tag{29}$$

$$\cos 36^\circ = \Phi/2,\ sin 36^\circ = \frac{1}{2}\sqrt{\sqrt{5}\phi}. \tag{30}$$

The sines and cosines of 54° and 72° follow immediately. If the "basic pentagon" has unit edge length, then its radius is

$$\rho_p = \sqrt{\Phi/\sqrt{5}}. \tag{31}$$

The radius of the decagon as described above is just Φ times the radius of the pentagon with unit edge length.

If we use pentagons which have circumcircles of radius 1, then we just put $\rho_p = 1$ for both H and H, and the radius of the decagon is Φ.

In our non-standard scheme, we have a "base" pentagon and a "cap" pentagon. The cap is the base rotated by 180°, and they are both parallel to the (x, y) plane. If we choose the origin of a Cartesian coordinate system to be at the centre of the inscribed sphere or circumsphere (which coincide) of the dodecahedron, then the z coordinates of the "base" and "cap" are at heights

$$H = \pm\rho_p(\Phi + 1)/2. \tag{32}$$

In this scheme, the heights of the nodes between the two caps are at $\pm h$ where

$$H = \pm\rho_p \times \phi/2; \tag{33}$$

All the nodes for the icosahedron and truncated icosahedron follow easily. The radius of the inscribed circle for a pentagon with unit edge length is

$$\rho_i = \sqrt{\Phi/\sqrt{5} - 1/4}. \tag{34}$$

If we label the location of the nodes of such a pentagon as A, B, C, D and E, then

$$\overrightarrow{OX} = \overrightarrow{OA} + 1/2\ \overrightarrow{AB} \tag{35}$$

is the centre of the first edge. Here O is some arbitrary centre of our coordinate system and *not* the centre of the pentagon, which we shall call Z. The centre of the pentagon lies in the direction of the third node located at Y.

$$\overrightarrow{XY} = 1/2\ \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}\ . \tag{36}$$

Then

$$\overrightarrow{OZ}=\overrightarrow{OA}+1/2\ \overrightarrow{AB}+\frac{\overrightarrow{XY}}{|\overrightarrow{XY}|}\times\rho_i. \tag{37}$$

The fine details are contained in the document that comes with the openGL programmes for the dodecahedron, icosahedron, and truncated icosahedron. You can download these from this site. At any rate, the centres of the 12 pentagons form the nodes of the icosahedron which consists of 20 equilateral triangles.

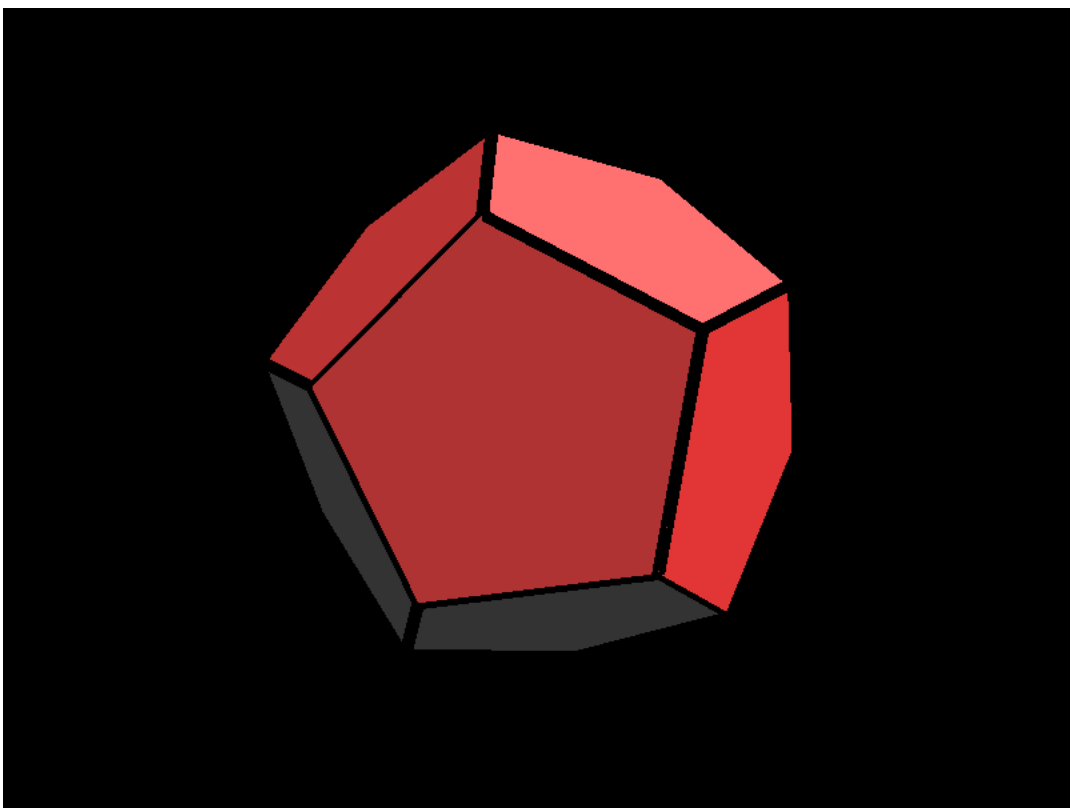

Figure 9: A computer perspective drawing of the dodecahedron.

