Quadratic Equations

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1 Introduction

Many secondary school students find the humble quadratic a real pig of an equation. (Sorry pigs!) However, they really aren't that bad, honest. The graph, $y = ax^2 + bx + c$ is an example of a conic section. (In this article, each of the letters a, b, c, d just denote some fixed number.) Finding where it crosses the x axis ($y = 0$) gives a quadratic equation.

2 Conic Sections

Look at this (roughly drawn) sketch of the quadratic equation $y = ax^2 + bx + c$.

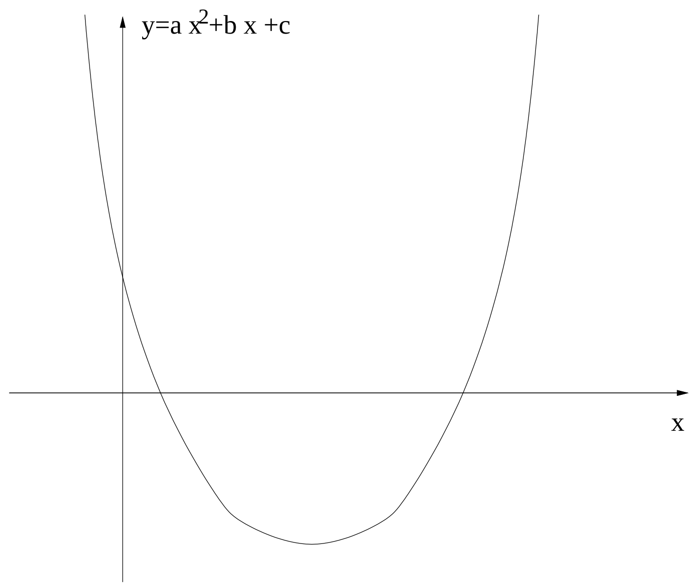

Figure 1: A sketch of a parabola.

The parabola is an example of a *conic section.* If you take an ice cream cone, you see a circular end and a pointed end. You can imagine that it's been sliced off a bigger cone. I shall imagine this bigger cone to be placed on a table, and the smaller cone to have been made by taking a slice in a plane parallel to the table's surface.

If the slice hadn't been quite parallel, the shape wouldn't be a circle, it would be a closed curve like a circle, but would be elongated. This curve is the *ellipse.* If you cut the cone straight down through the point, you get a triangle, but if you cut straight down away from the point, you get a curve called an *hyperbola.* You notice, especially if you are cutting down quite close to the point, that the curve looks close to a pair of straight lines apart from the curvy bit at the top. This is the hallmark of any hyperbola. It gets closer and closer to a pair of straight lines.

If you don't cut quite straight down, you still get an hyperbola. The thing to do now is forget the table, and imagine that the cone carries on forever down from the point. (Technically, in the mathematical cone, there is a mirror "upside down cone" going up from the point as well).

So, if you cut a cone straight across, you get a circle. As you cut less and less parallel to the table, you always get a more elongated ellipse each time. But if you cut at a certain angle, the ellipse won't close at all. However, this curve isn't an hyperbola either. You get a curve that doesn't look more and more like a pair of straight lines as you get away from the start of the cut. If you cut at a slightly steeper angle again, you get the hyperbola.

There is a critical cutting angle where you get a curve that isn't a circle, or an ellipse, or an hyperbola. This unique curve is the *parabola*. The graph $y = ax^2 + bx + c$ is always a parabola.

3 Quadratic Equations: Completing the Square

Changing the number b in the equation $y = ax^2 + bx + c$ will shift the curve left or right, and at the same time move the minimum (or minimum) down or up. This depends whether the change is positive or negative. If if a is positive, the curve has a minimum, if a is negative the curve has a minimum. Try plotting a few graphs with only b differing. If your good at sketching graphs this is easy to see. Changing just the number c just gives an up or down shift. Changing a makes the parabola "sharper" or "blunter" if the reader will permit it.

Now, suppose we need to find at what two values of x (if there are two values) that y is zero. This kind of problem crops up all over the place, you might find it in a problem about accelerating cars, and even the shape of energy bands in crystals.

