Topology on a Sphere Using a Truncated Icosahedron

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Contents

1 Introduction

We shall start off with a "template" pentagon, then introduce the nodes and faces of a dodecahedron. (There is an article in "MathSchool" here discussing the construction of the the dodecahedron discussed here. From this dodecahedron, we construct an icosahedron, and form the "bucky-ball" or truncated icosahedron from that. This in turn we decompose into triangles.

We go into some detail as to node numbers, face numbers, edge numbers, and the connections between the faces of the polyhedra. The main purpose of this article is to accompany a computer code, explaining all the details on nodes, faces, edges and connections between faces.

2 A Template Pentagon

We start off with a template pentagon. For the actual positions of the vertices in terms of Golden Ratios, see "MathSchool" or just use very basic trig.

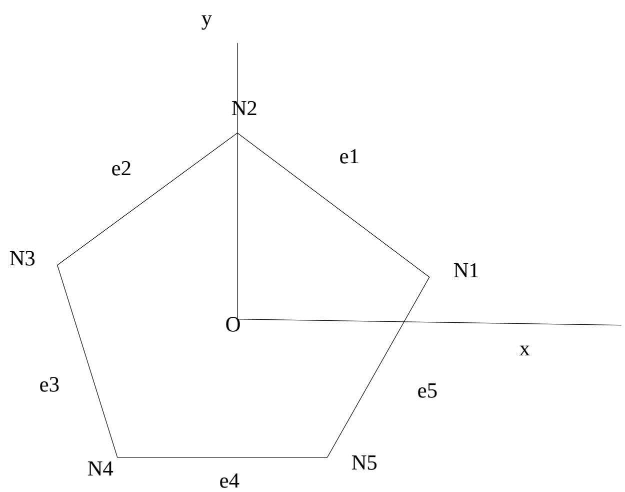

Figure 1: We envisage a template pentagon centred at the origin in the (x, y) plane. Any pentagon shall use this convention for node and edge numbering. Note the normal (edge 1 vector cross edge 2 vector) points in the positive z direction. So if this is used as a "base" for a dodecahedron, the normal is an *inward normal.*

We shall add a pentagon to each edge of a "base" pentagon in the (x, y) plane and "fold them up" as discussed in "MathSchool". This shall give us the "bottom half" of a dodecahedron. We shall insist all out normals be inward pointing. (It makes no real difference as to whether the normals are inward or outward, however they must either be *all* inward or *all* outward. When it comes to the upper half we must have a mirror image template pentagon as shown below.

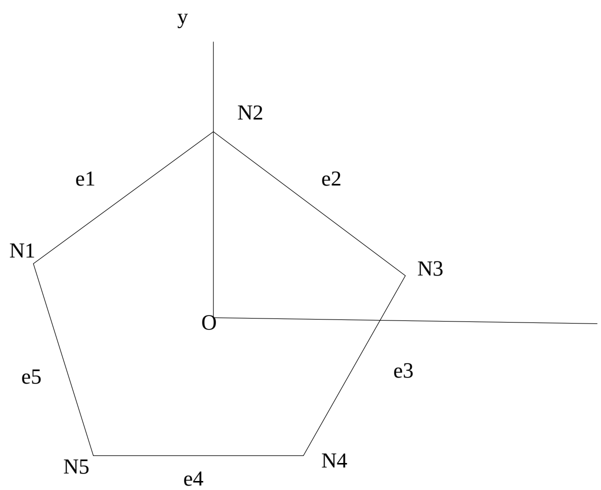

Figure 2: A mirror image template for the "upper half" of a dodecahedron.

3 The Dodecahedron

So, we start with a base pentagon (face A), We "glue" identical pentagons to the edges, and "fold them up" so that there are no gaps between the outer ring pentagons. This gives us a "lower half" for a dodecahedron. We have six faces and fifteen nodes numbered as below.

