The Humble Tetrahedron

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July 2, 2009

In this article, it is assumed that the reader understands Cartesian coordinates, basic vectors, trigonometry, and a bit of algebra. In case the reader does not know,

$$\sin 30^\circ = \frac{1}{2},\ \cos 30^\circ = \frac{\sqrt{3}}{2},\ \tan 30^\circ = \frac{1}{\sqrt{3}}$$
$$\sin 60^\circ = \frac{\sqrt{3}}{2},\ \cos 60^\circ = \frac{1}{2},\ \tan 60^\circ = \sqrt{3}. \tag{1}$$

The tetrahedron is a solid figure with the least possible number of sides. It is made of four triangles, and has four vertices. If you take any triangle, and add a point outside of the plane of a triangle, then you have the vertices of a tetrahedron.

Let's have a look at the regular tetrahedron in a Cartesian coordinate system. We shall make the origin one vertex of a tetrahedron, and the point $\mathbf{r} = L\,\mathbf{i}$ another vertex. Here $\mathbf{i}$ is just a unit vector in the direction that the x axis is pointing, and $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the y and z axes respectively.

If the base of the tetrahedron is an equilateral triangle of side length L, and lies in the (x, y) plane, then the then the third vertex is at

$$\mathbf{r} = \frac{1}{2}L\,\mathbf{i} + \frac{\sqrt{3}}{2}L\,\mathbf{j}. \tag{2}$$

Now if we want a regular tetrahedron, the (x, y) components will describe the centre of the triangle. To get there all we need to know is that the x component will be half way between the first to vertices. The centre of the triangle is at

$$\mathbf{r} = \frac{L}{2}\mathbf{i} + \frac{L}{2\sqrt{3}}\mathbf{j}. \tag{3}$$

This can be calculated by finding the centre of the inscribed circle. Now, the fourth vertex of the tetrahedron will be placed at some height h above the (x, y) plane. That is at,

$$\mathbf{r} = \frac{L}{2}\mathbf{i} + \frac{L}{2\sqrt{3}}\mathbf{j} + h\mathbf{k}. \tag{4}$$

Now, what height should the last vertex be, so that the resulting tetrahedron is a regular one. Now, in three dimensions, it is easy to show that the distance from the origin to any point (x, y, z) is just $d = \sqrt{x^2 + y^2 + z^2}$. Of course, the distance to the point (x, y) in the $(x, y, 0)$ plane is found from Pythagoras' theorem. Then Pythagoras' theorem is applied again for the triangle consisting of the points $(0, 0, 0)$, $(x, y, 0)$, and (x, y, z). From this we see that

$$|\mathbf{r}|^2 = \frac{1}{4}L^2 + \frac{1}{12}L^2 + h^2 = L^2. \tag{5}$$

This is the condition that the distance from the origin to the last vertex is L. From this we see that

$$h = \sqrt{\frac{2}{3}}L. \tag{6}$$

So now we have the four vertices of a tetrahedron. We shall name them

$$\mathbf{r}_0 = (0, 0, 0)$$

$$\mathbf{r}_1 = (L, 0, 0)$$

$$\mathbf{r}_2 = \left(\frac{L}{2}, \frac{\sqrt{3}}{2}L, 0\right).$$

$$\mathbf{r}_3 = \left(\frac{L}{2}, \frac{1}{2\sqrt{3}}L, \sqrt{\frac{2}{3}}L\right). \tag{7}$$

Now, we shall be interested in the circumcentre of the tetrahedron. Suppose that we have a sphere passing through all the vertices. The centre of this sphere is the circumcentre. We want to find the radius of the circumsphere and its centre. By symmetry, it shall be directly over the centre of the base triangle. We shall denote the vector from the origin to the circumcentre as $\mathbf{c}$.

$$\mathbf{c} = \frac{L}{2}\mathbf{i} + \frac{L}{2\sqrt{3}}\mathbf{j} + x\mathbf{k}. \tag{8}$$

