Fields and Spheres

C. Godsalve

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email seagods@hotmail.com

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1 Introduction

We examine some simple problems involving electrostatic and gravitational fields and spherical geometry. This is a *revision* article at undergraduate level. Any reader unfamiliar with the maths involved might still find it useful (at least in parts), but the article is not *intended* as a first introduction to the subject.

We start with a problem of power law attraction (or repulsion) from a uniform spherical shell and discuss two special cases: the inverse square law case is of physical importance, and we obtain the important and remarkable result that there is no force whatsoever inside a hollow spherical shell due to either its gravity, or uniform electric charge. We also examine the curious case where the force is proportional to the distance.

In §3, we revise the what is known as the *divergence* of a vector field, in spherical, cylindrical, and Cartesian coordinates, and finish off the section by revising Gauss' divergence theorem. Using this theorem we can polish off the problems of §2 with a sentence or two rather than actually "doing the sums".

In §4, we look at the problem of the vector field due to an inverse square law attraction to a circular ring. This problem has only cylindrical symmetry has a solution in terms of complete elliptic integrals. (The axial force along the axis of the ring is trivial by way of contrast.)

Lastly, we look at the problem of the field due to a spherical shell holding a non uniform charge distribution. We take a brief look at what the problem looks like in the direct approach of §4, and find it that the problem looks hopeless when tackled this way.

We then look at this problem from the point of view of potential theory, and find the solution for a simple case using the Laplace equation. Despite the complicated looking maths that crops up when attempting to solve partial differential equations, one important problem[1] turns out to have a remarkably simple solution. Namely that if the surface charge density is proportional only to the cosine of the zenith angle, then the solution outside the sphere looks like the electric potential of an ideal electric dipole. Inside the sphere, the electric field is constant, with the direction given by the sense of the sphere's dipole moment.

[1] The problem discussed is that used to derive the Claussius-Mossotti and Lorentz-Lorentz relations relating the dielectric constants and refractive indexes of dense materials to the polarisability of individual atoms.

2 A Spherical Shell with an r^n Force Law

We suppose we have some spherical shell, and that it has some kind of "charge" on it which has a uniform charge density. We imagine we have a "test particle" P, and if there is a "charge" at some point Q, then P experiences as a force in the direction $\overrightarrow{PQ}$, and that the magnitude of the force is proportional to $\mid \overrightarrow{PQ} \mid^n$. That is, the n^th power of the distance from P to Q.

As this is the first section, we introduce the basic spherical coordinates and notation. A general point P has coordinates as illustrated in Fig.2.1. Given the origin O of the coordinates, the position of any point P is

$$\overrightarrow{OP}= \rho \sin\theta(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) + \rho\cos\theta\mathbf{k}. \tag{2.1}$$

Here, $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are the usual unit vectors in the Cartesian x, y, and z directions.

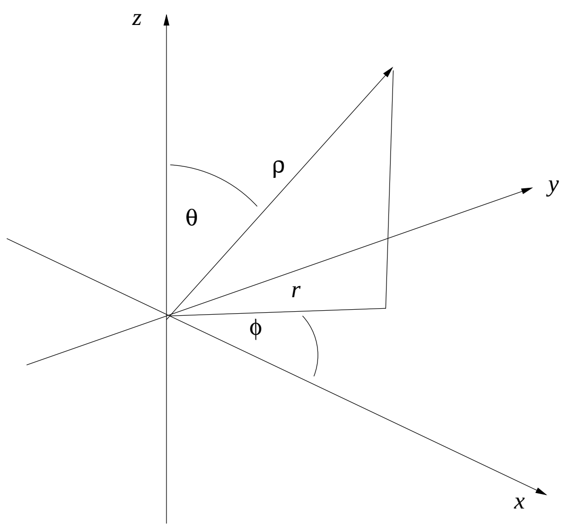

Figure 2.1: Spherical coordinates, the position vector has a distance ρ from the origin, and has a projection onto the (x, y) plane at $(r\cos\phi, r\sin\phi)$ where $r = \rho\sin\theta$.

In Fig.2.1, we have one such point, where we can define a new Cartesian system with the origin at P with unit vectors

$$\mathbf{e}_\rho = \sin\theta(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) + \cos\theta\mathbf{k},$$

$$\mathbf{e}_\phi = (-\sin\phi\mathbf{i} + \cos\phi\mathbf{j})$$
$$\mathbf{e}_\theta = \cos\theta(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) - \sin\theta\mathbf{k}. \tag{2.2}$$

We use r to denote the length of the projection of the radius vector onto the (x, y) plane, so $r = \rho\sin\theta$. The angle θ which we call the *zenith* angle is sometimes called the co-latitude, and if latitude is used instead of θ, the adjustments are obvious.

As far as this spherical shell problem is concerned, the shell is just defined by $\rho = constant$. A surface area element at the point Q is obtained by infinitesimal increments in ϕ alone, or θ alone, and as such the surface element has area

$$dA = (\rho sin\theta d\phi) \times (\rho d\theta) = \rho^2 \sin\theta d\theta d\phi. \tag{2.3}$$

Now, in spherical coordinates any vector field can be written

$$\mathbf{F} = f_\rho(\rho,\theta,\phi)\mathbf{e}_\rho + f_\theta(\rho,\theta,\phi)\mathbf{e}_\theta + f_\phi(\rho,\theta,\phi)\mathbf{e}_\phi. \tag{2.4}$$

So, what about our problem? The first thing to notice is the *symmetry.* If we were to rotate the sphere about any axis we might imagine, the force on our particle will not change. Similarly, if we rotate you test particle about the origin in any way we choose, the magnitude of the force will be the same. This symmetry automatically tells us that the force at P shall be directly toward (or away from) the origin. This tells us we can can simplify our problem. We can place P a distance R from the origin on the z axis $(\theta = 0)$. We don't need to encumber the problem by giving the point P a zenith angle (θ_P) or an azimuth angle ϕ_P. The simplified situation is depicted in Fig.2.2.

