Matrices

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1 Introduction

The subject of matrices is very large indeed, and I will not pretend that I am "covering" the subject here, I shall just briefly introduce some notion of a matrix representing a transformation.

So, what is a Matrix? In the nineteenth century mathematicians were very interested in the subject of determinants. These were involved in the solutions of simultaneous equations and many other things. They regarded the arrays of numbers that they formed the determinants from as their mothers, and matrix is one Latin word for mother. So, lets say that a matrix is an array of numbers, and write the matrix A like so

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Here the *matrix elements* a_{ij} are just numbers. The subscript pairs ij tell you exactly where an element is in the array. So i, the first subscript, is the number of the row, and j, the second subscript tells you the column number. So, a_{12} is the number in row number 1 (counting top down) and column 2 (counting left to right). Matrices can have any number of rows and columns, but we shall stick to square matrices where the number of rows and columns are equal. In fact the matrices here will be just two by two or three by three rows and columns.

All this of course is no use to anyone at all, what makes matrices useful is that they can multiply vectors, the result of the multiplication being a vector. This is the multiplication rule.

$$A\mathbf{b} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_{11}b_x + a_{12}b_y + a_{13}b_z \\ a_{21}b_x + a_{22}b_y + a_{23}b_z \\ a_{31}b_x + a_{32}b_y + a_{33}b_z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \mathbf{c}.$$

That is the matrix A multiplies the vector $\mathbf{b}$ and the result is another vector $\mathbf{c}$. The pattern becomes even clearer if we write b_1, b_2, and b_3 for the x, y, and z components of $\mathbf{b}$.

Now, suppose you apply this matrix to a triangle. The vertices of the triangle are at position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The product of the matrix with each vector is a new vector, and that forms a new triangle. This new triangle may be not be the same shape as the original, it might be a rotated, scaled, stretched, and reflected version of the original. So, our matrix performs what is known as an *affine transformation*.

Now, you might think a vector is just defined by its components. This is not the case. The components of a vector depend on what coordinates are used. The vector itself is the same in all coordinate systems, it is an *invariant*. So while its components change with coordinate system, the vector itself means precisely the same no matter what coordinate system is.

So, there is a completely different viewpoint to a matrix transformation. Suppose, instead of using unit vectors $\mathbf{i}$ and $\mathbf{j}$ parallel to the x and y axes, I decide that there are some other basis vectors describing a new coordinate system. These might not be unit vectors, and might not even be perpendicular. From this point of view, I can construct a matrix that, when it multiplies vectors given in terms of components relating to one coordinate system, the resulting components are for the same vectors, but whose coordinates relate to the new coordinate system. Given three vectors for the vertices of a triangle, the actual triangle is unchanged, but the coordinates now relate to a different coordinate system.

2 A reflection matrix

Let's look at a very simple example. Suppose a matrix R is to perform a reflection about the y axis. In two dimensions the matrix looks like

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So in the active sense, this matrix will reflect a vector about the y axis. If three vectors form the positions of the vertices of the original triangle, then the new positions $\mathbf{a}' = R\mathbf{a}$, $\mathbf{b}' = R\mathbf{b}$, $\mathbf{c}' = R\mathbf{c}$, form a reflected triangle. We have used the matrix to transform a triangle into a new reflected triangle.

Now suppose that we have used the usual convention of the positive x axis pointing to the right, but someone else always uses the opposite convention. That is, our friend's x

axis always points left. What coordinates do we give our friend so that our friend ends up with exactly the same shaped triangle as we have? The answer is that we give our friend the coordinates for $\mathbf{a}'$, $\mathbf{b}'$, and $\mathbf{c}'$. If we gave our friend $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the result on the graph paper would be a reflected version of our triangle. However, now our friend will reflect the reflection, giving us the original.

So in this example, we have seen a matrix transformation in the active sense, where the triangle has become reflected; and in the passive sense, where the same triangle is described by a different coordinate system.

3 Matrix Multiplication, the Identity and the Inverse

What happens when you take a vector, transform it according to some matrix A, and then transform the resulting vector by another matrix B? That is we have

$$\mathbf{a}' = A\mathbf{a}$$

and then

$$\mathbf{a}'' = B\mathbf{a}'.$$

Clearly we can get to the vector $\mathbf{a}''$ in one step.

$$\mathbf{a}'' = C\mathbf{a}.$$

The matrix C is called the matrix product of A and B. To find the components of C in terms of the components of A and B we need a rule for matrix *multiplication*. This is written as $C = AB$. (A times symbol can be used, but it is mostly dropped.) This we shall come across shortly. Note that just as when a vector is multiplied by a number, every component of the vector is multiplied by that number. The same thing goes for matrices.

