Some Basics on Vectors

C. Godsalve
email:seagods@hotmail.com

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1 introduction

The word ”vector” sounds strange doesn't it? The word comes from the Latin for ”to convey”. So, what exactly is a vector? A mathematician might give you a list of axioms that define a ”vector space”, and then say that any mathematical ”object” that satisfies these axioms is a vector. That mathematician would be right. Our mathematician would not have mentioned anything to do with distances or angles. Indeed, all sorts of things are vectors. Things like distances and angles are introduced if the vector space has a something called a *metric*.

We shall ignore the mathematicians, and go straight out and say that (for our purposes here) a vector is something that has magnitude *and* direction. This is where the ”to convey” bit comes from in the first place. Look at the figure below. It's a standard x, y graph. You are at the origin, you want to get to point A? Simply march off a certain distance in that particular direction and there you are. This, according to our definition is a vector. It has magnitude (the distance) and it certainly has direction. Let's call this vector $\overrightarrow{OA}$. It means "go from O to A, hence the arrow. Doing the opposite takes you back

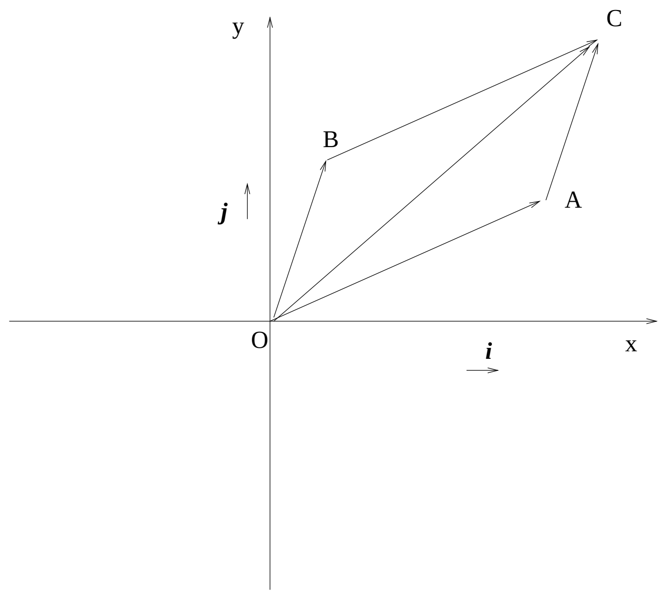


Figure 1: Two vectors are added, one takes you from the origin to A, another takes you from the origin to B. The result of adding the two vectors takes you from the origin to C.

to the origin. So, $\overrightarrow{AO}$ is also a vector. Now, we can write $\overrightarrow{OA}$+$\overrightarrow{AO}$= 0. In other words, going from O to A and then from A to O has taken you nowhere. We have introduced the idea that vectors can be added. This should be no surprise. Clearly, according to our definition, "go five miles" is not a vector, but "go five miles North-East is" because a direction is specified. Given another instruction, "then go six miles North", takes us somewhere else. We might have got there with just one instruction, "go so many miles that way". This instruction would be a vector too. So, we can add vectors, and the result of adding two vectors is another vector.

Let's go back to our diagram. Two arrows come out from the origin. One takes you to a point we call A, another takes us to B. If we "complete the parallelogram" by drawing extra lines parallel to $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as shown in Fig.1, we get to a point, we call it C, so $\overrightarrow{OA}$+ $\overrightarrow{OB}$= $\overrightarrow{OC}$. We shall call this a parallelogram law of vector addition. Now clearly $\overrightarrow{OA}$+ $\overrightarrow{AO}$= $\overrightarrow{OO}$. Going from O to A and back is going nowhere. So, we can write $\overrightarrow{AO}$=-$\overrightarrow{OA}$, and $\overrightarrow{OA}$ $-\overrightarrow{OA}$= $\overrightarrow{OO}$.

