Model:

$$Z_{ij} = a + b_A S_{ij}^A + b_B S_{ij}^B + d_{AB} U_{ij} + e_{ij} \tag{1}$$

Model:

$$Y_i = X_i \gamma + H_i^A \beta^A + H_i^B \beta^B + T_i \alpha + e_i \tag{2}$$

where $X_i = (X_{i1}, X_{i2}, \dots, X_{i,n})$ is the covariant, $H_i^A = (H_{i1}^A, H_{i2}^A, \dots, H_{i,l_A}^A)$ is the haplotype for Gene A and $H_i^B = (H_{i1}^B, H_{i2}^B, \dots, H_{i,l_B}^B)$ is the haplotype for Gene B, $\beta^A = (\beta_1^A, \beta_2^A, \dots, \beta_{l_A}^A)^T$ follows $MN(0, \tau_A R_A)$, $\beta^B = (\beta_1^B, \beta_2^B, \dots, \beta_{l_B}^B)^T$ follows $MN(0, \tau_B R_B)$, $T_i = (H_{i1}^A H_{i1}^B, \dots, H_{i1}^A H_{i,l_B}^B, H_{i2}^A H_{i1}^B, \dots, H_{i,l_A}^A H_{i,l_B}^B)$ records the interaction terms for the Gene A and Gene B. $\alpha = (\alpha_1, \dots, \alpha_{l_A, l_B})^T$ follows $MN(0, \phi Q)$

Re-parameter:

Let $H_i = (H_i^A, H_i^B, T_i)$, $\beta = (\beta^A, \beta^B, \alpha)^T$, so (1) can be rewrite as following model:

$$Y_i = X_i \gamma + H_i \beta + e_i \tag{3}$$

Let$R_\beta = \begin{pmatrix} \tau_A R_A & 0 & 0 \\ 0 & \tau_B R_B & 0 \\ 0 & 0 & \phi Q \end{pmatrix}$, so β follows $MN(0, R_\beta)$.

The rest derivation is almost the same with the model for just one gene:

Let $u_i = E(Y_i | \beta, X_i, H_i)$, so $g(u_i) = X_i \gamma + H_i \beta$. Define $u_i^0 = g^{-1}(X_i \gamma)$, we have

$$\begin{aligned} E(Y_i | X_i, H_i) &= E[E(Y_i | \beta, X_i, H_i)] \\ &= E(u_i) \\ &\approx E u_i^0 + [g'(u_i^0)]^{-1} H_i \beta \\ &= u_i^0 \end{aligned} \tag{4}$$

For two individuals i and j, we have

$$\begin{aligned} cov(Y_i, Y_j | X, H) &= E[Y_i - E(Y_i)][Y_j - E(Y_j)] \\ &\approx E[(Y_i - u_i^0)(Y_j - u_j^0)] \end{aligned} \tag{5}$$

The connection between the VC model and the similarity model is the definition of Z_{ij} which is

$$Z_{ij} = w_i (Y_i - u_i^0) w_j (Y_j - u_j^0)$$

$$\begin{aligned} E(Z_{ij} | X, H) &= E w_i (Y_i - u_i^0) w_j (Y_j - u_j^0) \\ &= w_i w_j E[(Y_i - u_i^0)(Y_j - u_j^0)] \\ &\approx cov(Y_i, Y_j | X, H) \end{aligned} \tag{6}$$

using the law of covariance, we have

$$cov(Y_i, Y_j | X, H) = E[cov(Y_i, Y_j | \beta, X, H)] + cov[E(Y_i | \beta, X, H), E(Y_j | \beta, X, H)] \tag{7}$$

Conditional on β,$cov(Y_i, Y_j | \beta, X, H)$ equals to 0, so we mainly focus on the second part of formula (7).

$$\begin{aligned}
cov[E(Y_i|\beta,X,H),E(Y_j|\beta,X,H)] &\approx cov[u_i^0+[g'(u_i^0)]^{-1}H_i\beta, u_j^0+[g'(u_j^0)]^{-1}H_j\beta] \\
&= cov[g'(u_i^0)]^{-1}H_i\beta,[g'(u_j^0)]^{-1}H_j\beta \\
&= [g'(u_i^0)g'(u_j^0)]^{-1}H_iR_\beta H_j^T
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
H_iR_\beta H_j^T &= H_i^AR_A(H_j^A)^T+H_i^BR_B(H_j^B)^T+T_iQT_j^T \\
&= \tau_A\sum_{h_A,k_A}H_{i,h_A}H_{j,k_A}r_{h_A,k_A}+\tau_B\sum_{h_B,k_B}H_{i,h_B}H_{j,k_B}r_{h_B,k_B}+\phi\sum_{h_A,k_A,h_B,k_B}H_{i,h_A}H_{i,h_B}H_{j,k_A}H_{j,k_B}r_{h_A,k_A}r_{h_B,k_B}
\end{aligned} \tag{9}$$

so if the $l_Al_B\times l_Al_B$ matrix Q is

$$Q=\begin{pmatrix}
r_{11}^Ar_{11}^B & \dots & r_{11}^Ar_{1l_B}^B & \dots & r_{1l_A}^Ar_{1l_B}^B \\
\vdots & \vdots & \vdots & \vdots & \\
r_{l_A1}^Ar_{11}^B & \dots & r_{l_A1}^Ar_{1l_B}^B & \dots & r_{l_Al_A}^Ar_{1l_B}^B \\
\vdots & \vdots & \vdots & \vdots & \\
r_{l_A1}^Ar_{l_B1}^B & \dots & r_{l_A1}^Ar_{l_Bl_B}^B & \dots & r_{l_Al_A}^Ar_{l_Bl_B}^B
\end{pmatrix}$$

then we choose $w_i=g'(u_i^0)$ then the two model are the same and ϕ equals to d_{AB}.