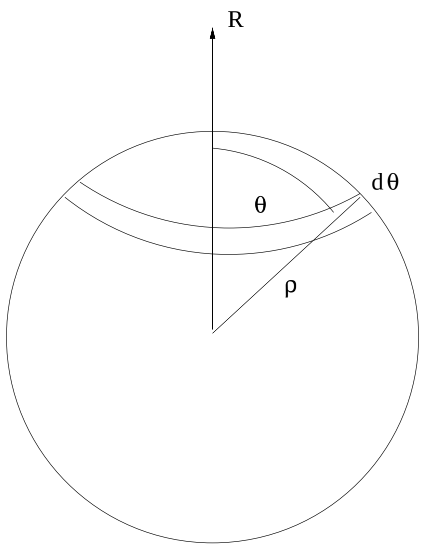

Figure 2.2: The simplified problem

If we have a constant charge density σ, we can place our element of charge of magnitude $\sigma\rho^2 \sin\theta d\theta d\phi$ at $\overrightarrow{OQ} = \rho\sin\theta\mathbf{i} + \rho\cos\theta\mathbf{k}$. Then

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \rho\sin\theta\mathbf{i} + (\rho\cos\theta - R)\mathbf{k}. \tag{2.5}$$

The distance between P and Q is given by the cosine rule as

$$s^2 = \rho^2 + R^2 - 2R\rho\cos\theta, \tag{2.6}$$

and we note that

$$2s\frac{ds}{d\theta} = 2R\rho\sin\theta. \tag{2.7}$$

So, with our power law force, the element at Q exerts a force

$$d\mathbf{F} = s^n \frac{\overrightarrow{PQ}}{\mid \overrightarrow{PQ} \mid} \times \sigma\rho^2\sin\theta d\theta d\phi = s^n(\sin\psi\mathbf{i} - \cos\psi\mathbf{k}) \times \sigma\rho^2\sin\theta d\theta d\phi. \tag{2.8}$$

Now we can make a further simplification, and add the mirror image element Q'. Symmetry tells us that the $\mathbf{i}$ components of the forces due to Q and Q' cancel, so the force on P is just

$$d\mathbf{F} = -2s^n\sigma\rho^2\sin\theta d\theta d\phi\cos\psi\mathbf{k}. \tag{2.9}$$

Now the cosine rule also tells us that

$$\rho^2 = R^2 + s^2 - 2Rs\cos\psi,$$

or

$$\cos\psi = \frac{R^2 + s^2 - \rho^2}{2Rs}. \tag{2.10}$$

So the magnitude of the force of Q and it's mirror image Q' is

$$dF = |d\mathbf{F}| = -s^{n-1}\sigma\rho^2 R^{-1}\sin\theta d\theta d\phi(R^2 + s^2 - \rho^2), \tag{2.11}$$

but we also need to recall from eqn.2.7 that $\sin\theta d\theta = sds/\rho R$, so

$$dF = |d\mathbf{F}| = -s^n\sigma\rho R^{-2}\, ds\, d\phi(R^2 + s^2 - \rho^2), \tag{2.12}$$

At last we integrate with respect to ϕ (from 0 to π because of our mirror image element) and

$$dF = |d\mathbf{F}| = -\frac{\sigma\pi\rho}{R^2}\left((R^2 - \rho^2)s^n + s^{n+2}\right)\, ds. \tag{2.13}$$

Of course, the case of the inverse square law ($n = -2$) is of special interest. If P is outside the spherical shell, then s varies between $R - \rho$ and $R + \rho$. The total force for all the elements of the sphere for the inverse square law is that

$$F = -\frac{\sigma\pi\rho}{R^2}\left[(R^2 - \rho^2)\int_{R-\rho}^{R+\rho}\frac{ds}{s^2} + \int_{R-\rho}^{R+\rho} ds\right] = -\frac{\sigma\pi\rho}{R^2}[2\rho + 2\rho] = -\sigma \times 4\pi\rho^2 \times \frac{1}{R^2}$$

or

$$F = -\frac{\Sigma}{R^2}. \tag{2.14}$$

In eqn.2.14, we have put $\sigma \times 4\pi\rho^2 = \Sigma$ to emphasise the nature of the result. From the point of view of P, and the case of an inverse square law, a "charged" sphere acts like a point at the origin with a charge Σ which is the total charge (charge per area σ times area $4\pi\rho^2$). The particle P is attracted to (or repelled from) the origin with a force inversely proportional to the square of the distance R from the origin.

Curiously, as the reader may verify, there is only one other power law that does this trick. This is the case $N = 1$. This weird law makes the attraction increase with distance — a bit like the force due to an infinitely extensible and always linear spring of infinitesimal natural length[2]. In this case the term in the bracket is $2R\rho(R^2 - \rho^2) + 2R\rho(R^2 + \rho^2)$ or $4R^3\rho$. Again, from the outside the spherical shell looks like a single point at the origin with charge σ, and the attraction or repulsion of the test charge P is linear in the distance R.

[2]When this kind of force is attractive, bodies can have circular orbits around the origin, and the period of the orbit is independent of the orbit radius.

What about a particle inside the spherical shell for the inverse square law? Inside the shell, the term $R^2 - \rho^2$ is negative, and the force changes sign when $s = \sqrt{\rho^2 - R^2}$, so all we need to do is change the limits of integration from $[R - \rho, R + \rho]$ to $[\rho - R, R + \rho]$. When we do this for the case of the inverse square law we find that *the particle sees no net force at all.* All the forces cancel wherever the particle is inside the spherical shell.