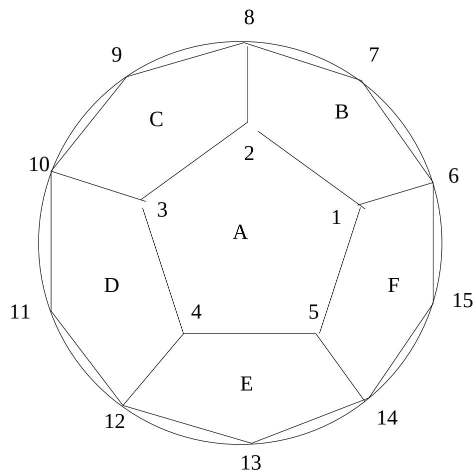

Figure 3: Node numbering for pentagons A, B, C, D, E and F. Here we shall use the first template in Fig.1.

Next we swap to the mirror template, and rotate it through 180∘. We glue on five extra pentagons round the edges and fold them down. This forms the upper half of the dodecahedron.

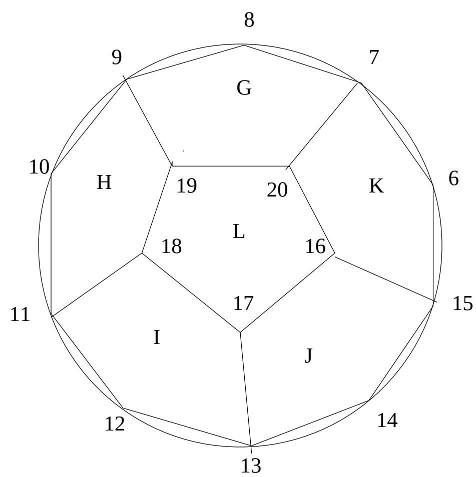

Figure 4: Node numbering for pentagons G, H, I, J, K and L. Here we use the second "mirror template" in Fig.2.

First, we write down the nodes of the first six pentagons labelled A to F which have the same convention as in Fig.1.

Pentagon	Node 1	Node 2	Node 3	Node 4	Node 5
A	1	2	3	4	5
B	6	7	8	2	1
C	8	9	10	3	2
D	10	11	12	4	3
E	12	13	14	5	4
F	13	15	6	1	5

The remaining pentagons G to L are labelled using the mirror image of Fig.2. Then the cross product $\mathbf{e}_1 \times \mathbf{e}_2$ is always an *inward* normal. Just reverse for outward normals. Here

e stands for an edge vector, and the subscript is just an edge number. Here are the nodes of the upper half of the dodecahedron.

Pentagon	Node 1	Node 2	Node 3	Node 4	Node 5
G	9	8	7	20	19
H	11	10	9	19	18
I	13	12	11	18	17
J	15	14	13	17	16
K	7	6	15	16	20
L	16	17	18	19	20

Naturally enough, edges 1 to 5 go from nodes 1 to 2, 2 to 3, 3 to 4, 4 to 5, and 5 to 1 in the template pentagon. We shall tabulate which pentagons are connected to edges 1 to 5 for each pentagon. We shall use subscripts. The subscript number matches the edge number in the neighbouring pentagon. For instance for pentagon A, the first entry is B_4. This means that edge 1 of A is connected to pentagon B. The subscript 4 means that this edge in A corresponds to edge 4 in B.Note that nodes are swapped in the neighbouring edge. Edge 1 of A connects to edge 4 of B, but edge 1 in A is $(node1, node2)$ while edge 4 in B is $(node2node1)$.