So, we have a "law" for the addition of two vectors, we have negative vectors, and a zero vector. If we add a zero vector to a vector it is unchanged. There are different notations in use for vectors. Some write them using the letters and arrows notation as we have done so far. Some write them down as single lower case letters with 'mats' underneath them, some authors write them as single lower case letters with arrows over them. Often, in books, they are written as single bold face letters. We shall use that notation here. So, $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OA}=\mathbf{b}$, and $\overrightarrow{OC}=\mathbf{a}+\mathbf{b}=\mathbf{c}$.

Now, we can multiply our vectors by numbers. Go three miles North-East, then doing the same thing five times more gives us six lots of going three miles North-East and takes us eighteen miles from North-East of where we started. So, vectors can be added, and multiplied by numbers to give other vectors.

Because of this, we can have special *unit* vectors. By this we mean that the length, in whatever system is used, is just one.

In the Cartesian coordinates of Fig.1, one of the unit vectors shall be called $\mathbf{i}$, it is parallel to the x-axis, and pointing right. Another unit vector shall point "up", and is parallel to the y-axis. This one we shall call $\mathbf{j}$. Now, think of some vector that we shall call $\mathbf{a}$. We could march directly from the origin to the point A, but we could have got there by travelling along the x axis, and then turning left, and travelling parallel to the y axis. It is very common, because it is convenient, to represent vectors in terms of unit vectors. So, we can write $\mathbf{a}=x_a\mathbf{i}+y_a\mathbf{j}$. We call the numbers x_a and y_a the x and y *components* of the vector $\mathbf{a}$. Note that

$$\mathbf{a}+\mathbf{b}=x_a\mathbf{i}+y_a\mathbf{j}+x_b\mathbf{i}+y_b\mathbf{j}=(x_a+x_b)\mathbf{i}+(y_a+y_b)\mathbf{j}$$

.

A vector $\mathbf{a}$ is often written out in these forms

$$\mathbf{a}=(x_a,y_a)=\begin{pmatrix} x_a \\ y_a \end{pmatrix}$$

.

Before going on to vector products, we note that the *norm* of a vector is its length. It is written between two vertical bars, so the norm of $\mathbf{a}$ is written $|\mathbf{a}|$. Pythagoras' theorem immediately tells us that $|\mathbf{a}|^2=x_a^2+y_a^2$. Then of course $\tilde{\mathbf{a}}=\frac{\mathbf{a}}{|\mathbf{a}|}$ is a unit vector in the direction of $\mathbf{a}$.

2 The dot product

The dot (or inner) product of two vectors is not a vector at all, it's just a number. So, what is the dot product between two vectors $\mathbf{a}$ and $\mathbf{b}$?. Suppose you take a unit vector along $\mathbf{a}$, march a particular distance and turn off at right angles. Suppose you pick the distance so that you can now get to $\mathbf{b}$ by marching off in this direction. This distance is $|\mathbf{a}||\mathbf{b}|\ cos\theta$. The Greek letter "theta" stands for the angle between the two vectors. This is written

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\ cos\theta.$$

This is the projection of one vector on to another.

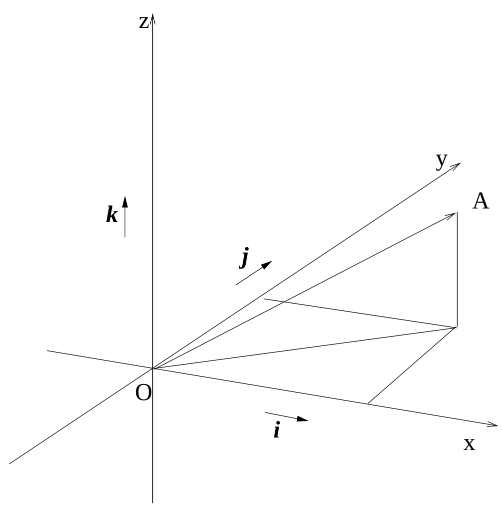
Now, clearly the projection of a unit vector onto itself is one, and so we have $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$. Also the cosine of 90 degrees is zero, so $\mathbf{i}\cdot\mathbf{j} = 0$. Using this we can now write out

$$\mathbf{a}\cdot\mathbf{b} = (x_a\mathbf{i} + y_a\mathbf{j})\cdot(x_b\mathbf{i} + y_b\mathbf{j}) = x_a x_b + y_a y_b.$$

So we can work out the dot product from the components.