In the case of our curious linear force law, the force does not vanish if P is inside the shell. Instead, it is still attracted toward the origin with a force proportional to R as if there were a central charge[3]. The attraction is toward the origin, even though there is no "stuff" at the origin doing this attraction or repulsion at the origin.

In the case of the inverse square law, a particle inside the shell acts just as if the shell wasn't even there. Whether the inverse square force is due to gravity, and the "charge" is the mass, or whether we have an electric charge and the force is the Coulomb law, the surrounding shell may as well not be there as far as forces on P are concerned, no matter how highly charged, or no matter how dense, that shell may be. We can build up a solid spherical body out of shells of infinitesimal thicknesses $d\rho$. We shall find that for a spherically symmetric distribution of matter, the details of the density of the of the mass/charge doesn't matter: for a particle outside the sphere, the body looks like a point of mass M or charge q at the origin.

In physics of course, the main case of interest is the inverse square law because of gravity and electricity. The linear force idea is of passing interest only as a curiosity, but does have a vague resemblance to the behaviour of the cosmological constant in the general theory of relativity. It is a field where the divergence is everywhere the same.

3 The Divergence of a Field, and Gauss' Theorem

The problems that we saw in §2 can be tackled in a simpler and entirely different way if we use vector calculus rather than direct geometrical calculation. If we have a vector field $\mathbf{F}$ so that at any point in space has a vector associated with it, then for Cartesian coordinates we can write

$$\mathbf{F}(x, y, z) = F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{i} + F_z(x, y, z)\mathbf{i}. \tag{3.1}$$

[3] A body looking at the origin "sees" more "charge" on the shell across the origin than the nearer "charge" behind it. The further parts of the shell pull more strongly and the net force is toward the origin.

In general, each component of (F_x, F_y, F_z) is an entirely different function of the three spatial coordinates.

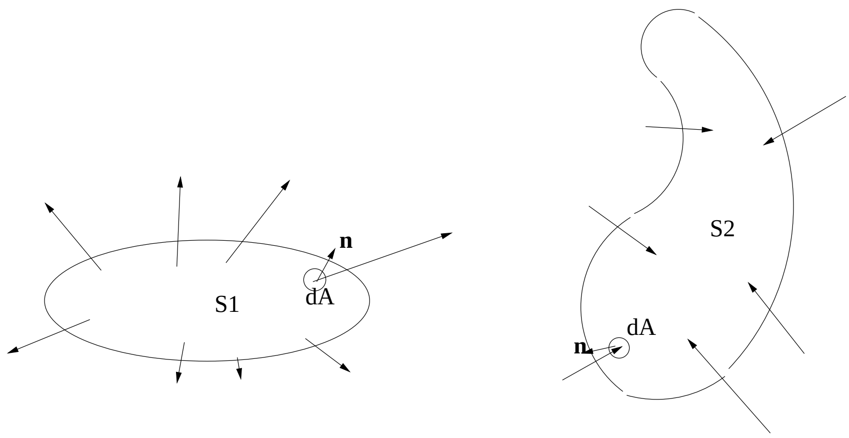

Figure 3.1: We depict two closed surfaces $S1$ and $S2$ say. Each can be thought of as consisting of patches of area dA, and a unit vector $\mathbf{n}$ at right angles to each patch can be defined. We imagine that there is some position dependent vector field $\mathbf{F}$.

In the following, we shall be interested in *closed surfaces* by which we mean that the surface has a definite inside and outside and encloses a volume completely. Some such surfaces can be defined as satisfying $f(x, y, z) = constant$, for instance $f(x, y, z) = x^2 + y^2 + z^2 = 9$ is a sphere of radius 3. Given some point (x, y, z) on the surface, then we denote $\mathbf{n}(x, y, z)$ as a unit vector which is perpendicular to that surface. If we also have some vector field $\mathbf{F}$, then

$$I = \oint\!\!\oint \mathbf{F} \cdot d\mathbf{n} \tag{3.2}$$

is of interest.

In eqn.3.2, $d\mathbf{n}$ is the (unit) normal vector $\mathbf{n}$ weighted by the surface area element dA as in Fig.3.1. (The loop through the integral signs tells us that the double integral is over a closed surface.)

If $\mathbf{F}$ in Fig.3.1 were a gas flow velocity, then there would be a net outflow through the surface $S1$ and I would be positive. The surface $S2$ sees a net inflow and $\mathbf{F} \cdot \mathbf{n}$, is negative for each patch and so I is negative. If on the other hand $I = 0$, either there is no flow through any patch, or as much flows in through some patches as flows out through others. There are many practical problems where integrals of the form of eqn.3.2 crop up, and I can take on different meanings for different problems.

It turns out that the following scalar field is very useful, it is called the *divergence* of the vector field $\mathbf{F}$, and is defined by

$$\mathrm{div}.\mathbf{F} = \nabla \cdot \mathbf{F} = \lim_{Volume \to 0} \frac{\oiint \mathbf{F} \cdot d\mathbf{n}}{\iiint_S dV} \tag{3.3}$$

The denominator is just the volume of the closed surface S. If we can take an infinite sequence of closed surfaces enclosing some point at (x, y, z) where the volume inside each successive surface becomes ever smaller then eqn.3.3 defines the divergence. It doesn't take much imagination demolish this "definition" as nonsense, but we can ignore rigorous mathematics here, and use the simplistic heuristic without further ado.