Say we need to find the roots of $3x^2 + 4x - 38 = 0$. We have three numbers here, 2, 4, and 38. We can reduce this by dividing both sides by 2. So we get $x^2 + 2x - 19 = 0$. So now we have only two numbers to deal with. (You would be correct in objecting that there is an invisible one times in front of the x^2). This division by the coefficient of x^2 *is always the first step*, we always want it in the form so that the coefficient of x^2 is one.a Of course, we can't do this if $a = 0$, but then we don't really have a quadratic in the first place. Also, if $c = 0$ we have $x(ax + b) = 0$ so we don't need to solve a quadratic. In all the following we assume that the coefficient of the highest power of x is not zero, nor is the constant term zero.

Next we add 38 to both sides. Why? We shall see shortly. If we do we now have

$x^2 + 2x = 19$. Now look at Fig.2.

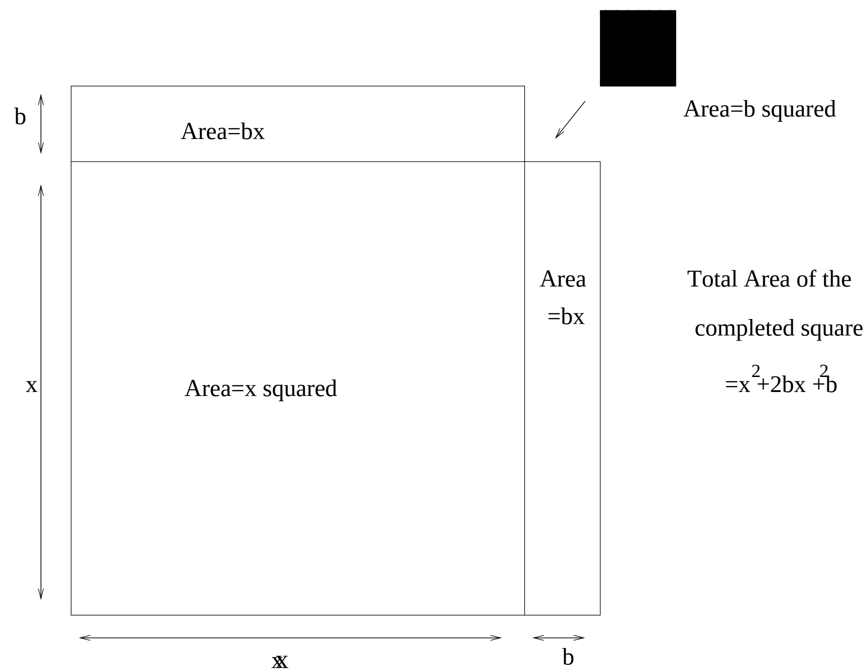

Figure 2: Adding the black square "completes the bigger white square.

In Fig.2 we have a blank square, with two borders of width s. The area of the blank shape is $x \times x + b \times x + b \times x$, or $x^2 + 2bx$. The blank shape is certainly not a square, but we can make it into a square by adding the small black square. That is, if we have $x^2 + 2 \times \textit{something} \times x$, it isn't easily represented by a square, but if we add the little square of side *something* we have made it into a *perfect square*. The side of the perfect square in Fig.2 is $x + a$.

Now back to our example, $x^2 + 2x = 19$ is where we had got to so far. In this case our $2 \times \textit{something} = 2$, so that the $\textit{something} = 1$. Now we add that $\textit{something}^2$ $(1^2 = 1)$ to the left hand side. In order to keep the equals sign balanced, we must add one to the right hand side as well.

So now we have $x^2 + 2x + 1 = 20$. The left hand side is a perfect square of side $x + 1$ so we can write this as $(x + 1)^2 = 20$. Now we can take the square root of both sides. $x + 1 = \pm\sqrt{20}$. Remember that $(-1) \times (-1) = 1 \times 1 = 1$, so that there are always two square roots, one positive, one negative. Each one will represent a different solution to $y = 0$. The $\pm$ sign means "plus *or* minus. So, now we can subtract 1 from both sides of $x + 1 = \pm\sqrt{20}$ to get two solutions. One is $x = -1 + \sqrt{20}$ and the other is $x = -1 - \sqrt{20}$.

It turns out that if the coefficient of x is negative the same thing applies. This is because we have $(x - b)^2 = x^2 - 2bx + b^2)$ as well as $(x + b)^2 = x^2 + 2bx + b^2$, so we can get a perfect square either way.