Pentagon	Edge 1	Edge 2	Edge 3	Edge 4	Edge 5
A	B_4	C_4	D_4	E_4	F_4
B	K_1	G_2	C_5	A_1	F_3
C	G_1	H_2	D_5	A_2	B_3
D	H_1	I_2	E_5	A_3	C_3
E	I_1	J_2	F_5	A_4	D_3
F	J_1	K_2	B_5	A_5	E_3
G	C_1	B_2	K_5	L_4	H_3
H	D_1	C_2	G_5	L_3	I_3
I	E_1	D_2	H_5	L_2	J_3
J	F_1	E_2	I_5	L_1	K_3
K	B_1	F_2	J_5	L_5	G_3
L	K_4	J_4	I_4	H_4	G_4

We could go on to split the pentagons into triangles, but instead we shall move on to the icosahedron.

4 The Icosahedron

The Icosahedron is the *Platonic Dual* of the dodecahedron. Instead of 20 vertices and 12 faces there are 12 vertices and 20 faces, each of which is an equilateral triangle. The vertices are at the centres of each pentagon of the dodecahedron. In this section, the letters A to L shall represent the position vectors of the nodes at the centres of pentagons A to L. It should be apparent from the context whether A is pentagon or a node. Again, a triangle has nodes $(N1, N2, N3)$ going and edges $(N1, N2), (N2, N3), (N3, N1)$. the cross product $edge1 \times edge2$ shall be an *inward* pointing normal. Without further ado we can tabulate the nodes of the 20 triangles T_1 through to T_20. (See Figs.3 and 4.)

Triangle	Node 1	Node 2	Node 3
T1	A	B	C
T2	A	C	D
T3	A	D	E
T4	A	E	F
T5	A	F	B
T6	B	G	C
T7	C	H	D
T8	D	I	E
T9	E	J	F
T10	F	K	B
T11	B	G	K
T12	C	H	G
T13	D	I	H
T14	E	J	I
T15	F	K	J
T16	L	J	I
T17	L	K	J
T18	L	G	K
T19	L	H	G
T20	L	I	H

We shall also write down how the triangles are connected. In the following, an entry such as $T_8(3)$ means edge 3 of triangle 8. Any minus sign indicates that the nodes of the are swapped. So, in the table below, edge 1 of triangle one connects to edge 2 of the triangle 5, but the nodes are the opposite way round. Here is the full list.

Triangle	Edge 1	Edge 2	Edge 43
T1	$-T5(3)$	$-T6(3)$	$-T2(1)$
T2	$-T1(3)$	$-T7(3)$	$-T3(1)$
T3	$-T2(3)$	$-T8(3)$	$-T4(1)$
T4	$-T3(3)$	$-T9(3)$	$-T5(1)$
T5	$-T4(3)$	$-T10(3)$	$-T1(1)$
T6	$T11(1)$	$T12(3)$	$-T1(2)$
T7	$T12(1)$	$T13(3)$	$-T2(2)$
T8	$T13(1)$	$T14(3)$	$-T3(2)$
T9	$T14(1)$	$T15(3)$	$-T4(2)$
T10	$T15(1)$	$T11(3)$	$-T5(2)$
T11	$T6(1)$	$T18(2)$	$T10(2)$
T12	$T7(1)$	$T19(2)$	$T6(2)$
T13	$T8(1)$	$T20(2)$	$T7(2)$
T14	$T9(1)$	$T16(2)$	$T8(2)$
T15	$T10(1)$	$T17(2)$	$T9(2)$
T16	$-T17(3)$	$T14(2)$	$-T20(1)$
T17	$-T18(3)$	$T15(2)$	$-T16(1)$
T18	$-T19(3)$	$T11(2)$	$-T17(1)$
T19	$-T20(3)$	$T12(2)$	$-T18(1)$
T20	$-T16(3)$	$T13(2)$	$-T19(1)$

5 Buckminster!

Now, we shall want to construct a truncated icosahedron. There are five "spokes" at each node. If we go to any node, and then mark off one third of the length of the spokes from that node, we have the nodes of small pentagon. For instance, node A in the icosahedron is at the centre of pentagon A in the original dodecahedron. The spokes radiating from node A radiate to nodes B through to F. If we mark off these one third lengths, we have the nodes of a pentagon which looks like the template pentagon rotated through 180^circ but smaller.