In Cartesian coordinates, things are especially simple. If we use a "rectangular box" of sides dx, dy and dz, then

$$\nabla \cdot \mathbf{F} = \frac{dydz \times \left(\frac{\partial F_x}{\partial x} dx\right) + dxdz \times \left(\frac{\partial F_y}{\partial y} dy\right) + dxdy \times \left(\frac{\partial F_z}{\partial x} dz\right)}{dxdydz} \tag{3.4}$$

To see this, two of the area elements of the box have the same area $dA = dydz$ and can be placed at a distance x and $(x + dx)$ from the origin. The unit normals are $\pm\mathbf{i}$ and the values of F_x at the two facets are $F_x(x, y, z)$ and $F_x(x + dx, y, z)$. The divergence of $\mathbf{F}$ of eqn.3.4 can be written as

$$\nabla \cdot \mathbf{F} = \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial x}\right] \cdot [F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}] \tag{3.5}$$

We can think of eqn.3.5 as the dot product of $\mathbf{F}$ and the vector operator ∇. Always remember that F_x emphatically is *not* by definition "a function of x" and is generally a function of x, y, and z. F_x is the x component of the vector field F. Note also that we haven't said anything at all about the thorny subject of limits and continuity in more than one dimension. We shall just assume we shall "get away with it" as it were. Note also, that if we have three arbitrary functions $F_x = F_x(y, z)$, $F_y = F_y(x, z)$, and $F_z = F_z(x, y)$, then $\nabla \cdot \mathbf{F} = 0$ everywhere in this case. The vector field F might be very complicated, but still have zero divergence everywhere in space. Obviously, a vector field where F_x, F_y and F_z are all constants has zero divergence. A field that has zero divergence everywhere is called *solenoidal.*

There is an added complication in spherical coordinates: not only is the volume element size a function of the coordinates, but the areas of the facets of a volume element vary too as we can see if we re-visit Fig.2.1 in this light. The volume element is $dV = \rho \sin\theta d\phi \times \rho d\theta \times d\rho$. The facets of the surface of the volume element have areas $\rho \sin\theta d\phi \times d\rho$, $\rho d\theta \times d\rho$ and $\rho \sin\theta d\phi \times \rho d\theta$. The last area corresponds to two facets of the volume element's surface. One at (ρ, θ, ϕ) and the other at $(\rho + d\rho, \theta, \phi)$. In this last pair of facets, the differential areas are different by $2\rho d\rho \sin\theta d\phi d\theta$. Also, there is another pair of facets which have different areas, these are the areas $\rho \sin\theta d\phi \times d\rho$ at θ and $\theta + d\theta$. The differential areas are different by $cos\theta d\rho d\theta d\phi$.

In spherical coordinates the field looks like $\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi$ (again, see Fig.2.1). The change in area across the surface elements gives rise to an element of the divergence that depends on F_ρ rather than its derivative w.r.t. ρ. Similarly we see that the change in areas across facets gives rise to a term dependent on F_θ. When we consider this we see that in spherical coordinates we can define the divergence via the vector operator

$$\begin{aligned}\nabla &= \left(\frac{\partial}{\partial \rho} + \frac{2}{\rho}\right)\mathbf{e}_\rho + \left(\frac{1}{\rho \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{\rho}\cot\theta\right)\mathbf{e}_\theta + \left(\frac{1}{\rho \sin\theta}\frac{\partial}{\partial \phi}\right)\mathbf{e}_\phi \\ &= \left(\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\rho^2\right)\mathbf{e}_\rho + \left(\frac{1}{\rho \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\right)\mathbf{e}_\theta + \left(\frac{1}{\rho \sin\theta}\frac{\partial}{\partial \phi}\right)\mathbf{e}_\phi. \end{aligned} \tag{3.6}$$

Another useful operation is to take the gradient. In spherical coordinates, if we have a scalar field $f(\rho, \theta, \phi)$, then the vector field called the gradient of f is just

$$\text{grad} f = \nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi. \tag{3.7}$$

The Laplacian of a scalar field is just the divergence of the gradient and is denoted $\nabla^2 f$.

While we are at it, we may as well throw in the gradient and divergence in terms of cylindrical coordinates. Things look simpler, and

$$\nabla = \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\right)\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\mathbf{e}_\phi + \frac{\partial}{\partial z}\mathbf{e}_z. \tag{3.8}$$

The gradient is given by

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z. \tag{3.9}$$

3.1 Gauss' Theorem

Among the multitude of important theorems discovered by Gauss, the theorem we are interested in here is Gauss' divergence theorem, and this is often just referred to as Gauss' theorem. It is almost just a consequence of the definition of the divergence of a vector field, and states that

$$\oiint_S \mathbf{F} \cdot d\mathbf{n} = \iiint_V \nabla \cdot \mathbf{F} dV. \tag{3.10}$$

This, coupled with the nature of $\nabla \cdot \mathbf{F}$ being associated with sources or sinks in a vector field, make the statement in eqn.3.10 a very powerful analytical tool. To see this, we shall proceed by example and consider the problems of §2 again.

We shall start by looking at a field with spherical symmetry. Then the form of the vector field must be $\mathbf{F} = F_\rho(\rho)\mathbf{e}_\rho$. Any other form cannot have spherical symmetry. Consider $F_\rho = \rho^n$, then

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\rho^{n+2} = (n+2)\rho^{n-1}. \tag{3.11}$$

Two particular cases stand out, one is for $n = -2$, and the other is for $n = 1$. In both these cases, the divergence is independent of ρ. The inverse square force is important for physics, and in this case $\nabla \cdot \mathbf{F} = 0$. Ah, but is it?