3 Three Dimensions, the Cross Product

So far, we have talked only in terms of the x,y plane. But we can have "up" as well as North and East, so too, we can have an extra z axis pointing "up" just as below. This



introduces an extra unit vector, $\mathbf{k}$. Everything that went before still applies, so now

$$\mathbf{a}\cdot\mathbf{b} = (x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k})\cdot(x_b\mathbf{i} + y_b\mathbf{j} + z_a\mathbf{k}) = x_a x_b + y_a y_b + z_a z_b.$$

Also,

$$\mathbf{a} = (x_a, y_a, z_a) = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix},$$

and

$$\begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} + \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} x_a + x_b \\ y_a + y_b \\ z_a + z_b \end{pmatrix}.$$

The norm squared of a vector is still just the dot product of the vector with itself.

There is something new in three dimensions though, the cross product, which is denoted with a cross. Unlike the dot product, this product of two vectors is itself a vector at right-angles to the initial two. So if

$$\mathbf{c} = \mathbf{a} \times \mathbf{b},$$

then

$$|\mathbf{c}| = |\mathbf{a}||\mathbf{b}|\, sin\theta,$$

and

$$\mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b} = 0.$$

Now, it was mentioned that $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, (hence the zero dot products above) but which way does it point? This direction is provided by the right hand rule. If you point your index finger along the direction of $\mathbf{a}$ and your second finger along the direction $\mathbf{b}$ then, your sticking out thumb is the direction (roughly) of $\mathbf{c}$. Note according to this definition

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a},$$

so the cross product does not commute.

Now, we immediately have $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, $-\mathbf{j} = \mathbf{i} \times \mathbf{k}$, and $\mathbf{i} = \mathbf{j} \times \mathbf{k}$. So if we have the components of two vectors, we can use the above to work out the components of the cross product. It is found that

$$\begin{pmatrix} x_c \\ y_c \\ zc \end{pmatrix} = \begin{pmatrix} (y_a z_b - z_a y_b) \\ -(x_a z_b - z_a x_b) \\ (x_a y_b - y_a x_b) \end{pmatrix}.$$

Now, we have made no mention as to what should motivate us to define such a product. Part of the motivation comes from physics and the study of electromagnetism, but in geometry it makes it easy to define local right handed sets of unit vectors all at right angles to each other no matter what the orientation. Note, that all this is for *right handed coordinate systems* as drawn above. If we swap the labels of the y and z axes, we would now have a left handed system. You can see for yourself what the effect would be on our cross product.

There is a useful result called the vector triple product. (The cross product is also sometimes called the vector product and sometimes the outer product.) Suppose we need $\mathbf{d} = \mathbf{a} \times \mathbf{b} \times \mathbf{c}$. Now we know that $\mathbf{d}$ must be in the same plane as $\mathbf{b}$ and $\mathbf{c}$ since the resulting vector must be perpendicular to $\mathbf{a}$, and $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{c}$. In fact

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

4 Vectors and Geometry

There is a lot that can be written on this subject, I provide only a couple of examples here. The first is the distance from a point to a line. Here we suppose that $\mathbf{r_0}$ is a vector from the origin to the line, and that we have a vector $\mathbf{u}$ to describe the line's direction. Then, given some point $\mathbf{r}$, what is the point such that the distance from $\mathbf{r}$ to the line is at its shortest? Clearly, any point on the line can be written as $\mathbf{r_0} + \lambda\mathbf{u}$. So the vector from the point to the line $\mathbf{v}$ is

$$\mathbf{v} = -\mathbf{r} + \mathbf{r_0} + \lambda\mathbf{u}.$$

So,

$$|\mathbf{v}|^2 = |\mathbf{r_0} - \mathbf{r}|^2 + 2\lambda(\mathbf{r} - \mathbf{r_0}) \cdot \mathbf{u} + \lambda^2\mathbf{u} \cdot \mathbf{u}.$$

If we differentiate w.r.t. λ we find that

$$\frac{d|\mathbf{v}|^2}{d\lambda} = 2(\mathbf{r_0} - \mathbf{r}) \cdot \mathbf{u} + 2\lambda\mathbf{u} \cdot \mathbf{u}.$$

which is zero when

$$\lambda = -\frac{(\mathbf{r} - \mathbf{r_0}) \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}.$$

Now that we have λ we have the the vector that gives the shortest distance from the point to the line.