We need to find x. If $\mathbf{c}$ is the circumcentre, then

$$|\mathbf{c}|^2 = \frac{L^2}{4} + \frac{L^2}{12} + x^2. \tag{9}$$

Now, this can be equated with

$$|-\mathbf{c} + \mathbf{r}_3|^2 = \left(\sqrt{\frac{2}{3}L - x}\right)^2 = \frac{2}{3}L^2 - 2\sqrt{\frac{2}{3}}Lx + x^2. \tag{10}$$

so that

$$\frac{2}{3}L^2 - 2\sqrt{\frac{2}{3}}Lx = \frac{L^2}{4} + \frac{L^2}{12}. \tag{11}$$

On a little tidying up

$$x = \frac{1}{6}\sqrt{\frac{3}{2}}L. \tag{12}$$

Plugging this back into eqn.9, we get

$$|\mathbf{c}| = L\sqrt{\frac{9}{24}} = \frac{L}{2}\sqrt{\frac{3}{2}}. \tag{13}$$

So, now we have the location of the circumcentre, and the radius of the circumsphere. By symmetry, we have $|\mathbf{c}|= |-\mathbf{c}+\mathbf{r}_3|= |-\mathbf{c}+\mathbf{r}_2|= |-\mathbf{c}+\mathbf{r}_1|$.

Now, we let's rename $\mathbf{r}_1$ and $\mathbf{r}_3$ as $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the normal vector of the facet defined by these two vectors and the origin is

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ L & 0 & 0 \\ \frac{1}{2}L & \frac{1}{2\sqrt{3}}L & \sqrt{\frac{2}{3}}L \end{vmatrix}. \tag{14}$$

A unit vector in the direction of the normal is

$$\mathbf{n}' = \frac{-2\sqrt{2}}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}. \tag{15}$$

A vector that is normal to both of these is

$$\mathbf{u} = \mathbf{a}\times\mathbf{b}\times\mathbf{a}. \tag{16}$$

If the reader does not understand the following bit, it doesn't matter, as it's the result that counts.

$$\begin{aligned} \mathbf{u} &= \mathbf{a}\times\mathbf{b}\times\mathbf{a}, \\ &= e_{ijk}a_j e_{krs}b_r a_s = e_{ijk}e_{krs}a_j b_r a_s \\ &= (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})a_j b_r a_s \\ & a_j b_i a_s - a_i b_j a_j \\ &= (\mathbf{a}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{a}. \end{aligned} \tag{17}$$

Setting $\mathbf{u}' = \mathbf{u}/|\mathbf{u}|$, we find

$$\mathbf{u}' = \frac{1}{3}\mathbf{j} + \frac{2\sqrt{2}}{3}\mathbf{k}. \tag{18}$$

Now, we can inscribe a sphere such that the sphere is tangential to each facet. Where is the centre of such a sphere.

Now, we can ask what is the distance from the circumcentre to this point. We put

$$\frac{L}{2}\mathbf{i} + \frac{L}{2\sqrt{2}}\mathbf{j} + h\mathbf{k} + \lambda\mathbf{n}' = \alpha\mathbf{i} + \beta\mathbf{u}' \tag{19}$$

That is, the centre will be some height h above the centre of the base facet.

$$\frac{L}{2\sqrt{3}} - \frac{2\sqrt{2}\lambda}{3} = \frac{\beta}{3}$$

$$h + \frac{1}{3}\lambda = \frac{2\sqrt{2}\beta}{3}. \tag{20}$$

On eliminating β, we find

$$\lambda\left(\frac{1}{2\sqrt{2}} + 2\sqrt{2}\right) = \frac{3}{2}\left(\frac{L}{\sqrt{3}} - \frac{h}{\sqrt{2}}\right) \tag{21}$$

Now, if $\lambda = 0$ then

$$h = \sqrt{\frac{2}{3}}L. \tag{22}$$

That is to say, we are at the top vertex. λ is the distance from any point of a vertical line passing through the top vertex, to the plane containing our facet. The point is parameterised by the value of h, the distance from the point to the base facet, or the (x, y) plane. If we set $h = \lambda$, the distance from the point to the base is the same as the distance from the point to the facet, then

$$\frac{9\lambda}{2\sqrt{2}} + \frac{3L}{2}\frac{\lambda}{\sqrt{2}} = \frac{3L}{2}\sqrt{3}. \tag{23}$$

On tidying up we find that, and recalling that $|\mathbf{c}|$ is the radius of the circumsphere.

$$\lambda = \frac{1}{4}\sqrt{\frac{2}{3}}$$

$$\frac{|\mathbf{c}|}{\lambda} = 3. \tag{24}$$

So, the radius of the circumsphere is three times that of the radius of the inscribed sphere.