In the case of the inverse square law ($n = -2$), the force and its divergence are undefined at $\rho = 0$. However we can look at the definition of the divergence, and suppose we have some sphere of radius a. Then at the surface $F_r = 1/a^2$, and

$$\oiint \mathbf{F} \cdot \mathbf{dn} = \int_0^{2\pi} \int_0^{\pi} \frac{\mathbf{e}_\rho}{a^2} \cdot (\ a^2 \sin\theta d\theta d\phi \mathbf{e}_\rho) = \frac{4\pi a^2}{a^2} = 4\pi = \iiint \nabla \cdot \mathbf{F} dV. \tag{3.12}$$

This can only mean one thing: the divergence of the inverse square law field is a three dimensional Dirac delta function. That is to say,

$$\nabla \cdot \frac{\mathbf{e}_\rho}{\rho^2} = 4\pi\delta^3(\mathbf{r}). \tag{3.13}$$

By definition $\delta^3(\mathbf{r}) = 0$ if the position vector $\mathbf{r} \neq 0$. Also, by definition the volume integral of $\delta^3(\mathbf{r})$ over *any* completely arbitrary closed surface containing the origin is equal to one and the volume of *any* completely arbitrary closed surface excluding the origin is zero.

Equation 3.13 tells us a lot to tell us. Suppose we are talking about electricity, the divergence of the field due to a single electron is a delta function weighted by the electric charge of one electron. We might look at a trillion electrons at a trillion different positions, and the divergence of the field will be a trillion (equally weighted) delta functions one at each electron position. Take any surface whatsoever, as long as it encloses them all, and the surface integral of the dot product of the electric field and the differential surface normal element is just the sum of the charges.

So, if we look at a spherical gravitating body, with a mass distribution that has spherical symmetry, and integrate $\mathbf{F} \cdot \mathbf{n}$ over *any* surface containing the entire body, the resulting number will just be the mass of the sphere. If we happen to make our surface a sphere concentric with the spherical body, and having a bigger radius than the body, then $|\mathbf{F}|$ will be constant. So we know that outside this body, the field has to follow an inverse square law, and that the strength of the field is given by the mass. This is true no matter what the density is as a function of ρ or what the radius of the body happens to be. The force outside the body looks *identical* to that force generated by a point mass with the same mass as the entire body.

If we have a hollow spherical shell body, then the integral of $\mathbf{F} \cdot \mathbf{n}$ over any concentric sphere inside the shell *must be zero.* The sphere contains no mass. The only function F_ρ that will do that trick is $F_\rho = 0$. The force inside a spherical shell, or indeed the gravitational force inside a hollowed out planet is zero, as long as the mass of the planet has spherical symmetry.

The divergence theorem lets us solve the problems posed in §2 almost immediately, as the reader can confirm for the linear force law where the divergence of the field is constant over all space.

4 The Field due to a Charged Ring

We want the field around a uniformly charged circular ring. The problem is that depicted in Fig.4.1, where we suppose the charge on any element Q of the ring of radius r at $\mathbf{r} = r\cos\phi\mathbf{i} + r\sin\phi\mathbf{j}$ is $\sigma d\phi$. The problem has cylindrical symmetry, and we do not lose generality if we place P in the (x, z) plane at $P = c\sin\theta_1\mathbf{i} + c\cos\theta_1\mathbf{k}$ where $c = \sqrt{a^2 + b^2}$.

The distance $s = |\overrightarrow{PQ}|$ is found from the cosine rule, so

$$s^2 = b^2 + t^2 = b^2 + a^2 + r^2 - 2ar\cos\phi = c^2 + r^2 - 2ar\cos\phi. \tag{4.1}$$

We need

$$\overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = (r\cos\phi - a)\mathbf{i} + \rho\sin\phi\mathbf{j} - b\mathbf{k}, \tag{4.2}$$

so we can write that the force at P due to the charge $\sigma\rho d\phi$ as

$$d\mathbf{F} = \frac{\sigma r d\phi \times [(r\cos\phi - a)\mathbf{i} + \rho\sin\phi\mathbf{j} - b\mathbf{k}]}{[c^2 + r^2 - 2ar\cos\phi]^{3/2}}. \tag{4.3}$$

Of course, we can add the mirror image force so there is no net $\mathbf{j}$ component and integrate from 0 to π rather than from 0 to 2π, and

$$d\mathbf{F} = \frac{2\sigma r d\phi \times [(r\cos\phi - a)\mathbf{i} - b\mathbf{k}]}{[c^2 + r^2 - 2ar\cos\phi]^{3/2}}. \tag{4.4}$$

We can tidy this up a bit, and use $A = a/r$, $B = b/r$ and $C = c/r$ so that

$$d\mathbf{F} = \frac{2\sigma r^2 d\phi \times [(\cos\phi - A)\mathbf{i} - B\mathbf{k}]}{r^3[1 + C^2 - 2A\cos\phi]^{3/2}}. \tag{4.5}$$

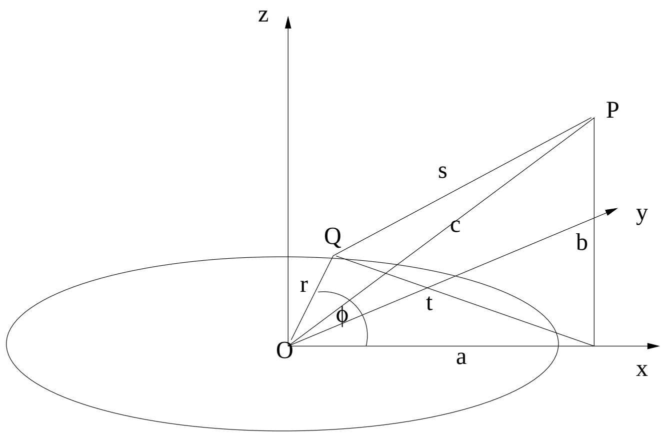

Figure 4.1: We have a charged ring in the (x, y) plane centred at the origin, and we want to know the force on a test particle at $P = a\mathbf{i} + b\mathbf{k}$