Now we shall consider the distance from a point to the plane. The equation of a plane may be written as

$$z = ax + by + c$$

However, it is convenient to rearrange it into

$$a_1x + a_2y + a_3z = K$$

In terms of vectors this is $\mathbf{a} \cdot \mathbf{r} = K$, so the equation of a plane is such that some vector $\mathbf{a}$ dot the position vector is a constant K. Now, we can always make the vector $\mathbf{a}$ a unit vector, so we shall just assume that this step has already been done. Then we know that $\mathbf{r} = K\mathbf{a}$ is a point on the plane. (Take the dot product of both sides with $\mathbf{a}$ and recall that $\mathbf{a}$ is a unit vector.) We now suppose that we have some other distinct point $\mathbf{s}$ on the plane, so that $\mathbf{u} = -K\mathbf{a} + \mathbf{s}$. In fact, we can find some other vector $\mathbf{v}$ that lies in the plane and is perpendicular to $\mathbf{u}$, and we can always make these unit vectors. Then the distance from the origin to the point on the plane can be written as

$$d^2 = |K\mathbf{a} + \mu\mathbf{u} + \nu\mathbf{v}|^2 = K^2 + \mu^2 + \nu^2 + 2(\mu\mathbf{u} + \nu\mathbf{v}) \cdot \mathbf{a}.$$

Now, the first three terms are Pythagoras' theorem, so we know that the term in the brackets is always zero. For this to be true in general we have $\mathbf{u} \cdot \mathbf{a} = 0$ and also $\mathbf{b} \cdot \mathbf{a} = 0$.This tells us that the vector $\mathbf{a}$ which describes the plane is in fact a normal to it. So, given that K has been modified so that $\mathbf{a}$ is a unit normal to the plane, what vector from any point to the plane has the shortest distance to the plane? Well, given any vector **b** not perpendicular to **a**, then there is some λ such that

$$(\mathbf{b} + \lambda \mathbf{a}) \cdot \mathbf{a} = K$$

which immediately gives us $\lambda = K - \mathbf{b} \cdot \mathbf{a}$

Suppose we are give three vertices of a triangle which defines a plane, and a line segment. Does the line segment intersect with the plane of the triangle? The answer is yes if λ is between zero and one. Suppose we don't just want to know if the the line segment intersects the plane, but we want to know if and where it falls in some triangle. We know the point that the plane intersects the line, we can make two new triangles between the point and the vertices of the original triangle. That gives us two angles at the intersection point. If the angles add up to more than 180 degrees we are in the triangle.

If we want to know where the intersection point is in the terms of the given triangle, then we can set up any point in the triangle as $\mathbf{r} + \mu\mathbf{a} + \nu\mathbf{b} = \mathbf{r_0} + \lambda$. Here we mean that **r** is the position of the first vertex, and **a** and **b** are the vectors from the first vertex to second and the third vertex. Then $\mathbf{r_0}$ is a vector from the origin to one end of the line segment, and **u** is a vector from there to the other end of the line segment. This gives us immediately λ, μ, and ν from three simultaneous equations. (In fact we already have λ so the problem can be reduced to a pair of simultaneous equations.) So if λ is between zero and one, and both μ and ν are positive such that $\mu + \nu \leq 1$ the intersection point is in the triangle and μ and ν are the internal triangle coordinates of that point.

5 Afterword

This little piece on vectors is in no way a complete study of what is actually a large subject, but it might be of some use to someone who has just come across vectors and wants to have at least some idea of what they're about. Also I have now gone through this document once to correct errors, but there may be some mistakes that I have missed.