Now, there is a regular tetrahedron lurking within every cube. To see this, look how a cube can be taken apart. In Fig.1, we see a cube dissected by cutting off four irregular corner tetrahedra. The remaining shape is also a tetrahedron. The length of each edge is the length of the diagonal of the a square facet that makes up the cube. All the lengths

Figure 1: Four irregular tetrahedron are cut from a cube. (The remaining tetrahedron isn't drawn).

are equal, and so we have a regular tetrahedron. For instance, the coordinates of a regular tetrahedron are given by (1,1,1), (-1,-1,1), (-1,1,-1), and (1,-1,-1). This makes it simpler to calculate the volume of the regular tetrahedron.

Supposing we have a cube with one corner at the origin, and with two of the base edges along the positive x and y axes. We now cut out the corner tetrahedron. If we take a thin slice horizontally through this tetrahedron then, at a height z we get a right angle triangle of area $1/2(1-z)^2$. So

$$V_1 = \frac{1}{2}\int_0^L (L-z)^2\,dz = \frac{L^3}{6}. \tag{25}$$

Once we cut out the other three quarter corner tetrahedra, the remaining regular tetrahedron has a volume of one third of that of the cube. If we set L' to be the edge length of the tetrahedron. then

$$V = \frac{L'^3}{6\sqrt{2}} = \frac{\sqrt{2}L'^3}{12}. \tag{26}$$

Now, let's begin to consider a completely arbitrary irregular tetrahedron, how can we find the inscribed sphere and the circumsphere for this beast?

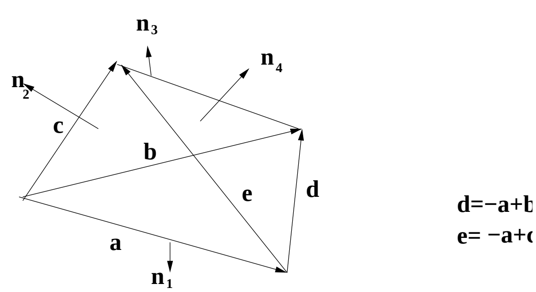

Figure 2: The vectors **a**, **b**, and **c** define a basis set, and any point can be described by three numbers, α, β, and γ.

First, we shall suppose that one corner of the tetrahedron is at the origin, and the tetrahedron is defined be three arbitrary (but not coplanar) vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Any point can be described by three numbers.

$$\mathbf{p} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}. \tag{27}$$

Now, consider a point where $\gamma = 0$. This is clearly in the plane of the facet described by the origin, $\mathbf{a}$, and $\mathbf{b}$. For a point within the facet we must have $\alpha > 0$, and $\beta > 0$, and also $\alpha + \beta \leq 1$. If $\alpha + \beta = 1$, we are on the far edge of the facet. On the facet with normal $\mathbf{n}_4$, we have $\alpha + \beta + \gamma = 1$.

We define the normals as

$$\begin{aligned}
\mathbf{n_1} &= \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b} \times \mathbf{a}|}, \\
\mathbf{n_2} &= \frac{\mathbf{a} \times \mathbf{c}}{|\mathbf{a} \times \mathbf{c}|}, \\
\mathbf{n_3} &= \frac{\mathbf{c} \times \mathbf{b}}{|\mathbf{c} \times \mathbf{b}|}, \\
\mathbf{n_4} &= \frac{\mathbf{d} \times \mathbf{e}}{|\mathbf{d} \times \mathbf{e}|}.
\end{aligned} \tag{28}$$

We have made the normals unit vectors so that λ takes on the meaning of distance. Now, to find the centre of the inscribed sphere, we need to find an α, a β and a γ, such that the point $\mathbf{p}$ is equidistant from all four facets. Looking at the first facet,

$$\mathbf{p} + \lambda\mathbf{n_1} = \alpha_1\mathbf{a} + \beta_1\mathbf{b}. \tag{29}$$

The α_1 and β_1 are the coordinates of the point where the line, in the direction of the normal $\mathbf{n}_1$ touches the $\mathbf{a}$, $\mathbf{b}$ plane. Now we shall take the dot product of both sides with $\mathbf{n}_1$, so that

$$\mathbf{p} \cdot \mathbf{n}_1 + \lambda = \gamma\mathbf{c} \cdot \mathbf{n}_1 + \lambda = 0. \tag{30}$$