We can tidy things up further so that if we put

$$\mu = \frac{2A}{1 + C^2} = \frac{2C\sin\theta_1}{1 + C^2}. \tag{4.6}$$

and then

$$d\mathbf{F} = \frac{2 \times 2\pi r\sigma}{2\pi r^2(1 + C^2)^{3/2}} \frac{[(\cos\phi - C\sin\theta_1)\mathbf{i} - C\cos\theta_1\mathbf{k}]}{[1 - \mu\cos\phi]^{3/2}} d\phi. \tag{4.7}$$

(We have arranged things so that the total mass/charge, $2\pi r\sigma$, occurs in the numerator.) For a given C, the maximum value that μ can take is when $\sin\theta_1 = 1$ (or $B = 0$) and

$$\frac{d\mu}{dA} = \frac{(1 - A)^2}{(1 + A^2)^2}. \tag{4.8}$$

This is zero at maximum μ which occurs when $A = 1$. When this is the case $\mu = 1$. So whatever happens, $0 < \mu < 1$, which is as it should be. The maximum value $\mu = 1$ only occurs when the point P is actually on the charged ring.

Now let's have a look at one of the integrals that occurs when integrating eqn.4.7, namely

$$I = \int_0^\pi \frac{d\phi}{(1-\mu\cos\phi)^{3/2}}. \tag{4.9}$$

If we put $\phi = 2\psi$ so that $\cos 2\psi = 2\cos^2\psi - 1$ then

$$I = 2\int_0^{\pi/2} \frac{d\psi}{(1+\mu-2\mu\cos^2\psi)^{3/2}} = \frac{2}{(1+\mu)^{3/2}}\int_0^{\pi/2} \frac{d\psi}{(1-k^2\cos^2\psi)^{3/2}}. \tag{4.10}$$

We have put

$$k^2 = \frac{2\mu}{1+\mu}. \tag{4.11}$$

Since $0 < \mu < 1$, we see that k^2 is always positive.

4.1 Elliptic Integrals

The study of elliptic integrals goes back as far as calculus itself, as Kepler's laws and the orbits of planets were among the first practical problems that the new calculus was applied to. It soon became apparent that certain integrals involved could not be solved in terms of elementary functions, and the elliptic integrals had to be defined as functions in themselves.

The elliptic integrals of the first and second kind are well documented [1] [2]. The function $F(\phi, k)$ such that

$$F(\phi, k) = \int_0^\phi \frac{dx}{\sqrt{1-k^2\sin^2 x}} \tag{4.12}$$

is an *incomplete* elliptical integral of the first kind, and the function $E(\phi, k)$ such that

$$E(\phi, k) = \int_0^\phi \sqrt{1-k^2\sin^2 x}dx \tag{4.13}$$

is an *incomplete* elliptical integral of the second kind. When the argument $\phi = \pi/2$, the functions $F(k)$ and $E(k)$ are called *complete* elliptic integrals of the first and second kinds.

Many results are known regarding these functions and are listed in standard hand books and tables [2]. For instance, if we put $\Delta(\phi, k) = \sqrt{1-k^2\sin^2\phi}$ and $k' = \sqrt{1-k^2}$, then it follows from the definition that

$$\int \frac{dx}{\Delta^3} = \frac{1}{k'^2}E(x,k) - \frac{k^2}{k'^2}\frac{\sin x\cos x}{\Delta}. \tag{4.14}$$

This result, along with

$$\int \frac{\cos^2 x\,dx}{\Delta^3} = \frac{1}{k^2}F(x,k) - \frac{1}{k^2}E(x,k) + \frac{\sin x \cos x}{\Delta} \tag{4.15}$$

enables us to write down the field due to a charged ring in terms of standard elliptic integrals.

5 The Electric Field Inside a Spherical Shell With a Non-Uniform Charge Distribution

The Coulomb force is an inverse square law, so the field inside a uniformly charged sphere has already been done in §2. We shall be interested in a sphere where the charge distribution σ is non uniform. We shall restrict ourselves to the case $\sigma = f(\cos\theta)$.

First we shall attempt the problem using a "direct attack" by adapting the results of the previous section: this fails miserably. However it is nonetheless interesting to just look at happens in this formulation. We then go on to formulate the problem as a partial differential equation.

5.1 Failure of the Direct Approach

In Fig.5.1, the charge distribution has cylindrical symmetry around the z axis, and the point in the sphere at $a\mathbf{i} + b\mathbf{k}$ can be considered as a general point because of this.

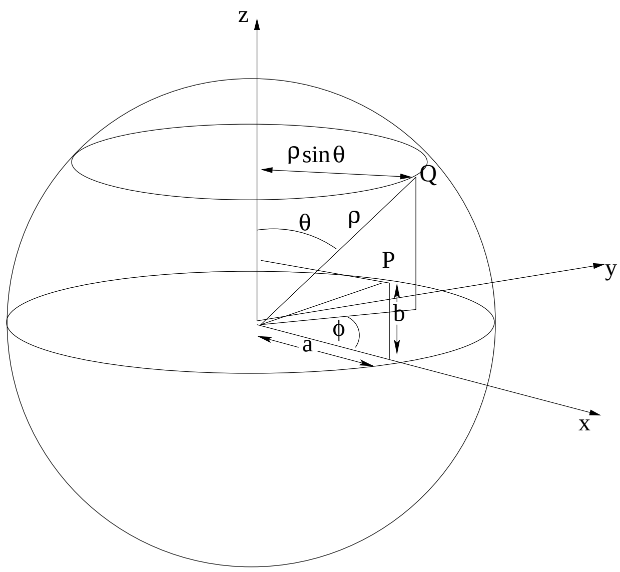

Figure 5.1: The charge density on a sphere of radius ρ is a function only of θ.