Now, a matrix does some kind of transformation on a vector. Given the matrix and just the *transformed* vector, can we get back to the original vector? The answer is yes, *if* the matrix has an inverse. This is a matrix A^{-1} such that $A^{-1}A$ is the identity matrix I. For a two by two matrix, the identity I has the components

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and for a three by three matrix it is

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that a vector is left unchanged when it is multiplied by the identity matrix. Now, we mentioned that we need to be able to multiply matrices together. How is that done? If you know how to multiply a vector by a matrix, you know how to multiply two matrices already. In the product AB, simply look on B as consisting of three column vectors (or two if A and B are two by two matrices). Then the resulting matrix is three column vectors, consisting of A times the first one on the left in B, then A times the middle column vector of B, and lastly A times the last one. The rule is the same in two by two matrices, or indeed for any square matrix, no matter how large. To be specific, if $C = AB$ then the components of C are

$$C = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

in the case of a two by two matrix. You should satisfy yourself that in general AB is not the same as BA, that is, *matrix multiplication does not commute.*

Now, the problem is, *does the inverse of a matrix always exist*? Obviously for a reflection it exists, it's just another reflection. Check it out that $R \times R = I$ in the above. Similarly for a simple rotation, a rotation in the opposite sense is the inverse. But look at this matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Clearly

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x + y \end{pmatrix}.$$

Every point in the plane is mapped onto a line at 45 degrees to the x axis. Look at the point (5,5), every point on the line $x + y = 5$ gets transformed to just this one point. After the transformation there is absolutely no indication of where the original point (x, y) was. This matrix has no inverse, as you cannot get back to the original position vector. Note that the matrix here with each element being one, is *not* a unit matrix. A unit matrix is just another phrase for the identity.

We shall now state, with no further discussion on the matter, what the conditions are for the inverse of a matrix to exist, and given that it exists, how to calculate it. However before doing so, I shall introduce what the transpose of a matrix is. It is written A^T, and given the components of A, then

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Clearly, the rows and the columns in the transpose have been swapped. That is, the components of A^T are a_{ji} rather than a_{ij}.

The condition that a two by two matrix has an inverse is that there is a number D called its *determinant*, and that this number is *not zero.* That is

$$D = a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

In the example of a matrix with no inverse that was discussed above, clearly $D = 0$. For a three by three matrix,

$$D = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \neq 0,$$

is the condition for an inverse to exist. Again, D is the determinant of the three by three matrix.

The inverse of a two by two matrix is given by

$$A^{-1} = \frac{1}{D}\begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}^T .$$

Again, the meaning of the T is to take the transpose of this matrix. For a three by three matrix

$$A^{-1} = \frac{1}{D}\begin{pmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{21}a_{33} - a_{23}a_{31}) & (a_{21}a_{32} - a_{22}a_{31}) \\ -(a_{12}a_{33} - a_{13}a_{32}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) \\ (a_{12}a_{23} - a_{13}a_{22}) & -(a_{11}a_{23} - a_{13}a_{21}) & (a_{11}a_{22} - a_{12}a_{21}) \end{pmatrix}^T .$$

So the inverse of a three by three matrix is pretty much a job for a computer! It can be done by hand, but it would be tiresome.

What is the recipe here? Suppose you want to calculate a component that goes in the ith row and jth column in the matrix under the transpose sign. Go to the original matrix, cover up that row and column and see what you have left. For a two by two matrix that will be a number. So for instance you cover up row 1 and column 1, all that is left is a_{22}. That is what goes in row one and column one. For the three by three matrix what is left is a two by two matrix. What goes in the component is the determinant of that two by two matrix. After this, there is a chequer board of pluses and minuses to add, and the transpose to take. Of course, after that, everything is divided by D. So if D is not zero, we are not in trouble.

4 Rotations

Suppose we have a vector in the x, y plane, and we want to rotate it clockwise through an angle θ. I am sure you can satisfy yourself that the rotated vector is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Now, draw a new set of axes, sharing the same origin. The new x and y axes are to be the old ones, but rotated *anti-clockwise* through an angle theta. Now, look at the coordinates of the old unrotated vector in the case above. You should be able to find that its coordinates in the new rotated system are given by just the same matrix multiplication as above. Transformations like reflections and rotations always have an inverse. The inverse in this case is to rotate again by $-\theta$. Remember that $\cos(-\theta) = \cos(\theta)$ and that $\sin(-\theta) = -\sin(\theta)$.

What about rotations in three dimensions. Well, our rotation above might be considered as a rotation about the z axis. So

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Similarly, a rotation through an angle ϕ about the y axis would result from

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and a rotation about the x axis is

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Now, Euler's theorem for rotations states that *any* three dimensional rotation can be done by three rotations about the axes in this manner. The following convention is often used. The first rotation is through ϕ about the z axis. The next is an angle θ about the new x axis, and the third is an angle ψ about the new z axis. Yes, that is two about the z axis, but that changed on the second rotation. These three angles are called the Euler angles. Note there are other conventions, so always check the definitions wherever you come across them.

There is an awful lot to learn out there about this subject of matrices, the purpose of this little document is just to introduce the idea of matrices as both active transformations where the objects are transformed, and as passive transformations where the objects remain the same, but the coordinate system that describes the object is changed.