We shall abuse notation a little here for the sake of brevity. By pentagon A' = nodes B, C, D, E F we mean that node 1 is $\overrightarrow{A}$ $+1/3$ $\overrightarrow{AB}$. That is, the prime denotes the centre, and the letter denotes the node toward which the spoke is pointing to. The divisions by 3 shall still be there, but they shall be implicit. The 60 nodes of these twelve smaller pentagons are *all* of the nodes of the truncated icosahedron. This happens to be the structure of the $C60$ molecule, which is also known as Buckminsterfullerene after the architect Buckminster who utilised the such structures in geodesic domes. These are 20 hexagons sharing these nodes. Each pentagon shares edges with five hexagons, and each hexagon shares nodes with three pentagons and three other hexagons. The twelve pentagons, using the notation just described are given next.

Pentagon	Node 1	Node 2	Node 3	Node 4	Node 5
A′	B	C	D	E	F
B′	A	F	K	G	C
C′	A	B	G	H	D
D′	A	C	H	I	E
E′	A	D	I	J	F
F′	A	E	J	K	B
G′	L	H	C	B	K
H′	L	I	D	C	G
I′	L	J	E	D	H
J′	L	K	F	E	I
K′	L	G	B	F	J
L′	K	J	I	H	G

We use the node listings above to define edges 1 to 5 of each pentagon in the truncated icosahedron. That is edge 1 consists of nodes 1 and 2, edge 2 consists of nodes 2 and 3, up until edge 5 which consists of nodes 5 and 1. We go back to the lower half of the dodecahedron in Fig.3. Fig.4 below shows a rough sketch of pentagons A, B and C. Within each of these is a smaller pentagon which is part of the truncated icosahedron, as marked by the prime. Triangle T_1 is shown. The three primed pentagons "shave off" the edges of T_1 to form the hexagon shown in the thicker line. Each triangle in the icosahedron is now a hexagon in the truncated icosahedron. We will want to list which edge of which pentagon corresponds to which edge of which hexagon.

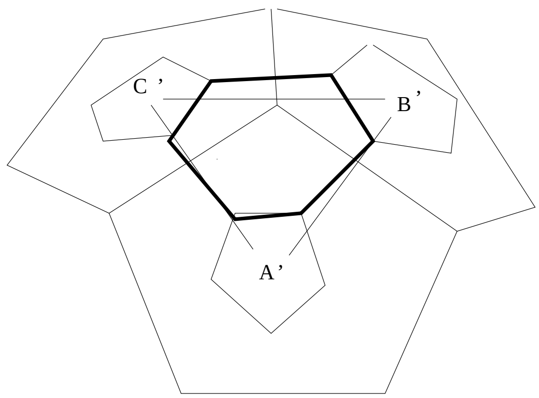

Figure 5: Triangle T_1 is formed by the centres of pentagons A, B, and C. These spokes between the pentagon centres define the pentagons A', B', and C'. One edge of each of these pentagons define the hexagon in T_1.

It is natural to start off by defining the hexagons in terms of the pentagons listed above.We proceed by defining a hexagon in terms of three pentagon edges. (A different pentagon for each edges. We shall denote these as hexagon edges 1, 3, and 5. The edges numbered 2, 4, and 6 follow automatically. The nodes and edges of a hexagon in some triangle X, Y, Z are depicted in Fig.6.

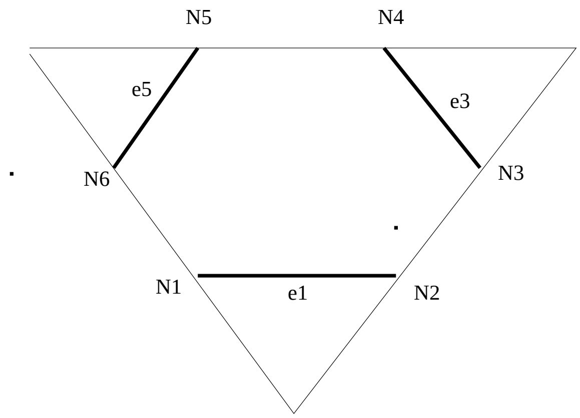

Figure 6: The hexagon nodes and edges in a triangle with nodes (X, Y, Z) are defined.