We have already done the geometry for this in §4. A horizontal ring through the sphere is given by $\theta = const.$, and has radius $\rho \sin\theta$. The charge density on the ring is then $\sigma = \sigma(\cos\phi)$. The force at P due to the charges at Q and it's mirror image across the (x, z) plane is then

$$d\mathbf{F} = 2\sigma(\cos\theta) \times (\rho^2 \sin\theta d\theta d\phi)$$
$$\times \frac{(\rho\sin\theta\cos\phi - a)\mathbf{i} + (\rho\cos\theta - b)\mathbf{k}}{[(b-\rho\cos\theta)^2 + a^2 + \rho^2\sin^2\theta - 2a\rho\sin\theta\cos\phi]^{3/2}} \tag{5.1}$$

For the moment, we shall look just at the z component (the force on P in the $\mathbf{k}$ direction), this is

$$dF_z = 2\sigma(\cos\theta) \times (\rho^2 \sin\theta d\theta d\phi)$$
$$\times \frac{(\cos\theta - C\cos\theta_1)}{\sqrt{\rho}[1 + C^2 - 2C\cos\theta_1\cos\theta]^{3/2}[1-\mu\cos\phi]^{3/2}}, \tag{5.2}$$

where

$$\mu = \frac{2C\sin\theta\sin\theta_1}{1 + C^2 - 2C\cos\theta\cos\theta_1}. \tag{5.3}$$

We have used a similar notation to that of §4, and $C = \sqrt{a^2+b^2}/\rho$. Obviously, since P is inside the sphere $C < 1$.

In fact, we may as well put $z = \cos\theta$ and $dz = -\sin\theta d\theta$ so that we now have changed to cylindrical coordinates and

$$\begin{aligned} -dF_z &= 2\sigma(z) \times (\rho^2 dz d\phi) \\ &\times \frac{(z-B)}{\sqrt{\rho}[1+A^2-B(2z-B)]^{3/2}[1-\mu\cos\phi]^{3/2}}, \end{aligned} \tag{5.4}$$

where

$$\mu = \frac{2A\sqrt{1-z^2}}{1+A^2-B(2z-B)}. \tag{5.5}$$

Well, we can integrate w.r.t ϕ and write this down as $E(\mu) = E(f(z))$, but where does this get us? We have to integrate a rather complicated function of z over the interval $(-1, 1)$. But note that if we put $\sigma = \sigma_0$, so that the charge density is a constant, we know the answer must be zero! Of course, in this case, we have spherical symmetry and could have put P on the z axis (where $\mu = 0$) and we have the same situation as in §2. What eqn.5.4 is telling us that may as well abandon all hope of solving for a non uniform charge distribution this way. Just try showing $F_z = 0$ if the charge distribution is constant!

5.2 The Laplace Equation

We know we have a conservative field, and can write down the electric field as $\mathbf{E} = -\nabla\phi$, that is as the gradient of some scalar potential function. Since the divergence of this field is zero everywhere outside or inside the sphere, we know that

$$\nabla^2\phi = 0. \tag{5.6}$$

This is Laplace's equation. We have a spherical surface for our problem, so in spherical coordinates we write eqn.5.6 as

$$\nabla^2\phi(r,\theta) = \left[\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right) + \left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right)\right]\phi(r,\theta) = 0. \tag{5.7}$$

The usual approach is to "separate variables" so that we express $\phi(r,\theta) = R(r)\Theta(\theta)$. Then we find that if α is some constant, that

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \alpha R = 0 \tag{5.8}$$

and

$$\sin\theta\frac{d^2\Theta}{d\theta^2} + \cos\theta\frac{d\Theta}{d\theta} + \alpha\sin\theta\Theta = 0. \tag{5.9}$$

Equation 5.8 is of the form of the Frobenius type. Though this has solutions of different forms, we shall hope for solutions in terms of *integer* powers of r. If we substitute r^l into eqn.5.8, we shall see that this has a solution with $\alpha = l(l+1)$ for positive powers, and the general solution is

$$R(r) = Ar^l + Br^{-(l+1)}. \tag{5.10}$$

Although l doesn't *have* to be an integer, our knowledge of electrostatics and the inverse square law suggests l will be and integer. Now we can go back to eqn.5.9, and put $x = cos\theta$, so that

$$\frac{d}{d\theta} = -\sin\theta\frac{d}{dx}, \tag{5.11}$$

whereupon we find that

$$(1-x^2)\frac{d\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + l(l+1)\Theta = 0. \tag{5.12}$$

This is *Legendre's equation* and for integer values of l has solutions which are Legendre polynomials P_l, where the first few orders are

$$P_0 = 1,$$

$$P_1 = x,$$

$$P_2 = \frac{1}{2}(3x^2 - 1),$$

$$P_3 = \frac{1}{2}(5x^3 - 3x),$$

$$(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x). \tag{5.13}$$

So, in general, the solution for Laplace's equation in spherical coordinates that has no azimuthal dependence is

$$\phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(\cos\theta). \tag{5.14}$$