Similarly we find

$$\begin{aligned}
\beta\mathbf{b} \cdot \mathbf{n}_2 + \lambda &= 0, \\
\beta\mathbf{a} \cdot \mathbf{n}_3 + \lambda &= 0.
\end{aligned} \tag{31}$$

Lastly we have

$$\mathbf{p} + \lambda\mathbf{n}_4 = \mathbf{a} + \epsilon_4\mathbf{e} + \delta_4\mathbf{e}. \tag{32}$$

Again, we take the dot product with $\mathbf{n_4}$, so that

$$\mathbf{p} \cdot \mathbf{n}_4 + \lambda = \mathbf{a} \cdot \mathbf{n}_4$$

or

$$\alpha\mathbf{a} \cdot \mathbf{n}_4 + \beta\mathbf{b} \cdot \mathbf{n}_4 + \gamma\mathbf{c} \cdot \mathbf{n}_4 + \lambda = \mathbf{a} \cdot \mathbf{n}_4. \tag{33}$$

So now we have a system of four linear equations in α, β, γ, andλ. However, it is more efficient to obtain three linear equations in α, β, andγ by substituting the first three equations in the last one and eliminating λ. Then λ can be found at the end by one last substitution. At any rate, we can calculate the location of the centre, and the radius of the inscribed sphere for any tetrahedron. We calculate the vectors from the coordinates of the corners, then calculate the necessary dot and cross products to find the coefficients of the system of equations. These can then be solved numerically or by hand. We note that given the coordinates of any four points, we must label the vectors in the same sense as in Fig.2 for these formula to work. If they are labelled in a different sense, the cross products as written here might be inward pointing rather than outward.

The circumsphere of an arbitrary tetrahedron can be gotten directly from Cartesian coordinates. Again, to simplify matters, we change coordinates so that the origin coincides with one of the corners of the tetrahedron. Suppose we already have the circumsphere, and its radius is R, and it is centred at (x, y, z). Then

$$x^2 + y^2 + z^2 = R^2. \tag{34}$$

Now, the square of the distance from the vertex at $\mathbf{a} = (x_1, y_1, z_1)$ is also R^2. So,

$$\begin{aligned}(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = R^2 \\ = x^2 - 2x_1 + x_1^2 + y^2 - 2y_1 + y_1^2 + z^2 - 2z_1 + z_1^2\end{aligned} \tag{35}$$

Subtracting eqn.35 from eqn.34 then gives

$$\begin{aligned}x_1x + y_1y + z_1z &= \frac{1}{2}\mathbf{a}\cdot\mathbf{a} \\ x_2x + y_2y + z_2z &= \frac{1}{2}\mathbf{b}\cdot\mathbf{b} \\ x_3x + y_3y + z_1z &= \frac{1}{2}\mathbf{c}\cdot\mathbf{c}.\end{aligned} \tag{36}$$

So the position of the centre of the circumsphere, and its radius, are trivial to calculate.

To finish off, we shall look at the volume of the tetrahedron depicted in Fig.2. First of all, the area of the base triangle in the $\mathbf{a}$, $\mathbf{b}$ plane is $|\mathbf{a}||\mathbf{b}|\,\sin\theta$. In vector notation this is $|\mathbf{a}\times\mathbf{b}|$. If we denote $\mathbf{k}$ as

$$\mathbf{k} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}, \tag{37}$$

then the volume is

$$V = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|\int_0^{\mathbf{k}\cdot\mathbf{c}}\left(1 - \frac{z}{\mathbf{k}\cdot\mathbf{c}}\right)^2 dz. \tag{38}$$

Here, z is the coordinate on the axis defined by $\mathbf{k}$. The square comes from the fact that as we take slices parallel to the $(\mathbf{a}, \mathbf{b})$ plane, the side lengths of the similar triangles scale linearly. So

$$V = \frac{1}{6}|\mathbf{a}\times\mathbf{b}|\;\mathbf{k}\cdot\mathbf{c} = \frac{1}{6}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}. \tag{39}$$

Of course, eqn.37 was used in the last step. The reader may verify the formula for the volume of a regular tetrahedron.