Now, this method works for *any* quadratic, so what if we drop the specific numbers and look at the *general* quadratic.

$$y = ax^2 + bx + c = 0. \tag{1}$$

As in the example we want the coefficient of x to be unity (or one if you prefer). Then we have to divide through by $a \neq 0$, so

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{2}$$

As before we get rid of the constant term on the left so that

$$x^2 + \frac{b}{a}x = -\frac{c}{a}. \tag{3}$$

Now, we see that the thing that represents the area of the black square in Fig.2 is in fact $\left(\frac{b}{2a}\right)^2$. (This is because $2 \times b/2a = b/a$.) So we add this to both sides, and the left hand side is now a complete square. That is

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2. \tag{4}$$

Check it, of you are in any doubt. Now we can write

$$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}}. \tag{5}$$

At this point, I assume the reader's algebra is good enough at least to verify that this is the same as

$$\left(x + \frac{b}{2a}\right) = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}. \tag{6}$$

So, after a little rearrangement

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

Again, I leave at least some algebra for the reader to figure out. If it's trivial, then fine, if the reader finds it tough, then it's good exercise, and if the reader finds it too tough, then it has to be "back to algebra revision" or indeed further reading and practise on algebra.

4 A More Geometrical Approach

There is another way of looking at the problem of solving the quadratic equation. If you are given the quadratic $x^2 - 10 = 0$, you know straight off that the solution is $x = \pm\sqrt{10}$. If you sketch the curve $y = x^2 - 10$ it is obvious that it is symmetrical about the y axis. There is no x term, or if you like, the coefficient of the x term is zero.

Suppose the coefficient of x isn't zero, then the curve is not symmetric about the y axis. But, suppose we draw a new axis, parallel to the y axis, which the curve *is* symmetric about. This has an origin O' to the left or right of the origin O of the original axes. The new axis shall be called y'. This is depicted in Fig.3.

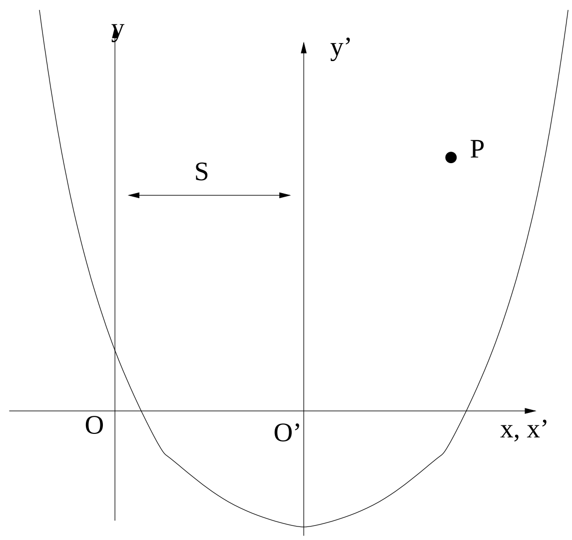

Figure 3: The point P can be described by two numbers (x, y) in the Oxy coordinates or two numbers $(O'x'y')$ in the $O'x'y'$ coordinates. Whatever the value of S, the value of x' is $x - S$.

We can see the recipe from jumping from one set of coordinates to another with the y axis shifted by some value S. The y value of any point is the same in both coordinates. All that happens is that the new "x" (which are calling x') is just the old x value minus S. That is to say $x' = x - S$ or $x = x' + S$.

Then, if we have

$$ax^2 + bx + c = 0, \tag{8}$$

we can shift the coordinates so that

$$a(x' + S)^2 + b(x' + S) + c = 0. \tag{9}$$

(Note that S can be negative if we choose, it just changes the direction of the shift.) If we expand the brackets we have

$$ax'^2 + 2aSx' + aS^2 + bx + bS + c = 0$$

or

$$ax'^2 + (b + 2aS)x' + c_bS + ad^2. \tag{10}$$

Now, we can choose the value of S so that

$$(b + 2aS) = 0. \tag{11}$$

So the equation now looks like like $ax'^2 + N = 0$, or $x'^2 + N/a = 0$, and we can write down the answer straight off. Note, that this gives the solution for x', not x. We have to shift the two results back to find the x solution. The shift is the familiar

$$S = -\frac{b}{2a}. \tag{12}$$

This isn't how you would normally solve the quadratic, but the "trick" of shifting the coordinates to ger rid of the term that is linear in x is a neat one.