If we had an azimuthal term (variation of the charge with ϕ, we would have Fourier components for $\Phi(\phi)$ and a solution in terms of associated Legendre functions. That is, the more general solution looks like

$$\phi(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l^m(\cos\theta)\cos m\phi \tag{5.15}$$

if Φ is always even in ϕ. There are many recurrence properties for the associated Legendre functions, we list only

$$(l-m+1)P_{l+1}^m = (2l+1)xP_l^m - (l+m)P_{l-1}^m$$

and

$$\sqrt{1-x^2}P_l^{m+1} = (l-m)xP_l^m - (l+m)P_{l-1}^m. \tag{5.16}$$

We shall now consider the simple case that the charge density is given by $\sigma = \sigma_0 \cos\theta$. Outside the sphere, as $r \to \infty$ we know that the potential will tend to that of an electric dipole. We don't expect the field to change rapidly with θ than the charge distribution, so that suggests using just $l = 1$ might give us the solution. The force will decrease as $r \to \infty$, so outside the sphere we can put

$$\phi \propto \frac{\cos\theta}{r^2}. \tag{5.17}$$

Inside the sphere, the $\cos\theta$ charge distribution also suggests $l = 1$, and

$$\phi \propto r\cos\theta. \tag{5.18}$$

In eqn.s 5.17 and 5.18, we shall call the unknown constants of proportionality A and B. We don't yet know if this really is the form of the solution, however we carry on in the hope that it is, and use eqn.3.7 to write the electric field as

$$\mathbf{E} = -\nabla\phi = 2A\frac{\cos\theta}{r^3}\mathbf{e}_r + A\frac{\sin\theta}{r^3}\mathbf{e}_\theta \tag{5.19}$$

outside the sphere. Inside the sphere we have

$$\mathbf{E} = -\nabla\phi = -B\cos\theta\mathbf{e}_r + B\sin\theta\mathbf{e}_\theta. \tag{5.20}$$

The result looks remarkably simple, it looks like an electric dipole field outside the sphere. Even more remarkable, is that inside the sphere, we have a constant field in the z direction! But, the solution we have written is just no solution at all unless it satisfies the boundary conditions. But, exactly what are the boundary conditions?

The first is continuity, there can be no no infinite potential gradients. This gives us $2A/\rho^2 = B\rho$ (the sphere's radius is now ρ). The second comes from Gauss' theorem. We can apply Gauss' theorem over a differential volume as in Fig.5.1. The facets both have surface area $\sim \rho^2 \sin\theta d\theta d\phi$ and the surface normals are parallel and anti-parallel to the sphere's unit normal.

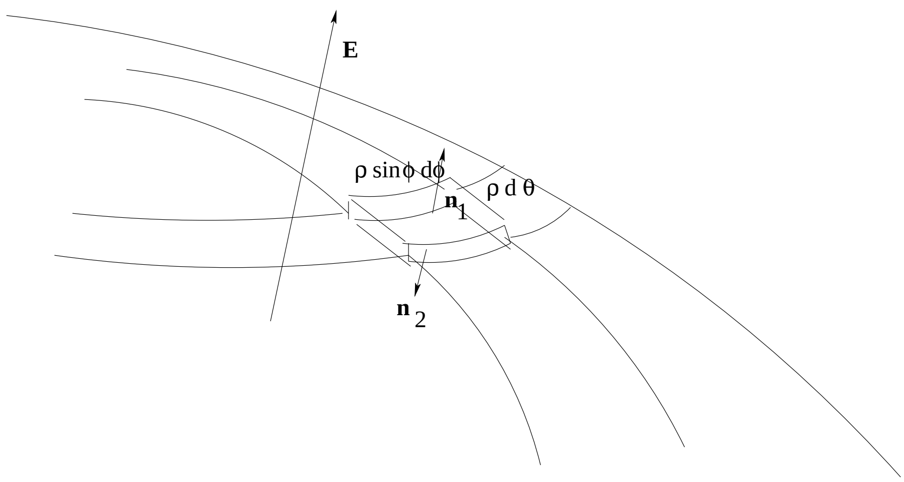

Figure 5.2: The differential volume with two main facets one just inside and one just outside the sphere. We make the these facets so close that the other facets have negligible area by comparison

If we take the surface integral of $\mathbf{E} \cdot d\mathbf{n}$ over this volume element, then the volume element contains the charge $\cos\theta dr \times \rho^2 \sin\theta d\theta d\phi$. The way we have set things up, the charge density is a delta function in the radial variable, and we shall find that the constant A is given by $A = \sigma_0 a^3/3$. (We have been careless of units, so there will be things like $4\pi\epsilon_0$ floating about if $S.I.$ units are used.)

At any rate, we have a potential and a field satisfying the boundary condition, and the solution really is that simple. Outside the sphere

$$\phi(r, \theta) = \sigma_0 \frac{\rho^3}{3} \frac{\cos\theta}{r^2}, \tag{5.21}$$

and

$$\mathbf{E} = \frac{\sigma_0 \rho^3}{3} \left[\frac{2}{r^3} \cos\theta \mathbf{e}_r + \frac{1}{r^3} \sin\theta \mathbf{e}_\theta \right]. \tag{5.22}$$

and inside the sphere

$$\mathbf{E} = -\frac{\sigma_0}{3} \mathbf{k}. \tag{5.23}$$

This result is of interest since it provides an idea of the field inside a small cavity inside a polarised dielectric material, which is the basis of the Claussius-Mossotti and Lorentz-Lorentz relations for the relationship between the polarisability of an individual atom and the dielectric constant and refractive index of a dense material [3].

References

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[2] I. Gradshteyn and I. Ryzhik, *Tables of Integrals, Series, and Products (Corrected and enlarged edition)* (Academic Press, Inc, San Diego, New York, Berkely, Boston, London Sydney, Tokyo, Toronto, 1980).

[3] R. Feynman, R. B. Leighton, and M. Sands, *The Feyman Lectures on Physics* (Addison-Wesley Publishing Company, Reading Massachusetts, Menlo Park, California, London, Sydney, Manilla, 1964), Vol. 2.