In this way, we define the twenty hexagons below. To make the notation clear, the first line means that edge 1 of hexagon 1 is an edge in pentagon A' In $A^p rime$ it is consists of the node pair (C, B) which is edge 1, but in reverse order, so it is given a minus sign. We shall split the tables up into four groups of five

Hexagon	edge	Pentagon	Node 1	Node 2	edge
H1	e1	A'	C	B	-e1
	e3	B'	A	C	-e5
	e5	C'	A	B	+e1
H2	e1	A'	D	C	-e2
	e3	C'	A	D	-e5
	e5	D'	A	C	+e1
H3	e1	A'	E	D	-e3
	e3	D'	A	E	-e5
	e5	E'	A	D	+e1
H4	e1	A'	F	E	-e4
	e3	E'	A	F	-e5
	e5	F'	A	E	+e1
H5	e1	A'	B	F	-e5
	e3	F'	A	B	-e5
	e5	B'	A	F	+e1

Hexagon	edge	Pentagon	Node 1	Node 2	edge
H6	e1	B'	C	G	-e4
	e3	G'	B	C	-e3
	e5	C'	B	G	+e2
H7	e1	C'	D	H	-e4
	e3	H'	C	D	-e3
	e5	D'	C	H	+e2
H8	e1	D'	E	I	-e4
	e3	I'	D	E	-e3
	e5	E'	D	I	+e2
H9	e1	E'	F	J	-e4
	e3	J'	E	F	-e3
	e5	F'	E	J	+e2
H10	e1	F'	B	K	-e4
	e3	K'	F	B	-e3
	e5	B'	F	K	+e2

Hexagon	edge	Pentagon	Node 1	Node 2	edge
H11	e1	B'	G	B	-e2
	e3	G'	K	B	-e4
	e5	K'	G	K	-e3
H12	e1	C'	C	H	-e2
	e3	H'	G	C	-e4
	e5	G'	H	G	-e3
H13	e1	D'	D	I	-e2
	e3	I'	H	D	-e4
	e5	H'	I	H	-e3
H14	e1	E'	E	J	-e2
	e3	J'	I	E	-e4
	e5	I'	J	I	-e2
H15	e1	F'	F	K	-e2
	e3	K'	J	F	-e4
	e5	J'	K	J	-e3

Hexagon	edge	Pentagon	Node 1	Node 2	edge
H16	e1	L'	I	J	+e3
	e3	J'	L	I	-e5
	e5	I'	L	J	+e1
H17	e1	L'	J	K	+e4
	e3	K'	L	J	-e5
	e5	J'	L	K	+e1
H18	e1	L'	K	G	+e5
	e3	G'	L	K	-e5
	e5	K'	L	G	+e1
H19	e1	L'	G	H	+e1
	e3	H'	L	G	-e5
	e5	G'	L	H	+e1
H20	e1	L'	H	I	+e2
	e3	I'	L	H	-e5
	e5	H'	L	I	+e1

We can write down the inverse table, which tells us which hexagon lies across any edge of any pentagon, and what the corresponding edge number and edge orientation is in the hexagon.

The question as to how the connectivity between the hexagons has already been done. The connections are the same as for the triangles in the icosahedron section, except that triangle edges 1, 2, and 3, are now hexagon edges 2, 4, and 5. So we have the nodes of the truncated icosahedron, the nodes of each pentagon and connection, how the pentagons and hexagons fit to each other, and how the hexagons fit with each other.