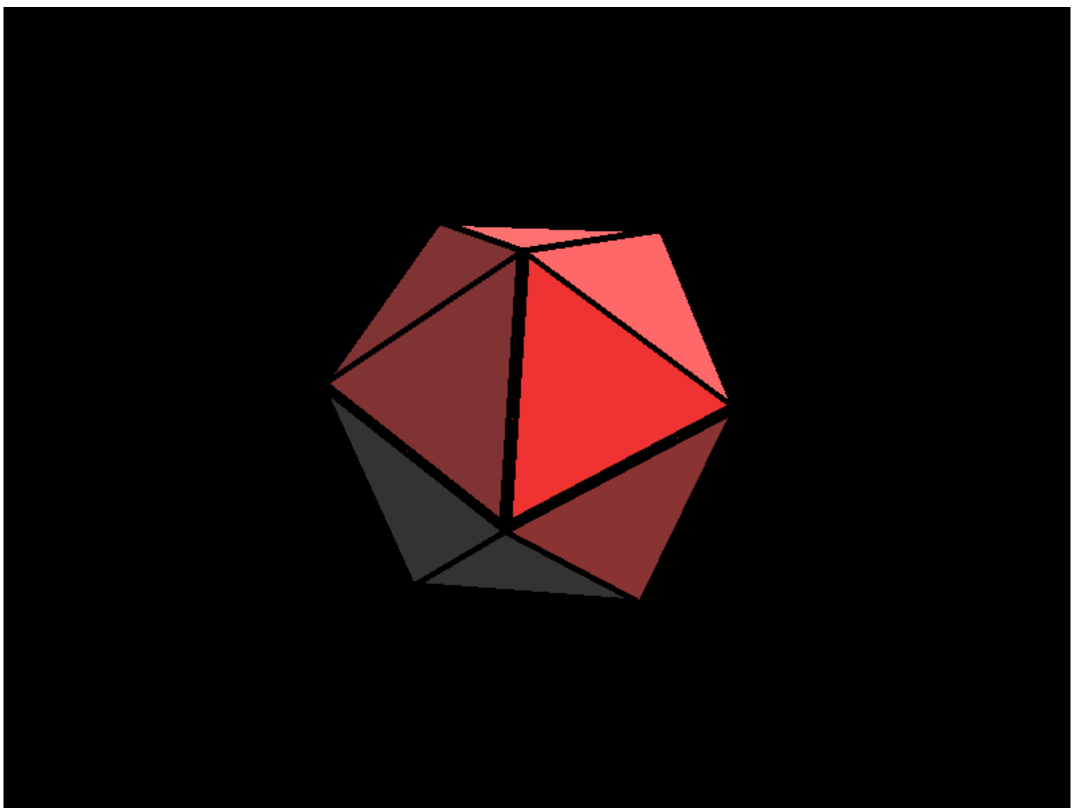

Figure 10: A computer perspective drawing of the icosahedron.

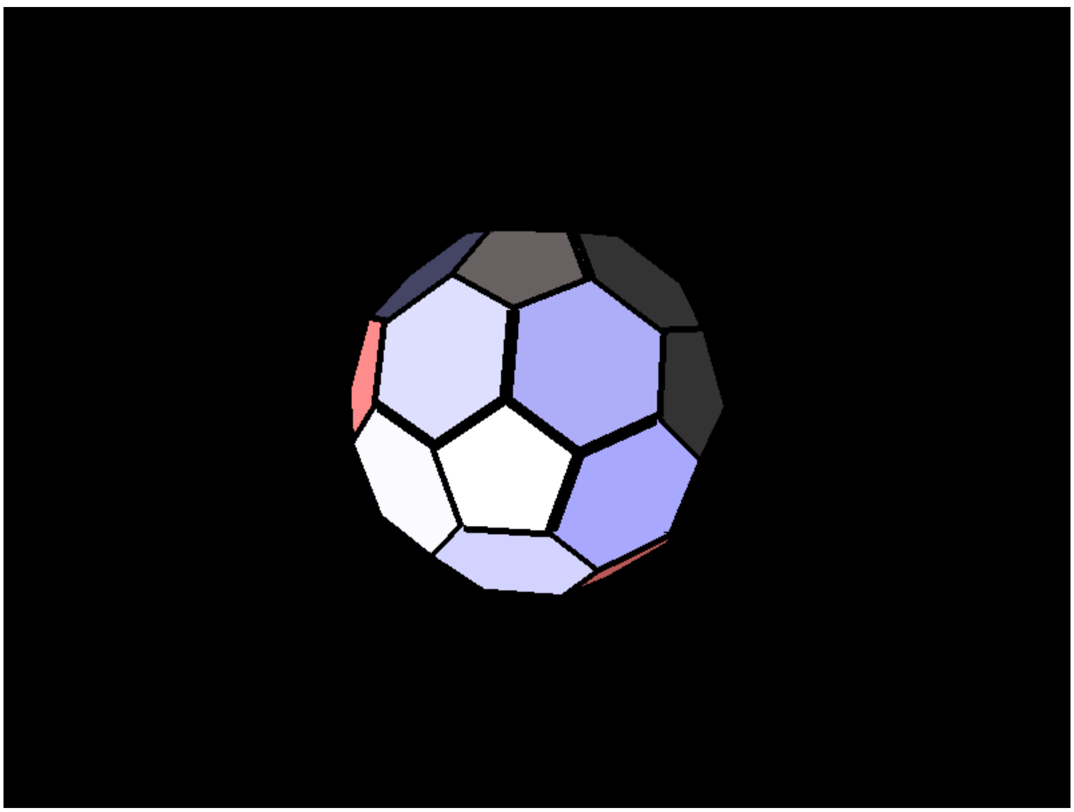

Figure 11: A computer perspective drawing of the truncated icosahedron.